STABILITY AND CONVERGENCE OF HIGH ORDER NUMERICAL METHODS FOR NONLINEAR HYPERBOLIC CONSERVATION LAWS

A Thesis

by

ORHAN MEHMETOGLU

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2012

Major Subject: Mathematics

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ABSTRACT

Stability and Convergence of High Order Numerical Methods for Nonlinear Hyperbolic Conservation Laws. (August 2012)

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Chair of Advisory Committee: Dr. Bojan Popov

Recently there have been numerous advances in the development of numerical algorithms to solve conservation laws. Even though the analytical theory (existence–uniqueness) is complete in the case of scalar conservation laws, there are many numerically robust methods for which the question of convergence and error estimates are still open. Usually high order schemes are constructed to be Total Variation Diminishing (TVD) which only guarantees convergence of such schemes to a weak solution. The standard approach in proving convergence to the entropy solution is to try to establish cell entropy inequalities. However, this typically requires additional non-homogeneous limitations on the numerical method, which reduces the modified scheme to first order when the mesh is refined. There are only a few results on the convergence which do not impose such limitations and all of them assume some smoothness on the initial data in addition to L^∞ bound.

The Nessyahu-Tadmor (NT) scheme is a typical example of a high order scheme. It is a simple yet robust second order non-oscillatory scheme, which relies on a non-linear piecewise linear reconstruction. A standard reconstruction choice is based on the so-called minmod limiter which gives a maximum principle for the scheme. Unfortunately, this limiter reduces the reconstruction to first order at local extrema. Numerical evidence suggests that this limitation is not necessary. By using MAPR-like limiters, one can allow local nonlinear reconstructions which do not reduce to first order at local extrema. However, use of such limiters requires a new approach when trying to prove a maximum principle for the scheme. It is also well known that the NT scheme does not satisfy the so-called strict cell entropy inequalities, which is the main difficulty in proving convergence to the entropy solution.

In this work, the NT scheme with MAPR-like limiters is considered. A maximum principle result for a conservation law with any Lipschitz flux and also with any k-monotone flux is proven. Using this result it is also proven that in the case of strictly convex flux, the NT scheme with a properly selected MAPR-like limiter satisfies an one-sided Lipschitz stability estimate. As a result, convergence to the unique entropy solution when the initial data satisfies the so-called one-sided Lipschitz condition is obtained. Finally, compensated compactness arguments are employed to prove that for any bounded initial data, the NT scheme based on a MAPR-like limiter converges strongly on compact sets to the unique entropy solution of the conservation law with a strictly convex flux.

To my family.

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CHAPTER I

INTRODUCTION

A conservation law states that the rate of change of the total amount of a particular measurable quantity in a fixed domain $\Omega \subset \mathbb{R}^d$ is governed by the flux of the quantity across the boundary of Ω. The general set-up for the conservation law consists of m equations in d spatial dimensions. Denoting the density vector of the conserved quantities by $u = u(x,t) = (u_1(x,t), \ldots, u_m(x,t))$ and the flux by $f = f(u) = (f_1(u), \ldots, f_d(u))$, the conservation law is

$$\frac{d}{dt}\int_\Omega u\,dx = -\int_{\partial\Omega} f(u)\cdot n\,dS, \tag{1.1}$$

where n denotes the outward unit normal along Ω, so that the integral on the right measures the outflow. From (1.1) we derive the initial-value problem for a general system of conservation laws:

$$\begin{cases} u_t + \nabla_x \cdot f(u) = 0, & (x,t) \in \mathbb{R}^d \times (0,\infty), \\ u = u^0, & (x,t) \in \mathbb{R}^d \times \{t=0\}, \end{cases} \tag{1.2}$$

where $u^0 = (u_1^0, \ldots, u_m^0)$ is the given function describing the initial distribution of u. The system is called hyperbolic when for each $\tau \in \mathbb{R}^d$ and $\omega \in \mathbb{R}^m$, the $m \times m$ matrix $\sum_{j=1}^d \tau_j f_j'(\omega)$ is diagonalizable with real eigenvalues.

In general, even for smooth initial data the classical solutions of nonlinear conservation laws (1.2) fail to exist for all time because of the formation of shock discontinuities. For this reason, it is necessary to extend the notion of solution to (1.2) to the class of weak solutions.

The journal model is SIAM Journal of Numerical Analysis.

Definition 1. A function $u \in L^\infty_{\text{loc}}(\mathbb{R}^d \times (0,\infty))^m$ is said to be a weak solution of the initial value problem (1.2) if

$$\int_0^\infty \int_{\mathbb{R}^d} \left\{ u \cdot \frac{\partial \phi}{\partial t} + \sum_{j=1}^d f_j(u) \cdot \frac{\partial \phi}{\partial x_j} \right\} dx\, dt + \int_{\mathbb{R}^d} u^0(x) \cdot \phi(x,0)\, dx = 0, \tag{1.3}$$

for all $\phi \in C^1_0(\mathbb{R}^d \times (0,\infty))^m$.

However, it turns out that there can be infinitely many weak solutions of (1.2) with the same initial data. In order to select a unique weak solution, that has physical significance, a viscosity limit solution may be considered. For this, we associate with (1.2) the following parabolic system

$$u^\varepsilon_t + \nabla_x \cdot f(u^\varepsilon) = \varepsilon\, \Delta u^\varepsilon, \quad \varepsilon > 0, \tag{1.4}$$

where $\varepsilon\, \Delta u^\varepsilon$ is the viscosity term. For a more general form of parabolic regularization, see [61].

Definition 2. A function u is said to be a viscosity limit solution of (1.2), if for a given compact set $K \in \mathbb{R}^d \times [0,\infty)$, there exists a sequence of sufficiently smooth solution $(u^\varepsilon)_\varepsilon$ of (1.4) such that

$$||u^\varepsilon||_{L^\infty(K)} \leq C(K), \tag{1.5}$$

where $C(K)$ is a constant that might depend on K, but is independent of ε, and

$$u^\varepsilon \to u \;\text{ as }\; \varepsilon \to 0 \;\; a.e. \;\text{ on }\; K. \tag{1.6}$$

The notion of viscosity solution is closely related to the concept of entropy solution. A convex function $S : \Omega \to \mathbb{R}$ is called an entropy for the system of conservation laws (1.2) if there exist d entropy fluxes $Q_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, d$ such that

$$S'(u)^t f'_j(u) = Q'_j(u)^t, \quad j = 1, \ldots, d. \tag{1.7}$$

The entropy condition is described as the following inequality

$$S_t(u) + \nabla_x \cdot Q(u) \leq 0, \tag{1.8}$$

which is satisfied in the distributional sense, where $Q(u) := (Q_1(u), \ldots, Q_d(u))$.

Definition 3. A weak solution u of (1.2) is called an entropy solution, if the inequality

$$\int_0^\infty \int_{\mathbb{R}^d} \left\{ S(u)\frac{\partial \phi}{\partial t} + \sum_{j=1}^{d} Q_j(u)\frac{\partial \phi}{\partial x_j} \right\} dx\, dt + \int_{\mathbb{R}^d} S(u^0(x))\phi(x,0)\, dx \geq 0, \tag{1.9}$$

is satisfied for all test functions $\phi \in C_0^1(\mathbb{R}^d \times (0,\infty))$, $\phi \geq 0$ and for all possible entropy pairs (S, Q).

It is easy to see that a viscosity limit solution of (1.4) is a weak entropy solution of (1.2). However, the reverse implication, that a weak entropy solution, is a viscosity limit solution requires a special treatment. Although it is known to be correct in the scalar case, there is a limited success of its extension to the case of systems in one dimension, not to mention general systems.

Very little is known about the multidimensional systems of conservation laws. The fundamental questions of existence, uniqueness and stability remain open for general systems, see [9, 13, 59]. One of the few achievements in the area is due to Kato [30] who proved a short time existence of H^s-solution for a time interval $[0, T]$, with $T = T(||u^0||_{H^s})$.

More is known for one-dimensional systems. The first existence result in this context is due to Glimm [19] who proved convergence of his random choice method to a weak entropy solution of strictly hyperbolic systems of conservation laws subject to initial condition with sufficiently small total variation. This result is based on a compactness argument which, by itself, does not guarantee the uniqueness. The uniqueness of the solutions obtained as limits of Glimm was proven by Bressan for a special case in [6], where he also showed that these solutions depend Lipschitz continuously on the initial data, in the L^1 norm. There are many more results based on the ideas of Glimm that we do not discuss here. More recently, Bianchini and Bressan [4] proved the existence of a viscosity limit solution to the strictly hyperbolic system under the assumption that the initial data is of small total variation and that the vanishing viscosity limit is precisely the limit of the Glimm solution of (1.2) in one dimension ($d = 1$). Another approach to prove existence, using the theory of compensated compactness, was introduced by Tartar and Murat [46, 62] and further developed by DiPerna [15], Chen [7] and many others. The solutions found in this setting are in the much larger space L^∞, and since the known uniqueness results apply only to BV solutions with small variations, see [4, 19], it remains a difficult open problem to prove the uniqueness in the class of large initial data (BV or L^∞). In [15], DiPerna established convergence of the artificial viscosity method for the isentropic equations of gas dynamics (2×2 system) in one space dimension for γ belonging to the sequence $(2k + 3)/(2k + 1)$ with $k = 2, 3, \dots$. This result was later extended for $\gamma \in (1, \infty)$ by Lions, Perthame and Souganidis in [40]. Similar to Glimm's method, there are many results based on the compensated compactness arguments which we don't mention here.

The analytical theory is complete in the case of scalar conservation laws. The existence, uniqueness and global stability of vanishing viscosity and "entropy" solutions for a scalar conservation law in one space dimension was first established by Oleinik in [49]. Oleinik proved that in the case of a strictly convex flux f, there exists a unique solution u of (1.2) with $m = 1$, $d = 1$ and $u^0 \in L^\infty(\mathbb{R})$, such that

$$u(x + z, t) - u(x, t) \leq \frac{C}{t} z, \tag{1.10}$$

for all $t > 0$ and $x, z \in \mathbb{R}$, $z > 0$. The inequality (1.10) is called Oleinik entropy condition. This result can also be considered as a regularity result since an L^∞ solution u satisfying (1.10) is of locally bounded total variation for $t > 0$ and therefore, the solution immediately becomes more regular in time (BV), even though the initial data is merely bounded (L^∞). A more general existence and uniqueness result of a weak entropy solution, see (1.8), in several space dimensions was proven by Volpert in [67] for BV initial data. He also showed that the unique entropy solution in this class coincides with the viscosity limit solution. Later, these results were extended by Kruzkov [33] to the class of L^∞ solutions. The proof relies on doubling the variables and using the family of Kruzkov's entropy pairs

$$S(u) = |u - c|, \quad Q(u) = \operatorname{sgn}(u - c)(f(u) - f(c)), \quad c \in \mathbb{R}, \tag{1.11}$$

to show an L^1-contraction property and thus, uniqueness. A more recent result due to Panov [53] shows that in the case of convex flux and one space dimension, a single entropy inequality is enough to select the unique entropy solution.

Next let us mention some numerical methods for hyperbolic conservation laws. Early constructions of approximate solutions for conservation laws relied on non-physical stabilization techniques such as artificial viscosity proposed by von Neumann and Richtmyer [68]. In 1959 Godunov [20] proposed a new approach for approximating solutions of one dimensional compressible fluid flow. His scheme, as originally presented, is based on solving Riemann problem exactly for one time step, and then averaging the exact solution over each cell. The time step in the scheme needs to be chosen sufficiently small so that the Riemann fans emerging from the interfaces between two cells do not interact. This can be achieved by employing a Courant-Friedrichs-Lewy (CFL) condition [11]. Godunov's method preserves monotonicity, but it is only first order accurate, and this fact was explained in his theorem that monotonicity preserving constant coefficient schemes can be no better than first order accurate. Also in 1965, Glimm [19] introduced a first order method which instead of averaging, randomly samples the exact solution of piecewise constant initial data.

In order to construct higher order methods a lot of research has been done in the area of nonlinear schemes for conservation laws. The pioneering works are due to Boris and Book [5] and van Leer [63, 64, 65, 66]. In his series of papers, van Leer, developed a second order Godunov-type scheme, monotone upstream-centered scheme for conservation laws (MUSCL), which uses Riemann solvers on piecewise linear reconstructions. Later, Woodward and Colella [69], developed the piecewise parabolic method (PPM), which may be considered as a further refinement of MUSCL. Parallel to these developments Roe [57], Osher and Solomon [51], Harten, Lax and van Leer [24], Einfeldt [16] and many others proposed approximate Riemann solvers and generated variants of the original Godunov method.

The construction of high order total-variation-diminishing (TVD) schemes was initiated by Harten [21]. The TVD property guarantees convergence to a weak solution. Earlier constructions of approximate solutions in the finite-difference setting used monotonicity property to guarantee convergence, see [12, 58]. However, as it was proven by Harten, Hyman and Lax [23], these schemes can be at most first order accurate. TVD schemes, on the other hand, may be higher order accurate away from the extrema, see [50]. The conditions on the limiters used in MUSCL and TVD schemes to establish the desired properties were considered by Sweby in [60]. The development of higher order non-oscillatory schemes based on different limiters started with the introduction of essentially non-oscillatory (ENO) schemes by Harten, Engquist, Osher and Chakravarthy [22, 25]. In order to improve the order of accuracy of these type of schemes weighted ENO (WENO) schemes were introduced in [27, 42].

The main difficulty in Godunov-type schemes is the requirement of a detailed solution of Riemann problem at each time step. A well-known method to avoid the solution of a Riemann problem is grid staggering in time. A prototype of a central difference approximation that uses this approach is the first order Lax-Friedrichs (LxF) scheme [18] which relies on piecewise constant reconstruction. Although LxF scheme is robust and stable, it suffers from excessive dissipation. To circumvent this problem Nessyahu and Tadmor [47] used the same philosophy with piecewise linear reconstruction and developed a second order non-oscillatory central scheme for one dimensional scalar conservation law. This method was later extended to two dimensional case by Jiang and Tadmor [28] and higher order of accuracy by Liu and Tadmor [43]. Due to their simplicity and stability there is a continued interest in the development of high order non-oscillatory central schemes. Some of the successful implementations and improvements are due to Levy and Tadmor [39], Kurganov and Tadmor [35], Kurganov, Noelle and Petrova [34] for Euler equations, Tadmor and Wu [2] for magneto-hydrodynamics equations, Bereux and Sainsaulieu for hyperbolic systems with relaxation source terms [3].

The central type schemes are numerically efficient and observed to have better accuracy than first order schemes, but there are only a few theoretical results known about them, see for example [22, 25, 26, 27, 28, 35, 47, 55]. This study considers the NT scheme with modified minmod limiter inspired by the so-called minimum angle reconstruction (MAPR) introduced by Christov and Popov in [10], and intends to prove stability and convergence results for the largest possible class of initial conditions.

Chapter II briefly describes the NT scheme with the MAPR-like limiter. In Chapter III, a maximum principle for the NT scheme with the new limiter is proven for a conservation law with a Lipschitz flux and also with any k-monotone flux for $k \geq 2$. Chapter IV considers a conservation law with strictly convex flux, and establishes an one-sided Lipschitz stability estimate for the NT scheme with a properly selected MAPR-like limiter. In Chapter V, the compensated compactness arguments together with the stability results from previous chapters are used to prove that for any bounded initial data, the NT scheme converges strongly on compact sets to the unique entropy solution. Chapter VI summarizes the results and open problems.

CHAPTER II

NESSYAHU-TADMOR (NT) SCHEME

In this chapter, we are going to recall the setup of second order non-oscillatory central difference approximations to the scalar conservation law

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u = u^0, & (x,t) \in \mathbb{R} \times \{t = 0\}. \end{cases} \tag{2.1}$$

We restrict our attention to the one-dimensional staggered Nessyahu-Tadmor (NT) scheme [47], which was the motivation for the construction of many other central staggered schemes, see for example [1, 10, 28, 35]. Unlike the upwind schemes, central schemes avoid the intricate and time consuming Riemann solvers. A fundamental step in the design of such numerical algorithms is a piecewise linear slope reconstruction, see for example [25, 47]. In order to guarantee the overall non-oscillatory nature of the approximate solution, nonlinear limiters such as minmod, generalized minmod, UNO, ENO and WENO are widely used. An unfortunate requirement of all proofs is that the piecewise linear reconstruction used must reduce to first order (zero slope) at local extrema in order to prove a maximum principle for the scheme, see for example [47]. Imposing such a limitation could deteriorate the performance of the methods. When this limitation is not imposed, there are typically no theoretical results for the methods, see for example [10, 22, 25, 47]. We consider a class of nonlinear reconstructions which include and are motivated by the so-called minimum angle plane reconstruction (MAPR) introduced in [10]. The key idea is that at local extrema the slope of the reconstruction is not set to zero but it is limited by the smallest local slope.

Let $v(x,t)$ be an approximate solution to (2.1), and assume that the space mesh Δx and the time mesh Δt are uniform. Let $x_j := j\,\Delta x$, $j \in \mathbb{Z}$ and $t^n = n\,\Delta t$, $n \in \mathbb{N}$. We define

$$v_j^n := \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v(x,t^n)\,dx \tag{2.2}$$

to be the average of v at time t^n over $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$. Next, assume that $v(\cdot,t^n)$ is a piecewise linear function of the form

$$v(x,t^n) = \sum L_j(x,t^n) := \sum \left(v_j^n + (x-x_j)\frac{1}{\Delta x}v_j'\right)\chi_j(x), \tag{2.3}$$

where χ_j is the characteristic function over $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ and $\frac{1}{\Delta x}v_j'$ is the numerical derivative of $v(x=x_j,t^n)$ which is yet to be determined. We proceed by integrating (2.1) over $(x_j,x_{j+1})\times(t^n,t^{n+1})$ which yields

$$\begin{aligned} v_{j+\frac{1}{2}}(t^{n+1}) = {} & \frac{1}{\Delta x}\left(\int_{x_j}^{x_{j+\frac{1}{2}}} L_j(x,t^n)\,dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} L_{j+1}(x,t^n)\,dx\right) \\ & - \frac{1}{\Delta x}\left(\int_{t^n}^{t^{n+1}} f(v(x_{j+1},\tau))\,d\tau - \int_{t^n}^{t^{n+1}} f(v(x_j,\tau))\,d\tau\right). \end{aligned} \tag{2.4}$$

The first two integrands on the right-hand side of (2.4), $L_j(x,t^n)$ and $L_{j+1}(x,t^n)$, can be integrated exactly and if the CFL condition (with mesh ratio $\lambda := \frac{\Delta t}{\Delta x}$)

$$\lambda \max_{x_j\le x\le x_{j+1}} |f'(v(x,t))| \le \frac{1}{2}, \quad j\in\mathbb{Z}, \tag{2.5}$$

is satisfied, then the last two integrands on the right-hand side of (2.4), $f(v(x_j,\tau))$ and $f(v(x_{j+1},\tau))$, can be integrated approximately by the midpoint rule at the expense of $O(\Delta t)^3$ local truncation error. Thus, we arrive at

$$v_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}(v_j^n + v_{j+1}^n) + \frac{1}{8}(v_j' - v_{j+1}') - \lambda\left(f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})\right). \tag{2.6}$$

The approximate midpoint value in time $v_j^{n+\frac{1}{2}}$, satisfying second order accuracy requirement, can be chosen as

$$v_j^{n+\frac{1}{2}} = v_j^n - \frac{1}{2}\lambda f'_j, \tag{2.7}$$

owing to Taylor expansion and (2.1). Here, $\frac{1}{\Delta x} f'_j$ stands for an approximate numerical derivative of the flux $f(v(x = x_j, t^n))$, which is yet to be specified. Although there are many different recipes to construct v'_j and f'_j, in this paper we only consider the following approximations of the numerical derivatives

$$v'_j = \mathrm{m}(v_{j+1}^n - v_j^n, v_j^n - v_{j-1}^n), \tag{2.8}$$

$$f'_j = f'(v_j^n)v'_j, \tag{2.9}$$

where $\mathrm{m}(\cdot, \cdot)$ is the MAPR-like minmod limiter

$$\mathrm{m}(a, b) := \begin{cases} \mathrm{sgn}(a)\ \min(|a|, |b|), & ab \geq 0, \\ \sigma \min(|a|, |b|), & ab < 0, \end{cases} \tag{2.10}$$

with $\sigma \in \mathbb{R}, |\sigma| \leq 1$.

Remark 1. Note that, the standard minmod limiter is included as a special case of (2.10) for $\sigma = 0$ and the choice

$$\sigma = \mathrm{sgn}(s), \quad \text{where} \quad s = \begin{cases} a, & |a| \leq |b|, \\ b, & |b| \leq |a|, \end{cases} \tag{2.11}$$

recovers the MAPR limiter introduced in [10].

Using the approximate slopes (2.8) and the approximate flux derivatives (2.9), we end up with a family of central differencing schemes in the predictor-corrector form

$$v_j^{n+\frac{1}{2}} = v_j^n - \frac{1}{2}\lambda f'_j, \tag{2.12}$$

$$v_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}(v_j^n + v_{j+1}^n) + \frac{1}{8}(v'_j - v'_{j+1}) - \lambda \left(f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}}) \right). \tag{2.13}$$

The global approximate solution $v(x,t)$ is defined to be piecewise constant in time: $v(x,t) = v_j^n$ for $(x,t) \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times [t^n, t^{n+1})$, where $j \in \mathbb{Z}$ if n is even and $j + \frac{1}{2} \in \mathbb{Z}$ if n is an odd integer.

CHAPTER III

MAXIMUM PRINCIPLE

A. Introduction

The advantage of our limiters introduced in Chapter II is that we do not need to restrict the slope of the reconstruction to zero at local extrema. However, using a second order reconstruction at local extrema requires a new approach when trying to prove a maximum principle. Note that (2.10) allows the predicted values $\{v_j^{n+\frac{1}{2}}\}_j$, see (2.12), to violate maximum principle. That is, the minimum/maximum of the sequence $\{v_j^{n+\frac{1}{2}}\}_j$ could be smaller/larger than that of $\{v_j^n\}_j$. This is going to be the main difficulty in proving maximum principle under a fixed CFL.

This chapter is organized as follows. In Section B, first a maximum principle for globally Lipschitz flux is proven, see Theorem 1, and then, the same kind of result is proven for a more general class of flux functions, namely for k-monotone flux functions, which is the main result of this chapter, see Theorem 2. The class of k-monotone functions include all strictly convex functions (for $k = 2$) and also any polynomial flux of a fixed degree $\leq k$, see Definition 4 in Section B or [32] for a definition of k-monotonicity. In Section C, it is shown that this type of maximum principle implies the usual Total Variation Diminishing (TVD) bound as described in [47], see Lemma 2. Finally, in Section D, we try to find an "optimal" σ for the MAPR-like limiters defined in (2.10).

B. Maximum principle of the NT scheme

We begin with a maximum principle result in a simpler setting when the flux is globally Lipschitz continuous.

Theorem 1. *Let v'_j be chosen by (2.8) and $f'_j = f'(v^n_j)v'_j$. If f is globally Lipschitz continuous, then the scheme described by (2.12)–(2.13) under the CFL condition*

$$\lambda \left\|f'\right\|_{L^\infty(\mathbb{R})} \leq \kappa \leq \frac{\sqrt{2}-1}{2} \tag{3.1}$$

satisfies the maximum principle

$$\min(v^n_j, v^n_{j+1}) \leq v^{n+1}_{j+\frac{1}{2}} \leq \max(v^n_j, v^n_{j+1}). \tag{3.2}$$

Proof. First, we rewrite the term $f(v^{n+\frac{1}{2}}_{j+1}) - f(v^{n+\frac{1}{2}}_j)$ in (2.13) as

$$f(v^{n+\frac{1}{2}}_{j+1}) - f(v^{n+\frac{1}{2}}_j) = f'(\xi^{n+\frac{1}{2}}_{j+\frac{1}{2}})(v^{n+\frac{1}{2}}_{j+1} - v^{n+\frac{1}{2}}_j), \tag{3.3}$$

where $\min(v^{n+\frac{1}{2}}_j, v^{n+\frac{1}{2}}_{j+1}) \leq \xi^{n+\frac{1}{2}}_{j+\frac{1}{2}} \leq \max(v^{n+\frac{1}{2}}_j, v^{n+\frac{1}{2}}_{j+1})$. Observe that,

$$\begin{aligned}\left|v^{n+\frac{1}{2}}_{j+1} - v^{n+\frac{1}{2}}_j\right| &= \left|v^n_{j+1} - v^n_j - \frac{\lambda}{2}(f'(v^n_{j+1})v'_{j+1} - f'(v^n_j)v'_j)\right| \\ &\leq \left(1 + \frac{\lambda}{2}\left(|f'(v^n_{j+1})| + |f'(v^n_j)|\right)\right)|v^n_{j+1} - v^n_j| \\ &\leq (1+\kappa)|v^n_{j+1} - v^n_j|.\end{aligned} \tag{3.4}$$

Using (3.1), (3.3) and (3.4) we find an upper bound for (2.13),

$$\begin{aligned}v^{n+1}_{j+\frac{1}{2}} &\leq \frac{1}{2}(v^n_j + v^n_{j+1}) + \frac{1}{8}|v'_j - v'_{j+1}| + \lambda\left|f(v^{n+\frac{1}{2}}_{j+1}) - f(v^{n+\frac{1}{2}}_j)\right| \\ &\leq \frac{1}{2}(v^n_j + v^n_{j+1}) + \left(\frac{1}{2} + \kappa\right)^2 |v_{j+1} - v_j| \\ &\leq \frac{1}{2}(v^n_j + v^n_{j+1}) + \frac{1}{2}|v_{j+1} - v_j| = \max(v^n_j, v^n_{j+1}),\end{aligned} \tag{3.5}$$

and similarly a lower bound,

$$\begin{aligned}v^{n+1}_{j+\frac{1}{2}} &\geq \frac{1}{2}(v^n_j + v^n_{j+1}) - \frac{1}{8}|v'_j - v'_{j+1}| - \lambda\left|f(v^{n+\frac{1}{2}}_{j+1}) - f(v^{n+\frac{1}{2}}_j)\right| \\ &\geq \frac{1}{2}(v^n_j + v^n_{j+1}) - \left(\frac{1}{2} + \kappa\right)^2 |v_{j+1} - v_j| \\ &\geq \frac{1}{2}(v^n_j + v^n_{j+1}) - \frac{1}{2}|v_{j+1} - v_j| = \min(v^n_j, v^n_{j+1}).\end{aligned} \tag{3.6}$$

The above two bounds prove the theorem. □

Remark 2. This proof also fixes an inaccuracy in the maximum principle proof given in [47] which is correct for the flux choice

$$f_j' = \mathrm{m}(f_j - f_{j-1}, f_{j+1} - f_j), \tag{3.7}$$

but the proof as written in [47] does not hold for the Jacobian form (2.8)–(2.9) even for the standard minmod limiter ($\sigma = 0$ in (2.10)).

Next, we are going to prove the maximum principle for a more general class of flux functions. We start with the following definition and properties of k-monotone functions, see [32, 54, 56] for more details on k-monotone functions.

Definition 4. A function $f : [a, b] \to \mathbb{R}$ is said to be k-monotone, $k \geq 1$, on $[a, b]$ if and only if for all choices of $(k + 1)$ distinct nodes $x_0, ..., x_k$ in $[a, b]$ the inequality

$$[x_0, ..., x_k]\, f \geq 0 \tag{3.8}$$

holds, where $[x_0, ..., x_k]\, f := \sum_{j=0}^{k} (f(x_j)/w'(x_j))$ denotes the k^{th} divided difference of f at $x_0, ..., x_k$ and $w(x) = \prod_{j=0}^{k} (x - x_j)$. f is said to be k-strictly monotone if the inequality (3.8) is strict.

Remark 3. If $f \in C^k[a, b]$, then f is k-strictly monotone if and only if there exists $\gamma_1 \in \mathbb{R}$ such that $0 < \gamma_1 \leq f^{(k)}(x),\ x \in [a, b]$.

Example 1. Some well known examples of k-monotone fluxes are

1. Polynomial fluxes: $f(u) = \sum_{i=0}^{k} a_i\, u^i \in \mathbb{P}_k,\ a_k > 0$. A special case is the Burgers' flux, $f(u) = \frac{u^2}{2}$, which is 2-strictly monotone, i.e., strictly convex.

2. Buckley-Leverett flux: $f(u) = u^2/(u^2 + a(1 - u)^2)$, $a > 0$, which is 1-monotone for $0 \leq u \leq 1$.

We also need the following definition:

Definition 5. The range of a function $g : \mathbb{R} \to \mathbb{R}$ is defined to be the interval

$$R(g) := [\operatorname{essinf}_{x\in\mathbb{R}} g(x), \operatorname{esssup}_{x\in\mathbb{R}} g(x)]. \tag{3.9}$$

Now, we are ready to state and prove the main result of this section.

Theorem 2. *Let v'_j be chosen as in (2.8) and $f'_j = f'(v^n_j)v'_j$. If f satisfies the following properties*

1. *f or $(-f)$ is $(m+2)$-strictly monotone on $R(u^0)$,*
2. *$f \in C^{m+2}(\mathbb{R})$ and there exists a constant $\gamma_2 \in \mathbb{R}$ such that $\left|f^{(m+2)}(x)\right| \leq \gamma_2$, for all $x \in \mathbb{R}$,*

then the NT-scheme described by (2.12)–(2.13) satisfies the maximum principle

$$\min(v^n_j, v^n_{j+1}) \leq v^{n+1}_{j+\frac{1}{2}} \leq \max(v^n_j, v^n_{j+1}), \tag{3.10}$$

under the CFL condition

$$\lambda \max_{w\in R(u^0)} |f'(w)| \leq \kappa, \tag{3.11}$$

where κ is a fixed constant which depends only on γ_1, γ_2 and m, see (3.25).

Remark 4. Note that the CFL condition (3.11) depends only on the maximum initial speed and smoothness properties of the flux. The CFL condition, see (3.25), amounts to solving a quadratic inequality similar to the one in [47]. For example, for the Burgers' equation, (3.11) is valid with $\kappa = \frac{1}{8}$, see [47] for details.

Proof. It suffices to prove the theorem for the case when f is $(m+2)$-strictly monotone on $R(u^0)$. By Remark 3 and the second property of f, we have that there exist

$\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\begin{aligned} f^{(m+2)}(x) \leq \gamma_2, \quad \text{for all } x \in \mathbb{R}, \\ 0 < \gamma_1 \leq f^{(m+2)}(x), \quad \text{for all } x \in R(u^0). \end{aligned} \tag{3.12}$$

We recall the definition of the open Newton-Cotes quadrature formula. For any given interval $[a, b]$, we define $h = (b-a)/(m+2)$ and introduce the nodes $x_i = a+(i+1)h$, where $i = -1, 0, ..., m+1$. Note that each $x_i,\ 0 \leq i \leq m$, is a convex combination of a and b, that is

$$x_i = \theta a + (1-\theta)b, \quad \theta \in \left\{ \frac{1}{m+2}, \frac{2}{m+2}, ..., \frac{m+1}{m+2} \right\}. \tag{3.13}$$

Next, we define

$$\ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{m} \frac{x-x_i}{x_i-x_j}, \quad L_m(x) = \sum_{i=0}^{m} f'(x_i)\ell_i(x), \text{ and } A_i = \int_a^b \ell_i(x)dx, \tag{3.14}$$

and estimate

$$\begin{aligned} \left| \int_a^b f'(x)dx \right| &\leq \left| \frac{1}{(m+1)!} \int_a^b f^{(m+2)}(\xi_x) \prod_{i=0}^{m} (x-x_i)\, dx \right| + \left| \sum_{i=0}^{m} A_i f'(x_i) \right| \\ &\leq \left| \frac{\gamma_2}{(m+1)!} \int_a^b \prod_{i=0}^{m} (x-x_i)\, dx \right| + \sum_{i=0}^{m} |A_i|\, |f'(x_i)| \\ &\leq c_m \gamma_2 |b-a|^{m+2} + \sum_{i=0}^{m} |A_i|\, |f'(x_i)| \\ &\leq c_m |b-a| \left(\gamma_2 |b-a|^{m+1} + \max_{0\leq i\leq m} |f'(x_i)| \right), \end{aligned} \tag{3.15}$$

where c_m is a constant depending only on m, whose value may change at each occurrence. With this convention, let $x_{\frac{1}{2}} = x_0 + \frac{h}{2}$ and observe that

$$\begin{aligned} \left| f'(x_{\frac{1}{2}}) - L_m(x_{\frac{1}{2}}) \right| &= \left| \frac{f^{(m+2)}(\xi)}{(m+1)!} \right| \frac{h}{2} \prod_{i=1}^{m} (2i-1)\frac{h}{2} \\ &= c_m \left| f^{(m+2)}(\xi) \right| |b-a|^{m+1}, \end{aligned} \tag{3.16}$$

for some $\xi \in [x_0, x_m]$. From (3.12) and (3.16) it follows that

$$\begin{aligned} |b-a|^{m+1} &\leq \frac{c_m}{\gamma_1} \left| f'(x_{\frac{1}{2}}) - L_m(x_{\frac{1}{2}}) \right| \\ &\leq \frac{c_m}{\gamma_1} \max_{0 \leq i \leq m} |f'(x_i)| \,. \end{aligned} \tag{3.17}$$

Together (3.15) and (3.17) imply,

$$\left| \int_a^b f'(x)\, dx \right| \leq c_m \left(\frac{\gamma_2}{\gamma_1} + 1 \right) \max_{0 \leq i \leq m} |f'(x_i)| \, |b-a|. \tag{3.18}$$

Let $a = v_j^{n+\frac{1}{2}}$ and $b = v_{j+1}^{n+\frac{1}{2}}$ in the above calculations and introduce the notation $v_{j+\frac{1}{2}}^{\theta} := \theta v_j^{n+\frac{1}{2}} + (1-\theta) v_{j+1}^{n+\frac{1}{2}}$. Then,

$$v_{j+\frac{1}{2}}^{\theta} = \theta v_j^n + (1-\theta) v_{j+1}^n - \frac{\lambda}{2} \left(\theta f'(v_j^n) v_j' + (1-\theta) f'(v_{j+1}^n) v_{j+1}' \right). \tag{3.19}$$

Note that,

$$\begin{aligned} v_{j+\frac{1}{2}}^{\theta} &\geq \min(v_j^n, v_{j+1}^n) + \Big(\min(\theta, 1-\theta) - \max(\theta, 1-\theta) \lambda \, \|f'\|_{L^\infty(R(v^n))} \Big) \, |\Delta v| \\ &\geq \min(v_j^n, v_{j+1}^n) + \left(\min(\theta, 1-\theta) - \max(\theta, 1-\theta)\kappa \right) |\Delta v| \,, \\ v_{j+\frac{1}{2}}^{\theta} &\leq \max(v_j^n, v_{j+1}^n) - \Big(\min(\theta, 1-\theta) - \max(\theta, 1-\theta) \lambda \, \|f'\|_{L^\infty(R(v^n))} \Big) \, |\Delta v| \\ &\leq \max(v_j^n, v_{j+1}^n) - \left(\min(\theta, 1-\theta) - \max(\theta, 1-\theta)\kappa \right) |\Delta v| \,, \end{aligned} \tag{3.20}$$

where $\Delta v := v_{j+1}^n - v_j^n$. Under the CFL condition $\kappa \leq \frac{1}{m+1}$ the above inequalities imply

$$\min(v_j^n, v_{j+1}^n) \leq v_{j+\frac{1}{2}}^{\theta} \leq \max(v_j^n, v_{j+1}^n). \tag{3.21}$$

Therefore, we conclude that $\min(v_j^n, v_{j+1}^n) \leq x_i \leq \max(v_j^n, v_{j+1}^n)$, for $i = 0, ..., m$. Next we rewrite (3.18) with $a = v_j^{n+\frac{1}{2}}$ and $b = v_{j+1}^{n+\frac{1}{2}}$,

$$\begin{aligned} \left| f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}}) \right| &\leq c_m \left(\frac{\gamma_2}{\gamma_1} + 1 \right) \max_{0 \leq i \leq m} |f'(x_i)| \left| v_{j+1}^{n+\frac{1}{2}} - v_j^{n+\frac{1}{2}} \right| \\ &\leq c_m \left(\frac{\gamma_2}{\gamma_1} + 1 \right) (1+\kappa) \, \|f'\|_{L^\infty(R(v^n))} \left| v_{j+1}^n - v_j^n \right| . \end{aligned} \tag{3.22}$$

Using this result in (2.13) yields the estimates

$$\begin{aligned} v_{j+\frac{1}{2}}^{n+1} &\leq \frac{1}{2}(v_j^n + v_{j+1}^n) + \frac{1}{8}\left|v_j' - v_{j+1}'\right| + \lambda \left| f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})\right| \\ &\leq \frac{1}{2}(v_j^n + v_{j+1}^n) + \left(\frac{1}{4} + \kappa(1+\kappa)c_m\left(\frac{\gamma_2}{\gamma_1}+1\right)\right)\left|v_{j+1}^n - v_j^n\right|, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} v_{j+\frac{1}{2}}^{n+1} &\geq \frac{1}{2}(v_j^n + v_{j+1}^n) - \frac{1}{8}\left|v_j' - v_{j+1}'\right| - \lambda \left| f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})\right| \\ &\geq \frac{1}{2}(v_j^n + v_{j+1}^n) - \left(\frac{1}{4} + \kappa(1+\kappa)c_m\left(\frac{\gamma_2}{\gamma_1}+1\right)\right)\left|v_{j+1}^n - v_j^n\right|. \end{aligned} \tag{3.24}$$

Thus, under the CFL condition

$$\kappa \leq \frac{1}{m+1} \quad \text{and} \quad \kappa(1+\kappa)c_m\left(\frac{\gamma_2}{\gamma_1}+1\right) \leq \frac{1}{4}, \tag{3.25}$$

we get the maximum principle (3.10). □

C. Total Variation Diminishing (TVD) bound

In this section, we show that the usual TVD bound, see [47], follows from the maximum principle (3.2). We need the following two lemmas to prove this result.

Lemma 1. *Given any non-constant sequence* $\{x_i\}_{i=m}^n$ *(i.e. not all* x_i*'s are equal) with* $n > m$*, there exist piecewise monotone subsequences* $\{x_{i_j}\}_{j=1}^k$ *(with* $k \geq 2$*) such that if* $\{x_{i_j}, x_{i_j+1}, \ldots, x_{i_{j+1}}\}$ *is non-decreasing, then* $\{x_{i_{j+1}}, x_{i_{j+1}+1}, \ldots, x_{i_{j+2}}\}$ *is non-increasing, and vice versa, with the property that* $\left|x_{i_j} - x_{i_{j+1}}\right| > 0$ *for all* j*.*

Proof. We will construct a subsequence $S_k := \{x_{i_j}\}_{j=1}^k$ of $\{x_i\}_{i=m}^n$ that satisfies the conditions in the lemma. Let $S = \{\bar{x}_{i_s}\}_{s=1}^l$ denote the set of all local extrema of $\{x_i\}_{i=m}^n$ such that all the $\bar{x}_{i_s} \in S$ excluding the endpoints, namely x_n and x_m, satisfy

$$|\bar{x}_{i_s} - \bar{x}_{i_s-1}| + |\bar{x}_{i_s} - \bar{x}_{i_s+1}| > 0, \tag{3.26}$$

$$\mathrm{sgn}(\bar{x}_{i_s} - \bar{x}_{i_s-1}) \neq \mathrm{sgn}(\bar{x}_{i_{s+1}+1} - \bar{x}_{i_{s+1}}). \tag{3.27}$$

Observe that $\bar{x}_{i_1} = x_m$ and $\bar{x}_{i_l} = x_n$. In order to construct $S_k \subset S$ we remove $\bar{x}_{i_{s+1}}$ from S if $\bar{x}_{i_s} = \bar{x}_{i_{s+1}}$ for $1 \leq s \leq l-2$, and we remove $\bar{x}_{i_{l-1}}$ from S if $\bar{x}_{i_{l-1}} = \bar{x}_{i_l}$. Finally, we reindex the remaining elements of S to get $S_k = \{x_{i_j}\}_{j=1}^k$. □

Lemma 2. *Given sequences* $\{x_j\}_{j=m}^n$ *and* $\{y_j\}_{j=m}^{n-1}$ *with* $m \leq n-1$*, satisfying*

$$\min(x_j, x_{j+1}) \leq y_j \leq \max(x_j, x_{j+1}), \quad m \leq j \leq n-1, \tag{3.28}$$

we have

$$\sum_{j=m}^{n-2} |y_{j+1} - y_j| \leq \sum_{j=m}^{n-1} |x_{j+1} - x_j|. \tag{3.29}$$

Proof. We consider the subsequence $\{x_{i_j}\}_{j=1}^k$ of the sequence $\{x_i\}_{i=m}^n$ described in Lemma 1. For any given j with $2 \leq j \leq k-1$, without loss of generality, assume that $\{x_{i_{j-1}}, x_{i_{j-1}+1}, \ldots, x_{i_j}\}$ is non-decreasing. That is, $\{y_{i_{j-1}}, y_{i_{j-1}+1}, \ldots, y_{i_j-1}\}$ is non-decreasing, and $\{x_{i_j}, x_{i_j+1}, \ldots, x_{i_{j+1}}\}$ and $\{y_{i_j}, y_{i_j+1}, \ldots, y_{i_{j+1}-1}\}$ are non-increasing. There are two possible cases: $y_{i_j-1} \leq y_{i_j}$ and $y_{i_j-1} \geq y_{i_j}$. Since the proofs are analogous, we will only carry out the first case.

We consider the sequences $\{y_{i_{j-1}}, y_{i_{j-1}+1}, \ldots, y_{i_j}\}$ and $\{y_{i_j+1}, \ldots, y_{i_{j+1}-1}\}$, which are non-decreasing and non-increasing, respectively. Then,

$$\begin{aligned}
\sum_{k=i_{j-1}}^{i_{j+1}-2} |y_{k+1} - y_k| &= \sum_{k=i_{j-1}}^{i_j-1} (y_{k+1} - y_k) + \sum_{k=i_j}^{i_{j+1}-2} (y_k - y_{k+1}) \\
&= 2y_{i_j} - y_{i_{j-1}} - y_{i_{j+1}-1} \leq 2x_{i_j} - x_{i_{j-1}} - x_{i_{j+1}} \\
&= \sum_{k=i_{j-1}}^{i_j-1} (x_{k+1} - x_k) + \sum_{k=i_j}^{i_{j+1}-1} (x_k - x_{k+1}) \\
&= \sum_{k=i_{j-1}}^{i_{j+1}-1} |x_{k+1} - x_k|,
\end{aligned} \tag{3.30}$$

which implies $\mathrm{Var}\left(\{y_k\}_{k=i_{j-1}}^{i_{j+1}-1}\right) \leq \mathrm{Var}\left(\{x_k\}_{k=i_{j-1}}^{i_{j+1}}\right)$. Similar arguments apply for the case when $\{x_{i_{j-1}}, x_{i_{j-1}+1}, \ldots, x_{i_j}\}$ is non-increasing. Therefore,

$$\mathrm{Var}\left(\{y_k\}_{k=i_{j-1}}^{i_{j+1}-1}\right) \leq \mathrm{Var}\left(\{x_k\}_{k=i_{j-1}}^{i_{j+1}}\right), \quad \text{for } 2 \leq j \leq k-1, \tag{3.31}$$

which implies

$$\mathrm{Var}\left(\{y_j\}_{j=m}^{n-1}\right) = \sum_{j=m}^{n-2} |y_{j+1} - y_j| \leq \sum_{j=m}^{n-1} |x_{j+1} - x_j| = \mathrm{Var}\left(\{x_j\}_{j=m}^{n}\right). \tag{3.32}$$

□

Remark 5. Lemma 2 applied to the sequences $\{v_j^n\}_j$ and $\{v_{j+\frac{1}{2}}^{n+1}\}_j$ gives that our maximum principle (Theorem 2) implies the usual TVD bound for the NT scheme.

D. Optimal choice for σ

In this section, we will try to find an "optimal" σ for the MAPR-like limiters defined in (2.10). The goal is to make the l_∞-norm of the staggered averages when the MAPR-like limiter is used bigger than the l_∞-norm of the staggered averages when standard minmod limiter is used for the same initial data. The following lemma motivates our "optimal" choice of σ, see (3.33).

Lemma 3. *Let* $\left\{v_j^n\right\}_j$ *be a sequence with isolated local extrema. Let* $\left\{v_{j+\frac{1}{2}}^{n+1}\right\}_j$ *and* $\left\{\bar{v}_{j+\frac{1}{2}}^{n+1}\right\}_j$ *denote the staggered averages for standard MAPR-like limiter and minmod limiter, respectively. If the parameter* σ *in the MAPR-like limiter (2.10) is chosen as*

$$\sigma = \mathrm{sgn}\left(\bar{v}_{j+\frac{1}{2}}^{n+1} - \bar{v}_{j-\frac{1}{2}}^{n+1}\right), \tag{3.33}$$

then we have

$$\left\| \left\{\bar{v}_{j+\frac{1}{2}}^{n+1}\right\}_j \right\|_{l_\infty} \leq \left\| \left\{v_{j+\frac{1}{2}}^{n+1}\right\}_j \right\|_{l_\infty}. \tag{3.34}$$

Proof. We will only consider the case of isolated local maxima with the case of isolated local minima being analogous. If the sequence $\left\{\bar{v}_{i+\frac{1}{2}}^{n+1}\right\}_i$ assumes its local maximum at $\bar{v}^{n+1}_{j+\frac{1}{2}}$, then we need to choose σ such that the difference $\Delta v^{n+1}_{j+\frac{1}{2}} := v^{n+1}_{j+\frac{1}{2}} - \bar{v}^{n+1}_{j+\frac{1}{2}}$ is positive. Let us rewrite the formula (2.13) for the staggered averages,

$$\begin{aligned} v^{n+1}_{j+\frac{1}{2}} &= \frac{1}{2}(v^n_j + v^n_{j+1}) + \frac{1}{8}(v'_j - v'_{j+1}) - \lambda\left(f(v^{n+\frac{1}{2}}_{j+1}) - f(v^{n+\frac{1}{2}}_j)\right), \\ \bar{v}^{n+1}_{j+\frac{1}{2}} &= \frac{1}{2}(v^n_j + v^n_{j+1}) + \frac{1}{8}(\bar{v}'_j - \bar{v}'_{j+1}) - \lambda\left(f(\bar{v}^{n+\frac{1}{2}}_{j+1}) - f(\bar{v}^{n+\frac{1}{2}}_j)\right). \end{aligned} \tag{3.35}$$

Subtracting the above equations and using $\bar{v}'_j = 0$, and $\bar{v}'_{j+1} = v'_{j+1}$, we obtain

$$\begin{aligned} \Delta v^{n+1}_{j+\frac{1}{2}} &= \frac{1}{8}v'_j + \lambda\left(f(v^{n+\frac{1}{2}}_j) - f(\bar{v}^{n+\frac{1}{2}}_j)\right) \\ &= \frac{1}{2}\left(\frac{1}{4} - \lambda^2 f'(\xi^{n+\frac{1}{2}}_j)f'(v^n_j)\right)v'_j, \end{aligned} \tag{3.36}$$

where $\min(\bar{v}^{n+\frac{1}{2}}_j, v^{n+\frac{1}{2}}_j) \leq \xi^{n+\frac{1}{2}}_j \leq \max(\bar{v}^{n+\frac{1}{2}}_j, v^{n+\frac{1}{2}}_j)$. By our maximum principle for MAPR-like limiters, see Theorem 2, we conclude that $\left|\lambda^2 f'(\xi^{n+\frac{1}{2}}_j)f'(v^n_j)\right| \leq \frac{1}{4}$. Thus, for $\Delta v^{n+1}_{j+\frac{1}{2}} > 0$ we need $v'_j > 0$.

If the sequence $\left\{\bar{v}^{n+1}_{i+\frac{1}{2}}\right\}_i$ assumes its local maximum at $\bar{v}^{n+1}_{j-\frac{1}{2}}$, we need to choose σ such that the difference $\Delta v^{n+1}_{j-\frac{1}{2}}$ is positive. By similar arguments we get

$$\Delta v^{n+1}_{j-\frac{1}{2}} = \frac{1}{2}\left(\lambda^2 f'(\xi^{n+\frac{1}{2}}_j)f'(v^n_j) - \frac{1}{4}\right)v'_j. \tag{3.37}$$

Hence, for $\Delta v^{n+1}_{j-\frac{1}{2}} > 0$ we need $v'_j < 0$. This motivates our MAPR-like limiter choice which takes σ such that we maximize the l_∞ norm of the staggered averages in one time step. □

Remark 6. The inequality (3.34) is strict if the l_∞- norm of the sequence $\left\{v^n_j\right\}_j$ is achieved in one cell only. That is, if $|v^n_i| = \|\left\{v^n_j\right\}_j\|_{l_\infty}$, $|v^n_i - v^n_{i-1}| > 0$ and $|v^n_i - v^n_{i+1}| > 0$, then

$$\left\|\left\{\bar{v}^{n+1}_{j+\frac{1}{2}}\right\}_j\right\|_{l_\infty} < \left\|\left\{v^{n+1}_{j+\frac{1}{2}}\right\}_j\right\|_{l_\infty}. \tag{3.38}$$

Remark 7. Note that the choice of σ in (3.33) is not just a function of cell averages (like a standard slope reconstruction) but also depends on the flux in a nonlinear way.

Our numerical experiments suggest that the above choice of σ always gives bigger l_∞-norms of the numerical solutions, but at this point it is not feasible to prove this result for multiple time steps.

CHAPTER IV

STABILITY

A. Introduction

Usually second order schemes are constructed to be Total Variation Diminishing (TVD) but that property only guarantees the convergence of such schemes to a weak solution, see for example [37]. The standard approach for proving convergence towards the entropy solution is to try to establish cell entropy inequalities. However, this usually leads to additional non-homogeneous limitations on the numerical method in order to fit it into the existing convergence theory. Unfortunately, this typically means that the modified method is reducing to a first order method when the mesh is refined. There are only few results on convergence of non-oscillatory second order schemes which do not require non-homogeneous limitations [38, 41, 70, 71]. All of the above references are tied in one way or another with local properties of the schemes and restrict to initial data with bounded total variation or even piecewise smooth data with finitely many extrema.

In this chapter, it is proven that in the case of strictly convex flux the NT scheme satisfies an one-sided Lipschitz stability estimate and converges to the entropy solution for any initial data satisfying some type of an one-sided condition, see [55], when the MAPR-like limiter is properly selected. This is a generalization of the result in [55] for minmod slope reconstruction.

B. One-sided l_2 stability and convergence of the NT scheme

Recall that $\{v_j^n\}_{j\in\mathbb{Z}}$ and $\{v_{j+\frac{1}{2}}^{n+1}\}_{j\in\mathbb{Z}}$ are the sequences of cell averages of the numerical solution of NT scheme at time t^n and t^{n+1}, respectively. Let us introduce the following notation for the jump sequences of the NT solution

$$\Delta v_{j+\frac{1}{2}} := v^n_{j+1} - v^n_j \ \text{ and } \ \Delta v_j := v^{n+1}_{j+\frac{1}{2}} - v^{n+1}_{j-\frac{1}{2}}, \quad j \in \mathbb{Z}, \tag{4.1}$$

at times t^n and t^{n+1}, respectively. With this notation we have the following theorem which is the main result of this section.

Theorem 3. *Let $u^0 \in L^\infty(\mathbb{R})$, f be strictly convex in $R(u^0)$ and f'' be bounded on $\mathbb{R}$. That is, there exist constants γ_1 and γ_2 such that*

$$0 < \gamma_1 \le f''(w), \ \textit{for all } w \in R(u^o), \tag{4.2}$$

$$0 \le f''(x) \le \gamma_2, \ \textit{for all } x \in \mathbb{R}. \tag{4.3}$$

Then there exists a constant κ which depends only on the ratio γ_1/γ_2 such that under the CFL condition

$$\lambda \max_{w \in R(u^0)} |f'(w)| \le \kappa, \tag{4.4}$$

the NT scheme with the limiter (2.10) and $0 \le \sigma \le 1$ satisfies the following one-sided Lipschitz condition

$$||\{(\Delta v_j)_+\}_{j\in\mathbb{Z}}||_{l_2} \le ||\{(\Delta v_{j+\frac{1}{2}})_+\}_{j\in\mathbb{Z}}||_{l_2}, \tag{4.5}$$

where we use the standard "+" notation: $x_+ = \max(x, 0)$. In other words, the l_2 norm of the positive jumps does not increase in time.

Remark 8. Taking $\sigma = 0$ in (2.10), we obtain the original minmod limiter. It is easy to see that among all σ, $|\sigma| \le 1$, the choice $\sigma = 1$ in (2.10) minimizes the size of the positive jumps in the piecewise linear numerical solution. These are the so-called entropy violating jumps for convex flux, see [52] and Remark 1 on page 422 in [47], and one needs to have control of their size in order to prove convergence to the entropy solution.

Proof. It is an easy exercise in real analysis to show that every bounded sequence can be decomposed into a union of monotone subsequences. That is, given the sequence $\{v_j^n\}_{j\in\mathbb{Z}}$ there exists a non-empty collection of index sets $\Lambda_k := \{j \mid j_{\min}^k \le j \le j_{\max}^k\}$ such that $j_{\max}^k = j_{\min}^{k+1}$ for all $k \in \mathbb{Z}$, and $\{v_j^n\}_{j\in\Lambda_k}$ is non-decreasing if k is even, and non-increasing if k is odd. This decomposition is not necessarily unique. To fix one, we choose $\{\Lambda_k\}_k$ such that Λ_k has maximum number of terms for each even k. That is, for all even k such that Λ_k is non-empty, we have

$$v_{j_{\min}^k - 1} \quad > \quad v_{j_{\min}^k} \;, \tag{4.6}$$

$$v_{j_{\max}^k + 1} \quad < \quad v_{j_{\max}^k} \;. \tag{4.7}$$

Note that we have a single set Λ_k if the data is monotone and that (4.6) and (4.7) only make sense if $-\infty < j_{\min}^k$ and $j_{\max}^k < \infty$, respectively. With this notation, there are only two possibilities to generate non-negative jumps Δv_j in the new sequence $\{v_{j+1/2}^{n+1}\}_{j\in\mathbb{Z}}$ starting from the old sequence $\{v_j^n\}_{j\in\mathbb{Z}}$:

(1) If $v_{j-1}^n \le v_j^n \le v_{j+1}^n$, i.e. $j-1,\ j,\ j+1 \in \Lambda_k$ for some even k, then we have an internal jump, i.e., generated from the interior of a non-decreasing monotone subsequence Λ_k.

(2) If $(v_j^n - v_{j-1}^n)(v_{j+1}^n - v_j^n) \le 0$ and at least one of these jumps ($\Delta v_{j-\frac{1}{2}}$ or $\Delta v_{j+\frac{1}{2}}$) is not zero. That is, $j-1, j \in \Lambda_k$ and $j, j+1 \in \Lambda_{k+1}$ for some k, then we have a boundary jump, i.e., generated on the boundary of Λ_k and Λ_{k+1}.

The jumps generated in (1) are always non-negative, whereas the jumps generated in (2) may have different signs.

Next, without loss of generality, we assume that there exists at least one non-decreasing subsequence of $\{v_j^n\}_{j\in\mathbb{Z}}$, say with index set $\Lambda_0 = \{0, \ldots, m\}$. Otherwise, there is nothing to prove. We define the modified cell averages $\{\bar{v}_j^n\}_{j\in\mathbb{Z}}$, as

$$\bar{v}_j^n := \begin{cases} v_0^n, & j \le 0, \\ v_j^n, & 0 < j < m, \\ v_m^n, & m \le j, \end{cases} \tag{4.8}$$

which is the so-called constant extension of $\{v_j^n\}_{j=0}^m$, see [55]. The jumps of the modified cell averages are given by

$$\Delta\bar{v}_{j+\frac{1}{2}} := \bar{v}_{j+1}^n - \bar{v}_j^n \text{ and } \Delta\bar{v}_j := \bar{v}_{j+\frac{1}{2}}^{n+1} - \bar{v}_{j-\frac{1}{2}}^{n+1}, \quad j \in \mathbb{Z}. \tag{4.9}$$

The following facts follow from the definition of $\{\bar{v}_j^n\}_{j\in\mathbb{Z}}$:

$$\begin{aligned} \bar{v}_{j+\frac{1}{2}}^{n+1} &= v_0^n,\ j \le -1, \\ \bar{v}_{j+\frac{1}{2}}^{n+1} &= v_m^n,\ j \ge m, \\ \bar{v}_{j+\frac{1}{2}}^{n+1} &= v_{j+\frac{1}{2}}^{n+1},\ 1 \le j \le m-2. \end{aligned} \tag{4.10}$$

Using the above we get

$$\begin{aligned} \Delta\bar{v}_j &= (\dots, 0, \Delta\bar{v}_0, \Delta\bar{v}_1, \Delta v_2, \dots, \Delta v_{m-2}, \Delta\bar{v}_{m-1}, \Delta\bar{v}_m, 0, \dots), \\ \Delta\bar{v}_{j+\frac{1}{2}} &= (\dots, 0, \Delta v_{\frac{1}{2}}, \Delta v_{\frac{3}{2}}, \dots, \Delta v_{m-\frac{1}{2}}, 0, \dots), \end{aligned} \tag{4.11}$$

with the convention that we drop any terms that do not make sense when $m \le 3$. In view of (1) and (2) the non-negative jumps of the new sequence can be decomposed into (interior and boundary) jumps generated by each Λ_k, for even k. Therefore, to prove the theorem it is enough to show that

$$||\{(\Delta v_j)_+\}_{j=0}^m||_{l_2} \le ||\{(\Delta v_{j+\frac{1}{2}})_+\}_{j=0}^{m-1}||_{l_2}, \tag{4.12}$$

because the left hand side includes all non-negative jumps that may be generated by Λ_0. For the sequence (4.8) the limiter (2.10) coincides with the minmod limiter.

Hence, we can apply the one-sided stability result from [55] for a single non-decreasing sequence

$$\sum_{j=0}^{m}(\Delta \bar{v}_j)^2 \leq \sum_{j=0}^{m-1}(\Delta v_{j+\frac{1}{2}})^2. \tag{4.13}$$

Having (4.13) it suffices to show that

$$\sum_{j=0}^{m}(\Delta v_j)_+^2 \leq \sum_{j=0}^{m}(\Delta \bar{v}_j)^2. \tag{4.14}$$

We split the proof of (4.14) into four cases.

Case 1: $m \geq 4$.

We need the following lemma.

Lemma 4. *The following inequality holds when* $\sigma \geq 0$ *in (2.10)*

$$(\Delta v_1)_+^2 + (\Delta v_0)_+^2 \leq (\Delta \bar{v}_1)^2 + (\Delta \bar{v}_0)^2. \tag{4.15}$$

Proof. We will prove the argument in two steps:

Step 1: First, we will show that $(\Delta v_1)^2 - (\Delta \bar{v}_1)^2 \leq 0$. Observe that,

$$\begin{aligned}(\Delta v_1)^2 - (\Delta \bar{v}_1)^2 &= (\Delta v_1 + \Delta \bar{v}_1)(\Delta v_1 - \Delta \bar{v}_1)\\ &= k_1^{n+1}\left((v_{\frac{3}{2}}^{n+1} - v_{\frac{1}{2}}^{n+1}) - (\bar{v}_{\frac{3}{2}}^{n+1} - \bar{v}_{\frac{1}{2}}^{n+1})\right) \\ &= k_1^{n+1}(\bar{v}_{\frac{1}{2}}^{n+1} - v_{\frac{1}{2}}^{n+1}),\end{aligned} \tag{4.16}$$

where $k_1^{n+1} := \Delta v_1 + \Delta \bar{v}_1 > 0$ as Δv_1 and $\Delta \bar{v}_1$ are both positive. Next, we need to check the sign of $\bar{v}_{\frac{1}{2}}^{n+1} - v_{\frac{1}{2}}^{n+1}$, where

$$\bar{v}_{\frac{1}{2}}^{n+1} = \frac{1}{2}(\bar{v}_0^n + \bar{v}_1^n) + \frac{1}{8}(\bar{v}_0' - \bar{v}_1') - \lambda\left(f(\bar{v}_1^{n+\frac{1}{2}}) - f(\bar{v}_0^{n+\frac{1}{2}})\right), \tag{4.17}$$

$$v_{\frac{1}{2}}^{n+1} = \frac{1}{2}(v_0^n + v_1^n) + \frac{1}{8}(v_0' - v_1') - \lambda\left(f(v_1^{n+\frac{1}{2}}) - f(v_0^{n+\frac{1}{2}})\right). \tag{4.18}$$

Note that $v_1' = \mathrm{m}(v_1^n - v_0^n, v_2^n - v_1^n) = \bar{v}_1'$, and $v_1^{n+\frac{1}{2}} = v_1^n - \frac{\lambda}{2}f'(v_1^n)v_1' = \bar{v}_1^{n+\frac{1}{2}}$, which follows from (4.10). Also observe that $\bar{v}_0' = \mathrm{m}(0, v_1^n - v_0^n) = 0$ and $v_0' \geq 0$ since $\sigma \geq 0$

in (2.10). Subtract (4.18) from (4.17) to get

$$
\begin{aligned}
\bar{v}_{\frac{1}{2}}^{n+1} - v_{\frac{1}{2}}^{n+1} &= \frac{1}{8}(\bar{v}_0' - v_0') + \lambda\left(f(\bar{v}_0^{n+\frac{1}{2}}) - f(v_0^{n+\frac{1}{2}})\right) \\
&= -\frac{1}{8}v_0' + \lambda\left(f(v_0^n) - f(v_0^{n+\frac{1}{2}})\right) \\
&\le -\frac{1}{8}v_0' + \frac{\lambda^2}{2}|f'(v_0^n)|v_0' \max\left(|f'(v_0^n)|, \left|f'(v_0^n - \frac{\lambda}{2}f'(v_0^n)v_0')\right|\right).
\end{aligned} \tag{4.19}
$$

There are two possibilities:

(I) $\max\left(|f'(v_0^n)|, |f'(v_0^n - \frac{\lambda}{2}f'(v_0^n)v_0'))|\right) = |f'(v_0^n)|$. Then, we rewrite (4.19) as

$$
\begin{aligned}
\bar{v}_{\frac{1}{2}}^{n+1} - v_{\frac{1}{2}}^{n+1} &\le -\frac{1}{8}v_0' + \frac{1}{2}|\lambda f'(v_0^n)|^2 v_0' \\
&\le \left(-\frac{1}{8} + \frac{\kappa^2}{2}\right) v_0' \le 0.
\end{aligned} \tag{4.20}
$$

(II) $\max\left(|f'(v_0^n)|, |f'(v_0^n - \frac{\lambda}{2}f'(v_0^n)v_0')|\right) = |f'(v_0^n - \frac{\lambda}{2}f'(v_0^n)v_0')|$. Then, by the mean value theorem and (4.2) we have

$$
|v_1^n - v_0^n| \le \frac{2}{\gamma_1}\max(|f'(v_1^n)|, |f'(v_0^n)|). \tag{4.21}
$$

Now, by the above inequality, Taylor expansion and (4.3) we have

$$
\begin{aligned}
\left|f'(v_0^n - \frac{\lambda}{2}f'(v_0^n)v_0')\right| &\le |f'(v_0^n)|\left(1 + \frac{\lambda}{2}\gamma_2 v_0'\right) \\
&\le |f'(v_0^n)|\left(1 + \lambda\frac{\gamma_2}{\gamma_1}\max(|f'(v_1^n)|, |f'(v_0^n)|\right) \\
&\le |f'(v_0^n)|\left(1 + \kappa\frac{\gamma_2}{\gamma_1}\right).
\end{aligned} \tag{4.22}
$$

We use this result in (4.19) to get

$$
\begin{aligned}
\bar{v}_{\frac{1}{2}}^{n+1} - v_{\frac{1}{2}}^{n+1} &\le -\frac{1}{8}v_0' + \frac{\lambda^2}{2}\left|f'\left(v_0^n - \frac{\lambda}{2}f'(v_0^n)v_0'\right)\right||f'(v_0^n)|v_0' \\
&\le -\frac{1}{8}v_0' + \frac{\lambda^2}{2}\left(1 + \kappa\frac{\gamma_2}{\gamma_1}\right)|f'(v_0^n)|^2 v_0' \\
&\le \left(-\frac{1}{8} + \frac{\kappa^2}{2}\left(1 + \kappa\frac{\gamma_2}{\gamma_1}\right)\right) v_0' \le 0.
\end{aligned} \tag{4.23}
$$

Therefore, in both cases (I) and (II), we conclude that

$$\bar{v}^{n+1}_{\frac{1}{2}} - v^{n+1}_{\frac{1}{2}} \leq 0. \tag{4.24}$$

We finish the proof of Step 1 by observing that the above inequality and (4.16) imply

$$(\Delta v_1)^2 - (\Delta \bar{v}_1)^2 \leq 0. \tag{4.25}$$

Step 2: We now prove that $(\Delta v_0)^2_+ - (\Delta \bar{v}_0)^2 \leq 0$. If $\Delta v_0 < 0$, then $(\Delta v_0)_+ = 0$ and there is nothing to prove. If $\Delta v_0 > 0$, we proceed as in the proof of Step 1 as follows,

$$\begin{aligned}(\Delta v_0)^2 - (\Delta \bar{v}_0)^2 &= (\Delta v_0 + \Delta \bar{v}_0)(\Delta v_0 - \Delta \bar{v}_0) \\ &= k_0^{n+1}\left((v^{n+1}_{\frac{1}{2}} - v^{n+1}_{-\frac{1}{2}}) - (\bar{v}^{n+1}_{\frac{1}{2}} - \bar{v}^{n+1}_{-\frac{1}{2}})\right) \\ &= k_0^{n+1}\left((v^{n+1}_{\frac{1}{2}} - \bar{v}^{n+1}_{\frac{1}{2}}) + (v^n_0 - v^{n+1}_{-\frac{1}{2}})\right),\end{aligned} \tag{4.26}$$

where $k_0^{n+1} := \Delta v_0 + \Delta \bar{v}_0 > 0$. We still need to check the sign of $\Delta v_0 - \Delta \bar{v}_0$ so we rewrite (3.24) for $v^{n+1}_{-\frac{1}{2}}$ and subtract it from v^n_0 to get

$$\begin{aligned}v^n_0 - v^{n+1}_{-\frac{1}{2}} &\leq \frac{1}{2}(v^n_0 - v^n_{-1}) + \left(\frac{1}{4} + \kappa(1+\kappa)c_m\left(\frac{\gamma_2}{\gamma_1} + 1\right)\right)|v^n_0 - v^n_{-1}| \\ &\leq \left(\frac{1}{4} - \kappa(1+\kappa)c_m\left(\frac{\gamma_2}{\gamma_1} + 1\right)\right)(v^n_0 - v^n_{-1}).\end{aligned} \tag{4.27}$$

By similar arguments used in (4.20) and (4.23), we have

$$v^{n+1}_{\frac{1}{2}} - \bar{v}^{n+1}_{\frac{1}{2}} \leq \left(\frac{1}{8} + \frac{\kappa^2}{2}c\right)v'_0, \tag{4.28}$$

where $c = 1$ if the assumption in (I) holds, or $c = (1 + \kappa\gamma_2/\gamma_1)$ if the assumption in (II) holds. Now we apply these bounds to $\Delta v_0 - \Delta \bar{v}_0$,

$$\begin{aligned}\Delta v_0 - \Delta \bar{v}_0 &\leq \left(\frac{1}{8} + \frac{\kappa^2}{2}c\right)v'_0 + \left(\frac{1}{4} - \kappa(1+\kappa)c_m\left(\frac{\gamma_2}{\gamma_1} + 1\right)\right)(v^n_0 - v^n_{-1}) \\ &\leq \left(\frac{1}{8} - \frac{\kappa^2}{2}c - \kappa(1+\kappa)c_m\left(\frac{\gamma_2}{\gamma_1} + 1\right)\right)(v^n_0 - v^n_{-1}) \leq 0.\end{aligned} \tag{4.29}$$

Finally, (4.26) and (4.29) imply

$$(\Delta v_0)_+^2 - (\Delta \bar{v}_0)^2 \leq 0. \tag{4.30}$$

This proves Step 2 and Lemma 4. □

By symmetric arguments it can also be shown that $(\Delta v_{m-1})_+^2 - (\Delta \bar{v}_{m-1})^2 \leq 0$ and $(\Delta v_m)_+^2 - (\Delta \bar{v}_m)^2 \leq 0$. Gathering all the results we have established so far, we end up with

$$\begin{aligned}(\Delta v_0)_+^2 + (\Delta v_1)_+^2 + (\Delta v_{m-1})_+^2 + (\Delta v_m)_+^2 \leq (\Delta \bar{v}_0)_+^2 + (\Delta \bar{v}_1)_+^2 \\ + (\Delta \bar{v}_{m-1})_+^2 + (\Delta \bar{v}_m)_+^2,\end{aligned} \tag{4.31}$$

which leads us to

$$\begin{aligned}\sum_{j=0}^{m} (\Delta v_j)_+^2 &= (\Delta v_0)_+^2 + (\Delta v_1)_+^2 + (\Delta v_{m-1})_+^2 + (\Delta v_m)_+^2 + \sum_{j=2}^{m-2} (\Delta v_j)_+^2 \\ &\leq (\Delta \bar{v}_0)_+^2 + (\Delta \bar{v}_1)_+^2 + (\Delta \bar{v}_{m-1})_+^2 + (\Delta \bar{v}_m)_+^2 + \sum_{j=2}^{m-2} (\Delta \bar{v}_j)_+^2 \\ &= \sum_{j=0}^{m} (\Delta \bar{v}_j)^2.\end{aligned} \tag{4.32}$$

This completes the proof for $m \geq 4$. Next, we consider the remaining cases.

Case 2: $m = 3$.

It is the same as Case 1 except that there are no middle jumps $(\Delta v_2, \ldots, \Delta v_{m-2})$ in (4.11). Hence, we have

$$(\Delta v_0)_+^2 + (\Delta v_1)_+^2 + (\Delta v_2)_+^2 + (\Delta v_3)_+^2 \leq (\Delta \bar{v}_0)^2 + (\Delta \bar{v}_1)^2 + (\Delta \bar{v}_2)^2 + (\Delta \bar{v}_3)^2. \tag{4.33}$$

Case 3: $m = 2$.

We have already proved that

$$(\Delta v_0)_+^2 - (\Delta \bar{v}_0)^2 \leq 0. \tag{4.34}$$

By symmetric arguments it is also true that

$$(\Delta v_2)_+^2 - (\Delta \bar{v}_2)^2 \leq 0. \tag{4.35}$$

Next, we need to show that $\Delta v_1 \leq \Delta \bar{v}_1$ since both are positive. From (4.19) we have $\bar{v}_{\frac{1}{2}}^{n+1} \leq v_{\frac{1}{2}}^{n+1}$ and by analogous arguments $\bar{v}_{\frac{3}{2}}^{n+1} \geq v_{\frac{3}{2}}^{n+1}$. Together they imply

$$\Delta v_1 = v_{\frac{3}{2}}^{n+1} - v_{\frac{1}{2}}^{n+1} \leq \bar{v}_{\frac{3}{2}}^{n+1} - \bar{v}_{\frac{1}{2}}^{n+1} = \Delta \bar{v}_1. \tag{4.36}$$

Therefore, we conclude that

$$(\Delta v_0)_+^2 + (\Delta v_1)_+^2 + (\Delta v_2)_+^2 \leq (\Delta \bar{v}_0)^2 + (\Delta \bar{v}_1)^2 + (\Delta \bar{v}_2)^2. \tag{4.37}$$

Case 4: $m = 1$.

We already have $(\Delta v_0)_+^2 \leq (\Delta \bar{v}_0)^2$. We still need to show that $(\Delta v_1)_+^2 \leq (\Delta \bar{v}_1)^2$. If $\Delta v_1 \leq 0$, then we are done. So suppose $\Delta v_1 > 0$ and observe that

$$\Delta v_1 = v_{\frac{3}{2}}^{n+1} - v_{\frac{1}{2}}^{n+1} \leq v_1^n - v_{\frac{1}{2}}^{n+1} = \bar{v}_{\frac{3}{2}}^{n+1} - v_{\frac{1}{2}}^{n+1} \leq \bar{v}_{\frac{3}{2}}^{n+1} - \bar{v}_{\frac{1}{2}}^{n+1} = \Delta \bar{v}_1, \tag{4.38}$$

and hence,

$$(\Delta v_0)_+^2 + (\Delta v_1)_+^2 \leq (\Delta \bar{v}_0)^2 + (\Delta \bar{v}_1)^2. \tag{4.39}$$

Therefore, in all four cases ($m = 1, 2, 3, 4$), we have

$$\sum_{j=0}^{m} (\Delta v_j)_+^2 \leq \sum_{j=0}^{m} (\Delta \bar{v}_j)^2. \tag{4.40}$$

□

Now, we restrict the choice of σ in the modified minmod limiter (see (2.10)), to be like the one in MAPR (see (2.11)). Namely, we define $\mathrm{m}(a, b)$ in (2.10) with σ

$$\mathrm{sgn}(\sigma) = \mathrm{sgn}(s), \quad \text{where} \quad s = \begin{cases} a, & |a| \leq |b|, \\ b, & |b| \leq |a|. \end{cases} \tag{4.41}$$

Under this assumption on the limiter, we have the following result.

Theorem 4. *Let* $u^0 \in L^\infty(\mathbb{R})$*,* f *be strictly convex in* $R(u^0)$ *and* f'' *be bounded on* $\mathbb{R}$*. Then there exists constant* κ *which depends only on the ratio* γ_1/γ_2 *and an absolute constant* c*,* $c \geq \frac{1}{9000}$*, such that under the CFL condition*

$$\lambda \max_{w \in R(u^0)} |f'(w)| \leq \kappa, \tag{4.42}$$

the NT scheme with the minmod limiter $m(a, b)$ *defined as in (2.10)–(4.41) satisfies the one-sided Lipschitz condition (4.5) provided that*

$$-\min\left(1, \frac{c}{2}\frac{\max(a_+, b_+)}{\min(|a|, |b|)}\right) \leq \sigma \leq 1. \tag{4.43}$$

Proof. The case $\sigma \geq 0$ follows from Theorem 3. Hence, we only consider the case $\sigma \leq 0$. This implies that we take a negative slope reconstruction at local minima and local maxima. Following the same steps as in the proof of Theorem 3, in the general case ($m \geq 4$) we get

$$\Delta v_1 - \Delta \bar{v}_1 \leq -\frac{1}{4} v_0' \quad \text{and} \quad \Delta v_0 - \Delta \bar{v}_0 \leq 0, \tag{4.44}$$

where we recall that $\Delta v_{-\frac{1}{2}} < 0 \leq \Delta v_{\frac{1}{2}}$ and $\sigma < 0$ implies that $|\Delta v_{-\frac{1}{2}}| \leq \Delta v_{\frac{1}{2}}$. Thus,

$$(\Delta v_0)_+^2 + (\Delta v_1)_+^2 \leq (\Delta \bar{v}_0)^2 + \left(\Delta \bar{v}_1 + \frac{1}{4}|\sigma| d\right)^2, \tag{4.45}$$

where $d := \min\left(|\Delta v_{-\frac{1}{2}}|, |\Delta v_{\frac{1}{2}}|\right)$. Notice that by (4.43), we obtain

$$\begin{aligned} \frac{1}{2}\Delta \bar{v}_1 |\sigma| d + \frac{1}{16}|\sigma|^2 d^2 &\leq c\left(\frac{5}{16}(\Delta v_{\frac{1}{2}})^2 + \frac{1}{4}\Delta v_{\frac{1}{2}} \Delta v_{\frac{3}{2}}\right) \\ &\leq c\left((\Delta v_{\frac{1}{2}})^2 + (\Delta v_{\frac{3}{2}} - 2\Delta v_{\frac{1}{2}})^2\right), \end{aligned} \tag{4.46}$$

which implies

$$(\Delta v_0)_+^2 + (\Delta v_1)_+^2 \leq (\Delta \bar{v}_0)^2 + (\Delta \bar{v}_1)^2 + c\left((\Delta^2 \Delta \bar{v}_{-\frac{1}{2}})^2 + (\Delta^2 \Delta \bar{v}_{\frac{1}{2}})^2\right). \tag{4.47}$$

By symmetric arguments, it can also be shown that

$$(\Delta v_{m-1})_+^2 + (\Delta v_m)_+^2 \le (\Delta \bar{v}_{m-1})^2 + (\Delta \bar{v}_m)^2 + c\left((\Delta^2 \Delta \bar{v}_{m-\frac{1}{2}})^2 + (\Delta^2 \Delta \bar{v}_{m+\frac{1}{2}})^2\right). \quad (4.48)$$

The remaining cases, $1 \le m \le 3$, can be handled in a similar way and we skip their proofs. We conclude that

$$\sum_{j=0}^{m} (\Delta v_j)_+^2 \le \sum_{j=0}^{m} (\Delta \bar{v}_j)^2 + c \sum_j (\Delta^2 \Delta \bar{v}_{j+\frac{1}{2}})^2 \le \sum_{j=0}^{m-1} (\Delta v_{j+\frac{1}{2}})^2, \quad (4.49)$$

where the last inequality follows from the one-sided stability result proven in [55] for any non-negative jump sequence, see (56) on page 553 in [55]. □

Remark 9. If the local jumps a, b in the minmod limiter (2.10)–(4.41) are such that

$$\frac{c}{2} \frac{\max(a_+, b_+)}{\min(|a|, |b|)} \ge 1, \quad (4.50)$$

we can recover the MAPR limiter taking σ as in (2.11). Note that (4.50) is always true if $c \ge 2$. However, even though the bound $c \ge \frac{1}{9000}$ can be improved, the current approach does not allow to prove the one-sided Lipschitz condition (4.5) with $c \ge 2$. Therefore, the minmod limiter (2.10)–(4.41) is more restrictive than MAPR in some cases.

Analogous to [55], a maximum principle and an one-sided stability result implies a convergence result. To state the convergence theorem we briefly introduce the space of functions of bounded variation and one-sided Lipschitz classes which are used in the context of conservation laws.

Definition 6. The space $\mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R}))$ consists of all functions $g \in L^1(\mathbb{R})$ such that the seminorm

$$|g|_{\mathrm{Lip}(1,\mathrm{L}^1(\mathbb{R}))} := \limsup_{y>0} \frac{1}{y} \int_{\mathbb{R}} |g(x+y) - g(x)|\, dx \quad (4.51)$$

is finite.

For functions $g \in \mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R}))$ we consider the classes $\mathrm{Lip}(\mathrm{s}, \mathrm{L}^\mathrm{p}(\mathbb{R}))+$ defined by

$$\| \left(g(\cdot - y) - g(\cdot)\right)_+ \|_{L^p(\mathbb{R})} \leq M y^s, \quad y > 0. \tag{4.52}$$

The smallest $M \geq 0$ for which (4.52) holds is denoted by $|g|_{\mathrm{Lip}(\mathrm{s}, \mathrm{L}^\mathrm{p}(\mathbb{R}))+}$. When we set $p = \infty$ and $s = 1$, we obtain the class $\mathrm{Lip}(1, L^\infty(\mathbb{R}))+$ which is the usual one-sided Lipschitz class used in conservation laws denoted by Lip+, see for example [47]. With this notation, by repeating exactly the same steps as in section 4 in [55], we obtain the following convergence result.

Theorem 5. *Let $u^0 \in \mathrm{Lip}(1, \mathrm{L}^1(\mathbb{R})) \cap \mathrm{Lip}(1, \mathrm{L}^2(\mathbb{R}))+$. Then, there exists $\kappa > 0$ such that under the CFL condition $\lambda ||f'||_{L^\infty(\mathbb{R})} \leq \kappa$ the NT scheme based on the limiter (2.10)–(4.41)–(4.43) converges to the unique entropy solution of (2.1).*

CHAPTER V

CONVERGENCE

A. Introduction

In the previous chapter, we proved an one-sided stability result for the NT scheme based on the minmod limiter and also on MAPR-like limiters. That implies the convergence of the second order NT scheme for any initial data satisfying some type of an one-sided condition, see [55]. This seemed to be unavoidable restriction when proving convergence for methods satisfying the so-called one-sided Lipschitz condition, see for example [41, 48]. In this chapter, we use the one-sided stability results from Chapter IV and [55], and prove, using compensated compactness arguments, that the NT scheme converges strongly to a weak solution and the limit satisfies a weak form of an entropy inequality. Because a single entropy inequality is enough to select the unique entropy solution in the case of a scalar strictly convex flux [14, 53], we conclude that the NT scheme converges to the unique entropy solution for any bounded initial data. The main contribution of this result is that we prove convergence of the NT scheme without imposing any one-sided conditions on the initial data and the result holds for the largest possible class of initial conditions, i.e., the class of initial data where we have existence-uniqueness of the entropy solution of the PDE and finite global speed of propagation. It should be possible to generalize the results of this paper to other second order schemes because there is a lot of numerical evidence that many other schemes also satisfy the one-sided stability property from [55] but we do not explore this here.

Section B proves our main result, Theorem 6: the NT scheme converges strongly on compact sets to the entropy solution of a scalar strictly convex conservation law with bounded initial data. Also in Subsection 1, an entropy production bounds for the NT scheme are derived, which are interesting in their own right.

B. Convergence of the NT scheme

We recall the definition of the range of a function, see Definition 5: for a given function $g : \mathbb{R} \to \mathbb{R}$ the range of g is defined to be the interval

$$R(g) := [\operatorname{essinf}_{x\in\mathbb{R}} g(x), \operatorname{esssup}_{x\in\mathbb{R}} g(x)]. \tag{5.1}$$

The following theorem is the main result of this chapter.

Theorem 6. *Let $f \in C^4(R(u^0))$. Then, under the assumptions of Theorem 3, there exists a constant $\kappa_0 > 0$ such that under the CFL condition*

$$\lambda \max_{w\in R(u^0)} |f'(w)| \leq \kappa_0, \tag{5.2}$$

the NT scheme described by (2.12)-(2.13) converges strongly on compact sets of $\mathbb{R}\times\ [0,\infty)$ to the unique entropy solution of (2.1).

Remark 10. Note that the result is also valid for σ (see (2.10)) chosen as in (4.43). Even in the special case of the minmod limiter, $\sigma = 0$, this is the first convergence result for the NT scheme for conservation laws with just bounded initial data.

Proof. In order to prove the strong convergence result we are going to employ the following compensated compactness lemma, see [8, 44] for details on compensated compactness.

Lemma 5. *Suppose $\{u^\varepsilon\}$ is a sequence of measurable functions on $\mathbb{R}\times(0,\infty)$ that satisfies the following two conditions:*

(1) There exist constants $-\infty < m < M < \infty$*, both independent of* ε*, such that*

$$m \le u^\varepsilon \le M \quad \text{for a.e.} \quad (x,t) \in \mathbb{R} \times (0,\infty);$$

(2) The two sequences

$$\{S_i(u^\varepsilon)_t + Q_i(u^\varepsilon)_x\}_{\varepsilon>0}, \quad i = 1, 2, \tag{5.3}$$

belong to a compact subset of $W^{-1,2}_{loc}(\mathbb{R} \times (0,\infty))$*, where*

$$S_1(u) = u - k, \qquad Q_1(u) = f(u), \tag{5.4}$$

and

$$S_2(u) = f(u) - f(k), \qquad Q_2(u) = \int (f'(u))^2 du, \tag{5.5}$$

and k *is an arbitrary constant. Then, there exists a subsequence of* $\{u^\varepsilon\}_{\varepsilon>0}$ *that converges a.e. to a function* $u \in L^\infty(\mathbb{R} \times (0,\infty))$*.*

We already know that the NT scheme with the limiter (2.10), under the assumptions of Theorem 6, satisfies a maximum principle, see Theorem 2. To verify that (5.3) is a compact subset of $W^{-1,2}_{loc}(\mathbb{R} \times (0,\infty))$ we are going to use the following well-known functional analysis lemma:

Lemma 6 (Murat Lemma)**.** *Let* $\Omega \in \mathbb{R}^d$ *be a bounded open set. Let* q *and* r *be constants satisfying* $1 < q \le 2 < r < \infty$*. Then*

$$\left\{ \text{ compact set of } W^{-1,q}_{loc}(\Omega) \right\} \cap \left\{ \text{ bounded set of } W^{-1,r}_{loc}(\Omega) \right\}$$

$$\subset \left\{ \text{ compact set of } W^{-1,2}_{loc}(\Omega) \right\}.$$

In the proof we will use the the following index set:

Definition 7. Let Ω be a fixed subset of $\mathbb{R} \times [0,\infty)$. Given n and mesh size $(\Delta x, \Delta t)$, we define an index set

$$\Lambda_n(\Omega) := \{j : (j\Delta x, n\Delta t) \in \Omega, \text{ and } j \in \mathbb{Z},\ n = even, \text{ and } j \in \mathbb{Z} + 1/2,\ n = odd\}.$$

When there is no ambiguity we will denote $\Lambda_n(\Omega)$ simply by Λ_n. The compensated compactness proof follows the framework of [29]. In order to make the presentation simpler we break the rest of the proof into three main steps.

1. Entropy production bounds

We are going to establish an entropy dissipation estimate as follows: (i) a cubic bound involving the l_3 norm of the jumps $\{\Delta v^n_{j+\frac{1}{2}}\}$; (ii) a possibly degenerate quadratic bound involving a weighted l_2 norm of the jumps. The approach is based on a discrete entropy production representation first established by Lax for the first order LxF scheme [36] and later extended by Nessyahu and Tadmor for the second order NT scheme [47], see Appendix B for details. We start with the cubic bound.

Lemma 7. *Let $\Omega = [-X, X] \times [0, T]$ where $X > 0$ and $T > 0$, and let $N = \lfloor T/\Delta t \rfloor$. Under the assumptions of Theorem 6 the NT scheme described by(2.12)-(2.13) satisfies the bound*

$$\Delta x \sum_{n=0}^{N} \sum_{j \in \Lambda_n} |\Delta v^n_{j+\frac{1}{2}}|^3 \leq C, \tag{5.6}$$

where C is a constant that may depend on $\gamma_1, \gamma_1/\gamma_2, |\Omega|, \lambda, \kappa_0$ and $||u^0||_{L^\infty(\mathbb{R})}$ but it is independent of the mesh size.

Proof. Following [47], we will establish a discrete entropy production identity, see (5.38), and then use it to prove the lemma. We define,

$$g(v) := \frac{\Delta g}{\Delta v}(v - v_j) + g_j, \quad \text{where} \quad g_j := f(v_j^{n+\frac{1}{2}}) + \frac{1}{8\lambda} v'_j \quad \text{and} \quad \Delta g := g_{j+1} - g_j, \tag{5.7}$$

where $\Delta v := \Delta v^n_{j+\frac{1}{2}}$. Next, we rewrite (2.13) as

$$v^{n+1}_{j+\frac{1}{2}} = \frac{1}{2}(v^n_j + v^n_{j+1}) - \lambda(g_{j+1} - g_j). \tag{5.8}$$

For a given entropy pair (S, Q) we approximate the entropy production $S_t(v) + Q_x(v)$ at $(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}})$ by

$$\begin{aligned} &S(v^{n+1}_{j+\frac{1}{2}}) - \frac{1}{2}\left(S(v^n_j) + S(v^n_{j+1})\right) + \lambda\left(Q(v^n_{j+1}) - Q(v^n_j)\right) \\ &= -\int_0^1\int_0^1 \frac{s}{4}\, S''(u(r,s))\,(1 - 2\lambda g'(w(r,s)))\,(1 + 2\lambda g'(v(s)))\,(\Delta v)^2\, dr\, ds \\ &+ \int_0^1 \lambda S'(v(s))(f'(v(s)) - g'(v(s)))\Delta v\, ds, \end{aligned} \tag{5.9}$$

where

$$v(s) = s v^n_j + (1 - s) v^n_{j+1}, \tag{5.10}$$

$$w(r, s) = r(v(s)) + (1 - r) v^n_{j+1}, \tag{5.11}$$

$$u(r, s) = \frac{v(s) + w(r, s)}{2} + \lambda(g(v(s)) - g(w(r, s))). \tag{5.12}$$

We refer the reader to the Appendix B for all details of the derivation of (5.9), see also [36, 47]. Let $S(u) = \frac{u^2}{2}$ and let us denote the integrals in (5.9) by I and J, respectively. After exact integration, we obtain

$$I = -\frac{1}{8}\left[1 - \left(2\lambda\frac{\Delta g}{\Delta v}\right)^2\right](\Delta v)^2. \tag{5.13}$$

For the second integral J, we use integration by parts and the trapezoidal rule to get

$$\begin{aligned} J &= \lambda \int_{v^n_j}^{v^n_{j+1}} z\left(f'(z) - g'(z)\right)\, dz \\ &= \lambda z(f(z) - g(z))\Big|_{v^n_j}^{v^n_{j+1}} - \frac{\lambda\Delta v}{2}(f(v^n_j) - g_j + f(v^n_{j+1}) - g_{j+1}) \\ &+ \frac{\lambda(\Delta v)^3}{12} f''(\xi_1), \end{aligned} \tag{5.14}$$

for some ξ_1 between v_j^n and v_{j+1}^n. From now on ξ_i will be used in the same spirit as an arbitrary point in the remainder term of an expansion. Next, we introduce a numerical entropy flux following the NT paper [47],

$$G(v_j^n) := Q(v_j^n) - v_j^n(f(v_j^n) - g_j), \tag{5.15}$$

and define

$$\bar{J} := J - \lambda z(f(z) - g(z))\Big|_{v_j^n}^{v_{j+1}^n}. \tag{5.16}$$

We restrict our attention to the numerical entropy production $E_{j+\frac{1}{2}}^{n+\frac{1}{2}} = S_t + G_x$ at $(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}})$, which is given by

$$\begin{aligned}
E_{j+\frac{1}{2}}^{n+\frac{1}{2}} &:= S(v_{j+\frac{1}{2}}^{n+1}) - \frac{1}{2}\left(S(v_j^n) + S(v_{j+1}^n)\right) + \lambda\left(G(v_{j+1}^n) - G(v_j^n)\right) \\
&= -\frac{1}{8}\left[1 - \left(2\lambda\frac{\Delta g}{\Delta v}\right)^2\right](\Delta v)^2 \\
&- \frac{\lambda\Delta v}{2}(f(v_j^n) - g_j + f(v_{j+1}^n) - g_{j+1}) + \frac{\lambda(\Delta v)^3}{12}f''(\xi_1) \\
&= I + \bar{J}.
\end{aligned} \tag{5.17}$$

We introduce the notation $a_j := f'(v_j^n)$ and start working with the term $\bar{J}$. By Taylor expansion, we have

$$\begin{aligned}
g_j &= f(v_j^n) - \frac{v_j'}{8\lambda}\left((2\lambda a_j)^2 - 1\right) + \frac{1}{8}\left(\lambda a_j v_j'\right)^2 f''(\xi_2), \\
g_{j+1} &= f(v_{j+1}^n) - \frac{v_{j+1}'}{8\lambda}\left((2\lambda a_{j+1})^2 - 1\right) + \frac{1}{8}\left(\lambda a_{j+1} v_{j+1}'\right)^2 f''(\xi_3).
\end{aligned} \tag{5.18}$$

Thus, $\bar{J}$ can be rewritten as

$$\begin{aligned}
\bar{J} = &\ \frac{\Delta v}{16}\left[v_j' + v_{j+1}' - ((2\lambda a_j)^2 v_j' + (2\lambda a_{j+1})^2 v_{j+1}')\right] \\
&+ \frac{\lambda(\Delta v)^3}{16}\left[(\lambda a_j)^2\left(\frac{v_j'}{\Delta v}\right)^2 f''(\xi_2) + (\lambda a_{j+1})^2\left(\frac{v_{j+1}'}{\Delta v}\right)^2 f''(\xi_3) + \frac{4}{3}f''(\xi_1)\right] \\
&=: \frac{\Delta v}{16}A_1 + \frac{\lambda(\Delta v)^3}{16}A_2.
\end{aligned} \tag{5.19}$$

We now transform A_1. Applying Taylor expansion to $f'(v_j^n)$ and $f'(v_{j+1}^n)$ yields

$$\begin{aligned} a_j = f'(v_j^n) = f'(v_{j+\frac{1}{2}}) - \frac{\Delta v}{2} f''(\xi_4) = a_{j+\frac{1}{2}} - \frac{\Delta v}{2} f''(\xi_4), \\ a_{j+1} = f'(v_{j+1}^n) = f'(v_{j+\frac{1}{2}}) + \frac{\Delta v}{2} f''(\xi_5) = a_{j+\frac{1}{2}} + \frac{\Delta v}{2} f''(\xi_5), \end{aligned} \tag{5.20}$$

where $v_{j+\frac{1}{2}} = \frac{v_j^n + v_{j+1}^n}{2}$. This enables us to write A_1 as

$$\begin{aligned} A_1 &= v_j' + v_{j+1}' - \Big((2\lambda a_{j+\frac{1}{2}} - \lambda \Delta v f''(\xi_4))^2 v_j' + (2\lambda a_{j+\frac{1}{2}} + \lambda \Delta v f''(\xi_5))^2 v_{j+1}' \Big) \\ &= (1 - 4\beta^2)(v_j' + v_{j+1}') - \lambda(\Delta v)^2 \Big[4\beta f''(\xi_5) \left(\frac{v_{j+1}'}{\Delta v} \right) - 4\beta f''(\xi_4) \left(\frac{v_j'}{\Delta v} \right) \\ &\qquad + f''(\xi_4)(\lambda f''(\xi_4) v_j') + f''(\xi_5)(\lambda f''(\xi_5) v_{j+1}') \Big] \\ &=: (1 - 4\beta^2)(v_j' + v_{j+1}') - \lambda(\Delta v)^2 A_3, \end{aligned} \tag{5.21}$$

where $\beta = \lambda a_{j+\frac{1}{2}}$. Thus, we obtain

$$\bar{J} = \frac{(\Delta v)^2}{8}(1 - 4\beta^2) \frac{v_j' + v_{j+1}'}{2\Delta v} + \frac{\lambda(\Delta v)^3}{16}(A_2 - A_3). \tag{5.22}$$

Observe that (5.20) implies

$$|\lambda f''(\xi_4)\Delta v|, \ |\lambda f''(\xi_5)\Delta v| \leq 4\kappa, \quad \text{where} \ \ \kappa := \lambda \max_{w \in R(u^0)} |f'(w)|. \tag{5.23}$$

Using the above estimate and the convexity of f, we have the following bounds for A_2 and A_3,

$$\frac{4}{3}\gamma_1 \leq A_2 \qquad \text{and} \qquad |A_3| \leq 8\gamma_2(\beta + \kappa). \tag{5.24}$$

Next, we turn our attention to I in (5.17). By Taylor expansion we have,

$$\begin{aligned} f(v_j^n) = f(v_{j+\frac{1}{2}}) - \frac{\Delta v}{2} a_{j+\frac{1}{2}} + \frac{(\Delta v)^2}{8} f''(\xi_6) \\ f(v_{j+1}^n) = f(v_{j+\frac{1}{2}}) + \frac{\Delta v}{2} a_{j+\frac{1}{2}} + \frac{(\Delta v)^2}{8} f''(\xi_7). \end{aligned} \tag{5.25}$$

Using (5.20) and (5.25) in (5.18), we derive

$$\begin{aligned}\frac{\lambda\Delta g}{\Delta v} =&\ \beta + \frac{1}{8}(1-4\beta^2)\frac{\Delta v'}{\Delta v} + \frac{\lambda\Delta v}{8}(f''(\xi_7) - f''(\xi_6))\\ &- \frac{\beta}{2}(\lambda f''(\xi_5)v'_{j+1} + \lambda f''(\xi_4)v'_j)\\ &- \frac{1}{8}\left[(\lambda\Delta v f''(\xi_5))^2\,\frac{v'_{j+1}}{\Delta v} - (\lambda\Delta v f''(\xi_4))^2\,\frac{v'_j}{\Delta v}\right]\\ &+ \frac{\lambda\Delta v}{8}\left[(\lambda a_{j+1})^2\left(\frac{v'_{j+1}}{\Delta v}\right)^2 f''(\xi_3) - (\lambda a_j)^2\left(\frac{v'_j}{\Delta v}\right)^2 f''(\xi_2)\right]\\ =:&\ \beta + \frac{1}{8}(1-4\beta^2)\frac{\Delta v'}{\Delta v} + A_4,\end{aligned} \tag{5.26}$$

where $\Delta v' := v'_{j+1} - v'_j$. Recall that

$$\begin{aligned}f(v_j^{n+\frac{1}{2}}) &= f(v_j^n - \frac{\lambda}{2}a_j v'_j) = f(v_j^n) - \frac{\lambda}{2}a_j^2 v'_j + \frac{1}{8}(\lambda a_j v'_j)^2 f''(\xi_2),\\ f(v_{j+1}^{n+\frac{1}{2}}) &= f(v_{j+1}^n - \frac{\lambda}{2}a_{j+1}v'_{j+1}) = f(v_{j+1}^n) - \frac{\lambda}{2}a_{j+1}^2 v'_{j+1}\\ &\qquad + \frac{1}{8}(\lambda a_{j+1}v'_{j+1})^2 f''(\xi_3).\end{aligned} \tag{5.27}$$

Then we obtain

$$\begin{aligned}&\frac{\lambda\Delta v}{8}\left[(\lambda a_{j+1})^2\left(\frac{v'_{j+1}}{\Delta v}\right)^2 f''(\xi_3) - (\lambda a_j)^2\left(\frac{v'_j}{\Delta v}\right)^2 f''(\xi_2)\right]\\ &= -\frac{\lambda}{\Delta v}\left[f(v_{j+1}^n) - f(v_j^n)\right] + \frac{\lambda}{\Delta v}\left[f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})\right]\\ &+ \frac{1}{2}\left[(\lambda a_{j+1})^2\frac{v'_{j+1}}{\Delta v} - (\lambda a_j)^2\frac{v'_j}{\Delta v}\right].\end{aligned} \tag{5.28}$$

Now, we need to bound $|f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})|$. In Appendix A, we have derived such a bound which is needed to prove a maximum principle of the NT scheme with convex flux , see (A.10). We recall that estimate here:

$$|f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})| \le \left(\frac{\gamma_2(1+\kappa)}{2\gamma_1} + 1\right)(1+\kappa)\max_j |f'(v_j^n)||\Delta v|. \tag{5.29}$$

Using the above in (5.28) we get

$$|A_4| \le C\kappa, \tag{5.30}$$

and similar to (5.24) we have

$$|A_4| \le C\gamma_2\lambda|\Delta v|, \tag{5.31}$$

where C denotes a constant independent of the mesh size, which might change on each occurrence later in this paper. We proceed by substituting (5.26) into (5.13) to get

$$\begin{aligned} I = -\frac{1}{8}\Bigg[&(1-4\beta^2)\left(1-\frac{1}{16}(1-4\beta^2)\left(\frac{\Delta v'}{\Delta v}\right)^2-\beta\frac{\Delta v'}{\Delta v}\right)\\ &-\left(8\beta+(1-4\beta^2)\frac{\Delta v'}{\Delta v}\right)A_4-4(A_4)^2\Bigg](\Delta v)^2. \end{aligned} \tag{5.32}$$

Using this bound together with (5.22), we have

$$\begin{aligned} I+\bar{J} = -\frac{1}{8}(\Delta v)^2(1-4\beta^2)&\left[1-\frac{1}{16}(1-4\beta^2)\left(\frac{\Delta v'}{\Delta v}\right)^2\right.\\ &\left.-(\beta+A_4)\frac{\Delta v'}{\Delta v}-\frac{v'_j+v'_{j+1}}{2\Delta v}\right]+\frac{\lambda(\Delta v)^3}{16}\left(A_2-A_3+\frac{16\beta A_4}{\lambda\Delta v}+\frac{8(A_4)^2}{\lambda\Delta v}\right)\\ &=: -\frac{1}{8}\mu_{j+\frac{1}{2}}(\Delta v)^2+\frac{1}{16}\lambda\nu_{j+\frac{1}{2}}(\Delta v)^3. \end{aligned} \tag{5.33}$$

Next, we are going to show that $\mu_{j+\frac{1}{2}}$ and $\nu_{j+\frac{1}{2}}$ are non-negative. We start with $\mu_{j+\frac{1}{2}}$ and use the inequality (5.30) to derive

$$\begin{aligned} \frac{\mu_{j+\frac{1}{2}}}{1-4\beta^2} &\ge 1-\frac{1}{16}(1-4\beta^2)\left(\frac{\Delta v'}{\Delta v}\right)^2-(\beta+C\kappa)\left|\frac{\Delta v'}{\Delta v}\right|-\frac{v'_j+v'_{j+1}}{2\Delta v}\\ &=\left(\frac{1}{2}-(\beta+C\kappa)-\frac{1}{16}(1-4\beta^2)\left|\frac{\Delta v'}{\Delta v}\right|\right)\left|\frac{\Delta v'}{\Delta v}\right|\\ &+1-\left(\frac{|v'_{j+1}-v'_j|}{2|\Delta v|}+\frac{v'_j+v'_{j+1}}{2\Delta v}\right)\ge 0. \end{aligned} \tag{5.34}$$

Note that, using (5.30) and (5.31) we obtain

$$\left|\frac{8(A_4)^2}{\lambda\Delta v}\right| \le C\gamma_2\kappa \quad \text{and} \quad \left|\frac{16\beta A_4}{\lambda\Delta v}\right| \le C\gamma_2\beta. \tag{5.35}$$

Using (5.24) and (5.35), we get the following bound for $\nu_{j+\frac{1}{2}}$:

$$\nu_{j+\frac{1}{2}} = A_2 - A_3 + \frac{16\beta A_4}{\lambda\Delta v} + \frac{8(A_4)^2}{\lambda\Delta v} \ge \frac{4}{3}\gamma_1 - C\gamma_2(\beta+\kappa) \ge 0. \tag{5.36}$$

Hence, there exists $\kappa_0 > 0$ such that for all $\kappa \le \kappa_0$ we have

$$0 < \gamma_1 \le \nu^n_{j+\frac{1}{2}} \le C\gamma_2. \tag{5.37}$$

So far, we have shown that

$$\begin{aligned} E^{n+\frac{1}{2}}_{j+\frac{1}{2}} &= S(v^{n+1}_{j+\frac{1}{2}}) - \frac{1}{2}\left(S(v^n_j) + S(v^n_{j+1})\right) + \lambda\left(G(v^n_{j+1}) - G(v^n_j)\right) \\ &= -\frac{1}{8}\mu_{j+\frac{1}{2}}(\Delta v)^2 + \frac{1}{16}\lambda\nu_{j+\frac{1}{2}}(\Delta v)^3, \end{aligned} \tag{5.38}$$

where $\mu_{j+\frac{1}{2}}, \nu_{j+\frac{1}{2}} \ge 0$ for sufficiently small κ_0.

Before we proceed, let us recall the so-called cone of dependence for the numerical solution, see [31]. The numerical solution $v(\cdot, T)$ on $[-X, X]$ depends only on the values of u^0 on $[-X-d, X+d]$, where $d = \lfloor \frac{3T}{2\lambda} \rfloor + 2h$. Next, let us define $M := \lfloor \frac{X+2d}{\Delta x} \rfloor + 1$ and $\bar{\Omega} := [-M\Delta x, M\Delta x] \times [0, (N+1)\Delta t]$, and introduce the numerical solution $w(x,t)$ on $\bar{\Omega}$, with the initial condition

$$w^0(x) = \begin{cases} u^0(x), & -X - d \le x \le X + d, \\ u^0(X+d), & X + d < x, \\ u^0(-X-d), & x < -X - d. \end{cases} \tag{5.39}$$

Observe that the numerical solutions $w(x,t)$ and $v(x,t)$ agree on the domain Ω. Next, we rewrite (5.38) for $w(x,t)$,

$$\begin{aligned}\tilde{E}^{n+\frac{1}{2}}_{j+\frac{1}{2}} &= S(w^{n+1}_{j+\frac{1}{2}}) - \frac{1}{2}\left(S(w^n_j) + S(w^n_{j+1})\right) + \lambda\left(G(w^n_{j+1}) - G(w^n_j)\right) \\ &= -\frac{1}{8}\tilde{\mu}_{j+\frac{1}{2}}(\Delta w)^2 + \frac{1}{16}\lambda\tilde{\nu}_{j+\frac{1}{2}}(\Delta w)^3,\end{aligned} \tag{5.40}$$

where $\tilde{\mu}_{j+\frac{1}{2}} \geq 0$ and $\tilde{\nu}_{j+\frac{1}{2}} \geq 0$ are the corresponding quantities for $w(\cdot,\cdot)$.

We proceed by summing over all n and j such that $(j\Delta x, n\Delta t) \in \bar{\Omega}$,

$$\sum_{n=0}^{N}\sum_{j\in\Lambda_n(\bar{\Omega})} \tilde{E}^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \sum_{n=0}^{N}\sum_{j\in\Lambda_n(\bar{\Omega})} -\frac{1}{8}\tilde{\mu}^n_{j+\frac{1}{2}}(\Delta w)^2 + \frac{1}{16}\lambda\tilde{\nu}^n_{j+\frac{1}{2}}(\Delta w)^3. \tag{5.41}$$

Note that the left-hand side can be written as

$$\begin{aligned}\sum_{n=0}^{N}\sum_{j\in\Lambda_n(\bar{\Omega})} \tilde{E}^{n+\frac{1}{2}}_{j+\frac{1}{2}} = &\sum_{j=-M+1}^{M} S(w^{N+1}_j) - S(w^0_j) \\ &+ \lambda(N+1)\left(G(w^0_{M+1}) - G(w^0_{-M})\right),\end{aligned} \tag{5.42}$$

if N is odd and

$$\begin{aligned}\sum_{n=0}^{N}\sum_{j\in\Lambda_n(\bar{\Omega})} \tilde{E}^{n+\frac{1}{2}}_{j+\frac{1}{2}} = &\sum_{j=-M}^{M} S(w^{N+1}_{j+\frac{1}{2}}) - \frac{1}{2}\left(S(w^0_{-M}) + S(w^0_{M+1})\right) - \sum_{j=-M+1}^{M} S(w^0_j) \\ &+ \lambda(N+1)\left(G(w^0_{M+1}) - G(w^0_{-M})\right),\end{aligned} \tag{5.43}$$

if N is even. Thus,

$$\begin{aligned}&-\sum_{j=-M}^{M+1} S(w^0_j) + \lambda(N+1)\left(G(w^0_{M+1}) - G(w^0_{-M})\right) \\ &\leq \sum_{n=0}^{N}\sum_{j\in\Lambda_n(\bar{\Omega})} -\frac{1}{8}\tilde{\mu}^n_{j+\frac{1}{2}}(\Delta w)^2 + \frac{1}{16}\lambda\tilde{\nu}^n_{j+\frac{1}{2}}(\Delta w)^3 \\ &= \sum_{n=0}^{N}\sum_{j\in\Lambda_n(\bar{\Omega})} \left[-\frac{1}{8}\tilde{\mu}^n_{j+\frac{1}{2}}(\Delta w)^2 + \frac{1}{16}\lambda\tilde{\nu}^n_{j+\frac{1}{2}}(\Delta w)^3_- + \frac{1}{16}\lambda\tilde{\nu}^n_{j+\frac{1}{2}}(\Delta w)^3_+\right],\end{aligned} \tag{5.44}$$

where we use the standard $+$ and $-$ notation: $x_+ = \max(x,0)$ and $x_- = \min(x,0)$.

Thus, we obtain,

$$-\sum_{n=0}^{N}\sum_{j\in\Lambda_n(\bar\Omega)}\frac{1}{16}\lambda\tilde\nu^n_{j+\frac12}(\Delta w)^3_- \le \sum_j S(w^0_j) - \lambda(N+1)\left(G(w^0_{M+1}) - G(w^0_{-M})\right) + \sum_{n=0}^{N}\sum_{j\in\Lambda_n(\bar\Omega)}\frac{1}{16}\lambda\tilde\nu^n_{j+\frac12}(\Delta w)^3_+. \tag{5.45}$$

We now need the key one-sided stability result for the NT scheme from [55], see (56) on page 553 in [55]:

$$c\,\gamma_1\lambda\sum_j(\Delta w)^3_+ \le \sum_j(\Delta w)^2_+ - \sum_j(\Delta\bar w)^2_+, \tag{5.46}$$

where $\Delta\bar w := \Delta\bar w^{n+1}_j = \bar w^{n+1}_{j+\frac12} - \bar w^{n+1}_{j-\frac12}$ are the jumps generated by the limiter (2.10) with $\sigma = 0$, and $c > 0$ is an absolute constant. Note that (5.46) holds for any $\kappa \le \kappa_0$ when κ_0 is sufficiently small but fixed. In the case of a MAPR-like minmod limiter, as described in Theorem 3, if the jump sequence is generated starting with the same $\{w^n_j\}$, we have that the l_2 norm of the jumps generated with the MAPR-like limiter is dominated by the l_2 norm of the jumps generated with the minmod limiter. That is

$$\sum_j(\Delta\tilde w)^2_+ \le \sum_j(\Delta\bar w)^2_+, \tag{5.47}$$

where $\Delta\tilde w := \Delta w^{n+1}_j = w^{n+1}_{j+\frac12} - w^{n+1}_{j-\frac12}$ and $\{w^{n+1}_{j+\frac12}\}$ is generated with the MAPR-like limiter. This result is proven in [45], see (3.13) and Theorem 3.1 there. Therefore, we have

$$c\,\gamma_1\lambda\sum_j(\Delta w)^3_+ \le \sum_j(\Delta w)^2_+ - \sum_j(\Delta\tilde w)^2_+. \tag{5.48}$$

Summing (5.48) over n yields

$$c\gamma_1\lambda\sum_{n=0}^{N}\sum_{j\in\Lambda_n(\bar\Omega)}(\Delta w)^3_+ \le \sum_j(\Delta w^0)^2_+ - \sum_j(\Delta\tilde w^N)^2_+ \le \sum_j S(w^0_j). \tag{5.49}$$

Using (5.49) in (5.45), we derive

$$-\lambda \sum_{n=0}^{N} \sum_{j\in\Lambda_n(\bar{\Omega})} (\Delta w)_-^3 \leq C \sum_j S(w_j^0) - C\lambda(N+1)\left(G(w_{M+1}^n) - G(w_{-M}^n)\right). \quad (5.50)$$

Thus, we obtain

$$\lambda\, \Delta x \sum_{n=0}^{N} \sum_{j\in\Lambda_n(\bar{\Omega})} |\Delta w|^3 \leq C\,\Delta x \sum_j (w_j^0)^2 + C \leq C \int_{-M\Delta x}^{(M+1)\Delta x} (u^0(x))^2 dx + C$$

$$\leq 2C(M+1)\Delta x ||u^0||_{L^\infty}^2 + C \leq C\left(||u^0||_{L^\infty(\mathbb{R})}^2 + 1\right). \quad (5.51)$$

Since $w(x,t)$ and $v(x,t)$ agree on the domain $\Omega \subset \bar{\Omega}$, we get

$$\lambda\, \Delta x \sum_{n=0}^{N} \sum_{j\in\Lambda_n(\Omega)} |\Delta v|^3 \leq C\left(||u^0||_{L^\infty(\mathbb{R})}^2 + 1\right). \quad (5.52)$$

□

Now we proceed with the quadratic entropy production bound.

Lemma 8. *Under the assumptions of Lemma 7, the NT scheme described by (2.12)-(2.13) satisfies the bound*

$$\Delta x \sum_{n=0}^{N} \sum_{j\in\Lambda_n} \bar{\mu}_{j+\frac{1}{2}} |\Delta v_{j+\frac{1}{2}}^n|^2 \leq C, \quad (5.53)$$

where C is a constant independent of the mesh size and $\bar{\mu}_{j+\frac{1}{2}}$ is defined by

$$\bar{\mu}_{j+\frac{1}{2}} := \frac{1}{8}(1-4\beta^2) f''(v_{j+\frac{1}{2}}) \left(1 - \frac{1}{16}(1-4\beta^2)\left(\frac{\Delta v'}{\Delta v}\right)^2 - \beta\frac{\Delta v'}{\Delta v} - \frac{v_j' + v_{j+1}'}{2\Delta v}\right),$$

and we recall that $\beta = \lambda a_{j+\frac{1}{2}} = \lambda f'(v_{j+\frac{1}{2}})$.

Proof. As in the proof of Lemma 7, for any given entropy pair (S, Q) we approximate the entropy production $S_t(v) + Q_x(v)$ at $(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}})$ by

$$\begin{aligned} & S(v^{n+1}_{j+\frac{1}{2}}) - \frac{1}{2}(S(v^n_j) + S(v^n_{j+1})) + \lambda(Q(v^n_{j+1}) - Q(v^n_j)) \\ & = -\int_0^1 \int_0^1 \frac{s}{4} S''(u(r,s))\,(1 - 2\lambda g'(w(r,s)))\,(1 + 2\lambda g'(v(s)))\,(\Delta v)^2\,dr\,ds \\ & + \int_0^1 \lambda S'(v(s))(f'(v(s)) - g'(v(s)))\Delta v\,ds. \end{aligned} \tag{5.54}$$

Let k be the unique constant such that $f(k) = \min_{u \in R(u^0)} f(u)$ and let $S(u) = f(u) - f(k)$. As before, we denote the first and second term in the right-hand side of (5.54) by I and J, respectively. Using (5.31) in (5.32) and the definition of $u(r,s)$, see (5.10)–(5.12), we transform I as follows:

$$\begin{aligned} I &= -\frac{1}{4}\left(1 - \left(2\lambda\frac{\Delta g}{\Delta v}\right)^2\right)(\Delta v)^2 \int_0^1 \int_0^1 s f''(u(r,s))\,dr\,ds \\ &= -\frac{1}{8}\left(1 - \left(2\lambda\frac{\Delta g}{\Delta v}\right)^2\right) f''(v_{j+\frac{1}{2}})(\Delta v)^2 + O((\Delta v)^3) \\ &= -\frac{1}{8} f''(v_{j+\frac{1}{2}})(1 - 4\beta^2)\left(1 - \frac{1}{16}(1 - 4\beta^2)\left(\frac{\Delta v'}{\Delta v}\right)^2 - \beta\frac{\Delta v'}{\Delta v}\right)(\Delta v)^2 \\ & + O((\Delta v)^3), \end{aligned} \tag{5.55}$$

where we use the standard $O((\Delta v)^3)$ notation, $|O((\Delta v)^3)| \leq c|\Delta v|^3$, with constant c only depending on $R(u^0), \kappa_0$ and $||f||_{C^4(R(u^0))}$. For the term J we use integration by parts and the trapezoidal integration formula to get

$$\begin{aligned} J &= \lambda \int_{v^n_j}^{v^n_{j+1}} f'(z)\,(f'(z) - g'(z))\,dz \\ &= \lambda f'(z)(f(z) - g(z))\Big|_{v^n_j}^{v^n_{j+1}} \\ & - \frac{\lambda\Delta v}{2}(f''(v^n_j)(f(v^n_j) - g_j) + f''(v^n_{j+1})(f(v^n_{j+1}) - g_{j+1})) + O((\Delta v)^3), \end{aligned} \tag{5.56}$$

where the error of the numerical integration is absorbed in the $O((\Delta v)^3)$ term. Similar to the proof of Lemma 7 we introduce a numerical entropy flux

$$G(v_j^n) := Q(v_j^n) - f'(v_j^n)(f(v_j^n) - g_j), \tag{5.57}$$

and define

$$\bar{J} := J - \lambda f'(z)(f(z) - g(z))\Big|_{v_j^n}^{v_{j+1}^n}. \tag{5.58}$$

We define the numerical entropy production $E_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ (which approximates $S_t + G_x$ at $(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}})$) as follows

$$E_{j+\frac{1}{2}}^{n+\frac{1}{2}} := S(v_{j+\frac{1}{2}}^{n+1}) - \frac{1}{2}\left(S(v_j^n) + S(v_{j+1}^n)\right) + \lambda\left(G(v_{j+1}^n) - G(v_j^n)\right) = I + \bar{J}. \tag{5.59}$$

By using the same arguments as before, see (5.18)–(5.22), and the smoothness of f, we write $\bar{J}$ as

$$\begin{aligned}\bar{J} &= \frac{1}{8}(1-4\beta^2)\frac{v_j' f''(v_j^n) + v_{j+1}' f''(v_{j+1}^n)}{2\Delta v}(\Delta v)^2 + O((\Delta v)^3)\\ &= \frac{1}{8}(1-4\beta^2) f''(v_{j+\frac{1}{2}})\frac{v_j' + v_{j+1}'}{2\Delta v}(\Delta v)^2 + O((\Delta v)^3).\end{aligned} \tag{5.60}$$

Then,

$$\begin{aligned}I + \bar{J} = -\frac{1}{8}(1-4\beta^2) f''(v_{j+\frac{1}{2}}) &\left[1 - \frac{1}{16}(1-4\beta^2)\left(\frac{\Delta v'}{\Delta v}\right)^2\right.\\ &\left. - \beta\frac{\Delta v'}{\Delta v} - \frac{v_j' + v_{j+1}'}{2\Delta v}\right](\Delta v)^2 + O((\Delta v)^3)\\ &= -\bar{\mu}_{j+\frac{1}{2}}(\Delta v)^2 + O((\Delta v)^3).\end{aligned} \tag{5.61}$$

Exactly as in (5.34), we derive again that $\bar{\mu}_{j+\frac{1}{2}} \geq 0$ for sufficiently small κ_0. Thus, the numerical entropy production defined in (5.59) can be written as

$$\begin{aligned}&S(v_{j+\frac{1}{2}}^{n+1}) - \frac{1}{2}\left(S(v_j^n) + S(v_{j+1}^n)\right) + \lambda\left(G(v_{j+1}^n) - G(v_j^n)\right)\\ &= -\bar{\mu}_{j+\frac{1}{2}}(\Delta v)^2 + O((\Delta v)^3).\end{aligned} \tag{5.62}$$

Similar to the quadratic case ($S(u) = \frac{u^2}{2}$), see (5.44) and (5.45), we obtain

$$\sum_{n=0}^{N} \sum_{j\in\Lambda_n(\bar{\Omega})} \hat{\mu}_{j+\frac{1}{2}}(\Delta w)^2 \le \sum_j (f(w_j^0) - f(k)) + C \sum_{n=0}^{N} \sum_{j\in\Lambda_n(\bar{\Omega})} |\Delta w|^3 \\ - \lambda(N+1)\left(G(w_{M+1}^n) - G(w_{-M}^n)\right), \tag{5.63}$$

where $\bar{\Omega}$, $w(\cdot,\cdot)$ and M are defined in Lemma 7, and $\hat{\mu}$ is the corresponding quantity for $w(\cdot,\cdot)$. Thus, using Lemma 7 and that G is bounded on $R(u^0)$, we obtain

$$\Delta x \sum_{n=0}^{N} \sum_{j\in\Lambda_n} \bar{\mu}_{j+\frac{1}{2}}(\Delta v)^2 \le \Delta x \sum_j (f(w_j^0) - f(k)) + C. \tag{5.64}$$

Next, we employ Jensen's inequality to get

$$f\left(\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^0(x)\,dx\right) - f(k) \le \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (f(u^0(x)) - f(k))\,dx. \tag{5.65}$$

Therefore,

$$\Delta x \sum_{n=0}^{N} \sum_{j\in\Lambda_n} \bar{\mu}_{j+\frac{1}{2}}(\Delta v)^2 \le \int_{-M\Delta x}^{(M+1)\Delta x} (f(u^0(x)) - f(k))\,dx + C \le C\,||f||_{L^\infty(R(u^0))}. \tag{5.66}$$

□

2. Strong convergence via compensated compactness

The following two lemmas imply that the NT solution is a compact subset of $W_{loc}^{-1,2}(\mathbb{R} \times (0,\infty))$, see Lemma 6, and thus, converges strongly on compact sets, Lemma 5. Recall that $v(x,t)$ is the NT solution defined in Chapter II.

Lemma 9. *Let Ω be a fixed open subset of $\mathbb{R} \times (0,\infty)$. For the entropy pairs (S,Q) given by (5.4) and (5.5), the sequence of distributions*

$$L(v) := S(v)_t + Q(v)_x \tag{5.67}$$

lies in a compact subset of $W_{loc}^{-1,q}(\Omega)$, $1 < q \le \frac{6}{5}$.

Proof. Let $\phi \in C_0^\infty(\Omega)$ and $\overline{\Omega} := [-M\Delta x, M\Delta x] \times [0, N\Delta t]$ be the smallest rectangle such that $\Omega \subset \overline{\Omega}$ and M, N are positive integers. We consider

$$-\langle L, \phi \rangle = \int_0^\infty \int_{\mathbb{R}} \left(S(v)\phi_t + Q(v)\phi_x\right)\, dx\, dt, \tag{5.68}$$

and decompose it as follows

$$\begin{aligned}
-\langle L, \phi \rangle &= \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \int_{t^n}^{t^{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(S(v)\phi_t + Q(v)\phi_x\right)\, dx\, dt \\
&= \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} S_j^n(\phi(x, t^{n+1}) - \phi(x, t^n))\, dx \\
&+ \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \int_{t^n}^{t^{n+1}} Q_j^n(\phi(x_{j+\frac{1}{2}}, t) - \phi(x_{j-\frac{1}{2}}, t))\, dt \\
&= -\sum_{n=1}^{N} \sum_{j \in \Lambda_n} \int_{x_{j-\frac{1}{2}}}^{x_j} \left(S_j^n - S_{j-\frac{1}{2}}^{n-1}\right) \phi(x, t^n)\, dx \\
&- \sum_{n=1}^{N} \sum_{j \in \Lambda_n} \int_{x_j}^{x_{j+\frac{1}{2}}} \left(S_j^n - S_{j+\frac{1}{2}}^{n-1}\right) \phi(x, t^n)\, dx \\
&- \sum_{n=1}^{N} \sum_{j \in \Lambda_n} \int_{t^{n-1}}^{t^n} \left(Q_{j+\frac{1}{2}}^{n-1} - Q_{j-\frac{1}{2}}^{n-1}\right) \phi(x_j, t)\, dt,
\end{aligned} \tag{5.69}$$

where $S_j^n := S(v_j^n)$, $Q_j^n := Q(v_j^n)$ and $\Lambda_n = \Lambda_n(\overline{\Omega})$. After some rearrangements,

$$\begin{aligned}
-\langle L, \phi \rangle &= \sum_{n=1}^{N} \sum_{j \in \Lambda_n} \int_{x_{j-\frac{1}{2}}}^{x_j} \left(S_j^n - S_{j-\frac{1}{2}}^{n-1}\right) \left(\phi_j^n - \phi(x, t^n)\right)\, dx \\
&+ \sum_{n=1}^{N} \sum_{j \in \Lambda_n} \int_{x_j}^{x_{j+\frac{1}{2}}} \left(S_j^n - S_{j+\frac{1}{2}}^{n-1}\right) \left(\phi_j^n - \phi(x, t^n)\right)\, dx \\
&+ \sum_{n=1}^{N} \sum_{j \in \Lambda_n} \int_{t^{n-1}}^{t^n} \left(Q_{j+\frac{1}{2}}^{n-1} - Q_{j-\frac{1}{2}}^{n-1}\right) \left(\phi_j^n - \phi(x_j, t)\right)\, dt \\
&- \sum_{n=1}^{N} \sum_{j \in \Lambda_n} \Delta x\, \phi_j^n \left[S_j^n - \frac{1}{2}\left(S_{j+\frac{1}{2}}^{n-1} + S_{j-\frac{1}{2}}^{n-1}\right) + \lambda\left(Q_{j+\frac{1}{2}}^{n-1} - Q_{j-\frac{1}{2}}^{n-1}\right)\right] \\
&=: \langle L_1 + L_2 + L_3 + L_4, \phi \rangle.
\end{aligned} \tag{5.70}$$

As in Lemma 8 we take k to be the unique constant such that $f(k) = \min_{u \in R(u^0)} f(u)$. In order to apply the compensated compactness arguments, we need to consider the following two entropies: $S(u) = u - k$ and $S(u) = f(u) - f(k)$, see (5.4) and (5.5).

Case 1: $S(u) = u - k$. The treatment of all the terms $\langle L_i, \phi \rangle$, $i = 1, \ldots, 4$ is the same as in Case 2.

Case 2: $S(u) = f(u) - f(k)$, $||S''||_{L^\infty(\Omega)} = \gamma_2$. We transform the term $\langle L_4, \phi \rangle$ as follows

$$\begin{aligned}
\langle L_4, \phi \rangle &= \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \left[S_{j+\frac{1}{2}}^{n+1} - \frac{1}{2} \left(S_{j+1}^n + S_j^n \right) + \lambda \left(Q_{j+1}^n - Q_j^n \right) \right] \\
&= - \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \int_0^1 \int_0^1 \frac{s}{4} S''(u(r,s))(1 - 2\lambda g'(w(r,s))) \\
&\qquad\qquad (1 + 2\lambda g'(v(s)))(\Delta v)^2 \, dr \, ds \\
&+ \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \int_0^1 \lambda S'(v(s)) \left(f'(v(s)) - g'(v(s)) \right) \Delta v \, ds \\
&= \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \frac{1}{4} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \left(\left(2\lambda \frac{\Delta g}{\Delta v} \right)^2 - 1 \right) (\Delta v)^2 \int_0^1 \int_0^1 s S''(u(r,s)) dr ds \\
&- \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \lambda \int_{v_j^n}^{v_{j+1}^n} S''(z)(f(z) - g(z)) \, dz \\
&+ \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \lambda S'(z)(f(z) - g(z)) \Big|_{v_j^n}^{v_{j+1}^n} .
\end{aligned} \tag{5.71}$$

We write $\langle L_4, \phi \rangle = \langle L_{4,1}, \phi \rangle + \langle L_{4,2}, \phi \rangle$ where

$$\begin{aligned}
\langle L_{4,1}, \phi \rangle &= \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \frac{1}{4} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \left(\left(2\lambda \frac{\Delta g}{\Delta v} \right)^2 - 1 \right) (\Delta v)^2 \int_0^1 \int_0^1 s S''(u(r,s)) dr \, ds \\
&- \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \lambda \int_{v_j^n}^{v_{j+1}^n} S''(z)(f(z) - g(z)) \, dz,
\end{aligned} \tag{5.72}$$

and

$$\begin{aligned}\left\langle L_{4,2}, \phi\right\rangle &= \sum_{n=0}^{N-1} \sum_{j\in\Lambda_n} \Delta x\, \phi_{j+\frac{1}{2}}^{n+1} \lambda S'(z)(f(z)-g(z))\Big|_{v_j^n}^{v_{j+1}^n} \\ &= \lambda \Delta x \sum_{n=0}^{N-1} \sum_{j\in\Lambda_n} (\phi_{j-\frac{1}{2}}^{n+1} - \phi_{j+\frac{1}{2}}^{n+1}) S'(v_j^n)(f(v_j^n) - g(v_j^n)).\end{aligned} \tag{5.73}$$

We recall the identity (5.61)

$$\begin{aligned}&-\frac{1}{4}\left(1-\left(2\lambda\frac{\Delta g}{\Delta v}\right)^2\right)(\Delta v)^2 \int_0^1 \int_0^1 s\, f''(u(r,s))\, dr\, ds \\ &+ \lambda \int_{v_j^n}^{v_{j+1}^n} f''(z)(f(z)-g(z))\, dz \\ &= -\bar{\mu}_{j+\frac{1}{2}} (\Delta v)^2 + O((\Delta v)^3),\end{aligned} \tag{5.74}$$

where $\bar{\mu}_{j+\frac{1}{2}} \geq 0$. Using the above in (5.72) we get

$$\begin{aligned}\left|\left\langle L_{4,1}, \phi\right\rangle\right| &\leq ||\phi||_{L^\infty(\Omega)} \left(\Delta x \sum_{n=0}^{N-1} \sum_{j\in\Lambda_n} \bar{\mu}_{j+\frac{1}{2}} (\Delta v)^2 + C\Delta x \sum_{n=0}^{N-1} \sum_{j\in\Lambda_n} |\Delta v|^3\right) \\ &\leq C||\phi||_{L^\infty(\Omega)},\end{aligned} \tag{5.75}$$

where the last inequality follows from Lemma 7 and Lemma 8. To estimate $\left\langle L_{4,2}, \phi\right\rangle$, we use the bound

$$\begin{aligned}\lambda(f(v_j^n) - g(v_j^n)) &= \frac{v_j'}{8}((2\lambda a_j)^2 - 1) - \frac{\lambda}{8}(\lambda a_j v_j')^2 f''(\xi) \\ &\leq \frac{|\Delta v|}{8}((4\kappa^2+1)+\kappa^3) \leq C|\Delta v|,\end{aligned} \tag{5.76}$$

in (5.73) and derive

$$
\begin{aligned}
|\langle L_{4,2},\phi\rangle| &\le C\Delta x\sum_{n=0}^{N-1}\sum_{j\in\Lambda_n}|\Delta\bar{\phi}||\Delta v| \\
&\le C\left(\sum_{n=0}^{N-1}\sum_{j\in\Lambda_n}\Delta x|\Delta\bar{\phi}|^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(\sum_{n=0}^{N-1}\sum_{j\in\Lambda_n}\Delta x|\Delta v|^{3}\right)^{\frac{1}{3}} \\
&\le C\left(||\phi||_{C_0^\alpha}^{\frac{3}{2}}(\Delta x)^{\frac{3}{2}\alpha+1}\sum_{n=0}^{N-1}\sum_{j\in\Lambda_n}1\right)^{\frac{2}{3}} \\
&\le C||\phi||_{C_0^\alpha}\frac{(\Delta x)^{\alpha+\frac{2}{3}}}{(\Delta x)^{\frac{4}{3}}} = C||\phi||_{C_0^\alpha}(\Delta x)^{\alpha-\frac{2}{3}},
\end{aligned}
\tag{5.77}
$$

where $\Delta\bar{\phi} := \Delta\phi_j^{n+1} = \phi_{j+\frac{1}{2}}^{n+1} - \phi_{j-\frac{1}{2}}^{n+1}$. Combining (5.75) and (5.77), we conclude

$$
|\langle L_4,\phi\rangle| \le C\left(||\phi||_{L^\infty(\Omega)} + ||\phi||_{C_0^\alpha}(\Delta x)^{\alpha-2/3}\right). \tag{5.78}
$$

Next, we estimate the term $\langle L_1,\phi\rangle$. We have

$$
\begin{aligned}
|\langle L_1,\phi\rangle| &= \left|\sum_{n=1}^{N}\sum_{j\in\Lambda_n}\int_{x_{j-\frac{1}{2}}}^{x_j}(S_j^n - S_{j-\frac{1}{2}}^{n-1})(\phi_j^n - \phi(x,t^n))\,dx\right| \\
&\le \sum_{n=1}^{N}\sum_{j\in\Lambda_n}\left|S_j^n - S_{j-\frac{1}{2}}^{n-1}\right|\int_{x_{j-\frac{1}{2}}}^{x_j}\left|\phi_j^n - \phi(x,t^n)\right|\,dx \\
&\le \sum_{n=1}^{N}\sum_{j\in\Lambda_n}|S'(\xi_j^n)|\left|v_j^n - v_{j-\frac{1}{2}}^{n-1}\right|\int_{x_{j-\frac{1}{2}}}^{x_j}||\phi||_{C_0^\alpha}|\Delta x|^\alpha\,dx,
\end{aligned}
\tag{5.79}
$$

where $\xi_j^n \in R(u^0)$. Observe that

$$
\begin{aligned}
|v_{j+\frac{1}{2}}^{n+1} - v_j^n| &\le \frac{1}{2}|\Delta v| + \lambda|\Delta g| \\
&\le \frac{1}{2}|\Delta v| + \lambda\left|f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})\right| + \frac{1}{8}|v'_{j+1} - v'_j| \\
&\le \frac{3}{4}|\Delta v| + \lambda\left|f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})\right| \le C|\Delta v|,
\end{aligned}
\tag{5.80}
$$

where the last estimate follows from (5.29). Hence,

$$\begin{aligned}|\langle L_1,\phi\rangle| &\le C\sum_{n=1}^{N}\sum_{j\in\Lambda_n}|\Delta v|\,||\phi||_{C_0^\alpha}|\Delta x|^{\alpha+1}\\ &\le C||\phi||_{C_0^\alpha}|\Delta x|^{\alpha+\frac{2}{3}}\left(\Delta x\sum_{n=1}^{N}\sum_{j\in\Lambda_n}|\Delta v|^3\right)^{\frac{1}{3}}\left(\sum_{n=1}^{N}\sum_{j\in\Lambda_n}1\right)^{\frac{2}{3}}\\ &\le C||\phi||_{C_0^\alpha}|\Delta x|^{\alpha-\frac{2}{3}}.\end{aligned} \tag{5.81}$$

Similarly, we have the following estimates for the remaining two terms

$$|\langle L_2,\phi\rangle|+|\langle L_3,\phi\rangle|\le C||\phi||_{C_0^\alpha(\Omega)}|\Delta x|^{\alpha-\frac{2}{3}}. \tag{5.82}$$

Now, we use the above two estimates, together with (5.78), to get

$$|\langle L_1+L_2+L_3+L_4,\phi\rangle|\le|\langle M_1,\phi\rangle|+|\langle M_2,\phi\rangle|, \tag{5.83}$$

where $M_1=L_{4,1}$, $M_2=L_1+L_2+L_3+L_{4,2}$ and the following two bounds hold

$$|\langle M_1,\phi\rangle|\le C||\phi||_{L^\infty(\Omega)}\ \text{ and }\ |\langle M_2,\phi\rangle|\le C||\phi||_{C_0^\alpha(\Omega)}(\Delta x)^{\alpha-\frac{2}{3}}. \tag{5.84}$$

The estimate of the term $\langle M_1,\phi\rangle$ yields a uniform bound

$$||M_1||_{C_0^\star}(\Omega)\le C, \tag{5.85}$$

where $C_0^\star(\Omega)$ denotes the space of bounded measures on Ω. Using the Sobolev embedding theorem, we have $C_0^\star(\Omega)\subset W^{-1,q_1}(\Omega)$ with compact injection for any $q_1\in(1,2)$. Therefore, we have

$$\{M_1\}\ \text{ is compact in }\ W^{-1,q_1}(\Omega). \tag{5.86}$$

For the term $\langle M_2,\phi\rangle$, we use the Sobolev embedding $W^{1,p}(\Omega)\hookrightarrow C_0^\alpha(\Omega)$ for $p>\frac{2}{1-\alpha}$ and $\alpha\in(\frac{2}{3},1)$. Hence,

$$||M_2||_{W^{-1,q_2}(\Omega)}\le C\Delta x^{\alpha-\frac{2}{3}}, \tag{5.87}$$

where $q_2 = \frac{p}{p-1} \in \left(1, \frac{2}{\alpha+1}\right)$. Thus, we get

$$\{M_2\} \text{ is compact in } W^{-1,q_2}(\Omega). \tag{5.88}$$

We conclude that the sequence of distributions $\{L\}$ is compact in $W^{-1,q}(\Omega)$, $1 < q = \min(q_1, q_2) < \frac{2}{\alpha+1} < \frac{6}{5}$ and this finishes the proof of Lemma 9. □

Now, we are ready to prove the following lemma which ends this section.

Lemma 10. *Let Ω be a fixed open subset of $\mathbb{R} \times (0, \infty)$. For the entropy pairs (S, Q) given by (5.4) and (5.5), the sequence of distributions*

$$L(v) = S(v)_t + Q(v)_x \tag{5.89}$$

lies in a bounded subset of $W^{-1,r}_{loc}(\Omega)$, $1 < r < \infty$.

Proof. Let $\phi \in W^{1,q}_0(\Omega)$, where q is the conjugate exponent to r, and consider

$$\langle L, \phi \rangle = -\int_0^\infty \int_{\mathbb{R}} (S(v)\phi_t + Q(v)\phi_x) dxdt. \tag{5.90}$$

We choose the two entropy pairs (S, Q) as before, see (5.4)–(5.5). Using the uniform bounds $||S(v)||_{L^\infty(\Omega)} \leq C$ and $||Q(v)||_{L^\infty(\Omega)} \leq C$, we conclude

$$|\langle L, \phi \rangle| \leq C \int\int_\Omega |\phi_t| + |\phi_x| \, dx\, dt \leq C \; |\Omega|^{1/r} ||\phi||_{W^{1,q}_0(\Omega)}. \tag{5.91}$$

□

3. Convergence towards the unique entropy solution

In the previous subsection, we proved that NT solution converges on compact sets. Now, we will prove that the limit is the unique entropy solution of the conservation law (2.1).

Lemma 11. *For the entropy pair* (S, Q) *such that* $S(u)=\frac{u^2}{2}$ *and* $Q(u)=\int^u wf'(w)dw$, *we have the following inequality*

$$\big\langle L(u), \phi\big\rangle = \big\langle S(u)_t + Q(u)_x, \phi\big\rangle \leq 0$$

for all $\phi \geq 0$, $\phi \in C_0^\infty(\mathbb{R} \times (0, \infty))$, *where* u *is the strong limit of the NT scheme.*

Proof. Let's fix a test function $\phi \geq 0$ and define $\Omega := [-M\Delta x, M\Delta x] \times [0, N\Delta t]$ to be the smallest rectangle such that $\operatorname{supp}(\phi) \subset \Omega$ with M, N positive integers. Given the numerical solution $v(x,t)$, after integration by parts we rewrite $\big\langle L(v), \phi\big\rangle$ as

$$\big\langle L, \phi\big\rangle = -\int_0^\infty \int_{\mathbb{R}} \left(\frac{v^2}{2}\phi_t + Q(v)\phi_x\right) dx\, dt = -\big\langle L_1 + L_2 + L_3 + L_4, \phi\big\rangle, \tag{5.92}$$

where $\big\langle L_i, \phi\big\rangle$ for $i = 1, \dots, 4$ are the same as in (5.70). We know that

$$\big\langle L_1 + L_2 + L_3, \phi\big\rangle = o(1), \tag{5.93}$$

where we use the standard $o(1)$-notation. Thus, we restrict our attention to $\big\langle L_4, \phi\big\rangle$,

$$-\big\langle L_4, \phi\big\rangle = \sum_{n=0}^{N-1} \sum_{j\in\Lambda_n} \Delta x\, \phi_{j+\frac{1}{2}}^{n+1}\, E_{j+\frac{1}{2}}^{n+\frac{1}{2}} + \lambda \sum_{n=0}^{N-1} \sum_{j\in\Lambda_n} \Delta x\, \phi_{j+\frac{1}{2}}^{n+1}\, \Delta H_{j+\frac{1}{2}}^n, \tag{5.94}$$

where we recall that

$$\begin{aligned} E_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= -\frac{1}{8}(\Delta v)^2 \mu_{j+\frac{1}{2}}^n + \lambda(\Delta v)^3 \nu_{j+\frac{1}{2}}^n \\ &= \lambda(\Delta v)_+^3 \nu_{j+\frac{1}{2}}^n + \left[-\frac{1}{8}(\Delta v)^2 \mu_{j+\frac{1}{2}}^n + \lambda(\Delta v)_-^3 \nu_{j+\frac{1}{2}}^n\right] \\ &=: (E_{j+\frac{1}{2}}^{n+\frac{1}{2}})_+ + (E_{j+\frac{1}{2}}^{n+\frac{1}{2}})_- \end{aligned} \tag{5.95}$$

and we define

$$\Delta H_{j+\frac{1}{2}}^n = v_{j+1}^n(f(v_{j+1}^n) - g(v_{j+1}^n)) - v_j^n(f(v_j^n) - g(v_j^n)). \tag{5.96}$$

Note that the expression $\lambda \sum_{n=0}^{N-1} \sum_{j\in\Lambda_n} \Delta x\, \phi_{j+\frac{1}{2}}^{n+1}\, \Delta H_{j+\frac{1}{2}}^n$ is equal to (5.73) with

$S'(u) = u$. Hence, using the bound (5.77) with $S'(u) = u$, we write

$$-\langle L_4, \phi \rangle = \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \, (E_{j+\frac{1}{2}}^{n+\frac{1}{2}})_- + \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \, (E_{j+\frac{1}{2}}^{n+\frac{1}{2}})_+ + o(1), \quad (5.97)$$

where $o(1)$ absorbs the term $O(\Delta x^{\alpha - \frac{2}{3}})$. For the second summand above we use the estimate (5.49),

$$c\gamma_1 \lambda \sum_{n=0}^{N} \sum_{j \in \Lambda_n} (\Delta v)_+^3 \leq \sum_j (\Delta v^0)_+^2, \quad (5.98)$$

which gives

$$\begin{aligned} \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \, (E_{j+\frac{1}{2}}^{n+\frac{1}{2}})_+ &= \Delta x \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \lambda (\Delta v)_+^3 \, \phi_{j+\frac{1}{2}}^{n+1} \, \nu_{j+\frac{1}{2}}^n \\ &\leq C ||\phi||_{L^\infty(\Omega)} \Delta x \sum_j (\Delta v^0)_+^2. \end{aligned} \quad (5.99)$$

Therefore, we have

$$-\langle L_4, \phi \rangle \leq \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \, (E_{j+\frac{1}{2}}^{n+\frac{1}{2}})_- + C ||\phi||_{L^\infty(\Omega)} \Delta x \sum_j (\Delta v^0)_+^2 + o(1). \quad (5.100)$$

Observe that

$$\sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \, (E_{j+\frac{1}{2}}^{n+\frac{1}{2}})_- \leq 0 \quad (5.101)$$

and

$$\begin{aligned} \sum_j \Delta x (v_{j+1}^0 - v_j^0)_+^2 &\leq \Delta x \sum_{j \in Z} \left(\frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (v^0(x + \Delta x) - v^0(x)) dx \right)^2 \\ &\leq \frac{1}{\Delta x} \sum_j \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (v^0(x + \Delta x) - v^0(x))^2 dx \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} 1 dx \\ &= \int_{\mathbb{R}} |v^0(x + \Delta x) - v^0(x)|^2 dx = o(1), \end{aligned} \quad (5.102)$$

where the last part follows from continuity of translation in the L^2 norm, see for example [17]. Hence, we conclude

$$-\left\langle L_1 + L_2 + L_3 + L_4, \phi \right\rangle \leq \sum_{n=0}^{N-1} \sum_{j \in \Lambda_n} \Delta x \, \phi_{j+\frac{1}{2}}^{n+1} \, (E_{j+\frac{1}{2}}^{n+\frac{1}{2}})_- + o(1), \tag{5.103}$$

and when $\Delta x \to 0$, for any strong limit u, we have

$$\left\langle L(u), \phi \right\rangle \leq 0. \tag{5.104}$$

□

Based on the result of E. Yu. Panov [53] , see also [14], a single entropy inequality is enough to select the unique entropy solution. Thus, we conclude that the NT scheme converges strongly and the limit is the unique entropy solution of (2.1). □

CHAPTER VI

CONCLUSION

In this thesis, stability and convergence results were proven for the second order non-oscillatory Nessyahu-Tadmor (NT) scheme with a modified minmod limiter inspired by the so-called minimum angle plane reconstruction (MAPR), see [10].

The advantage of MAPR-like limiters is that at local extrema, the slope of the local reconstruction is not set to zero and the local approximation is second order. It is shown that the NT scheme with the new limiter satisfies a maximum principle for a conservation law with any Lipschitz flux and also with any k-monotone flux for $k \geq 2$. It is also proven that the usual TVD bound follows from the local maximum principle. The maximum principle result is later employed to prove that, in the case of strictly convex flux, the NT scheme with a properly selected MAPR-like limiter satisfies an one-sided Lipschitz stability estimate. As a result, the numerical solution converges to the unique entropy solution when the initial data satisfies some type of an one-sided Lipschitz condition. Finally, using the stability results from previous chapters and compensated compactness arguments, it is proven that the NT scheme with the modified limiter converges strongly on compact sets to a weak solution and the limit satisfies a weak form of an entropy inequality. Based on the result that a single entropy inequality is enough to select the unique entropy solution in the case of a scalar strictly convex flux, see [53], we conclude that the NT scheme converges to the entropy solution of the conservation law for any given bounded initial data. The main contribution of this thesis is that convergence of the NT scheme is proven without imposing any non-homogenous limitations on the numerical method or one-sided conditions on the initial data, and the result holds for the largest possible class of initial conditions, that is, the class of initial data where we have existence-uniqueness of the entropy solution of the PDE. This is a new result even in the case of the classical minmod limiter.

It should be possible to generalize these results in the case of scalar conservation laws to other second and higher order schemes as numerical evidence suggests that many other schemes also satisfy the one-sided stability property, see Lemma 7 and Lemma 8. In the case of systems, one of the fundamental problems is to establish the existence of an invariant domain and prove that the numerical solution stays bounded for a large class of initial data. For instance, once having this kind of result for the LxF scheme for the Lagrangian p-system, DiPerna's compensated compactness arguments will imply that up to a subsequence, the numerical solution converges to an entropy solution. Trying to prove the existence of an invariant domain for the NT scheme for the systems such as chromatography and p-system, for example, suggests that characteristic-wise limiting is needed. Thus, one possible future direction of study is to devise numerical schemes using flux and space dependent limiters, which allow to define an invariant domain or prove compactness without reducing the accuracy order of the scheme.

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APPENDIX A

MAXIMUM PRINCIPLE OF NT SCHEME WITH CONVEX FLUX

Here, we give an alternative proof for the maximum principle, see Theorem 2, for the case when the flux is convex.

Theorem 7. *Let v'_j be chosen as in (2.8) and $f'_j = f'(v^n_j)v'_j$. If f is strictly convex, that is, there exists constants $\gamma_1 \leq \gamma_2$ such that*

$$0 < \gamma_1 \leq f'' \leq \gamma_2 \tag{A.1}$$

then the scheme described by (2.12), (2.13) satisfies the maximum principle

$$\min(v^n_j, v^n_{j+1}) \leq v^{n+1}_{j+\frac{1}{2}} \leq \max(v^n_j, v^n_{j+1}), \tag{A.2}$$

under the CFL condition

$$\lambda \max_{w \in R(u^0)} |f'(w)| \leq \kappa, \tag{A.3}$$

where κ is a fixed constant which depends only on γ_1 and γ_2, see (A.13).

Proof. First, we observe

$$\begin{aligned} |f(b) - f(a)| = \left| \int_a^b f'(t)dt \right| &= \left| \int_a^b \left(\int_{\frac{a+b}{2}}^t f''(s)ds + f'\left(\frac{a+b}{2}\right) \right) dt \right| \\ &\leq \left(\frac{\gamma_2}{4}|b-a| + \left| f'\left(\frac{a+b}{2}\right) \right| \right) |b-a|. \end{aligned} \tag{A.4}$$

Let $a = v_j^{n+\frac{1}{2}}$ and $b = v_{j+1}^{n+\frac{1}{2}}$ in the above inequality,

$$\begin{aligned} \left| f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}}) \right| \leq &\left| f'\left(\frac{v_j^{n+\frac{1}{2}} + v_{j+1}^{n+\frac{1}{2}}}{2} \right) \right| \left| v_{j+1}^{n+\frac{1}{2}} - v_j^{n+\frac{1}{2}} \right| \\ &+ \frac{\gamma_2}{4} \left| v_{j+1}^{n+\frac{1}{2}} - v_j^{n+\frac{1}{2}} \right|^2. \end{aligned} \tag{A.5}$$

Next, we bound the terms appearing on the right hand side. We start with,

$$\begin{aligned}\left|v_{j+1}^{n+\frac{1}{2}} - v_j^{n+\frac{1}{2}}\right| &= \left|v_{j+1}^n - v_j^n - \frac{\lambda}{2}(f'(v_{j+1}^n)v'_{j+1} - f'(v_j^n)v'_j)\right| \\ &\le \left(1 + \frac{\lambda}{2}\left(|f'(v_{j+1}^n)| + |f'(v_j^n)|\right)\right)|v_{j+1}^n - v_j^n| \\ &\le (1+\kappa)|v_{j+1}^n - v_j^n| \le \frac{1+\kappa}{\gamma_1}|f'(v_{j+1}^n) - f'(v_j^n)| \\ &\le \frac{2(1+\kappa)}{\gamma_1}\max_j\left|f'(v_j^n)\right|.\end{aligned} \tag{A.6}$$

Note that the inequality

$$\left|\frac{\lambda}{4}\left(f'(v_j^n)v'_j + f'(v_{j+1}^n)v'_{j+1}\right)\right| \le \frac{\kappa}{2}|v_{j+1}^n - v_j^n| \tag{A.7}$$

implies

$$\min(v_j^n, v_{j+1}^n) \le \frac{v_j^{n+\frac{1}{2}} + v_{j+1}^{n+\frac{1}{2}}}{2} \le \max(v_j^n, v_{j+1}^n), \tag{A.8}$$

for all $\kappa \le 1$. From (A.1) and (A.8) it follows that

$$\left|f'\left(\frac{v_{j+1}^{n+\frac{1}{2}} + v_j^{n+\frac{1}{2}}}{2}\right)\right| \le \max_j\left|f'(v_j^n)\right|. \tag{A.9}$$

We use (A.5), (A.6) and (A.9) and derive

$$\begin{aligned}\left|f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})\right| &\le \left(\frac{\gamma_2(1+\kappa)}{2\gamma_1} + 1\right)\max_j|f'(v_j^n)|\left|v_{j+1}^{n+\frac{1}{2}} - v_j^{n+\frac{1}{2}}\right| \\ &\le \left(\frac{\gamma_2(1+\kappa)}{2\gamma_1} + 1\right)(1+\kappa)\max_j|f'(v_j^n)||v_{j+1}^n - v_j^n|.\end{aligned} \tag{A.10}$$

Using (A.10) in (2.13) gives the estimates

$$\begin{aligned}v_{j+\frac{1}{2}}^{n+1} &\le \frac{1}{2}(v_j^n + v_{j+1}^n) + \frac{1}{8}|v'_j - v'_{j+1}| + \lambda\left|f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}})\right| \\ &\le \frac{1}{2}(v_j^n + v_{j+1}^n) + \left(\frac{1}{4} + \kappa(1+\kappa)\left(\frac{\gamma_2(1+\kappa)}{2\gamma_1} + 1\right)\right)|v_{j+1}^n - v_j^n|,\end{aligned} \tag{A.11}$$

and

$$\begin{aligned} v_{j+\frac{1}{2}}^{n+1} &\geq \frac{1}{2}(v_j^n + v_{j+1}^n) - \frac{1}{8}|v_j' - v_{j+1}'| - \lambda \left| f(v_{j+1}^{n+\frac{1}{2}}) - f(v_j^{n+\frac{1}{2}}) \right| \\ &\geq \frac{1}{2}(v_j^n + v_{j+1}^n) - \left(\frac{1}{4} + \kappa(1+\kappa) \left(\frac{\gamma_2(1+\kappa)}{2\gamma_1} + 1 \right) \right) |v_{j+1}^n - v_j^n|. \end{aligned} \tag{A.12}$$

Hence, under the CFL condition

$$\kappa(1+\kappa) \left(\frac{\gamma_2(1+\kappa)}{2\gamma_1} + 1 \right) \leq \frac{1}{4}, \tag{A.13}$$

we have

$$\min(v_j^n, v_{j+1}^n) \leq v_{j+\frac{1}{2}}^{n+1} \leq \max(v_j^n, v_{j+1}^n). \tag{A.14}$$

□

APPENDIX B

ENTROPY PRODUCTION ESTIMATES

In this appendix, we will restate the proof given in [36]. That is, in the case of systems of conservation laws in one space dimension, the first order Lax-Friedrichs(LxF) scheme satisfies a discrete entropy inequality for any given entropy pair, see Lemma 12. Following [47], we will also derive a discrete entropy production identity for the second order Nessyahu-Tadmor scheme in the case of scalar conservation laws in one space dimension, see Lemma 13. The goal is to provide the reader with a reference which is free of typos and unifies the notations in [36] and [47].

First, we consider systems of conservation laws in one space dimension:

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u = u^0, & (x,t) \in \mathbb{R} \times \{t=0\}, \end{cases} \tag{B.1}$$

where $u = (u_1, \ldots, u_m) : \mathbb{R} \to \mathbb{R}^m$ is the conservative variable, $f = (f_1, \ldots, f_m) : \mathbb{R}^m \to \mathbb{R}^m$ is the flux function, and $u^0(x) : \mathbb{R} \to \mathbb{R}^m$ is the given function describing the initial distribution of $u = (u_1, \ldots, u_m)$.

The Lax-Friedrichs scheme for (B.1) is given by

$$v^{n+1}_{j+\frac{1}{2}} = \frac{1}{2}(v^n_j + v^n_{j+1}) - \lambda(f(v^n_{j+1}) - f(v^n_j)), \tag{B.2}$$

which is initialized with

$$v^0_j := \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^0(x)\, dx. \tag{B.3}$$

The global approximate solution $v(x,t)$ is defined to be piecewise constant in time: $v(x,t) = v^n_j$ for $(x,t) \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times [t^n, t^{n+1})$, where $j \in \mathbb{Z}$ if n is even and $j + \frac{1}{2} \in \mathbb{Z}$ if n is an odd integer.

Lemma 12. *For a given entropy pair (S, Q), there exists a constant $\kappa_0 > 0$, which only depends on the maximum and minimum eigenvalues of S'', see (B.20), such that under the CFL condition*

$$\lambda \max_{w \in R(u^0)} ||f'(w)||_2 \leq \kappa_0, \tag{B.4}$$

where $|| \cdot ||_2$ is the l_2-matrix norm, the Lax-Friedrichs scheme described by (B.2) satisfies the following discrete entropy inequality

$$\frac{1}{\Delta t}\left(S(v^{n+1}_{j+\frac{1}{2}}) - \frac{S(v^n_j) + S(v^n_{j+1})}{2}\right) + \frac{1}{\Delta x}(Q(v^n_{j+1}) - Q(v^n_j)) \leq 0. \tag{B.5}$$

Proof. Given entropy pair (S, Q), define

$$E^{n+\frac{1}{2}}_{j+\frac{1}{2}} := S(v^{n+1}_{j+\frac{1}{2}}) - \frac{S(v^n_j) + S(v^n_{j+1})}{2} + \lambda(Q(v^n_{j+1}) - Q(v^n_j)). \tag{B.6}$$

We introduce the notations $u := v^{n+1}_{j+\frac{1}{2}}$, $v := v^n_j$, $w := v^n_{j+1}$ and rewrite $E^{n+\frac{1}{2}}_{j+\frac{1}{2}}$ as

$$E^{n+\frac{1}{2}}_{j+\frac{1}{2}} = S(u) - \frac{S(v) + S(w)}{2} + \lambda(Q(w) - Q(v)). \tag{B.7}$$

We define

$$\begin{aligned} v(s) &:= sv + (1-s)w, \\ u(s) &:= \frac{v(s) + w}{2} + \lambda(f(v(s)) - f(w)), \end{aligned} \tag{B.8}$$

and set

$$\begin{aligned} G(s) &:= \frac{S(v(s)) + S(w)}{2} + \lambda(Q(v(s)) - Q(w)), \\ H(s) &:= S(u(s)). \end{aligned} \tag{B.9}$$

Observe that $G(0) = H(0) = S(w)$, $H(1) = S(u)$ and

$$G(1) = \frac{S(v) + S(w)}{2} + \lambda(Q(w) - Q(v)). \tag{B.10}$$

Thus,

$$-E_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{S(v)+S(w)}{2} + \lambda(Q(w)-Q(v)) - S(u)$$
$$= G(1) - H(1) = \int_0^1 \frac{d}{ds}(G(s)-H(s))\,ds. \tag{B.11}$$

We now consider

$$\frac{d}{ds}(G(s)-H(s)) = \frac{d}{ds}\left(\frac{S(v(s))}{2} - S(u(s)) + \lambda Q(v(s))\right)$$
$$= \frac{1}{2}S'(v(s))\cdot v'(s) - S'(u(s))\cdot u'(s) + \lambda Q'(v(s))\cdot v'(s)$$
$$= \frac{1}{2}S'(v(s))\cdot(v-w) - S'(u(s))\cdot\left(\frac{v'(s)}{2} + \lambda f'(v(s))v'(s)\right)$$
$$+ \lambda S'(v(s))\cdot f'(v(s))(v-w) \tag{B.12}$$
$$= \frac{1}{2}S'(v(s))\cdot(I+2\lambda f'(v(s)))(v-w)$$
$$- \frac{1}{2}S'(u(s))\cdot(I+2\lambda f'(v(s)))(v-w)$$
$$= \frac{1}{2}(S'(v(s))-S'(u(s)))\cdot(I+2\lambda f'(v(s)))(v-w).$$

Next, we introduce

$$w(r,s) := rv(s) + (1-r)w = rsv + (1-rs)w,$$
$$u(r,s) := \frac{v(s)+w(r,s)}{2} + \lambda(f(v(s)) - f(w(r,s))), \tag{B.13}$$

and note that

$$w(1,s) = v(s), \quad w(0,s) = w,$$
$$u(1,s) = v(s), \quad u(0,s) = u(s). \tag{B.14}$$

Using the above definitions and properties we write

$$S'(v(s)) - S'(u(s)) = \int_0^1 \frac{\partial}{\partial r}S'(u(r,s))\,dr, \tag{B.15}$$

and consider

$$\begin{aligned}\frac{\partial}{\partial r}S'(u(r,s)) &= S''(u(r,s))\frac{\partial}{\partial r}u(r,s)\\ &= S''(u(r,s))\left(\frac{1}{2}s(v-w)-\lambda f'(w(r,s))s(v-w)\right)\\ &= \frac{s}{2}S''(u(r,s))(I-2\lambda f'(w(r,s)))(v-w).\end{aligned} \tag{B.16}$$

Substituting the above into (B.12) we get

$$\begin{aligned}&\frac{d}{ds}(G(s)-H(s))\\ &= \int_0^1 \frac{s}{4}S''(u(r,s))(I-2\lambda f'(w(r,s)))(v-w)\,dr\cdot(I+2\lambda f'(v(s)))(v-w)\\ &= \int_0^1 \frac{s}{4}(v-w)^t(I-2\lambda f'(w(r,s)))^t(S''(u(r,s)))^t(I+2\lambda f'(v(s)))(v-w)\,dr.\end{aligned} \tag{B.17}$$

Using this identity in (B.11) yields

$$\begin{aligned}E_{j+\frac{1}{2}}^{n+\frac{1}{2}} = -\int_0^1\int_0^1 &\frac{s}{4}(v-w)^t(I-2\lambda f'(w(r,s)))^t(S''(u(r,s)))^t\\ &(I+2\lambda f'(v(s)))(v-w)\,dr\,ds.\end{aligned} \tag{B.18}$$

Let us denote the minimum and maximum eigenvalues of S'' by m and M, respectively. Since S'' is symmetric positive definite matrix we have

$$E_{j+\frac{1}{2}}^{n+\frac{1}{2}} \le -\frac{1}{4}m + M(\lambda||f'||_2+\lambda^2||f'||_2^2)||v-w||^2 \le 0, \tag{B.19}$$

where the last inequality is true under the CFL condition

$$\lambda||f'||_2 \le \frac{\sqrt{1+\frac{m}{M}}-1}{2}. \tag{B.20}$$

□

Next, we will establish an entropy production identity for the Nessyahu-Tadmor scheme described by (2.12)–(2.13). We will use the formulation in (5.8) for the staggered averages at time t^{n+1}:

$$v_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}(v_j^n + v_{j+1}^n) - \lambda(g(v_{j+1}^n) - g(v_j^n)). \tag{B.21}$$

Lemma 13. *For a given entropy pair (S, Q), the NT scheme satisfies the following entropy production identity*

$$\begin{aligned} E_{j+\frac{1}{2}}^{n+\frac{1}{2}} = &-\int_0^1\int_0^1 \frac{s}{4} S''(u(r,s))(1-2\lambda g'(w(r,s)))(1+2\lambda g'(v(s)))(v-w)^2 dr ds \\ &+ \int_0^1 \lambda S'(v(s))(f'(v(s)) - g'(v(s)))(w-v)\, ds. \end{aligned} \tag{B.22}$$

where $E_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ is defined as in (B.6):

$$E_{j+\frac{1}{2}}^{n+\frac{1}{2}} = S(v_{j+\frac{1}{2}}^{n+1}) - \frac{S(v_j^n) + S(v_{j+1}^n)}{2} + \lambda(Q(v_{j+1}^n) - Q(v_j^n)). \tag{B.23}$$

Proof. We use the same notations $u = v_{j+\frac{1}{2}}^{n+1},\ v = v_j^n,\ w = v_{j+1}^n$ and approximate the entropy production as in (B.6),

$$E_{j+\frac{1}{2}}^{n+\frac{1}{2}} = S(u) - \frac{S(v) + S(w)}{2} + \lambda(Q(w) - Q(v)). \tag{B.24}$$

We define

$$\begin{aligned} v(s) &:= sv + (1-s)w, \\ u(s) &:= \frac{v(s) + w}{2} + \lambda(g(v(s)) - g(w)), \end{aligned} \tag{B.25}$$

and use the same definitions for $G(s)$ and $H(s)$ as in (B.9). From (B.11) we know

$$-E_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \int_0^1 \frac{d}{ds}(G(s) - H(s))\, ds. \tag{B.26}$$

Now, we consider

$$
\begin{aligned}
&\frac{d}{ds}(G(s)-H(s))\\
&=\frac{d}{ds}\left(\frac{S(v(s))}{2}-S(u(s))+\lambda Q(v(s))\right)\\
&=\frac{1}{2}\big(S'(v(s))(1+2\lambda f'(v(s)))-S'(u(s))(1+2\lambda g'(v(s)))\big)(v-w)\\
&=\frac{1}{2}\big((S'(v(s))-S'(u(s)))(1+2\lambda g'(v(s)))\big)(v-w)\\
&+\lambda S'(v(s))(f'(v(s))-g'(v(s)))(v-w).
\end{aligned} \tag{B.27}
$$

Similar to (B.13) we set

$$
\begin{aligned}
w(r,s)&:=rv(s)+(1-r)w=rsv+(1-rs)w,\\
u(r,s)&:=\frac{v(s)+w(r,s)}{2}+\lambda(g(v(s))-g(w(r,s))),
\end{aligned} \tag{B.28}
$$

and use (B.14) to write

$$
\begin{aligned}
S'(v(s))-S'(u(s))&=\int_0^1\frac{\partial}{\partial r}S'(u(r,s))\,dr\\
&=\int_0^1\frac{s}{2}S''(u(r,s))(1-2\lambda g'(w(r,s)))(v-w)\,dr.
\end{aligned} \tag{B.29}
$$

Substituting the above identity into (B.27) and rewriting (B.26), we derive

$$
\begin{aligned}
E_{j+\frac{1}{2}}^{n+\frac{1}{2}}=&-\int_0^1\int_0^1\frac{s}{4}S''(u(r,s))(1-2\lambda g'(w(r,s)))(1+2\lambda g'(v(s)))(v-w)^2drds\\
&+\int_0^1\lambda S'(v(s))(f'(v(s))-g'(v(s)))(w-v)\,ds.
\end{aligned} \tag{B.30}
$$

□

VITA

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