WEIGHTED BERGMAN KERNEL FUNCTIONS

AND THE LU QI-KENG PROBLEM

A Dissertation

by

ROBERT LAWRENCE JACOBSON

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2012

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Approved by:

Chair of Committee,	Harold P. Boas
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ABSTRACT

Weighted Bergman Kernel Functions and the Lu Qi-keng Problem. (May 2012)

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The classical Lu Qi-keng Conjecture asks whether the Bergman kernel function for every domain is zero free. The answer is no, and several counterexamples exist in the literature. However, the more general Lu Qi-keng Problem, that of determining which domains in $\mathbb{C}^n$ have vanishing kernels, remains a difficult open problem in several complex variables. A challenge in studying the Lu Qi-keng Problem is that concrete formulas for kernels are generally difficult or impossible to compute. Our primary focus is on developing methods of computing concrete formulas in order to study the Lu Qi-keng Problem.

The kernel for the annulus was historically the first counterexample to the Lu Qi-keng Conjecture. We locate the zeros of the kernel for the annulus more precisely than previous authors. We develop a theory giving a formula for the weighted kernel on a general planar domain with weight the modulus squared of a meromorphic function. A consequence of this theory is a technique for computing explicit, closed-form formulas for such kernels where the weight is associated to a meromorphic kernel with a finite number of zeros on the domain. For kernels associated to meromorphic functions with an arbitrary number of zeros on the domain, we obtain a weighted version of the classical Ramadanov's Theorem which says that for a sequence of nested bounded domains exhausting a limiting domain, the sequence of associated kernels converges to the kernel associated to the limiting domain. The relationship between the zeros of the weighted kernels and the zeros of the corresponding unweighted kernels is investigated, and since these weighted kernels are related to unweighted kernels in $\mathbb{C}^2$, this investigation contributes to the study of the Lu Qi-keng Problem. This theory provides a much easier technique for computing certain weighted kernels than classical techniques and provides a unifying explanation of many previously known kernel formulas. We also present and explore a generalization of the Lu Qi-keng Problem.

DEDICATION

To my family and friends whose love and support I will cherish forever.

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I would like to express my gratitude to my advisor Dr. Harold Boas for his tireless dedication to mathematics and to my success, and for his extraordinary patience.

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1. INTRODUCTION

Our primary object of study is the so-called Bergman kernel function, a reproducing kernel for the Hilbert space of square-integrable holomorphic functions on a given nonempty connected open set in $\mathbb{C}^n$. Of present interest are questions related to determining when this function takes the value zero on its domain of definition. The Bergman kernel function is an object of considerable study in complex analysis. The problems of computing explicit formulas for this function and determining its zero set are classical problems in complex analysis.

1.1 The mathematical setting: the definition of the Bergman kernel function

If Ω is a domain (a nonempty connected open set) in n-dimensional complex space, then the Bergman space for Ω, denoted $A^2(\Omega)$ and named after the venerable twentieth century complex analyst Stefan Bergman, is the set of holomorphic square integrable functions on Ω. When supplied with the inner product defined by

$$\langle f, g \rangle := \int_\Omega f(w)\overline{g(w)}\, dw \quad \text{for all } f, g \in A^2(\Omega), \tag{1.1}$$

where dw is the real $2n$-dimensional Lebesgue volume (or area) measure, the Bergman space $A^2(\Omega)$ becomes a Hilbert space and is precisely the set

This dissertation follows the style of the *Michigan Mathematical Journal*.

$$A^2(\Omega) \equiv \{f \mid f \text{ is holomorphic on } \Omega \text{ and } \langle f, f\rangle < \infty\}.$$

If $\{\phi_j\}_{j=0}^{\infty}$ is an orthonormal Hilbert space basis for $A^2(\Omega)$ then the Bergman kernel function $K^\Omega\colon \Omega \times \Omega \to \mathbb{C}$ is defined by

$$K^\Omega(z,w) := \sum_{j=0}^{\infty} \phi_j(z)\overline{\phi_j(w)}. \tag{1.2}$$

The kernel $K^\Omega(z,w)$ is called a reproducing kernel for the Hilbert space $A^2(\Omega)$ because $K^\Omega(z,w)$ as a function of z is the unique function in $A^2(\Omega)$ such that for every $f \in A^2(\Omega)$, $f(w) = \left\langle f, K^\Omega(\cdot, w)\right\rangle$. The details of this classical theory can be found in many texts on complex analysis, in particular in [1, 18].

1.2 The historical setting: a short history of the Lu Qi-keng Problem

In his 1966 paper "On Kaehler manifolds with constant curvature," Lu Qi-keng writes,

> "But there seems to be nobody yet who has proved that for a bounded domain D in a general $\mathbb{C}^n$ [the Bergman kernel function] $K(z,\overline{\zeta})$ has no zero point in D, although there are many concrete examples justifying this statement." [22, p. 293]

While this statement falls short of being a conjecture, the statement that the kernel for a bounded domain is always zero free has nonetheless come to be known as the

Lu Qi-keng Conjecture. (A detailed survey of this problem is found in [4].) Domains for which the associated Bergman kernel is zero-free are now commonly called Lu Qi-keng domains, and the problem of determining which domains are Lu Qi-keng is known as the Lu Qi-keng Problem. This problem is of interest in the study of so-called Bergman representative coordinates which require the kernel to be zero-free (see [16, 17]). Indeed, it is in a discussion of Bergman representative coordinates that Lu Qi-keng first raises the issue. Also, as a consequence of the transformation formula for Bergman kernels under biholomorphic mappings [1, 4], which relates the kernels associated to two biholomorphic domains, the property of having a zero-free kernel is a biholomorphic invariant. This property is another tool in the study of biholomorphic equivalence classes of domains.

The classical Lu Qi-keng Conjecture is false. Skwarczyński was the first to give a negative answer to the Lu Qi-keng Conjecture in 1969 by showing that the kernel for an annulus with sufficiently small inner radius vanishes [26]. In the same year, Rosenthal [25] showed that every doubly connected non Lu Qi-keng domain is biholomorphic to an annulus, indicating a connection between the topology of a domain and the zeros of its kernel. This connection was illuminated by Suita and Yamada who found in 1976 that for bounded domains with smooth boundary in the complex plane the kernel is zero free if and only if the domain is simply connected [27].

Attention shifted to whether or not a result similar to that of Suita and Yamada holds for domains in higher dimensions. Greene and Krantz observed that Skwarczyński and Rosenthal's example lifts to higher dimensions [14] and showed that smoothly bounded domains in $\mathbb{C}^n$ satisfying a certain geometric condition that are sufficiently close to a Lu Qi-keng domain under a suitable metric are themselves Lu Qi-keng [14, 15]. It was thought that topologically trivial domains (perhaps with additional assumptions on the boundary) would be Lu Qi-keng [18, p. 58]. In 1986, Boas found a smooth bounded strongly pseudoconvex domain in $\mathbb{C}^2$ whose closure is diffeomorphic to the ball and whose Bergman kernel function has zeros. Thus, in higher dimensions an analogous topological characterization fails to hold even when a high degree of regularity is assumed of the domain [2]. Boas subsequently proved that the Lu Qi-keng domains form a nowhere dense set in a modified Hausdorff topology and are in that sense exceptional sets [3], and Boas, Fu, and Straube have described a convex non Lu Qi-keng domain [5]. Many descriptions of other domains for which the Bergman kernel vanishes exist in the literature. (Several such examples are collected by Jarnicki and Pflug in [17, p. 146ff].) Even so, the general problem of characterizing such domains, the Lu Qi-keng Problem, remains unsolved.

1.3 A brief motivation

To study the Lu Qi-keng Problem in higher dimensions, we would like concrete examples of kernels on domains in n-dimensional complex space, but obtaining a closed-form formula for the kernel from (1.2) is possible only for domains with a high degree of symmetry. There are, however, several techniques for relating the kernel of one domain to the kernel of another domain of different complex dimension (see [5]). We can therefore study the kernel of a domain by studying the kernel of a related domain about which we have more information.

A particular instance of these techniques relates the kernel for domains in two complex dimensions of the form

$$\Omega := \{(z,w) \in \mathbb{C}^2 \mid |z| < 1, |w| < \varphi(z)\} \subset \mathbb{C}^2, \tag{1.3}$$

where $\varphi(z)$ is some real-valued nonnegative measurable function defined on the unit disk in the plane, to so-called weighted Bergman kernel functions in the complex plane. (See Section 3.1.1.) For a nonnegative real-valued measurable function ψ defined on any domain Ω, one can replace the inner product in (1.1) with a *weighted* inner product:

$$\langle f, g \rangle_\psi := \int_\Omega f(w)\overline{g(w)}\psi(w)\, dw.$$

One then obtains the weighted Bergman space

$$A^2_\psi(\Omega) := \{f \mid f \text{ is holomorphic on } \Omega \text{ and } \langle f, f \rangle_\psi < \infty\}$$

and a weighted Bergman kernel function $K^\Omega_\psi(z,w)$ defined by (1.2) but now with the Hilbert space basis orthonormal with respect to the weighted inner product. The (unweighted) kernel for the domain Ω of (1.3) is related to the weighted kernel for the unit disk $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ with weight $\pi\varphi^2$ via the identity

$K^{\mathbb{D}}_{\pi\varphi^2}(z,w) \equiv K^{\Omega}((z,0),(w,0))$. Therefore, if $K^{\mathbb{D}}_{\pi\varphi^2}(z,w)$ has zeros, then so does $K^{\Omega}((z_1,w_1),(z_2,w_2))$.

1.4 A note on notation

We denote by $\mathbb{C}$ the space of complex numbers and by $\mathbb{C}^n$ the space of complex n-vectors. The open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$ is denoted by $\mathbb{D}$. We make occasional use of the Möbius transformation $\mu_c(z) := \frac{z-c}{1-\bar{c}z}$ which, when c is fixed in $\mathbb{D}$, is an automorphism of the disk taking c to the origin. We call a real-valued nonnegative measurable function which is not almost everywhere zero a weight function. A domain is a nonempty connected open set, although we may refer to a disconnected domain by which we mean a nonempty disconnected open set. We denote the unweighted Bergman kernel on a domain Ω by $K^{\Omega}(z,w)$ and the weighted Bergman kernel with respect to a weight function $\psi(z)$ by $K^{\Omega}_{\psi(z)}(z,w)$, where it is understood that the independent variable of the function ψ in the subscript is unrelated to the argument of the Bergman kernel function. If the domain is clear from context it will be suppressed in the notation. Note that z and w may be complex vectors. The usual unweighted Bergman space associated to a domain Ω is denoted $A^2(\Omega)$ with norm $\|\cdot\|$, while the weighted Bergman space with respect to a weight $\psi(z)$ is denoted $A^2_{\psi(z)}(\Omega)$ with norm $\|\cdot\|_{\psi(z)}$. In the case of the Bergman kernel, the Bergman space, and the norm and inner product on the Bergman space, the independent variable of the weight function in the subscript will be suppressed whenever possible to avoid confusion. If a domain is such that its associated unweighted kernel is zero free the domain is said to be Lu Qi-keng; otherwise the domain is non Lu Qi-keng or, equivalently, not Lu Qi-keng. We use the generic word kernel to refer to either a weighted or an unweighted Bergman kernel function. An unweighted Bergman kernel function is of course just a weighted Bergman kernel function $K_\psi(z, w)$ with trivial weight $\psi(z) \equiv 1$. The measure dz will denote the real $2n$-dimensional Lebesgue volume (or area) measure.

2. THE BERGMAN KERNEL ON AN ANNULUS

Historically, the first example of a non Lu Qi-keng domain discovered was annuli in the plane, the Bergman kernel of which was shown by Rosenthal [25] and Skwarczyński [26] to have a zero. Suita and Yamada subsequently showed that every smoothly bounded multiply connected planar domain is non Lu Qi-keng and gave an explicit relationship between the connectivity and the number of zeros of the Bergman kernel [27]. The zero of the Bergman kernel of the annulus found by Rosenthal and Skwarczyński turns out to be the only zero. However, they did not locate this zero with any precision.

Throughout Section 2, we set $\Omega := \{z \in \mathbb{C} \mid r < |z| < 1\} \subset \mathbb{C}$ for $r \in (0, 1)$. Following [26], we adopt the notation $\rho := r^2$.

2.1 Computing the Bergman kernel $K(z, w)$ for the annulus

We compute two different series representations for $K(z, w)$. This is a standard exercise.

By Definition (1.2), if $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis for $A^2(\Omega)$, then the Bergman kernel $K(z, w)\colon \Omega \times \Omega \to \mathbb{C}$ for Ω is given by

$$K(z, w) = \sum_{j=1}^{\infty} \phi(z)\overline{\phi(w)}, \tag{2.1}$$

where the sum converges uniformly on compact subsets of Ω. Since each holomorphic function on the annulus can be represented by a Laurent series centered at zero which converges uniformly on compact sets, and since functions of the form z^j ($j \in \mathbb{Z}$) are holomorphic, square integrable, and orthogonal in the Bergman space $A^2(\Omega)$ for Ω, the set $\{z^j\}_{j\in\mathbb{Z}}$ is an orthogonal basis for $A^2(\Omega)$. We need only normalize these basis functions to obtain a series representation for $K(z,w)$.

For $z^{n-1} = s^{n-1}e^{i\theta(n-1)}$, with $n \in \mathbb{Z} \setminus \{0\}$, $s = |z|$, and $\theta = \arg(z)$, we have

$$\begin{aligned}\left\|z^{n-1}\right\|^2 &= \int_\Omega |z^{n-1}|^2\, dz = \int_0^{2\pi}\int_r^1 s^{2n-1}\, ds\, d\theta = \int_0^{2\pi}\left[\frac{1}{2n}s^{2n}\right]_r^1 d\theta \\ &= \int_0^{2\pi}\left(\frac{1}{2n} - \frac{1}{2n}r^{2n}\right) d\theta = \frac{\pi}{n}(1-r^{2n}).\end{aligned}$$

Hence $z^{n-1}\cdot\left[\frac{n}{\pi}\left(\frac{1}{1-r^{2n}}\right)\right]^{1/2}$ is normalized.

For z^{-1} (the $n = 0$ case excluded above),

$$\begin{aligned}\left\|z^{-1}\right\|^2 &= \int_\Omega |z|^{-2}\, dz = \int_0^{2\pi}\int_r^1 \frac{1}{s}\, ds\, d\theta = \int_0^{2\pi}[\log(s)]_r^1\, d\theta \\ &= -2\pi\log(r) = 2\pi\log(1/r).\end{aligned}$$

Hence $\frac{1}{z}\left(2\pi\log(1/r)\right)^{-1/2}$ is normalized.

The functions z^j, $j \in \mathbb{Z}$, are orthogonal by the symmetry of Ω. Formula (2.1) yields

$$\begin{aligned} K(z,w) &= \frac{1}{z\overline{w}} \cdot \frac{-1}{2\pi \log(r)} + \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{m(z\overline{w})^{m-1}}{\pi(1-r^{2m})} \\ &= \frac{1}{z\overline{w}} \cdot \frac{-1}{\pi \log(r^2)} + \sum_{m=1}^{\infty} \left(\frac{m(z\overline{w})^{m-1}}{\pi(1-r^{2m})} - \frac{m(z\overline{w})^{-m-1}}{\pi(1-r^{-2m})} \right) \\ &= \frac{1}{\pi z\overline{w}} \left[\frac{-1}{\log(r^2)} + \sum_{m=1}^{\infty} \left(\frac{m(z\overline{w})^{m}}{1-r^{2m}} - \frac{m(z\overline{w})^{-m}}{1-r^{-2m}} \right) \right]. \end{aligned}$$

Note that $K(z,w)$ is really a function of $z\overline{w}$. Writing $q = z\overline{w}$, $\rho = r^2$, and $L_\rho(z\overline{w}) := K(z,w)$, we have

$$L_\rho(q) = \frac{1}{\pi q} \left[\frac{-1}{\log(\rho)} + \sum_{m=1}^{\infty} \left(\frac{mq^m}{1-\rho^m} - \frac{m(1/q)^m}{1-\rho^{-m}} \right) \right]. \tag{2.2}$$

In [26] we have the formula

$$L_\rho(q) = \frac{1}{\pi q} \left[\frac{-1}{\log(\rho)} + \sum_{m=0}^{\infty} \left(\frac{q\rho^m}{(1-q\rho^m)^2} + \frac{(\rho/q)\rho^m}{(1-(\rho/q)\rho^m)^2} \right) \right]. \tag{2.3}$$

To prove that (2.2) and (2.3) are the same (an exercise omitted from [26]), we compute as follows:

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{q\rho^m}{(1-q\rho^m)^2} &= \sum_{m=0}^{\infty} \frac{(q\rho^m - 1) + 1}{(1-q\rho^m)^2} \\ &= \sum_{m=0}^{\infty} \frac{-1}{1-q\rho^m} + \frac{1}{(1-q\rho^m)^2} =: \sum_{m=0}^{\infty} A(q\rho^m) + B(q\rho^m). \end{aligned}$$

Observing that $B(x) = -A'(x)$ and that $A(x) = \sum_{k=0}^{\infty} -x^k$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} A(q\rho^m) + B(q\rho^m) &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[-(q\rho^m)^k + k(q\rho^m)^{k-1}\right] \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left[-q^k(\rho^m)^k + (k+1)q^k(\rho^m)^k\right] \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} kq^k(\rho^k)^m = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} kq^k(\rho^k)^m \\ &= \sum_{k=1}^{\infty}\frac{kq^k}{1-\rho^k}.\end{aligned}$$

Replacing q with ρ/q in the above computation, we obtain

$$\sum_{m=0}^{\infty}\frac{(\rho/q)\rho^m}{(1-(\rho/q)\rho^m)^2} = \sum_{m=1}^{\infty}\frac{m(\rho/q)^m}{1-\rho^m}.$$

Factoring out ρ^m from the numerator and the denominator of this last sum and canceling, we get other term in the sum in (2.3).

2.2 Locating the zeros of $L_\rho(q)$

Our goal is to locate the zeros of $L_\rho(q)$. We first recite a theorem recorded in [1], originally due to Nobuyuki Suita and Akira Yamada [27]. (See [1, p. 132] for a proof.)

Theorem 2.1. *Let $D \subset \mathbb{C}$ be an n-connected, smooth, bounded domain. For $w \in D$ sufficiently close to the boundary of D, the Bergman kernel $K^D(z, w)$ for D has exactly $n-1$ zeros in D as a function of z.*

The annulus Ω is doubly connected, so when $w \in D$ is close to the boundary, $K^{\Omega}(z,w)$ has a single zero as a function of $z \in \Omega$. Let $\widetilde{\Omega} := \{r^2 = \rho < |z| < 1\}$. For $z, w \in \Omega$, $z\overline{w} =: q \in \widetilde{\Omega}$, and when $|w|$ is close to 1 (i.e., w is close to the outer boundary of Ω), $q \in \Omega \subset \widetilde{\Omega}$. Hence $L_\rho(q)$ has a single zero in Ω; call it q_0. Now consider the map $q \mapsto \rho/q$, a holomorphic automorphism of the annulus $\widetilde{\Omega}$ that reflects q about the circle $\{|q| = r\}$. Since one can relate the Bergman kernel of a domain to the Bergman kernel of its holomorphic image under a biholomorphism via a nonzero factor involving the derivative of the biholomorphism (see [1]), one sees that ρ/q_0 is also a zero of $L_\rho(q)$, and q_0 and ρ/q_0 are the only zeros of $L_\rho(q)$. (That ρ/q_0 is a zero of $L_\rho(q)$ can also be seen by inspecting (2.3).)

Setting $\phi_\rho(q) := \sum_{m=0}^{\infty}\left(\frac{q\rho^m}{(1-q\rho^m)^2} + \frac{(\rho/q)\rho^m}{(1-(\rho/q)\rho^m)^2}\right)$, we have that

$$\pi q L_\rho(q) = \frac{-1}{\log(\rho)} + \phi_\rho(q).$$

Observe that if $q \in (-1, 0)$, then $\phi_\rho(q)$ is real, negative, and bounded above by $q/(1-q)^2$. Thus if $-1/\log(\rho) < -q/(1-q)^2$, or, equivalently, $\rho < e^{(1-q)^2/q}$, then $\pi q L_\rho(q) < -1/\log(\rho) + q/(1-q)^2 < 0$. In particular, $\pi q L_\rho(q)$ is negative when $\rho < e^{(1-q)^2/q}$.

Theorem 2.2. *Let $\gamma, \rho \in (0,1)$. Then if ρ is sufficiently small depending on γ, $-\rho^\gamma L_\rho(-\rho^\gamma) > 0$.*

Proof. Since $\pi q L_\rho(q) = \frac{-1}{\log(\rho)} + \phi_\rho(q)$, we seek to show that $-\phi_\rho(-\rho^\gamma) < -1/\log(\rho)$ for ρ small enough. We have

$$\begin{aligned}
\phi_\rho(-\rho^\gamma) &= \sum_{m=0}^{\infty}\left(\frac{-\rho^\gamma\rho^m}{(1-(-\rho^\gamma)\rho^m)^2} + \frac{[\rho/(-\rho^\gamma)]\rho^m}{(1-[\rho/(-\rho^\gamma)]\rho^m)^2}\right) \\
\implies -\phi_\rho(-\rho^\gamma) &\leq \rho^\gamma\sum_{m=0}^{\infty}\rho^m + \rho^{1-\gamma}\sum_{m=0}^{\infty}\rho^m \\
&\leq \frac{2\rho^\alpha}{1-\rho},
\end{aligned}$$

where $\alpha := \min(\gamma, 1-\gamma)$. Now,

$$\frac{2\rho^\alpha}{1-\rho} < \frac{-1}{\log(\rho)} = \frac{1}{\log(\frac{1}{\rho})} \iff \frac{1}{2}\frac{1}{\rho^\alpha} - \frac{1}{2}\rho^{(1-\alpha)} > \log(\frac{1}{\rho}), \tag{2.4}$$

and clearly the right-hand inequality holds for ρ sufficiently small. □

Remark. Note that "sufficiently small" is quantifiable by Equation (2.4).

Corollary 2.3. *Let $q' \in (-1, 0)$ and $\gamma \in (0, 1)$. Then $L_\rho(q)$ has a zero on the interval $(q', -\rho^\gamma)$ for ρ sufficiently small.*

Proof. Clearly $\pi q L_\rho(q)$ is real on the negative real axis. By the paragraph preceding Theorem 2.2, if $q \in (-1, q']$ and $\rho < e^{(1-q')^2/q'}$, then $\pi q L_\rho(q)$ is negative. By Theorem 2.2, $\pi q L_\rho(q)$ is positive at $q = -\rho^\gamma$ for ρ sufficiently small. By the Intermediate Value Theorem, $\pi q L_\rho(q)$ is zero for some $q \in (q', -\rho^\gamma)$, establishing the corollary. □

In [26], Skwarczyński observes that $qL_\rho(q)$ is positive for positive q and shows that $qL_\rho(q) < 0$ at $q = -1$. Corollary 2.3 locates both zeros of $L_\rho(q)$ in particular intervals on the negative real axis.

2.3 Locating the zeros of $K(z, w)$

We found $q_0 \in (-1, -r)$ such that $L_\rho(q_0) = 0$ and proved that q_0 and $q_1 := \rho/q_0$ are the only zeros of $L_\rho(q)$. The ordered pair (z_0, w_0) is a zero of $K(z, w)$ if and only if $q = z_0\overline{w}_0$ is a zero of $L_\rho(q)$. Hence if (z_0, w_0) is a zero of $K(z, w)$, then z_0 and w_0 lie on the same line through the origin, and $(\lambda z_0, \frac{w_0}{\overline{\lambda}})$ is also a zero of $K(z, w)$ for $\lambda \in \mathbb{C}$ such that $\lambda z_0, \frac{w_0}{\overline{\lambda}} \in \Omega$. By Theorem 2.1, for fixed $w \in \Omega$ near the boundary of Ω, $K(z, w)$ has a single zero as a function of z. What about when w is not near the boundary? A priori, $K(z, w)$ may have at most two zeros (one for q_0 and one for q_1), or possibly none at all. We answer this question presently.

Because of the circular symmetry described in the previous paragraph, we may restrict attention to $w \in (r, 1) \subset \mathbb{R}$ without loss of generality.

Lemma 2.4. *Let $w \in (r, 1)$, and suppose $q_0 \in (-1, -r)$.*

1. *If $-q_0 < w$ then there exists a $z_0 \in \Omega$ such that $z_0 w = q_0$.*

2. *If $w < -r/q_0$ then there exists a $z_1 \in \Omega$ such that $z_1 w = q_1 = r^2/q_0$.*

Proof. Once (1) is established, (2) follows from the inversion automorphism on Ω. One can also repeat an argument symmetric to the proof of (1), which we do below.

Proof of 1. We need to show that $z_0 = q_0/w$ is in Ω. We have

$$|z_0| < 1 \iff |q_0/w| < 1 \iff -q_0 < w,$$

and

$$r < |z_0| \implies -r > z_0 \implies -r > q_0/w \implies w < -q_0/r.$$

Hence z_0 is in Ω when $-q_0 < w < -q_0/r$. But $q_0 \in (-1, -r)$ implies $-q_0/r > 1$, and $w > -q_0$ by hypothesis. Hence $z_0 \in \Omega$.

Proof of 2. We proceed as in the proof of (1):

$$\begin{aligned}
&z_1 w = r^2/q_0 \implies z_1 = \frac{r^2}{q_0 w};\\
&z_1 < -r \implies z_1 w < -rw \implies \frac{r^2}{q_0 w} w < -rw \implies w < -r/q_0; \text{ and}\\
&z_1 > -1 \implies \frac{\rho}{z_0 w} > -1 \implies w > -r^2/q_0.
\end{aligned}$$

Hence z_1 is in Ω when $-r^2/q_0 < w < -r/q_0$. But $-r^2/q_0 < r$, and $w < -r/q_0$ by hypothesis, so $z_1 \in \Omega$. □

Corollary 2.5. *As a function of z, $K(z, w)$ has*

1. *one zero if $w \in (r, -r/q_0)$ or $w \in (-q_0, 1)$, but $w \notin (r, -r/q_0) \cap (-q_0, 1) = (-q_0, -r/q_0)$;*

2. *two zeros if $w \in (-q_0, -r/q_0)$, which is nonempty whenever $r > q_0^2$;*

3. no zeros if $w \in [-r/q_0, -q_0]$ which is nonempty whenever $r \leq q_0^2$.

Moreover, $r < q_0^2$ for r sufficiently small.

Proof. Parts (1)-(3) are obvious from the previous proposition. To prove the last statement we observe that it is equivalent to $\sqrt{r} < |q_0| = -q_0$. Setting $\gamma = 1/4$ in Theorem 2.2, we have that $\pi q L_\rho(q) > 0$ for $q = -\sqrt{r}$ and ρ small enough, and since $\pi q L_\rho(q) < 0$ near $q = -1$, this locates $q_0 \in (-1, -\sqrt{r})$; hence $q_0 < -\sqrt{r}$, which is equivalent to $r < q_0^2$. $\square$

Remark. Computer evidence suggests that case (2) in the corollary never happens.

2.4 Additional remarks about the Bergman kernel on the annulus

Proposition 2.6. *$qL_\rho(q)$ extends to a holomorphic function on $\mathbb{C} \setminus S$ where S is the set of singular points $S := \{1/\rho^m, \rho^m \;:\; m = 0, 1, 2, \dots\}$.*

Proof. We show that the series $\sum_{m=0}^{\infty} \frac{q\rho^m}{(1-q\rho^m)^2} + \frac{(1/q)\rho^{m+1}}{(1-(1/q)\rho^{m+1})^2}$ converges uniformly on compact sets which are disjoint from S.

Let $A \subset \mathbb{C}$ be compact and disjoint from S, and let $c \in (0,1)$. Let $m_0 \in \mathbb{N}$ such that $|q| < (1-c)/\rho^m$ for all $m \in \mathbb{N}$ with $m \geq m_0$, for all $q \in A$. Then for $m \geq m_0$,

$$-|q|\rho^m > c - 1 \implies 1 - |q|\rho^m > c.$$

Let $m_1 \in \mathbb{N}$ with $m_1 \geq m_0$ such that $1/|q| \leq (1-c)/\rho^{m+1}$ for all $m \geq m_1$ and all $q \in A$. Then whenever $m \geq m_1$ and $q \in A$, $1 - 1/|q|\rho^{m+1} > c$. Thus,

$$\begin{aligned}
&\sum_{m=0}^{\infty} \left| \frac{q\rho^m}{(1-q\rho^m)^2} + \frac{(1/q)\rho^{m+1}}{(1-(1/q)\rho^{m+1})^2} \right| \\
&\leq \sum_{m=0}^{\infty} \frac{|q|\rho^m}{c^2} + \frac{|1/q|\rho^{m+1}}{c^2} \\
&= \frac{|q|}{c^2} \frac{\rho_1^m}{1-\rho} + \frac{1}{|q|c^2} \frac{\rho^{m_1+1}}{1-\rho} < \infty.
\end{aligned}$$

□

The Bergman kernel for both the disk $\mathbb{D}$ and the punctured disk $\mathbb{D}\backslash\{0\}$ is $\frac{1}{\pi(1-z\overline{w})^2}$. (See, for example, Section A.1.1 for a proof of this fact.) We expect the kernel for the annulus with inner radius r to converge to the kernel for the punctured disk as $r \to 0$. The following proposition establishes this fact.

Proposition 2.7. $\pi q L_\rho(q) \to \frac{q}{(1-q)^2}$ *as* $\rho \to 0$ *uniformly on compact subsets of* $\mathbb{D} \setminus \{0\}$ *and pointwise on* $\mathbb{D}$.

Proof. This proposition follows easily from Ramadanov's Theorem (Theorem 3.21). Below is an alternative direct proof.

From Formula (2.3),

$$\pi q L_\rho(q) = \frac{-1}{\log \rho} + \frac{q}{(1-q)^2} + \frac{\rho/q}{(1-\rho/q)^2} + \sum_{m=1}^{\infty} \left(\frac{q\rho^m}{(1-q\rho^m)^2} + \frac{(\rho/q)\rho^m}{(1-(\rho/q)\rho^m)^2} \right).$$

Observe that the singularity at $q = 0$ is removable. If A is a compact subset of $\mathbb{D}\backslash\{0\}$, then, when ρ is sufficiently small, A is disjoint from the set S of Proposition 2.6. Taking $\rho \to 0$, the conclusion is evident.

□

The paragraph preceding Theorem 2.2 shows that if $q' \in (-1, 0)$ and $\rho < e^{\left[(1-q')^2/q'\right)]}$, then $\pi q L_\rho(q)$ is negative for all $q \in (-1, q')$. Thus as $\rho \to 0$, the zeros of $\pi q L_\rho(q)$ inside the unit circle converge to zero.

Definition.

$$A_n := \{w \in \Omega \mid \#\{z \in \Omega \mid K(z, w) = 0\} = n\}$$

$$B_n := \{w \in \Omega \mid \#\{z \in \Omega \mid K(z, w) = 0\} > n\}$$

$$C_n := \{w \in \Omega \mid \#\{z \in \Omega \mid K(z, w) = 0\} \leq n\} = \Omega \setminus B_n.$$

Recall the following classical theorem of Hurwitz.

Theorem 2.8. *Let $\Omega \subset \mathbb{C}$ be a domain, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on Ω such that the f_n have at most m zeros in Ω, and $f_n \to f$ uniformly on compact subsets of Ω. Then either $f \equiv 0$ or f has at most m zeros in Ω.*

From this theorem the following corollaries are obvious.

Corollary 2.9. *The set C_n is closed in the relative topology on Ω.*

Corollary 2.10. *The set B_n is open.*

3. WEIGHTED KERNELS RELATED TO MEROMORPHIC FUNCTIONS

We now consider weighted kernels for weights that are the modulus of a meromorphic function raised to some power. Computing a concrete formula for such a kernel will assist in investigating its zero set. Because of a well-known construction described in Section 3.1.1, information about the zero set of the weighted kernel will yield information about the zero set of an associated unweighted kernel on a domain in higher dimensions.

3.1 Preliminary theory

The goal of Sections 3.1 and 3.2 is to express a weighted kernel in terms of another weighted kernel that is in some sense simpler than the first. We first motivate our study of weighted kernels on the plane by illuminating a connection between weighted kernels and the Lu Qi-keng problem in higher dimensions.

3.1.1 Relating weighted kernels in $\mathbb{C}$ to unweighted kernels in $\mathbb{C}^2$

Weighted kernels for domains in lower dimensions can sometimes be related to unweighted kernels for domains in higher dimensions. Consider domains of the form

$$\Omega := \{(z, w) \in \mathbb{C}^2 \mid z \in D, |w| < \phi(z)\} \subset \mathbb{C}^2, \tag{3.1}$$

where $\phi(z)$ is some real-valued nonnegative measurable function on a bounded planar domain D. By the symmetry of Ω in the ω coordinate, the kernel $K^{\Omega}(z, w, \zeta, \omega)$ for Ω is really a function of $w\overline{\omega}$, that is, $K^{\Omega}(z, w, \zeta, \omega) =: \widetilde{K}^{\Omega}(z, \zeta, w\overline{\omega})$. For each $f \in A^2_{\pi\phi^2}(D)$, extend f to a holomorphic function F on Ω via $F(z, w) = f(z)$. We have

$$\begin{aligned}
\|F\|^2_{A^2(\Omega)} &= \int_D \int_{|w|<\phi(z)} |f(z)|^2 \, dw \, dz \\
&= \int_D |f(z)|^2 \pi |\phi(z)|^2 \, dz \\
&= \|f\|^2_{A^2_{\pi\phi^2}(D)} < \infty,
\end{aligned}$$

and so $F \in A^2(\Omega)$. By the reproducing property of $K^{\Omega}(z, w, \zeta, \omega)$,

$$\begin{aligned}
F(z, 0) &= \int_{\zeta \in D} \int_{|\omega|<\phi(\zeta)} F(\zeta, \omega) K^{\Omega}(z, 0, \zeta, \omega) \, d\omega \, d\zeta \\
&= \int_{\zeta \in D} \int_{|\omega|<\phi(\zeta)} F(\zeta, 0) K^{\Omega}(z, 0, \zeta, 0) \, d\omega \, d\zeta \\
&= \int_{\zeta \in D} F(\zeta, 0) K^{\Omega}(z, 0, \zeta, 0) \pi \phi^2(\zeta) \, d\zeta.
\end{aligned}$$

But $F(z, 0) = f(z)$, so by the uniqueness of the (weighted) kernel on $A^2_{\pi\phi^2}(D)$, we must have that $K^D_{\pi\phi^2}(z, w) \equiv K^{\Omega}(z, 0, w, 0)$. Thus studying the function on the left hand side of this equivalence yields information about the function on the right hand side. We summarize this discussion in the following theorem.

Theorem 3.1. *Let D be a bounded domain in $\mathbb{C}$, let $\phi(z)\colon D \to [0,\infty]$ be a weight function on D, and let Ω be defined by* (3.1). *Then $K_{\pi\phi^2}(z,w) \equiv K(z,0,w,0)$.*

The idea behind this theorem appears in the literature in various forms. Theorem 3.1 is essentially Corollary 2.1 of [20] which Ligocka, generalizing an idea found in a proof due to Forelli and Rudin in [11], calls the Forelli–Rudin construction. The term Forelli–Rudin construction appears elsewhere in subsequent literature (for example, in [29]) in reference to similar techniques. Such techniques are surveyed in [5].

3.1.2 Elementary theorems

In the first theorem in this section, Theorem 3.2, we express a kernel with weight the modulus squared of a zero-free holomorphic function in terms of a simpler kernel. This theorem can be thought of as the simplest case of expressing a weighted kernel in terms of a simpler kernel and is fundamental to the rest of the theory.

Theorem 3.2. *Let $\Omega \subset \mathbb{C}^n$, let $K_\varphi(z,w)$ be the weighted Bergman kernel on Ω with respect to a weight function φ, and let g be meromorphic on Ω. Suppose that, after possibly removing singularities, $\frac{K_\varphi(z,w)}{g(z)}$ is holomorphic in z. Then $K_{\varphi\cdot|g|^2}(z,w) = \frac{K_\varphi(z,w)}{g(z)\overline{g(w)}}$.*

Remark. If $g(z_0) = 0$ for some $z_0 \in \Omega$, then z_0 must be a zero of $K_\varphi(z,w)$ of the same order if $\frac{K(z,w)}{g(z)\overline{g(w)}}$ is to have a removable singularity at z_0 and hence be in

$A^2_{|g|^2}(\Omega)$. (A nonexample is the case $\Omega := \mathbb{D}$, $g(z) := z$. Then, as we shall soon see, $K_{|z|^2}(z,w) = (2 - z\overline{w})K(z,w) \neq \frac{K(z,w)}{z\overline{w}}$, the right hand side of which is not even holomorphic on $\mathbb{D}$.) Moreover, $g(z)$ may have poles, in which case $K_{|g|^2}(z,w)$ will have zeros. As explained in Section 3.1.1, $K_{|g|^2}(z,w)$ having zeros means that a related unweighted kernel for a corresponding domain in higher dimensions has zeros.

Proof. To save us some writing, we will assume $\varphi(z) \equiv 1$, as the proof is the same. We have that $\int_\Omega \left|\frac{K(z,w)}{g(z)}\right|^2 |g(z)|^2 \, dz = \|K(\cdot,w)\|^2 < \infty$, so

$$\frac{K(z,w)}{g(z)} \in A^2_{|g|^2}(\Omega) \text{ as a function of } z. \tag{3.2}$$

Also, $\int_\Omega |K_{|g|^2}(z,w)|^2 |g(z)|^2 \, dz = \left\|K_{|g|^2}(\cdot,w)\right\|^2_{|g|^2} < \infty$, so

$$K_{|g|^2}(z,w)g(z) \in A^2(\Omega) \text{ as a function of } z. \tag{3.3}$$

By (3.2) and the reproducing property of $K_{|g|^2}(z,w)$, we have

$$\begin{aligned}
\frac{K(z,w)}{g(z)} &= \int_\Omega \frac{K(\zeta,w)}{g(\zeta)} K_{|g|^2}(z,\zeta)|g(\zeta)|^2 \, d\zeta \\
&= \int_\Omega K(\zeta,w)K_{|g|^2}(z,\zeta)\overline{g(\zeta)} \, d\zeta \\
&= \overline{\int_\Omega K(w,\zeta)K_{|g|^2}(\zeta,z)g(\zeta) \, d\zeta} =: I.
\end{aligned}$$

By (3.3) and the reproducing property of $K(z,w)$,

$$I = \overline{K_{|g|^2}(w,z)g(w)} = \overline{g(w)}K_{|g|^2}(z,w).$$

We have shown that $\frac{K(z,w)}{g(z)} = \overline{g(w)}K_{|g|^2}(z,w)$, from which the theorem follows. $\square$

This theorem provides a recipe for constructing non Lu Qi-keng domains, as illustrated by the following example.

Example 3.3. Let $c \in \mathbb{D}$, and define $\varphi(z) := z - c$ and $\Omega := \{(z,w) \in \mathbb{C}^2 \mid z \in \mathbb{D}, |w| < \left|\frac{1}{\sqrt{\pi}\varphi(z)}\right|\} \subset \mathbb{C}^2$. Then, following the discussion in Section 3.1.1, the kernel $K^{\mathbb{D}}_{\varphi^{-2}}(z,w)$ satisfies $K^{\mathbb{D}}_{\varphi^{-2}}(z,w) \equiv K^{\Omega}(z,0,w,0)$. Theorem 3.2 gives $K^{\mathbb{D}}_{\varphi^{-2}}(z,w) = (z-c)K^{\mathbb{D}}(z,w)(\overline{w}-\overline{c})$, which clearly has zeros whenever $z = c$ or $w = c$. Hence Ω is not Lu Qi-keng.

The technique of Example 3.3, though elementary, appears to be absent from the literature. This example justifies the claim at the beginning of this chapter that, to the extent that we seek an explicit formula for a weighted kernel when the weight is the modulus squared of a meromorphic function, it suffices to study the case of the modulus squared of a holomorphic function, as the poles appear as zeros of the same order in the formula for the weighted kernel given by Theorem 3.2. On the other hand, if the meromorphic function associated to the weight has zeros, then clearly those zeros cannot appear as poles in the formula for the weighted kernel since the kernel is holomorphic. We return to this issue after establishing some facts about Bergman spaces seen as a vector spaces.

Recall the definition of the Bergman kernel: for a weight φ (possibly trivial, $\varphi \equiv 1$) on a domain Ω, if $\{\psi_j\}_{j=1}^{\infty}$ is an orthonormal basis for $A^2_{\varphi}(\Omega)$, then $K^{\Omega}_{\varphi}(z,w)$ is defined by

$$K^{\Omega}_{\varphi}(z,w) = \sum_{j=1}^{\infty} \psi_j(z)\overline{\psi_j(w)}. \tag{3.4}$$

This fact along with a consideration of the vector space structure of $A^2_{\varphi}(\Omega)$ will achieve the goal described at the beginning of this section.

Theorem 3.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain, let $S \subset \Omega$ be a set that is locally the zero set of a nonconstant holomorphic function, and let ψ be meromorphic on Ω such that $\psi|_{\Omega\setminus S}$ is both nonvanishing and holomorphic on $\Omega \setminus S$. Then $f \in A^2_{|\psi|^2}(\Omega \setminus S)$ if and only if $f \cdot \psi$ extends to a holomorphic function on Ω with $f \cdot \psi \in A^2(\Omega)$, and $g \in A^2(\Omega)$ if and only if $\frac{g}{\psi}|_{\Omega\setminus S} \in A^2_{|\psi|^2}(\Omega \setminus S)$.*

Proof. We have

$$\|f \cdot \psi\|^2_{A^2(\Omega\setminus S)} = \int_{\Omega\setminus S} |f(z)|^2 |\psi(z)|^2 \, dz = \|f\|^2_{A^2_{|\psi|^2}(\Omega\setminus S)} . \tag{3.5}$$

If $f \in A^2_{|\psi|^2}(\Omega\setminus S)$, then the right hand side of Equation (3.5) is finite by definition. Since S is a Lebesgue null set, for every $z_0 \in S$ and every neighborhood $U \subset \Omega$ of z_0, $\|f \cdot \psi\|_{L^2(U)} < \infty$, that is, $f \cdot \psi \in L^2(U)$. Hence by the L^2-version of the Riemann Removable Singularity Theorem (see [24, E.3.2]), $f \cdot \psi$ can be extended holomorphically to all of Ω. Moreover, $\|f \cdot \psi\|_{\Omega \setminus S} = \|f \cdot \psi\|_{\Omega}$. Hence $f(z)\psi(z) \in A^2(\Omega)$.

On the other hand, if $f(z)\psi(z) \in A^2(\Omega)$, then

$$\|f(z)\psi(z)\|_{A^2(\Omega)} = \|f(z)\psi(z)\|_{A^2(\Omega\setminus S)} < \infty,$$

and hence the right hand side of Equation (3.5) is finite. Thus $f \in A^2_{|\psi|^2}(\Omega \setminus S)$.

For the second half of the conclusion, set $f = g/\psi$ and apply the argument of the preceding two paragraphs to see that $g \in A^2(\Omega)$ if and only if $\frac{g}{\psi}|_{\Omega\setminus S} \in A^2_{|\psi|^2}(\Omega\setminus S)$. $\square$

Corollary 3.5. *Let Ω, S, and ψ be as in Theorem 3.4, and let φ be a weight function on Ω that is bounded away from zero in a neighborhood of S. Then $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis for $A^2_{\varphi}(\Omega)$ if and only if $\{\frac{\phi_j}{\psi}\}_{j=1}^{\infty}$ is an orthonormal basis for $A^2_{|\psi|^2\varphi}(\Omega \setminus S)$.*

Proof. By the proof of Theorem 3.4, the map from $A^2_{\varphi}(\Omega)$ to $A^2_{|\psi|^2\varphi}(\Omega \setminus S)$ given by $g \mapsto \frac{g}{\psi}|_{\Omega\setminus S}$ is an isometric surjection. Since $\langle \phi_j/\psi, \phi_k/\psi\rangle_{|\psi|^2\varphi} = \langle \phi_j, \phi_k\rangle_{\varphi} = \delta_{j,k}$, this map is an isometric isomorphism of Hilbert spaces. $\square$

Remark. The basis elements ϕ_j/ψ of $A^2_{|\psi|^2}(\Omega \setminus S)$ may have poles at the zeros of ψ. Thus the new basis may not intersect the previous basis; in fact the new basis may not intersect the previous Bergman space. However, in the case that ψ is zero-free on Ω and S is empty, using Equation (3.4) to express $K_{|\psi|^2\varphi}(z, w)$ in terms of the basis $\{\phi_j/\psi\}_{j=1}^{\infty}$ represents another proof of Theorem 3.2.

3.2 Decomposition theorems

Theorem 3.2 allows us to express a weighted kernel with a pole in the weight in terms of another weighted kernel with the pole in the weight removed. (See Example 3.3.) In this section we develop theorems that allow us to express a weighted kernel having a zero in the weight in terms of another weighted kernel with the zero in the weight removed. Recall that the obstruction to using Theorem 3.2 in the case that the meromorphic function in the weight vanishes is that kernels are holomorphic; kernels cannot have poles.

To understand the strategy of the theorems in this section, consider the punctured disk $\mathbb{D}\setminus\{0\}$ and weight function $|z|^2$. The space $A^2_{|z|^2}(\mathbb{D}\setminus\{0\})$ may contain functions with a singularity at the origin. However, functions with a pole of order 2 or greater or an essential singularity at the origin are not square integrable in $A^2_{|z|^2}(\mathbb{D}\setminus\{0\})$. Thus, an orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ for $A^2_{|z|^2}(\mathbb{D})$ can be extended to a basis for $A^2_{|z|^2}(\mathbb{D}\setminus\{0\})$ by adding a single (normalized) basis function ϕ_0 orthogonal to the others which has a single pole of order one at the origin. From the orthonormal basis representation for weighted kernels given by Equation (3.4) we obtain

$$K^{\mathbb{D}\setminus\{0\}}_{|z|^2}(z,w) = K^{\mathbb{D}}_{|z|^2}(z,w) + \phi_0(z)\overline{\phi_0(w)},$$

that is,

$$K^{\mathbb{D}}_{|z|^2}(z,w) = K^{\mathbb{D}\setminus\{0\}}_{|z|^2}(z,w) - \phi_0(z)\overline{\phi_0(w)}. \tag{3.6}$$

By the L^2-version of the Riemann Removable Singularity Theorem [24, E.3.2], every function in $A^2(\mathbb{D} \setminus \{0\})$ extends uniquely to a function in $A^2(\mathbb{D})$; hence $A^2(\mathbb{D} \setminus \{0\}) \equiv A^2(\mathbb{D})$. It follows that $K^{\mathbb{D}}(z,w) \equiv K^{\mathbb{D}\setminus\{0\}}(z,w)$. Using this fact and applying Theorem 3.2 to the kernel on the right hand side of (3.6), we obtain the formula

$$K^{\mathbb{D}}_{|z|^2}(z,w) = \frac{K^{\mathbb{D}}(z,w)}{z\overline{w}} - \phi_0(z)\overline{\phi_0(w)}. \tag{3.7}$$

A simple computation (which we omit) shows that $\phi_0(z) = \frac{1}{z\sqrt{\pi}}$. Rearranging the right hand side of (3.7) and using the fact that $K^{\mathbb{D}}(z,w) = \frac{1}{\pi(1-z\overline{w})^2}$, we get that $K^{\mathbb{D}}_{|z|^2}(z,w) = \frac{2-z\overline{w}}{\pi(1-z\overline{w})^2} = (2 - z\overline{w})K^{\mathbb{D}}(z,w)$, justifying the nonexample of the remark following Theorem 3.2. (A more detailed computation using classical techniques is given in Section A, including a computation of $K^{\mathbb{D}}(z,w)$.)

For a general planar domain Ω and holomorphic function f, we are able to express $K^{\Omega}_{|f|^2}(z,w)$ in terms of the kernel associated to a "simpler" weight function and the basis functions for the orthogonal complement of $A^2_{|f|^2}(\Omega)$ in a larger space of functions.

Theorem 3.6. *Let $\Omega \subset \mathbb{C}$ be a domain, $c \in \Omega$, and φ be a weight on Ω which is bounded in a neighborhood of c. Then*

$$K^{\Omega}_{|z-c|^2\varphi}(z,w) = \frac{K^{\Omega}_{\varphi}(z,w)}{(z-c)(\overline{w}-\overline{c})} - \frac{K^{\Omega}_{\varphi}(z,c)K^{\Omega}_{\varphi}(c,w)}{(z-c)(\overline{w}-\overline{c})K^{\Omega}_{\varphi}(c,c)}. \tag{3.8}$$

Remark. The requirement that φ be bounded in a neighborhood of c excludes cases such as $\varphi(z) = \frac{1}{|z|^2}$ with $c = 0$. The right hand side of Equation (3.8) has singularities at $z = c$ and $w = c$, but these singularities are removable.

Proof. Let $\psi(z) := \frac{K_\varphi^\Omega(z,c)}{z-c}$. Clearly $\psi \in A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$. Our strategy is as follows:

1. $\frac{K_\varphi^\Omega(z,w)}{(z-c)(\overline{w}-\overline{c})}$ reproduces elements of $A^2_{|z-c|^2\varphi}(\Omega)$ in $A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$.

2. $\psi(z)$ is orthogonal to $A^2_{|z-c|^2\varphi}(\Omega)$ in $A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$; as a consequence,

3. $\psi(z)$ is orthogonal to $K^\Omega_{|z-c|^2\varphi}(z,w)$ in $A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$.

4. From (1) and (2), $Q(z,w) := \frac{K_\varphi^\Omega(z,w)}{(z-c)(\overline{w}-\overline{c})} - c_0(w)\psi(z)$ also reproduces elements of $A^2_{|z-c|^2\varphi}(\Omega)$ in $A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$, where $c_0(w)$ is arbitrary.

5. Setting $c_0(w) := \overline{\psi(w)}/K_\varphi^\Omega(c,c)$, we have $Q \in A^2_{|z-c|^2\varphi}(\Omega)$; it follows from (4) and the uniqueness of the Bergman kernel that $Q(z,w) \equiv K^\Omega_{|z-c|^2\varphi}(z,w)$.

Once (1) and (2) are proven, (3) and (4) are obvious.

Proof of (1): Let $f \in A^2_{|z-c|^2\varphi}(\Omega)$. We have

$$
\begin{aligned}
&\int_{\Omega\setminus\{c\}} f(w)\frac{K_\varphi^\Omega(z,w)}{(z-c)(\overline{w}-\overline{c})}|w-c|^2\varphi(w)\,dw \\
&\quad = \frac{1}{z-c}\int_\Omega K_\varphi^\Omega(z,w)f(w)(w-c)\varphi(w)\,dw \\
&\quad = \frac{1}{z-c}f(z)(z-c) \qquad\qquad \left(\text{since } f(z)(z-c) \in A^2_\varphi(\Omega)\right) \\
&\quad = f(z).
\end{aligned}
$$

This proves (1).

Proof of (2)*:* Let $f \in A^2_{|z-c|^2\varphi}(\Omega)$. We have

$$
\begin{aligned}
&\int_{\Omega\setminus\{c\}} f(w)\overline{\psi(w)}|w-c|^2\varphi(w)\,dw \\
&\quad= \int_{\Omega\setminus\{c\}} f(w)\frac{\overline{K^\Omega_\varphi(w,c)}}{\overline{w}-\overline{c}}|w-c|^2\varphi(w)\,dw \\
&\quad= \int_{\Omega} f(w)(w-c)K^\Omega_\varphi(c,w)\varphi(w)\,dw \\
&\quad= 0 \qquad\qquad \text{(since } f(z)(z-c)\in A^2_\varphi(\Omega)).
\end{aligned}
$$

This proves (2).

To finish the proof, observe that for $c_0(w) := \overline{\psi(w)}/K^\Omega_\varphi(c,c)$, we have that

$$
Q(z,w) \equiv \frac{K^\Omega_\varphi(z,w)}{(z-c)(\overline{w}-\overline{c})} - \frac{K^\Omega_\varphi(z,c)K^\Omega_\varphi(c,w)}{(z-c)(\overline{w}-\overline{c})K^\Omega_\varphi(c,c)},
$$

which has a removable singularity at $z=c$ and $w=c$. Thus (5) holds, and the theorem is proven. $\square$

Equation (3.7) is a special case of Equation (3.8). When φ is both bounded and bounded away from zero near c, the function $\frac{K^\Omega_\varphi(z,c)}{(z-c)\sqrt{K^\Omega_\varphi(c,c)}}$ turns out to be the orthonormal basis for the orthogonal complement of $A^2_{|z-c|^2\varphi(z)}(\Omega)$ in $A^2_{|z-c|^2\varphi(z)}(\Omega\setminus\{c\})$, which is ϕ_0 in Equation (3.7). Theorem 3.6 combined with Theorem 3.2 allows one to produce an explicit formula for $K^\Omega_{|f|^2}(z,w)$ in terms of $K^\Omega(z,w)$ in the case that f is meromorphic on Ω with a finite number of zeros. One just iterates the formula of Equation (3.8).

Theorem 3.6 is an illustrative special case of a more general theorem the proof of which is similar. Instead of considering a single linear factor in the weight in Equation (3.8), we can prove the theorem with an arbitrary number of zeros—including an infinite number of zeros as long as we assume additionally that everything converges appropriately.

Theorem 3.7. *Let Ω be a planar domain, $\{c_j\}_{j=1}^m$ a sequence of m distinct points in Ω, $\{\alpha_j\}_{j=1}^m$ a sequence of positive integers, and φ a weight such that for all j, φ is both bounded and bounded away from zero in a neighborhood of c_j. Define the following polynomials:*

$$\begin{aligned} p(z) &:= (z-c_1)^{\alpha_1}(z-c_2)^{\alpha_2}\cdots(z-c_m)^{\alpha_m}; \\ p_{j,k}(z) &:= (z-c_1)^{\alpha_1}(z-c_2)^{\alpha_2}\cdots(z-c_{j-1})^{\alpha_{j-1}}(z-c_j)^k, \quad (1\le j\le m, 1\le k\le\alpha_j); \\ q_{j,k}(z) &:= p(z)/p_{j,k}(z) \\ &\quad = (z-c_j)^{\alpha_j-k}(z-c_{j+1})^{\alpha_{j+1}}(z-c_{j+2})^{\alpha_{j+2}}\cdots(z-c_m)^{\alpha_m}. \end{aligned}$$

Then

$$K^{\Omega}_{|p(z)|^2\varphi}(z,w) = \frac{K^{\Omega}_{\varphi}(z,w)}{p(z)\overline{p(w)}} - \sum_{j=1}^{m}\sum_{k=1}^{\alpha_j} \frac{K^{\Omega}_{|q_{j,k}|^2\varphi}(z,c_j)K^{\Omega}_{|q_{j,k}|^2\varphi}(c_j,w)}{p_{j,k}(z)\overline{p_{j,k}(w)}K^{\Omega}_{|q_{j,k}|^2\varphi}(c_j,c_j)}.$$

Remark 3.8. By the L^2-version of the Riemann Removable Singularity Theorem [24, E.3.2], when a weight ψ is both bounded and bounded away from zero in a neighborhood of c, then $K_\psi^{\Omega\setminus\{c\}}(z,w) \equiv K_\psi^\Omega(z,w)$. This hypothesis appears in several subsequent theorems.

Proof. We wish to show that the functions $\psi_{j,k}(z) := \frac{K^\Omega_{|q_{j,k}|^2\varphi}(z,c_j)}{p_{j,k}(z)}$ form a basis for the orthogonal complement of $A^2_{|p|^2\varphi}(\Omega)$ in $A^2_{|p|^2\varphi}(\Omega \setminus \{c_j\}_{j=1}^m)$. We prove only that the $\psi_{j,k}$ are mutually orthogonal, the rest being an easy exercise.

For ψ_{j_0,k_0} and ψ_{j_1,k_1} distinct, we may assume $j_0 > j_1$ or else $j_0 = j_1$ and $k_0 > k_1$. Then

$$p_{j_0,j_1}(z) = p_{j_1,k_1}(z)(z-c_{j_1})^{\alpha_{j_1}-k_1}(z-c_{j_1+1})^{\alpha_{j_1+1}} \cdots (z-c_{j_0})^{k_0},$$

and

$$\begin{aligned}
&\langle \psi_{j_0,k_0}(z), \psi_{j_1,k_1}(z)\rangle_{|p|^2\varphi} \\
&\quad = \int_{\Omega\setminus\{c_j\}_{j=1}^m} \frac{K^\Omega_{|q_{j_0,k_0}|^2\varphi}(z,c_{j_0})}{p_{j_0,k_0}(z)} \frac{\overline{K^\Omega_{|q_{j_1,k_1}|^2\varphi}(c_{j_1},z)}}{\overline{p_{j_1,k_1}(z)}} |p(z)|^2\varphi(z)\,dz \\
&\quad = \int_\Omega K^\Omega_{|q_{j_0,k_0}|^2\varphi}(z,c_{j_0}) \\
&\qquad \times K^\Omega_{|q_{j_1,k_1}|^2\varphi}(c_{j_1},z)\overline{(z-c_{j_1})^{\alpha_{j_1}-k_1}(z-c_{j_1+1})^{\alpha_{j_1+1}}\cdots(z-c_{j_0})^{k_0}} \\
&\qquad \times |q_{j_0,k_0}(z)|^2\varphi(z)\,dz \\
&\quad = 0.
\end{aligned}$$

$\square$

Remark. Observe that m may be infinite as long as we have convergence of all of the functions involved. That is to say, the proof does not depend on m being finite; we can still construct an orthonormal basis for the orthogonal complement of $A^2_{|p|^2\varphi}(\Omega)$ in $A^2_{|p|^2\varphi}(\Omega \setminus \{c_j\}_{j=1}^m)$. However, this is of limited practical value since in that case Theorem 3.7 fails to give a closed form formula for the original weighted kernel.

Combining Theorem 3.7 with Theorem 3.2, one can express $K^{\Omega}_{|f|^2\varphi}(z, w)$ as an algebraic expression in terms of $K^{\Omega}_{\varphi}(z, w)$ for any f meromorphic on Ω having a finite number of zeros and any weight function φ that is both bounded and bounded away from zero near the zeros of f. For example, Theorem 3.7 yields closed form formulas for weighted kernels on the disk $\mathbb{D}$ for any weight of the form $|f|^2$ where f is holomorphic on $\mathbb{D}$ with a finite number of zeros. Note however that the formula of Theorem 3.6 and the corresponding formula in Theorem 3.7 have removable singularities at the zeros of the original weights and that we are identifying the functions represented by the right hand sides of those formulas with their holomorphic extensions to the singular points. In practice, this identification manifests itself as an algebraic simplification, though the expressions quickly become too complicated to manipulate by hand when more than one or two zeros are removed from the weight.

3.3 Zeros of the weighted kernels

Now that we have the tools of Section 3.1 and Section 3.2 that give formulas for weighted kernels in terms of simpler kernels, we can study the relationship the zeros of these weighted kernels have to the zeros of the simpler kernels.

Theorem 3.9. *Let Ω be a domain in $\mathbb{C}$, let $c, z_0, w_0 \in \Omega$, and let φ be a weight on Ω that is bounded and bounded away from zero in some neighborhood of c. Suppose $K_{|z-c|^2\varphi}(z_0, w_0) = 0$. Then $K_\varphi(z_0, w_0) = 0$ if and only if either $K_\varphi(z_0, c) = 0$ or $K_\varphi(c, w_0) = 0$.*

Proof. By the hypothesis and Theorem 3.6,

$$0 = \frac{K_\varphi(z_0, w_0)}{(z_0 - c)(\overline{w_0} - \overline{c})} - \frac{K_\varphi(z_0, c)K_\varphi(c, w_0)}{(z_0 - c)(\overline{w_0} - \overline{c})K_\varphi(c, c)},$$

from which the theorem is evident. □

The hypothesis that φ be bounded in a neighborhood of c ensures that c really is a zero of the weight $|z - c|^2\varphi(c)$. Requiring that φ be bounded away from zero in a neighborhood of c determines the order of the zero of the weight $|z-c|^2\varphi(c)$ to be two, a fact to which there are two significant consequences. First, as a consequence of the L^2-version of the Riemann Removable Singularity Theorem, $K_\varphi^\Omega(z, w) \equiv K_\varphi^{\Omega\setminus\{c\}}(z, w)$ on $\Omega \setminus \{c\}$. We employ this fact in the next several theorems without comment. Second, for zeros of higher orders in the weight, we would need to use Theorem 3.7, which does not give the conclusion, rather than Theorem 3.6.

Theorem 3.9 says the value of $K_\varphi^\Omega(z,w)$ at c affects the zero set of $K_{|z-c|^2\varphi}^\Omega(z,w)$. Compare this to the case that $c \notin \Omega$, in which case Theorem 3.2 says that the zero sets of both kernels coincide.

Theorem 3.9 assumes $K_{|z-c|^2\varphi}^\Omega(z,w)$ has a zero and then says when $K_\varphi^\Omega(z,w)$ has a zero. The next theorem assumes $K_\varphi(z,w)$ has a zero and then says when $K_{|z-c|^2\varphi}^\Omega(z,w)$ has a zero.

Theorem 3.10. *Let Ω be a domain in $\mathbb{C}$, let $z_0, c \in \Omega$ with $z_0 \neq c$, and let φ be a weight on Ω that is bounded and bounded away from zero in some neighborhood of c. Suppose $K_\varphi(z_0, c) = 0$. Then $K_{|z-c|^2\varphi}(z_0, w)$ has a zero of order $m-1$ at $w = c$ if and only if $K_\varphi(z_0, w)$ has a zero of order m at $w = c$.*

Proof. By Theorem 3.6,

$$\begin{aligned} K_{|z-c|^2\varphi}(z_0, w) &= \frac{K_\varphi(z_0, w)}{(z_0 - c)(\overline{w} - \overline{c})} - \frac{K_\varphi(z_0, c) K_\varphi(c, w)}{(z_0 - c)(\overline{w} - \overline{c}) K_\varphi(c, c)} \\ &= \frac{1}{z_0 - c} \cdot \frac{K_\varphi(z_0, w)}{\overline{w} - \overline{c}}. \end{aligned}$$

If m is the order of the zero of $K_\varphi(z_0, w)$ at $w = c$, then this last expression has a zero of order $m - 1$ at $w = c$. □

Theorem 3.11. *Let Ω be a domain in $\mathbb{C}$, let $c_0, c_1, c_2 \in \Omega$ be distinct, and let φ be a weight on Ω that in some neighborhood of c_0 is bounded and bounded away from zero. Suppose either $K_\varphi(c_0, c_1) = 0$ or $K_\varphi(c_0, c_2) = 0$. Then $K_{|z-c_0|^2\varphi}(c_1, c_2) = 0$ if and only if $K_\varphi(c_1, c_2) = 0$.*

Proof. By Theorem 3.6,

$$\begin{aligned} K_{|z-c_0|^2\varphi}(c_1, c_2) &= \frac{K_\varphi(c_1, c_2)}{(c_1 - c_0)(\overline{c_2} - \overline{c_0})} - \frac{K_\varphi(c_1, c_0)K_\varphi(c_0, c_2)}{(c_1 - c_0)(\overline{c_2} - \overline{c_0})K_\varphi(c_0, c_0)} \\ &= \frac{1}{(c_1 - c_0)(\overline{c_2} - \overline{c_0})} \cdot K_\varphi(c_1, c_2), \end{aligned}$$

from which the theorem is evident. □

The next theorem attempts to find a c, z_0, and w_0 so that $\frac{K_\varphi(z_0,w_0)}{K_\varphi(z_0,c)} = \frac{K_\varphi(c,w_0)}{K_\varphi(c,c)}$ by making the right hand side small through some hypothesis, then adjusting the z variable on the left hand side (assumed to be near a zero of the left hand side) to make the equality true. It will then follow from Theorem 3.6 that $K_{|z-c_0|^2\varphi}(z, w) = 0$ at some point (z, w).

Theorem 3.12. *Let Ω be a domain in $\mathbb{C}$, and let φ be a weight on Ω. Suppose that for some $c_0 \in \partial\Omega$ and some sequence $\{c_j\}_{j=1}^\infty$ in Ω converging to c_0, we have $\frac{K_\varphi(z,c_j)}{K_\varphi(c_j,c_j)} \to 0$ as $j \to \infty$ for all fixed $z \in \Omega$. Suppose also that there exist $z_0, w_0 \in \Omega$ such that $K_\varphi(z_0, w_0) = 0$ and that $K_\varphi(z, c_j)$ is bounded away from 0 when j is large enough and z is in a compact subset of Ω. Then for sufficiently large j (i.e., for c_j sufficiently close to $c_0 \in \partial\Omega$), there exists a $z_1 = z_1(c_j) \in \Omega$ near z_0 such that $K_{|z-c_j|^2\varphi}(z_1, w_0) = 0$.*

Proof. Define the following for all $\zeta, \omega, z \in \Omega$ and $\varepsilon > 0$:

$$
\begin{aligned}
g_{\zeta,\omega}(z) &:= \frac{K_\varphi(z,\omega)}{K_\varphi(z,\zeta)};\\
\alpha(\zeta) &:= |g_{\zeta,w_0}(\zeta)| = \left|\frac{K_\varphi(\zeta,w_0)}{K_\varphi(\zeta,\zeta)}\right|; \text{ and}\\
B(z,\varepsilon) &:= \{w \in \Omega \mid |z-w| < \varepsilon\} \text{ (the usual open } \varepsilon\text{-ball about } z).
\end{aligned}
$$

Observe that by hypothesis, $\alpha(c_j) \to 0$ as $j \to \infty$. Let $d := \frac{1}{2}\mathrm{dist}(z_0, \partial\Omega)$. Choose a $j_0 \in \mathbb{N}$ so that the following hold:

1. $\frac{1}{j_0} < d$, and

2. $|c_j - c_0| < \frac{1}{j_0}$ for all $j > j_0$.

By (1) and the definition of d,

3. the closed ball $\overline{B\left(z_0, \frac{1}{j_0}\right)}$ is contained in Ω.

By hypothesis, for j large enough, $K_\varphi(z, c_j)$ is bounded away from zero for $z \in \overline{B\left(z_0, \frac{1}{j_0}\right)}$. Thus for j large enough, the zeros of $g_{c_j,w_0}(z) := \frac{K_\varphi(z,w_0)}{K_\varphi(z,c_j)}$ correspond to the zeros of $K_\varphi(z, w_0)$ on $\overline{B\left(z_0, \frac{1}{j_0}\right)}$. So by possibly increasing j_0, we can choose j_0 large enough so that we also have

4. $\overline{B\left(z_0, \frac{1}{j_0}\right)}$ contains a single zero of $g_{c_j,w_0}(z)$ when $j > j_0$, namely z_0.

Now choose $j_1 \geq j_0$ such that

5. $\alpha(c_j) < \frac{1}{j_0}$ for all $j \geq j_1$, and

6. $\alpha(c_j) < \inf \left\{ |g_{c_{j_1},w_0}(z)| \mid z \in \partial B\left(z_0, \frac{1}{j_0}\right) \right\}$ for all $j \geq j_1$.

Now we argue that $C_0 := g_{c_{j_1},w_0}(\partial B\left(z_0, \frac{1}{j_0}\right))$ is a closed curve about the origin and the point $g_{c_{j_1},w_0}(c_{j_1})$. Since z_0 is a zero of the holomorphic function $g_{c_{j_1},w_0}(z)$ and $\partial B\left(z_0, \frac{1}{j_0}\right)$ is a closed curve about z_0, it follows from the argument principle of the elementary theory of holomorphic functions that C_0 is a closed curve about the origin. Moreover, $\alpha(c_{j_1}) < \inf \left\{ |g_{c_{j_1},w_0}(z)| \mid z \in \partial B\left(z_0, \frac{1}{j_0}\right) \right\}$ by (6), and so C_0 also encloses a region containing $g_{c_{j_1},w_0}(c_{j_1})$, that is, $|g_{c_{j_1},w_0}(c_{j_1})| < |g_{c_{j_1},w_0}(z)|$ on $\partial B\left(z_0, \frac{1}{j_0}\right)$. By Rouché's Theorem [6, p. 110], it follows that $g_{c_{j_1},w_0}(z) - g_{c_{j_1},w_0}(c_{j_1})$ has a zero in $B\left(z_0, \frac{1}{j_0}\right)$. Hence for some $z_1 \in B\left(z_0, \frac{1}{j_0}\right)$, we have $g_{c_{j_1},w_0}(z_1) = g_{c_{j_1},w_0}(c_{j_1})$, which is equivalent to $\frac{K_\varphi(z_1,w_0)}{K_\varphi(z_1,c_{j_1})} = \frac{K_\varphi(c_{j_1},w_0)}{K_\varphi(c_{j_1},c_{j_1})}$. Since both $|z_0 - z_1| < d$ and $|c_0 - c_{j_1}| < d$, it must be that $z_1 \neq c_{j_1}$. Therefore $K_{|z-c_{j_1}|^2\varphi}(z_1, w_0) = 0$. $\square$

When $c \notin \Omega$, then $K_{|z-c|^2\varphi}(z,w) = \frac{K_\varphi(z,w)}{(z-c)(\overline{w}-\overline{c})}$ by Theorem 3.2, so the zero set of $K_{|z-c|^2\varphi}(z,w)$ corresponds to the zero set of $K_\varphi(z,w)$ in that case. An interpretation of Theorem 3.12 is that for $c \in \Omega$ as c approaches the boundary of Ω, the zero set of $K_{|z-c|^2\varphi}(z,w)$ approaches the zero set of $K_\varphi(z,w)$. The following corollary to Theorem 3.10 does not assume that c is near the boundary of Ω, though unlike in Theorem 3.12 we assume c is adapted to a zero of the kernel.

Corollary 3.13 (Corollary to Theorem 3.10)**.** *Let Ω be a domain in $\mathbb{C}$, and let φ be a weight on Ω. Suppose $c, w_0 \in \Omega$ such that $K_\varphi(z,w_0)$ has a zero of order $m > 1$ at $z = c$. Then there exist $z_1, z_2, \ldots, z_{m-1}, w_1 \in \Omega$ with the z_j near z_0 and w_1 near w_0 such that $K_{|z-c|^2\varphi}(z_j, w_1) = 0$ for $j = 1, \ldots, m-1$.*

Proof. Just apply Hurwitz's Theorem to the conclusion of Theorem 3.10. □

Theorem 3.14.

A. *Suppose $\Omega \subset \mathbb{C}$ is a domain, and φ a weight, and $\{c_j\}_{j=1}^{\infty}$ is a sequence in Ω converging to a point $c_0 \in \partial\Omega$ such that for fixed z, $\frac{K_\varphi(z,c_j)}{\sqrt{K_\varphi(c_j,c_j)}} \to 0$ as $j \to \infty$. Suppose also that $K_{|z-c|^2\varphi}(z_0, w_0) = 0$ for all $c \in \Omega$. Then either*

   (a) both $K_\varphi(z_0, w) \equiv 0$ and $K_{|z-c|^2\varphi}(z_0, w) \equiv 0$ as functions of w for all c; or

   (b) both $K_\varphi(z, w_0) \equiv 0$ and $K_{|z-c|^2\varphi}(z, w_0) \equiv 0$ as functions of z for all c.

B. *For any domain Ω and weight φ, if $K_\varphi(z, w_0) \equiv 0$ as a function of z, then for all $c \in \mathbb{C}$, $K_{|z-c|^2\varphi}(z, w_0) \equiv 0$ as well.*

Remark. Part (B) is similar to Theorem 3.9 and follows from Theorem 3.9, the hypothesis that $K_\varphi(z, w_0) \equiv 0$, and continuity.

Proof. We prove part (A) first; the proof of part (B) will be obvious from the proof of part (A).

Let $c \in \Omega$. Assume first that $z_0 \neq c$ and $w_0 \neq c$. Then by Theorem 3.6 we must have

$$K_\varphi(z_0, w_0) = \frac{K_\varphi(z_0, c)K_\varphi(c, w_0)}{K_\varphi(c, c)}. \tag{3.9}$$

The right hand side of Equation (3.9) vanishes when we replace c with c_j and let $j \to \infty$. Hence $K_\varphi(z_0, w_0) = 0$, and therefore either $K_\varphi(z_0, c) = 0$ or $K_\varphi(c, w_0) = 0$. One of these two conditions must hold for a set of values of c having an accumulation point, hence for all c. Assume without loss of generality that the condition holding for all c is $K_\varphi(c, w_0) = 0$. Thus $K_\varphi(z, w_0) \equiv 0$ as a function of z. But then

$$K_\varphi(z, w_0) = \frac{K_\varphi(z, c)K_\varphi(c, w_0)}{K_\varphi(c, c)} = 0 \text{ for all } z,$$

and hence (by Theorem 3.6) $K_{|z-c|^2\varphi}(z, w_0) \equiv 0$ as a function of z. □

Since Theorems 3.12 and 3.14 have a hypothesis requiring that or implied by $\frac{K_\varphi(z,c)}{\sqrt{K_\varphi(c,c)}} \to 0$ as $c \to c_0 \in \partial\Omega$, we give sufficient conditions on a domain for these hypotheses to be satisfied. Below is [12, Lemma 4.1 part 2] which is "implicit in work of Pflug (see [17, Section 7.6]) and Ohsawa [21] on the completeness of the Bergman metric" according to Fu and Straube [12].

Theorem 3.15. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Suppose p_0 is a point in the boundary of Ω satisfying the following outer cone condition:*

> *there exist $r \in (0, 1]$, $a \geq 1$, and a sequence $\{w_\ell\}_{\ell=1}^\infty$ of points $w_\ell \notin \Omega$ with $\lim_{\ell\to\infty} w_\ell = p_0$ and $\Omega \cap B(w_\ell, r\,\|w_\ell - p_0\|^a) = \emptyset$.*

Then for any sequence $\{p_j\}_{j=1}^\infty \subset \Omega$ converging to p_0,

$$\lim_{j\to\infty} \frac{K^\Omega(z, p_j)}{\sqrt{K^\Omega(p_j, p_j)}} = 0.$$

The outer cone condition of Theorem 3.15 is satisfied when Ω has C^1 boundary, for example. Pseudoconvexity is a central notion in several complex variables which reduces to a triviality for domains of a single complex dimension: every domain in the plane is pseudoconvex [18]. We will therefore say no more about pseudoconvexity. Because we wish to also have the conclusion of the above theorem for certain weighted kernels, we show that the property addressed by the theorem is preserved when the weight of a kernel is multiplied by the modulus squared of a linear factor.

Theorem 3.16. *Suppose $\Omega \subset \mathbb{C}$ is a domain, $p_0 \in \partial\Omega$, and $\{p_j\}_{j=1}^{\infty} \subset \Omega$ is a sequence with $p_j \to p_0$ as $j \to \infty$ such that $\frac{K_\varphi(z,p_j)}{\sqrt{K_\varphi(p_j,p_j)}} \to 0$ as $j \to \infty$ locally uniformly. Then for any $c \in \Omega$ with $K_\varphi(c,c) \neq 0$, $\frac{K_{|z-c|^2\varphi}(z,p_j)}{\sqrt{K_{|z-c|^2\varphi}(p_j,p_j)}} \to 0$ as $j \to \infty$ locally uniformly.*

Proof. From Theorem 3.6 we get

$$\frac{K_{|z-c|^2\varphi}(z,p_j)}{\sqrt{K_{|z-c|^2\varphi}(p_j,p_j)}} = \frac{\frac{K_\varphi(z,p_j)K_\varphi(c,c)-K_\varphi(z,c)K_\varphi(c,p_j)}{(z-c)(\overline{p_j}-\overline{c})K_\varphi(c,c)}}{\left(\frac{K_\varphi(p_j,p_j)K_\varphi(c,c)-|K_\varphi(p_j,c)|^2}{|p_j-c|^2K_\varphi(c,c)}\right)^{1/2}}$$

$$= \frac{|p_j-c|^2K_\varphi(c,c)^{1/2}}{(z-c)(\overline{p_j}-\overline{c})K_\varphi(c,c)} \cdot \frac{K_\varphi(z,p_j)K_\varphi(c,c)-K_\varphi(z,c)K_\varphi(c,p_j)}{(K_\varphi(p_j,p_j)K_\varphi(c,c)-|K_\varphi(p_j,c)|^2)^{1/2}}$$

$$= \frac{(p_j-c)}{(z-c)K_\varphi(c,c)^{1/2}} \cdot \frac{\frac{K_\varphi(z,p_j)K_\varphi(c,c)}{K_\varphi(p_j,p_j)^{1/2}} - \frac{K_\varphi(z,c)K_\varphi(c,p_j)}{K_\varphi(p_j,p_j)^{1/2}}}{\left(K_\varphi(c,c) - \frac{|K_\varphi(p_j,c)|^2}{K_\varphi(p_j,p_j)}\right)^{1/2}}.$$

The first factor approaches a constant as $j \to \infty$. In the second factor, every fraction in the numerator and the denominator approaches zero as $j \to \infty$ locally uniformly by hypothesis, so the second factor approaches zero as $j \to \infty$ locally uniformly. This proves the theorem. $\square$

3.4 Convergence of kernels in terms of convergence of weights

Under reasonable hypotheses (which are guaranteed by Theorem 3.15 and Theorem 3.16), the formulas given by Theorem 3.2 and Theorem 3.6 agree as the zero in the weight approaches the boundary.

Proposition 3.17. *Let $\Omega \subset \mathbb{C}$ be a domain, let $c_0 \in \partial\Omega$, and let φ be a weight on Ω that is bounded and bounded away from zero in a neighborhood of c_0 intersected with Ω. Suppose that for any sequence $\{c_j\}_{j=1}^{\infty}$ in Ω converging to c_0, the expression $\frac{K_\varphi(z,c_j)}{\sqrt{K_\varphi(c_j,c_j)}} \to 0$ as $j \to \infty$ either pointwise or uniformly on compact sets. Then $K_{|z-c_j|^2\varphi}(z,w) \to K_{|z-c_0|^2\varphi}(z,w)$ as $j \to \infty$ pointwise or uniformly on compact sets respectively.*

Proof. By Theorem 3.6, for any j we have

$$K_{|z-c_j|^2\varphi}(z,w) = \frac{K_\varphi(z,w)}{(z-c_j)(\overline{w}-\overline{c_j})} - \frac{K_\varphi(z,c_j)K_\varphi(c_j,w)}{(z-c_j)(\overline{w}-\overline{c_j})K_\varphi(c_j,c_j)}.$$

The second term on the right hand side is

$$\frac{1}{(z-c_j)(\overline{w}-\overline{c_j})} \cdot \frac{K_\varphi(z,\, c_j)}{\sqrt{K_\varphi(c_j,\, c_j)}} \cdot \frac{K_\varphi(c_j,\, w)}{\sqrt{K_\varphi(c_j,\, c_j)}},$$

the first factor of which stays bounded as $c_j \to c_0$, while the second two factors vanish as $c_j \to c_0$ by hypothesis. Hence as $c_j \to c_0$ we have

$$K_{|z-c_j|^2\varphi}(z,w) \to \frac{K_\varphi(z,w)}{(z-c_0)(\overline{w}-\overline{c_0})},$$

the right hand side of which is the representation for $K_{|z-c_0|^2\varphi}(z,w)$ given by Theorem 3.4. $\square$

One can exploit Theorem 3.2 and Theorem 3.7 to prove convergence of a sequence of weighted kernels $\{K_{|f_j|^2\varphi}\}_{j=1}^\infty$ in terms of convergence of the sequence of holomorphic functions $\{f_j\}_{j=1}^\infty$ by showing that the formulas given by these theorems converge.

Theorem 3.18. *Let $\Omega \subset \mathbb{C}$ be a bounded domain, and let A be a neighborhood of Ω with $\Omega \subset\subset A$. Suppose f is holomorphic on A with $f \not\equiv 0$, φ is a weight on A which is bounded and bounded away from zero in a neighborhood of the zeros of f, and $\{f_j\}_{j=0}^\infty$ is a sequence of holomorphic functions on A converging uniformly on compact subsets of A to f. If f has zeros on $\partial\Omega$, then also suppose that for any sequence $\{c_j\}_{j=1}^\infty$ converging to a point $c_0 \in \partial\Omega$, the expression $\frac{K^\Omega_{|f|^2\varphi}(z,c_j)}{\sqrt{K^\Omega_{|f|^2\varphi}(c_j,c_j)}}$ converges to zero as $j \to \infty$. Then $K^\Omega_{|f_j|^2\varphi}(z,w)$ converges uniformly on compact subsets of Ω to $K^\Omega_{|f|^2\varphi}(z,w)$.*

Proof. Let $B \subset\subset A$ be a bounded neighborhood of Ω such that f has no zeros on ∂B. Since f is holomorphic on A and $B \subset\subset A$, f can only have a finite number of zeros on B. Moreover, $\{f_j\}_{j=0}^{\infty}$ converges uniformly to f on any neighborhood of Ω that is relatively compact in A. By Hurwitz's Theorem, eventually the f_j have the same number of zeros on B as f, say m zeros counting multiplicity; without loss of generality, assume f_j has m zeros on B for all j. We may factor f and the f_j as $f(z) = (z-c_1)\cdots(z-c_m)g(z)$ and $f_j(z) = (z-c_1^j)\cdots(z-c_m^j)g_j(z)$ where g and g_j are nonvanishing on $\overline{B}$. Since $g_j \to g$ uniformly on Ω, it follows from Theorem 3.2 that $K^{\Omega}_{|g_j|^2\varphi}(z,w) \to K^{\Omega}_{|g|^2\varphi}(z,w)$ uniformly on Ω. Now applying Theorem 3.7 and, if $c_m \in \partial\Omega$ and $c_m^j \in \partial\Omega$, Theorem 3.17, we have that $K^{\Omega}_{|z-c_m^j|^2|g_j|^2\varphi}(z,w)$ converges uniformly on compact subsets of Ω to $K^{\Omega}_{|z-c_m|^2|g|^2\varphi}(z,w)$. Iterating this argument $m-1$ more times yields that $K^{\Omega}_{|f_j|^2\varphi}(z,w)$ converges uniformly on Ω to $K^{\Omega}_{|f|^2\varphi}(z,w)$. $\square$

Remark. The hypotheses of this theorem may seem overwrought. We can show convergence when the holomorphic functions in the weight are zero free with Theorem 3.2. Also, we can show convergence when the holomorphic functions are products of the same number of linear factors with zeros in the domain using Theorem 3.7. The hypotheses of Theorem 3.18 essentially reduce the proof to these cases.

Note that it is trivial to extend this theorem to the case that the functions g and g_j in the proof are nonvanishing meromorphic functions.

We have proven theorems that give a formula for the weighted kernel $K^{\Omega}_{\psi}(z,w)$ on a planar domain Ω when ψ is of the form $\psi(z) = |\varphi(z)|^2$ (φ holomorphic on Ω), but if the holomorphic function φ has an infinite number of zeros in Ω, these formulas do not yield closed-form representations for $K^{\Omega}_{\psi}(z,w)$. Indeed, we expect these formulas to depend on every point in the zero set of φ, and so we should not expect a simple closed-form representation in general. To overcome this difficulty we may try to approximate $K_{\psi}^{\Omega}(z,w)$ with a weighted kernel for which we do have a closed form expression, in particular a weighted kernel for which the weight is the modulus squared of a holomorphic function with finitely many zeros.

To this end, we study the case that φ is a convergent Blaschke product,

$$\varphi(z) := \prod_{j=0}^{\infty} \overline{a_j} \frac{a_j - z}{1 - \overline{a_j} z}, \tag{3.10}$$

where $\{a_j\}_{j=0}^{\infty}$ is a sequence in $\mathbb{D}$ satisfying

$$\sum_{j=0}^{\infty} (1 - |a_j|) < \infty. \tag{3.11}$$

Recall the theory of Blaschke products says that the infinite product in (3.10) converges if and only if the associated infinite sum in (3.11) converges, and this convergence is uniform on compact subsets of $\mathbb{D}$. We also define

$$\varphi_k(z) := \prod_{j=0}^{k} \overline{a_j} \frac{a_j - z}{1 - \overline{a_j} z}.$$

The theory of Blaschke products gives that $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$. We also have $|\varphi_k(z)| < 1$ for all $z \in \mathbb{D}$.

Theorem 3.19. *Let φ_k and φ be defined as above, and let $S := \{a_j \mid j = 1, 2, \dots\}$, the zero set of φ. Then $K^{\mathbb{D}}_{|\varphi_k|^2}(z,w)$ converges uniformly on compact subsets of $\mathbb{D}$ to $K^{\mathbb{D}}_{|\varphi|^2}(z,w)$.*

Proof. Define $S_k := \{a_0, a_1, \dots, a_k\}$, the zero set ϕ_k. By Theorem 3.2, $K^{\mathbb{D}\setminus S}_{|\varphi|^2}(z,w) = \frac{K(z,w)}{\varphi(z)\overline{\varphi(w)}}$ and $K^{\mathbb{D}\setminus S_k}_{|\varphi_k|^2}(z,w) = \frac{K(z,w)}{\varphi_k(z)\overline{\varphi_k(w)}}$. The functions $K^{\mathbb{D}}_{|\varphi_k|^2}(z,w)$ and $K^{\mathbb{D}}_{|\varphi|^2}(z,w)$ are the projections from $L^2_{|\varphi|^2}(\mathbb{D})$ onto $A^2_{|\varphi|^2}(\mathbb{D})$ of $K^{\mathbb{D}\setminus S_k}_{|\varphi_k|^2}(z,w)$ and $K^{\mathbb{D}\setminus S}_{|\varphi|^2}(z,w)$ respectively. This projection is realized by computing the inner product in $L^2_{|\varphi|^2}(\mathbb{D})$ of a function in $L^2_{|\varphi|^2}(\mathbb{D})$ and $K^{\mathbb{D}}_{|\varphi|^2}(z,w)$. Therefore, to show that $K^{\mathbb{D}}_{|\varphi_k|^2}(z,w)$ converges to $K^{\mathbb{D}}_{|\varphi|^2}(z,w)$, it suffices to show that $\left\langle K^{\mathbb{D}\setminus S_k}_{|\varphi_k|^2}(\cdot,w),\ K^{\mathbb{D}}_{|\varphi|^2}(\cdot,z)\right\rangle_{|\varphi|^2}$ converges to $\left\langle K^{\mathbb{D}\setminus S}_{|\varphi|^2}(\cdot,w),\ K^{\mathbb{D}}_{|\varphi|^2}(\cdot,z)\right\rangle_{|\varphi|^2}$. We first show convergence for w in a compact subset of $\mathbb{D}\setminus S$ and z in a compact subset of $\mathbb{D}$. This is equivalent to showing that

$$\int_{\mathbb{D}} \left(\frac{K(\zeta,w)}{\varphi_k(\zeta)\overline{\varphi_k(w)}} - \frac{K(\zeta,w)}{\varphi(\zeta)\overline{\varphi(w)}} \right) K^{\mathbb{D}}_{|\varphi|^2}(z,\zeta)|\varphi(\zeta)|^2\, d\zeta \to 0$$

as $k \to \infty$, for w in a compact subset of $\mathbb{D}\setminus S$ and z in a compact subset of $\mathbb{D}$. After some standard algebraic manipulation and an application of Hölder's inequality, we reduce to the problem of showing (for w, z as above) that

$$\int_{\mathbb{D}} \left| \frac{\psi_k(\zeta)}{\overline{\varphi_k(w)}} - \frac{1}{\overline{\varphi(w)}} \right|^2 d\zeta \to 0 \text{ as } k \to 0,$$

where we define $\psi_k(z) := \varphi(z)/\varphi_k(z) = \prod_{j=k+1}^{\infty} \overline{a_j}a_j \frac{a_j - z}{1-\overline{a_j}z}$. By the theory of Blaschke products, $\varphi_k(z)$ converges uniformly on compact subsets of $\mathbb{D}$ to $\varphi(z)$, so $\psi_k(z)$

converges uniformly on compact subsets of $\mathbb{D} \setminus S$ to one. We have that $(1/\varphi_k(w) - 1/\varphi(w)) \to 0$ as $k \to \infty$ (since w is in a compact subset of $\mathbb{D}\setminus S$). Now take $\delta \in (0,1)$ (thinking of δ as close to 1), and split up the integral as follows:

$$\begin{aligned}
&\int_{\mathbb{D}} \left| \frac{\overline{\psi_k(\zeta)}}{\overline{\varphi_k(w)}} - \frac{1}{\overline{\varphi(w)}} \right|^2 d\zeta \\
&\quad = \int_{|\zeta|\le\delta} \left| \frac{\overline{\psi_k(\zeta)}}{\overline{\varphi_k(w)}} - \frac{1}{\overline{\varphi(w)}} \right|^2 d\zeta + \int_{\delta<|\zeta|<1} \left| \frac{\overline{\psi_k(\zeta)}}{\overline{\varphi_k(w)}} - \frac{1}{\overline{\varphi(w)}} \right|^2 d\zeta \\
&\quad := I_1 + I_2.
\end{aligned}$$

The second integral is controlled by the fact that $|\psi_k(z)| < 1$ on $\mathbb{D}$ (by the theory of convergent Blaschke products), so as $\delta \to 1^-$, $I_2 \to 0$. For I_1, the set $B_\delta := \{\zeta \in \mathbb{D} \mid |\zeta| \le \delta\}$ is compact, and so $\psi_k(z)$ converges uniformly to 1 on B_δ. Letting $k \to \infty$ and $\delta \to 1$ we obtain the result for w in a compact subset of $\mathbb{D}\setminus S$ and z in a compact subset of $\mathbb{D}$.

Now let $w_0 \in S$. Since the points in S are isolated, there exists some ε-ball B in $\mathbb{D}$ centered at w_0 such that $\overline{B} \cap S = \emptyset$. By the above, $K^{\mathbb{D}}_{|\varphi_k|^2}(z,w) - K^{\mathbb{D}}_{|\varphi|^2}(z,w)$ converges uniformly to zero for z in an arbitrary compact subset of $\mathbb{D}$ and w in the compact set ∂B. By the Maximum Principle, it follows that $K^{\mathbb{D}}_{|\varphi_k|^2}(z,w) - K^{\mathbb{D}}_{|\varphi|^2}(z,w)$ converges uniformly to zero for z in an arbitrary compact subset of $\mathbb{D}$ and w in the compact set $\overline{B}$. Hence $K^{\mathbb{D}}_{|\varphi_k|^2}(z,w)$ converges uniformly on compact subsets of $\mathbb{D}$ to $K^{\mathbb{D}}_{|\varphi|^2}(z,w)$. $\square$

Computing explicit formulas of weighted kernels with the theory we are developing depends on our ability to compute an explicit formula for the unweighted kernel, and this is only possible for domains with a high degree of symmetry. In the case that our weight is the square of the modulus of a holomorphic function with an infinite number of zeros on the domain, a straightforward application of Theorem 3.7 at best produces an infinite series, not a closed-form formula. However, we may be able to approximate our domain from the inside by relatively compact domains for which we can compute the unweighted kernel. The holomorphic function associated to the weight, when the weight is restricted to these approximating domains, will have only a finite number of zeros, and hence our theory will yield a closed form formula for the weighted kernel on the approximating domain with weight the square of the modulus of the holomorphic function restricted to the subdomain. We may then apply Ramadanov's Theorem (Theorem 3.21 in section 3.5.1) to argue that these weighted kernels converge uniformly on compact subsets of the original domain to the weighted kernel we seek. Since the usual statement of Ramadanov's Theorem applies only to unweighted kernels, we must shift attention to unweighted kernels on domains in $\mathbb{C}^2$ associated to the weighted kernels on domains in $\mathbb{C}$. This is essentially an explanation of the following weighted version of Ramadanov's Theorem and its proof.

Theorem 3.20 (Ramadanov's Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let φ be a bounded continuous function on Ω. Suppose $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \cdots$ is a sequence* *of relatively compact domains such that* $\cup_{j=1}^{\infty}\Omega_j = \Omega$. *Then* $K^{\Omega_j}_{|\varphi|^2}(z,w)$ *converges uniformly on compact subsets of* $\Omega \times \Omega$ *to* $K^{\Omega}_{|\varphi|^2}(z,w)$ *as* $j \to \infty$.

Proof. Set $\Omega_0 := \Omega$ and define

$$\widetilde{\Omega}_j := \{(z,w) \in \mathbb{C}^2 \mid z \in \Omega_j,\ |w| < |\varphi(z)|/\sqrt{\pi}\}, \qquad j = 0,1,2,\dots .$$

By Ramadanov's Theorem (Theorem 3.21), $K^{\widetilde{\Omega}_j}(z,\zeta,w,\omega)$ converges uniformly on compact subsets of $\widetilde{\Omega}_0 \times \widetilde{\Omega}_0$ to $K^{\widetilde{\Omega}_0}(z,\zeta,w,\omega)$ as $j \to \infty$. Therefore, $K^{\widetilde{\Omega}_j}(z,0,w,0)$ as a function of z and w converges uniformly on compact subsets of $\Omega_0 \times \Omega_0$ to $K^{\widetilde{\Omega}_0}(z,0,w,0)$ as $j \to \infty$. By Theorem 3.1, $K^{\Omega_j}_{|\varphi|^2}(z,w)$ converges uniformly on compact subsets of $\Omega \times \Omega$ to $K^{\Omega}_{|\varphi|^2}(z,w)$ as $j \to \infty$. $\square$

Remark. In the case that φ in the theorem is holomorphic, φ may or may not have an infinite number of zeros on Ω. Note, however, that if φ has an infinite number of zeros on Ω, it still has only a finite number of zeros on each Ω_j, and so we have an explicit formula for each $K^{\Omega_j}_{|\varphi|^2}(z,w)$ in terms of $K^{\Omega_j}(z,w)$ from our theory.

3.5 An interpretation in terms of domains in $\mathbb{C}^2$

3.5.1 A brief history of stability theorems

An old question in the field of several complex variables is, if a sequence of domains converges in some sense to a limiting domain, do the associated kernel functions also converge in some sense to the kernel function for the limiting domain? One of the earliest answers to this question is attributed to Ramadanov [23].

Theorem 3.21 (Ramadanov's Theorem, 1967)**.** *Let* $\{\Omega_j\}_{j=1}^{\infty}$ *be a sequence of domains in* $\mathbb{C}^n$ *with* $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \cdots$*, and set* $\Omega := \cup_{j=1}^{\infty}\Omega_j$*. Then* $K^{\Omega_j}(z,w) \to K^{\Omega}(z,w)$ *uniformly on compact subsets of* $\Omega \times \Omega$*.*

Boas observes that "Ramadanov gave the theorem for bounded domains in $\mathbb{C}^1$ and stated only convergence in one variable with the other fixed, but the more general result requires only a slight modification of his proof" [2].

Theorem 3.21 requires the sequence of domains to be increasing and does not assume any boundary regularity. Leveraging an asymptotic expansion for the Bergman kernel on certain smooth domains due to Fefferman [10], Greene and Krantz proved the following [13–15].

Theorem 3.22 (Greene and Krantz, 1981)**.** *If* $\{\Omega_j\}_{j=1}^{\infty}$ *is a sequence of* C^∞ *strongly pseudoconvex domains converging in* C^∞ *to a* C^∞ *pseudoconvex domain* Ω*, then* $K^{\Omega_j}(z,w) \to K^{\Omega}(z,w)$ *uniformly on compact subsets of* $\Omega \times \Omega$*.*

Theorem 3.22 has strong assumptions on the boundaries of the domains and uses a notion of convergence that is quite strong (see [13, sec. 1]). Subsequent results primarily focus on convergence of domains with respect to some variation of the so-called Hausdorff distance (see [19, A.1]) and have seen successive loosening of the boundary regularity hypothesis.

Definition. The *Hausdorff distance* between two domains S and T is the quantity

$$\mathcal{HD}(S,T) := \max\{\ \sup_{s\in S} \mathrm{dist}(s,T),\ \sup_{t\in T} \mathrm{dist}(t,S)\ \}.$$

The Hausdorff distance by itself is typically inadequate for the study of the convergence of kernels of converging domains as two different domains with different kernels can be separated by a Hausdorff distance of 0. For example, if S is the unit disk and T is the unit disk with the nonnegative real axis removed, then $\mathcal{HD}(S,T) = 0$, yet $K^S(z,w) \neq K^T(z,w)$. In [3], Boas defines the following two metrics on bounded nonempty open sets.

Definition. For any bounded nonempty open sets S and T, define

$$\rho_1(S,T) := \mathcal{HD}(S,T) + \mathcal{HD}(\partial S, \partial T), \text{ and}$$

$$\rho_2(S,T) := \mathrm{vol}(S \setminus T) + \mathrm{vol}(T \setminus S) + \sup_{z\in\mathbb{C}^n} |d_S(z) - d_T(z)|,$$

where $d_X(z) := \mathrm{dist}(z, \mathbb{C}^n \setminus X)$ is defined for any open set X and vol is the usual Lebesgue volume.

Convergence with respect to ρ_1 is often referred to as "convergence in the sense of Boas" in the literature. In [3] Boas states and proves the following stability theorem, which he describes as a folk theorem, remarking that the idea of the proof is in the literature.

Theorem 3.23. *Let $\{\Omega_j\}$ be a sequence of bounded pseudoconvex domains with C^∞ boundary that converges, in the sense of either ρ_1 or ρ_2, to a bounded domain Ω; in the case of ρ_2, assume also that the Ω_j have uniformly bounded diameters (this is automatic in the case of ρ_1). Then the Bergman kernel functions of the Ω_j converge to the Bergman kernel function of Ω uniformly on compact subsets of $\Omega \times \Omega$.*

The hypotheses of Theorem 3.22 imply those of Theorem 3.23, so Theorem 3.23 is a more powerful theorem. (See also [9] which proves a special case of Theorem 3.23.) Boas observes in [3] that the C^∞ regularity condition can be weakened to C^2 regularity. B. Chen and J. Zhang weaken the boundary regularity requirement further: one only need assume that $\partial\Omega$ can be described locally as the graph of a Hölder-continuous function [7] (in particular, of a continuous function [8]).

3.5.2 A comparison to previous stability theorems

For a domain $\widetilde{\Omega}$ and a meromorphic function ϕ on $\widetilde{\Omega}$, the weighted kernel $K^{\widetilde{\Omega}}_{|\phi|^2}(z,w)$ is related to the unweighted kernel for the domain

$$\Omega := \{(z,w) \in \mathbb{C}^2 \mid z \in \widetilde{\Omega}, |w| < |\phi(z)|/\sqrt{\pi}\} \subset \mathbb{C}^2. \tag{3.12}$$

by the relation $K^{\widetilde{\Omega}}_{|\phi|^2}(z,w) \equiv K^{\Omega}(z,0,w,0)$ by Theorem 3.1 in Section 3.1.1.

There are two significant characterizing components of each of the stability theorems in Section 3.5.1: the assumptions about the geometry of the domains (pseudoconvexity, boundary regularity, etc.), and the notion of convergence employed. In Ramadanov's Theorem 3.21, the domains need only be bounded (except for the limiting domain), while the notion of convergence is restrictive: the domains in the sequence need to be increasing and exhaust the limiting domain. The subsequent stability theorems relax the requirement that the domains in the sequence be increasing but at the significant cost of additional geometric requirements for the domains. In our present study, the definition in (3.12) implies a particular geometry which differs from the requirements in the stability theorems of Section 3.5.1; domains defined as in (3.12) need not have a boundary which is locally the graph of a continuous function, need not be bounded, and need not be pseudoconvex. The natural definition of convergence of a sequence of domains of the form (3.12) is in terms of convergence of the meromorphic functions in their definitions. This notion of convergence implies convergence with respect to ρ_1.

Note, however, that our theory applied to domains in $\mathbb{C}^2$ only gives information about a two complex-dimensional subspace of the domain of definition of the unweighted kernels for $\mathbb{C}^2$ domains. For the purpose of studying the Lu Qi-Keng problem, this is often enough (see [5]).

3.6 Applications to the disk

To see how our theory can be applied to easily compute weighted kernels which are difficult to compute using other techniques, we apply the machinery of Section 3.2 for a few weighted kernels on the disk. As a comparison, we also compute $K^{\mathbb{D}}_{|z(z-c)|^2}(z,w)$ without using the techniques from Section 3.2 but rather using a new method. The technique employed, while novel, is limited to situations where a spanning set for the Bergman space is "almost orthogonal" in some sense. However, the new method retains the classical strategy of summing an infinite series to find an explicit formula for the kernel.

3.6.1 $K^{\mathbb{D}}_{|z|^{2p}}(z,w),\ p \in \mathbb{N}$

We apply Theorem 3.7 to $K_{|z|^{2p}}(z,w)$:

$$K_{|z|^{2p}}(z,w) = \frac{K(z,w)}{z^p\overline{w}^p} - \sum_{j=1}^{p} \frac{K_{|z|^{2(p-j)}}(z,0)K_{|z|^{2(p-j)}}(0,w)}{z^j\overline{w}^j K_{|z|^{2(p-j)}}(0,0)}. \tag{3.13}$$

For the case $p = 1$, we have

$$\begin{aligned} K_{|z|^2}(z,w) &= \frac{K(z,w)}{z\overline{w}} - \frac{K(z,0)K(0,w)}{z\overline{w}K(0,0)} = \frac{1}{\pi z\overline{w}(1-z\overline{w})^2} - \frac{1}{\pi z\overline{w}} \\ &= \frac{(2-z\overline{w})}{\pi(1-z\overline{w})^2} \end{aligned}$$

which corresponds to the formula computed in Section A.1.4.

For $p \in \mathbb{N}$ it can be shown by induction that the formula above also agrees with that which is computed in Section A.1.3.

3.6.2 $K^{\mathbb{D}}_{|\mu_c|^{2p}}(z,w),\ p \in \mathbb{N}$

Let $c \in \mathbb{D}$. The Möbius transformation $\mu_c(z) := \frac{z-c}{1-\bar{c}z}$ is a biholomorphic automorphism of $\mathbb{D}$. By applying the biholomorphic transformation rule for Bergman kernels [1] to Equation (3.13) (or to Theorem 3.7), we obtain

$$K_{|\mu_c|^{2p}}(z,w) = \frac{K(z,w)}{\mu_c(z)^p\overline{\mu_c(w)}^p} - \sum_{j=1}^{p} \frac{K_{|\mu_c|^{2(p-j)}}(z,c)K_{|\mu_c|^{2(p-j)}}(c,w)}{\mu_c(z)^j\overline{\mu_c(w)}^j K_{|\mu_c|^{2(p-j)}}(c,c)}.$$

Alternatively, observe that $g(z) := (1-\bar{c}z)^{-1}$ is holomorphic on $\mathbb{D}$, and hence by Theorem 3.2, $K_{|\mu_c|^{2p}}(z,w) = (z-c)K_{|z-c|^{2p}}(z,w)(\overline{w}-\overline{c})$. Now proceed as in the previous section to obtain the formula above.

3.6.3 $K^{\mathbb{D}}_{|z(z-c)|^2}(z,w),\ c \in \mathbb{D}$

Using Decomposition Theorems

The standard methods of computing weighted Bergman kernels on the disk fail in the case of the weight $|z(z-c)|^2$. Because $|z(z-c)|^2$ is not radially symmetric, the monomials z^j are not orthogonal in $A^2_{|z(z-c)|^2}(\mathbb{D})$, and finding an orthogonal basis for $A^2_{|z(z-c)|^2}(\mathbb{D})$ is a challenge. One can overcome this hurdle by exploiting the fact that the monomials are *nearly* orthogonal in the sense that $\left\langle z^j, z^k \right\rangle_{|z(z-c)|^2} = 0$ when

$|j-k|>1$. We compute $K_{|z(z-c)|^2}(z,w)$ using this fact in Section 3.6.3. However, using Theorem 3.6 is far easier in this case:

$$K_{|z(z-c)|^2}(z,w) = \frac{K_{|z|^2}(z,w)}{(z-c)(\overline{w}-\overline{c})} - \frac{K_{|z|^2}(z,c)K_{|z|^2}(c,w)}{(z-c)(\overline{w}-\overline{c})K_{|z|^2}(c,c)}$$
$$= \frac{(2-z\overline{w})}{\pi(1-z\overline{w})^2(z-c)(\overline{w}-\overline{c})} - \frac{(2-z\overline{c})(2-c\overline{w})(1-|c|^2)^2}{\pi(1-\overline{c}z)^2(1-c\overline{w})^2(2-|c|^2)(z-c)(\overline{w}-\overline{c})}.$$

An Alternative Method

In Section 3.6.3 we compute $K^{\mathbb{D}}_{|z(z-c)|^2}(z,w)$, $c\in\mathbb{D}$, with great ease using Theorem 3.6. We present another approach in this section. To save some writing, for this section only we omit the weight in the subscript of the weighted inner product, that is, we denote by $\langle\cdot,\cdot\rangle$ the inner product in $A^2_{|z(z-c)|^2}(\mathbb{D})$. Similarly, we omit the subscript in the norm. Observing that

1. $|z|^2|z-c|^2 = |z|^4 - \overline{c}z|z|^2 - c\overline{z}|z|^2 + |c|^2|z|^2$, and
2. $\int_{\mathbb{D}} |z|^{2j}\,dz = \frac{\pi}{j+1}$ for $j\in\mathbb{N}$,

we have that for $j,k\in\mathbb{N}$,

$$\left\langle z^j, z^k\right\rangle = \begin{cases} -\overline{c}\frac{\pi}{j+3} & \text{if } j=k-1 \\ \frac{\pi}{j+3} + |c|^2\frac{\pi}{j+2} & \text{if } j=k \\ -c\frac{\pi}{j+2} & \text{if } j=k+1 \\ 0 & \text{if } |j-k|>1 \end{cases}. \tag{3.14}$$

In this sense, the monomials are *almost* orthogonal in $A^2_{|z(z-c)|^2}(\mathbb{D})$. Our goal will be to find two nonzero functions ϕ_0 and ϕ_1 in $A^2_{|z(z-c)|^2}(\mathbb{D} \setminus \{0, c\})$ of unit norm orthogonal to $A^2_{|z(z-c)|^2}(\mathbb{D})$ and to each other. Once they are found, since $A^2_{|z(z-c)|^2}(\mathbb{D})$ has codimension two in $A^2_{|z(z-c)|^2}(\mathbb{D}\setminus\{0, c\})$, the elementary theory of Bergman spaces gives that

$$K^{\mathbb{D}\setminus\{0,c\}}_{|z(z-c)|^2}(z, w) = K^{\mathbb{D}}_{|z(z-c)|^2}(z, w) + \phi_0(z)\overline{\phi_0(w)} + \phi_1(z)\overline{\phi_1(w)}.$$

We desire ϕ_0 to be orthogonal to the monomials z^k, $k \in \mathbb{N}$. From (3.14) we see that $cz^{k-1} + z^k + \frac{1}{c}z^{k+1}$ is orthogonal to z^k for $k \in \mathbb{N}$. So then

$$g(z) := \frac{1}{z} \sum_{\ell=0}^{\infty} \left(\frac{z}{c}\right)^{\ell} = \frac{c}{z(c-z)}$$

is orthogonal to z^k for every $k \in \mathbb{N}$. Since $g \in A^2_{|z(z-c)|^2}(\mathbb{D} \setminus \{0, c\})$, we may set $\phi_0(z) := g(z)/\|g\|$. We have

$$\|g\|^2 = \int_{\mathbb{D}} \frac{c}{z(c-z)} \frac{\overline{c}}{\overline{z}(\overline{c}-\overline{z})} |z|^2 |z-c|^2 \, dz = |c|^2 \pi,$$

so

$$\phi_0(z) = \frac{c}{z(c-z)|c|\sqrt{\pi}} = \frac{c\, K(z, 0)}{z(z-c)|c|\sqrt{K(0, 0)}}.$$

Because $\phi_1 \notin A^2_{|z(z-c)|^2}(\mathbb{D})$, we expect ϕ_1 to have a pole of order one at $z = 0$ or $z = c$ (or both). Since ϕ_0 has a pole at both $z = 0$ and $z = c$, we look for a function in $A^2_{|z(z-c)|^2}(\mathbb{D} \setminus \{0, c\})$ of the form $h(z)/(z-c)$ such that

1. $\frac{h(z)}{z-c}$ has a pole at $z = c$, and
2. $h(z) = \sum_{j=0}^{\infty} b_j z^j$, $(b_j \in \mathbb{C})$, i.e. $h(z)$ has no pole at $z = 0$.

From

$$\left\langle \frac{z^j}{z-c}, z^k \right\rangle = \int_{\mathbb{D}} \frac{z^j}{z-c} \overline{z}^k |z|^2 (\overline{z} - \overline{c})(z-c)\, dz = \int_{\mathbb{D}} z^j \overline{z}^k |z|^2 (\overline{z} - \overline{c})\, dz,$$

we obtain

$$\left\langle \frac{z^j}{z-c}, z^k \right\rangle = \begin{cases} -\overline{c}\frac{\pi}{j+2} & \text{if } j = k \\ \frac{\pi}{j+2} & \text{if } j = k+1 \\ 0 & \text{else} \end{cases} \cdot \tag{3.15}$$

Since ϕ_1 is to be orthogonal to the monomials, we must have

$$0 = \left\langle \frac{h(z)}{z-c}, z^k \right\rangle = -\overline{c}\, b_k \frac{\pi}{k+2} + b_{k+1} \frac{\pi}{k+3},$$

and hence $b_{k+1} = b_k\, \overline{c}\, \frac{k+3}{k+2}$. So then

$$b_k = b_{k-1}\, \overline{c}\, \frac{k+2}{k+1} = \overline{c}^k \frac{(k+2)(k+1)\cdots 3}{(k+1)k\cdots 2} b_0 = \frac{1}{2} \overline{c}^k (k+2) b_0.$$

Since we will normalize later, we may assume that $b_0 = 1$. Thus we have computed that

$$h(z) = \frac{1}{2}\sum_{k=0}^{\infty} \overline{c}^k (k+2) z^k = \frac{2 - \overline{c}z}{2(1-\overline{c}z)^2}.$$

We compute the norm of $h(z)/(z-c)$:

$$\begin{aligned}
\left\| \frac{h(z)}{z-c} \right\|^2 &= \int_{\mathbb{D}} \frac{2-\overline{c}z}{2(1-\overline{c}z)^2(z-c)} \frac{2-c\overline{z}}{2(1-c\overline{z})^2(\overline{z}-\overline{c})} |z|^2 |z-c|^2 \, dz \\
&= \frac{\pi^2}{4} \int_{\mathbb{D}} K_{|z|^2}(z,c) K_{|z|^2}(c,z) |z|^2 \, dz \\
&= \frac{\pi^2}{4} K_{|z|^2}(c,c) = \frac{\pi}{4} \frac{2-|c|^2}{(1-|c|^2)^2},
\end{aligned}$$

where both the second and last equalities follow from the formula in Section A.1.4. Thus we may set

$$\phi_1(z) := \frac{\pi K_{|z|^2}(z,c)}{2(z-c)} \frac{2}{\pi\sqrt{K_{|z|^2}(c,c)}} = \frac{K_{|z|^2}(z,c)}{(z-c)\sqrt{K_{|z|^2}(c,c)}}.$$

We check that ϕ_0 is orthogonal to ϕ_1:

$$\begin{aligned}
\langle \phi_0, \phi_1 \rangle &= c_0 \int_{\mathbb{D}} \frac{K(z,0)}{z(z-c)} \frac{K_{|z|^2}(c,z)}{(\overline{z}-\overline{c})} |z|^2 |z-c|^2 \, dz \quad \text{(for some } c_0 \in \mathbb{C}\text{)} \\
&= c_0 \int_{\mathbb{D}} K(z,0) K_{|z|^2}(c,z) \overline{z} \, dz = 0.
\end{aligned}$$

In summary, we have computed that

$$\begin{aligned}
K^{\mathbb{D}\setminus\{0,c\}}_{|z(z-c)|^2}(z,w) &= K^{\mathbb{D}}_{|z(z-c)|^2}(z,w) + \phi_0(z)\overline{\phi_0(w)} + \phi_1(z)\overline{\phi_1(w)} \\
&= K^{\mathbb{D}}_{|z(z-c)|^2}(z,w) \\
&\quad + \frac{K(z,0)K(0,w)}{z(z-c)\overline{w}(\overline{w}-\overline{c})K(0,0)} \\
&\quad + \frac{K_{|z|^2}(z,c)K_{|z|^2}(c,w)}{(z-c)(\overline{w}-\overline{c})K_{|z|^2}(c,c)}.
\end{aligned}$$

Applying Theorem 3.2 to the leftmost expression, we have

$$\begin{aligned}
K^{\mathbb{D}}_{|z(z-c)|^2}(z,w) = \frac{K(z,w)}{z(z-c)\overline{w}(\overline{w}-\overline{c})} \\
- \frac{K(z,0)K(0,w)}{z(z-c)\overline{w}(\overline{w}-\overline{c})K(0,0)} \\
- \frac{K_{|z|^2}(z,c)K_{|z|^2}(c,w)}{(z-c)(\overline{w}-\overline{c})K_{|z|^2}(c,c)},
\end{aligned}$$

the right-hand side of which is an explicit formula which agrees with the formula computed in Section 3.6.3.

4. A GENERALIZED LU QI-KENG PROBLEM

The Lu Qi-keng Conjecture asks whether $K(z, w)$ is zero free on every domain. The answer is no, and the annulus presented previously was historically the first counterexample [25, 26]. Since then, many examples have been given, even examples which are pseudoconvex with analytic boundary [2]. Boas proved that Lu Qi-keng domains, as domains whose kernel is zero free have come to be called, are nowhere dense in a suitable topology, and thus are exceptional domains [3]. In this sense most domains have the property that $K(z, w)$ takes the value zero.

One way to generalize the Lu Qi-keng Problem is to ask, for a fixed $k \in \mathbb{N}$, on which domains can one find k distinct points $z_1, z_2, \ldots, z_k$ in the domain such that $K(z_\ell, z_m) = 0$ for $\ell \neq m$. The original Lu Qi-keng Problem is the special case $k = 2$ (though without the requirement that the z_ℓ be distinct).

Definition. A domain $\Omega \subset \mathbb{C}^n$ has property $P(k)$ ($k \in \mathbb{N}$, $k \geq 2$) if there exist k distinct points $z_1, z_2, \ldots, z_k \in \Omega$ such that $K(z_\ell, z_m) = 0$ for $\ell \neq m$. If Ω has property $P(k)$, then we shall call Ω a $P(k)$ domain, or say that Ω is $P(k)$.

4.1 Elementary facts about property $P(k)$

A consideration of simple examples and propositions will establish some elementary facts about property $P(k)$.

Example 4.1. In Section 2 we proved that the annulus $\Omega := \{z \in \mathbb{C} \mid 0 < r < |z| < 1\}$ is $P(2)$ when r is sufficiently small. Indeed, for any $r \in (0,1)$, Ω is doubly connected, and hence by Theorem 2.1, Ω is $P(2)$. It follows immediately from Corollary 2.5 that when $r < q_0^2$ (where q_0 is the zero of $L_\rho(q)$ with largest magnitude), Ω is not $P(3)$. In fact, Ω is not $P(3)$ for any $r \in (0,1)$. To see this, suppose $\zeta, z, w \in \Omega$ such that $z\overline{w} = q_0$ and $\zeta\overline{w} = q_1$. Then $z = q_0/\overline{w}$ and $\zeta = r/(q_0\overline{w})$, and so $L_\rho(z\overline{\zeta}) = L_\rho(r/|w|^2)$. Since $r/|w|^2 > 0$ and L_ρ is positive on the positive real axis, $L_\rho(z\overline{\zeta}) \neq 0$.

Example 4.2. In two dimensions, $\Omega \times \mathbb{D}$ is an example of a $P(2)$ domain which is not $P(3)$, where $\mathbb{D}$ is the unit disk and Ω is the annulus as in the last example. To see this, let $z^{(j)} = (z_1^{(j)}, z_2^{(j)})$ with $z^{(j)} \in \Omega \times \mathbb{D}$ $(j = 1, 2, 3)$ be distinct. Recall that the kernel $K^{\mathbb{D}}(z, w)$ for the unit disk is zero free. Suppose $K^{\Omega\times\mathbb{D}}(z^{(1)}, z^{(2)}) = 0$ and $K^{\Omega\times\mathbb{D}}(z^{(1)}, z^{(3)}) = 0$. Then $K^{\Omega}(z_1^{(1)}, z_1^{(2)}) = K^{\Omega}(z_1^{(1)}, z_1^{(3)}) = 0$, and since $K^{\Omega}(z_1^{(1)}, w)$ can only have at most one zero in w we must have $z_1^{(2)} = z_1^{(3)}$. Hence $K^{\Omega\times\mathbb{D}}(z^{(2)}, z^{(3)}) \neq 0$.

Example 4.3. On the other hand, $\Omega \times \Omega$ is a $P(3)$ domain. Let (z_0, z_1) be a zero of $K^{\Omega}(z, w)$, and set $z^{(1)} = (z_0, z_1)$, $z^{(2)} = (z_1, z_0)$, and $z^{(3)} = (z_1, z_1)$. Then $K^{\Omega\times\Omega}(z^{(j)}, z^{(k)}) = 0$ for $j \neq k$. In fact, setting $z^{(4)} = (z_0, z_0)$, we see that $\Omega \times \Omega$ is a $P(4)$ domain. This example illustrates the following propositions:

Proposition 4.4. *If $\Omega \in \mathbb{C}^n$ is a $P(m)$ domain where $m \geq 2$, then Ω is also a $P(k)$ domain for all k with $2 \leq k \leq m$.*

Proof. This is obvious. □

Proposition 4.5. *If $\Omega_1, \Omega_2, \ldots, \Omega_j$ are domains (possibly of different dimensions) which are $P(k_1), P(k_2), \ldots, P(k_j)$ respectively, then $\Omega := \Omega_1 \times \Omega_2 \times \cdots \times \Omega_j$ is $P(k_1 \cdot k_2 \cdots k_j)$.*

Proof of Proposition 4.5. The kernel for Ω is $K^{\Omega}(z,w) := K^{\Omega_1}(z_1,w_1) \times K^{\Omega_2}(z_2,w_2) \times \cdots \times K^{\Omega_j}(z_j,w_j)$, where $K^{\Omega_\ell}(z_\ell,w_\ell)$ is the kernel for Ω_ℓ, $\ell = 1,2,\ldots,j$. The kernel $K^{\Omega}(z,w) = 0$ if and only if one of the $K^{\Omega_\ell}(z_\ell,w_\ell)$ is zero. To finish the proof, count how many ways there are to arrange the coordinates of z and w using the corresponding zeros of $K^{\Omega_\ell}(z_\ell,w_\ell)$. □

Proposition 4.6. *If $\Omega_1, \Omega_2, \ldots, \Omega_j$ are disjoint domains in $\mathbb{C}^n$ and for each ℓ, Ω_ℓ is $P(k_\ell)$, then $\cup_\ell \Omega_\ell$ is $P(\sum_\ell k_\ell)$.*

Proof. The kernel for $\cup_\ell \Omega_\ell$ is

$$K(z,w) = \begin{cases} K^{\Omega_\ell}(z,w) & \text{if } z,w \in \Omega_\ell \text{ for some } \ell \\ \\ 0 & \text{if } z \in \Omega_\ell \text{ and } z \in \Omega_m \text{ for } \ell \neq m, \end{cases}$$

where $K^{\Omega_\ell}(z,w)$ is the kernel for Ω_ℓ. The rest of the proof is obvious. □

4.2 Stability of property $P(k)$

A natural question is to ask how property $P(k)$ is preserved or not as one creates a new domain from an old one. For example, Theorem 3.11 can be interpreted as saying that three points realize property $P(3)$ for a certain weighted kernel only if those same three points realize property $P(3)$ for the associated unweighted kernel. (If one prefers to consider unweighted kernels, one may use the machinery of Section 3.1.1 to interpret Theorem 3.11 as a statement about unweighted kernels on domains in $\mathbb{C}^2$.)

Another way of taking a $\mathbb{C}^n$ domain and producing another $\mathbb{C}^n$ domain is to perturb the domain in some way. We shall see that this preserves property $P(k)$ only in special cases. One may also look at lower dimensional "slices" of a domain, that is, intersections of the domain with lower dimensional affine subspaces which have nonempty intersection with the domain. We shall see from a survey of simple examples that there is no simple relationship between property $P(k)$ of the domain and the corresponding property of a slice of the domain.

4.2.1 Stability of $P(2)$ domains

Suppose $\Omega \subset \mathbb{C}^n$ is $P(2)$ with kernel $K(z,w)$, and $\Omega_j \subset \mathbb{C}^n$ is a sequence of domains with kernels $K^{\Omega_j}(z,w)$ such that the $K^{\Omega_j}(z,w)$ converge uniformly on compact subsets of Ω to $K(z,w)$. That is, for $w_0 \in \Omega$ fixed, $K^{\Omega_j}(z,w_0) \to K(z,w_0)$ uniformly on compact subsets of Ω as a function of z. (This happens, for example, if the Ω_j are increasing monotonically to Ω by Theorem 3.21.) Let $z^0, w^0 \in \Omega$ such that $K(z^0,w^0) = 0$ but $K(z,w^0)$ is not the zero function, and let $\widetilde{\Omega} \subset \mathbb{C}$ be a one complex-dimensional slice of $\Omega \subset \mathbb{C}^n$ containing the points z^0 and w^0. Then the restrictions of $K^{\Omega_j}(z, w^0)$ to $\widetilde{\Omega}$ are holomorphic functions converging normally on $\widetilde{\Omega}$ to the restriction of $K(z, w^0)$ to $\widetilde{\Omega}$. By Hurwitz's Theorem, eventually the restrictions of $K^{\Omega_j}(z, w^0)$ to $\widetilde{\Omega}$ all have a zero near z^0. Hence the Ω_j are $P(2)$ for large enough j. This shows the following:

Proposition 4.7. *Property $P(2)$ is preserved whenever $\Omega_j \to \Omega$ such that $K^{\Omega_j} \to K$ normally.*

Other kinds of convergence do not necessarily preserve property $P(2)$. For example, connect two disjoint disks with a very thin corridor, and allow that corridor to shrink, or equivalently, place a very small disc very close to the boundary of a larger disjoint disc (both ideas found in [3]). The disconnected discs (seen as together forming a single disconnected open set) are $P(2)$, but any simply connected planar domain is Lu Qi-keng. In the case of the shrinking corridor, we have $K(z, w) \neq 0$ for all $z, w \in \Omega$. In the case of the small and large disc, $K(z, w) \equiv 0$ for all z and w in separate components. More generally, for certain notions of "close", a $P(k)$ domain can be very "close" to a $P(j)$ domain for very different k and j.

Remark. By the previous proposition, we easily have smooth examples of $P(2)$ domains in $\mathbb{C}^n$ for any n. Examples of smooth $P(3)$ domains seem to be hard to construct.

4.2.2 Slices of $P(k)$ domains

Example 4.8. In [2], Boas studied the domain $D = \{(z,w) \in \mathbb{C}^2 \colon |w| < 1/(1+|z|)\}$ which is a complete Reinhardt domain on which the only square integrable monomials have a factor of w. Thus every function in the Bergman space vanishes on the z axis when $w = 0$. Such a domain is $P(k)$ for every $k \geq 2$, for $K((z,0),(\zeta,0)) = 0$ for every $z, \zeta \in D$. Choosing points $(0, w_0), (z_0, 0) \in D$, one can slice D with the complex line $((1-\lambda)z_0, \lambda w_0), \lambda \in \mathbb{C}$. Then $\widetilde{D} := \{\lambda \in \mathbb{C} \colon (1-\lambda)z_0, \lambda w_0) \in D\} = \{\lambda \in \mathbb{C} \colon |\lambda w_0| + |\lambda(1-\lambda)z_0 w_0| < 1\}$. When w_0 and z_0 have sufficiently large magnitude, $\widetilde{D}$ is the union of two lakes, one near $\lambda = 0$ and another near $\lambda = 1$ (the size and shape of which depend on z_0, w_0). Thus $\widetilde{D}$ is $P(2)$ by virtue of being disconnected, whereas D is $P(k)$ for any $k \geq 2$.

Example 4.9. In [28], Wiegerinck studied domains of the form $\Omega_k = X_1 \cup X_2 \cup B_{4k} \cup \{(z,w) \in \mathbb{C}^2 \colon |z| < 2e, |w| < 2e\}$, where

$$X_1 := \{(z,w) \in \mathbb{C}^2 \colon |w| < 1/(|z| \cdot \log|z|), |z| > e\};$$

$$X_2 := \{(z,w) \in \mathbb{C}^2 \colon |z| < 1/(|w| \cdot \log|w|), |w| > e\}; \text{ and}$$

$$B_m := \{(z,w) \in \mathbb{C}^2 \colon ||z| - |w|| < 1/(|z| + |w|)^m\}.$$

These domains have Bergman space of dimension k. Thus Ω_1 has a constant nonzero kernel and hence is not $P(j)$ for any $j \geq 2$.

We slice Ω_k as follows. Let $\omega \in \mathbb{R}^+$ be sufficiently large so as to allow our slice to avoid the polydisc part of Ω_k, say, $\omega = 10$. Define the "slice" operator $\sim$ on sets: for a set $Y \in \mathbb{C}^2$, define $\widetilde{Y} := \{\zeta \in \mathbb{C} \colon (\zeta\omega, (1-\zeta)\omega) \in Y\} \subset \mathbb{C}$. Then we have that $\widetilde{X}_1$ is the disjoint union of two lakes, the smaller one about 0, the larger one about 1; $\widetilde{X}_2$ is the disjoint union of two lakes, the smaller one about 1, the larger one about 0; and $\widetilde{B}_{4k}$ is a thin onion shape symmetric about the x-axis and a vertical line through 1/2, and very thin even at its thickest point. So $\widetilde{\Omega}_k$ is $P(3)$ while Ω_k is Lu Qi-keng.

Remark. In Example (4.8), k decreases for $P(k)$ as we drop to a slice. In Example (4.9), k increases for $P(k)$ as we drop to a slice. In both cases, the slice is $P(k)$ by virtue of being disconnected. Examples of convex non Lu Qi-keng domains are given in [5]. A convex non Lu Qi-keng domain such as the one constructed in [5] is an example of a $P(2)$ domain such that every one-complex-dimensional slice is Lu Qi-keng.

5. CONCLUSION

Historically, Skwarczyński was the first to give a negative answer to the so-called Lu Qi-keng Conjecture by showing that the kernel for the annulus vanishes [26]. The zeros of the kernel for the annulus are investigated more deeply in Section 2. Section 3 develops a theory giving formulas for certain weighted kernels on the plane related to unweighted kernels in $\mathbb{C}^2$ and investigates the zero sets of those kernels, contributing to the study of the Lu Qi-keng problem. This theory provides a much easier technique for computing certain weighted kernels than classical techniques and explains many of the formulas computed in Appendix A. The example of the annulus provides a setting in which the generalization of the Lu Qi-keng Problem of Section 4 can be introduced, as it forms the basis for several illustrative example domains.

There remain many open problems related to these contributions.

While the zeros of the annulus are located with some accuracy in Section 2, a precise formula for the zeros remains unknown. Moreover, computer evidence suggests that case (2) in Corollary 2.5 never happens, but a proof has not been presented.

It might be possible to expand the theory of Section 3 in order to compute explicit formulas for kernels of the form $K_{|f|^\alpha}(z,w)$, where f is holomorphic and α is real (rather than an even integer). Kernels of this form are some of the few kernels in Appendix A which do not have formulas that follow from a simple application of the theorems of Section 3. Another possibility is to expand the theory of Section 3 to study kernels of the form $K_\varphi(z, w)$, where φ is nonnegative (sub)harmonic. Both possibilities represent intriguing directions for future research. The Lu Qi-keng Problem, a topic of active research in the field, and its generalization in Section 4 remain rich sources of open questions. Boas, Fu, and Straube found a convex domain in $\mathbb{C}^3$ for which the Bergman kernel vanishes [5]. Is there such a convex domain in $\mathbb{C}^2$? Is there a planar $P(3)$ domain? Is there a $P(k)$ domain in C^n with smooth boundary for arbitrary $k, n > 2$? The answer to these questions remain unknown.

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APPENDIX A

ENCYCLOPEDIA OF BERGMAN KERNELS

Here we collect some computations of weighted kernels on the disk. In this section we make use of the Möbius transformation $\mu_c(z) := \frac{z-c}{1-\bar{c}z}$ with $c \in \mathbb{C}$, which is an automorphism of the disk when $c \in \mathbb{D}$ taking c to the origin. We denote the set of positive integers by $\mathbb{N}$, and the set of nonzero integers by $\mathbb{Z}^*$.

The computations in this section are elementary exercises in the classic theory of Bergman kernel functions. We use Formula (3.4) extensively in this section to obtain a series representation for the kernel. We then sum the series to achieve a concrete formula for the kernel.

A.1 Weighted kernels on the disk

A.1.1 $K_{|z|^\alpha}$, $\alpha > -2, \alpha \in \mathbb{R}$

The monomials are orthogonal with respect to the weighted inner product and clearly span the weighted Berman space. To find an orthonormal basis we compute the norm of each monomial. For $j \in \mathbb{N}$,

$$\left\|z^j\right\| = \int_{\mathbb{D}} |z|^{2j+\alpha}\, dz = \int_0^{2\pi} \int_0^1 r^{2j+1+\alpha}\, dr\, d\theta = \frac{2\pi}{2(j+1)+\alpha},$$

which is finite when $2j+1+\alpha > -1 \Leftrightarrow j > -\frac{2+\alpha}{2} = -\frac{\alpha}{2} - 1$.

Remark. Because $\alpha > -2$, this weight gives the same Bergman space as no weight, $A^2_{|z|^\alpha} \equiv A^2(\mathbb{D})$.

It remains to sum the series representing the weighted kernel.

$$\begin{aligned} K_{|z|^\alpha}(z,w) &= \sum_{j=0}^{\infty} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j = \sum_{j=0}^{\infty} \frac{j+1}{\pi}(z\overline{w})^j + \frac{\alpha}{2\pi}\sum_{j=0}^{\infty}(z\overline{w})^j \\ &= K(z,w) + \frac{\alpha}{2\pi(1-z\overline{w})}. \end{aligned}$$

$$\boxed{K_{|z|^\alpha}(z,w) = K(z,w) + \frac{\alpha}{2\pi(1-z\overline{w})} = \left(1 + \frac{\alpha}{2} - \frac{\alpha}{2}z\overline{w}\right) K(z,w)}$$

A.1.2 $K_{|z|^\alpha}$, $\alpha \le -2, \alpha \in \mathbb{R}$

As in the previous section, $\|z^j\| = \frac{2\pi}{2(j+1)+\alpha}$, which is finite when $2j+1+\alpha > -1 \Leftrightarrow j > -\frac{2+\alpha}{2} = -\frac{\alpha}{2} - 1$.

Remark. When $\alpha \le -2m$, $m \in \mathbb{N}$ the monomial z^{m-1} is not in $A^2_{|z|\alpha}(\mathbb{D})$. The computation below differs from that of the previous section only in our accomodation of this fact.

Let $m \in \mathbb{N}$ such that $-2(m+1) < \alpha \leq -2m$. Then

$$\begin{aligned} K_{|z|^\alpha}(z,w) &= \sum_{j=m}^{\infty} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j \\ &= \sum_{j=0}^{\infty} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j - \sum_{j=0}^{m-1} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j \\ &= K(z,w) + \frac{\alpha}{2\pi(1-z\overline{w})} - \sum_{j=0}^{m-1} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j. \end{aligned}$$

$$\boxed{K_{|z|^\alpha}(z,w) = K(z,w) + \frac{\alpha}{2\pi(1-z\overline{w})} - \sum_{j=0}^{m-1} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j}$$

A.1.3 $K_{|z|^{2p}}$, $p \in \mathbb{N}$

This is a special case of the case $K_{|z|^\alpha}$, $\alpha > -2, \alpha \in \mathbb{R}$.

$$\boxed{K_{|z|^{2p}}(z,w) = K(z,w) + \frac{p}{\pi(1-z\overline{w})} = ((p+1) - pz\overline{w})K(z,w)}$$

A.1.4 $K_{|z|^2}$

This is a special case of the last formula.

$$\boxed{K_{|z|^2}(z,w) = K(z,w) + \frac{1}{\pi(1 - z\overline{w})} = (2 - z\overline{w})K(z,w)}$$

A.1.5 $K_{|z|^{-2p}}, p \in \mathbb{N}$

Method 1: Using the computation for $K_{|z|^\alpha}, \alpha \leq -2, \alpha \in \mathbb{R}$, one obtains

$$K_{|z|^{-2p}}(z,w) = \sum_{j=0}^{\infty} \frac{j+2}{\pi}(z\overline{w})^j - \sum_{j=0}^{m-1} \frac{j+2}{\pi}(z\overline{w})^j.$$

One then sums the series.

Method 2: Using Theorem 3.2 with $g(z) = z^{-p}$ one obtains the following.

$$\boxed{K_{|z|^{-2p}}(z,w) = z^p K(z,w)\overline{w}^p}$$

A.1.6 $K_{|z|^{-2}}$

This is a special case of the above.

$$\boxed{K_{|z|^{-2}}(z,w) = zK(z,w)\overline{w}}$$

A.1.7 $K_{|z|^{-2p}\psi}, p \in \mathbb{N}$

Using Theorem 3.2 with $g(z) = z^{-p}$ one obtains the following.

$$\boxed{K_{|z|^{-2p}\psi}(z,w) = z^p K_\psi(z,w)\overline{w}^p}$$

A.1.8 $K_{|z|^{-2}\psi}$

This is a special case of the above.

$$\boxed{K_{|z|^{-2}\psi}(z,w) = zK_\psi(z,w)\overline{w}}$$

A.1.9 $K_{|z|^\alpha}, 0 < \alpha \leq 2, \alpha \in \mathbb{R}$

This is a special case of $K_{|z|^\alpha}, \alpha > -2, \alpha \in \mathbb{R}$.

$$\boxed{K_{|z|^\alpha}(z,w) = K(z,w) + \frac{\alpha}{2\pi(1 - z\overline{w})} = \left(\left(1 + \frac{\alpha}{2}\right) - \frac{\alpha}{2}z\overline{w}\right) K(z,w)}$$

A.1.10 $K_{|\mu_c|^2}, c \in \mathbb{D}$

We do a change of variables $w = \mu_c(\omega)$ to write $K_{|\mu_c|^2}$ in terms of $K_{|z|^2}$. With this change of variables, $dw = |\mu_c'(\omega)|^2 \, d\omega$, where $|\mu_c'(\omega)|^2$ is the (real) Jacobian of the map, and dw and $d\omega$ are the usual real Lebesgue area measures. For $f \in A^2_{|w|^2}(\mathbb{D})$,

$$\begin{aligned} f(z) &= \int_{\mathbb{D}} f(w) K_{|w|^2}(z,w) |w|^2 \, dw \\ &= \int_{\mathbb{D}=\mu_c(\mathbb{D})} f(\mu_c(\omega)) K_{|w|^2}(z, \mu_c(\omega)) |\mu_c(\omega)|^2 |\mu_c'(\omega)|^2 \, d\omega. \end{aligned}$$

We evaluate at $z = \mu_c(\zeta)$ and multiply both sides by $\mu_c'(\zeta)$:

$$\mu_c'(\zeta) f(\mu_c(\zeta)) = \int_{\mathbb{D}} \mu_c'(\omega) f(\mu_c(\omega)) \mu_c'(\zeta) K_{|w|^2}(\mu_c(\zeta), \mu_c(\omega)) \overline{\mu_c'(\omega)} |\mu_c(\omega)|^2 \, d\omega.$$

Observe that for every $h \in A^2_{|\mu_c|^2}(\mathbb{D})$ there is an $f_h \in A^2_{|w|^2}(\mathbb{D})$ such that $h(\zeta) = \mu_c'(\zeta) f_h(\mu_c(\zeta))$. (Just set $f_h(\zeta) := \frac{h(\mu_{-c}(\zeta))}{\mu_c'(\mu_{-c}(\zeta))}$.) Thus for all $h \in A^2_{|\mu_c|^2}(\mathbb{D})$,

$$h(\zeta) = \int_{\mathbb{D}} h(\omega) \mu_c'(\zeta) K_{|w|^2}(\mu_c(\zeta), \mu_c(\omega)) \overline{\mu_c'(\omega)} |\mu_c(\omega)|^2 \, d\omega.$$

By the uniqueness property of reproducing Bergman kernels we have the following.

$$\boxed{K_{|\mu_c|^2}(z,w) = \mu_c'(z) K_{|w|^2}(\mu_c(z), \mu_c(w)) \overline{\mu_c'(w)}}$$

We can expand this formula using the formula for $K_{|z|^2}$, the definition of $\mu_c(z)$, and the fact that $\mu_c'(z) = \frac{1-|c|^2}{(1-\bar{c}z)^2}$:

$$\begin{aligned}
K_{|z|^2}(\mu_c(z), \mu_c(w)) &= \frac{2 - \mu_c(z)\overline{\mu_c(w)}}{\pi(1 - \mu_c(z)\overline{\mu_c(w)})^2} \\
&= \frac{1}{\pi} \cdot \frac{[2(1-\bar{c}z)(1-c\overline{w}) - (z-c)(\overline{w}-\bar{c})]\,(1-\bar{c}z)(1-c\overline{w})}{((1-\bar{c}z)(1-c\overline{w}) - (z-c)(\overline{w}-\bar{c}))^2} \\
&= \frac{1}{\pi} \cdot \frac{[2(1-\bar{c}z)(1-c\overline{w}) - (z-c)(\overline{w}-\bar{c})]\,(1-\bar{c}z)(1-c\overline{w})}{(1-|c|^2)^2(1-z\overline{w})^2} \\
&= \frac{[2(1-\bar{c}z)(1-c\overline{w}) - (z-c)(\overline{w}-\bar{c})]\,(1-\bar{c}z)(1-c\overline{w})}{(1-|c|^2)^2} K(z,w).
\end{aligned}$$

This computation yields

$$\boxed{\begin{aligned}
K_{|\mu_c|^2}(z,w) &= [2(1-\bar{c}z)(1-c\overline{w}) - (z-c)(\overline{w}-\bar{c})]\,\frac{K(z,w)}{(1-\bar{c}z)(1-c\overline{w})} \\
&= (2 - \mu_c(z)\overline{\mu_c(w)})K(z,w)
\end{aligned}}$$

A.1.11 $K_{|\mu_c|^{2p}}, c \in \mathbb{D}, p \in \mathbb{N}$ or $p \in \mathbb{R}$ with $p > -1$

By repeating the same change of variables argument as in $K_{|\mu_c|^2}$, one obtains

$$\boxed{K_{|\mu_c|^{2p}}(z,w) = \mu_c'(z) K_{|w|^{2p}}(\mu_c(z), \mu_c(w))\overline{\mu_c'(w)}}$$

As with our computation of $K_{|\mu_c|^2}$, we expand using the formula for $K_{|z|^{2p}}$ and μ_c', where here p is allowed to be real and greater than -1.

$$\boxed{\begin{aligned} K_{|\mu_c|^{2p}}(z,w) &= [(p+1)(1-\overline{c}z)(1-c\overline{w}) - p(z-c)(\overline{w}-\overline{c})]\,\frac{K(z,w)}{(1-\overline{c}z)(1-c\overline{w})} \\ &= ((p+1) - p\mu_c(z)\overline{\mu_c(w)})K(z,w) \end{aligned}}$$

A.1.12 $K_{|z-c|^2}, c \in \mathbb{D}$

Observe that $z - c = (1-\overline{c}z)\mu_c(z)$. Now apply Theorem 3.2 with $g(z) = 1 - \overline{c}z$ to get

$$\boxed{\begin{aligned} K_{|z-c|^2}(z,w) &= \frac{K_{|\mu_c|^2}(z,w)}{(1-\overline{c}z)(1-c\overline{w})} \\ &= (2 - \mu_c(z)\overline{\mu_c(w)})\frac{K(z,w)}{(1-\overline{c}z)(1-c\overline{w})} \\ &= [2(1-\overline{c}z)(1-c\overline{w}) - (z-c)(\overline{w}-\overline{c})]\,\frac{K(z,w)}{(1-\overline{c}z)^2(1-c\overline{w})^2} \end{aligned}}$$

A.1.13 $K_{|z-c|^{2p}}, c \in \mathbb{D}, p \in \mathbb{N}$

Repeating the previous argument with $g(z) = (1 - \overline{c}z)^p$ we get

$$\boxed{\begin{aligned} K_{|z-c|^{2p}}(z,w) &= \frac{K_{|\mu_c|^{2p}}(z,w)}{(1-\overline{c}z)^p(1-c\overline{w})^p} \\ &= \left((p+1) - p\mu_c(z)\overline{\mu_c(w)}\right)\frac{K(z,w)}{(1-\overline{c}z)^p(1-c\overline{w})^p} \end{aligned}}$$

A.1.14 Various formulas using Theorem 3.2

The following formulas are trivial to compute in the light of Theorem 3.2. Let $c \in \mathbb{D}$, $a \in \mathbb{C} \setminus \overline{\mathbb{D}}$, and $p \in \mathbb{N}$.

$$\boxed{\begin{aligned} K_{|z-c|^{-2}}(z,w) &= (z-c)K(z,w)(\overline{w}-\overline{c}) \\ K_{|z-c|^{-2p}}(z,w) &= (z-c)^p K(z,w)(\overline{w}-\overline{c})^p \\ K_{|z-a|}(z,w) &= \frac{K(z,w)}{(z-a)(\overline{w}-\overline{a})} \end{aligned}}$$

$$K_{|z-a|^{2p}}(z,w) = \frac{K(z,w)}{(z-a)^p(\overline{w}-\overline{a})^p}$$

$$K_{|z-a|^{-2}}(z,w) = (z-a)K(z,w)(\overline{w}-\overline{a})$$

$$K_{|z-a|^{-2p}}(z,w) = (z-a)^p K(z,w)(\overline{w}-\overline{a})^p$$

A.2 Weighted kernels on the punctured disk

We use the notation of the previous section and restrict our attention to the cases which differ from the case of the (unpunctured) disk.

A.2.1 $K_{|z|^\alpha}, \alpha > 0, \alpha \in \mathbb{R}$

$$\left\|z^j\right\|_{|z|^\alpha} = \int_{\mathbb{D}^*} |z|^{2j+\alpha}\, dz = \int_0^{2\pi}\int_0^1 r^{2j+1+\alpha}\, dr\, d\theta = \frac{2\pi}{2(j+1)+\alpha},$$

which is finite when $2j+1+\alpha > -1 \Leftrightarrow j > -\frac{2+\alpha}{2} = -\frac{\alpha}{2} - 1$.

Remark. Contrary to the case of $\mathbb{D}$, here the functions $\frac{1}{z^k}$, $k \in \mathbb{N}$ are holomorphic on $\mathbb{D}^*$. Thus the only obstruction to them being in the Bergman space is having a finite norm with respect to the weight $|z|^\alpha$.

Let $m \in \mathbb{Z}$ such that $-2(m+1) \le \alpha < -2m$. Then z^j is in $A^2_{|z|^\alpha}(\mathbb{D}^*)$ whenever $j > \frac{-\alpha}{2} - 1 \ge m-1$, where the right hand side is negative. We follow the computation we did for the (unpunctured) disk to obtain

$$\begin{aligned} K_{|z|^\alpha}(z,w) &= \sum_{j=m+1}^{\infty} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j \\ &= \sum_{j=0}^{\infty} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j + \sum_{j=m+1}^{-1} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j \end{aligned}$$

where the last sum is understood to be empty (and therefore zero) when $m+1 > -1$.

$$\boxed{K_{|z|^\alpha}(z,w) = K(z,w) + \frac{\alpha}{2\pi(1-z\overline{w})} - \sum_{j=m+1}^{-1} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j}$$

A.2.2 $K_{|z|^\alpha}, \alpha < 0, \alpha \in \mathbb{R}$

Let $m \in \mathbb{Z}$ such that $-2(m+1) < \alpha \le -2m$. Then

$$\boxed{K_{|z|^\alpha}(z,w) = K(z,w) + \frac{\alpha}{2\pi(1-z\overline{w})} - \sum_{j=0}^{m} \frac{2(j+1)+\alpha}{2\pi}(z\overline{w})^j}$$

A.2.3 $K_{|z|^{2p}}, p \in \mathbb{N}$ (or $p \in \mathbb{Z}^*$)

Method 1: From the general formula for $K_{|z|^\alpha}, \alpha < 0, \alpha \in \mathbb{R}$, we have

$$\boxed{K_{|z|^{2p}}(z,w) = K(z,w) + \frac{p}{\pi(1-z\overline{w})} + \frac{1}{\pi}\sum_{j=1}^{p}(p+1-j)(z\overline{w})^{-j} = \frac{K(z,w)}{z^p\overline{w}^p}}$$

Method 2: Observe that $\frac{K(z,w)}{z^p\overline{w}^p}$ is holomorphic on $\mathbb{D}^*$ and apply Theorem 3.2 with $g(z) = z^p$.

Remark. This second method works for $p < 0, p \in \mathbb{Z}$, too.

A.2.4 $K_{|z|^\alpha}, \alpha \in \mathbb{R}, 0 \le \alpha \le 2$

This is a special case of $K_{|z|^\alpha}$, $\alpha \in \mathbb{R}$.

$$\boxed{K_{|z|^{2p}}(z,w) = K(z,w) + \frac{\alpha}{2\pi z\overline{w}} + \frac{\alpha}{2\pi(1-z\overline{w})} = \frac{\left[\frac{\alpha}{2} + (1-\frac{\alpha}{2})z\overline{w}\right]}{z\overline{w}}K(z,w)}$$

VITA

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