PROJECT BIDDING STRATEGY

CONSIDERING CORRELATIONS BETWEEN BIDDERS

A Thesis

by

MINSOO KIM

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2011

Major Subject: Civil Engineering

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Approved by:

Chair of Committee,	Kenneth F. Reinschmidt
Committee Members,	Ivan Damnjanovic
	Rodrigo Velez
Head of Department,	John Niedzwecki

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ABSTRACT

Project Bidding Strategy

Considering Correlations between Bidders.

(August 2011)

Minsoo Kim, B.E., Kyungwon University, Korea

Chair of Advisory Committee: Dr. Kenneth F. Reinschmidt

One of the most important considerations in winning a competitive bid is the determination of an optimum strategy developed by predicting the competitor's most probable actions. There may be some common factors for different contractors in establishing their bid prices, such as references for cost estimating, construction materials, site conditions, or labor prices. Those dependencies from past bids can be used to improve the strategy to predict future bids. By identifying the interrelationships between bidders with statistical correlations, this study provides an overview of how correlations among bidders influence the bidders winning probability. With data available for over 7,000 Michigan Department of Transportation highway projects that can be used to calculate correlations between the different contractors, a Monte Carlo simulation is used to generate correlated random variables and the probability of winning from the results of the simulation. The primary focus of this paper outlines the use of conditional probability for predicting the probability of winning to establish a contractor's strategy for remaining bids with their estimated bid price and known information about competitors from past data. If a contractor estimated his/her bid price to be lower than his/her average bid, a higher probability of winning would be achieved with competitors who have a low correlation with the contractor. Conversely, the lower probability of winning decreases as the contractor bid with highly correlated contractors when their bid price is estimated to be higher than the average bid.

DEDICATION

To my parents, who always believed in me

ACKNOWLEDGEMENTS

I would like to express the deepest appreciation to my committee chair, Dr. Kenneth F. Reinschmidt, for all his guidance and motivation throughout the course of this research. I am grateful to my committee members, Dr. Damnjanovic and Dr. Velez, for their guidance and support.

I am also thankful to Kalene Curtis at the Michigan Department of Transportation for helping me understand the data. Thanks to my friends and colleagues, and the department faculty and staff for making my time at Texas A&M University a great experience. Especially, I appreciate Kyungki Kim for his help with Matlab and spending difficult times together.

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1. INTRODUCTION

1.1. Overview

Success in a project begins with winning a bid based on a rational strategy after estimating the appropriate bid price affected by the uncertainty of various factors. The estimation of a bidding price consists of two steps: cost estimation and the addition of an appropriate markup to the cost estimate. The cost estimation of a construction project is complicated, due to the characteristics of the construction industry, such as economic factors, the complexity of projects, and the long duration of projects. Bidding is a major source of financial risk in construction projects. Uncertainty remains, because competitors' bids are typically unpredictable. For example, sometimes contractors lose bids even when they think their bidding price is low enough or win bids with high bid prices. Hence, establishing a bidding strategy is a first step in developing a successful project and is regarded as one of the most important strategies of a company.

A competitive bidding process that selects a winner based on the lowest bid dominates the construction industry in the U.S. (Ioannou and Awwad 2010). With fierce competition, the most successful managers are able to make accurate predictions more successfully than their competitors in the determination of a bid price. Managers are able to make decisions based on available information, such as the size and type of project, number of competitors involved, owners' willingness to accept the bid price, distribution

This thesis follows the style of *Construction Management and Economics*.

of competitors' bids and the current industry workload.

The most widely used method in academic studies for setting a bid price is that managers estimate the probability of winning in competitive bidding against each competitor according to their markup, after analyzing the ratio between the competitors' bid price and their estimated price. This approach was first suggested by Friedman (1956). The main idea behind this approach is that each bid submitted on a project is considered to be independent, with the probability distributions of all potential bidders derived from the past data. However, in reality, each bidder's strategy is established on the basis of others' strategies, and bidder's strategies have an interdependent relationship.

Reinschmidt (2010) points out that bids are not necessarily independent, but are instead correlated. In construction projects, there are many factors that can impact bidding prices. From those factors, there may be some common factors for different contractors used to establish bid prices, such as the reference for cost estimating, construction materials, site conditions, or labor prices. These dependencies from past bids can be used to improve a strategy to predict future bids.

This study was conducted in two phases. The first phase explored the relationship between correlations and the probability of winning, allowing for the development of a model to explain how correlations with other competitors affect the probability of winning. The second phase involved investigating the conditional probability for predicting the probability of winning in different conditions. The analysis was used to model bidding strategies and assist companies in competing in an effective way.

A Monte Carlo model was developed to establish a relationship between correlations for different contractors and the probability of winning that affects bidding strategies. A Monte Carlo simulation that can be used to generate correlated random variables was used to model bidding strategies and their effects on the probability of winning. The data collection process was performed to validate the simulation results by obtaining bidding information for each firm from the website of Michigan Department of Transportation's (MDOT) (http://mdotwas1.mdot.state.mi.us/public/bids/).

1.2. Contribution of this study

There are two significant issues addressed in this study. (1) the first consideration of interrelationships between contractors, and (2) the application with enough practical data. Most bidding studies focus on individual firms, while pointing out that bidding involves a complex relationship between potential bidders. In complex probabilistic systems, the interrelationships between random variables are specified by their correlations. Therefore, the explicit consideration of correlations should lead to a better understanding of the bidding process. This study is the first to compute and study these interactions by applying an extensive amount of real data.

The correlation between contractors is measured by the degree of variation between their bidding prices on previous bidding opportunities. It measures how much bidding prices between contractors changes together and how much the bidding price of a contractor varies with the other contractors. For example, using historical data, it was determined that the correlation between the bidding price of Bidder 1 and Bidder 2 may be positive, implying that when the bidding price of Bidder 1 is higher, Bidder 1 would expect that the bidding price of Bidder 2 would be higher as well.

Laryea and Hughes (2008) point out that one of the major problems with developing bidding models is a lack of empirical data. Even if someone establishes a good model to predict bid prices theoretically, one needs data to prove the model and validate the conclusions. One point of significance in this study is that all analyses are conducted with actual project data. The correlations used in this study are computed from real bidding histories of contractors and validation of the model is proven by the data. Over 7,000 projects were used to model the relationship between correlations of different contractors and their probability of winning.

1.3. Research objectives

The six hypotheses in this study are as follows:

1. Bids for actual construction projects are not independent, rather are highly correlated.

2. Winning bids divided by the engineer's estimate is normally distributed.

3. A larger number of bidders are associated with a lower winning bid price.

4. Correlations affect the probability that any bid will be the winning bid.

5. The probability of winning, estimated by the Monte Carlo simulation for generating correlated random variables, is more accurate than the independent case compared with the historical probability of winning.

6. Contractors with low correlations with competing contractors have a higher probability of winning than contractors with high correlations with competing bidders in the case that their estimated bid is lower than their average bid. Conversely, contractors that have higher correlations with competing bidders have higher probability of winning when their estimated bid price is higher than their average bid.

The overall goal of this study is to help state Departments of Transportation (DOT) predict the probability of winning for different bidders and help contractors set up their bidding strategies by predicting the probability of winning at a certain bid. The goals were achieved by accomplishing the following objectives:

1. To identify and describe the correlation of various firms

2. To clarify and define the relationship between the probability of winning and the correlation

3. To model the relationship between the correlation and the probability of winning

4. To model the probability of winning by the change in the bidding price

5. To validate the model using the data from the MDOT

6. To offer conclusions and recommendations and state limitations of the study

2. LITERATURE REVIEW

Decision-making for the process of bidding on a project is complex, due to the characteristics of the construction industry, such as a long duration and the large size of projects. A long duration and large size is affected by the uncertainty of various factors that affect a contractor's bidding decision. Bidding strategies are difficult to model and predict, due to this uncertainty. This section reviews three sets of related literature: factors that impact bidding decisions, historical bidding models, and the uses of the Monte Carlo simulation with the consideration of correlations among uncertainties.

2.1. Factors that impact on bid

Many factors affect the bidding decisions of contractors in the construction industry. These factors are comprised of hard facts, experience, judgment, and perceptions (Ahmad and Minkarah 1988). Bidding strategies are difficult to model and predict, since what is attractive to bidders is not constant and that the set of bidders changes from bid to bid. Individual factors, such as the size of the contract, bidding time, backlog, and available resources, weigh differently based on the attractiveness of the job.

Ahmad and Minkarah (1988) conducted a questionnaire survey of the top-400, general contractor/construction firms in the U.S. The objective of the survey was to identify the decisive factors for bidding decisions and the percentage of profit markup. The relative importance of the factors was found to differ depending on the stage of the bidding process. Overall, the type of the job, need for work, experience with the owner, historic profits and degree of hazards were the top five factors affecting the bid/no-bid decision. However, the degree of hazard, degree of difficulty, type of job, uncertainty in the estimate and historic profit were the top five factors affecting the percent-markup decision. This study also illustrated that the client relationship, quality of design and reliability of subcontractors had a substantial influence on the bidding decisions.

Shash (1993) conducted a similar study and identified different factors influencing the bidding decisions and profit markup sizes made by top United Kingdom (UK) contractors. He found that the need for work, competition, and previous experience on similar projects were identified as the top three factors that affected a contractor's decision to bid for a project, while the degree of difficulty, the risk involved, and the current work load were the highest ranked factors affecting profit markup size decisions.

In proposing a neural network bid/no bid model, Wanous (2003) surveyed contractors in Syria to determine the factors that had the most influence on the bid/no bid decision-making process. The survey revealed that client characteristics had the largest influence. The conditions in the contract, financial capability of the client and relationship with clients had the greatest effect. Smith and Bohn (1999) also supported the view that owner capabilities are considered important factors in decisions to bid/no bid on construction projects. Project size and the amount of time required to bid were next in importance.

The contractor's estimate is vital to the success of a company. Bidding too low can lead to bad consequences, such as small profits, losses, or the termination of the business, in extreme cases. Shash and Abdul-Hadi (1993) investigated the factors that contractors in Saudi Arabia considered when deciding to bid for projects. They found that project cash flow, contract type, staff capability, familiarity with the project and project size had an influence.

In developing a bid reasoning model, Chua and Li (2000) identified four areas of consideration in bidding decisions by interviewing six experienced individuals. Through discussions with experts in the industry, they found that competition, risk, company's position in bidding, and the need for work were the determining factors in the contractor's bidding decision. By using the Analytic Hierarchy Process (AHP), the study was able to rank and determine their relative importance in achieving their sub-goals.

2.2. Bidding models

Kangari and Riggs (1989) divided the risk analysis model into two major categories: classical models, such as probability theory or Monte Carlo simulations, and conceptual models that assess risk in linguistic terms. They point out that most information for determining a risk analysis is not numerical that is necessary for a classical model, but linguistic expressed risk information in terms of words using a natural language, and there is not enough numerical data to develop statistical patterns. However, there are 35 classical models out of the available 45 risk models developed since 1990 that can be used to analyze the data (Laryea and Hughes 2008).

The earliest bidding model is Friedman's Model (1956), a probabilistic model that determines the probability of winning numerically. The main point of the model is that the probability of winning is estimated from the probability distribution of the ratio between the competitors' bids and the estimated bid price. Using an estimation of all potential competitors' previous bidding patterns, a known probability distribution is calculated. Then he predicted the probability of winning by considering each bid submitted on a project as independent. He also suggested the estimation of the number of bidders based on a regression analysis between the number of bidders and a contractor's cost estimates in the previous bidding history. Most models developed later have referred to Friedman's model (Mayo 1992).

Gate (1967) criticized the Friedman model and recommended a solution by calculating the probability of winning and the assumption of independence. Griffis (1971) also disputed the problem of Friedman's model, that bids must be independent. He maintained some common factors to bidders when they bid, such as labor, materials, or subcontractors. Morin and Clough (1969) also recommended minor changes in Friedman's assumption about the estimation of the number of bidders. They placed emphasis on major competitors and more recent bids compared with Friedman's average bidder. McCaffer (1976) recommended a way of estimating the normalized bid value from the relationship between the mean bid and the lowest bids from the overall distribution. From the normalized bid for each bidder, he established contractors' behavior patterns to identify major competitors.

There are many models available, but static assumptions are applied. Contractors' objectives can change over time, based on their current workload (Griffis 1971). In proposing a new and improved bidding model considering a static and non-static bidding environment, Christodoulou (2000) analyzed the deficiency of previous methods and suggested a bidding model that incorporated the historical bidding data of the competitors. He applied artificial neural networks for the analysis and determined patterns of factors that affect the bidding characteristics and method of Parzen Windows for estimating multi-dimensional probability distribution functions. The model was validated as an improvement to the previous methods by conducting testing using data from the earlier literature, as well as data from the New York City public agency.

2.3. Correlation between bidders

Varying degrees of imperfect knowledge that affect the bidding decision of each contractor and previous bidding models are presented. In this section, papers that use the Monte Carlo simulation as one of the methodologies to reflect the correlation among uncertainties are introduced.

One of the common errors in using the Monte Carlo simulation to estimate the total cost is the assumption of independence among cost components (Touran and Wiser 1992). To estimate the total construction cost, Touran and Wiser (1992) selected a Monte Carlo simulation by considering the dependence of the cost components with the multivariate lognormal distribution as a statistical distribution for the cost components. The model was validated by showing more accuracy in predictions using the cost data of a low-rise office building from R. S. Means (Touran and Wiser 1992). Chau (1995) also mentioned the problem of the widely accepted assumption that cost distribution is independent and triangular. The paper proved that the assumption of independence in cost components caused an underestimation of uncertainties, while the triangular assumption overestimates the uncertainties.

The consideration of correlations between uncertainties in a construction project is not only utilized with cost components, but also with activity durations. Wang and Demsetz (2000) insisted that the correlation between each activity increases with the variation of the project duration. They analyzed the impact of correlations on the full project duration by comparing some simulations that do not consider correlations in durations with their own model (networks under correlated uncertainty). The results revealed that the variability of project duration is enormous when the correlation is considered.

Correlations between uncertainties are also utilized to approximate the behavior of financial markets. A number of books about Value at Risk (VaR), including (Jorion 2007; Marrison 2002) deal with the Monte Carlo simulation by considering the correlation across risks, such as the market risk or the credit risk. For example, the credit quality of a firm may be changed by the market conditions or financial conditions of the firm. Other models considering correlations have also been developed (Okmen and Ozta, 2008; Skitmore and Ng 2002; Touran 1993; Touran and Suphot 1997).

3. DATA DESCRIPTION

All data used in this study were collected by obtaining the bidding data on the closing prices from the MDOT website (http://mdotwas1.mdot.state.mi.us/public/bids/). The data included all Michigan highway construction projects from January 2001 to December 2009, totaling 7,395 projects, excluding projects with missing information and projects with only one bidder, which are not in a competitive situation.

3.1. Engineer's estimate

The engineer's estimate (EE) is the in-house engineer's cost estimate of the amount that the DOT considers an acceptable contracting price. Each state DOT has a different policy about the timing in relation to releasing an EE. Some auction literature shows that publishing the owner's value before bid letting causes bidders to establish a more aggressive bidding behavior and leads to lower bids. As an empirical result, (De Silva, Dunne, Kankanamge, and Kosmopoulou 2008) showed that the average bid divided by the EE in Oklahoma declined by about 0.09, while the standard deviation decreased by about 0.068, after a change in the EE release policy that released their EE prior to the bidding process.

With regard to modeling bid values, Drew and Skitmore (1997) measured the value of each bidder as a ratio of the bid over the lowest bid. By expressing bids as a ratio to the lowest bid, the value of the bid can be normalized and made easier to understand. However, this process can only be done ex-post, after the bids are opened. Since the

MDOT releases the state's internal estimates of the costs to complete highway construction projects prior to bid (the first four digits of the EE), each bidder can have knowledge of the EE in advance. By using the EE as a baseline, it helps bidders to predict a percentage relative to the EE, prior to their own project estimation process. In extending Drew and Skitmore's study, an alternative normalized bid that is the ratio between the bids and the EE is measured.

$$X = \frac{Bid}{Engineer's Estimate} \tag{3.1}$$

Figure 3-1 illustrates a frequency histogram of the ratio of the low bid over the EE. The green line is a smoothed curve of the frequency distribution and the red line is a normal curve fitted to the data. Although there is slight difference between the two curves, it is apparent to the eye that the shape of the histogram approximately follows a normal distribution. The mean value of the ratio of the low bid to the EE for all 7,431 projects is 0.931, approximately 3% higher than that of the Oklahoma DOT from the study in De Silva, et al. (2008). In other words, the average awarded bid amount is about 7% below the EE; there is a 50% probability of winning when contractors bid with a price 7% below the EE. The standard deviation of the ratio of the low bid to the EE was 0.1457. More information about the histogram, such as a Q-Q plot which compares the distribution of the data with the normal distribution by plotting their quantiles or cumulative distribution functions, is presented in Appendix A.

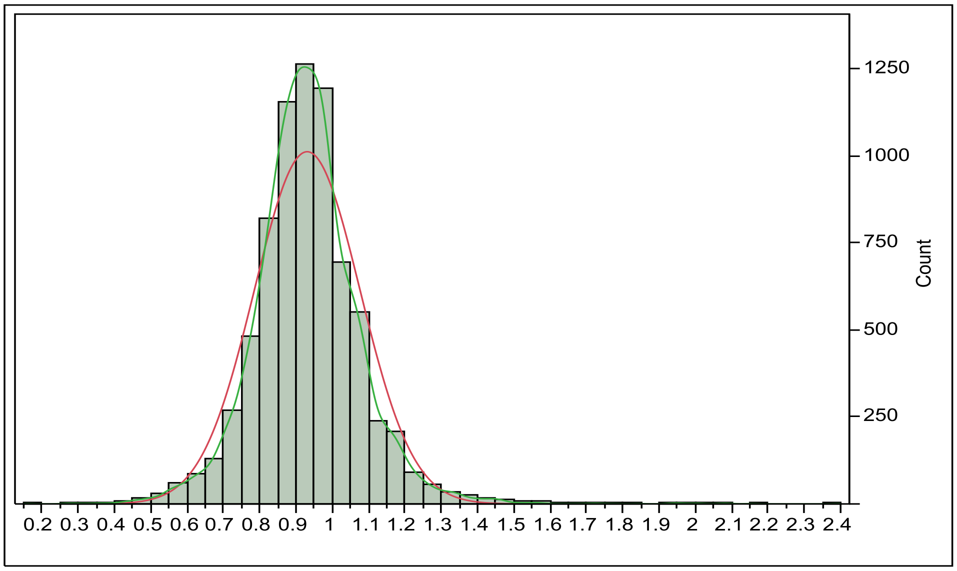

Figure 3-1. Low bid divided by engineer's estimate for 7395 projects

There were a total of 715 bidders who bid at least one time on the MDOT projects. Figure 3-2 illustrates the frequency histogram of the number of jobs bid per bidder in the 9 years. The firm who bid the most on the MDOT projects participated 1,315 times; the least amount of times a company bid was once. The average number of bids for each company was 53; 90% of the contractors attended bids less than 158 times.

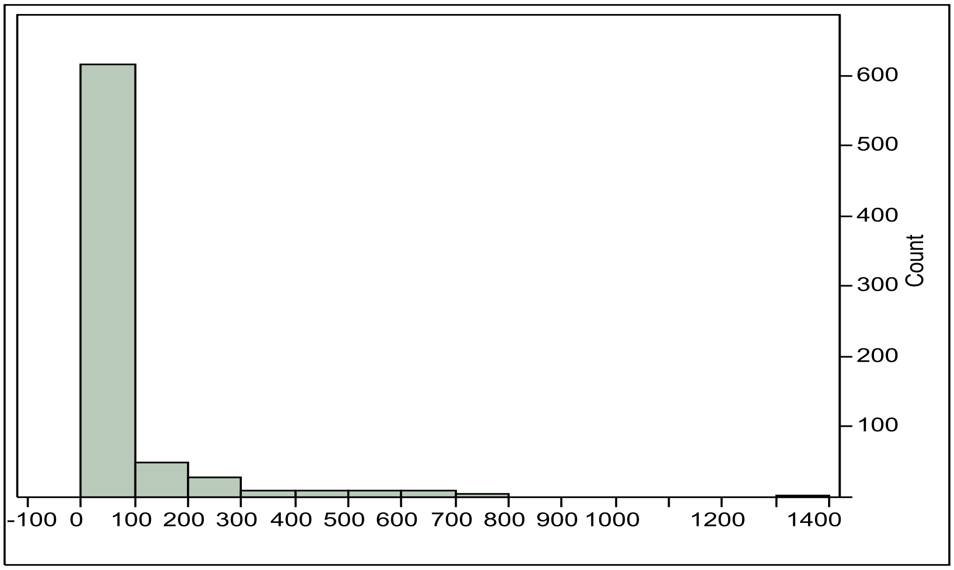

Figure 3-2. The histogram of the number of jobs contractors bid

3.2. Correlations between bidders

Although early bidding models, such as Friedman's model, considers each bid submitted by different contractors independently, it is apparent that there are correlations between bids in the database. Table 3-1 shows the correlation matrix of the top 20 contractors who bid most on projects from the MDOT, excluding cases in which two contractors bid on the same projects fewer than 10 times. The maximum number of bids that two contractors in top 20 bid on the same projects was 464 times. With information about the top 20 bidders identified from the Excel spreadsheet, pairwise bid information between the two bidders in the top 20 was extracted by Matlab when the two bidders had bidding histories for the same project. Table 3-1 illustrates the mean correlation coefficient between the top 20 contractors as 0.6 with a maximum of 0.865 and minimum of 0.101.

The method used to calculate the correlations between the two bidders is introduced in Section 4.2.1.

3.3. The number of bidders

Many studies suggest that competition is another key factor that influences the bidding price decision. Shash (1993) conducted a contractor questionnaire, determining that the number of competitors tendering is the second most important factor affecting the bidding price. Shash (1993) then determined that the intensity of competition is regarded as the second most significant element (following the need for work) among the top UK contractors.

Many bidding strategies are based on the knowledge that increasing the number of bidders causes more intense competition. The observation here is that the number of competitors participating in the project bidding process is inversely related to the winning bidding price. Kuhlman and Johnson (1983), in studying the relationship between competition and the winning price divided by EE, identified two methods that can be used to measure the intensity of competition. One method is to consider the number of expected bidders before the bidding opening with a public list of potential bidders. The other method is to consider the number of actual bidders after the bidding opening. This study concluded that the number of actual bidders who participated in the bidding is a more appropriate variable, as it has a higher significant level in the regression analysis than the number of expected bidders.

Table 3-1. Correlation matrix between top 20 bidders

	540	**45**	**103**	**459**	**441**	**349**	**431**	**24**	**499**	**180**	**510**	**439**	**64**	**43**	**182**	**109**	**341**	**20**	**69**	**244**
540	1		0.554	0.365	0.471	0.617	0.536	0.543	0.626				0.560		0.416			0.500	0.767	
45		1		0.583	0.791	0.741				0.780	0.754	0.653		0.639	0.661	0.757				
103	0.554		1	0.596	0.571	0.507	0.722	0.654		0.816				0.667	0.316	0.430	0.865	0.407	0.551	0.674
459	0.365	0.583	0.596	1	0.612	0.569	0.707			0.604	0.817			0.375	0.525			0.390	0.477	
441	0.471	0.791	0.571	0.612	1	0.718	0.537			0.514	0.644	0.504		0.101	0.620	0.614		0.226		
349	0.617	0.741	0.507	0.569	0.718	1	0.584							0.375	0.435			0.459		0.558
431	0.536		0.722	0.707	0.537	0.584	1	0.779										0.598	0.607	
24	0.543		0.654				0.779	1		0.710				0.603			0.693	0.480	0.605	0.643
499	0.626								1				0.601							
180		0.780	0.816	0.604	0.514			0.710		1	0.618	0.693		0.778	0.754	0.623	0.766		0.666	0.700
510		0.754		0.817	0.644					0.618	1	0.583		0.603	0.536	0.643	0.462			0.698
439		0.653			0.504					0.693	0.583	1			0.636	0.658				
64	0.560								0.601				1							
43		0.639	0.667	0.375	0.101	0.375		0.603		0.778	0.603			1	0.539	0.559	0.692		0.758	0.730
182	0.416	0.661	0.316	0.525	0.620	0.435				0.754	0.536	0.636		0.539	1	0.558		0.324		
109		0.757	0.430		0.614					0.623	0.643	0.658		0.559	0.558	1	0.529			
341			0.865					0.693		0.766	0.462			0.692		0.529	1		0.791	0.614
20	0.500		0.407	0.390	0.226	0.459	0.598	0.480							0.324			1	0.654	
69	0.767		0.551	0.477			0.607	0.605		0.666				0.758			0.791	0.654	1	0.817
244			0.674			0.558		0.643		0.700	0.698			0.730			0.614		0.817	1

Figure 3-3 illustrates the relationship between the MDOT award cost over the EE and the number of bidders. The non-linear regression line in the scatter plot shows that the award cost over the EE decreases as the number of bidders increases. Even though the R-squared value is low (0.048), there is reasonable evidence that there is an effect due to the number of bidders from the result of a t-test (Appendix B). Using the parameter from the regression analysis, it can be concluded that the ratio between the low bid and EE decreases with each additional competitor. Since different contractors have different bid distributions, the variance of the bid value will be high when the number of competitors increases; this may lead to a lower winning bid value. Increasing the number of bidders is advantageous to the DOT, who makes great efforts to get more contractors to bid.

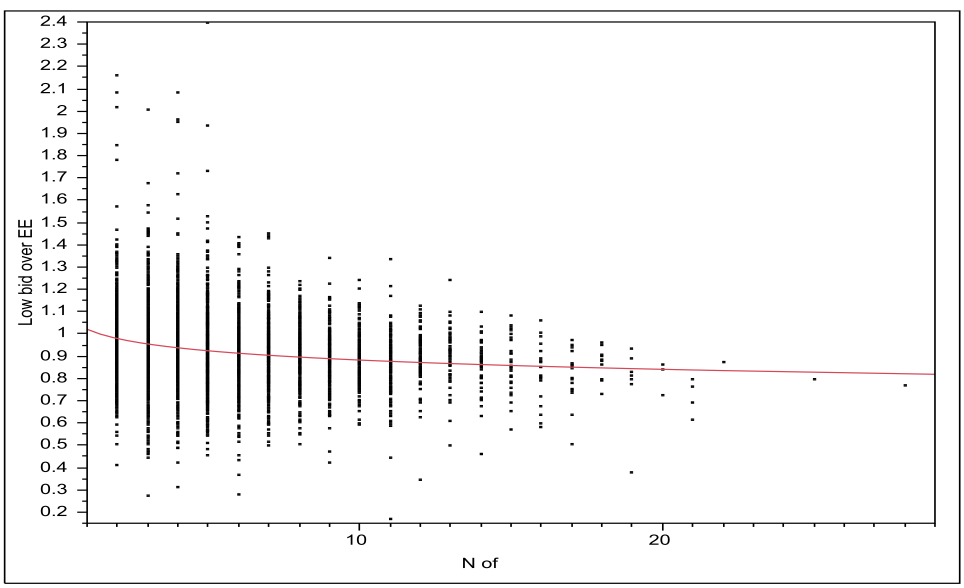

Figure 3-3. The relationship between bid award and the number of bidders

Figure 3-4 shows the histogram of the number of bidders per a bid for 7395 projects with the average number of bidders of 5.15. The maximum number of bidders was 28 and the minimum number was 2, because the case for only one bidder was excluded. From the probability distribution, it was determined that 90% of the projects had fewer than 9 bidders.

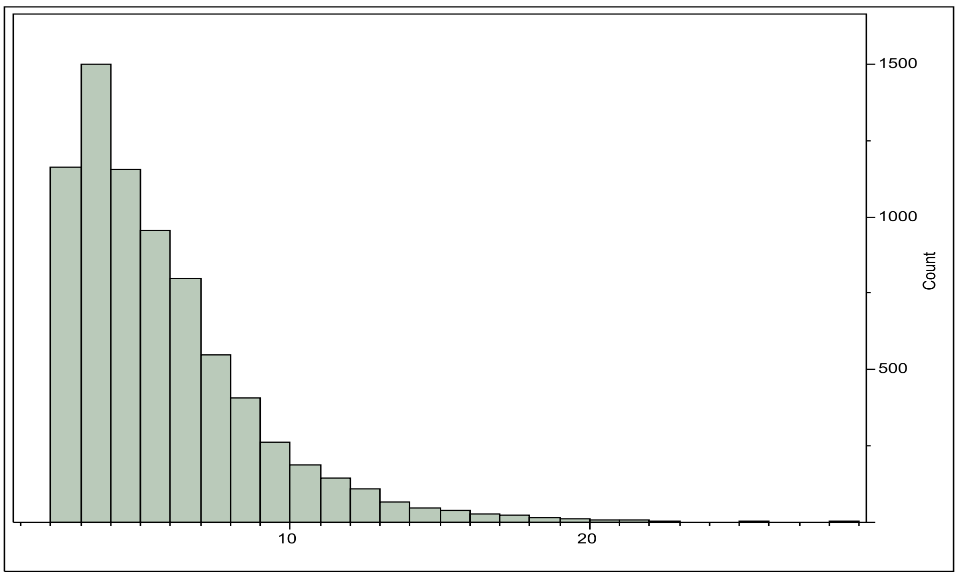

Figure 3-4. The histogram of the number of bidders per bid

4. MONTE CARLO SIMULATION FOR CORRELATED BIDDERS

4.1. The probability of winning with mean and standard deviation

Before investigating the effect of correlations between bidders on the probability of winning thorough the Monte Carlo simulation for generating correlated random variables, one may ask a question how varying mean and standard deviation influence on the probability of winning. Reinschmidt (2010) examined the effect of varying mean and standard deviation on the probability of winning. By assuming that all bidders are independent and their distributions are normally distributed with known means and standard deviations, the Monte Carlo simulation is conducted with 100000 trials for each case. Table 4-1 shows the base case of three bidders.

Table 4-1. The base information of three bidders

	Bidder 1	Bidder 2	Bidder 3
Mean	1	1	1
Std. Dev	0	0	0

As a first case, in order to explore how the probability of winning changes with respect to bidder 1's average bids, mean values of probability distribution of bidder 1 are varied from 0.8 to 1.2, while the standard deviations remain equal. It is expected that the probability of winning by bidder 1 is larger when the average bid of bidder 1 is smaller and vice versa. Figure 4-1 shows the same result as it was expected.

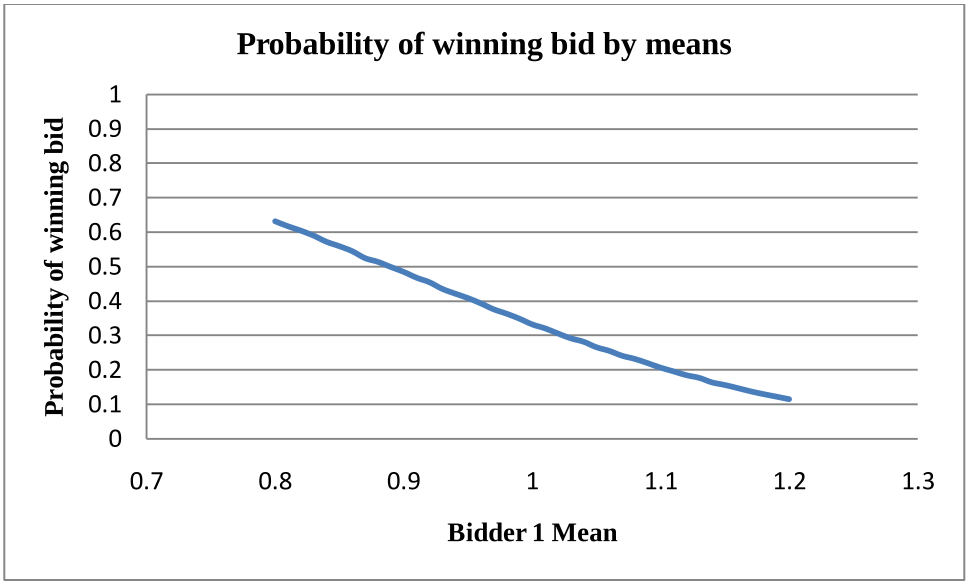

Figure 4-1. The probability of winning bid by varying means

Figure 4-2 shows the influence of changing the standard deviation of the bids by bidder 1 while all others are equal to the base case in table 4-1. The standard deviation of bidder 1 ranges from 0.01 to 0.4. If the variance of the bids by bidder 1 is larger, the probability of winning also is larger. On the other hand, the probability of winning decreases as bidder 1's bids have less variation. If the standard deviation of bidder 1 is close to zero, his winning probability is approximately 25% which is the probability that two independent bidders bid higher than bidder 1.

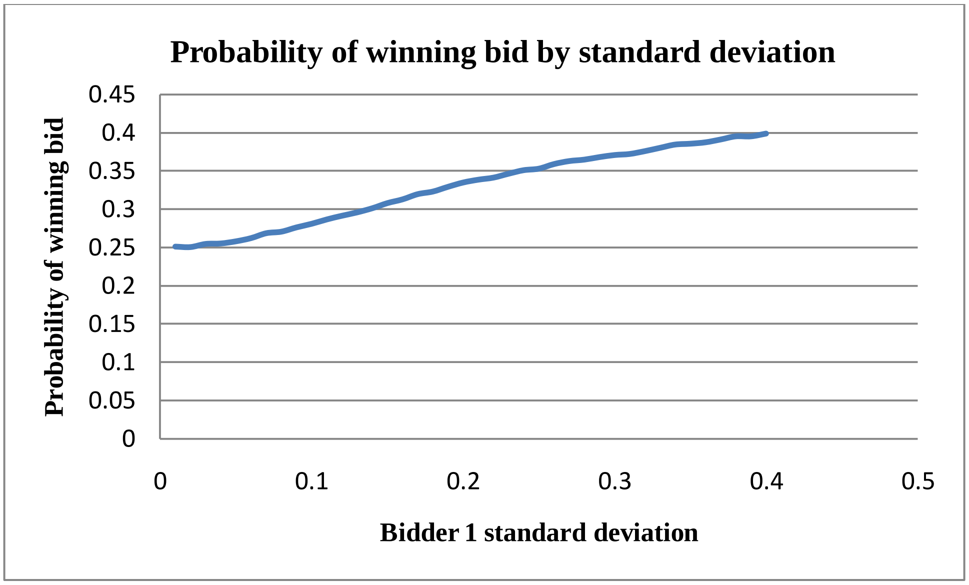

Figure 4-2. The probability of winning bid by varying standard deviations

4.2. Process for formulation of simulation

From this section, the Monte Carlo simulation for creating correlated random variables using Cholesky decomposition of the covariance matrix is described. The Monte Carlo simulation generates random variables with known probability distributions. With decomposition of the covariance matrix, the Monte Carlo approach allows one to create random variables with correlations from historical data. The overall process of the simulation is shown in below.

1. Calculate covariance matrix among bidders from bidding history
2. Decompose covariance matrix
3. Decide on the probability distributions to be used for each bidder

4. Generate random variables with decomposed covariance matrix
5. Count the number of jobs for which each bidder is low bidder

4.2.1. **Review of covariance matrix**

Unlike independent variables, some variables tend to vary in the same direction or in the opposite direction. Suppose that if the bidding prices of two bidders are affected by the same factors, their bidding prices increase or fall at the same time, then it can be said that their bidding prices have positive correlation. Let $X = [x_1, x_2, \cdots, x_n]$ be a vector of bid prices for contractors. Given any pair of bidding prices, the covariance is defined by

$$\mathrm{cov}(x_i, x_j) = \sigma_{i,j} = E[(x_i - \mu_i)(x_j - \mu_j)] \tag{4.1}$$

where μ_i and μ_j are mean bid values of contractor i and j.

With the covariance between two contractors, the Covariance matrix of C is

$$C = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix} \tag{4.2}$$

The correlation coefficient between contractor X and Y is given by

$$corr(x_i, x_j) = \rho_{ij} = \frac{\mathrm{cov}(x_i, x_j)}{\sigma_i \sigma_j} \tag{4.3}$$

where σ_i and σ_j are standard deviations of the bids of contractor i and j.

4.2.2. **Decomposition of covariance matrix**

4.2.2.1. Cholesky decomposition

One widely used way to transform independent random variables into correlated random variables is Cholesky decomposition discovered by André-Louis Cholesky. The Cholesky decomposition generates a new matrix A such that the given covariance matrix C is the product of a triangular matrix A and the transposed matrix A^T :

$$C = AA^T \tag{4.4}$$

In order to generate correlated random variables, one needs to take the lower triangular matrix A and standard normal random variable Z which has mean zero and standard deviation 1. By post multiplying A by Z we can produce correlated standard normal variables Y:

$$Y = AZ \tag{4.5}$$

The confirmation that the values of Y have the desired correlation can be proved by calculating the covariance matrix of Y:

$$\mathrm{cov}(Y) = E[YY^T] = E[AZZ^T A^T] = AE[ZZ^T]A^T = AA^T = C$$

where $E[ZZ^T] = I$ (identity matrix) because Z is a matrix of independent variables, so E[Z]=0.

It is easy to compute Cholesky decomposition in MATLAB using *chol* function. Cholesky decomposition is relatively straight forward to program, but the Cholesky matrix does not work if the matrix is not positive definite. A true covariance matrix must be positive definite, in which all eigenvalues of the covariance matrix must be greater than zero, and it is not a true covariance matrix if the matrix is not positive definite. However, in practice, there are usually not enough data to accurately compute the true covariance matrix (Lurie and Goldberg 1998). In order to compute the true covariance matrix, a huge amount of data points is required to get enough subsets of contractors that bid against each other. Therefore, in this study, correlation coefficients are computed pairwise in order to use the maximum number of data points available for each pair and to maximize the accuracy of each computed correlation. The positive definite problem resulting from not having a true covariance matrix is avoided by altering the eigenvalues to be positive.

4.2.2.2. Eigenvalue problem

In spite of the negative eigenvalue problem, there are alternative methods to decompose the covariance matrix such as adjustment of eigenvalues in the matrix, eigenvalue decomposition, or singular value decomposition. Even though there are more ways to decompose the covariance matrix, the three ways mentioned above are applied in this study.

If someone prefers to use Cholesky decomposition even though some of the eigenvalues are negative, it is possible to create a positive definite matrix by relatively small adjustments to the eigenvalue matrix by forcing negative eigenvalues to be zero or slightly positive. With some functions in MATLAB, the decomposition processes go through by changing some negative eigenvalues to positive and then back-computing the covariance matrix.

Also there is another solution, which is eigenvalue decomposition when faced with the positive definite problem in the covariance matrix. It is more difficult than Cholesky decomposition, but it can work for covariance matrices that are not positive definite. Eigenvalue decomposition works by looking for two matrices, Λ and E, to satisfy the following equation:

$$C = E^T \Lambda E \tag{4.6}$$

C is the covariance matrix and E is a matrix of eigenvectors from the covariance matrix. Since Λ is a diagonal matrix such that all the elements are zero except for elements on the main diagonal, it allows us to decompose the covariance matrix as shown below. After decomposing the covariance matrix, the process of generating correlated random variables is the same as in the Cholesky decomposition case.

$$C = B^T B \tag{4.7}$$

where $B = \sqrt{\Lambda} E$

Lastly, the singular value decomposition can be an alternative method. That is

$$C = U \Lambda V^T \tag{4.8}$$

The left singular vector U is composed of the eigenvectors of CC^T and the right singular vector V is composed of the eigenvectors of C^TC. The process to generate correlated random variables with singular value decomposition after decomposition of the covariance matrix is same as the eigenvalue decomposition case.

4.3. Generation of correlated random variables

MATLAB *randn* function generates random numbers from the standard Normal distribution with any number of indicated trials. Since the statistical distribution for low bid price over EE is seen to be a Normal distribution, Normal random variables Z with specific mean μ and variance v can be obtained from:

$$X = Zv + \mu \tag{4.9}$$

Similar to the way to generate independent Normal random variables, correlated dependent variables Y can be generated from the matrix multiplication of lower triangular matrix A from decomposition method and Normal independent variates Z:

$$Y = AZ + \mu \tag{4.10}$$

To compare with the correlation and probability of winning, a correlation matrix of six contractors was computed from MDOT data. The six contractors are taken from the top 10 bidders who bid on the most projects in MDOT for the past 9 years. Also, they are selected when they have enough bidding history between two contractors. Correlations were computed pairwise between contractors for all bids by listing all contracts that were bid by both of them. As a reminder, correlation coefficients are calculated from the normalized values of X from equation (3.1). The correlation coefficient matrix and average correlation coefficients with other five contractors for each contractor are shown in Table 4-2. The contractor 431 has the highest average correlation with other five contractors and contractor 540 has the lowest correlation with others.

Table 4-2. Correlation matrix among six bidders

Contractor number	540	103	459	441	349	431	Average
540	1.0000	0.5539	0.3650	0.4711	0.6170	0.5361	0.5086
103	0.5539	1.0000	0.5956	0.5708	0.5070	0.7223	0.5899
459	0.3650	0.5956	1.0000	0.6120	0.5688	0.7074	0.5697
441	0.4711	0.5708	0.6120	1.0000	0.7184	0.5368	0.5818
349	0.6170	0.5070	0.5688	0.7184	1.0000	0.5836	0.5989
431	0.5361	0.7223	0.7074	0.5368	0.5836	1.0000	0.6172

The Monte Carlo simulation is conducted by drawing X at random for each of the six bidders. In order to analyze the relationship between correlation with others and the probability of winning, the condition is given that the only difference among bidders in this simulation is the correlation coefficient while other things remain equal (means and standard deviations for all bidders are 1 and 0.2). Figure 4-3 shows the results after 100,000 random trials by considering each trial as a bid situation by the six bidders. It plots the percentage of contracts won by each bidder and the average correlation coefficient for each bidder from Table 4-2. With R-square of 0.98 in the figure, the very strong linear relationship between those two factors can be seen. With the result of this simulation, one would conclude that contractors that are highly correlated with other contractors win fewer bidding competitions than firms that are less highly correlated. Conversely, it would be advantageous for contractors to be as little correlated with other bidders as possible. The result was also same as the case that mean and standard deviation is above or less than base case (mean=1 Std. Dev=0.2). The back computed mean, standard deviation and correlation matrix from random variables are given in Appendix C.

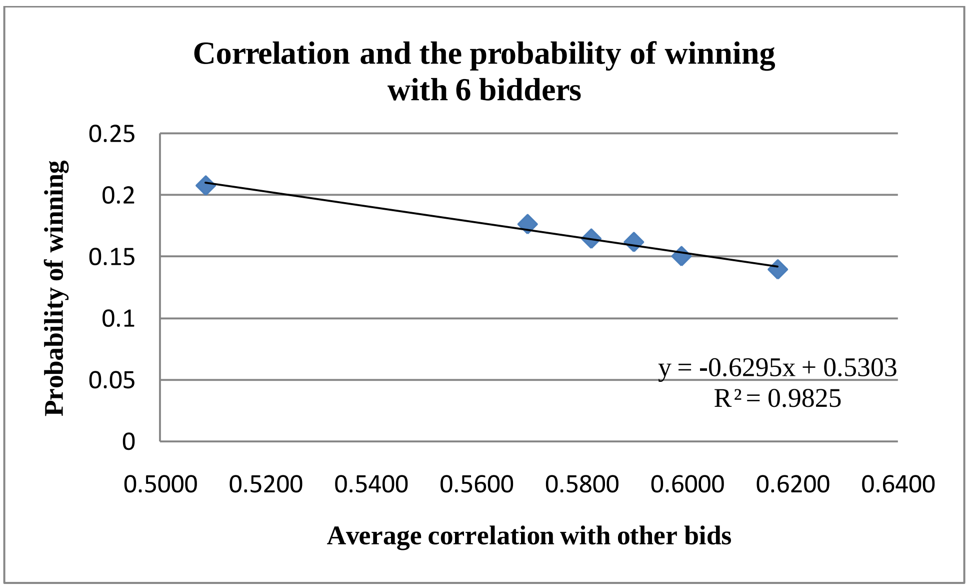

Figure 4-3. Average correlation vs. probability of winning

At this time the number of contractors is increased to seven. In contrast to the six bidders' case, the negative eigenvalue problem arises when the number of contractors is increased. In order to solve this problem, three different kinds of methods discussed in the previous section were conducted. With comparison of back computed covariance matrix from random variables and original matrix shown in Table 4-3, revised original covariance matrix to avoid negative eigenvalue problem is selected and shown in Table 4-4. The back computed covariance matrices from random variables with three kinds of methods can be checked in Appendix D.

Table 4-3. Original covariance matrix (Not positive definite)

Contractor number	540	45	103	459	441	349	431
540	0.0400	0.0251	0.0222	0.0146	0.0188	0.0247	0.0214
45	0.0251	0.0400	0.0371	0.0233	0.0317	0.0296	0.0299
103	0.0222	0.0371	0.0400	0.0238	0.0228	0.0203	0.0289
459	0.0146	0.0233	0.0238	0.0400	0.0245	0.0228	0.0283
441	0.0188	0.0317	0.0228	0.0245	0.0400	0.0287	0.0215
349	0.0247	0.0296	0.0203	0.0228	0.0287	0.0400	0.0233
431	0.0214	0.0299	0.0289	0.0283	0.0215	0.0233	0.0400

Table 4-4. Revised covariance matrix (Positive definite)

Contractor number	540	45	103	459	441	349	431
540	0.0399	0.0249	0.0219	0.0147	0.0189	0.0246	0.0214
45	0.0249	0.0400	0.0369	0.0234	0.0317	0.0295	0.0300
103	0.0219	0.0369	0.0399	0.0238	0.0228	0.0203	0.0290
459	0.0147	0.0234	0.0238	0.0400	0.0246	0.0228	0.0285
441	0.0189	0.0317	0.0228	0.0246	0.0402	0.0288	0.0216
349	0.0246	0.0295	0.0203	0.0228	0.0288	0.0399	0.0234
431	0.0214	0.0300	0.0290	0.0285	0.0216	0.0234	0.0402

Using Cholesky decomposition with the revised covariance matrix, seven correlated random numbers are generated and the probability of winning for each contractor is calculated as in the six bidders' case. Even though the R-square goes down slightly by 0.93, there is still strong relationship between correlation and percentage of wins. Figure 4-4 shows the result that supports the same conclusion as in the previous case, in which low correlation coefficients with competing bidders are associated with higher probability of winning the bid.

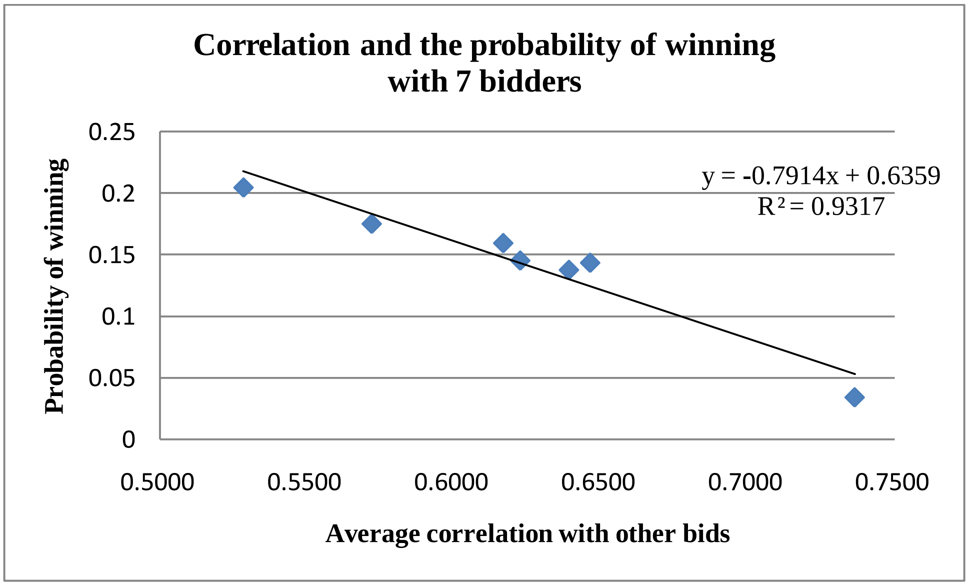

Figure 4-4. Average correlation vs. probability of win

5. CONDITIONAL PROBABILITY DISTRIBUTION

Throughout the previous section, the correlations between contractors and their probability of winning was introduced by generating correlated random variables under limitation that the only difference between bidders is correlation and all other parameters are equal. So, how can bidders establish their strategy for remaining bids with their estimated bid price and known information from past data? The probability of winning needs to be reevaluated after bidding price data becomes available. The probability of winning a given or assumed contractor's bidding price for a bid is called conditional probability.

This section examines how a contractor's estimated bid price affects its probability of winning. The same assumption with the previous section is used in this section, in which each contractor's bid is approximately Normally distributed from historical data and is correlated. Using Bayes' Law and the assumption above, it is possible to compute the conditional probability distribution on bids (Reinschmidt 2010).

5.1. Bivariate normal

As a first simple bidding model, the consideration of this model is the relationship between one contractor and other contractors. This model considers the case in which one contractor only knows the correlation with other contractors and does not know the correlation among others due to the lack of information about others. Suppose that contractors have their average bids with μ_i and standard deviation of σ_i from normalized past data with equation (3.1). With that information, a Univariate Normal distribution of each contractor can be represented by

$$f_i(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\{-\frac{1}{2}(\frac{x_i - \mu_i}{\sigma_i})^2\} \tag{5.1}$$

Two random variables of x_1 and x_2 from bids of contractor 1 and 2 can be represented by the joint probability distribution denoted with correlation coefficient of ρ_{12}.

$$\begin{aligned} f_{1,2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}} \exp[-\frac{1}{2(1-\rho_{12}^2)}\{(\frac{x_1 - \mu_1}{\sigma_1})^2 \\ -\frac{2\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + (\frac{x_2 - \mu_2}{\sigma_1})^2\}] \end{aligned} \tag{5.2}$$

With Bayes' theorem

$$P(A \cap B) = P(A|B)P(B) = P(B \cap A) = P(B|A)P(A)$$

the conditional probability density function of x_1 given $x_2 = X_2$ is

$$f_{(1|2)}(x_1 | x_2 = X_2) = \frac{f_{1,2}(x_1, x_2)}{f_2(x_2)} \tag{5.3}$$

With correlation coefficient, ρ_{12}, between two contractors, the joint probability density function for the two bids is Bivariate normal.

$$f_{(x_2|x_1)}(x_2) = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}}\exp[-\frac{1}{2(1-\rho_{12}^2)}\{(\frac{x_1-\mu_1}{\sigma_1})^2 - \frac{2\rho_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + (\frac{x_2-\mu_2}{\sigma_1})^2\}]}{\frac{1}{2\pi\sigma_1}\exp\{-\frac{1}{2}(\frac{x_1-\mu_1}{\sigma_1})^2\}} \tag{5.4}$$

with mean

$$\mu_1 + \rho_{12}\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)$$

and standard deviation

$$\sigma_1\sqrt{1-\rho_{12}^2}$$

5.2. Pairwise comparison

As a first example, how correlation impacts the probability of winning according for variance bid prices is studied. The condition is given that the only difference among bidders is the correlation coefficient while other things are equal. Data for contractors come from top 10 contractors who bid most on projects from MDOT. Based on contractor 459 who bid most on projects with other bidders in top 10, correlations with other 7 contractors are applied to estimate the probability of winning by using conditional probability distributions except for two contractors who rarely have information about bids with contractor 459. The following Table 5-1 shows the correlation coefficients between contractor 459 and other 7 contractors.

Table 5-1. The number of bidders against bidder 459

Contractor number	Correlation coefficient with bidder 459
540	0.3650
45	0.5830
103	0.5956
441	0.6120
349	0.5688
431	0.7074
180	0.6041

In order to understand the effect of correlation, in this section, suppose that all bidders' means and standard deviations are equal as $\mu_i = 1, \sigma_i = 0.2$, which means the only difference among bidders are their correlation coefficients. Let's consider that contractor 459 bids on a project with the price that is 5% higher and 5% lower than his mean that can be represented by conditional distribution of other i competitors such as $f_{(i|459)}(x_i | x_{459} = 1.05)$ and $f_{(i|459)}(x_i | x_{459} = 0.95)$. Changed means and standard deviations of other 7 contractors interacting with the bidding price of contractor 459 are shown in Appendix E. The relationship between correlation coefficient and bidder 459's probability of winning when bidder 459 has different strategies is shown in Figure 5-1.

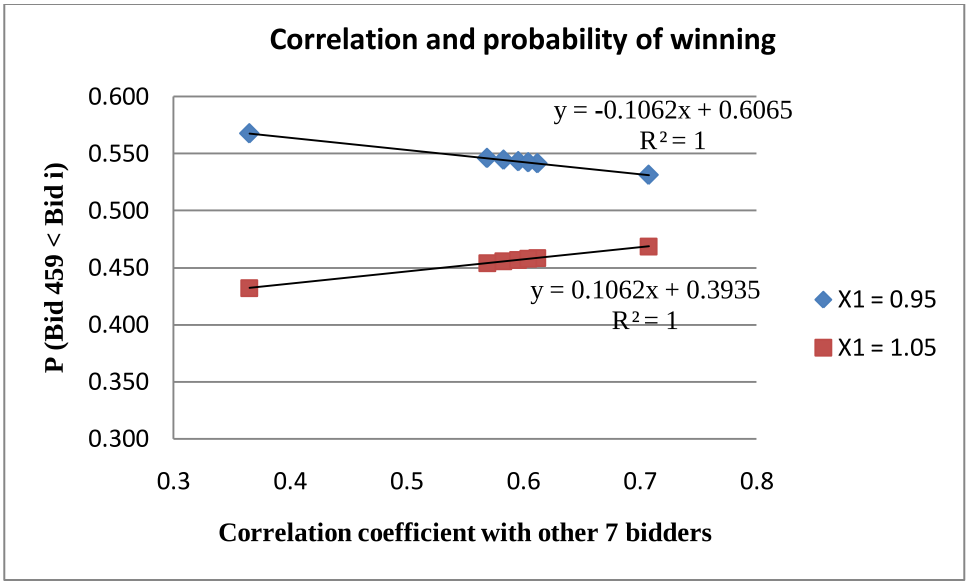

Figure 5-1. The relationship between correlation and probability of winning with given conditions

From Figure 5-1, it is obvious that lower bids ($X_{459} = 0.95$) have higher probabilities of winning than higher bids ($X_{459} = 1.05$). However, in the case in which contractor 459 estimated his bid price to be lower (0.95EE) than his average bid, a higher probability of winning would be achieved with competitors who have low correlation with the contractor. Conversely, the probability of Contractor 459 winning increases when the contractor bids against highly correlated contractors and the contractor's bid price (1.05EE) is higher than his average bid. When the contractor 459 is bidding below his historical average (0.95EE), that contractor would benefit if his competitors were uncorrelated (or even negatively correlated), so they would be less likely to bid low as well, and potentially underbid contractor 459. Conversely, when the contractor 459 is bidding above his historical average (1.05EE), that contractor would benefit if his competitors were highly correlated, so they would be more likely to bid high as well and less likely to underbid contractor 459.

5.3. Pairwise comparison with real data

To validate the points mentioned previously, a comparison between the probability of winning from the theoretical model and the historical result is conducted. In order to apply more information than correlation coefficients, means and variances of 8 contractors are calculated from their whole bidding history that is the total number of jobs bid by the contractors. It can be considered by two cases about getting information about their competitors. First case is that a firm only knows competitors' bid information when they bid on the same projects. Second one is that contractors can have access to all bidding information about other bidders through the posted bids on the DOTs' websites. The second case is applied in this study because the more data we have, the more accurate information contractors can get about their competitors. Also, in the first case, if we use only the information that both contractors bid on, there would be a different mean and variance for every pair of bidders, which would be even more complicated. In second case, there is one mean and one variance for each contractor. Moreover, since many DOTs release their bid results from their Website these days, it is relatively easy to get data on all competitors' previous bids. We do not have information on which contractors may be using these data to improve their winning percentages. The comparison between case one and case two is attached in Appendix E with the following example.

Table 5-2 shows the number of bids and correlation coefficients with contractor 459 for 7 other contractors, and means and standard deviations of eight contractors for all normalized bids from 2001 to 2009.

Table 5-2. Contractors' information about the number of bids and correlation coefficients with contractor 459, and means and standard deviations

Contractor number	Mean	Std. Dev	The number of bids with bidder 459	Correlation coefficients with bidder 459
459	1.073	0.156	-	-
540	1.025	0.183	55	0.365
45	1.108	0.268	37	0.583
103	1.032	0.160	327	0.596
441	1.015	0.166	329	0.612
349	1.047	0.159	426	0.569
431	0.990	0.161	65	0.707
180	1.007	0.196	17	0.604

With means, standard deviations, and correlation coefficients in Table 5-2, the simulation is conducted to compare the probabilities from simulations with independent case and correlated case and prior history. From the results in Figure 5-2 and Table 5-3, correlated cases are closer to the probabilities from prior history than independent case except for the one exceptional case with contractor 45. Even though average value of contractor 45 is higher than contractor 459 from their whole bidding history, the contractor 45 bid lower many times than the contractor 459 when the projects that the two bidders bid on is considered. Thus the estimated probability of winning of contractor 459 is higher than contractor 45.

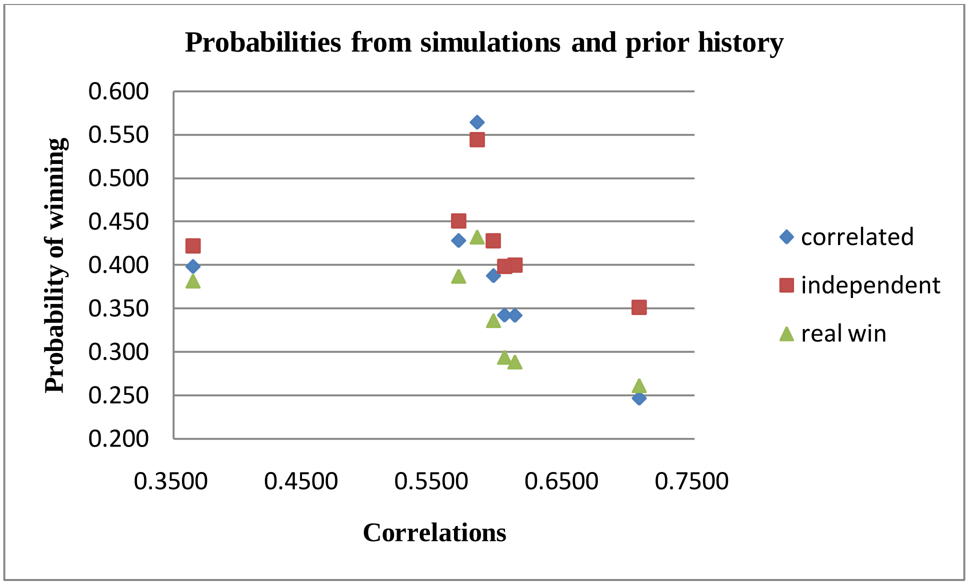

Figure 5-2. Comparison of probabilities from simulation and prior history

Table 5-3. Comparison of probabilities from simulation and prior history

Contractor number	Probabilities of win		
	Prior history	Correlated case (difference with Prior history)	Independent case (difference with Prior history)
540	0.382	0.398 (0.016)	0.422 (0.04)
45	0.432	0.564 (0.132)	0.545 (0.113)
103	0.336	0.388 (0.052)	0.428 (0.092)
441	0.289	0.342 (0.053)	0.400 (0.111)
349	0.387	0.428 (0.041)	0.451 (0.064)
431	0.262	0.246 (-0.016)	0.351 (0.089)
180	0.294	0.342 (0.048)	0.399 (0.105)

Consider the case in which contractor 459 bids with $X_{459} = 1.126$ and $X_{459} = 1.019$ that is about 5% higher and less than the mean from historical data and others bids $x_i = x_{540}, x_{45}, x_{103}, x_{441}, x_{349}, x_{431}, x_{180}$ are also same as prior history. The conditional probability density functions on X_i given condition on the value of $X_{459} = 1.126$ and $X_{459} = 1.019$ are

$$f_{(i|459)}(x_i | x_{459} = 1.126, 1.019) = \frac{f_{i,459}(x_i, x_{459})}{f_{459}(x_{459})}$$

with means $(\mu_i + \rho_{i459} \frac{\sigma_i}{\sigma_{459}}(x_{459} - \mu_{459}))$ and standard deviations $\sigma_i \sqrt{1-\rho_{459i}^2}$.

Table 5-4 shows the changed probability of winning conditional on bid prices of bidder 459, which are 5% higher and lower than his average. The conditional probabilities of winning for contractor 459 are estimated with pairwise comparison. In both cases that the contractor 459 bids 5% higher and lower than his average, the highest chance of winning is expected when he bids with contractor 45. Also, the contractor 540 shows biggest change in probability of winning with regard to conditional bid value of contractor 459.

Table 5-4. The relationship between simulation results and prior history given means and standard deviations

Contractor number	**P(Bid 459 < Bid | Bid 459=1.019) (increase)**	**P(Bid 459 < Bid | Bid 459=1.073)**	**P(Bid 459 < Bid | Bid 459=1.126) (decrease)**
540	46.0% (7.1%)	39.0%	32.3% (6.7%)
45	56.3% (0.0%)	56.3%	56.3% (0.0%)
103	43.9% (6.3%)	37.6%	31.6% (6.0%)
441	38.3% (5.3%)	33.0%	28.0% (5.0%)
349	48.9% (6.9%)	42.0%	35.4% (6.6%)
431	27.4% (4.0%)	23.3%	19.6% (3.7%)
180	36.7% (3.0%)	33.7%	30.8% (2.9%)

Figure 5-3 shows the comparison between the probabilities of winning estimated from conditional probability with average bid price and the prior history probability that the bidder 459 bids lower than the others when they bid on the same project. The relatively high R-square tells that the analysis with conditional distribution and historical results are similar, which also means 7 contractors bid history is close to normal distribution. The estimated probability of winning of bidder 459 from the Monte Carlo simulation with the past data and conditional probability given $X_{459} = 1.07$ with are shown in Appendix F.

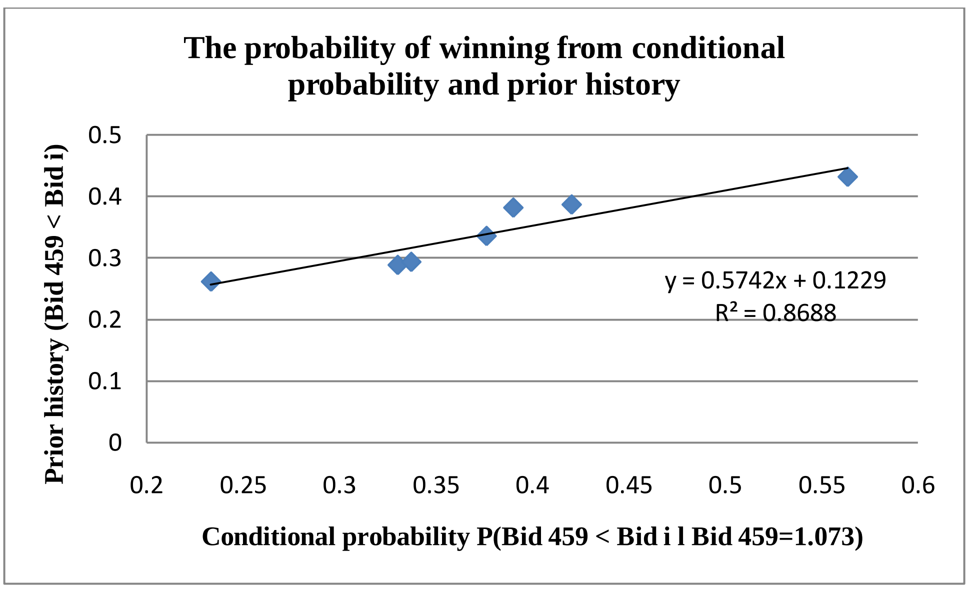

Figure 5-3. The comparison between conditional probability and real history

5.4. Trivariate case

In addition to two bidders' case, this section considers the case in which there are three bidders. The same assumption that bidding price for each bidder is normally distributed and correlated is still in effect. With their average bids of μ_i and standard deviation of σ_i, a Univariate Normal distribution of each contractor is represented in the equation (5.1). In order to explain three bidders' case with three marginal density functions, the expression about trivariate joint distribution is referenced from (Reinschmidt 2010) that is originally derived from (Burrington and May 1953).

Three random variables of x_1, x_2 and x_3 from bids of contractor 1, 2 and 3 can be represented by the trivariate joint probability density function:

$$f_{123}(x_1, x_2, x_3) = \frac{e^{-\phi/2}}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3\sqrt{|R|}} \tag{5.5}$$

where:

$$R = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{pmatrix}$$

$$\phi = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{R_{ij}(x_i-\mu_i)(x_j-\mu_j)}{|R|\sigma_i\sigma_j}$$

$|R|$ = the determinant of R $(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})$

R_{ij} = the cofactor of i^{th} row and j^{th} column of R

Suppose that bidder 3 estimates a bid price as X_3 and he wants to know the probability of winning with other two bidders based on the estimated bid price. Using Baye's Law mentioned in the previous section, the conditional probability density function on x_1, x_2 given $x_3 = X_3$:

$$\begin{aligned} f_{12|3}(x_1, x_2 \mid x_3 = X_3) &= \frac{f_{123}(x_1, x_2, x_3)}{f_3(x_3)} \\ &= \frac{\exp(\frac{-\phi_{123} + \phi_3}{2})}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}}} \end{aligned} \tag{5.6}$$

with the condition $1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23} > 0$

Two random variables of x_1 and x_2 from bids of contractor 1 and 2 conditional on $x_3 = X_3$ can be represented by the joint probability distribution shown in equation (5. 2) with revised means and standard deviations:

$$\mu_{1|3} = \mu_1 + \rho_{13}\frac{\sigma_1}{\sigma_3}(X_3 - \mu_3)$$

$$\mu_{2|3} = \mu_2 + \rho_{23}\frac{\sigma_2}{\sigma_3}(X_3 - \mu_3)$$

$$\sigma_{1|3} = \sqrt{\sigma_1^2(1-\rho_{13}^2)}$$

$$\sigma_{2|3} = \sqrt{\sigma_2^2(1-\rho_{23}^2)}$$

$$\rho_{12|3} = \sqrt{\frac{(\rho_{12} - \rho_{13}\rho_{23})^2}{(1-\rho_{13}^2)(1-\rho_{23}^2)}}$$

The conditional probability distribution on x_1, x_2 given $x_3 = X_3$ can be generated by Monte Carlo simulation with the revised means, standard deviations and correlation coefficients. By comparing those two random variables and fixed X_3, the probability of winning conditional on $x_3 = X_3$ can be estimated.

5.4.1. **The probability of winning with different conditions**

In order to study how bidders' conditional values affect the probability of winning, the probability of winning under given conditions is estimated with different means, standard deviations and correlation coefficients. Bidder 3's probability of winning for each conditional value of X_3 from 0.8 to 1.2 is estimated from conditional probability distribution with 100000 trials of Monte Carlo simulation. There are three different correlations, 0.01, 0.5, and 0.95, and all other conditions are equal having same means and standard deviations, which are 1.0 and 0.2.

Table 5-5. The conditions to generate conditional probability distribution

	Bidder 1	Bidder 2	Bidder 3
Mean	1.0	1.0	1.0
Standard deviation	0.2	0.2	0.2
Correlation coefficient among all bidders			0.1, 0.5, 0.95

According to the result shown in Figure 5-4, the sensitivity of probability of winning depending upon conditional value of bidder 3 is high when the correlation with others is low and vice versa. For independent case (correlation =0.01), the probability of winning increases substantially when bidder 3 has a smaller bid price than his average ($X_3 < 1.0$). However, for the high correlation case, if bidder 3's bid price is smaller, his probability of winning increases by a small amount. Let's assume that bidder 3 established his bid value at 0.95, which is 5% smaller than usual. The probability of winning is higher by 11.2% in the independent case but only 1.4% in highly correlated case. However, when the bidder 3 has bid value higher by 5%, his probability of winning is lower by 8.6% in the independent case and lower by 1.8% in the dependent case. Therefore, as a strategy of bidder 3, if he is trying to bid higher than normal, it is better to bid against highly correlated competitors and to avoid low correlated bidders. Conversely, when his estimated price is smaller than his average, it is better to bid with low correlated bidders. This result shows the same result with bivariate case. The specific table is shown in Appendix G.

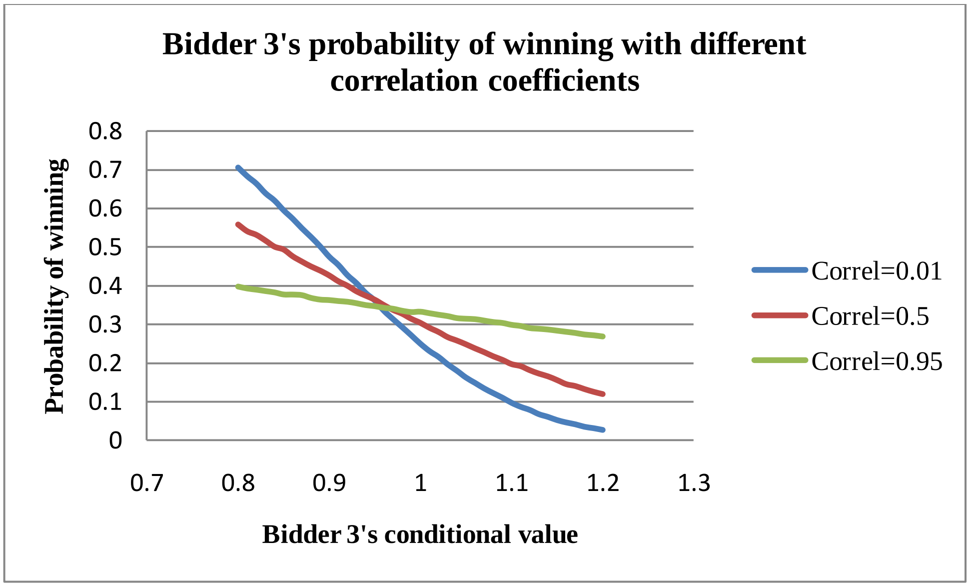

Figure 5-4. Conditional probability of winning with different correlation coefficients

Additionally, the effect of varying mean and standard deviation on the probability of winning with fixed correlation coefficient of 0.5 is shown in Figures 5-5 and 5-6. It is sure that if they have low means, the probability of winning is higher. In the case of different standard deviations case, if their bid price is lower than average, the probability of winning increases when their variation of bid price is high. In the other hand, for bidders who have small variation of bid price, the small amount of the probability of winning is lower if their bidding price is higher than normal.

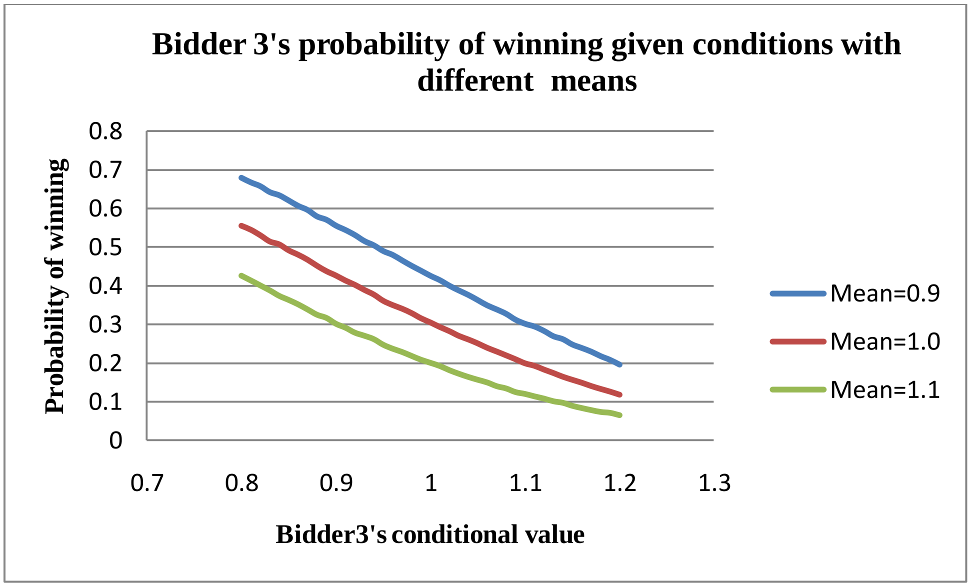

Figure 5-5. Conditional probability of winning with different means

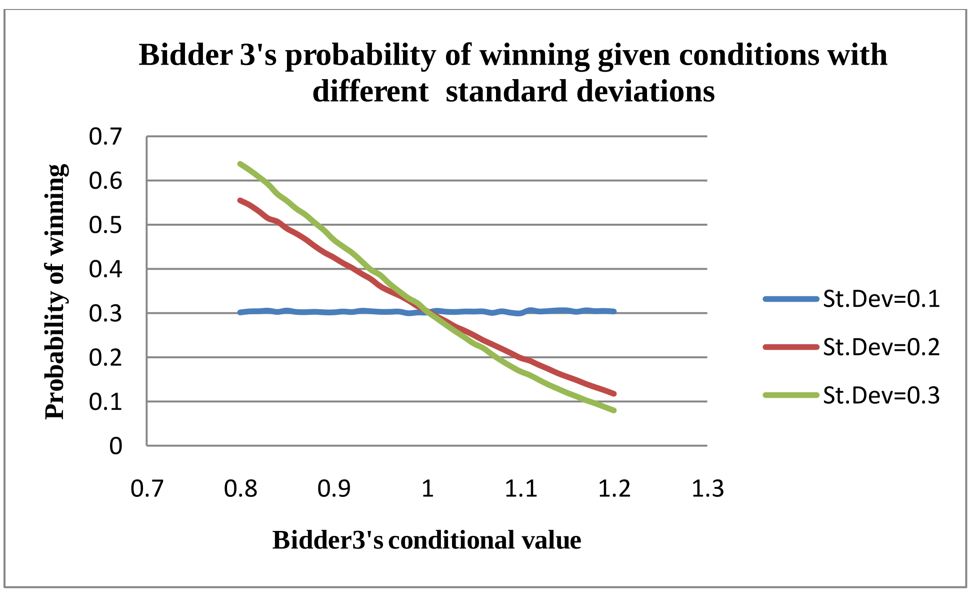

Figure 5-6. Conditional probability of winning with different standard deviations

5.4.2. **Trivariate case with real data**

In order to validate the trivariate case, a comparison between the probability of winning from the Monte Carlo simulation with three correlated bidders and their historical results when the three bidders bid on the same project is conducted. Correlation coefficients for three bidders are calculated by pairwise same as two bidders' case in the previous section. More information than correlation coefficients, which are means and variances of 3 contractors, are calculated from their whole bidding history that is the total number of jobs bid by the contractors. The Table 5-6 shows means, standard deviations and correlation coefficients of the three contractors for all normalized bids from 2001 to 2009.

Table 5-6. Three contractors' information about their correlation coefficients, means and standard deviations

	103	349	459	Mean	Std. Dev
103	1.0000	0.5070	0.5956	1.032	0.160
349	0.5070	1.0000	0.5688	1.047	0.159
459	0.5956	0.5688	1.0000	1.073	0.156

With means, standard deviations, and correlation coefficients in Table 5-6, the simulation is conducted to compare the probabilities from simulations with independent case and correlated case and prior history. From the results in Figure 5-7 and Table 5-7, correlated case is closer to the probabilities from prior history than independent case.

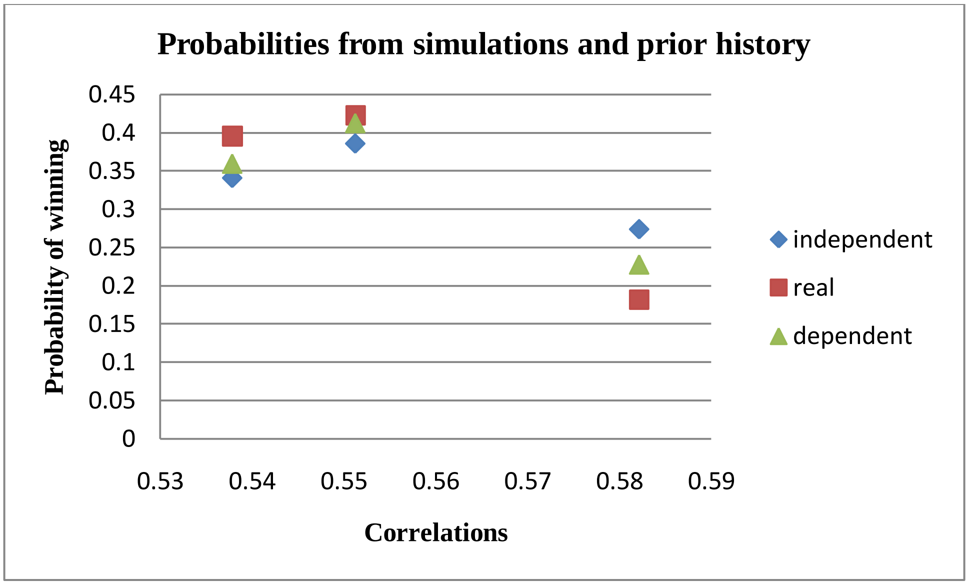

Figure 5-7. Comparison of probabilities from simulation and prior history

Table 5-7. Comparison of probabilities from simulation and prior history

Contractor number	Probabilities of winning		
	Prior history	Correlated case	Independent case
103	0.423	0.413	0.386
349	0.395	0.360	0.341
459	0.182	0.228	0.274

Similar to the pairwise case, suppose contractor 459 bids with $X_{459} = 1.126$ and $X_{459} = 1.019$, which is 5% higher and less than the average from historical data and others bids $x_i = x_{103}, x_{349}$ are also same as prior history. The conditional probability density functions on X_i given condition on the value of $X_{459} = 1.126$ and $X_{459} = 1.019$ are

$$f_{12|3}(x_{103}, x_{349} \mid x_{459} = 1.126, 1.019) = \frac{f_{103,349,459}(x_{103}, x_{349}, x_{459})}{f_{459}(x_{459})}$$

with means $(\mu_i + \rho_{i459} \frac{\sigma_i}{\sigma_{459}}(x_{459} - \mu_{459}))$, standard deviations $\sigma_i \sqrt{1-\rho_{459i}^2}$, and the correlation coefficient ($\rho_{103,349|459} = \sqrt{\frac{(\rho_{103,349} - \rho_{103,459}\rho_{349,459})^2}{(1-\rho_{103,459}^2)(1-\rho_{349,459}^2)}}$) between contractor 103 and 349.

The Table 5-8 shows the each contractor's changed probability of winning conditional on the bid price of bidder 459. With bidder 103 and 346, the probability of winning of bidder 459 increases 5.8% when they bid 5% below than their average bid price and decreases 4.9% when they bid 5% more than their average bid price.

Table 5-8. The relationship between simulation results and prior history given means and standard deviations

	103	349	459
Probability of winning conditional on Bid 459=1.073	43.1%	37.2%	19.7%
Probability of winning conditional on Bid 459=1.019 (increased probability of winning of bidder 459)	40.4%	34.1%	25.5% (5.8%)
Probability of winning conditional on Bid 459=1.126) (decreased probability of winning of bidder 459)	45.6%	39.6%	14.8% (4.9%)

Figure 5-8 shows the comparison of each bidder's probability of winning estimated from conditional probability with average bid price of bidder 459 and the probability of winning from the prior history (220 times) of three bidders. The relatively high R-square tells that the analysis with conditional distribution and historical results are similar.

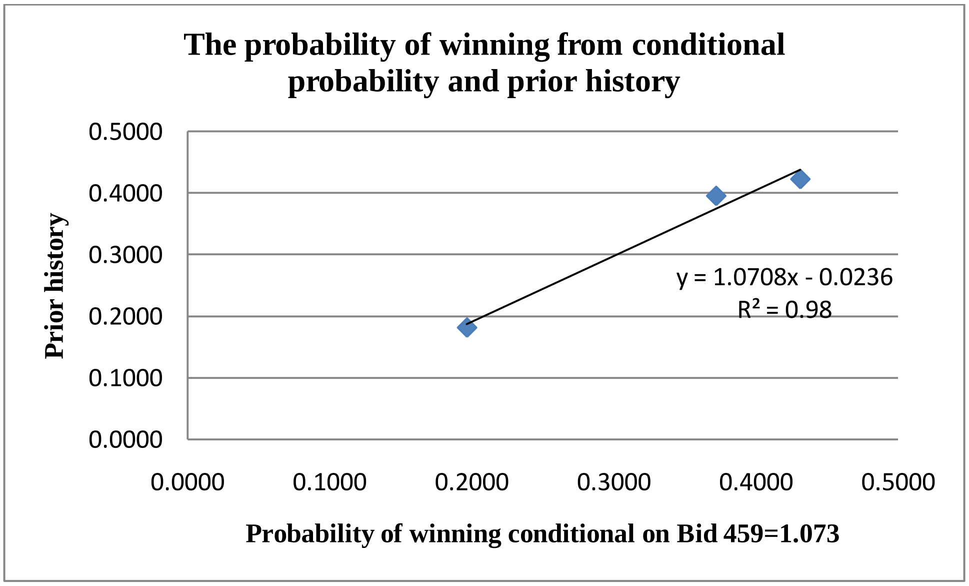

Figure 5-8. The comparison between conditional probability and real history

6. CONCLUSIONS AND RECOMMENDATIONS

This study began with the hypothesis that bids on construction projects for state DOTs are not necessarily independent, but instead are correlated. This hypothesis is supported by the MDOT data analysis, which illustrates actual bids correlated over a wide range of correlations, from 0 to 0.9. Before investigating how correlations affect the probability of winning, the characteristics of winning bids are examined using the following two hypotheses: winning bids divided by the EE is normally distributed and a larger number of bidders is associated with a lower winning bid price. With the shape of the Q-Q plot and cumulative distribution function, the hypothesis that the distribution of the winning bids divided by the EE follows a normal distribution is proven. In addition, the results of the regression analysis between the number of bidders per bid and low bids have demonstrated that the higher the number of bids, the lower is the winning bids divided by EE. It is advantageous for the DOT to have more bidders per project to decrease the awarded bid prices.

To prove the hypothesis that the correlations affect the probability that any bid will be the winning bid, a Monte Carlo simulation generating the correlated random variables was utilized to establish the relationship between the correlations and the probability of winning with the condition that the only difference among bidders in the simulation is the correlation coefficients, other things are equal. With 100,000 random trials by considering each trial as a bid situation, the percentage of contracts won by each bidder and the average correlation coefficient for each bidder was estimated. The results illustrate that the exact pattern for the correlations affects the probability of winning.

According to the hypothesis that the consideration of the correlation more accurately predicts the probability of winning than in the independent case, a Monte Carlo simulation with a means, standard deviation, and correlation coefficient from the historical data was conducted. From the results of the comparisons among the probabilities from the simulations with the independent case and correlated case and from the prior history, the case considering the correlations was more similar to the probability of winning from the prior history, than in the independent case.

The conditional probability for predicting the probability of winning in different conditions was explored. To help contractors establish their strategy for the remaining bids with their estimated bid prices and known information from the previous data, the probability of winning was found to be conditional on the knowledge of the contractors' estimated bidding price. From the results of the analysis with the conditional probability, the hypothesis that contractors with low correlations with competing contractors have a higher probability of winning than contractors with high correlations with competing bidders in the case that their estimated bid is lower than their average bid is proven. Conversely, contractors having higher correlations with competing bidders have a higher probability of winning when their estimated bid price is higher than their average bid.

This study contributes to the literature in two primary ways. One is that the consideration of interrelationships between contractors to contractors' bidding strategies and the application of enough practical data. In bidding, which involves a complex relationship between potential bidders, the consideration of interrelationships between contractors specified by correlations, in addition to considering the means and variances, would be a contribution in the competitive bidding literature. In addition, the distribution of bid values, correlations for different contractors and their probability of winning were estimated from data for over 7,000 projects.

There are a few limitations to this study, which provides an area of opportunity for improvements. Even though this study draws the general relationship between correlations and the probability of winning, the analysis is limited to relationships between two bidders and three bidders, because there is not enough bidding history to prove the case for more than three bidders continuously bidding on the same projects. In this study, correlations estimated by any contractor pair with less than 10 jobs are excluded due to the belief that fewer than 10 is a small sample, leading to inaccurate estimates of correlation. A Bootstrapping can be a way to alleviate small sample size difficulty when estimating correlations.

Similar to previous models, this study is based on the assumption that competitors will continue to bid in the same way as they have in the past. However, a contractors' bidding strategy can change, depending on their circumstances. Parameters considered in this study, such as the means, variances and correlations can change over the time span of the bidding data. It can be possible to develop dynamic models by taking into account the interrelationships between competitors, as well as the means and variances, with respect to time. In this study, it is assumed that all projects are essentially the same. However, the parameters used in this study can change with the type of project, such as pavement, bridge and miscellaneous projects. There is some evidence in the database that can be used to categorize types of projects.

The data for this study was gathered specifically from the state of Michigan, in which the bidders know the EE prior to bidding. Only eight states (LA, MA, MI, NV, OK, TX, UT, and NH) release their EE prior to bid letting (De Silva, et al. 2008). Therefore, the assumptions used in this study, that the bidders know the EE prior to bidding and the EE can be used to normalize bids, is limited to these eight states. If information about the EE is not available, the estimated bid price of a specific contractor interested in other competitors' bids can be used with the determination of an appropriate distribution, similar to Friedman's model. In this case, however, a large amount of historical data is needed to analyze other competitors' parameters, based on the estimated price of one contractor and the correlations with them. As such, the low bid value can be used to normalize the bids. With normalized values by low bids for each bidder from historical data, the same parameters used in this study are obtainable with the determination of the appropriate distribution of normalized bids, such as the lognormal distribution.

Further study may incorporate additional factors to enhance the model, such as the contractors' current work load (backlog) or economic indices. Another good topic includes how their bid/no bid decisions are associated with the correlations between contractors or with some specific contractors. Commercial projects would also be of interest. However, this study has determined that very large amounts of data are needed to obtain accurate parameters. To get a more accurate estimate of parameters with a few data points, bootstrapping would be useful to improve the accuracy of the computation of the correlation coefficients.

In conclusion, in the case that a contractor's estimated bid price is lower than his/her average bid price, it is better to bid with a high variance in the bidding price and to bid with less correlated contractors. Conversely, it is advantageous for a contractor to set the bid price with a low variance and to bid with highly correlated contractors, if the contractor's estimated bid price is lower than the average bid price.

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APPENDIX A

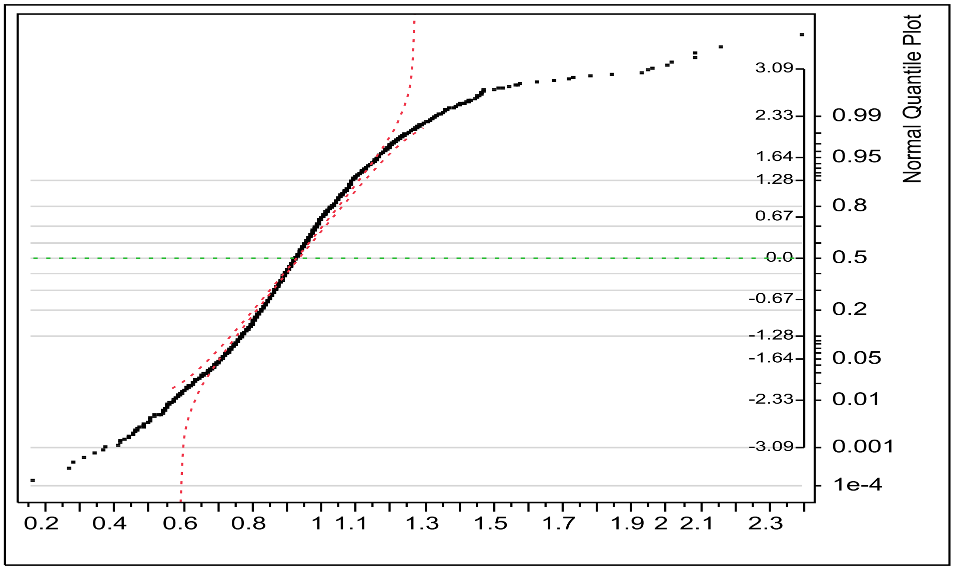

Figure A-1. Normal Quantile plot of low bids divided by EE

Table A-1. Information about the histogram in Figure 3-1

100.0%	maximum	2.39214	Mean	0.9309897
99.5%		1.43419	Std Dev	0.145655
97.5%		1.2285	Std Err Mean	0.0016897
90.0%		1.09408	Upper 95% Mean	0.9343019
75.0%	quartile	1.00455	Lower 95% Mean	0.9276774
50.0%	median	0.92562	N	7431
25.0%	quartile	0.84863	Sum	6918.1841
10.0%		0.76976	Variance	0.0212154
2.5%		0.64911	Skewness	0.8205661
0.5%		0.5271	Kurtosis	6.4069143
0.0%	minimum	0.1645	Mean	0.9309897

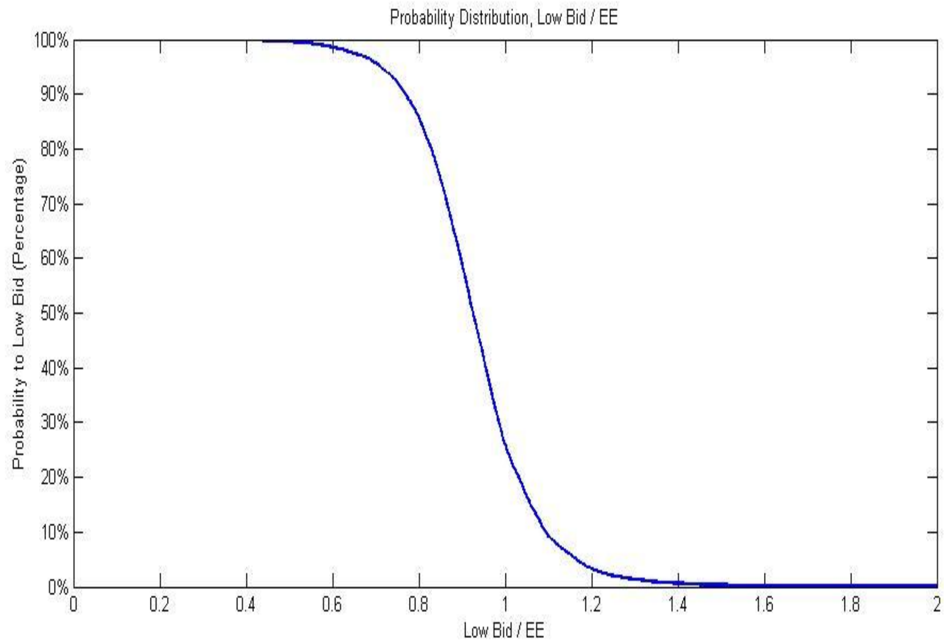

Figure A-2. Cumulative distribution of low bids divided by Engineer's Estimate for 7395 projects

APPENDIX B

Transformed Fit to Log
Low bid over EE = 1.0205791 - 0.059966*Log(N of bidder)

Summary of Fit

RSquare	0.048098
RSquare Adj	0.04797
Root Mean Square Error	0.142119
Mean of Response	0.93099
Observations (or Sum Wgts)	7431

Analysis of Variance

Source	**DF**	**Sum of Squares**	**Mean Square**	**F Ratio**
Model	1	7.58172	7.58172	375.3758
Error	7429	150.04866	0.02020	**Prob > F**
C. Total	7430	157.63039		<.0001*

Parameter Estimates

Term	**Estimate**	**Std Error**	**t Ratio**	**Prob>|t|**
Intercept	1.0205791	0.004909	207.89	<.0001*
Log(N of bidder)	-0.059966	0.003095	-19.37	<.0001*

APPENDIX C

Back computed correlation matrix, means and variances from the Monte Carlo simulation for 100,000 random trials in order to compare how close to the original parameters

Table C-1. Back computed correlation matrix

Contractor number	540	103	459	441	349	431
540	1.0000	0.5508	0.3596	0.4709	0.6150	0.5340
103	0.5508	1.0000	0.5929	0.5707	0.5009	0.7217
459	0.3596	0.5929	1.0000	0.6128	0.5675	0.7053
441	0.4709	0.5707	0.6128	1.0000	0.7185	0.5372
349	0.6150	0.5009	0.5675	0.7185	1.0000	0.5803
431	0.5340	0.7217	0.7053	0.5372	0.5803	1.0000

Table C-2. Back computed mean and Std. Dev (original mean=1, Std. Dev=0.2)

Contractor number	Mean	Std. Dev
540	0.9998	0.1994
103	0.9999	0.1999
459	1.0001	0.1997
441	0.9997	0.2000
349	1.0001	0.1998
431	1.0003	0.1993

APPENDIX D

Table D-1. Back computed covariance matrix by revised Cholesky decomposition

Contractor number	540	45	103	459	441	349	431
540	0.0399	0.0249	0.0219	0.0147	0.0189	0.0246	0.0214
45	0.0249	0.0400	0.0369	0.0234	0.0317	0.0295	0.0300
103	0.0219	0.0369	0.0399	0.0238	0.0228	0.0203	0.0290
459	0.0147	0.0234	0.0238	0.0400	0.0246	0.0228	0.0285
441	0.0189	0.0317	0.0228	0.0246	0.0402	0.0288	0.0216
349	0.0246	0.0295	0.0203	0.0228	0.0288	0.0399	0.0234
431	0.0214	0.0300	0.0290	0.0285	0.0216	0.0234	0.0402

Table D-2. Back computed covariance matrix by Eigenvalue decomposition

Contractor number	540	45	103	459	441	349	431
540	0.0399	0.0249	0.0221	0.0144	0.0185	0.0244	0.0211
45	0.0249	0.0401	0.0369	0.0232	0.0314	0.0293	0.0296
103	0.0221	0.0369	0.0402	0.0237	0.0227	0.0202	0.0288
459	0.0144	0.0232	0.0237	0.0398	0.0242	0.0225	0.0281
441	0.0185	0.0314	0.0227	0.0242	0.0398	0.0284	0.0213
349	0.0244	0.0293	0.0202	0.0225	0.0284	0.0397	0.023
431	0.0211	0.0296	0.0288	0.0281	0.0213	0.023	0.0398

Table D-3. Back computed covariance matrix by Singular value decomposition

Contractor number	540	45	103	459	441	349	431
540	0.0401	0.0252	0.0222	0.0145	0.0189	0.0248	0.0214
45	0.0252	0.0401	0.0368	0.0232	0.0316	0.0296	0.0298
103	0.0222	0.0368	0.0399	0.0236	0.0228	0.0203	0.0287
459	0.0145	0.0232	0.0236	0.0399	0.0244	0.0227	0.0282
441	0.0189	0.0316	0.0228	0.0244	0.0401	0.0287	0.0215
349	0.0248	0.0296	0.0203	0.0227	0.0287	0.04	0.0233
431	0.0214	0.0298	0.0287	0.0282	0.0215	0.0233	0.0399

APPENDIX E

Means and standard deviations from conditional probability density function of x_i given $x_{459} = 1.05, 0.95$ and probability of winning of bidder 459 from the conditional probability

Table E-1. Changed means and standard deviations by condition of $X_{459} = 1.05$

	Mean	Std. Dev	P(Bid459 < Bid i | Bid 459=1.05)
$F(x_{540} \mid x_{459} = 1.05)$	1.018	0.186	0.432
$F(x_{45} \mid x_{459} = 1.05)$	1.029	0.162	0.455
$F(x_{103} \mid x_{459} = 1.05)$	1.030	0.161	0.457
$F(x_{441} \mid x_{459} = 1.05)$	1.031	0.158	0.459
$F(x_{349} \mid x_{459} = 1.05)$	1.028	0.164	0.454
$F(x_{431} \mid x_{459} = 1.05)$	1.035	0.141	0.469
$F(x_{180} \mid x_{459} = 1.05)$	1.030	0.159	0.458

Table E-2. Changed means and standard deviations by condition of $X_{459} = 0.95$

	Mean	Std. Dev	P(Bid 459 < Bid i | Bid 459=0.95)
$F(x_{540} \mid x_{459} = 0.95)$	0.982	0.186	0.568
$F(x_{45} \mid x_{459} = 0.95)$	0.971	0.162	0.545
$F(x_{103} \mid x_{459} = 0.95)$	0.970	0.161	0.543
$F(x_{441} \mid x_{459} = 0.95)$	0.969	0.158	0.541
$F(x_{349} \mid x_{459} = 0.95)$	0.972	0.164	0.546
$F(x_{431} \mid x_{459} = 0.95)$	0.965	0.141	0.531
$F(x_{180} \mid x_{459} = 0.95)$	0.970	0.159	0.542

APPENDIX F

Figure F-1 shows the case one that used means and variances of competitors when they bid on the same projects in order to compare with the case two in the figure 5-3. The case two used in this study shows higher R-squares.

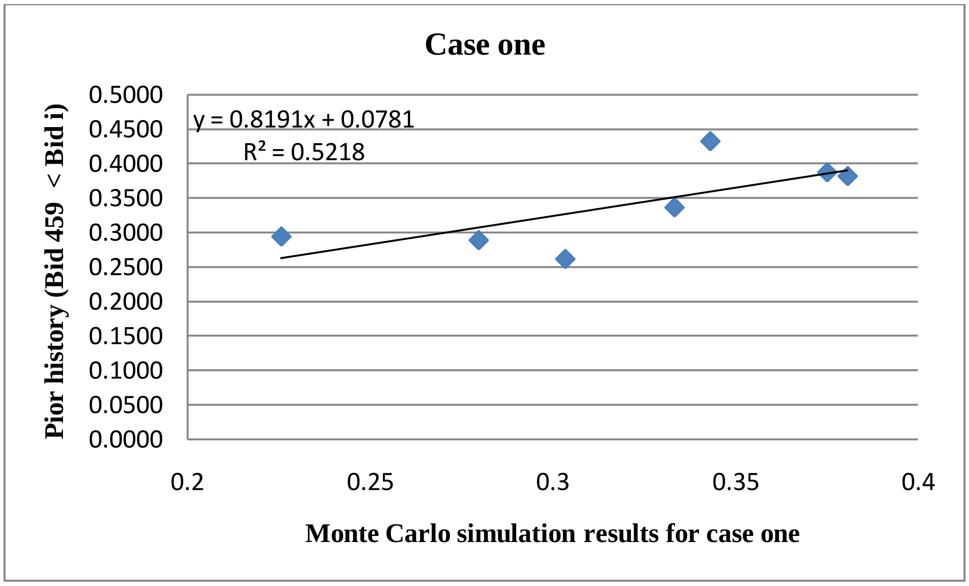

Figure F-1. The comparison of results between prior history and Monte Carlo simulation for case one

Figure F-2 shows the probability of winning estimated from the Monte Carlo simulation and prior history in order to compare with the figure 5-3 that shows the probability of winning estimated from joint probability density function conditional on the average bid value of contractor 459. From the figure F-3, it is proved that two results are almost same.

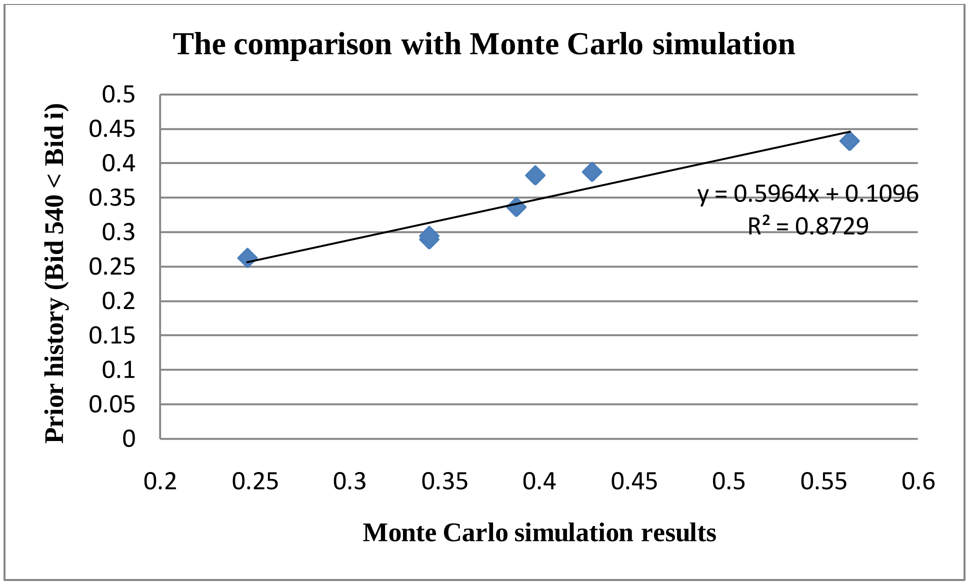

Figure F-2. The comparison of results between prior history and Monte Carlo simulation

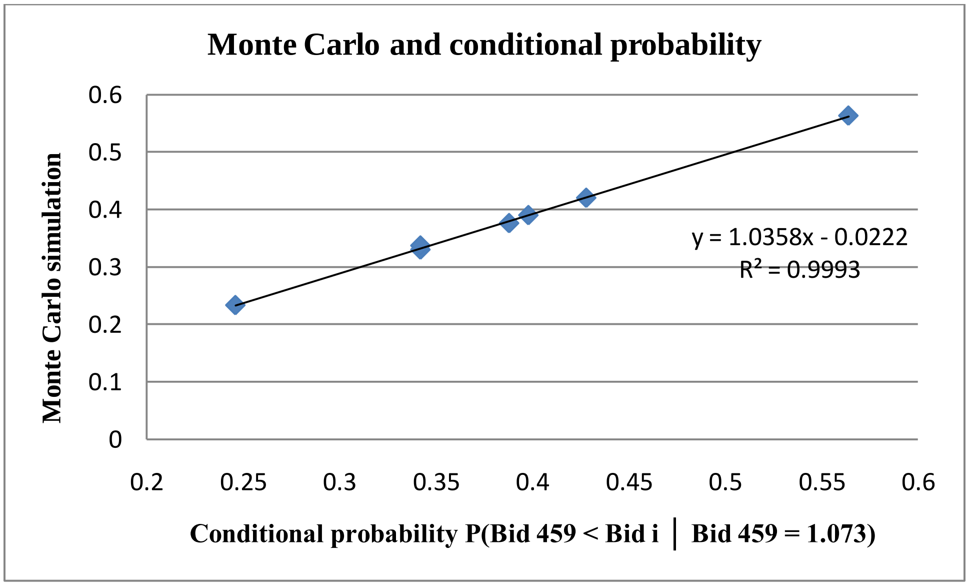

Figure F-3. The comparison of results between Monte Carlo simulation and conditional probability

APPENDIX G

Table G-1. Conditional probability of winning with different correlations

X3	Correl=0.01	Correl=0.5	Correl=0.95
0.8	0.70606	0.55857	0.39838
0.81	0.68363	0.5411	0.39339
0.82	0.6645	0.53177	0.39028
0.83	0.63963	0.51715	0.38672
0.84	0.62046	0.50123	0.38322
0.85	0.59529	0.49376	0.37789
0.86	0.57333	0.47623	0.37762
0.87	0.54887	0.46279	0.37612
0.88	0.5263	0.45017	0.36897
0.89	0.50193	0.4395	0.3646
0.9	0.4752	0.42693	0.36333
0.91	0.4539	0.41189	0.36088
0.92	0.42738	0.40033	0.35897
0.93	0.40651	0.38575	0.35511
0.94	0.38142	0.37474	0.35034
0.95	0.36193	0.36428	0.34761
0.96	0.33522	0.35034	0.34322
0.97	0.31375	0.33789	0.3409
0.98	0.29307	0.32812	0.33568
0.99	0.27184	0.315	0.33227
1	0.25006	0.30453	0.33326
1.01	0.23102	0.29172	0.32932
1.02	0.21577	0.2809	0.32539
1.03	0.19667	0.26756	0.3218
1.04	0.18033	0.25885	0.31667
1.05	0.16256	0.24912	0.31497
1.06	0.14866	0.2386	0.31389
1.07	0.13435	0.22882	0.31024
1.08	0.12211	0.21805	0.30641
1.09	0.11043	0.20899	0.30414
1.1	0.0972	0.19804	0.29923
1.11	0.08686	0.19277	0.29627
1.12	0.07871	0.18224	0.29078
1.13	0.06777	0.17354	0.28899
1.14	0.06072	0.16639	0.28709
1.15	0.05243	0.15674	0.28411
1.16	0.0464	0.14625	0.28115
1.17	0.04157	0.14134	0.27818
1.18	0.03522	0.13352	0.27417
1.19	0.03151	0.12652	0.27213
1.2	0.02705	0.12064	0.26905

Table G-2. Conditional probability of winning with different means

X3	Mean=1.1	Mean=1.0	Mean=0.9
0.8	0.42577	0.55521	0.67916
0.81	0.41342	0.54485	0.66723
0.82	0.40065	0.53066	0.65741
0.83	0.38787	0.51457	0.64217
0.84	0.37369	0.50705	0.634
0.85	0.36321	0.49167	0.62057
0.86	0.35181	0.48032	0.60667
0.87	0.3386	0.46731	0.596
0.88	0.32504	0.45172	0.57936
0.89	0.31679	0.43771	0.57067
0.9	0.30168	0.42646	0.55562
0.91	0.29166	0.41359	0.5444
0.92	0.27883	0.40249	0.53148
0.93	0.2709	0.38942	0.51594
0.94	0.2621	0.37731	0.5049
0.95	0.2476	0.36115	0.49007
0.96	0.23741	0.35006	0.48003
0.97	0.22892	0.34067	0.46588
0.98	0.2188	0.32931	0.45197
0.99	0.20894	0.31587	0.43919
1	0.20095	0.30503	0.4261
1.01	0.19268	0.29319	0.41475
1.02	0.18196	0.2826	0.40052
1.03	0.17283	0.27039	0.38825
1.04	0.16454	0.26115	0.37673
1.05	0.15727	0.25089	0.36337
1.06	0.1503	0.23949	0.34978
1.07	0.14077	0.22988	0.33933
1.08	0.13487	0.21998	0.32795
1.09	0.12546	0.20989	0.31273
1.1	0.12075	0.19937	0.30233
1.11	0.1145	0.1929	0.29515
1.12	0.10874	0.18324	0.28392
1.13	0.10187	0.17429	0.27014
1.14	0.09789	0.1647	0.26251
1.15	0.09015	0.15677	0.24882
1.16	0.08435	0.14917	0.23979
1.17	0.07904	0.14064	0.23033
1.18	0.07428	0.13317	0.21872
1.19	0.07196	0.12614	0.20898
1.2	0.06592	0.11822	0.19683

Table G-3. Conditional probability of winning with different standard deviations

X3	St.dev=0.1	St.dev=0.2	St.dev=0.3
0.8	0.30234	0.55521	0.638
0.81	0.30451	0.54485	0.62415
0.82	0.30484	0.53066	0.60848
0.83	0.30594	0.51457	0.59174
0.84	0.3034	0.50705	0.56957
0.85	0.30612	0.49167	0.55431
0.86	0.30344	0.48032	0.53679
0.87	0.30296	0.46731	0.52264
0.88	0.30368	0.45172	0.50452
0.89	0.30261	0.43771	0.48747
0.9	0.30262	0.42646	0.46652
0.91	0.30415	0.41359	0.45108
0.92	0.30324	0.40249	0.43638
0.93	0.30575	0.38942	0.41751
0.94	0.3048	0.37731	0.39831
0.95	0.30358	0.36115	0.38616
0.96	0.30359	0.35006	0.36651
0.97	0.30412	0.34067	0.34988
0.98	0.30079	0.32931	0.3343
0.99	0.30231	0.31587	0.32237
1	0.3023	0.30503	0.30363
1.01	0.30554	0.29319	0.2889
1.02	0.3039	0.2826	0.27389
1.03	0.30325	0.27039	0.25929
1.04	0.30428	0.26115	0.24562
1.05	0.30409	0.25089	0.2313
1.06	0.30442	0.23949	0.22119
1.07	0.30153	0.22988	0.2059
1.08	0.30466	0.21998	0.19206
1.09	0.30175	0.20989	0.17948
1.1	0.30065	0.19937	0.16806
1.11	0.30687	0.1929	0.15982
1.12	0.30435	0.18324	0.14877
1.13	0.30525	0.17429	0.13813
1.14	0.30651	0.1647	0.12892
1.15	0.30647	0.15677	0.11967
1.16	0.30368	0.14917	0.11159
1.17	0.30657	0.14064	0.10294
1.18	0.30505	0.13317	0.09591
1.19	0.30548	0.12614	0.08776
1.2	0.30428	0.11822	0.07976

VITA

Name Minsoo Kim

Address Zachry Department of Civil Engineering

Texas A&M University

College Station Texas 77843-3136

Email mins66@tamu.edu

Education B.S. 2009 Kyungwon University, South Korea

M.S. 2011 Texas A&M University, College Station, USA