STATIC, CYLINDRICAL SYMMETRY IN GENERAL RELATIVITY AND VACUUM ENERGY

A Senior Scholars Thesis

by

CYNTHIA TRENDAFILOVA

Submitted to the Office of Undergraduate Research
Texas A&M University
in partial fulfillment of the requirements for the designation as

UNDERGRADUATE RESEARCH SCHOLAR

April 2011

Majors: Mathematics
Physics

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Director for Undergraduate Research:	Sumana Datta

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ABSTRACT

Static, Cylindrical Symmetry in General Relativity and Vacuum Energy. (April 2011)

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In the first section of my research, in analogy with the standard derivation of the spherically symmetric Schwarzschild solution of the Einstein field equations, I find all static, cylindrically symmetric solutions of the Einstein equations for vacuum. These include not only the well known cone solution, which is locally flat, but others in which the metric coefficients are powers of the radial coordinate and the space-time is curved. These solutions appear in the literature, but in different forms, corresponding to different definitions of the radial coordinate. I find expressions for transforming between these different metric forms and examine some special points of interest. I then examine some special cases of non-vacuum solutions of the equations as well. Because all the vacuum solutions are singular on the axis, I match them to interior solutions with nonvanishing energy density and pressure. In addition to the well known cosmic string solution joining on to the cone, we find some numerical solutions that join on to the other exterior solutions. I then consider only a static, flat, cylindrically symmetric space-time. I calculate the components of the stress-energy tensor in terms of the cylinder kernel and its derivatives. The cylinder kernel in cylindrical coordinates has been previously calculated and can be used to find the energy density and pressure on various cylindrical boundaries; future work will include

finding these quantities for various cylindrically symmetric geometries.

DEDICATION

To my father

ACKNOWLEDGMENTS

I thank my faculty advisor, Dr. Stephen Fulling, for all of his help and guidance throughout the research and writing processes. I also thank David Garfinkle for helpful remarks and suggestions.

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CHAPTER I

INTRODUCTION

Gravity is one of the four fundamental forces which govern our universe. With Einstein's formulation of his general theory of relativity, we gained a deeper understanding of how matter shapes our universe and interacts according to the gravitational force. The presence of matter adds curvature to space-time, which can be described using the metric tensor. The Einstein field equations, given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu}, \tag{1.1}$$

describe how the presence of matter, described by the stress-energy-momentum tensor, $T_{\mu\nu}$, is related to the Einstein tensor, $G_{\mu\nu}$, which is a quantity calculated through derivatives of the metric tensor, $g_{\mu\nu}$, describing the space-time of interest.

A particular case commonly studied in textbooks is that of spherical symmetry, in which the metric is written as a general static, spherically symmetric expression, where the components are functions of the radial coordinate only. The Einstein tensor can be calculated from this, and one can solve the Einstein equations for vacuum, in which there are no sources and the stress-energy-momentum tensor is zero everywhere, yielding the well-known Schwarzschild solution which describes the region exterior to a star [1, p.262]. The Einstein equations for the spherically symmetric metric can also be solved exactly for certain other cases of $T_{\mu\nu}$.

However, a far less studied case is that of cylindrical symmetry; it does not possess the

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This thesis follows the style of the *European Journal of Physics*.

same simplifications and is not encountered as often as the spherical case. For a cylindrically symmetric metric, there is translational symmetry along an axis and also rotational symmetry around said axis. In the case when one is also dealing with a static situation, the metric can be written as

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\phi^2 + e^{2\Psi}dz^2, \tag{1.2}$$

where Φ, Λ, and Ψ are functions of r only, similarly to the spherically symmetric case. General solutions of the Einstein equations for vacuum have been found previously for this case, including for some non-static situations and cases breaking translational invariance, attributed originally to Weyl and Levi-Civita in the early 20th century [2, 3]. Analogous solutions which broke the condition of a static metric have also been studied by Rosen and Marder in the mid 20th century [4].

Furthermore, in the 1980s, attention was given to the study of cosmic strings, which are thin cylinders, usually filled with a non-Abelian gauge field, surrounded by vacuum. The space outside of such strings is described by a cone, which is regular Minkowski space-time minus a wedge characterized by a deficit angle, and the corresponding space-time is locally flat [5]. However, the static, cylindrically symmetric cases of the solutions discovered previously are not all restricted to this type, and there are more general solutions than just cones. It is possible that the other solutions correspond to some more general equation of state describing matter inside a string, or could be useful in some other physical situation.

In this research, I derive these solutions, but expressed in a slightly different form from what has been done previously, writing them instead using a convention which seems most natural to me. By writing a general expression for a static, cylindrically symmetric metric and calculating the corresponding Einstein tensor, I verify by solving the system of differential equations resulting from the vacuum Einstein equations that the previously found solutions are indeed the most general static, cylindrically symmetric ones. It is expected that all solutions involve metric coefficients which are powers of r; once the solutions are found, I also examine some of their properties. Previous authors found solutions using different conventions for writing the general form of a cylindrically symmetric metric. It is possible to transform between these various forms by rescaling coordinates, and I find and present the relationships for doing so.

Furthermore, I acquire some solutions of the Einstein equations for a cylindrically symmetric metric and some nonzero components of the stress-energy-momentum tensor. There are some simple cases which allow for easily obtainable exact solutions from the differential equations, which I solve in agreement with previous work on cosmic string solutions by Gott and others [6, 7]. For other cases I use Mathematica's numerical solving algorithms to calculate numerical solutions instead, after choosing appropriate boundary conditions. I expect to find solutions for a string of finite radius which can then be connected at the boundary to the already known vacuum solutions.

After studying the situation of cylindrical symmetry as related to general relativity, I also examine its relationship to another area of physics, quantum field theory. The theory of quantum mechanics states that certain physical quantities are quantized and can only occur in discrete amounts, and it describes the behavior of matter at very small scales. Although useful, the theory as developed in the 1920s lacked the scope to describe certain phenomena such as relativistic situations and production and annihilation of particles; by extending it to describe fields, rather than fixed numbers of particles, and taking the fields as the basic physical objects instead, such issues could then be addressed [8, p.48].

By applying quantum field theory, one discovers that vacuum itself usually has a nonzero vacuum energy even where there is no matter present [8, p.96]. This vacuum energy can be calculated for various geometries, and it has been done previously for the cases of flat plates and spherical geometries [9]. However, there are still problems with the theory which are not fully understood, such as those regarding the energy-balance equation,

$$\frac{\partial E}{\partial h} = -\int_S p_h, \tag{1.3}$$

where h is a general parameter, p_h is the pressure along h, and S is the area of interest. In the cutoff theory used to calculate the energy, this equation is violated [9]; perhaps examining some properties of cylindrical geometry may provide additional insight into these current paradoxes.

Schwartz-Perlov and Olum have calculated the components of the stress-energy tensor previously for the case of a static, spherically symmetric system [10], and the pressure on a boundary has been calculated previously for the spherical case as well [9]. This has also been done for flat, perfectly reflecting boundaries, but it becomes more complicated when the boundaries become curved, because the simple method of images no longer applies; however, there are several methods that can be used [11].

I proceed to calculate the components of the stress-energy tensor of a quantized scalar field for a static, cylindrically symmetric system, in the case of locally flat space rather than the general cylindrically symmetric case, using a method analogous to the one utilized by Schwartz-Perlov and Olum. They first calculate the components of the stress-energy tensor on the x-axis and then generalize this result to radial and tangential pressures; a similar procedure can be used for the case of cylindrical symmetry [10]. One can then take these components and express them in terms of the known cylinder kernel in order to calculate the pressure on a cylindrical, perfectly reflecting boundary. Future work will involve calculating the pressure and energy density for various situations. One can make certain approximations with the free cylinder kernel; one can make a first-order optical approximation by adding or subtracting a single term from the known free cylinder kernel, or one can use the Multiple-Reflection Expansion and a formula developed by Liu to get a better approximation [11]. A continuation of the work in this thesis will examine the properties that arise in these situations, in the hopes of shedding some insight into the apparent paradoxes present in the theory.

CHAPTER II

STATIC, CYLINDRICAL SYMMETRY IN GENERAL RELATIVITY

In this chapter, solutions of Einstein's equations are presented for various static, cylindrically symmetric cases.

Vacuum solution of Einstein equations

In writing a general expression for a metric exhibiting cylindrical symmetry, we require that it must have axial symmetry and thus the metric components must be independent of the angular coordinate, ϕ; we also require translational symmetry along z, so the coefficients must be independent of z as well. In our case we are also examining only the static situation, so the metric components must be independent of t, leaving any unknown functions to be functions of the radial variable, r, only. In analogy to the standard treatment of spherical symmetry [1], we define r so that the coefficient of $d\phi^2$ is equal to r^2. Later we discuss alternative conventions, and also the question of whether any generality is lost by this convention. Thus the metric can be written

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\phi^2 + e^{2\Psi}dz^2, \tag{2.1}$$

where Φ, Λ, and Ψ are the unknown functions of r for which we would like to solve. By writing our unknown functions in the form of exponentials, we guarantee that our coefficients will be positive as we would like them to be, and also mirror the standard textbook treatment of the spherically symmetric metric. The form in which we have written the metric does not restrict the range of ϕ to be from 0 to 2π; instead it runs from 0 to some angle ϕ_*. As we shall show later, ϕ can be forced to fill an angle of 2π by rescaling ϕ and bringing in an additional numerical factor multiplying the angular term, or by also rescaling r and bringing in a numerical factor multiplying the dr^2 term.

The standard known expressions for the Christoffel symbols ($\Gamma^{\gamma}_{\beta\mu}$), Riemann curvature tensor ($R^{\alpha}_{\beta\mu\nu}$), and Ricci tensor ($R_{\alpha\beta}$) associated with a given metric are as follows [1]:

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}), \tag{2.2}$$

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}, \tag{2.3}$$

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}, \tag{2.4}$$

(where for our static, cylindrically symmetric metric, $g_{tt} = -e^{2\Phi}$, $g_{rr} = e^{2\Lambda}$, $g_{\phi\phi} = r^2$, $g_{zz} = e^{2\Psi}$, and all other metric components are zero). All of the components of these objects can be calculated for this metric, and the results are presented below.

Nonzero Christoffel Symbols:

$$\begin{aligned} \Gamma^{t}_{tr} = \Gamma^{t}_{rt} &= \Phi' \\ \Gamma^{r}_{tt} &= \Phi' e^{2(\Phi-\Lambda)} \\ \Gamma^{r}_{rr} &= \Lambda' \\ \Gamma^{r}_{\phi\phi} &= -re^{-2\Lambda} \\ \Gamma^{r}_{zz} &= -\Psi' e^{2(\Psi-\Lambda)} \\ \Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} &= \frac{1}{r} \\ \Gamma^{z}_{rz} = \Gamma^{z}_{zr} &= \Psi' \end{aligned} \tag{2.5}$$

Nonzero Riemann Curvature Tensor Components:

$$\begin{aligned} R^{t}{}_{\phi\phi t} &= r\Phi' e^{-2\Lambda} \\ R^{r}{}_{\phi\phi r} &= -r\Lambda' e^{-2\Lambda} \\ R^{r}{}_{zzr} &= (\Psi'' + \Psi'^2 - \Psi'\Lambda')e^{2(\Psi-\Lambda)} \end{aligned}$$

$$R^r{}_{ttr} = -(\Phi'' + \Phi'^2 - \Phi'\Lambda')e^{2(\Phi-\Lambda)} \tag{2.6}$$
$$R^z{}_{ttz} = -\Psi'\Phi' e^{2(\Phi-\Lambda)}$$
$$R^z{}_{\phi\phi z} = r\Psi' e^{-2\Lambda}$$

Nonzero Ricci Tensor Components:

$$\begin{aligned}
R_{tt} &= (\Phi'' + \Phi'^2 - \Phi'\Lambda' + \frac{1}{r}\Phi' + \Psi'\Phi')e^{2(\Phi-\Lambda)} \\
R_{rr} &= -\Phi'' - \Phi'^2 + \Phi'\Lambda' + \frac{1}{r}\Lambda' - \Psi'' - \Psi'^2 + \Lambda'\Psi' \\
R_{\phi\phi} &= r(\Lambda' - \Phi' - \Psi')e^{-2\Lambda} \\
R_{zz} &= -(\Psi'' + \Psi'^2 - \Psi'\Lambda' + \Psi'\Phi' + \frac{1}{r}\Psi')e^{2(\Psi-\Lambda)}
\end{aligned} \tag{2.7}$$

Primes correspond to differentiation with respect to r, e.g., $\Phi' = \frac{d\Phi}{dr}$.

In general, space-time is described by the Einstein field equations [1],

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu}. \tag{2.8}$$

The presence of matter, described by the stress-energy-momentum tensor $T_{\mu\nu}$, affects the curvature of space-time as described by the metric tensor, $g_{\mu\nu}$, and its derivatives as contained in the Einstein tensor, $G_{\mu\nu}$. The quantity Λ is the cosmological constant, which we will take to be zero, and k is a constant which is equal to 8π, reducing the equations to

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{2.9}$$

We would like to solve the Einstein field equations for the vacuum solution, when there are no sources present and $T_{\mu\nu} = 0$; this corresponds to $G_{\alpha\beta} = 0$. It is easy to show, however, that it is sufficient to calculate the solutions for $R_{\alpha\beta} = 0$. We begin with the standard definition of the Einstein tensor, $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$ [1]. From this we can calculate the trace of the Einstein tensor $G := G^{\mu}{}_{\mu} = R^{\mu}{}_{\mu} - \frac{1}{2}Rg^{\mu}{}_{\mu} = R - 2R = -R$ and thus obtain the following relation between the Ricci and Einstein tensors: $R_{\alpha\beta} = G_{\alpha\beta} - \frac{1}{2}Gg_{\alpha\beta}$. Thus we see that if $R_{\alpha\beta} = 0$ then $G_{\alpha\beta} = 0$, and conversely, if $G_{\alpha\beta} = 0$ then $R_{\alpha\beta} = 0$. Thus the solutions to $R_{\alpha\beta} = 0$ are also the solutions to the vacuum Einstein field equations, $G_{\alpha\beta} = 0$.

By equating the nontrivial components of the Ricci tensor with zero, we obtain a set of four ordinary differential equations for Φ, Λ, and Ψ. We further note that the exponential function is never equal to zero, so the differential equations reduce to

$$(\Phi'' + \Phi'^2 - \Phi'\Lambda' + \frac{1}{r}\Phi' + \Psi'\Phi') = 0, \tag{2.10}$$

$$-\Phi'' - \Phi'^2 + \Phi'\Lambda' + \frac{1}{r}\Lambda' - \Psi'' - \Psi'^2 + \Lambda'\Psi' = 0, \tag{2.11}$$

$$(\Lambda' - \Phi' - \Psi') = 0, \tag{2.12}$$

$$-(\Psi'' + \Psi'^2 - \Psi'\Lambda' + \Psi'\Phi' + \frac{1}{r}\Psi') = 0. \tag{2.13}$$

We see that (2.12) can be solved for Λ' in terms of the other two unknown functions ($\Lambda' = \Phi' + \Psi'$), which can then be substituted into (2.10), (2.11), and (2.13) to eliminate Λ'. Thus this system can be reduced to

$$\Lambda' = \Phi' + \Psi', \tag{2.14}$$

$$\Phi'' + \frac{1}{r}\Phi' = 0, \tag{2.15}$$

$$\Psi'' + \frac{1}{r}\Psi' = 0, \tag{2.16}$$

$$\Phi'\Psi' + \frac{1}{r}\Phi' + \frac{1}{r}\Psi' = 0. \tag{2.17}$$

Now (2.14), (2.15), and (2.16) are linear second-order equations easily solved by separation of variables. For example, let $A = \frac{d\Phi}{dr}$; then (2.19) becomes $\frac{dA}{dr} = -\frac{1}{r}A$ or $dA\frac{1}{A} = -\frac{1}{r}dr$, which can be integrated to yield $\ln(A) = -\ln(r) + \ln(a_1)$. This yields $A = \frac{d\Phi}{dr} = a_1\frac{1}{r}$ which integrates to $\Phi = \ln(r_1^a) + \ln(a_2)$. Similarly, $\Psi = \ln(r^{b_1}) + \ln(b_2)$ and $\Lambda =$

$\ln(r^{a_1+b_1})+\ln(c)$. Also, substituting these solutions into (2.17) provides the additional constraint that $a_1b_1+a_1+b_1=0$. Thus the static, cylindrically symmetric metric is

$$ds^2 = -a_2^2 r^{2a_1}dt^2 + c^2 r^{2(a_1+b_1)}dr^2 + r^2 d\phi^2 + b_2^2 r^{2b_1}dz^2 \tag{2.18}$$

with $0 \le \phi < \phi_*$. The multiplicative constants a_2 and b_2 can easily be absorbed by a linear rescaling of t and z, resulting in

$$ds^2 = -r^{2a_1}dt^2 + c^2 r^{2(a_1+b_1)}dr^2 + r^2 d\phi^2 + r^{2b_1}dz^2; \tag{2.19}$$

after each change of variables in what follows, we shall carry out this procedure again without comment. Here we have shown that the coefficients must be powers of r as in (2.19), with $a_1b_1+a_1+b_1=0$. Since we no longer have to worry about the constants a_2 and b_2, we now drop the subscripts on a_1 and b_1, and simply write the constraint as

$$ab+a+b=0. \tag{2.20}$$

The form of the metric derived here is in agreement with equation (3) of Marder [4], with $C=-b_1$, and the metric coefficients are powers of r as also found in previous work by Weyl and Levi-Civita. The constant c can also be absorbed by rescaling r, which affects the $d\phi^2$ term by bringing out another constant in front, resulting in

$$ds^2 = -r^{2a}dt^2 + r^{2(a+b)}dr^2 + K^2 r^2 d\phi^2 + r^{2b}dz^2. \tag{2.21}$$

This leads to two natural conventions for the $d\phi^2$ term. One can now rescale ϕ so that the constant K^2 is absorbed, thus redefining the range ϕ_* of ϕ. One could instead rescale ϕ to fix its range to be from 0 to 2π, in which case the constant remains, multiplying either $d\phi^2$ as in (2.21) or dr^2 as in (2.19). In the work that follows, we use the first convention,

$$ds^2 = -r^{2a}dt^2 + r^{2(a+b)}dr^2 + r^2 d\phi^2 + r^{2b}dz^2, \tag{2.22}$$

where the arbitrary constant is hidden in the periodicity, ϕ_*.

Various forms for cylindrical metrics

We now examine in greater detail the relationship between a and b, which is given in (2.20) and illustrated in Figure 1.

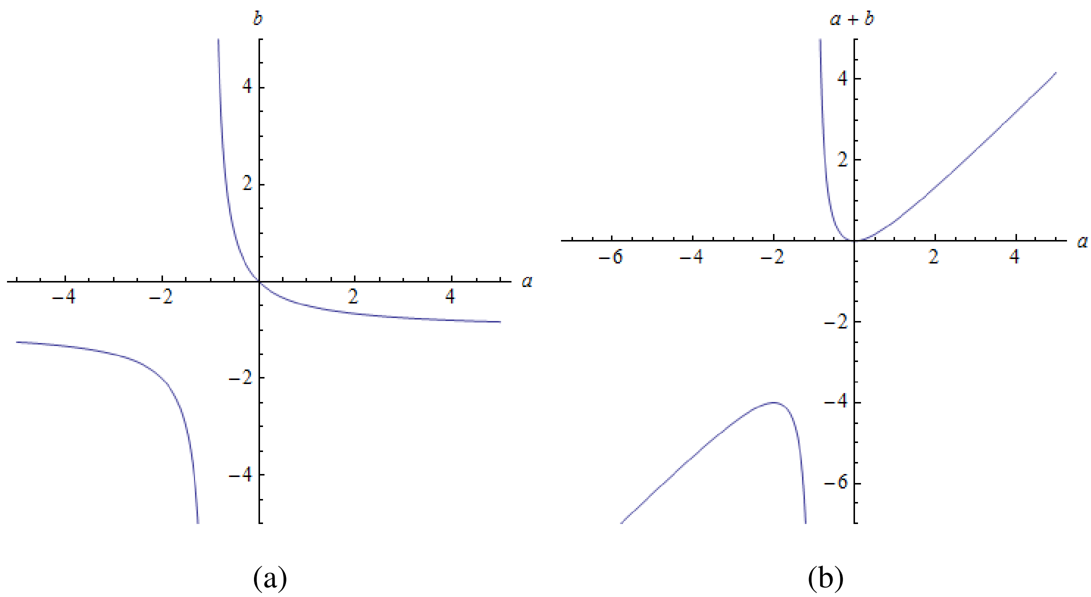


Figure 1. (a) Plot of the relationship between a and b; (b) plot of the relationship between a and $a+b$.

We note the existence of several special points on these graphs and examine their significance in various different forms of writing the cylindrical metric. One such point is $a = b = 0$, which reduces the metric of (2.22) to

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2. \tag{2.23}$$

This describes a cone; that is, flat space missing a wedge of deficit angle $\Delta\phi = 2\pi - \phi_*$. If $\phi_* > 2\pi$, a wedge is added. Ordinary Minkowski space arises as the very special case $\phi_* = 2\pi$. It is also useful to note the symmetry between the t and z coordinates, along with the Lorentz symmetry under boosts in the z direction.

Another point of interest is $b = -1$, in which case $a \to \infty$. The significance of this (apparently singular) case can be better demonstrated if we rescale r, t, and z of equation (2.20) (with the rescaling for r given explicitly later in this chapter) to write the metric in the form [4]:

$$ds^2 = -r^{-2b}dt^2 + r^{2(1+b)}d\phi^2 + A^2 r^{2b(1+b)}(dr^2 + dz^2) \tag{2.24}$$

with $A := c(1+b)$. If we now treat A as the arbitrary constant instead of b, the metric remains nonsingular when $b = -1$ in the other terms. After rescaling r with $\bar{r} = A^{1/[b(1+b)-1)]}r$ to absorb A, one gets

$$ds^2 = -\bar{r}^{-2b}d\bar{t}^2 + A^{(2(1+b))/(-b(1+b)-1)}\bar{r}^{2(1+b)}d\phi^2 + \bar{r}^{2b(1+b)}d\bar{r}^2 + \bar{r}^{2b(1+b)}d\bar{z}^2. \tag{2.25}$$

We must now rescale ϕ as well in order to absorb the final constant in front of the $d\phi^2$ term; this changes the range ϕ_*. The metric becomes

$$ds^2 = -\bar{r}^{-2b}d\bar{t}^2 + \bar{r}^{2(1+b)}d\bar{\phi}^2 + \bar{r}^{2b(1+b)}d\bar{r}^2 + \bar{r}^{2b(1+b)}d\bar{z}^2, \tag{2.26}$$

and at the point where $b = -1$ it reduces to

$$ds^2 = -\bar{r}^2 d\bar{t}^2 + d\bar{\phi}^2 + d\bar{r}^2 + d\bar{z}^2. \tag{2.27}$$

Under the transformation $T = \bar{r}\sinh\bar{t}$ and $R = \bar{r}\cosh\bar{t}$ we see that (2.27) is locally equivalent to flat space,

$$ds^2 = -dT^2 + d\bar{\phi}^2 + dR^2 + d\bar{z}^2 \tag{2.28}$$

with $\bar{\phi}$ a periodic coordinate.

We also note that the general relationship in (2.20) is symmetric when a and b are switched, corresponding to switching z and t. This observation suggests that the case $a = -1$, $b \to \infty$ is parallel to the foregoing one. To see its physical significance, we can write the metric of (2.19) in the form [4]

$$ds^2 = L^{-2b} r^{2a^2+2a}(-dt^2 + dr^2) + L^{2(1+b)} r^{2+2a} d\phi^2 + L^{-2(1+b)} r^{-2a} dz^2 \qquad (2.29)$$

with $L := ((1+b)/K)^{(1+b)^{-2}}$. At the point where $a = -1$ and $b \to \infty$, this reduces to

$$ds^2 = -dt^2 + dr^2 + d\phi^2 + r^2 dz^2. \qquad (2.30)$$

Under the transformation $Z = r\sin z$ and $R = r\cos z$ we get

$$ds^2 = -dt^2 + d\phi^2 + dR^2 + dZ^2, \qquad (2.31)$$

and this once again looks like flat space locally, but with ϕ still a periodic coordinate. We note that these locally flat solutions are not included in the general solution found previously because there we fixed the coefficient of $d\phi^2$ to be r^2, whereas in (2.28) and (2.31) that coefficient is a constant.

Transforming between metric conventions

We now examine how transforming from (2.19) to the other metric forms affects the radial coordinate r. To go from (2.19) to (2.24), which corresponds to the convention used by Weyl and Levi-Civita in solving for a general cylindrical metric in 1917 [2], we must use $\bar{r} = (c/(a+1))r^{a+1}$. We see that in this case, the exponent of r is negative whenever $a < -1$ or, equivalently, $b < -1$. Under this condition, $r = 0$ in our gauge choice corresponds to $\bar{r} = \infty$ in Weyl's gauge choice. To go from our form of the metric to that of

(2.29), attributed to Rosen [4], we require $\bar{r} = (c/(b+1))r^{b+1}$. Once again, the exponent of r is negative whenever $b < -1$ ($a < -1$), and in that case $r = 0$ corresponds to $\bar{r} = \infty$ and vice versa.

Another possible choice for writing the metric is used by Garfinkle [12], written in the form

$$ds^2 = -Adt^2 + Bd\phi^2 + dr^2 + Cdz^2, \tag{2.32}$$

where A, B, and C are once again functions of r only. In this case, to transform from our gauge to (2.32) we require $\bar{r} = (c/(a+b+1))r^{a+b+1}$. Under this transformation, the metric becomes

$$ds^2 = -(D\bar{r})^{(2a/(a+b+1))}dt^2 + d\bar{r}^2 + (D\bar{r})^{(2/(a+b+1))}d\phi^2 + (D\bar{r})^{(2b/(a+b+1))}dz^2 \tag{2.33}$$

with $D := (a+b+1)/c$. The exponent of r in our definition of $\bar{r}$ is negative whenever $a+b < -1$, which occurs whenever $b < -1$ ($a < -1$). Thus in all three alternate metric forms discussed here, $r = 0$ in our gauge corresponds to $\bar{r} = \infty$ in the new gauge whenever $a < -1$ and $b < -1$ (hence $a+b \leq -4$ from Figure 1 (b)). The other possibilities have $a+b \geq 0$ and r and $\bar{r}$ running in the same direction.

It is also interesting to calculate $W \equiv R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ for the vacuum solution, because this is the simplest nonzero curvature invariant; since $\frac{1}{2}Rg_{\alpha\beta} = R_{\alpha\beta} - G_{\alpha\beta}$ implies that the Ricci scalar $R = 0$. After raising and lowering indices on the Riemann tensor components as necessary, and using the identities that $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$ and $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$, we sum over the necessary indices and get the result that

$$W = R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} = 4Cr^{-4(a+b+1)} \tag{2.34}$$

where $C = 3a^2 + 3b^2 + 3a^2b^2 + 2ab + 2a^2b + 2ab^2$. We see that the exponent is negative whenever $a + b > -1$, and in that case, $W \to 0$ as $r \to \infty$. In the other case, when the exponent is negative instead, $W \to \infty$ as $r \to \infty$ but $W \to 0$ as $\bar{r} \to \infty$, and in that case there is some ambiguity as to which of these limits is the "outside" and which is the "inside".

Solutions of the Einstein equations with sources

We would like to find some cylindrical space-times that are not singular along the central axis. This requires solving the Einstein equations in cases where T has nonzero components. In order to proceed with this, we first require a few more basic quantities and tensors encountered in general relativity. We present here the results for the Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$, the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ [1], and the stress-energy tensor, $T_{\mu\nu}$, for the cylindrically symmetric metric given in (2.1). The components of the stress-energy tensor for a perfect fluid are given by $T_{\alpha\beta} = (\rho + p)U_\alpha U_\beta + pg_{\alpha\beta}$, where ρ is the matter density, p is the pressure, and U_α is the four-velocity of the matter; since we will be looking at static matter with no motion, all components of the four-velocity are zero except $U_t = -e^\Phi$ [1]. The results for these calculations are presented below.

Ricci Scalar:

$$\begin{aligned} R = e^{-2\Lambda}(&-2\Phi'' - 2\Phi'^2 + 2\Phi'\Lambda \; - 2\Psi'' - 2\Psi'^2 + 2\Psi'\Lambda' - 2\Psi'\Phi' + \\ &\frac{2}{r}\Lambda' - \frac{2}{r}\Phi' - \frac{2}{r}\Psi') \end{aligned} \tag{2.35}$$

Nonzero Einstein Tensor Components:

$$\begin{aligned} G_{tt} &= e^{2(\Phi-\Lambda)}(-\Psi'' - \Psi'^2 + \Psi'\Lambda' + \frac{1}{r}\Lambda' - \frac{1}{r}\Psi') \\ G_{rr} &= \Psi'\Phi' + \frac{1}{r}\Phi' + \frac{1}{r}\Psi' \\ G_{\phi\phi} &= r^2e^{-2\Lambda}(\Phi'' + \Phi'^2 - \Phi'\Lambda' + \Psi'' + \Psi'^2 - \Psi'\Lambda' + \Psi'\Phi') \\ G_{zz} &= e^{2(\Psi-\Lambda)}(\Phi'' + \Phi'^2 - \Phi'\Lambda' - \frac{1}{r}\Lambda' + \frac{1}{r}\Phi') \end{aligned} \tag{2.36}$$

Nonzero Stress Tensor Components:

$$
\begin{aligned}
T_{tt} &= \rho e^{2\Phi} \\
T_{rr} &= p_r e^{2\Lambda} \\
T_{\phi\phi} &= p_\phi r^2 \\
T_{zz} &= p_z e^{2\Psi}
\end{aligned}
\tag{2.37}
$$

From the Einstein field equations, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, and the conservation laws, $T^{\alpha\beta}{}_{;\beta} = 0$, we get the following five differential equations:

$$0 = \frac{\partial p_r}{\partial r} + p_r(\Phi' + \Psi' + \frac{1}{r}) + \rho\Phi' - p_z\Psi' - \frac{1}{r}p_\phi, \tag{2.38}$$

$$4\pi(\rho + p_r + p_\phi + p_z)e^{2\Lambda} = \Phi'' + \Phi'^2 - \Phi'\Lambda' + \Psi'\Phi' + \frac{1}{r}\Phi', \tag{2.39}$$

$$\begin{aligned} &4\pi(\rho + p_r - p_\phi - p_z)e^{2\Lambda} = \\ &-\Phi'' - \Phi'^2 + \Phi'\Lambda' + \frac{1}{r}\Lambda' - \Psi'' - \Psi'^2 + \Lambda'\Psi', \end{aligned} \tag{2.40}$$

$$4\pi(\rho - p_r + p_\phi - p_z)e^{2\Lambda} = \frac{1}{r}(\Lambda' - \Phi' - \Psi'), \tag{2.41}$$

$$4\pi(\rho - p_r - p_\phi + p_z)e^{2\Lambda} = -\Psi'' - \Psi'^2 + \Psi'\Lambda' - \Psi'\Phi' - \frac{1}{r}\Psi'. \tag{2.42}$$

We can simplify these by summing (2.39) and (2.40) and subtracting (2.42). This yields

$$4\pi(\rho + 3p_r + p_\phi - p_z)e^{2\Lambda} = 2\Psi'\Phi' + \frac{1}{r}\Phi' + \frac{1}{r}\Psi' + \frac{1}{r}\Lambda'. \tag{2.43}$$

We now add and subtract equation (2.41) from (2.43), resulting in

$$4\pi(2\rho + 2p_r + 2p_\phi - 2p_z)e^{2\Lambda} = 2\Psi'\Phi' + \frac{2}{r}\Lambda' \tag{2.44}$$

$$4\pi(4p_r)e^{2\Lambda} = 2\Psi'\Phi' + \frac{2}{r}(\Phi' + \Psi'). \tag{2.45}$$

We now have a system of differential equations where equations (2.38), (2.39), (2.41), and (2.42) can be solved for p_r, p_ϕ, p_z, Φ, Ψ, and Λ (given ρ and an equation of state relating

ρ and the various pressures), and (2.45), which contains only lower-order derivatives of the unknown functions, provides an additional constraint. The system of all five equations is second-order in Φ and Ψ and first-order in Λ and p_r.

Differentiating equation (2.45) with respect to r and using equations (2.38), (2.39), (2.42), and (2.44) to substitute for p_r', Λ', Φ'', and Ψ'' yields an expression which reduces to $0 = 0$; thus equation (2.45) must hold for all r if it holds at any r.

Solving these equations for arbitrary ρ, p_r, p_ϕ, and p_z is rather difficult, however, so a simpler case one can look at is when $\rho = -p_z$ and the other pressure components are zero. In this case, the differential equations reduce to

$$0 = \rho(\Phi' + \Psi'), \tag{2.46}$$

$$0 = \Phi'' + \Phi'^2 - \Phi'\Lambda' + \Psi'\Phi' + \frac{1}{r}\Phi', \tag{2.47}$$

$$4\pi(2\rho)e^{2\Lambda} = \frac{1}{r}(\Lambda' - \Phi' - \Psi'), \tag{2.48}$$

$$0 = -\Psi'' - \Psi'^2 + \Psi'\Lambda' - \Psi'\Phi' - \frac{1}{r}\Psi', \tag{2.49}$$

$$0 = 2\Phi'\Psi' + \frac{2}{r}(\Phi' + \Psi'). \tag{2.50}$$

From equation (2.46) we can see that $\Phi' + \Psi' = 0$, allowing us to solve equation (2.48) for $\Lambda' = 8\pi\rho r e^{2\Lambda}$, which can easily be solved using integration by parts to get Λ for a given ρ. From (2.50) and the fact that $\Phi' + \Psi' = 0$, we also see that $\Phi'\Psi' = 0$. Thus we can conclude that $\Phi' = \Psi' = 0$, yielding $\Phi = a_1$ and $\Psi = a_2$ (where a_1 and a_2 are constants). The metric of the solution can be written as

$$ds^2 = -dt^2 + e^{2\Lambda}dr^2 + r^2 d\phi^2 + dz^2. \tag{2.51}$$

This case has been solved previously by Gott [6], with $\rho = 1/(8\pi r_0^2)$ (where r_0 is a constant). Using this value of ρ, our solutions yield

$$ds^2 = -dt^2 + (r_0^2/(c-r^2))dr^2 + r^2 d\phi^2 + dz^2, \tag{2.52}$$

which agrees with Gott's solution after making the substitution $r = \sin(\theta)$ and then rescaling coordinates as necessary. This metric is Lorentz-invariant under boosts in the (z,t) plane, and thus ρ and p_z are independent of frame. If we did not have the condition that $\rho = -p_z$, the solution would not be Lorentz invariant in this way and ρ and p_z would not be frame-independent. Indeed, generically one would expect the density of the matter in a string source to be affected by Lorentz contraction when one moves out of the rest frame. But a Gott string, like cosmological dark energy (where all components of p equal $-\rho$), has no preferred rest frame.

We now present some numerical solutions (calculated with *Mathematica*) for the case where we take ρ to be constant out to a radius R and zero outside of this radius, and the pressure is isotropic in all directions, $p_r = p_\phi = p_z \equiv p$. This is analogous to isotropic pressure in the spherically symmetric case, even if considering it in the cylindrically symmetric case is not astrophysically realistic. Since our differential equations involve factors of $1/r$, they present problems when trying to solve the system numerically starting from $r = 0$. In order to deal with this, we first make power series expansions of p, Φ, Ψ, and Λ around $r = 0$. We keep terms up to order r in the p and Λ expansions (since our differential equations involve first-order derivatives of these functions) and keep terms up to order r^2 in the Φ and Ψ expansions (since the differential equations involve second-order derivatives of these functions), resulting in

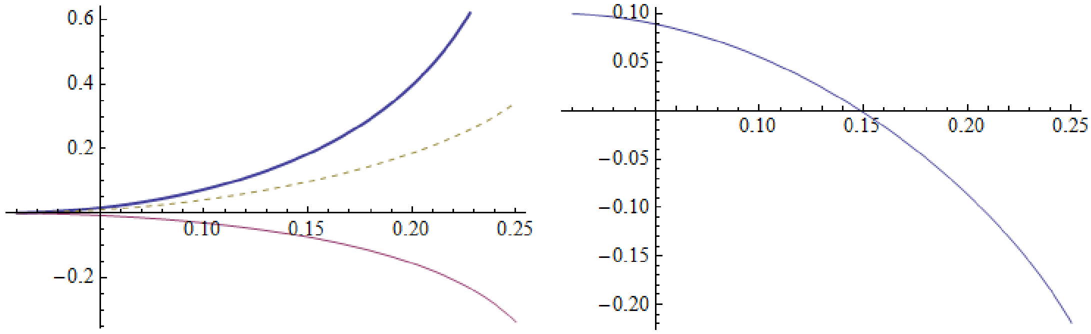
$$p = p_0 + p_1 r, \tag{2.53}$$

$$\Lambda = \Lambda_0 + \Lambda_1 r, \tag{2.54}$$

$$\Phi = \Phi_0 + \Phi_1 r + \Phi_2 r^2, \tag{2.55}$$

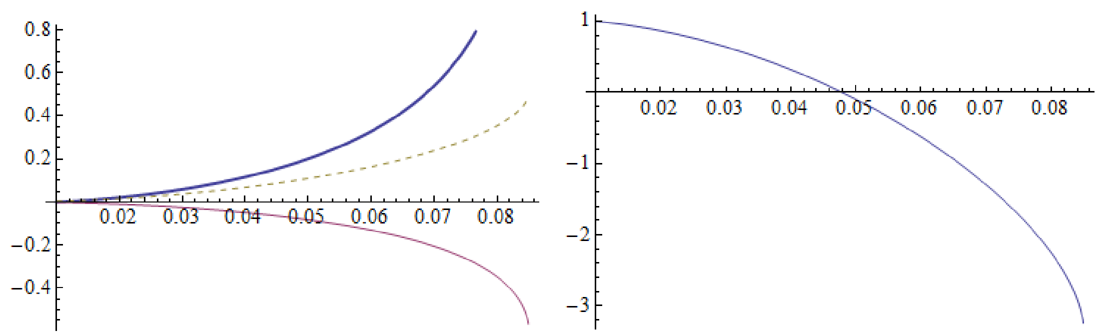
$$\Psi = \Psi_0 + \Psi_1 r + \Psi_2 r^2. \tag{2.56}$$

We also take the initial conditions $\Psi = \Phi = \Lambda = 0$ at $r = 0$ so that the corresponding metric coefficients are equal to 1 at that point, and choose $\Psi' = \Phi' = 0$ to get smooth solutions at the axis. Equations (2.53)–(2.56) should satisfy our differential equations near $r = 0$, so we substitute them into (2.38), (2.39), (2.41), (2.42), and (2.45) (taking $p_r = p_\phi = p_z = p$). Using our chosen initial conditions and looking at the lowest-order terms in the expansion of each equation, we obtain the relationships $p_1 = 0$, $\Lambda_1 = 0$, $\Phi_2 = \pi(\rho + 3p_0)$, and $\Psi_2 = -\pi(\rho - p_0)$. After choosing values for ρ and $p_0 = p(0)$, we can determine the values of p, Φ, Ψ, and Λ at some small r away from 0; we take $r = 0.01$. We use these as our initial conditions for the numerical calculations and obtain solutions for various values of ρ and p_0; several examples of such solutions are provided. For the case of $\rho = 1$, $p_0 = 0.1$, the results are given in Figure 2. When $\rho = 10$, $p_0 = 1$, the results are given in Figure 3. For $\rho = 1$, $p_0 = 0.2$, the results are in Figure 4, and when $\rho = 10$, $p_0 = 3$, the results are shown in Figure 5.



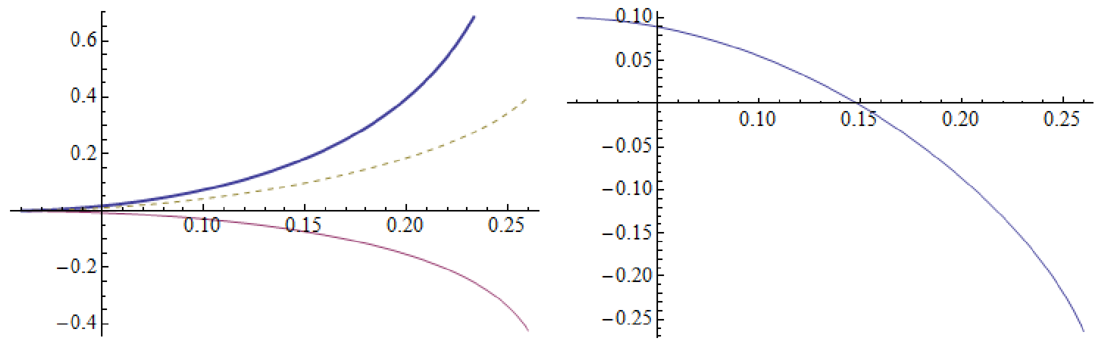
(a) (b)

Figure 2. For $\rho = 1$ and $p_0 = 0.1$ (interior): (a) plot of $\Lambda(r)$ (thick), $\Psi(r)$ (normal), and $\Phi(r)$ (dashed); (b) plot of $p(r)$.



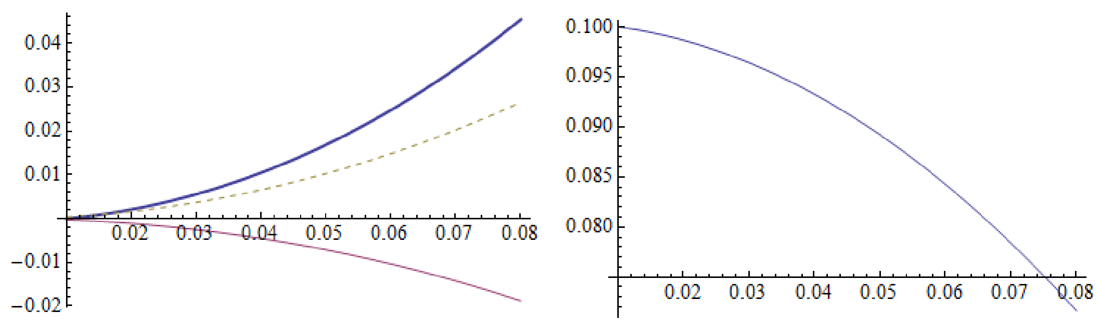
(a) (b)

Figure 3. For $\rho = 10$ and $p_0 = 1$ (interior): (a) plot of $\Lambda(r)$ (thick), $\Psi(r)$ (normal), and $\Phi(r)$ (dashed); (b) plot of $p(r)$.



(a) (b)

Figure 4. For $\rho = 1$ and $p_0 = 0.2$ (interior): (a) plot of $\Lambda(r)$ (thick), $\Psi(r)$ (normal), and $\Phi(r)$ (dashed); (b) plot of $p(r)$.



(a) (b)

Figure 5. For $\rho = 10$ and $p_0 = 3$ (interior): (a) plot of $\Lambda(r)$ (thick), $\Psi(r)$ (normal), and $\Phi(r)$ (dashed); (b) plot of $p(r)$.

After finding interior solutions numerically for $p_r = p_\phi = p_z = p$, we can then connect them to the exterior vacuum solution found previously. We take R to be the point where $p(r) = 0$, and use the values of $\Phi(R)$, $\Phi'(R)$, $\Psi(R)$, $\Psi'(R)$, and $\Lambda(R)$ from the numerical solutions as conditions to determine the unknown coefficients (a_1, a_2, b_1, b_2, and c) of Φ, Ψ, and Λ from the vacuum case. We must match our interior solution with the most general vacuum solution which includes all the arbitrary constants ($\Phi = \ln(r_1^a) + \ln(a_2)$, $\Psi = \ln(r^{b_1}) + \ln(b_2)$, and $\Lambda = \ln(r^{a_1+b_1}) + \ln(c)$), since we chose our initial conditions so that the interior metric coefficients are all 1 at $r = 0$. Because of this, we are not free to scale away a_2, b_2, and c, and we must keep them in the metric in order to match our two sets of solutions.

The results for the sample cases given above are presented in the figures. When $\rho = 1$, $p_0 = 0.1$, we get that $R = 0.1486$, and the resulting exterior solutions are plotted in Figure 6. For the case of $\rho = 10$, $p_0 = 1$, we get $R = 0.04791$, and the exterior solutions are plotted in Figure 7. The other two cases are given in Figures 8 and 9. Numerical constants for all of these solutions are given in table I. Also, after rescaling coordinates appropriately in order to put the metric in the form of equation (2.22), the range of ϕ changes as well; it had to be 2π for the inner solution to guarantee smoothness at the origin, and thus the outer solution initially has range 2π when matched with the inner solution. The new value of ϕ_* is given by $\phi_* = (2\pi)c^{-1/(a_1+b_1+1)}$. The values of ϕ_* for the two solutions described above are also given in the table.

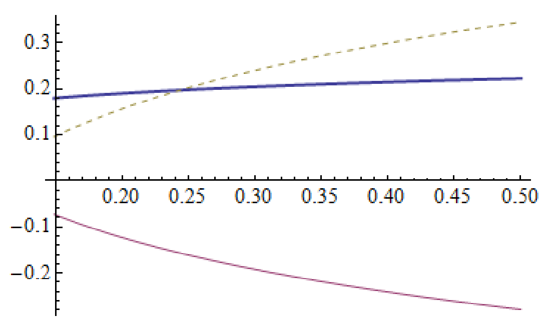


Figure 6. For $\rho = 1$ and $p_0 = 0.1$ (exterior): plot of $\Lambda(r)$ (thick), $\Psi(r)$ (normal), and $\Phi(r)$ (dashed).

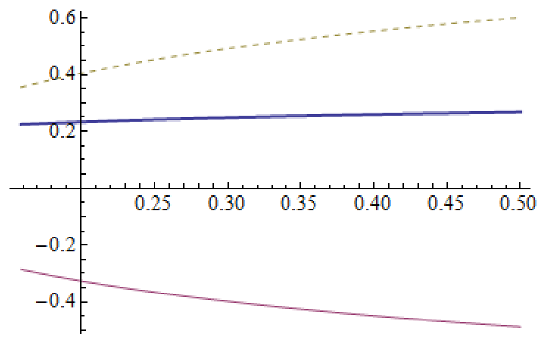


Figure 7. For $\rho = 10$ and $p_0 = 1$ (exterior): plot of $\Lambda(r)$ (thick), $\Psi(r)$ (normal), and $\Phi(r)$ (dashed).

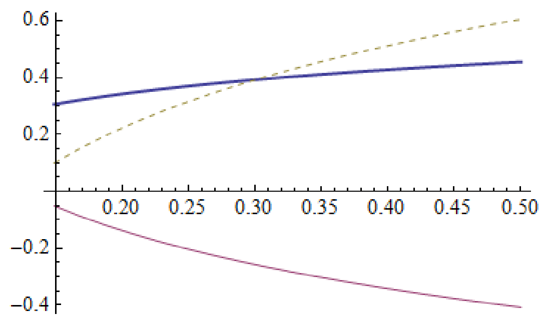


Figure 8. For $\rho = 1$ and $p_0 = 0.2$ (exterior): plot of $\Lambda(r)$ (thick), $\Psi(r)$ (normal), and $\Phi(r)$ (dashed).

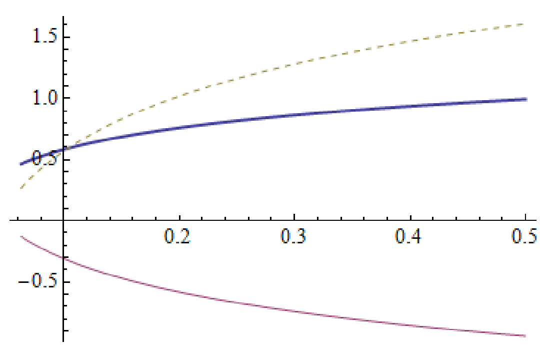


Figure 9. For $\rho = 10$ and $p_0 = 3$ (exterior): plot of $\Lambda(r)$ (thick), $\Psi(r)$ (normal), and $\Phi(r)$ (dashed).

Table I. Numerical constants for exterior solution when $\rho = 1$ and $p_0 = 0.1$ (second column), $\rho = 10$ and $p_0 = 1$ (third column), $\rho = 1$ and $p_0 = 0.2$ (fourth column), and $\rho = 10$ and $p_0 = 3$ (fifth column).

R	0.1486	0.04791	0.1815	0.06298
$R^2\rho$	0.02208	0.02296	0.03294	0.03966
a_1	0.2052	0.2136	0.4165	0.6480
b_1	−0.1703	−0.1761	−0.2941	−0.3933
a_2	1.627	2.114	2.444	7.846
b_2	0.6722	0.5435	0.5429	0.2969
c	1.279	1.342	1.7159	3.2192
ϕ_*	4.954	4.732	3.884	2.475

In the spherically symmetric case, the Buchdahl theorem requires that $R > \frac{9}{4}M$ for any stellar model, where $M = \frac{4}{3}\pi\rho R^3$ [1]. This implies that $R^2\rho < \frac{1}{10}$, where $R^2\rho$ is a dimensionless quantity. Although this inequality need not hold in the cylindrically symmetric case, it is interesting to note that in the examples studied above, it does in fact hold.

CHAPTER III

STATIC, CYLINDRICAL SYMMETRY IN VACUUM ENERGY

In order to calculate the pressure on cylindrical boundaries, one first needs expressions for the components of the stress-energy tensor, $T_{\mu\nu}$, in terms of the known cylinder kernel, $\bar{T}$. The cylinder kernel is given by $\frac{\partial \bar{T}}{\partial t} = T$, where $x = (r, \theta, z)$ and $x' = (r', \theta', z')$ are two different points in space. T is defined by $\frac{\partial^2 T}{\partial t^2} = -\nabla^2 T$ with appropriate boundary conditions, the initial condition $T(0,x,x') = \delta(x-x') = \frac{\partial \bar{T}}{\partial t}(0,x,x')$, and a requirement that $T(t,x,x')$ is bounded as $t \to +\infty$ [13].

To calculate the components of the stress-energy tensor in a static, flat, cylindrically symmetric spacetime, we utilize the same method used by Schwartz-Perlov and Olum to calculate the stress-energy tensor in a spherically symmetric spacetime [10]. We begin by taking the r, θ, and z unit vectors along the x, y, and z axes; due to the cylindrical symmetry of the situation, this is valid regardless of how we rotate in the x-y plane. We define the "$\perp$" coordinate to be along the θ direction, but with units of length, like the y coordinate, so that we may take straightforward derivatives of the form $\partial_\perp = \frac{\partial}{\partial \perp}$. Also, in the calculations that follow, "0" is used to refer to components and derivatives in the time coordinate. With this setup, and using the metric $\eta_{00} = -1$, $\eta_{rr} = \eta_{\perp\perp} = \eta_{zz} = 1$ (all other components equal to zero), we can make use of the stress-energy tensor formula given in Schwartz-Perlov and Olum's paper

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\eta_{\mu\nu}\partial^\lambda \phi \partial_\lambda \phi + \xi[\eta_{\mu\nu}\partial_\lambda \partial^\lambda (\phi^2) - \partial_\mu \partial_\nu (\phi^2)], \tag{3.1}$$

where ϕ is a massless real scalar field [10]. We can now plug in to find T_{00}, T_{rr}, $T_{\perp\perp}$, and T_{zz} in terms of the scalar field ϕ, raising and lowering indices as necessary. We also take

$\xi = \beta + \frac{1}{4}$ (ξ is the curvature coupling) and let $\dot{\phi} = \partial_0 \phi$. This results in

$$T_{00} = \frac{1}{2}[\dot{\phi}^2 - \phi\partial_r^2\phi - \phi\partial_\perp^2\phi - \phi\partial_z^2\phi]$$
$$-2\beta[(\partial_r\phi)^2 + \phi\partial_r^2\phi + (\partial_\perp\phi)^2 + \phi\partial_\perp^2\phi + (\partial_z\phi)^2 + \phi\partial_z^2\phi] \tag{3.2}$$
$$T_{rr} = \frac{1}{2}[-\phi\ddot{\phi} + (\partial_r\phi)^2 + \phi\partial_\perp^2\phi + \phi\partial_z^2\phi]$$
$$+2\beta[(\partial_\perp\phi)^2 + \phi\partial_\perp^2\phi + (\partial_z\phi)^2 + \phi\partial_z^2\phi - \dot{\phi}^2 - \phi\ddot{\phi}] \tag{3.3}$$
$$T_{\perp\perp} = \frac{1}{2}[-\phi\ddot{\phi} + (\partial_\perp\phi)^2 + \phi\partial_r^2\phi + \phi\partial_z^2\phi]$$
$$+2\beta[-\dot{\phi}^2 - \phi\ddot{\phi} + (\partial_r\phi)^2 + \phi\partial_r^2\phi + (\partial_z\phi)^2 + \phi\partial_z^2\phi \tag{3.4}$$
$$T_{zz} = \frac{1}{2}[-\phi\ddot{\phi} + (\partial_z\phi)^2 + \phi\partial_r^2\phi + \phi\partial_\perp^2\phi]$$
$$+2\beta[-\dot{\phi}^2 - \phi\ddot{\phi} + (\partial_r\phi)^2 + \phi\partial_r^2\phi + (\partial_\perp\phi)^2 + \phi\partial_\perp^2\phi. \tag{3.5}$$

Now from the wave equation which the field ϕ must satisfy,

$$0 = -\ddot{\phi} + \nabla^2\phi, \tag{3.6}$$

we can get several expressions which allow us to further simplify equations 3.2-3.5. This yields

$$T_{00} = \frac{1}{2}[\dot{\phi}^2 - \phi\ddot{\phi}] - 2\beta[\partial_r(\phi\partial_r\phi) + \partial_\perp(\phi\partial_\perp\phi) + \partial_z(\phi\partial_z\phi)] \tag{3.7}$$
$$T_{rr} = \frac{1}{2}[(\partial_r\phi)^2 - \phi\partial_r^2\phi] + 2\beta[\partial_\perp(\phi\partial_\perp\phi) + \partial_z(\phi\partial_z\phi) - (\dot{\phi}^2 + \phi\ddot{\phi})] \tag{3.8}$$
$$T_{\perp\perp} = \frac{1}{2}[(\partial_\perp\phi)^2 - \phi\partial_\perp^2\phi] + 2\beta[-(\dot{\phi}^2 + \phi\ddot{\phi}) + \partial_r(\phi\partial_z\phi) + \partial_z(\phi\partial_z\phi)] \tag{3.9}$$
$$T_{zz} = \frac{1}{2}[(\partial_z\phi)^2 - \phi\partial_z^2\phi] + 2\beta[-(\dot{\phi}^2 + \phi\ddot{\phi}) + \partial_r(\phi\partial_r\phi) + \partial_\perp(\phi\partial_\perp\phi)]. \tag{3.10}$$

Also, due to the symmetry of the situation, the expectation value of ϕ^2 does not depend on time or the z coordinate, so $\partial_0^2\langle\phi^2\rangle = \partial_z^2\langle\phi^2\rangle = 0$. We can rewrite the derivatives in the "$\perp$" direction using the relations $\partial_\perp^2\phi = \frac{1}{r}\partial_r\phi + \frac{1}{r^2}\partial_\theta^2\phi$ and $2\langle(\partial_\perp\phi)^2\rangle + 2\langle\phi\partial_\perp^2\phi\rangle =$

$\partial_\perp^2\langle\phi^2\rangle = \frac{1}{r}\partial_r(\phi^2) = \frac{2}{r}\phi\partial_r\phi$. These simplifications yield

$$T_{00} = \frac{1}{2}[\dot{\phi}^2 - \phi\ddot{\phi}] - 2\beta[\partial_r(\phi\partial_r\phi) + \frac{1}{r}\phi\partial_r\phi] \tag{3.11}$$

$$T_{rr} = \frac{1}{2}[(\partial_r\phi)^2 - \phi\partial_r^2\phi] + 2\beta[\frac{1}{r}\phi\partial_r\phi] \tag{3.12}$$

$$T_{\perp\perp} = -\frac{1}{2r}\phi\partial_r\phi + \frac{1}{r^2}(\partial_\theta\phi)^2 - \frac{1}{2r^2}\partial_\theta^2(\phi^2) + 2\beta[\partial_r(\phi\partial_r\phi)] \tag{3.13}$$

$$T_{zz} = \frac{1}{2}[(\partial_z\phi)^2 - \phi\partial_z^2\phi] + 2\beta[\partial_r(\phi\partial_r\phi) + \frac{1}{r}\phi\partial_r\phi]. \tag{3.14}$$

Now we can rewrite the components of $T_{\mu\nu}$ in terms of the cylinder kernel, $\bar{T}$, rather than ϕ. Since $\langle 0|\phi(x)\phi(x')|0\rangle = -\frac{1}{2}\bar{T}(x,x')$, we can easily calculate all necessary derivatives of ϕ in terms of derivatives of $\bar{T}$. For example,

$$\partial_r\bar{T} = -2\langle 0|\phi(x')\partial_r\phi(x)|0\rangle \tag{3.15}$$

$$\partial_r^2\bar{T} = -2\langle 0|\phi(x')\partial_r^2\phi(x)|0\rangle \tag{3.16}$$

$$\partial_{r'}\partial_r\bar{T} = -2\langle 0|\partial_r\phi(x)\partial_{r'}\phi(x')|0\rangle \tag{3.17}$$

$$\tag{3.18}$$

yield the relations

$$\langle\phi\partial_r\phi\rangle = -\frac{1}{2}\partial_r\bar{T} \tag{3.19}$$

$$\langle\phi\partial_r^2\phi\rangle = -\frac{1}{2}\partial_r^2\bar{T} \tag{3.20}$$

$$\langle(\partial_r\phi)^2\rangle = -\frac{1}{2}\partial_{r'}\partial_r\bar{T} \tag{3.21}$$

after taking $x = x'$. The derivatives for the other coordinates are computed similarly. The components of $T_{\mu\nu}$ finally become

$$T_{00} = -\frac{1}{2}\partial_0^2\bar{T} + \beta[\partial_r\partial_{r'}\bar{T} + \partial_r^2\bar{T} + \frac{1}{r}\partial_r\bar{T}] \tag{3.22}$$

$$T_{rr} = -\frac{1}{4}[\partial_r\partial_{r'}\bar{T} - \partial_r^2\bar{T}] - \frac{1}{r}\beta\partial_r\bar{T} \tag{3.23}$$

$$T_{\perp\perp} = \frac{1}{4r}\partial_r \bar{T} + \frac{1}{2r^2}\partial_\theta^2 \bar{T} - \beta[\partial_r \partial_{r'} \bar{T} + \partial_r \bar{T}] \tag{3.24}$$

$$T_{zz} = -\frac{1}{4}[\partial_z \partial_{z'} \bar{T} - \partial_z^2 \bar{T}] - \beta[\partial_r \partial_{r'} \bar{T} + \partial_r^2 \bar{T} + \frac{1}{r}\partial_r \bar{T}]. \tag{3.25}$$

The cylinder kernel for 4-dimensional spacetime in cylindrical coordinates has been found previously by Smith [14] and has also been calculated by Mai Truong [15], yielding

$$\bar{T}(t,r,\theta,z,r',\theta',z') = -\frac{1}{2\pi\theta_1 rr' \sinh(u)} \frac{\sinh(\frac{2\pi}{\theta_1}u)}{\cosh(\frac{2\pi}{\theta_1}u) - \cos(\frac{2\pi}{\theta_1}(\theta-\theta'))}, \tag{3.26}$$

with u given by $\cosh u = \frac{r^2+r'^2+z^2+t^2}{2rr'}$. From this expression, we can calculate all derivatives of $\bar{T}$ which are needed for the components of $T_{\mu\nu}$. For example,

$$\begin{aligned}
\frac{\partial \bar{T}}{\partial t} = (2\pi\theta_1 rr' \sinh u)^{-2}(2\pi\theta_1 rr' \cosh u \frac{\partial u}{\partial t}) \frac{\sinh(\frac{2\pi}{\theta_1}u)}{\cosh(\frac{2\pi}{\theta_1}u) - \cos(\frac{2\pi(\theta-\theta')}{\theta_1})} \\
+ \frac{-1}{2\pi\theta_1 rr' \sinh u}[\frac{\cosh(\frac{2\pi}{\theta_1}u)(\frac{2\pi}{\theta_1}\frac{\partial u}{\partial t})}{\cosh(\frac{2\pi}{\theta_1}) - \cos(\frac{2\pi}{\theta_1}(\theta-\theta'))} \\
- \sinh(\frac{2\pi}{\theta_1}u)(\cosh(\frac{2\pi}{\theta_1}u) - \cos(\frac{2\pi}{\theta_1}(\theta-\theta'))^{-2}(\sinh(\frac{2\pi}{\theta_1}u)\frac{\partial u}{\partial t})].
\end{aligned} \tag{3.27}$$

Now that we have the components of $T_{\mu\nu}$ in this form, they can be used to find the energy density and pressure for various cylindrically symmetric configurations, and future work will be done to calculate these quantities. The procedure outlined above can also be used for the general static, cylindrically symmetric metric, given in equation (2.1), to find the components of $T_{\mu\nu}$. This allows us to examine a wider variety of cylindrically symmetric situations and calculate the pressure and energy in those cases as well.

CHAPTER IV

CONCLUSION

Through my work in this thesis, I have examined the presence of cylindrical symmetry in various physical situations. The main portion of my work focused on solving Einstein's field equations of gravitation for various cases of $T_{\mu\nu}$. Although the vacuum solutions have been found previously and known for some time, I focused on a different metric convention from what most studies have usually used. I derived expressions for transforming between these different metric conventions and examined their relationships to certain "special points" of the general vacuum cylindrically symmetric solutions. Along with the vacuum solutions, I also examined several cases of solving the Einstein equations with sources. I was able to solve the equations with $\rho = -p_z$ and $p_r = p_\phi = 0$ and extract Gott's solution for $\rho = 1/(8\pi r_0^2)$, which is applicable to the study of cosmic strings. I did not succeed in solving them for arbirtrary pressure components, so the final portion of my work with Einstein's equations involved solving them for the simplified case of isotropic pressures and constant ρ. With the use of Mathematica, I succeeded in finding a way of solving them numerically, since I was unable to solve them analytically in this case.

Finally, in order to study cylindrical symmetry in vacuum energy, I calculated the components of the stress-energy tensor in the case of static, locally flat space. These can be used to calculate pressure and energy density in various arrangements, and this work will be continued next year to find these quantities. With this, hopefully I can provide some new information to help study paradoxes that currently exist in the theory.

REFERENCES

[1] Schutz B F 2009 *A First Course in General Relativity* (Cambridge, UK: Cambridge University Press)

[2] Weyl H 1917 *Ann. Phys., Lpz* **54** 117

[3] Levi-Civita T 1919 *Atti Acc. Lincei Rend.* **28** 101

[4] Marder L 1958 *Proc. R. Soc. Lond. A* **244** 524

[5] Futamase T and Garfinkle D 1988 *Phys. Rev. D* **37** 2086

[6] Gott J R 1985 *Ap. J* **288** 422

[7] Hiscock W A 1985 *Phys. Rev. D* **31** 3288

[8] Fulling S A 1989 *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge, UK: Cambridge University Press)

[9] Fulling S A 2010 *Internat. J. Mod. Phys.* **25** 2364

[10] Schwartz-Perlov D and Olum K D 2005 *Phys. Rev. D* **72** 065013

[11] Liu Z H 2009 *Closed Path Approach to Casimir Effect in Rectangular Cavities and Pistons*, Ph.D. dissertation (Texas A&M University) http://www.math.tamu.edu/ fulling/qvac09/liudiss.pdf

[12] Garfinkle D 1985 *Phys. Rev. D* **32** 1323

[13] Fulling S A, Kaplan L, Kirsten K, Liu Z H and Milton K A 2009 *J. Phys. A: Math. Theor.* **42** 155402

[14] Smith A G 1990 Gravitational effects of an infinite straight cosmic string on classical and quantum fields: Self-forces and vacuum fluctuations in *The Formation and Evolution of Cosmic Strings* ed. G Gibbons S Hawking and T Vachaspati (Cambridge, UK: Cambridge University Press) pp 263-292

[15] Truong M 2008 Vacuum Energy in Wedges and Around Cosmic Strings, Unpublished Manuscript (Texas A&M University) http://www.math.tamu.edu/~stephen.fulling/qvac08/truong/

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