# NEW RESULTS IN STABILITY, CONTROL, AND ESTIMATION OF FRACTIONAL ORDER SYSTEMS

A Dissertation

by

BONG SU KOH

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2011

Major Subject: Aerospace Engineering

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May 2011

Major Subject: Aerospace Engineering

# ABSTRACT

New Results in Stability, Control, and Estimation of

Fractional Order Systems. (May 2011)

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A review of recent literature and the research effort underlying this dissertation indicates that fractional order differential equations have significant potential to advance dynamical system methods broadly. Particular promise exists in the area of control and estimation, even for systems where fractional order models do not arise "naturally". This dissertation is aimed at further building of the base methodology with a focus on robust feedback control and state estimation.

By setting the mathematical foundation with the fractional derivative Caputo definition, we can expand the concept of the fractional order calculus in a way that enables us to build corresponding controllers and estimators in the state-space form. For the robust eigenstructure assignment, we first examine the conditioning problem of the closed-loop eigenvalues and stability robustnesss criteria for the fractional order system, and we find a unique application of an n-dimensional rotation algorithm developed by Mortari, to solve the robust eigenstructure assignment problem in a novel way. In contradistinction to the existing Fractional Kalman filter developed by using Gründwald-

Letnikov definition, the new Fractional Kalman filter that we establish by utilizing Caputo definition and our algorithms provide us with powerful means for solving practical state estimation problems for fractional order systems.

# DEDICATION

To my family

## ACKNOWLEDGEMENTS

My professors, friends, and family members offered critical support while I was writing this dissertation. I am particularly indebted to my advisor Dr. John. L. Junkins for his support and belief in me. I feel lucky to have a great teacher like him who has not only taught me how to lead my professional life but also inspired me through his unceasing passion for knowledge and creativity.

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# CHAPTER I

# INTRODUCTION AND BACKGROUND OF FRACTIONAL DIFFERENTIAL EQUATIONS

## A. INTRODUCTION

A review of recent literature and the research effort underlying this dissertation indicates that fractional order differential equations have significant potential to advance dynamical system methods broadly. Particular promise exists in the area of control and estimation, even for systems where fractional order models do not arise "naturally". This dissertation is aimed at further building of the base methodology with a focus on robust feedback control and state estimation.

The concept of the fractional order derivative was introduced by L'Hopital and Leibniz in 1695. While in-depth studies have expanded the concept mathematically, its application to engineering has begun only recently because of the complexity. One important mathematical study for the control engineering is given by Podlubny who opened up the possibility of applying fractional calculus to control engineering by establishing $PI^{\lambda}D^{\mu}$ controller, a generalized version of the P.I.D. (Proportional Integral Derivative) controller[1][2].

The implication of Podlubny's study for the fractional order system is that we can expand the concept of the fractional order derivative in a way that enables us to build corresponding controllers and estimators in the state-space form. In order to do this,

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This dissertation follows the style of *Journal of Guidance, Control and Dynamics*.

the most fundamental first step in setting the mathematical foundation is to choose among existing fractional derivative definitions such as Riemann-Liouville definition, Gründwald-Letnikov definition and Caputo's definition. Caputo's definition is our choice because it requires *only integer-order initial conditions* to obtain the solution of the fractional order differential equation[1]. This enables us to bypass the difficulty of defining the fractional-order initial conditions physically. In most engineering problems, a physical model does not give rise to non-integer derivatives, rather they arise in the quest for more general feedback control laws. In essence, using the fractional derivative concept, we have access to an *infinite family of linear closed loop behaviors* not available if we restrict feedback to integer order derivatives. Also, by utilizing the Caputo definition, we will build up the foundations of the needed mathematical properties and concepts such as the linearity, composition rules, etc, and in so doing we are able to achieve a more complete setting than using the Riemann-Liouville definition only.

With these foundations, the robust eigenstructure assignment and the Kalman filter for fractional order systems are developed. Algorithms for robust eigenstructure assignment and the Kalman filter have already been widely and successfully used in the integer order cases[3][4]. For the fractional order case, however, they have not been completely developed heretofore. In our study, we examine new problem set-ups needed for applying the algorithms to the fractional order case. For the robust eigenstructure assignment, we first examine the conditioning problem of the closed-loop eigenvalues and stability robustnesss criteria for the fractional order system, and we find a unique

application of an n-dimensional rotation algorithm developed by Mortari, to solve the robust eigenstructure assignment problem in a novel way[3][5][6]. In contradistinction to the existing Fractional Kalman filter developed by using Gründwald-Letnikov definition, the new Fractional Kalman filter that we establish by utilizing Caputo definition and our algorithms provide us with new and powerful means for solving practical state estimation problems for fractional order systems.

## B. BACKGROUND OF FRACTIONAL ORDER DERIVATIVES

We first examine three kinds of definitions of fractional derivatives[1]. Among them, we choose the Caputo derivative in stating our problems because of the initial condition problem explained to be later.

### Definitions of Factional Derivatives

### Riemann-Liouville Definition

Riemann-Liouville fractional derivative is defined by

$$ {}_{t_0}\mathbf{D}_t^{\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} \frac{x(\tau)d\tau}{(t-\tau)^{\alpha-m+1}} \tag{1.1} $$

where $t_0 < t \leq T < \infty$, $m-1 < \alpha \leq m, m \in \mathbb{N}$. In other short expression, $m = \lceil \alpha \rceil$ denotes the next integer larger than $\alpha$. Notice the apparent history dependence. The subscript $t_0$ denotes the lower limit and $t$ is the upper limit related to the operation of fractional differentiation. Ross called them *terminals* of fractional differentiation. If left

terminal $t_0$ is fixed and the right terminal is variable, this is called the *left fractional derivative of* $x(t)$. Also, mathematically, we can define the right fractional derivative as

$$ {}_{t}\mathbf{D}_{t_f}^{\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dt}\right)^{m} \int_{t}^{t_f} \frac{x(\tau)d\tau}{(t-\tau)^{\alpha-m+1}} \tag{1.2} $$

From the physical point of view, the current state $x(t)$ in the left fractional derivative depends on all past states $x(\tau)(t_0 < \tau < t)$. On the other hand, the right fractional derivative depends on the future states which cannot be known in the causal physical world. This leads us to consider only the causal(left) fractional derivative.

Riemann-Liouville fractional integral is defined by

$$ {}_{t_0}I_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{x(\tau)d\tau}{(t-\tau)^{1-\alpha}}, \qquad t > t_0 \tag{1.3} $$

With this Riemann-Liouville integral, Riemann-Liouvile derivative can also be expressed as

$$ {}_{t_0}\mathbf{D}_t^{\alpha} x(t) = {}_{t_0}D_t^{m}\, {}_{t_0}I_t^{m-\alpha} x(t), \qquad m = \lceil \alpha \rceil \tag{1.4} $$

The Riemann-Liouville definition has played a big role in the pure mathematics literature underwriting the development of fractional derivative and integral concepts. However, it requires the fractional initial condition which frequently cannot be interpreted physically for solving the inhomogeneous fractional differential equations. This leads us to recommend, and adopt for this dissertation, the Caputo's definition which is discussed in the following.

**Caputo's Definition**

Caputo's fractional derivative is defined by

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}\frac{x^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha-m+1}} \tag{1.5} $$

where $m = \lceil \alpha \rceil$ and $x^{(m)}(t) = \frac{d^m \mathbf{x}(t)}{dt^m}$. With this Riemann-Liouville integral, Caputo derivative can also be expressed as[7]

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = {}_{t_0}I_t^{m-\alpha}\,{}_{t_0}D_t^{m}x(t), \qquad m = \lceil \alpha \rceil \tag{1.6} $$

To show the advantage of Caputo definition for dealing with the initial conditions for solving the fractional differential equations, we use the Laplace transform for both definitions. The Laplace transform of the Riemann-Liouville derivative is given by

$$ \mathcal{L}\{\,{}_0\mathbf{D}_t^{\alpha}x(t)\} = \int_0^{\infty} e^{-st}\{\,{}_0\mathbf{D}_t^{\alpha}x(t)\}dt = s^{\alpha}X(s) - \sum_{k=0}^{m-1} s^{k}\big[\,{}_0\mathbf{D}_t^{\alpha-k-1}x(t)\big]_{t=0} \tag{1.7} $$

where $m = \lceil \alpha \rceil$. And, the Laplace transform of the Caputo derivative is given by

$$ \mathcal{L}\{{}_0^{C}D_t^{\alpha}x(t)\} = \int_0^{\infty} e^{-st}\{{}_0^{C}D_t^{\alpha}x(t)\}dt = s^{\alpha}X(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}x^{(k)}(0) \tag{1.8} $$

where $m = \lceil \alpha \rceil$. Obviously, the Caputo derivative requires only initial values with the integer-order derivatives which allow the physical interpretation. But, Riemann-Liouville derivatives needs initial values with the fractional-order form. In order to approach the issue of non-constant initial value problem Bagley and Calico introduced a type of the initialization function, which is more fully developed in Lorenzo and Hartley [8] [9].

The relationship between the Riemann-Liouville and the Caputo definitions can be given by

$$ {}_{t_0}\mathbf{D}_t^{\alpha} x(t) = {}_{t_0}^{C}D_t^{\alpha} x(t) + \sum_{k=0}^{m-1} \frac{(t-t_0)^{k-\alpha}}{\Gamma(k+1-\alpha)} x^{(k)}(t_0{}^{+}), \quad m = \lceil \alpha \rceil \tag{1.9a} $$

$$ {}_{t_0}^{C}D_t^{\alpha} x(t) = {}_{t_0}\mathbf{D}_t^{\alpha} \left( \left[ x(t) - \sum_{k=0}^{m-1} \frac{(t-t_0)^{k}}{k!} x^{(k)}(t_0{}^{+}) \right] \right), \quad m = \lceil \alpha \rceil \tag{1.9b} $$

Both the Riemann-Liouville and Caputo derivatives becomes the conventional n-th derivative as fractional order α becomes integer n[1], so in that sense, both definition generalize classical integer order derivatives to the non-integer case.

$$ {}_{t_0}\mathbf{D}_t^{n} x(t) = {}_{t_0}^{C}D_t^{n} x(t) = x^{(n)}, \quad n = 1,2, \ldots \in \mathbb{N} \tag{1.10} $$

**Gründwald -Letnikov Definition**

Gründwald-Letnikov derivative is given by

$$ \begin{aligned} {}_{t_0}D_t^{\alpha} x(t) &= \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor t/h \rfloor} (-1)^k \binom{\alpha}{k} x(t-kh) \\ &= \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor t/h \rfloor} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} x(t-kh) \end{aligned} \tag{1.11} $$

where $m = \lceil \alpha \rceil$ and $\binom{\alpha}{k} = \frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{k!}$ is the binomial coefficient. This formula can be described by

$$ \begin{aligned} {}_{t_0}D_t^{\alpha} x(t) &= \sum_{k=0}^{m} \frac{x^{(k)}(t_0)(t-t_0)^{-\alpha+k}}{\Gamma(\alpha-k+1)} + \frac{1}{\Gamma(-\alpha+k+1)} \int_{t_0}^{t} \frac{x^{(m+1)}(\tau)d\tau}{(t-\tau)^{\alpha-m}} \\ &= \left(\frac{d}{dt}\right)^{m+1} \int_{t_0}^{t} \frac{x^{(m+1)}(\tau)d\tau}{(t-\tau)^{\alpha-m}} = {}_{t_0}\mathbf{D}_t^{\alpha} x(t) \end{aligned} \tag{1.12} $$

where $m \leq \alpha < m+1, m \in \mathbb{N}$. Therefore, Gründwald-Letnikov derivative can be equal to the Riemann-Liouville derivative under the assumption of continuity and a sufficient number of continuous derivatives of x(t). Generally, Gründwald-Letnikov derivative has been widely used for obtaining the numerical solution of the fractional differential equations because of its implementation convenience. For the numerical approximation, the following formula can be used.

$$ {}_{t_0}D_t^{\alpha} x(t) \approx \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-t_0}{h}\right]} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} x(t-kh) \tag{1.13} $$

where [p] means the integer part of p.

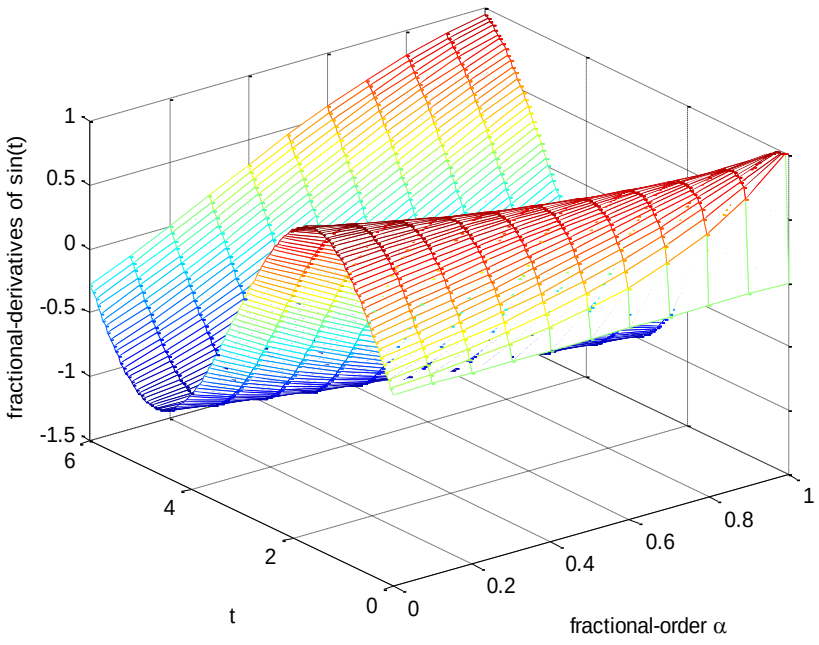


**Figure 1.1.** A Family of Fractional Derivatives of sin(t)

By using the numerical approximation in Eq. (1.13), we can show one simple example, ${}_0\mathrm{D}_\mathrm{t}^{\alpha}\sin(\mathrm{t})$ in Fig. (1.1).

**Properties of Fractional Derivatives**

The following properties give us the insight into the application of fractional derivatives.

**Linearity**

We can do the following general linear operation with any fractional derivative definitions.

$$\mathrm{D}^{\alpha}\big(\lambda \mathrm{f}(\mathrm{t}) + \mu \mathrm{g}(\mathrm{t})\big) = \lambda \mathrm{D}^{\alpha}\mathrm{f}(\mathrm{t}) + \mu \mathrm{D}^{\alpha}\mathrm{g}(\mathrm{t}) \tag{1.14}$$

where $\lambda$ and $\mu$ are constants. The proof can directly be established by using the definition. For Caputo fractional derivatives of order $\alpha(\mathrm{m}-1<\alpha\leq \mathrm{m})$, we have

$$\begin{aligned} {}_{\mathrm{t}_0}^{\mathrm{C}}\mathrm{D}_\mathrm{t}^{\alpha}\big(\lambda \mathrm{f}(\mathrm{t}) + \mu \mathrm{g}(\mathrm{t})\big) &= \frac{1}{\Gamma(\mathrm{m}-\alpha)}\int_{\mathrm{t}_0}^{\mathrm{t}} \frac{\big(\lambda \mathrm{f}(\tau) + \mu \mathrm{g}(\tau)\big)^{(\mathrm{m})}\mathrm{d}\tau}{(\mathrm{t}-\tau)^{\alpha-\mathrm{m}+1}} \\ &= \frac{1}{\Gamma(\mathrm{m}-\alpha)}\int_{\mathrm{t}_0}^{\mathrm{t}} \frac{\lambda \mathrm{f}^{(\mathrm{m})}(\tau)\mathrm{d}\tau}{(\mathrm{t}-\tau)^{\alpha-\mathrm{m}+1}} + \frac{1}{\Gamma(\mathrm{m}-\alpha)}\int_{\mathrm{t}_0}^{\mathrm{t}} \frac{\mu \mathrm{g}^{(\mathrm{m})}(\tau)\mathrm{d}\tau}{(\mathrm{t}-\tau)^{\alpha-\mathrm{m}+1}} \\ &= \lambda\, {}_{\mathrm{t}_0}^{\mathrm{C}}\mathrm{D}_\mathrm{t}^{\alpha}\mathrm{f}(\mathrm{t}) + \mu\, {}_{\mathrm{t}_0}^{\mathrm{C}}\mathrm{D}_\mathrm{t}^{\alpha}\mathrm{g}(\mathrm{t}) \end{aligned} \tag{1.15}$$

**Leibniz Rule**

For integer-order derivatives, we have the following formula.

$$\frac{d^n}{dt^n}\big(f(t)g(t)\big) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(t)\, g^{(n-k)}(t) \tag{1.16}$$

For fractional differentiation, we have the following analogous formula with the Gründwald-Letnikov and the Riemann-Liouville fractional derivatives[1].

$${}_{t_0}D_t^{\alpha}\big(f(t)g(t)\big) = \sum_{k=0}^{\infty} \binom{\alpha}{k} f^{(k)}(t)\, {}_{t_0}D_t^{\alpha-k}\, g(t) \tag{1.17}$$

It is useful for the product of the polynomial functions, but it is difficult to obtain for general functions.

**Interchange of the Differentiation Operators of Integer and Fractional Orders**

For the Caputo derivative,

$${}_{t_0}^{C}D_t^{\alpha}\left({}_{t_0}^{C}D_t^{m}x(t)\right) = {}_{t_0}^{C}D_t^{m}\left({}_{t_0}^{C}D_t^{\alpha}x(t)\right) = {}_{t_0}^{C}D_t^{\alpha+m}x(t) \tag{1.18}$$

where m is an arbitrary integer greater than or equal to zero, and $n-1 < \alpha < n$. And the following condition should be satisfied[1].

$$x^{(k)}(0) = 0, \quad k = n, n+1, \dots, m \tag{1.19}$$

This means that there is no restriction from the definition on the values of $x^{(k)}(0),\ k = 0,1,\dots,n-1$ and therefore these initial conditions can be used to satisfy physical initial conditions. On the contrary, for the Riemann-Liouville derivative,

$${}_{t_0}\mathbf{D}_t^{m}\left({}_{t_0}\mathbf{D}_t^{\alpha}x(t)\right) = {}_{t_0}\mathbf{D}_t^{\alpha}\left({}_{t_0}\mathbf{D}_t^{m}x(t)\right) = {}_{t_0}\mathbf{D}_t^{\alpha+m}x(t) \tag{1.20}$$

where m is an arbitrary integer greater than or equal to zero, and $n-1 < \alpha < n$. And the following restricted condition should be satisfied[1].

$$x^{(k)}(0) = 0, \quad k = 0,1,\dots,m \tag{1.21}$$

This imposes a very constraining condition with regards to physical applications. For the mixed operator of Reimann-Liouville derivative and the fractional integral, we have following property[1].

$${}_{t_0}\mathbf{D}_t^{\alpha}\left({}_{t_0}I_t^{\beta}x(t)\right) = \begin{cases} {}_{t_0}\mathbf{D}_t^{\alpha-\beta}x(t), & \alpha > \beta \\ {}_{t_0}I_t^{\beta-\alpha}x(t), & \alpha \le \beta \end{cases} \tag{1.22}$$

**Composition Rules**

In the fractional integrals defined in Eq. (1.3), the following composition rules are valid[1][10].

$${}_{t_0}I_t^{\alpha}\left({}_{t_0}I_t^{\beta}x(t)\right) = {}_{t_0}I_t^{\beta}\left({}_{t_0}I_t^{\alpha}x(t)\right) = {}_{t_0}I_t^{\alpha+\beta}x(t) \tag{1.23}$$

In general case, the composition rules of Riemann-Liouville derivatives are not valid[1].

$${}_{t_0}\mathbf{D}_t^{\alpha}\left({}_{t_0}\mathbf{D}_t^{\beta}x(t)\right) \neq {}_{t_0}\mathbf{D}_t^{\alpha+\beta}x(t) \neq {}_{t_0}\mathbf{D}_t^{\beta}\left({}_{t_0}\mathbf{D}_t^{\alpha}x(t)\right) \tag{1.24}$$

where $m-1 \le \alpha < m$ and $n-1 \le \beta < n$. Note that this composition rule is valid only if $\alpha = \beta$ or $x^{(k)}(t_0) = 0, k = 0,1,2,\dots,r-1, r = \max(n,m)$. In some literature, researchers ignore this property and its consequences[10].

*Theorem 1.1*

For the Caputo derivative, if m=n and $x^{(m)}(0) = x^{(n)}(0) = 0$ where $m-1 \le \alpha < m$ and $n-1 \le \beta < n$, the Caputo differential operator commutes.

$${}_{t_0}^{C}D_t^{\alpha}\left({}_{t_0}^{C}D_t^{\beta}x(t)\right) = {}_{t_0}^{C}D_t^{\beta}\left({}_{t_0}^{C}D_t^{\alpha}x(t)\right) = {}_{t_0}^{C}D_t^{\alpha+\beta}x(t) \tag{1.25}$$

Proof

By Eq. (1.6),

$$ {}_{t_0}^{C}D_t^{\alpha}\left({}_{t_0}^{C}D_t^{\beta}x(t)\right) = {}_{t_0}I_t^{m-\alpha}\,{}_{t_0}^{C}D_t^{m}\left({}_{t_0}^{C}D_t^{\beta}x(t)\right) \tag{1.26} $$

By the interchange property given in Eq. (1.18),

$$ {}_{t_0}I_t^{m-\alpha}\,{}_{t_0}^{C}D_t^{m}\left({}_{t_0}^{C}D_t^{\beta}x(t)\right) = {}_{t_0}I_t^{m-\alpha}\,{}_{t_0}^{C}D_t^{m+\beta}x(t) \tag{1.27} $$

To satisfy above equation, we need the following condition.

$$ x^{(k)}(0) = 0, \quad k = n, n+1, \dots, m \tag{1.28} $$

By Eq. (1.6) again,

$$ {}_{t_0}I_t^{m-\alpha}\,{}_{t_0}^{C}D_t^{m+\beta}x(t) = {}_{t_0}I_t^{m-\alpha}\,{}_{t_0}I_t^{n-\beta}\,{}_{t_0}^{C}D_t^{m+n}x(t) \tag{1.29} $$

By the composition rule of the fractional integral,

$$ {}_{t_0}I_t^{m-\alpha}\,{}_{t_0}I_t^{n-\beta}\,{}_{t_0}^{C}D_t^{m+n}x(t) = {}_{t_0}I_t^{m+n-\alpha-\beta}\,{}_{t_0}^{C}D_t^{m+n}x(t) \tag{1.30} $$

By Eq. (1.6),

$$ {}_{t_0}I_t^{m+n-\alpha-\beta}\,{}_{t_0}^{C}D_t^{m+n}x(t) = {}_{t_0}^{C}D_t^{\alpha+\beta}x(t) \tag{1.31} $$

We apply the above procedure for ${}_{t_0}^{C}D_t^{\beta}\left({}_{t_0}^{C}D_t^{\alpha}x(t)\right)$. Then,

$$ {}_{t_0}^{C}D_t^{\beta}\left({}_{t_0}^{C}D_t^{\alpha}x(t)\right) = {}_{t_0}I_t^{n-\beta}\,{}_{t_0}^{C}D_t^{n}\left({}_{t_0}^{C}D_t^{\alpha}x(t)\right) = {}_{t_0}I_t^{n-\beta}\,{}_{t_0}^{C}D_t^{n+\alpha}x(t) \tag{1.32} $$

To satisfy the last equality of above equation, we need the following condition.

$$ x^{(k)}(0) = 0, \quad k = m, m+1, \dots, n \tag{1.33} $$

And,

$$ \begin{aligned} {}_{t_0}I_t^{n-\beta}\,{}_{t_0}^{C}D_t^{n+\alpha}x(t) &= {}_{t_0}I_t^{n-\beta}\,{}_{t_0}I_t^{m-\alpha}\,{}_{t_0}^{C}D_t^{m+n}x(t) = {}_{t_0}I_t^{m+n-\alpha-\beta}\,{}_{t_0}^{C}D_t^{m+n}x(t) \\ &= {}_{t_0}^{C}D_t^{\alpha+\beta}x(t) \end{aligned} \tag{1.34} $$

If both conditions given in Eq. (1.28) and Eq. (1.33) are satisfied simultaneously, the composition rule is valid. Therefore, the Caputo differential operator commutes as long as m is equal to n with $x^{(m)}(0) = x^{(n)}(0) = 0$.

**Derivative of a Constant C**

The Caputo derivative of a constant is 0.

$$ {}_0^C D_t^\alpha C = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{C^{(m)} d\tau}{(t-\tau)^{\alpha-m+1}} = 0 \tag{1.35} $$

On the contrary, the Riemann-Liouville derivative of a constant is not 0.

$$ {}_{t_0}\mathbf{D}_t^\alpha C = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^{t} \frac{C d\tau}{(t-\tau)^\alpha} = \frac{C(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \tag{1.36} $$

If $t_0$ is $-\infty$, the Riemann-Liouville fractional derivative of a constant C also has 0. Note that when we put $t_0 = -\infty$ in both definitions, they come to have the same equation[1].

$$ {}_{-\infty}^C D_t^\alpha x(t) = {}_{-\infty}\mathbf{D}_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{x^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha-n+1}} \tag{1.37} $$

where $n - 1 < \alpha < n$.

**Short Memory Principle**

Let us consider the Gründwald-Letnikov derivative because it helps us see more easily how past values of $x(t)$ work for the fractional derivative. From the Gründwald-Letnikov derivative definition, if current time t is large, we need a lot of past values of $x(t)$ for calculating the fractional derivative. This property of fractional order derivatives is not attractable and useful for dealing with the practical engineering problem because

we have finite memory. Fortunately, for large t, we have the *short memory principle*[1]. This is described by

$$ {}_{t_0}D_t^{\alpha}x(t) \approx {}_{t-L}D_t^{\alpha}x(t), \qquad t > t_0 + L \tag{1.38} $$

where L is the "memory length". This means that the fractional derivative depends on mainly the "recent past" values of x(t). This approximation can attract many researchers to the fractional derivatives world. The estimated error associated with the approximation Eq. (1.38) has the following upper bound[1]

$$ \varepsilon(t) = \left| {}_{t_0}D_t^{\alpha}x(t) - {}_{t-L}D_t^{\alpha}x(t) \right| \leq \frac{ML^{-\alpha}}{|\Gamma(1-\alpha)|}, \qquad t_0 + L \leq t \leq T \tag{1.39} $$

where $x(t) \leq M$ for $t_0 \leq t \leq T$. In the case of the Caputo derivative, the *short memory principle* is frequently useful. The truncation error E is given by[11]

$$ E \leq \left| \frac{M}{\Gamma(1-\alpha)} \int_0^{t-L} \frac{1}{(t-\tau)^{\alpha}} d\tau \right| = \frac{M}{\Gamma(2-\alpha)} (t^{1-\alpha} - L^{1-\alpha}) \tag{1.40} $$

where $\sup_{\tau \in [0,t]} \left| x^{(1)}(\tau) \right| = M$ and $0 < \alpha < 1$.

## C. PROBLEM STATEMENT

Let us consider the following linear time-invariant fractional order system**.**

$$ aD^{4/2}x(t) + bD^{\alpha}x(t) + cx(t) = u(t) \tag{1.41} $$

If α is 3/2, this equation is known as the Bagley-Torvik equation[7]. More generally, $D^{\alpha}$ can be any fractional differential operator of order α among Riemann-Liouville definition, Caputo definition and Gründwald-Letnikov definition as explained previously. We choose the Caputo definition in order not to use the fractional order initial condition.

To interpret and examine the above fractional order system, the state-space representation can be used. For Caputo definition, Diethelm *et al* all showed the way to build up the state-space representation of the Bagley-Torvik equation. By applying his idea to the extension to the general fractional order case, the state-space representation algorithm can be done. Let us assume that α is 1/2. Now, we have the following form.

$$ {}_{t_0}^{C}D_t^{1/2}\mathbf{x}(t) = A\mathbf{x}(t) + Bu(t) \tag{1.42} $$

where $A \in \mathbb{R}^{4\times4}$, $B \in \mathbb{R}^{4\times1}$ and state vector are described by

$$ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c/a & -b/a & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/a \end{bmatrix} \text{ and } \mathbf{x}(t) = \begin{Bmatrix} x \\ D^{1/2}x \\ D^{2/2}x \\ D^{3/2}x \end{Bmatrix} \tag{1.43} $$

Input is given as the state feedback control form.

$$ \mathbf{u}(t) = F\mathbf{x}(t) + \mathbf{v}(t) \tag{1.44} $$

where $F \in \mathbb{R}^{1\times4}$ is fixed. Now, the system has the following closed-loop form.

$$ {}_{t_0}^{C}D_t^{1/2}\mathbf{x}(t) = (A + BF)\mathbf{x}(t) + B\mathbf{v}(t) \tag{1.45} $$

To consider more general case, let us assume that we have the following closed-loop form.

$$ {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = (A + BF)\mathbf{x}(t) + B\mathbf{v}(t) \tag{1.46} $$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $F \in \mathbb{R}^{m\times n}$. As the integer order case, we introduce the robust eigenstructure assignment problem which look for choosing real gain matrix F in order to put the closed-loop eigenvalues(poles) of the fractional order system into desired specific places. All closed-loop poles should be in the following stable region[12].

$$|\varphi| > \frac{\pi}{2}\alpha, \qquad \varphi = \arg\big(\mathrm{eig}(A + BF)\big) \tag{1.47}$$

where fractional order $0 < \alpha < 2$ and φ represents the argument of each eigenvalues of matrix A+BF. Also, for robust eigenstrucure assignment, assigned poles should be as insensitive as possible to the perturbations in A+BF.

After designing the gain matrix F, we need the estimator for the state. Hence, we propose the fractional Kalman filter defined by Caputo derivatives. The approach by Gründwald-Letnikov definition is already done by Sierociuk *et al*[13]. However, the Caputo derivative is more attractive to us for the reasons mentioned before, which leads us to derive the novel fractional Kalman filter defined by Caputo derivatives.

## D. OVERVIEW OF THE STUDY

This section overviews the generalized steps that will be followed in this dissertation. In this chapter we discussed the basic knowledge regarding the existing ideas in fractional calculus and explored the various properties such as solution forms, linearity, and short memory principle, and so on of fractional differential equations that utilize Caputo derivatives.

In chapter II we build formulations for state-space representation and linear system theory for the Caputo fractional order system. Also, we examine the stability, controllability and observability for fractional order systems

Chapter III deals with issues about eigenvalue and eigenvector sensitivity under the perturbations in fractional order systems. For the fractional order linear state-space models, we examine the eigenvalue sensitivity due to system matrix perturbation[3]. The

condition number for the closed loop eigenvector matrix, for the integer order case, is a widely appreciated metric for measuring the robustness of stability. For the fractional order case, we develop the robust stability formulations analogous to the integer derivative case[6]. We find the relationship between the perturbation of the closed-loop system matrix and the condition number of the closed loop systems modal(eigenvector) matrix.

In chapter IV we examine the robust eigenstructure assignment to obtain the real gain matrix F which places the closed-loop eigenvalues(poles) of the fractional order system into *a priori* prescribed locations. Also, we establish an algorithm to minimize the condition number of the modal matrix for the assigned poles to render them as insensitive as possible to the perturbations in closed-loop systems matrix. We compare our results with existing methods.

In chapter V we derive the Kalman filter for fractional order systems. For the Gründwald-Letnikov derivative, the fractional Kalman filter has already been introduced[13]. But we introduce the fractional order system Kalman filter defined by Caputo derivative. We derive the discrete-time fractional Kalman filter and investigate the various properties of the discrete-time fractional Kalman filter.

# CHAPTER II

# LINEAR CONTROL THEORY FOR FRACTIONAL ORDER SYSTEMS

## A. STATE-SPACE REPRESENTATION

Let us consider the following linear time-invariant fractional order system defined by the Caputo derivative.

$$a x^{(2)}(t) + b\,{}_{t_0}^{C}D_t^{\alpha} x(t) + c x(t) = u(t) \tag{2.1}$$

As mentioned in the previous chapter, we use the Caputo definition because it has the following two great advantages:

1) The Laplace transform of the Caputo derivative requires only the initial values of integer-order derivatives with known physical interpretations.
2) The Caputo derivative of a constant is 0, while the Riemann-Liouville derivative of a constant is not 0.

In this dissertation, we consider only the fractional order of $0 < \alpha < 1$. Therefore, we can solve Eq. (2.1) with the following initial conditions

$$x(0) = x_0, \qquad x'(0) = x_0' \tag{2.2}$$

How can we change Eq. (2.1) into state-space form? Diethelm and Ford suggest a good answer to the question of solving the Bagley-Torvik equation numerically using the Caputo definition[7]. For the Bagley-Torvik equation, they showed the state-space equations in the case of α=0.5 are as follows,

$${}_{t_0}^{C}D_t^{0.5} x_1(t) = x_2(t), \quad {}_{t_0}^{C}D_t^{0.5} x_2(t) = x_3(t), \quad {}_{t_0}^{C}D_t^{0.5} x_3(t) = x_4(t) \tag{2.3a}$$

$$ {}_{t_0}^{C}D_t^{0.5}x_4(t) = a^{-1}(-cx_1(t) - bx_2(t) + u(t)) \tag{2.3b} $$

For the matrix form, we can express it as

$$ {}_{t_0}^{C}D_t^{0.5}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a^{-1}c & -a^{-1}b & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + a^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{2.4} $$

with initial conditions $x_1(0) = x_0, x_2(0) = 0, x_3(0) = x_0', x_4(0) = 0$. We can extend this idea to the general case mentioned in [7][14].

Let us consider the following n-term linear fractional order differential equation.

$$ x^{(\beta_n)}(t) + p_{n-1}x^{(\beta_{n-1})}(t) + \cdots + p_1x^{(\beta_1)}(t) + p_0 = u(t) \tag{2.5} $$

where $x^{(\beta_k)}(t) = {}_{t_0}^{C}D_t^{\beta_k}x(t)$ and $\beta_i = \frac{a_i}{b_i}, a_i$ and $b_i \in \mathbb{N}$ for all i . $\beta_n > \beta_{n-1} > \cdots > \beta_1$ , $0 < \beta_1 \leq 1$, and $\beta_i - \beta_{i-1} \leq 1$ for all i. This differential equation is subject to the following initial conditions.

$$ x^{(k)}(t_0) = x_0^{(k)}, k = 0,1, \ldots, \lceil \beta_n \rceil - 1 \tag{2.6} $$

From the above equation, we want to find the state-space representation with the commensurate fractional order and the equivalent solution. To achieve this objective, we first need to know the following lemma shown in reference[7].

*Lemma 2.1*[14]

*Let $x(t) \in C^k[0,T]$ for some T>0 and some $k \in \mathbb{N}$, and let $\alpha \notin \mathbb{N}$ such that 0< $\alpha$ <k. Then,*

$$ {}_{t_0}^{C}D_t^{\alpha}x(t_0) = 0 \tag{2.7} $$

Proof

We rewrite Eq. (1.6).

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = {}_{t_0}I_t^{m-\alpha}x^{(m)}(t), \quad m = \lceil \alpha \rceil \tag{2.8} $$

Since $m \leq k$ and $x(t) \in C^k$, ${}_{t_0}D_t^m x(t)$ is a continuous function. Thus, we can show that the right side in Eq. (2.8) becomes zero as t goes to zero by

$$ \lim_{t\to 0} {}_{t_0}I_t^{m-\alpha}x^{(m)}(t) = \lim_{t\to 0} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-u)^{m-\alpha-1} x^{(m)}(u)\, du = 0 \tag{2.9} $$

And, from Eq. (1.9) another important relationship is given by

$$ {}_{t_0}\mathbf{D}_t^{\alpha}x(t) = {}_{t_0}^{C}D_t^{\alpha}x(t) \tag{2.10} $$

whenever $x(t_0^+) = 0$.

Now, we are going to prove the state-space representation of Eq. (2.5).

*Theorem 2.1*[14]

Equation (2.5) is equivalent to the following system of equations

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha}x_1(t) &= x_2(t) \\ {}_{t_0}^{C}D_t^{\alpha}x_2(t) &= x_3(t) \\ {}_{t_0}^{C}D_t^{\alpha}x_3(t) &= x_4(t) \\ &\vdots \\ {}_{t_0}^{C}D_t^{\alpha}x_N(t) &= -p_{n-1}x_{n-1}(t) - p_{n-2}x_{n-2}(t) - \cdots - p_1x_1(t) + u(t) \end{aligned} \tag{2.7} $$

subject to

$$x_i(t_0) = \begin{cases} x_0^{(k)} & \text{if } i = kM + 1 \text{ with } k = 0,1,\dots,\lceil \beta_n \rceil - 1, \in \mathbb{N} \\ 0 & \text{elsewhere} \end{cases} \tag{2.8}$$

where M is the least common multiple(LCM) of the denominators $(b_1, b_2, \cdots, b_n)$, $M = \mathrm{LCM}(b_1, b_2, \cdots, b_n)$. Then the fractional order of the corresponding commensurate state-space representation and the dimension N can be obtained by

$$\alpha = \frac{1}{M} \quad \text{and} \quad N = M \times \beta_n \tag{2.9}$$

Proof

First, from Lemma 2.1, we have ${}_{t_0}^{C}D_t^{\alpha} x(t_0) = 0$. Then let $x_1(t) = x(t)$ and let $x_2(t)$ have the following relationship.

$$x_2(t) = {}_{t_0}^{C}D_t^{\alpha} x_1(t) \tag{2.10}$$

By using ${}_{t_0}^{C}D_t^{\alpha} x(t_0) = 0$ and Eq. (2.10), the relationship between $x_3(t)$ and $x_1(t)$ is given by

$$x_3(t) = {}_{t_0}^{C}D_t^{\alpha} x_2(t) = {}_{t_0}^{C}D_t^{\alpha}\, {}_{t_0}^{C}D_t^{\alpha} x_1(t) = {}_{t_0}\mathbf{D}_t^{\alpha}\, {}_{t_0}^{C}D_t^{\alpha} x_1(t) \tag{2.11}$$

From the definition of the Caputo operator given in Eq. (1.6), we have

$${}_{t_0}\mathbf{D}_t^{\alpha}\, {}_{t_0}^{C}D_t^{\alpha} x_1(t) = {}_{t_0}\mathbf{D}_t^{\alpha}\, {}_{t_0}I_t^{1-\alpha} x^{(1)}(t) \tag{2.12}$$

By the property of the mixed operator of Riemann-Liouville derivative and the fractional integral given in Eq. (1.22), the right side of Eq. (2.12) becomes

$${}_{t_0}\mathbf{D}_t^{\alpha}\, {}_{t_0}I_t^{1-\alpha} x^{(1)}(t) = {}_{t_0}I_t^{1-2\alpha} x^{(1)}(t) = {}_{t_0}^{C}D_t^{2\alpha} x_1(t) \tag{2.13}$$

The last equality follows from Eq. (1.6). Therefore, we show $x_3(t) = {}_{t_0}^{C}D_t^{2\alpha} x_1(t)$. By applying the above procedure to $x_k(t), k = 4, \dots, N$, we have the following equations.

$$
\begin{aligned}
x_4(t) &= {}_{t_0}^{C}D_t^{\alpha}x_3(t) = {}_{t_0}^{C}D_t^{\alpha}\,{}_{t_0}^{C}D_t^{2\alpha}x_1(t) = {}_{t_0}\mathbf{D}_t^{\alpha}\,{}_{t_0}^{C}D_t^{2\alpha}x_1(t) \\
&= {}_{t_0}\mathbf{D}_t^{\alpha}\,{}_{t_0}I_t^{1-2\alpha}x^{(1)}(t) = {}_{t_0}I_t^{1-3\alpha}x^{(1)}(t) = {}_{t_0}^{C}D_t^{3\alpha}x_1(t) \\
&\vdots \\
x_{M+1}(t) &= {}_{t_0}^{C}D_t^{\alpha}x_M(t) = {}_{t_0}^{C}D_t^{\alpha}\,{}_{t_0}^{C}D_t^{(M-1)\alpha}x_1(t) = {}_{t_0}\mathbf{D}_t^{\alpha}\,{}_{t_0}^{C}D_t^{(M-1)\alpha}x_1(t) \\
&= {}_{t_0}\mathbf{D}_t^{\alpha}\,{}_{t_0}I_t^{1-(M-1)\alpha}x_1^{(1)}(t) = {}_{t_0}\mathbf{D}_t^{\alpha}\,{}_{t_0}I_t^{\alpha}x_1^{(1)}(t) = x_1^{(1)}(t) \\
&\vdots \\
x_N(t) &= {}_{t_0}^{C}D_t^{\alpha}x_{N-1}(t) = {}_{t_0}^{C}D_t^{\alpha}\,{}_{t_0}^{C}D_t^{(N-2)\alpha}x_1(t) = {}_{t_0}\mathbf{D}_t^{\alpha}\,{}_{t_0}^{C}D_t^{(N-2)\alpha}x_1(t) \\
&= {}_{t_0}\mathbf{D}_t^{\alpha}\,{}_{t_0}I_t^{\lceil\beta_n\rceil-1-(N-2)\alpha}x_1^{(\lceil\beta_n\rceil-1)}(t) \\
&= {}_{t_0}I_t^{\lceil\beta_n\rceil-1-N\alpha+\alpha}x_1^{(\lceil\beta_n\rceil-1)}(t) = {}_{t_0}I_t^{\lceil\beta_n\rceil-1-\beta_n+\alpha}x_1^{(\lceil\beta_n\rceil-1)}(t) \\
&= {}_{t_0}^{C}D_t^{\beta_n-\alpha}x_1(t) \\
{}_{t_0}^{C}D_t^{\alpha}x_N(t) &= {}_{t_0}^{C}D_t^{\alpha}\,{}_{t_0}^{C}D_t^{(N-1)\alpha}x_1(t) = {}_{t_0}\mathbf{D}_t^{\alpha}\,{}_{t_0}^{C}D_t^{(N-1)\alpha}x_1(t) \\
&= {}_{t_0}\mathbf{D}_t^{\alpha}\,{}_{t_0}I_t^{\lceil\beta_n\rceil-1-(N-1)\alpha}x_1^{(\lceil\beta_n\rceil-1)}(t) \\
&= {}_{t_0}I_t^{\lceil\beta_n\rceil-1-\beta_n}x_1^{(\lceil\beta_n\rceil-1)}(t) = {}_{t_0}^{C}D_t^{\beta_n}x_1(t)
\end{aligned}
\tag{2.14}
$$

From the above relationships and Eq. (2.5), we finally obtain the following equation.

$$
{}_{t_0}^{C}D_t^{\alpha}x_N(t) = -p_{n-1}x_{n-1}(t) - p_{n-2}x_{n-2}(t) - \cdots - p_1x_1(t) - p_0 + u(t) \tag{2.15}
$$

Now, we can transform any fractional-order equations defined by the Caputo definition into the state-space representation. As you can expect, this seems to have the *curse of dimensionality* if we have a very small common fractional order. To avoid this problem, one can use the incommensurate order[15]. However, in this dissertation, we only consider the commensurate order case for convenience of the analysis.

From Theorem 2.1, we have the corresponding matrix form by

$$ {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = A\mathbf{x}(t) + Bu(t) \tag{2.16} $$

where the state vector can be given by $\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]^{\mathbf{T}}$ and the matrices A and B are given by

$$ A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & \cdots & -p_{n-1} \end{bmatrix} \tag{2.17} $$

$$ B = [0 \quad 0 \quad \cdots \quad 1]^T $$

If we have a general linear output equation, it can be given by

$$ y(t) = H\mathbf{x}(t) + Du(t) \tag{2.18} $$

where $y \in \mathbb{R}^{1\times1}$, $H \in \mathbb{R}^{1\times n}$ and $D \in \mathbb{R}^{1\times m}$.

By using the Laplace transform with zero initial conditions, we can have the s-domain representations given by

$$ s^{\alpha}X(s) = AX(s) + BU(s) \tag{2.19a} $$

$$ Y(s) = HX(s) + DU(s) \tag{2.19b} $$

where s is the Laplace variable. In order to obtain the input-output transfer function, we substitute $X(s)$ solved from Eq (2.19a) into Eq. (2.19b), which yields

$$ Y(s) = (H(s^{\alpha}I - A)^{-1}B + D)U(s) \tag{2.20} $$

We call the determinant of $(s^{\alpha}I - A)$ the *characteristic equation* because it determines the stability and behavior of the system. Also, the solution of the *characteristic equation* gives us the poles of the transfer function in the s-domain. Generally, it is difficult to find these poles in the s-domain for fractional order systems because of the $s^{\alpha}$ form.

Instead, we transform the s-domain into the w-domain by $w = s^{\alpha}$. Then we call the solution of determinant of $(wI - A)$ poles for fractional order systems[16][17]. By examining these poles, we can check the stability of fractional order systems. We will examine the stability characteristics in a later section.

## B. EIGENVALUES AND EIGENVECTORS OF FRACTIONAL ORDER LINEAR SYSTEMS AND STABILITY ANALYSIS

In this section, we want to investigate the stability of the linear fractional order system. Let us consider Eq. (2.16). We assume that the input u(t) is zero. Then we have the following fractional order system

$$ {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = A\mathbf{x}(t) \tag{2.21}$$

This is the *eigenvalue problem* for the fractional order system. From now on, we will show that the eigenvalue matrix has the diagonal form of $\Lambda$ and the corresponding eigenvector matrix is $\Phi$ for the above fractional order system. Assume that A can be diagonalizable, which means that A is a non-defective matrix. Hence, there exists the coordinate transformation which makes the matrix A diagonal form. We can introduce the modal vector $\mathbf{z}$ which has the following transformation.

$$\mathbf{x}(t) = \Phi\mathbf{z}(t) \tag{2.22}$$

By our assumption, the coordinate transformation $\Phi$ has the following diagonalizing property.

$$\Phi^{-1}A\Phi = \Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\} \tag{2.23}$$

With Eq. (2.22), Eq. (2.21) can be written as

$$\Phi\, {}_{t_0}^{C}D_t^{\alpha}\mathbf{z}(t) = A\Phi\mathbf{z}(t) \tag{2.24}$$

By the pre-multiplication of $\Phi^{-1}$ on both sides, we have

$${}_{t_0}^{C}D_t^{\alpha}\mathbf{z}(t) = \Phi^{-1}A\Phi\mathbf{z}(t) = \Lambda\mathbf{z}(t) \tag{2.25}$$

Then each element $z_i(t)$ of $\mathbf{z}(t)$ has the following uncoupled differential equation.

$${}_{t_0}^{C}D_t^{\alpha}z_i(t) = \lambda_i z_i(t) \tag{2.26}$$

And the solution is given by

$$z_i(t) = E_{\alpha}(\lambda_i(t-t_0)^{\alpha})z_i(t_0) \tag{2.27}$$

Hence, $\mathbf{z}(t)$ can be expressed in a state transition matrix form.

$$\mathbf{z}(t) = \Psi(t,t_0)\mathbf{z}(t_0) \tag{2.28}$$

where

$$\Psi(t,t_0) \equiv \begin{bmatrix} E_{\alpha}(\lambda_1(t-t_0)^{\alpha}) & 0 & \cdots & 0 \\ 0 & E_{\alpha}(\lambda_2(t-t_0)^{\alpha}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{\alpha}(\lambda_n(t-t_0)^{\alpha}) \end{bmatrix} \tag{2.29}$$

Finally, the solution of the original fractional order system can be given by

$$\mathbf{x}(t) = \Phi\Psi(t,t_0)\Phi^{-1}\,\mathbf{x}(t_0) = \Xi(t,t_0)\,\mathbf{x}(t_0) \tag{2.30}$$

where $\Xi(t,t_0) = \Phi\Psi(t,t_0)\Phi^{-1}$ . To show that the eigenvalue matrix is $\Lambda$ and the eigenvector matrix is $\Phi$ for the original fractional order system, we use the solution obtained in Eq. (2.30). By substituting this solution into Eq. (2.21), the left hand side of Eq. (2.21) can be given by

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = {}_{t_0}^{C}D_t^{\alpha}\Xi(t,t_0)\mathbf{x}(t_0) &= \Phi\, {}_{t_0}^{C}D_t^{\alpha}\Psi(t,t_0)\Phi^{-1}\,\mathbf{x}(t_0) \\ &= \Phi\Lambda\Psi(t,t_0)\Phi^{-1}\,\mathbf{x}(t_0) \end{aligned} \tag{2.31}$$

And the right hand side of Eq. (2.21) can be given by

$$A\mathbf{x}(t) = A\Xi(t, t_0)\,\mathbf{x}(t_0) = A\Phi\Psi(t, t_0)\Phi^{-1}\,\mathbf{x}(t_0) \tag{2.32}$$

Eq. (2.31) and Eq. (2.32) should be equal.

$$A\Phi\Psi(t, t_0)\Phi^{-1}\,\mathbf{x}(t_0) = \Phi\Lambda\Psi(t, t_0)\Phi^{-1}\,\mathbf{x}(t_0) \tag{2.33}$$

Therefore, Eq. (2.24) yields

$$\Phi\Lambda = A\Phi \tag{2.34}$$

Now, this can be expressed as the following eigenvalue-eigenvector pair form.

$$\lambda_i\boldsymbol{\phi}_i = A\boldsymbol{\phi}_i \tag{2.35}$$

Then, $\Phi = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_n]$. Therefore, eigenvalues of the fractional order system given by Eq. (2.20) are $\lambda_i (i = 1, \dots, n)$. The corresponding eigenvectors are $\boldsymbol{\phi}_i (i = 1, \dots, n)$ determined by Eq. (2.35). This means that the coordinate transformation matrix $\Phi$ given in Eq. (2.21) is the eigenvector matrix for the original fractional order equation and the diagonalized matrix $\Lambda$ is the eigenvalue matrix.

Also, we can find the poles(eigenvalues) by using the Laplace transform. With zero initial conditions, Eq. (2.20) can be described by

$$(s^{\alpha}I - A)X(s) = 0 \tag{2.36}$$

For the s-domain, the characteristic equation is given by

$$\det(s^{\alpha}I - A) = 0 \tag{2.37}$$

We can verify the stability if the poles are in the left half plane of complex s-domain. For some $\alpha$, it is difficult to obtain the s values to satisfy the above equation. As such, we need a more convenient method to check the fractional order system. This can be done with another complex map given by the $s^{\alpha} = w$ transformation[16][17]. Then our characteristic equation is given by

$$\det(wI - A) = 0 \tag{2.38}$$

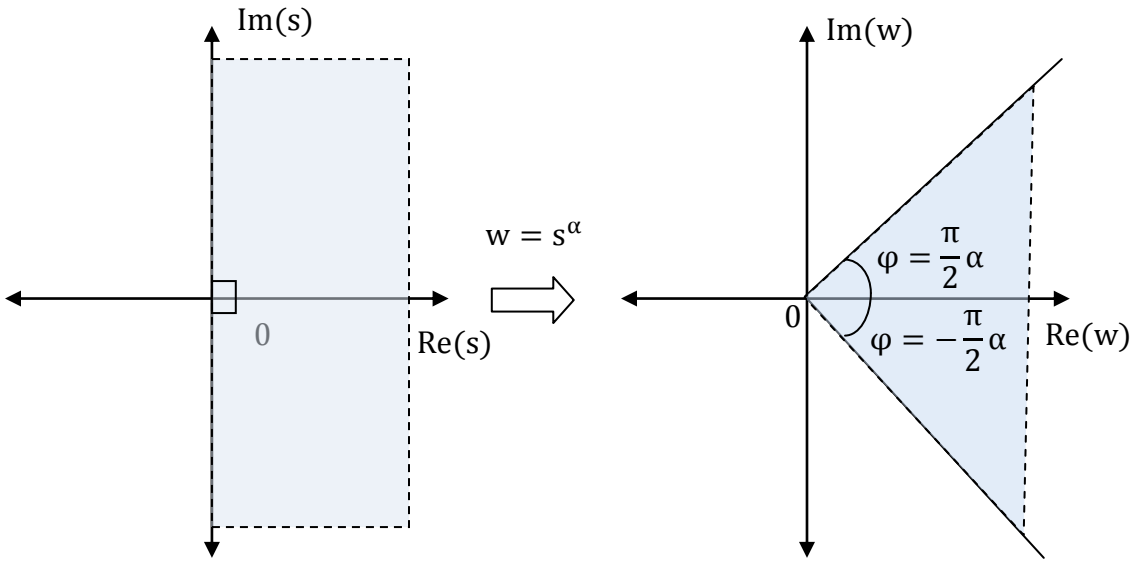


**Figure 2.1**. Unstable Regions of s-domain and w-domain

For checking the stability in the complex w-plane, we need to check the transformation of the stability boundary $\left(s = re^{\pm\frac{\pi}{2}i}, 0 \leq r \in \mathcal{R} \leq \infty\right)$ into the w-plane. This transformation equation can be given by

$$w = s^{\alpha} = r^{\alpha}e^{\pm\frac{\pi}{2}\alpha i} \tag{2.39}$$

This means that the imaginary axis in s-plane transforms to a wedge which consists of two lines with the corresponding arguments, $\varphi = \pm\frac{\pi}{2}\alpha$. And the right half plane in the s-plane maps into the region with $|\varphi| < \frac{\pi}{2}\alpha$ in Fig. (2.1). Therefore, the stable region in the

complex w-plane can be checked if the arguments of poles are satisfied with the following inequality.

$$|\varphi| > \frac{\pi}{2}\alpha \tag{2.40}$$

Also, this stability region has been examined with other researchers. The following theorem shows the stability of the fractional order system.

*Theorem 2.2*[12][15][16][17][18]

The fractional order system given in Eq. (2.21) is stable if and only if the following inequality is satisfied.

$$|\varphi| > \frac{\pi}{2}\alpha \tag{2.41}$$

where $0 < \alpha < 2$ and $\varphi = \arg\big(\mathrm{eig}(A)\big)$.

According to the argument φ of eigenvalues and the fractional order of a system, the step responses can be summarized in Table 2.1[16][17].

**Table 2.1**. Step Responses

| Argument φ | Stability and Time Response |
|---|---|
| $\lvert\varphi\rvert < \frac{\pi}{2}\alpha$ | Unstable and oscillatory |
| $\frac{\pi}{2}\alpha < \lvert\varphi\rvert < \pi\alpha$ | *Underdamped*(Stable and oscillatory) |
| $\lvert\varphi\rvert = \pi\alpha$ | *Overdamped*(Stable ) |
| $\pi\alpha < \lvert\varphi\rvert < \pi$ | *Hyperdamped*(Stable) |
| $\lvert\varphi\rvert = \pi$ | *Ultradamped*(or over-hyperdamped, Stable) |

Example 2.1. Let us consider the following fractional system.

$$x^{(1)}(t) - {}_0^C D_t^{0.5} x(t) + x(t) = u(t) \tag{2.42}$$

By theorem 2.1, the state space representation of above system can be expressed as

$${}_0^C D_t^{0.5} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.43}$$

We assume that the initial conditions are given as $x_1(0) = x(0)$ and $x_2(0) = 0$. To obtain the eigenvalues, we use the following characteristic equation.

$$\det(s^{0.5} I - A) = 0 \;\leftrightarrow\; S^1 - s^{0.5} + 1 = 0 \tag{2.44}$$

Let $w = s^{0.5}$. Then we have the characteristic equation in terms of w.

$$w^2 - w + 1 = 0 \tag{2.45}$$

The solutions are obtained by $w_{1,2} = \frac{1 \pm \sqrt{3}i}{2}$. To check the stability of these poles, we examine if the arguments of the eigenvalues are satisfied with the stability inequality $\left( \left( \left| \varphi_{1,2} \right| = 1.1462 \right) > \left( \frac{\pi}{2} \alpha = 0.7854 \right) \right)$. Therefore, this fractional order system is stable.

## C. SOLUTION OF LINEAR FRACTIONAL ORDER SYSTEMS

We want to find the solution of time-invariant linear fractional order systems with $0 < \alpha < 1$ given in Eq. (2.15). By the Laplace transform of Eq. (2,15), we have the following s-domain equation.

$$X(s) = (s^\alpha I - A)^{-1} s^{\alpha - 1} \mathbf{x}_0 + (s^\alpha I - A)^{-1} B U(s) \tag{2.46}$$

By the inverse Laplace transform of Eq. (2.46), the solution of Eq. (2.15) in the time domain can be given by

$$\mathbf{x}(t) = E_\alpha(A(t-t_0)^\alpha)\mathbf{x}_0 + \int_{t_0}^{t}(t-\xi)^{\alpha-1}E_{\alpha,\alpha}(A(t-\xi)^\alpha)\,Bu(\mathbf{x}(\xi),\xi)d\xi \qquad (2.47)$$

where the Mittag-Leffler function, a generalization of the exponential function, is defined as

$$E_\alpha(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad (z \in \mathbb{C}; \Re(\alpha) > 0) \qquad (2.48a)$$

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (z,\beta \in \mathbb{C}; \Re(\alpha) > 0) \qquad (2.48b)$$

We can see that $E_1(z) = \exp(z)$ and $E_\alpha(z) = E_{\alpha,1}(z)$. Also, we define the α-exponential function as

$$e_\alpha^{\lambda z} := z^{\alpha-1}\sum_{k=0}^{\infty}\lambda^k\frac{z^{\alpha k}}{\Gamma(\alpha(k+1))} = z^{\alpha-1}E_{\alpha,\alpha}(\lambda z^\alpha), \quad (z \in \mathbb{C}\backslash\{0\}, \lambda \in \mathbb{C}; \Re(\alpha) > 0) \qquad (2.49)$$

By using the α-exponential function, Eq. (2.47) can be described by

$$\mathbf{x}(t) = E_\alpha(A(t-t_0)^\alpha)\mathbf{x}_0 + \int_{t_0}^{t} e_\alpha^{A(t-\xi)}\,Bu(\xi)d\xi \qquad (2.50)$$

Let us consider '$t_0$' is equal to 0. In this case, Eq. (2.50) is expressed as

$$\mathbf{x}(t) = E_\alpha(At^\alpha)\mathbf{x}_0 + \int_{0}^{t} e_\alpha^{A(t-\xi)}\,Bu(\xi)d\xi \qquad (2.51)$$

## D. CONTROLLABILITY AND OBSERVABILITY

### Controllability

From Eq. (2.48) and Eq. (2.49), we can express the α-exponential function given in Eq. (2.51) as

$$e_{\alpha}^{A(t-\xi)} = (t-\xi)^{\alpha-1}\sum_{k=0}^{\infty}\frac{(A(t-\xi)^{\alpha})^{k}}{\Gamma(\alpha k+\alpha)} = \sum_{k=0}^{\infty}(t-\xi)^{\alpha-1}\frac{((t-\xi)^{\alpha})^{k}}{\Gamma(\alpha k+\alpha)}A^{k} \tag{2.52}$$

By Cayley-Hamilton Theorem[3], $A^n$ can be written as a linear combination of {I, A, …, $A^{n-1}$}. Also, $A^i$ for i>n can be written as a linear combination of {I, A, …, $A^{n-1}$}. Therefore, the above equation can be expressed as[19]

$$e_{\alpha}^{A(t-\xi)} = \sum_{k=0}^{n-1} p_k(t-\xi)A^k \tag{2.53}$$

where $p_k(t)$ is the sum of the coefficients related to the $A^k$ for $k = 0,1,\dots,n-1$, respectively. Then the solution at $t = t_f$ can be given as

$$\mathbf{x}(t_f) = E_{\alpha}(At_f^{\alpha})\mathbf{x}(0^+) + \int_0^{t_f}\sum_{k=0}^{n-1} p_k(t_f-\xi)\,A^kBu(\xi)d\xi \tag{2.54}$$

By spreading the summation, we have

$$\mathbf{x}(t_f) - E_{\alpha}(At_f^{\alpha})\mathbf{x}(0^+) = [B \quad AB \quad \cdots \quad A^{n-1}B]\begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1}\end{bmatrix} \tag{2.55}$$

where $r_k = \int_0^{t_f} p_k(t-\xi)\,u(\xi)d\xi$.. From the above equation, we can state that the system inputs can drive any initial state $\mathbf{x}(0^+)$ to the arbitrary final state $\mathbf{x}(t_f)$ in finite time($0 < t < t_f$) if n × nm matrix $[B \quad AB \quad \cdots \quad A^{n-1}B]$ has rank n. This means that the system is *controllable* if $[B \quad AB \quad \cdots \quad A^{n-1}B]$ has rank n, which is the same as integer order systems.

**Observability**

Observable canonical form can be described by

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}_o(t) &= A_o\mathbf{x}_o(t) + B_o u(t) \\ y_o &= H_o\mathbf{x}_o(t) + D_o u(t) \end{aligned} \tag{2.56}$$

where the matrices $A_o \in \mathbb{R}^{n\times n}$ , $B_o \in \mathbb{R}^{n\times m}$, $H_o \in \mathbb{R}^{1\times n}$ and $D_o \in \mathbb{R}^{1\times m}$ are given by

$$A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & \cdots & 0 & -p_1 \\ 0 & 1 & \cdots & 0 & -p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -p_{n-1} \end{bmatrix} \tag{2.57}$$

$$H_o = [0 \quad 0 \quad \cdots \quad 1]$$

If a general single-output system given by (A, B, H, D) can be transformed to the observer canonical form given in Eq. (2.56), it is "fully observable"[4]. By using a transformation of state, we can convert the state-space described by Eq. (2.16) to the observable canonical form. We can introduce the new vector which has the following transformation relationship.

$$\mathbf{x}(t) = T\mathbf{x}_o(t) \tag{2.58}$$

By substituting Eq. (2.58) into Eq. (2.16), we have

$$T\,{}_{t_0}^{C}D_t^{\alpha}\mathbf{x}_o(t) = AT\mathbf{x}_o(t) + Bu(t) \tag{2.59}$$

By pre-multiplication of the inverse of T on both sides, we have

$${}_{t_0}^{C}D_t^{\alpha}\mathbf{x}_o(t) = T^{-1}AT\mathbf{x}_o(t) + T^{-1}Bu(t) \tag{2.60}$$

Now, the coordinate transformation T should be satisfied with the following property.

$$T^{-1}AT = A_0 \tag{2.61}$$

By the pre-multiplication of T on both sides, we have

$$AT = TA_0 \tag{2.62}$$

With $T = [\mathbf{t}_1 \quad \mathbf{t}_2 \quad \cdots \quad \mathbf{t}_n]$, the above equation can be expressed by

$$A[\mathbf{t}_1 \quad \mathbf{t}_2 \quad \cdots \quad \mathbf{t}_n] = [\mathbf{t}_1 \quad \mathbf{t}_2 \quad \cdots \quad \mathbf{t}_n]\begin{bmatrix} 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & \cdots & 0 & -p_1 \\ 0 & 1 & \cdots & 0 & -p_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -p_{n-1} \end{bmatrix} \tag{2.63}$$

Then we have some relationships as

$$\begin{gathered} \mathbf{t}_2 = A\mathbf{t}_1 \\ \mathbf{t}_3 = A\mathbf{t}_2 = A^2\mathbf{t}_1 \\ \vdots \\ \mathbf{t}_n = A\mathbf{t}_{n-1} = A^n\mathbf{t}_1 \end{gathered} \tag{2.64}$$

By Eq. (2.58) and Eq. (2.18),

$$y(t) = y_o(t) \leftrightarrow HT\mathbf{x}_o(t) = H_o\mathbf{x}_o(t) \tag{2.65}$$

From the above equation, $HT = H[\mathbf{t}_1 \quad \mathbf{t}_2 \quad \cdots \quad \mathbf{t}_n] = H_o$. Then we have the following equation.

$$\begin{gathered} H\mathbf{t}_1 = 0 \\ H\mathbf{t}_2 = 0 \\ \vdots \\ H\mathbf{t}_n = 1 \end{gathered} \tag{2.66}$$

Substituting Eq. (2.64) into Eq. (2.66) yields

$$\mathbf{t}_1 = \begin{bmatrix} H \\ HA \\ \vdots \\ HA^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{2.67}$$

If $[H \quad HA \quad \cdots \quad HA^{n-1}]^T$ is nonsingular, there exists the state transformation to make the original system the observable canonical form. Therefore, we have to check if $[H \quad HA \quad \cdots \quad HA^{n-1}]^T$ has rank n for observability.

Now, we want to have the discrete time representation of Eq. (2.42). We will try two approaches and show the numerical simulations for both approaches.

## E. DISCRETE-TIME SYSTEMS

### Approach I

From Eq. (2.51), the solution has the following form at $t=t_{n+1}$.

$$\mathbf{x}(t_{n+1}) = E_\alpha(At_{n+1}{}^\alpha)\mathbf{x}(0^+) + \int_0^{t_{n+1}} e_\alpha^{A(t_{n+1}-\xi)} Bu(\xi)d\xi \tag{2.68}$$

And, the solution at $t=t_n$ has the following form.

$$\mathbf{x}(t_n) = E_\alpha(At_n{}^\alpha)\mathbf{x}(0^+) + \int_0^{t_n} e_\alpha^{A(t_n-\xi)} Bu(\xi)d\xi \tag{2.69}$$

where $t_{n+1} - t_n = \Delta t, \text{a sampling interval}$. We assume that the sampling interval is fixed. Subtracting Eq. (2.69) from Eq. (2.68) leads to

$$\begin{aligned}\mathbf{x}(t_{n+1}) = \mathbf{x}(t_n) + \left(E_\alpha(At_{n+1}{}^\alpha) - E_\alpha(At_n{}^\alpha)\right)\mathbf{x}(0^+) \\ - \int_0^{t_n} e_\alpha^{A(t_n-\xi)} Bu(\xi)d\xi + \int_0^{t_{n+1}} e_\alpha^{A(t_{n+1}-\xi)} Bu(\xi)d\xi\end{aligned} \tag{2.70}$$

The last term on the right hand side can be divided into two terms as

$$\int_0^{t_{n+1}} e_\alpha^{A(t_{n+1}-\xi)} Bu(\xi)d\xi = \int_0^{t_n} e_\alpha^{A(t_{n+1}-\xi)} Bu(\xi)d\xi + \int_{t_n}^{t_{n+1}} e_\alpha^{A(t_{n+1}-\xi)} Bu(\xi)d\xi \tag{2.71}$$

With Eq. (2.70) and Eq. (2.71), we have the following equation.

$$\mathbf{x}(t_{n+1}) = \mathbf{x}(t_n) + \left(E_\alpha(At_{n+1}{}^\alpha) - E_\alpha(At_n{}^\alpha)\right)\mathbf{x}(0^+) + \int_0^{t_n} (e_\alpha^{A(t_{n+1}-\xi)} - e_\alpha^{A(t_n-\xi)})Bu(\xi)d\xi + \int_{t_n}^{t_{n+1}} e_\alpha^{A(t_{n+1}-\xi)} Bu(\xi)d\xi \tag{2.72}$$

Now assume that the control input is constant during a sampling interval, $\Delta t$. Then we want to find the analytical solution to $\int_{t_n}^{t_{n+1}} e_\alpha^{A(t_{n+1}-\xi)} Bu(\xi)d\xi$. The following derivative equation is given in[1][20].

$$\frac{\partial}{\partial z} z^{\beta-1} E_{\alpha,\beta}(\lambda z^\alpha) = z^{\beta-2} E_{\alpha,\beta-1}(\lambda z^\alpha) \tag{2.73}$$

Letting $\beta = \alpha + 1$ and substituting it into Eq. (2.73), we have

$$\frac{\partial}{\partial z} z^\alpha E_{\alpha,\alpha+1}(\lambda z^\alpha) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha) \tag{2.74}$$

We can see that the right hand side of Eq. (2.74) is equal to $e_\alpha^{\lambda z}$. Integrating both sides in Eq. (2.74) with respect to z yields

$$\int_a^b \left(\frac{\partial}{\partial z} z^\alpha E_{\alpha,\alpha+1}(\lambda z^\alpha)\right) dz = \int_a^b z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha) dz$$

$$z^\alpha E_{\alpha,\alpha+1}(\lambda z^\alpha)\Big|_a^b = \int_a^b z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha) dz = \int_a^b e_\alpha^{\lambda z} dz \tag{2.75}$$

$$\int_a^b e_\alpha^{\lambda z} dz = b^\alpha E_{\alpha,\alpha+1}(\lambda b^\alpha) - a^\alpha E_{\alpha,\alpha+1}(\lambda a^\alpha)$$

By the change of variables, Eq. (2.75) becomes

$$\int_{t_n}^{t_{n+1}} e_\alpha^{A(t_{n+1}-\xi)} Bu(\xi)d\xi = (t_{n+1} - t_n)^\alpha E_{\alpha,\alpha+1}(A(t_{n+1} - t_n)^\alpha) Bu(t_n) = \Delta t^\alpha E_{\alpha,\alpha+1}(A\Delta t^\alpha) Bu(t_n) \tag{2.76}$$

By applying the above result to Eq. (2.72), we can obtain the following relationship.

$$\mathbf{x}(t_{n+1}) = \mathbf{x}(t_n) + \left(E_\alpha(At_{n+1}{}^\alpha) - E_\alpha(At_n{}^\alpha)\right)\mathbf{x}(0^+) + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (e_\alpha^{A(t_{n+1}-\xi)} - e_\alpha^{A(t_n-\xi)})\, d\xi\, Bu(t_k) + \Delta t^\alpha E_{\alpha,\alpha+1}(A\Delta t^\alpha)Bu(t_n) \tag{2.77}$$

Finally, we obtain the discrete-time representation.

$$\mathbf{x}(t_{n+1}) = \mathbf{x}(t_n) + \Psi(t_{n+1}, t_n)\mathbf{x}(0^+) + \sum_{k=0}^{n-1} \Lambda(t_{n+1}, t_n, t_{k+1}, t_k)\, u(t_k) + \Gamma(\Delta t)u(t_n) \tag{2.78}$$

where

$$\Psi(t_{n+1}, t_n) = E_\alpha(At_{n+1}{}^\alpha) - E_\alpha(At_n{}^\alpha) \tag{2.79}$$

$$\begin{aligned}\Lambda(t_{n+1}, t_n, t_{k+1}, t_k) &= \Big((t_{n+1} - t_k)^\alpha E_{\alpha,\alpha+1}(A(t_{n+1} - t_k)^\alpha) \\ &\quad - (t_{n+1} - t_{k+1})^\alpha E_{\alpha,\alpha+1}(A(t_{n+1} - t_{k+1})^\alpha)\Big) \\ &\quad - \Big((t_n - t_k)^\alpha E_{\alpha,\alpha+1}(A(t_n - t_k)^\alpha) \\ &\quad - (t_n - t_{k+1})^\alpha E_{\alpha,\alpha+1}(A(t_n - t_{k+1})^\alpha)\Big) B\end{aligned} \tag{2.80}$$

$$\Gamma(\Delta t) = \Delta t^\alpha E_{\alpha,\alpha+1}(A\Delta t^\alpha)B \tag{2.81}$$

**Approach II**

This approach aims at eliminating the presence of initial values, $\mathbf{x}(0^+)$, in the final discrete-time representation. From Eq. (2.69), we obtain the following equation.

$$\mathbf{x}(0^+) = \mathrm{E}_\alpha(\mathrm{At_n}^\alpha)^{-1}\left(\mathbf{x}(\mathrm{t_n}) - \int_0^{\mathrm{t_n}} \mathrm{e}_\alpha^{\mathrm{A}(\mathrm{t_n}-\xi)}\,\mathrm{Bu}(\xi)\mathrm{d}\xi\right) \tag{2.82}$$

Substituting above equation into Eq. (2.70) leads to

$$\begin{aligned}\mathbf{x}(\mathrm{t_{n+1}}) = {} & \mathrm{E}_\alpha(\mathrm{At_{n+1}}^\alpha)\mathrm{E}_\alpha(\mathrm{At_n}^\alpha)^{-1}\mathbf{x}(\mathrm{t_n}) \\ & - \mathrm{E}_\alpha(\mathrm{At_{n+1}}^\alpha)\mathrm{E}_\alpha(\mathrm{At_n}^\alpha)^{-1}\int_0^{\mathrm{t_n}} \mathrm{e}_\alpha^{\mathrm{A}(\mathrm{t_n}-\xi)}\,\mathrm{Bu}(\xi)\mathrm{d}\xi \\ & + \int_0^{\mathrm{t_{n+1}}} \mathrm{e}_\alpha^{\mathrm{A}(\mathrm{t_{n+1}}-\xi)}\,\mathrm{Bu}(\xi)\mathrm{d}\xi\end{aligned} \tag{2.83}$$

By splitting the last term and combining it with the second term on the right, we have

$$\begin{aligned}\mathbf{x}(\mathrm{t_{n+1}}) = {} & \mathrm{E}_\alpha(\mathrm{At_{n+1}}^\alpha)\mathrm{E}_\alpha(\mathrm{At_n}^\alpha)^{-1}\mathbf{x}(\mathrm{t_n}) \\ & + \int_0^{\mathrm{t_n}} \left(\mathrm{e}_\alpha^{\mathrm{A}(\mathrm{t_{n+1}}-\xi)} - \mathrm{E}_\alpha(\mathrm{At_{n+1}}^\alpha)\mathrm{E}_\alpha(\mathrm{At_n}^\alpha)^{-1}\mathrm{e}_\alpha^{\mathrm{A}(\mathrm{t_n}-\xi)}\right)\mathrm{Bu}(\xi)\mathrm{d}\xi \\ & + \int_{\mathrm{t_n}}^{\mathrm{t_{n+1}}} \mathrm{e}_\alpha^{\mathrm{A}(\mathrm{t_{n+1}}-\xi)}\,\mathrm{Bu}(\xi)\mathrm{d}\xi\end{aligned} \tag{2.84}$$

Then we change the continuous integral to the discrete-time summation.

$$\begin{aligned}\mathbf{x}(\mathrm{t_{n+1}}) = {} & \mathrm{E}_\alpha(\mathrm{At_{n+1}}^\alpha)\mathrm{E}_\alpha(\mathrm{At_n}^\alpha)^{-1}\mathbf{x}(\mathrm{t_n}) \\ & + \sum_{\mathrm{k}=0}^{\mathrm{n}-1}\int_{\mathrm{t_k}}^{\mathrm{t_{k+1}}} \left(\mathrm{e}_\alpha^{\mathrm{A}(\mathrm{t_{n+1}}-\xi)}\right. \\ & \left. - \mathrm{E}_\alpha(\mathrm{At_{n+1}}^\alpha)\mathrm{E}_\alpha(\mathrm{At_n}^\alpha)^{-1}\mathrm{e}_\alpha^{\mathrm{A}(\mathrm{t_n}-\xi)}\right)\mathrm{d}\xi\,\mathrm{Bu}(\mathrm{t_k}) \\ & + \Delta\mathrm{t}^\alpha\mathrm{E}_{\alpha,\alpha+1}(\mathrm{A}\Delta\mathrm{t}^\alpha)\mathrm{Bu}(\mathrm{t_n})\end{aligned} \tag{2.85}$$

Finally, we have the following discrete-time representation.

$$\mathbf{x}(\mathrm{t_{n+1}}) = \Phi(\mathrm{t_{n+1}}, \mathrm{t_n})\mathbf{x}(\mathrm{t_n}) + \sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{Y}(\mathrm{t_{n+1}}, \mathrm{t_n}, \mathrm{t_{k+1}}, \mathrm{t_k})\,\mathrm{u}(\mathrm{t_k}) + \Gamma(\Delta\mathrm{t})\mathrm{u}(\mathrm{t_n}) \tag{2.86}$$

where

$$\Phi(t_{n+1}, t_n) = E_\alpha(At_{n+1}{}^\alpha)E_\alpha(At_n{}^\alpha)^{-1} \tag{2.87}$$

$$\begin{aligned} &Y(t_{n+1}, t_n, t_{k+1}, t_k) \\ &= \Big((t_{n+1} - t_k)^\alpha E_{\alpha,\alpha+1}(A(t_{n+1} - t_k)^\alpha) \\ &- (t_{n+1} - t_{k+1})^\alpha E_{\alpha,\alpha+1}(A(t_{n+1} - t_{k+1})^\alpha)\Big) \\ &- E_\alpha(At_{n+1}{}^\alpha)E_\alpha(At_n{}^\alpha)^{-1}\Big((t_n - t_k)^\alpha E_{\alpha,\alpha+1}(A(t_n - t_k)^\alpha) \\ &- (t_n - t_{k+1})^\alpha E_{\alpha,\alpha+1}(A(t_n - t_{k+1})^\alpha)\Big)B \end{aligned} \tag{2.88}$$

$$\Gamma(\Delta t) = \Delta t^\alpha E_{\alpha,\alpha+1}(A\Delta t^\alpha)B \tag{2.89}$$

Equation (2.86) is equivalent to Eq. (2.78). From both equations, we can observe that the fractional derivative has a non-local property because of the second term on the right hand side.

## F. NUMERICAL SIMULATION

In this section, we want to compare the numerical solutions done by analytic solution, PECE(Predict, Evaluate, Correct, Evaluate) algorithm, and two discrete-time approaches to verify our discrete-time models developed in the previous section. The example is given by the first order linear differential equation.

$$\frac{dx(t)}{dt} = -x(t) + u(t) \tag{2.90}$$

where x(0) is given. We can change the above integer order system to the following fractional order system.

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{0.5}x_1(t) &= x_2(t) \\
{}_{t_0}^{C}D_t^{0.5}x_2(t) &= -x1(t) + u(t)
\end{aligned}
\tag{2.91}
$$

where $x_1(0) = x(0)$ and $x_2(0) = 0$. Eq. (2.91) can be represented as the matrix form.

$$
{}_{0}^{C}D_t^{0.5}\mathbf{x}(t) = A\mathbf{x}(t) + Bu(t) \tag{2.92}
$$

where

$$
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.93}
$$

Let $x_1(0) = 10$. And the control input is chosen as a step input, u(t)=1 with t>0.

**Analytic Solution**

Let us find the analytic solution. By taking the Laplace transform of Eq. (2.91), we have

$$
s^{0.5}X_1(s) - s^{-0.5}x(0) = X_2(s) \tag{2.94a}
$$

$$
s^{0.5}X_2(s) = -X_1(s) + \frac{1}{s} \tag{2.94b}
$$

Substituting $X_2(s)$ in Eq. (2.94a) into Eq. (2.94b) leads to

$$
X_1(s) = \frac{x(0) - 1}{s + 1} + \frac{1}{s} \tag{2.95}
$$

By Eq. (2.94a),

$$
X_2(s) = \frac{\big(1 - x(0)\big)}{s + 1} s^{-0.5} \tag{2.96}
$$

The inverse Laplace transform leads to

$$x_1(t) = 9e^{-t} + 1$$
$$x_2(t) = -9tE_{1,1.5}(-t) \tag{2.97}$$

**PECE(Predict, Evaluate, Correct, Evaluate) Algorithm**

This algorithm developed by Diethelm, Ford, and Freed is widely used for numerical simulation of fractional order differential equation defined by the Caputo derivative[21][22]. For the explanation of the general case, let us consider the following fractional differential equation.

$${}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = \mathbf{f}(t, \mathbf{x}(t)) \tag{2.98}$$

where $\mathbf{x}^{(k)} = \mathbf{x}_{0^+}^{(k)}, k = 0,1, \dots, \lceil\alpha\rceil$. Let the simulation time be T. And the simulation interval is [0,T]. The grid is given by

$$t_n = n \times h \tag{2.99}$$

where $n = 0,1,2, \dots, N$ and h is the time interval, $h = \frac{T}{N}$. And then, we want to obtain x(t) at time T. Now, we explain the procedure.

Step 1). Predict the $\mathbf{x}_N^P(h)$ at t=T.

$$\mathbf{x}_N^P(h) = \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{T^k}{k!}\mathbf{x}_{0^+}^{(k)} + \frac{h^{\alpha}}{\Gamma(1+\alpha)}\sum_{n=0}^{N-1} b_{n,N}\mathbf{f}(t_n, \mathbf{x}_n) \tag{2.100}$$

where

$$b_{n,N} = (N-n)^{\alpha} - (N-n-1)^{\alpha} \tag{2.101}$$

Step 2). Evaluate $\mathbf{f}\left(T, \mathbf{x}_N^P\right)$.

Step 3). Then, correct with

$$\mathbf{x}_N(h) = \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{T^k}{k!}\mathbf{x}_{0^+}^{(k)} + \frac{h^\alpha}{\Gamma(2+\alpha)}\left(\sum_{n=0}^{N-1} c_{n,N}\mathbf{f}(t_n, \mathbf{x}_n) + c_{N,N}\ \mathbf{f}\left(T, \mathbf{x}_N^P\right)\right) \tag{2.102}$$

$$\mathbf{x}(T) = \mathbf{x}_N(h) + O\left(h^{\min(1+\alpha,2)}\right)$$

where

$$c_{n,N} = \begin{cases} (1+\alpha)N^\alpha - N^{1+\alpha} + (N-1)^{1+\alpha} & \text{if } n = 0 \\ (N-n+1)^{1+\alpha} - 2(N-n)^{1+\alpha} + (N-n-1)^{1+\alpha} & \text{if } 0 < n < N \\ 1 & \text{if } n = N \end{cases} \tag{2.103}$$

Step 4). Re-evaluate $\mathbf{f}(T, \mathbf{x}_N)$ and save it as $\mathbf{f}(t_N, \mathbf{x}_N)$.

After finishing steps 1 to 4, we increase the simulation time T, then repeat above steps.

**Numerical Results**

From the following results(Figs. (2.2) to (2.5)), we can observe that the solution by the PECE algorithm and the discrete-time solutions(approach I and approach II developed in the previous section) give us a good approximation of the analytic solution. In this case, the discrete-time solutions were found to have smaller error than the PECE algorithm.

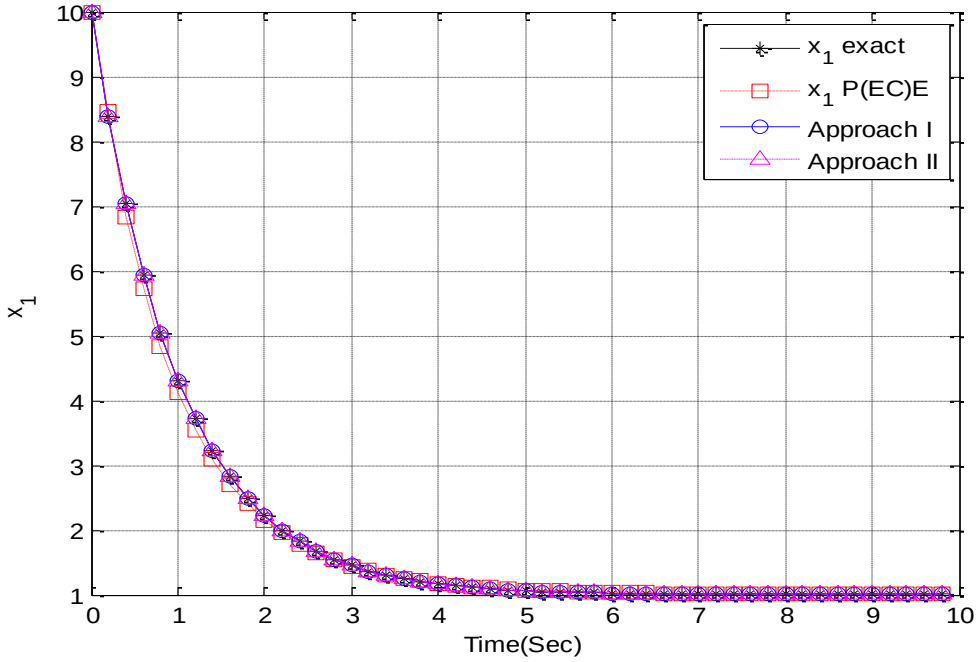


**Figure 2.2**. Time History of $x_1$

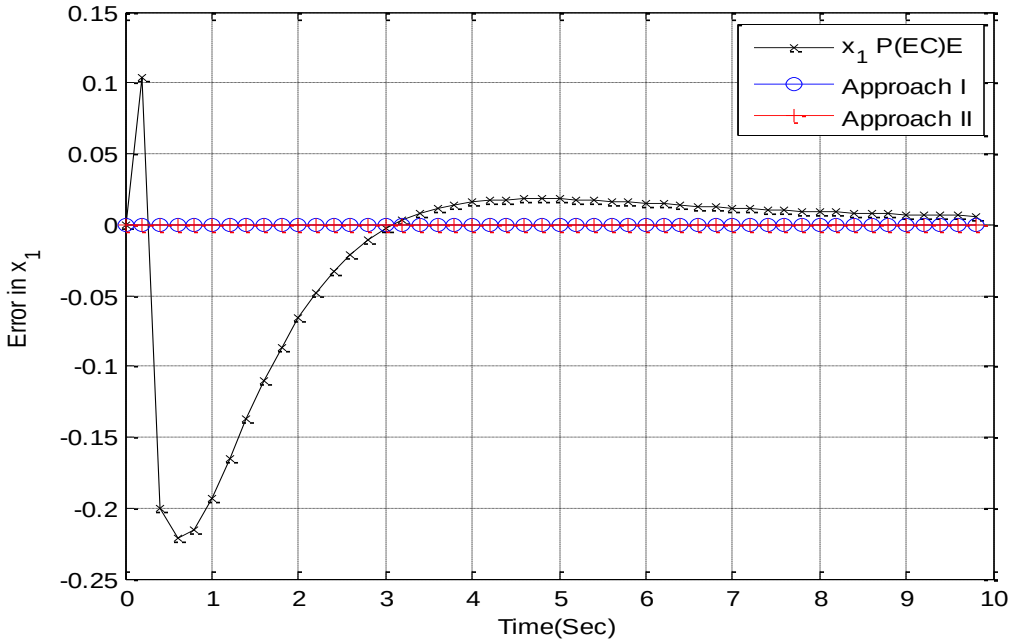


**Figure 2.3**. Time History of Error in $x_1$ from Analytical Solution

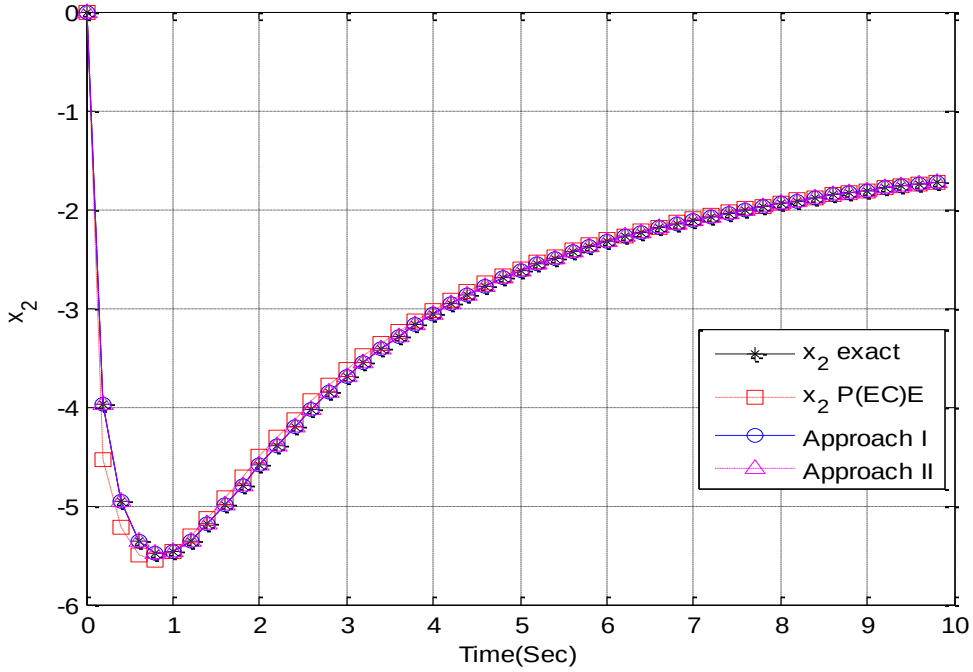


**Figure 2.4**. Time History of $x_2$

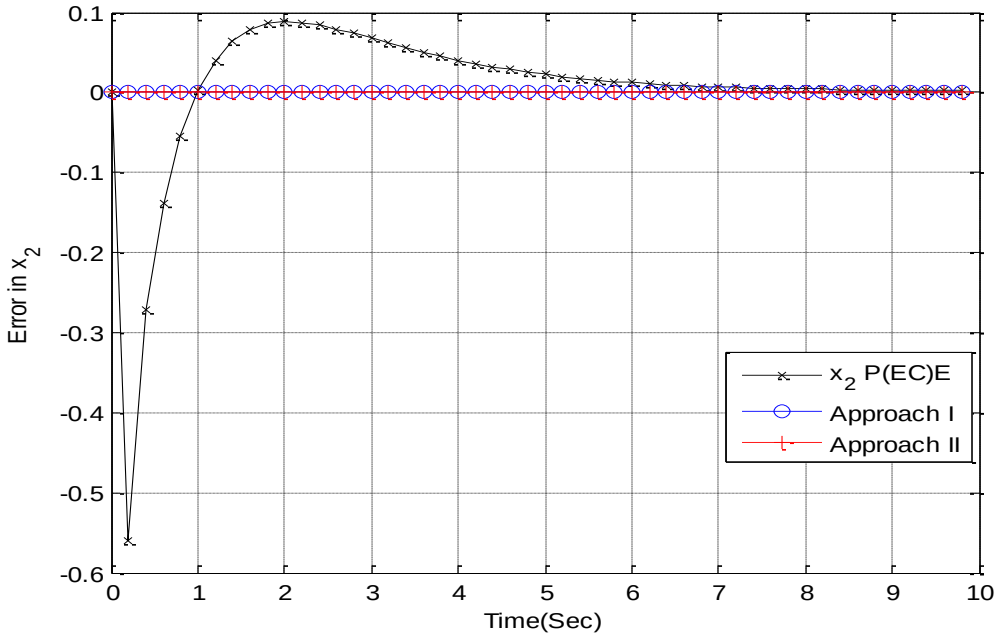


**Figure 2.5**. Time History of Error in $x_2$ from Analytical Solution

# CHAPTER III

# STABILITY ROBUSTNESS CRITERIA

## A. SENSITIVITY AND CONDITIONING

The existence of uncertainties or parameter variations can change the eigenvalues of the system affecting stability. This forces us to consider stability robustness. For the conventional integer order system, large amounts of research about stability robustness have been done. However, in the fractional order system, very few studies are available. The reason is because we cannot apply the Lyapunov stability theory, the major analysis tool for this stability area, to fractional order systems even if some results of it exist in references[23][24]. This makes it difficult to interpret the stability of fractional order systems and leads us to try to find a way of interpreting it in complex-domain.

In this chapter, we want to draw some useful results about stability from the analogy between the integer order system and the fractional order system.

### Eigenvalue and Eigenvector Sensitivities

First, we examine the *right and left eigenvalue problem* for the given non-defective integer order linear system. Similarly, we can derive the *right and left eigenvalue problem* for fractional order systems. Then the analytic expressions for the eigenvalue and eigenvector derivatives with respect to some arbitrary parameter can be used for both linear systems.

Let us consider a non-defective linear autonomous system given by

$$B\dot{\mathbf{x}}(t) = A\mathbf{x}(t) \tag{3.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n\times n}$. From above equation, the *right and left eigenvalue problems* can be expressed as[3]

$$\lambda_i B\boldsymbol{\phi}_i = A\boldsymbol{\phi}_i \quad \text{and} \quad \lambda_i B^T\boldsymbol{\psi}_i = A^T\boldsymbol{\psi}_i, \quad i = 1,2,\dots,n \tag{3.2}$$

with the biorthogonality conditions

$$\boldsymbol{\phi}_i^T B\boldsymbol{\phi}_i = 1 \quad \text{and} \quad \boldsymbol{\psi}_j^T B\boldsymbol{\phi}_i = \delta_{ij}, \quad i,j = 1,2,\dots,n \tag{3.3}$$

where $\delta_{ij}$ is the Kronecker delta and $\lambda_i$ presents the eigenvalue with the corresponding right and left eigenvectors $\boldsymbol{\phi}_i$ and $\boldsymbol{\psi}_i$. Note that if the eigenvalue is complex, the transpose should be changed to the Hermitian conjugate. By using Eq. (3.2) and Eq. (3.3), the right and left eigenvectors have the following relationship.

$$\boldsymbol{\psi}_j^T A\boldsymbol{\phi}_i = \lambda_i\delta_{ij}, \quad i,j = 1,2,\dots,n \tag{3.4}$$

Similarly, we want to find the eigenvalue-eigenvector pairs for the fractional order system. Let us consider an n-dimensional non-defective linear fractional order system.

$$B\,{}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = A\mathbf{x}(t) \tag{3.5}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n\times n}$. Now, we want to know *the right and left eigenvalue problems* of the above fractional order system. The solution form of the above equation can be given by $\mathbf{x}(t) = \boldsymbol{\phi}E_\alpha(\lambda t^\alpha)$. Substituting this form into Eq. (3.5) yields

$$B\,{}_{t_0}^{C}D_t^{\alpha}(\boldsymbol{\phi}E_\alpha(\lambda t^\alpha)) = A\boldsymbol{\phi}E_\alpha(\lambda t^\alpha) \tag{3.6}$$

By ${}_{t_0}^{C}D_t^{\alpha}(E_\alpha(\lambda t^\alpha)) = \lambda E_\alpha(\lambda t^\alpha)$, the above equation can be described by

$$\lambda B\boldsymbol{\phi}E_\alpha(\lambda t^\alpha) = A\boldsymbol{\phi}E_\alpha(\lambda t^\alpha) \tag{3.7}$$

Therefore, the right eigenvector is shown by

$$(\lambda_i B - A)\boldsymbol{\phi}_i = 0, \quad i = 1,2,\dots,n \tag{3.8}$$

From the above equation and the definition of the left eigenvector, the left eigenvector can be obtained by

$$\boldsymbol{\psi}_i^T(\lambda_i B - A) = 0, \quad i = 1,2,\dots,n \tag{3.9}$$

By taking the transpose on both sides, we have

$$(\lambda_i B - A)^T\boldsymbol{\psi}_i = 0, \quad i = 1,2,\dots,n \tag{3.10}$$

From Eq. (3.8) and Eq. (3.10), we can construct the *eigenvalue problems* for fractional order systems as

$$\lambda_i B\boldsymbol{\phi}_i = A\boldsymbol{\phi}_i \quad \text{and} \quad \lambda_i B^T\boldsymbol{\psi}_i = A^T\boldsymbol{\psi}_i, \quad i = 1,2,\dots,n \tag{3.11}$$

with the biorthogonoality conditions

$$\boldsymbol{\phi}_i^T B\boldsymbol{\phi}_i = 1 \quad \text{and} \quad \boldsymbol{\psi}_j^T B\boldsymbol{\phi}_i = \delta_{ij}, \quad i,j = 1,2,\dots,n \tag{3.12}$$

As a result, we obtain the same *eigenvalue problems* as for the integer order systems. From these observations, we can readily extend the eigenvalue and eigenvector sensitivities formula used for integer order systems to fractional order systems.

*Theorem 3.1*[3]. Eigenvalue and eigenvector sensitivities

Partial derivatives of eigenvalues and eigenvectors of systems described by Eq. (3.1) with respect to some arbitrary parameter,p, are given by

$$\frac{\partial\lambda_i}{\partial p} = \boldsymbol{\psi}_j^T\left(\frac{\partial A}{\partial p} - \lambda_i\frac{\partial B}{\partial p}\right)\boldsymbol{\phi}_i, \quad i,j = 1,2,\dots,n \tag{3.13}$$

$$\frac{\partial \boldsymbol{\phi}_i}{\partial p} = \sum_{j=1}^{n} \alpha_{ji} \boldsymbol{\phi}_j \quad \text{and} \quad \frac{\partial \boldsymbol{\psi}_i}{\partial p} = \sum_{j=1}^{n} \beta_{ji} \boldsymbol{\psi}_j, \quad i, j = 1,2, \dots, n$$

where $\alpha_{ij}$ and $\beta_{ij}$ are given by

$$\alpha_{ji} = \begin{cases} \dfrac{1}{\lambda_i - \lambda_j} \boldsymbol{\psi}_j^T \left( \dfrac{\partial A}{\partial p} - \lambda_i \dfrac{\partial B}{\partial p} \right) \boldsymbol{\phi}_i & j \neq i \\ -\dfrac{1}{2} \left[ \boldsymbol{\phi}_i^T \dfrac{\partial B}{\partial p} \boldsymbol{\phi}_i + \sum\limits_{\substack{k=1 \\ k \neq i}}^{n} \alpha_{ki} \boldsymbol{\phi}_k^T (B + B^T) \boldsymbol{\phi}_j \right] & j = i \end{cases} \tag{3.14}$$

$$\beta_{ji} = \begin{cases} \dfrac{1}{\lambda_i - \lambda_j} \boldsymbol{\psi}_i^T \left( \dfrac{\partial A}{\partial p} - \lambda_i \dfrac{\partial B}{\partial p} \right) \boldsymbol{\phi}_j & j \neq i \\ -\boldsymbol{\psi}_i^T \dfrac{\partial B}{\partial p} \boldsymbol{\phi}_i - \alpha_{ii} & j = i \end{cases}$$

From the previous observations, we can also use these formulas to find the eigenvalue and eigenvector sensitivity for fractional order systems given in Eq. (3.5). By using this eigenvalue sensitivity, a new robust stability measure was introduced in Reference [25].

**Conditioning of the Eigenvalue Problem**

In this section, we want to find the upper bound of the variation of eigenvalues due to the perturbation in the system matrix for fractional order systems. As we did in the previous section, we can extend the existent theorem for integer order systems to for fractional order systems.

*Theorem 3.2*[3][26]

Let us consider the integer order system given in Eq. (3.1) or the fractional order systems given in Eq. (3.5) with A which has the following eigenvalue and eigenvector matrix. We assume that B=I in Eq. (3.1).

$$\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\} \quad \text{and} \quad \Phi = [\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n] \tag{3.15}$$

And, consider a perturbative matrix E. The eigenvalue and the corresponding eigenvector of a perturbed system matrix A+E are given by $\mu$ and $\boldsymbol{\upsilon}$ with the normalized condition, $\|\boldsymbol{\upsilon}\| = 1$, respectively. Then, the following inequality is satisfied.

$$\min_i |\lambda_i - \mu| \le \|E\| \kappa(\Phi), \quad i = 1, \dots, n \tag{3.16}$$

where $\kappa(\Phi) \equiv \|\Phi\| \|\Phi^{-1}\|$ is called the condition number which is widely used to represent the quantitative measure of ill-conditioning of the system. For the fractional order case, the proof as shown in the integer order case can be used and there is no difference between them so that we can apply the theorem to the fractional order system case.

Also, this theorem can be interpreted as

$$\min_i |\lambda_i - \mu_j| \le \|E\| \kappa(\Phi), \quad j = 1, \dots, n \tag{3.17}$$

From this theorem, it is observed that the condition number plays a big role for the upper bound on the eigenvalue variations due to the perturbation. Norm of the perturbative matrix cannot be chosen, but the condition number can be an adjustable design parameter if we make the original system closed-loop form by putting the feedback controller. Therefore, the condition number is an important measure for the eigenvalue

sensitivity. By the definition of the spectral norm, the condition number has the alternative form as

$$\kappa(\Phi) = \frac{\sigma_1(\Phi)}{\sigma_n(\Phi)} \tag{3.18}$$

where $\sigma_1$ and $\sigma_n$ are the maximum and minimum singular values of the modal matrix $\Phi$.

This condition number has the following inequality property[27].

$$1 \leq \kappa(\Phi) \leq \infty \tag{3.19}$$

If the eigenvector matrix is an orthogonal(unitary) matrix, the condition number has the lower limit. If the eigenvector matrix is any rank-deficient matrix of $\Phi$, the condition number reaches the upper limit[3]. From these observations, when we design the feedback controller with multiple inputs, we can conclude that we should make the closed-loop eigenvector matrix close to an orthogonal(unitary) matrix for the closed-loop eigenvalues to be insensitive to perturbations. In other words, the condition number should be treated as the cost function to be minimized if we want to design the feedback controller for stability robustness.

**Measure of Robustness**

There are many kinds of definitions to measure the robustness of the linear integer order system. Let us have the right and left eigenvectors described in Eq. (3.2) with B=I. It is well known that the sensitivity of each eigenvalue $\lambda_i$ depends on the magnitude of the condition number $c_j$[6][27].

$$c_j = \frac{1}{s_j} = \frac{\|\boldsymbol{\psi}_j\|_2 \|\boldsymbol{\phi}_j\|_2}{|\boldsymbol{\psi}_j{}^T \boldsymbol{\phi}_j|} \geq 1, \quad j = 1, \ldots, n \tag{3.20}$$

where $s_j$ can be interpreted as the cosine of the angle between the right and left eigenvectors for real $\lambda_i$. Furthermore, this condition number $c_j$ has the following inequallity[6] [27].

$$\max_j c_j \leq \kappa_2(\Phi) \equiv \|\Phi\|_2 \|\Phi^{-1}\|_2 \tag{3.21}$$

Since the condition number $c_j$ is greater than or equal to 1 and is less than $\kappa_2(\Phi)$, $c_j$ for all j has the minimum value if and only if the eigenvector matrix is orthogonal(unitary). From Eq. (3.21), we can introduce two measures of robustness.

$$\nu_1 = \|\mathbf{c}\|_\infty = \max_j c_j \tag{3.22}$$

where $\mathbf{c}^T = [c_1, c_2, \cdots, c_n]$. And,

$$\nu_2 = \kappa_2(\Phi) \tag{3.23}$$

Now, we assume that the normalized right eigenvectors $\|\boldsymbol{\phi}_j\|_2 = 1$ for all j. This assumption can make the measures of robustness be the most simple and convenient expressions as possible. The left eigenvector matrix can be given by

$$\Psi^T = [\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2 \quad \cdots \quad \boldsymbol{\psi}_n] = \Phi^{-1} \tag{3.24}$$

$\Psi^T \Phi = I$, which gives us $\boldsymbol{\psi}_j{}^T \boldsymbol{\phi}_j = 1$. Then, the condition number defined in Eq. (3.20) can be expressed only in terms of the norm of the left eigenvectors.

$$c_j = \frac{\|\boldsymbol{\psi}_j\|_2 \|\boldsymbol{\phi}_j\|_2}{|\boldsymbol{\psi}_j{}^T \boldsymbol{\phi}_j|} = \|\boldsymbol{\psi}_j\|_2 \geq 1, \quad j = 1, \ldots, n \tag{3.25}$$

From our assumption of the right eigenvector's normalization, we have

$$\|\Phi\|_F = \sqrt{\sum_{j=1}^{n} \left\|\boldsymbol{\phi}_j\right\|_2^2} = n^{1/2} \tag{3.26}$$

And, from Eq. (3.24) and Eq. (3.25), we have

$$\left\|\Phi^{-1}\right\|_F \equiv \|\Psi^T\|_F = \sqrt{\sum_{j=1}^{n} \left\|\boldsymbol{\psi}_j\right\|_2^2} = \sqrt{\sum_{j=1}^{n} c_j^2} = \|\mathbf{c}\|_2 \tag{3.27}$$

Then another measure can be introduced by[6]

$$\nu_3 = n^{-1/2}\left\|\Phi^{-1}\right\|_F = n^{-1/2}\|\mathbf{c}\|_2 = n^{-1}\|\Phi\|_F\left\|\Phi^{-1}\right\|_F \equiv n^{-1}\kappa_F(\Phi) \tag{3.28}$$

Note that all measures from $\nu_1$ to $\nu_3$ have the minimal value if and only if all condition number $c_j = 1$, that is, if the eigenvector matrix $\Phi$ is orthogonal(unitary).

We want to find the inequalities among all measures. The vector norm inequality is given by[28]

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \sqrt{n}\|\mathbf{x}\|_\infty \tag{3.29}$$

where $\mathbf{x} \in \mathbb{C}^n$. Then, we have $\|\mathbf{c}\|_\infty \le \|\mathbf{c}\|_2 \le \sqrt{n}\|\mathbf{c}\|_\infty$, which gives us

$$\nu_1 \le \sqrt{n}\nu_3 \le \sqrt{n}\nu_1 \tag{3.30}$$

Pre-multiplication of $\sqrt{n}$ leads us to

$$\sqrt{n}\nu_1 \le n\nu_3 \le n\nu_1 \tag{3.31}$$

And the matrix norm is given by[28]

$$\begin{gathered} \|\Phi\|_2 \le \|\Phi\|_F \le \sqrt{n}\|\Phi\|_2 \\ \left\|\Phi^{-1}\right\|_2 \le \left\|\Phi^{-1}\right\|_F \le \sqrt{n}\left\|\Phi^{-1}\right\|_2 \end{gathered} \tag{3.32}$$

where $\Phi \in \mathbb{C}^{n\times n}$. Then we have $\|\Phi\|_2\left\|\Phi^{-1}\right\|_2 \le \|\Phi\|_F\left\|\Phi^{-1}\right\|_F \le n\|\Phi\|_2\left\|\Phi^{-1}\right\|_2$, which gives us

$$\nu_2 \leq n\nu_3 \leq n\nu_2 \tag{3.33}$$

From Eq. (3.30), Eq. (3.31) and Eq. (3.33), we have

$$1 \leq \nu_3 \leq \nu_1 \leq \nu_2 \leq n\nu_3 \tag{3.34}$$

**Geometrical Stability Measures**

As mentioned previously, we can have the minimum condition number when the eigenvector matrix is orthogonal or unitary. From this fact, we can set up the desired unitary eigenvector matrix. Because our eigenvector matrix has some admissible space considered as the constraint, we try to make our eigenvector matrix lie as close as possible to the prescribed unitary eigenvector matrix.

Suppose that we choose a desired unitary eigenvector matrix $\widehat{\Phi}$ and we have the achieved eigenvector matrix $\Phi$. In order to obtain the distance between $\widehat{\Phi}$ and $\Phi$, we can measure the angles between two eigenvectors selected individually from both eigenvector matrices.

$$\left\|\widehat{\boldsymbol{\phi}}_j^H \boldsymbol{\phi}_j\right\| = \sqrt{1 - \left(\sin\theta_j\right)^2} = \cos\theta_j \leq 1 \quad j = 1, \dots, n \tag{3.35}$$

where $\theta_j$ is the angle between $\boldsymbol{\phi}_j$ and $\widehat{\boldsymbol{\phi}}_j$. Then we define the new measure as[6]

$$\nu_4 = n^{-1/2}\left(\sum_{j=1}^{n}\left(\sin\theta_j\right)^2\right)^{\frac{1}{2}}, \quad 0 \leq \nu_4 \leq 1 \tag{3.36}$$

When $\Phi$ is the unitary, $\nu_4$ has the lower limit.

We also introduce a new measure, where the sum of angles between eigenvectors should be as large as possible to be close to the unitary matrix. In other words, the sum

of the dot product between eigenvectors should be zero in case of the orthogonal or unitary matrix. Therefore, the new measure is given by

$$\nu_5 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left\| \boldsymbol{\phi}_i^H \boldsymbol{\phi}_j \right\| \tag{3.37}$$

If we choose $\nu_5$ as the cost function, the minimum value of it is zero when the eigenvector matrix is orthogonal or unitary.

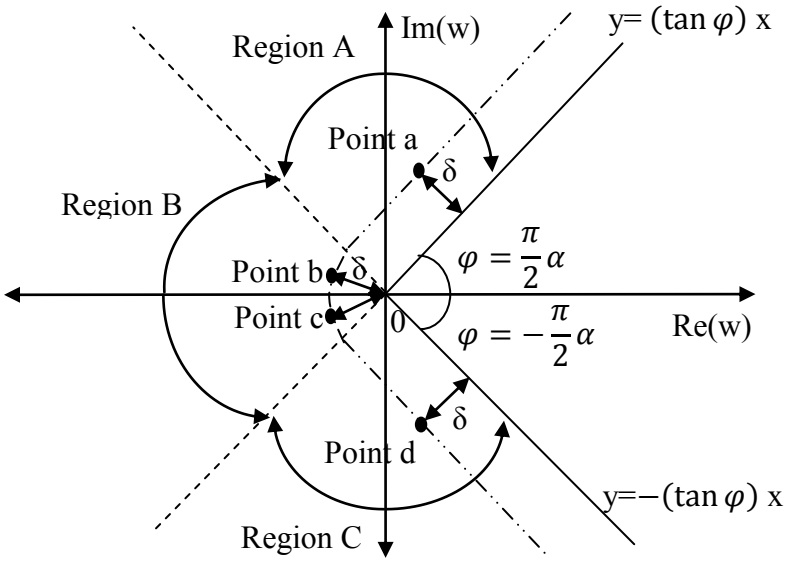


**Figure 3.1.** Distance between Poles and the Stability Lines

**Weighted Robust Stability Measures**

We still assume that the normalized right eigenvectors $\left\| \boldsymbol{\phi}_j \right\|_2 = 1$ for all j. There are two weighted measures of conditioning obtained by using $\nu_3$ in Eq. (3.28) and $\nu_4$ in Eq. (3.36)[6].

$$\nu_3(D) = \left\|D\Phi^{-1}\right\|_F / \|D\|_F \tag{3.38}$$

$$\nu_4(D) = \left(\sum_{j=1}^{n} d_j^2 \left(\sin\theta_j\right)^2\right)^{\frac{1}{2}} \Bigg/ \left(\sum_{j=1}^{n} d_j^2\right)^{\frac{1}{2}} \tag{3.39}$$

where $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$ is a real diagonal weight matrix with $d_j > 0$ for all j. The imaginary axis is the boundary line for the stability of integer order systems. Hence, if possible, the real part of the eigenvalue should be a large negative number for maximizing the stability margin. To weigh this property, we can choose $d_j^{-1} = \mathrm{Re}(-\lambda_j)$ for the integer order system. However, for the fractional order system, the imaginary axis is not the boundary line for stability any more. We can put weights $d_j^{-1} = \delta$ in Fig (3.1) for the fractional order system. Let us examine how to obtain δ. Assume that two complex conjugate eigenvalues are given by $\mathrm{Re}(\lambda) \pm \mathrm{Im}(\lambda)i$ and the fractional order of a system is given by α. Depending on the location of eigenvalues, we have two different δ formulas.

First case is when the eigenvalue is located in Region A or Region C in Fig (3.1). Let us assume that eigenvalues are given in Point a and Point b in Fig (3.1). The stability lines can be described by

$$y = \pm(\tan\varphi)x, \quad x \geq 0 \tag{3.40}$$

where $\varphi = \frac{\pi}{2}\alpha$. x and y are variables which represent Re(w) and Im(w) of the w-complex domain, respectively. Now, we want to find the closest distance δ between the location of the eigenvalue and the stability line. The closest distance between a point$(m, n)$ and a line$(px + qy + r = 0)$ is well-known by

$$\text{Distance} = \frac{|pm + qn + r|}{\sqrt{p^2 + q^2}} \tag{3.41}$$

Therefore, $\delta$ can be obtained by

$$\delta = \frac{\left|\left(\tan\frac{\pi}{2}\alpha\right)\text{Re}(\lambda) - \text{Im}(\lambda)\right|}{\sqrt{\left(\tan\frac{\pi}{2}\alpha\right)^2 + 1}} \tag{3.42}$$

By $(\tan\theta)^2 + 1 = (\sec\theta)^2$, we obtain the closest distance as

$$\delta = \left|\left(\sin\frac{\pi}{2}\alpha\right)\text{Re}(\lambda) - \left(\cos\frac{\pi}{2}\alpha\right)\text{Im}(\lambda)\right| \tag{3.43}$$

Now, we consider the second case which has the eigenvalues in Region B. So we assume that the eigenvalues are given in Point b and Point c. The closest distance $\delta$ between the location of the eigenvalue and zero point can be obtained by

$$\delta = \sqrt{\text{Re}(\lambda)^2 + \text{Im}(\lambda)^2} \tag{3.44}$$

In summary,

$$\delta = \begin{cases} \left|\left(\sin\frac{\pi}{2}\alpha\right)\text{Re}(\lambda) - \left(\cos\frac{\pi}{2}\alpha\right)\text{Im}(\lambda)\right|, & \lambda \text{ in Region A or Reigon C} \\ \sqrt{\text{Re}(\lambda)^2 + \text{Im}(\lambda)^2}, & \lambda \text{ in Region B} \end{cases} \tag{3.45}$$

## B. STABILITY ROBUSTNESS CRITERIA

### Problem Formulation

Let us consider the following equation.

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t) + f(t, \mathbf{x}(t)) \tag{3.46}$$

where $A \in \mathbb{R}^{n\times n}$ is a time-invariant stable matrix and $f(t, \mathbf{x}(t))$ is a time-varying nonlinear vector function with $f(t, \mathbf{0}) = 0$ for $\forall t$. We can consider $f(t, \mathbf{x}(t))$ as the

perturbation in the system. Even if we don't know it exactly, it might be possible to have an estimate of some bound on the perturbation. Therefore, we want to investigate this bound on the perturbation to make the system still stable. For the integer order system, the following theorem shows it. This bound is conservative, meaning that there are many methods showing better results. But the following theorem can be a good starting point to examine the bound on perturbation for fractional order systems.

*Theorem 3.3*[29]

The system described in Eq. (3.46) is stable if the matrix A is diagonalizable and

$$\frac{\|\mathrm{f}(\mathbf{z}(\mathrm{t}),\mathrm{t})\|}{\|\mathbf{z}(\mathrm{t})\|} \leq \mu \equiv -\frac{\sigma}{\kappa(\Phi)} \tag{3.47}$$

where $\sigma \equiv \mathrm{Max}\left(\mathrm{Re}\left(\lambda(\mathrm{A})\right)\right) < 0$. $\mathrm{Re}\left(\lambda(\mathrm{A})\right)$ represents the real part of eigenvalues of A. $\kappa(\Phi) \equiv \|\Phi\|_s\|\Phi^{-1}\|_s$ is the condition number where $\Phi$ is the similarity transformation matrix which diagonalizes matrix A.

This theorem can be proved by using the Lyapunov method or transition matrix approach. From this theorem, we can extract the bound of linear perturbation to guarantee the stability of the following system.

$$\frac{\mathrm{d}\mathbf{x}(\mathrm{t})}{\mathrm{dt}} = \mathrm{A}\mathbf{x}(\mathrm{t}) + \mathrm{E}(\mathrm{t})\mathbf{x}(\mathrm{t}) \tag{3.48}$$

*Corollary 3.1*[3][29]

The above system is stable if

$$\|E(t)\| < \frac{\min[-\mathrm{Re}\{\lambda_i(A)\}]}{\kappa(\Phi)}, \quad i = 1, \dots, n \tag{3.49}$$

Now, we want to know if we have a similar result for the fractional order system.

**Fractional Order System**

To show the stability robustness criteria for the fractional order system, we will use the transition matrix approach as shown in reference[29] because it is difficult to find the Lyapunov approach for the fractional order system. To achieve this objective, we have to investigate the Mittag-Leffler function which plays an important role in the solution of fractional order systems. In the exponential function, a special case of the Mittage-Leffler function, we have

$$\|\exp(\Lambda)\|_s = \|\exp(\sigma)\|_s \tag{3.50}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, all $\lambda_i, i = 1, \dots, n$ has the negative real part, $\mathrm{Re}\{\lambda_i\} < 0, i = 1,2, \dots, n$, and $\sigma \equiv \mathrm{Max}[\mathrm{Re}\{\lambda_i\}] < 0, i = 1,2, \dots, n$. But, the Mittag-Leffler function does not hold this property. We consider three cases to study the norm of the multivariable Mittag-Leffler function numerically. Assume that $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and $\alpha = 0.5$ for the parameter of Mittag-Leffler function.

1) Real parts of the Mittag-Leffler function remain the same: We choose $\Lambda = \mathrm{diag}\{-1, -1 + j, -1 + 2j, -1 + 3j, -1 + 4j\}$. So, $\mathrm{Re}\{E_{\alpha,\alpha}(\lambda_i)\} = -1, i = 1, \dots, 5$.
2) Norms of the Mittag-Leffler function are same: We choose $\lambda_i = \cos\theta_i + j\sin\theta_i, \theta_i = \frac{3i}{20}\pi + \frac{\pi}{4}, i = 1, \dots, 5$. So, $|\lambda_i| = 1, i = 1, \dots, 5$.

3) Case 2 with two different scales: We choose $\lambda_i = \rho(\cos\theta_i + j\sin\theta_i), i = 1,..,10, \theta_{5+i} = \theta_i, \theta_i = \frac{3i}{20}\pi + \frac{\pi}{4}, i = 1,\dots,5$ . $\rho = 1$ for $i = 1,..,5$ and $\rho = 1.2$ for $i = 6,..,10$.

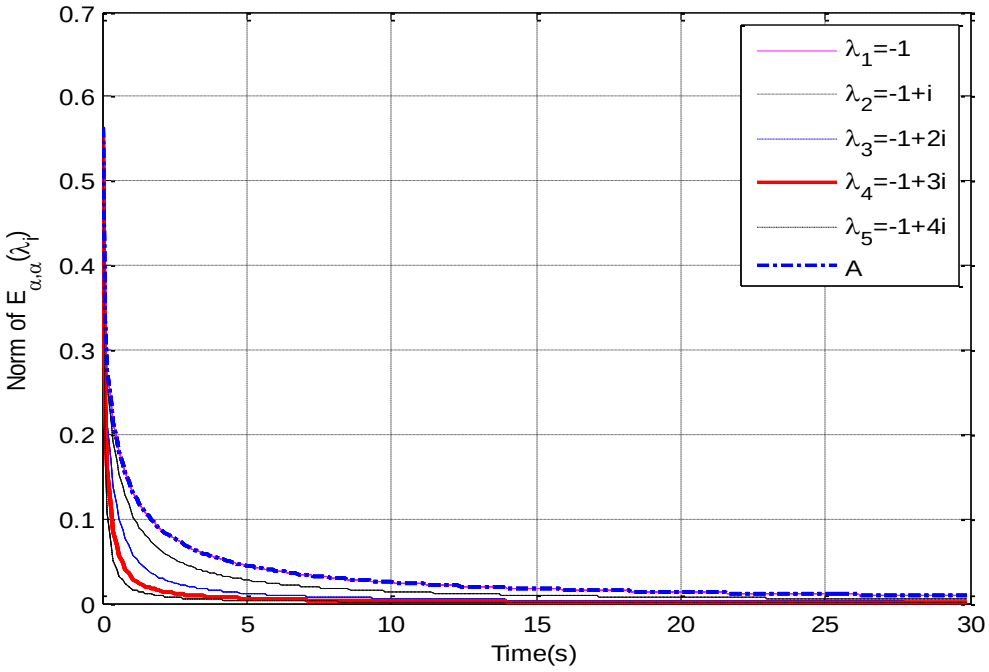


**Figure 3.2.** Norm of Mittag-Leffler Function, Case I

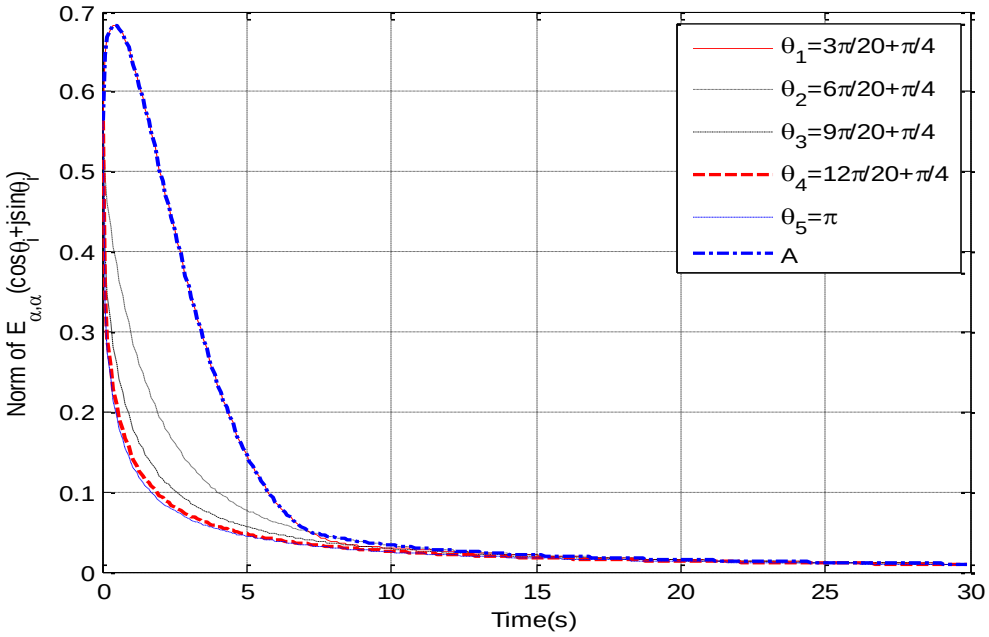


**Figure 3.3.** Norm of Mittag-Leffler Function, Case II

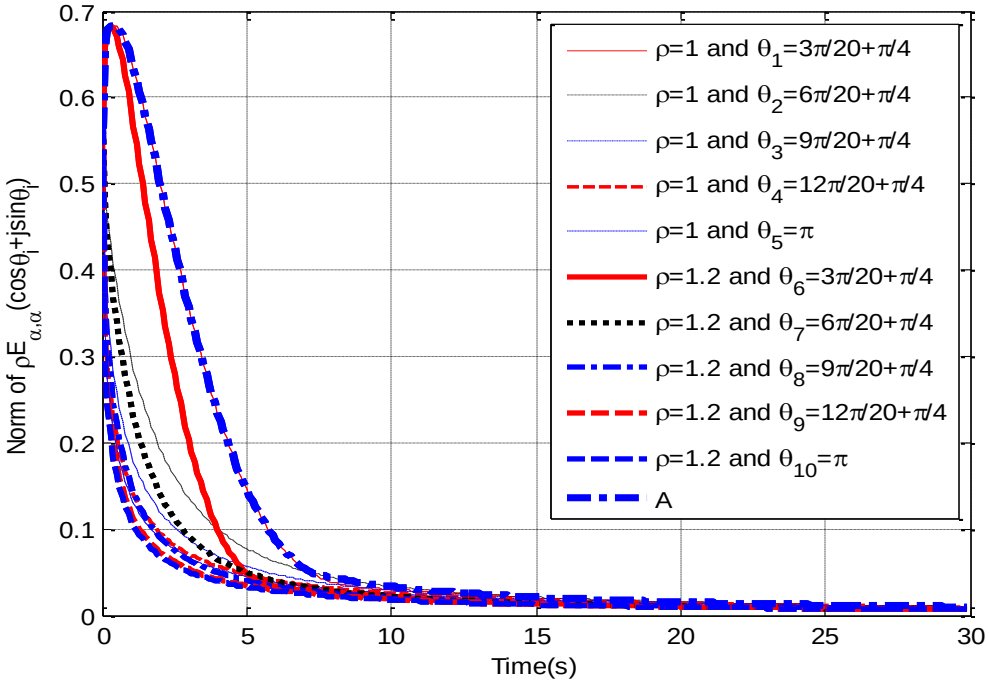


**Figure 3.4.** Norm of Mittag-Leffler Function, Case III

From Fig. (3.2), we can verify that the Mittag-Leffler function does not have the same norm even if the real parts of $\lambda_i, i = 1, \ldots, 5$ are the same. As the norm of the imaginary part of $\lambda_i$ increases, the norm of the Mittag-Leffler function tends to decrease. From Fig. (3.3), the Mittag-Leffler function does not have the same norm in a transition time even if $|\lambda_i|, i = 1, \ldots, 5$ remain the same. After a transition time, their norms seem to converge to the same value. From Fig. (3.4), the norm of the Mittag-Leffler function tends to have the small value when the norm of $\lambda_i$ has the smaller value. But this is not always true. From these observations, we cannot have the following equation except for the special case.

$$\left\|E_{\alpha,\alpha}(\Lambda)\right\|_s = \left\|E_{\alpha,\alpha}(\sigma)\right\|_s \tag{3.51}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, all $\lambda_i, i = 1, \ldots, n$ and $\sigma$ is one of $\lambda_i, i = 1, \ldots, n$ to make $\left\|E_{\alpha,\alpha}(\Lambda)\right\|_s$ maximum. So we need a useful relation about the norm of Mittag-Leffler function.

*Theorem 3.4*[1]

If $\alpha < 2$, $\beta$ is an arbitrary real number, $\mu$ is such that $\frac{\pi\alpha}{2} < \mu < min\{\pi, \pi\alpha\}$ and $C$ is a real constant, then

$$\left|E_{\alpha,\beta}(z)\right| \leq \frac{C}{1 + |z|}, \quad (\mu \leq |\arg(z)| \leq \pi), |z| \geq 0 \tag{3.52}$$

From the above theorem, we can introduce the following corollary.

*Corollary 3.2*

If $\alpha < 2$, $\beta$ is an arbitrary real number, $\Lambda \in \mathbb{C}^{n\times n}$ is the diagonal matrix, $\mu$ is such that $\frac{\pi\alpha}{2} < \mu < min\{\pi, \pi\alpha\}$, then

$$\left\|E_{\alpha,\beta}(\Lambda)\right\|_s \leq \frac{C_{max}}{1+\rho}, \quad (\mu \leq |arg(\lambda_i)| \leq \pi), i = 1, .., n \tag{3.53}$$

where $\Lambda = diag\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and $\rho \equiv Min[|\lambda_i|]$. And $C_{max} \equiv Max[C_i]$ where $C_i$ can be given by

$$\left\|E_{\alpha,\beta}(\lambda_i)\right\|_s \leq \frac{C_i}{1+|\lambda_i|}, \quad i = 1, \dots, n \tag{3.54}$$

Proof

Since $\Lambda$ is the diagonal matrix, we have

$$\begin{aligned}\left\|E_{\alpha,\beta}(\Lambda)\right\|_s &= \left\|diag\left\{E_{\alpha,\beta}(\lambda_1), E_{\alpha,\beta}(\lambda_2), \dots, E_{\alpha,\beta}(\lambda_n)\right\}\right\|_s \\ &\leq diag\left\{\left\|E_{\alpha,\beta}(\lambda_1)\right\|_s, \left\|E_{\alpha,\beta}(\lambda_2)\right\|_s, \dots, \left\|E_{\alpha,\beta}(\lambda_n)\right\|_s\right\}\end{aligned} \tag{3.55}$$

By using Theorem 3.4, the norm of each Mittag-Leffler function has the inequality described in Eq. (3.54). Thus, if we take the $\rho \equiv Min[|\lambda_i|]$ and $C_{max} \equiv Max[C_i]$, we have

$$\left\|E_{\alpha,\beta}(\Lambda)\right\|_s \leq \frac{C_{max}}{1+\rho} \tag{3.56}$$

*Theorem 3.5*

We consider the system described by the following fractional differential equations

$$ {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = A\mathbf{x}(t) + f(t, \mathbf{x}(t)) \tag{3.57} $$

where $0 < \alpha < 1$ and A is a time-invariant stable matrix and $f\big(t, \mathbf{x}(t)\big)$ is a time-varying nonlinear vector function with $f(t, \mathbf{0}) = 0$ for $\forall t$. The system is stable if the matrix A is diagonalizable and

$$ \frac{\|f(\mathbf{z}(t), t)\|}{\|\mathbf{z}(t)\|} \leq \mu < \mu_1 \equiv \frac{\rho\alpha}{\kappa(\Phi)C_{max}} \tag{3.58} $$

where $A = \Phi\Lambda\Phi^{-1}$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, $\rho \equiv \mathrm{Min}[|\lambda_i|]$, $\kappa(\Phi) \equiv \|\Phi\|_s\|\Phi^{-1}\|_s$ and $C_{max} \equiv \mathrm{Max}[C_i]$. $C_i$ satisfies

$$ \left\|E_{\alpha,\alpha}(\lambda_i(t - t_0)^{\alpha})\right\|_s \leq \frac{C_i}{1 + |\lambda_i|(t - t_0)^{\alpha}}, \quad i = 1, \dots, n \text{ and } \forall t \geq t_0 \geq 0 \tag{3.59} $$

Proof

Reference [30] gives us an important idea to have the above theorem. By using the Laplace transform of Eq. (3.37) and taking the inverse Laplace transform of it, the solution is given by

$$ \mathbf{x}(t) = E_{\alpha}(A(t - t_0)^{\alpha})\mathbf{x}(t_0^{+}) + \int_{t_0}^{t}(t - \xi)^{\alpha-1}E_{\alpha,\alpha}(A(t - \xi)^{\alpha})\, f(\mathbf{x}(\xi), \xi)d\xi \tag{3.60} $$

Taking the Euclidean norms of both sides, we obtain

$$ \begin{aligned} \|\mathbf{x}(t)\| \leq{}& \|E_{\alpha}(A(t - t_0)^{\alpha})\|_s\|\mathbf{x}(t_0^{+})\| \\ &+ \int_{t_0}^{t}\left\|(t - \xi)^{\alpha-1}E_{\alpha,\alpha}(A(t - \xi)^{\alpha})\right\|_s \|f(\mathbf{x}(\xi), \xi)\|d\xi \end{aligned} \tag{3.61} $$

Since we have $\|f(\mathbf{x}(\xi), \xi)\| \leq \mu\|\mathbf{z}(t)\|$ from the assumption,

$$\|\mathbf{x}(t)\| \le \|E_\alpha(A(t-t_0)^\alpha)\|_s \|\mathbf{x}(t_0^+)\| + \int_{t_0}^{t} \left\|(t-\xi)^{\alpha-1} E_{\alpha,\alpha}(A(t-\xi)^\alpha)\right\|_s \mu \|\mathbf{x}(\xi)\| d\xi \tag{3.62}$$

There is an inequality relationship between $E_\alpha(At^\alpha)$ and $E_{\alpha,\alpha}(At^\alpha)$. By the definition of Mittag-Leffler function, each function can be expressed as

$$E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(\alpha k+1)} \quad \text{and} \quad E_{\alpha,\alpha}(At^\alpha) = \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(\alpha k+\alpha)} \tag{3.63}$$

where $\Gamma(m)$ is the gamma function, $\mathrm{Re}\{m\} > 0$ for convergence. The relationship between the beta function and the gamma function is given by

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{3.64}$$

where $\mathrm{Re}\{m\} > 0$ and $\mathrm{Re}\{n\} > 0$. Then we can obtain the following inequality[31]

$$\frac{\Gamma(\alpha k+\alpha)}{\Gamma(\alpha k+1)} = \frac{B(\alpha k+\alpha, 1-\alpha)}{\Gamma(1-\alpha)} \le \frac{B(\alpha, 1-\alpha)}{\Gamma(1-\alpha)} = \Gamma(\alpha) \tag{3.65}$$

Taking Eq. (3.63) and Eq. (3.65) into account gives

$$E_\alpha(At^\alpha) \le \Gamma(\alpha) E_{\alpha,\alpha}(At^\alpha) \tag{3.66}$$

Thus, Eq. (3.62) can be given by

$$\|\mathbf{x}(t)\| \le \Gamma(\alpha)\left\|E_{\alpha,\alpha}(A(t-t_0)^\alpha)\right\|_s \|\mathbf{x}(t_0^+)\| + \int_{t_0}^{t} \left\|(t-\xi)^{\alpha-1} E_{\alpha,\alpha}(A(t-\xi)^\alpha)\right\|_s \mu \|\mathbf{x}(\xi)\| d\xi \tag{3.67}$$

Since A is diagonalizable from the assumption, we have

$$E_{\alpha,\alpha}(A(t-t_0)^\alpha) = \Phi E_{\alpha,\alpha}(\Lambda(t-t_0)^\alpha)\Phi^{-1} \tag{3.68}$$

where $A = \Phi\Lambda\Phi^{-1}$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Taking the norm of both sides of Eq. (3.68) gives us

$$\begin{aligned} \left\| E_{\alpha,\alpha}(A(t-t_0)^\alpha) \right\|_s &\le \|\Phi\|_s \|\Phi^{-1}\|_s \left\| E_{\alpha,\alpha}(\Lambda(t-t_0)^\alpha) \right\|_s \\ &= \kappa(\Phi) \left\| E_{\alpha,\alpha}(\Lambda(t-t_0)^\alpha) \right\|_s \end{aligned} \tag{3.69}$$

By using Corollary 3.2, Eq. (3.69) can be given by

$$\left\| E_{\alpha,\alpha}(A(t-t_0)^\alpha) \right\|_s \le \kappa(\Phi) \frac{C_{max}}{1+\rho(t-t_0)^\alpha} \tag{3.70}$$

Combination of Eq. (3.70) and Eq. (3.67) yields

$$\begin{aligned} \|\mathbf{x}(t)\| \le{}& \frac{\kappa(\Phi)\Gamma(\alpha)C_{max}}{1+\rho(t-t_0)^\alpha} \|\mathbf{x}(t_0^+)\| \\ &+ \kappa(\Phi) \int_{t_0}^{t} (t-\xi)^{\alpha-1} \frac{C_{max}}{1+\rho(t-\xi)^\alpha} \mu \|\mathbf{x}(\xi)\| d\xi \end{aligned} \tag{3.71}$$

*Mahmudov and Musaev's Inequality*[32][33]

Let x(t), p(t) and q(t) be nonnegative continuous functions defined on J. w(t,s) be a continuous and nonnegative function on the rectangle : $\alpha \le \xi \le t \le \beta$ and nondecreasing in t for each $s \in J$. If

$$u(t) \le p(t) + q(t) \int_{\alpha}^{t} w(t,\xi) u(\xi) d\xi, \quad t \in J \tag{3.72}$$

then

$$u(t) \le p(t) + q(t) \int_{\alpha}^{t} w(t,\xi) p(\xi) \exp\left( \int_{\xi}^{t} w(t,r) q(r) dr \right) d\xi, \quad t \in J \tag{3.73}$$

To apply the above inequality to Eq. (3.71), let $u(t) = \|\mathbf{x}(t)\|$, $q(t) = 1$, $p(t) = \frac{\kappa(\Phi)\Gamma(\alpha)C_{max}}{1+\rho(t-t_0)^{\alpha}}\|\mathbf{x}(t_0^+)\|$, and $w(t,\xi) = \kappa(\Phi)\mu C_{max}\frac{(t-\xi)^{\alpha-1}}{1+\rho(t-\xi)^{\alpha}}$. Then,

$$\begin{aligned}&\|\mathbf{x}(t)\| \\ &\leq \frac{\kappa(\Phi)\Gamma(\alpha)C_{max}}{1+\rho(t-t_0)^{\alpha}}\|\mathbf{x}(t_0^+)\| \\ &+\int_{t_0}^{t}\frac{(t-\xi)^{\alpha-1}}{(1+\rho(t-\xi)^{\alpha})}\frac{K}{(1+\rho(\xi-t_0)^{\alpha})}\exp\left(\int_{\xi}^{t}\kappa(\Phi)\mu C_{max}\frac{(t-r)^{\alpha-1}}{1+\rho(t-r)^{\alpha}}dr\right)d\xi\end{aligned} \tag{3.74}$$

where $K = \kappa(\Phi)^2\mu C_{max}^2\Gamma(\alpha)\|\mathbf{x}(t_0^+)\|$**.** Integrating the inside of the exponential function in Eq. (3.74) yields

$$\int_{\xi}^{t}\kappa(\Phi)\mu C_{max}\frac{(t-r)^{\alpha-1}}{1+\rho(t-r)^{\alpha}}dr = \frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}\ln(1+\rho(t-\xi)^{\alpha}) \tag{3.75}$$

By using Eq. (3.74) and Eq. (3.75), we have

$$\begin{aligned}\|\mathbf{x}(t)\| &\leq \frac{\kappa(\Phi)\Gamma(\alpha)C_{max}}{1+\rho(t-t_0)^{\alpha}}\|\mathbf{x}(t_0^+)\| \\ &+\int_{t_0}^{t}\frac{(t-\xi)^{\alpha-1}}{\left((1+\rho(t-\xi)^{\alpha})^{1-\frac{\kappa\mu C_{max}}{\rho\alpha}}\right)}\frac{K}{(1+\rho(\xi-t_0)^{\alpha})}d\xi\end{aligned} \tag{3.76}$$

We can apply the same integral technique in reference [30] to integrate Eq. (3.76).

$$\begin{aligned}&\int_{t_0}^{t}\frac{(t-\xi)^{\alpha-1}}{\left((1+\rho(t-\xi)^{\alpha})^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}\right)}\frac{K}{(1+\rho(\xi-t_0)^{\alpha})}d\xi \\ &= \int_{t_0}^{\frac{1}{2}(t+t_0)}\frac{(t-\xi)^{\alpha-1}}{\left((1+\rho(t-\xi)^{\alpha})^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}\right)}\frac{K}{(1+\rho(\xi-t_0)^{\alpha})}d\xi\end{aligned} \tag{3.77}$$

$$+\int_{\frac{1}{2}(t+t_0)}^{t}\frac{(t-\xi)^{\alpha-1}}{\left((1+\rho(t-\xi)^{\alpha})^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}\right)}\frac{K}{(1+\rho(\xi-t_0)^{\alpha})}d\xi$$

Now, we assume

$$\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}<1 \tag{3.78}$$

Note that this assumption is going to be an important inequality condition explained later.

Since $\alpha<1$ and $(t-\xi)\geq(\xi-t_0)$ for $t_0\leq\xi\leq\frac{1}{2}(t+t_0)$ with the above assumption, the first integral on the right hand side of Eq. (3.77) becomes

$$\begin{aligned}&\int_{t_0}^{\frac{1}{2}(t+t_0)}\frac{(t-\xi)^{\alpha-1}}{\left((1+\rho(t-\xi)^{\alpha})^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}\right)}\frac{K}{(1+\rho(\xi-t_0)^{\alpha})}d\xi\\&\leq\int_{t_0}^{\frac{1}{2}(t+t_0)}\frac{(\xi-t_0)^{\alpha-1}}{\left((1+\rho(\xi-t_0)^{\alpha})^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}\right)}\frac{K}{(1+\rho(\xi-t_0)^{\alpha})}d\xi\end{aligned} \tag{3.79}$$

Substituting $r=\xi-t_0$ into Eq. (3.79), the right integral term in Eq. (3.79) becomes

$$\begin{aligned}&\int_{0}^{\frac{1}{2}(t-t_0)}\frac{r^{\alpha-1}}{\left((1+\rho r^{\alpha})^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}\right)}\frac{K}{(1+\rho r^{\alpha})}dr\\&=\int_{0}^{\frac{1}{2}(t-t_0)}\frac{K\times r^{\alpha-1}}{\left((1+\rho r^{\alpha})^{2-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}\right)}dr\end{aligned} \tag{3.80}$$

Similarly, since $\alpha<1$ and $(t-\xi)\leq(\xi-t_0)$ for $\frac{1}{2}(t+t_0)\leq\xi\leq t$ with the inequality in Eq. (3.78), we obtain

$$\int_{\frac{1}{2}(t+t_0)}^{t} \frac{(t-\xi)^{\alpha-1}}{\left((1+\rho(t-\xi)^{\alpha})^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}\right)} \frac{K}{(1+\rho(\xi-t_0)^{\alpha})} d\xi$$

$$\leq \int_{\frac{1}{2}(t+t_0)}^{t} \frac{(t-\xi)^{\alpha-1}}{\left((1+\rho(t-\xi)^{\alpha})^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}\right)} \frac{K}{(1+\rho(t-\xi)^{\alpha})} d\xi \tag{3.81}$$

$$= \int_{0}^{\frac{1}{2}(t-t_0)} \frac{K \times s^{\alpha-1}}{(1+\rho s^{\alpha})^{2-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}} ds$$

Thus, Eq. (3.76) becomes

$$\|\mathbf{x}(t)\| \leq \frac{\kappa(\Phi)\Gamma(\alpha)C_{max}}{1+\rho(t-t_0)^{\alpha}} \|\mathbf{x}(t_0^+)\| + \int_{0}^{\frac{1}{2}(t-t_0)} \frac{2K \times s^{\alpha-1}}{(1+\rho s^{\alpha})^{2-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}} ds \tag{3.82}$$

Then,

$$\|\mathbf{x}(t)\| \leq \frac{\kappa(\Phi)\Gamma(\alpha)C_{max}}{1+\rho(t-t_0)^{\alpha}} \|\mathbf{x}(t_0^+)\| + \int_{0}^{\frac{1}{2}(t-t_0)} \frac{2K \times s^{\alpha-1}}{(1+\rho s^{\alpha})^{2-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}} ds \tag{3.83}$$

Integral of the second term on the right hand side of Eq. (3.83) is given by

$$\begin{aligned} \int_{0}^{\frac{1}{2}(t-t_0)} \frac{2K \times s^{\alpha-1}}{(1+\rho s^{\alpha})^{2-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}} ds &= \frac{2K/(\kappa(\Phi)\mu C_{max}-\rho\alpha)}{(1+\rho s^{\alpha})^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}} \Bigg]_{s=0}^{s=\frac{1}{2}(t-t_0)} \\ &= \frac{2K/(\kappa(\Phi)\mu C_{max}-\rho\alpha)}{\left(1+\rho\left(\frac{1}{2}(t-t_0)\right)^{\alpha}\right)^{1-\frac{(\Phi)\mu C_{max}}{\rho\alpha}}} - \frac{2K}{(\kappa(\Phi)\mu C_{max}-\rho\alpha)} \end{aligned} \tag{3.84}$$

Substituting Eq. (3.84) and K into Eq. (3.83) yields

$$\|\mathbf{x}(t)\| \leq \frac{\kappa(\Phi)\Gamma(\alpha)C_{max}}{1+\rho(t-t_0)^{\alpha}} \|\mathbf{x}(t_0^+)\| + \frac{2\kappa(\Phi)^2\mu C_{max}^2\Gamma(\alpha)\|\mathbf{x}(t_0^+)\|}{(\rho\alpha-\kappa(\Phi)\mu C_{max})} \tag{3.85}$$

$$-\frac{2\kappa(\Phi)^2\mu C_{max}^2\Gamma(\alpha)\|\mathbf{x}(t_0^+)\|}{(\rho\alpha-\kappa(\Phi)\mu C_{max})\left(1+\rho\left(\frac{1}{2}(t-t_0)\right)^{\alpha}\right)^{1-\frac{\kappa(\Phi)\mu C_{max}}{\rho\alpha}}}$$

The first function on the right hand side is monotonously decreasing function to zero. And, the sum of the second and third functions is greater than or equal to zero when $t \geq t_0$, but it increases monotonously to the finite value, $\frac{2\kappa(\Phi)^2\mu C_{max}^2\Gamma(\alpha)\|\mathbf{x}(t_0^+)\|}{(\rho\alpha-\kappa(\Phi)\mu C_{max})}$ as t increases. This means that the upper bound of $\|x(t)\|$ is not the infinite value in $t_0^+ \leq t \leq \infty$, which implies that the system is stable. Therefore, this fractional order system is stable if our assumption described in Eq. (3.78) is valid. This implies that the fractional order system is stable if the norm of the perturbation is less than the upper bound given by

$$\mu < \mu_1 \equiv \frac{\rho\alpha}{\kappa(\Phi)C_{max}} \tag{3.86}$$

*Corollary 3.3*

Let us consider the linear perturbations of $f(t,\mathbf{x}(t))$. Then we can write the fractional order system given in Eq. (3.57) as

$${}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = A\mathbf{x}(t) + E(t)\mathbf{x}(t) \tag{3.87}$$

The above system is stable if the following condition is satisfied.

$$\|E(t)\| < \mu_1 \tag{3.88}$$

Proof

By using the norm inequality[29],

$$\|\mathrm{E}(t)\mathbf{x}(t)\| \le \|\mathrm{E}(t)\|_s\|\mathbf{x}(t)\| \le \|\mathrm{E}(t)\|\|\mathbf{x}(t)\| = \left(\sqrt{\sum_{i,j=1}^{n}\left|\mathrm{E}_{ij}(t)\right|^2}\right)\|\mathbf{x}(t)\| \tag{3.89}$$

where $\mathrm{E}_{ij}(t)$ is the $(i,j)^{th}$ element of $\mathrm{E}(t)$. From Theorem 3.5, we obtain directly

$$\|\mathrm{E}(t)\|_s \le \|\mathrm{E}(t)\| < \mu_1 \quad \text{and} \quad \left|\mathrm{E}_{ij}(t)\right| < \frac{\mu_1}{n} \tag{3.90}$$

Example 3.1. Let us consider that the fractional order α is 0.5 and the matrix A in Eq. (3.87) is given by

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \tag{3.91}$$

From Corollary 3.3, the upper bound $\mu_1$ is given by

$$\mu_1 = \frac{0.5\rho}{\kappa(\Phi)\mathrm{C}_{max}} \tag{3.92}$$

By the following procedure, we can obtain the variables in Eq. (3.92).

1) By the similarity transformation, $\mathrm{A}=\Phi\Lambda\Phi^{-1}$, the eigenvalue and the eigenvector matrices can be obtained.

$$\Phi = \begin{bmatrix} 0.7071 & 0.7071 \\ 0.3536 + 0.6124\mathrm{j} & 0.3536 - 0.6124\mathrm{j} \end{bmatrix} \tag{3.93a}$$

$$\Lambda = \begin{bmatrix} 0.5 + 0.8660\mathrm{j} & 0 \\ 0 & 0.5 - 0.8660\mathrm{j} \end{bmatrix} \tag{3.93b}$$

2) By the definition of the condition number,

$$\kappa(\Phi) = \|\Phi\|\|\Phi^{-1}\| = 1.7321 \tag{3.94}$$

3) By the definition of ρ described in Theorem 3.5,

$$\rho = \mathrm{Min}\{|0.5 + 0.8660\mathrm{j}\,|, \quad |0.5 - 0.8660\mathrm{j}\,|\} = 1.$$

4) By trial and error numerically, we obtain $C_{max} = 2.4$ from Fig. (3.5) and the following Eq. (3.95).

$$\left\|E_{0.5,0.5}((0.5 \pm 0.8660j)t^{0.5})\right\|_s \leq \frac{2.4}{1+t^{0.5}} \tag{3.95}$$

Therefore, from Eq. (3.92), we have $\mu_1 = 0.1203$. From Corollary 3.3, the linear perturbations have the following upper bounds.

$$\|E(t)\| < \mu_1 = 0.1203 \text{ and } \left|E_{ij}(t)\right| < \frac{\mu_1}{n} = 0.06015 \tag{3.96}$$

By Fig. (3.6), the above result is verified because all eigenvalues do not cross the stability lines.

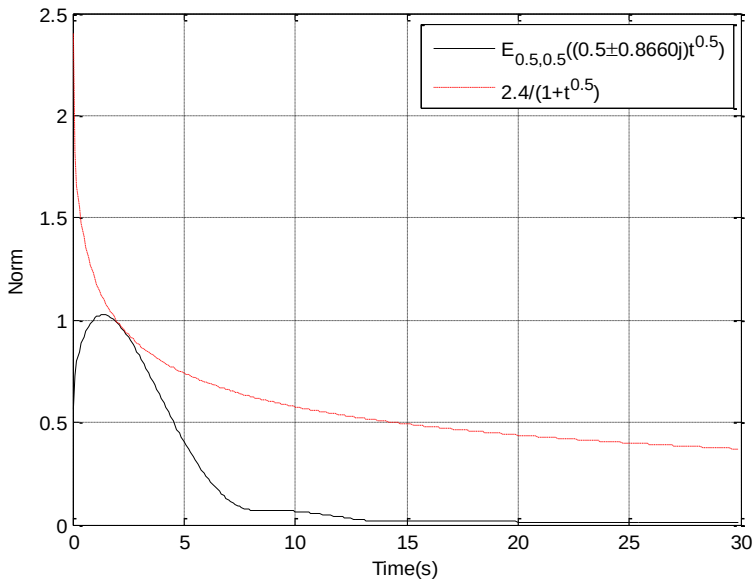


**Figure. 3.5.** Norms of $E_{0.5,0.5}((0.5 \pm 0.8660j)t^{0.5})$ and $\frac{2.4}{1+t^{0.5}}$

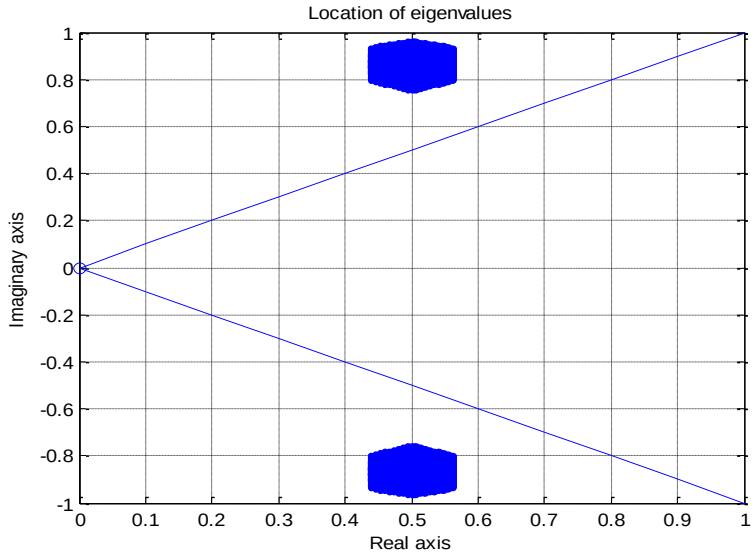


**Figure 3.6.** Locations of Eigenvalues and the Stability Lines

**Bilinear Systems**

Reference [29] showed the stability region of bilinear systems for integer order systems, which gives us the useful application of obtaining the stability region of the satellite dynamics. Similarly, we will consider bilinear systems governed by the fractional order derivative. Then the stability region of it will be achieved. Let us consider the following bilinear system.

$$\frac{d\mathbf{x}(t)}{dt} = F\mathbf{x}(t) + \sum_{i=1}^{m} B_i\, u_i \mathbf{x}(t) + C\mathbf{u}(t) \tag{3.97}$$

where $(t) = [u_1(t), u_2(t), \dots, u_m(t)]^T \in \mathbb{R}^m$ . If we design the controller by using the state feedback form as $\mathbf{u}(t) = K\mathbf{x}(t)$, the above system can be given by

$$\frac{d\mathbf{x}(t)}{dt} = (F + CK)\mathbf{x}(t) + \sum_{i=1}^{m} B_i \left(k_i^T \mathbf{x}(t)\right)\mathbf{x}(t) \tag{3.98}$$

where $K = [k_1(t), k_2(t), \dots, k_m(t)]^T \in \mathbb{R}^{m\times n}$.

*Corollary 3.4*[29]

The stability region of the above system near the origin is given by

$$\|\mathbf{x}(t)\| \le \frac{\mu}{(\sum_{i=1}^{n}\|D_i\|_s^2)^{\frac{1}{2}}} \tag{3.99}$$

where μ is defined in Eq. (3.47) with $A = F + CK$ and $D_j = \sum_{i=1}^{m} k_i b_{ij}^T$ denoting the $j^{th}$ row of $B_i$ by $b_{ij}^T$.

Let us consider the fractional order state feedback given by $\mathbf{u}(t) = K_1\mathbf{x}(t) + K_2\, {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t)$. For the simplicity, we choose $\alpha = 0.5$. Letting $\mathbf{w}(t) = {}_{t_0}^{C}D_t^{0.5}\mathbf{x}(t)$ and using the technique to make the state-space representation for fractional order system in previous chapter, the state-space representation can be expressed by

$$\begin{aligned} {}_{t_0}^{C}D_t^{0.5}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{w}(t)\end{bmatrix} &= \begin{bmatrix} 0 & I \\ F + CK_1 & CK_2\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{w}(t)\end{bmatrix} \\ &\quad + \sum_{i=1}^{m}[B_i \quad 0]\left(\begin{bmatrix}k_{1i}\\ k_{2i}\end{bmatrix}^T\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{w}(t)\end{bmatrix}\right)\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{w}(t)\end{bmatrix} \end{aligned} \tag{3.100}$$

where $k_{ji}$ presents the $i^{th}$ row of $K_j, j = 1 \text{ or } 2$. Assuming that $A \equiv \begin{bmatrix} 0 & I \\ F + CK_1 & CK_2 \end{bmatrix}$ is asymptotically stable, we consider the region of stability around the stationary point $\mathbf{p}(t) \equiv \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{w}(t) \end{bmatrix} = \mathbf{0}$. Then, we rewrite Eq. (3.100) as

$$ {}_{t_0}^{C}D_t^{0.5}\mathbf{p}(t) = A\mathbf{p}(t) + f(\mathbf{p}(t)) \tag{3.101} $$

where

$$ f(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}^T D_1 \mathbf{p} \\ \mathbf{p}^T D_2 \mathbf{p} \\ \vdots \\ \mathbf{p}^T D_n \mathbf{p} \end{bmatrix} \quad \text{and} \quad D_j = \sum_{i=1}^{m} \begin{bmatrix} k_{1i} b_{ij}^T & 0 \\ k_{2i} b_{ij}^T & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \tag{3.102} $$

denoting the $j^{th}$ row of $B_i$ by $b_{ij}^T$. Therefore,

$$ \left\| f(\mathbf{p}(t)) \right\|^2 = \sum_{i=1}^{n} |\mathbf{p}^T D_i \mathbf{p}|^2 \leq \sum_{i=1}^{n} (\|D_i\|_s \|\mathbf{p}(t)\|^2)^2 = \left( \sum_{i=1}^{n} \|D_i\|_s^2 \right) \|\mathbf{p}(t)\|^4 \tag{3.103} $$

Then we obtain

$$ \frac{\left\| f(\mathbf{p}(t)) \right\|}{\|\mathbf{p}(t)\|} \leq \left( \sum_{i=1}^{n} \|D_i\|_s^2 \right)^{\frac{1}{2}} \|\mathbf{p}(t)\| \tag{3.104} $$

From Theorem 3.5, we can obtain the sufficient condition for stability as

$$ \left( \sum_{i=1}^{n} \|D_i\|_s^2 \right)^{\frac{1}{2}} \|\mathbf{p}(t)\| < \mu_1 \tag{3.105} $$

*Corollary 3.5*

The stability region of the above system near the origin is given by

$$\|\mathbf{p}(t)\| < \frac{\mu_1}{(\sum_{i=1}^{n}\|D_i\|_s^2)^{\frac{1}{2}}} \tag{3.106}$$

**Spacecraft Dynamics**

Euler's rotational equations of motion for a cylindrical spacecraft are given by choosing the body coordinate system to coincide with the principal axes of the spacecraft as[4][29]

$$\frac{d\omega_1}{dt} = \frac{(J_2 - J_3)}{J_1}\omega_2\omega_3 + \frac{L_1}{J_1} \tag{3.107a}$$

$$\frac{d\omega_2}{dt} = \frac{(J_3 - J_1)}{J_2}\omega_3\omega_1 + \frac{L_2}{J_2} \tag{3.107b}$$

$$\frac{d\omega_3}{dt} = \frac{L_3}{J_3} \tag{3.107c}$$

where $\omega_i, i = 1,2,3$ is the angular velocity and $L_i, i = 1,2,3$ is the applied torque and $J_i, i = 1,2,3$ is the moment of inertia with respect to $i^{th}$ principal axis.

*Corollary 3.6*

By using the negative state feedback form, $L_i(t) = -k_i\omega_t(t)(k_i > 0)$, the stability region of Eq. (3.91) is given by[29]

$$(\omega_1^2 + \omega_2^2 + \omega_3^2)^{\frac{1}{2}} \leq \frac{\min_{i=1,2,3}\frac{k_i}{J_i}}{(J_1^2 + J_2^2)^{\frac{1}{2}}} \tag{3.108}$$

Let us consider the fractional order feedback.

$$\mathbf{L}(t) = -K_1\boldsymbol{\omega}(t) - K_2\,{}_{t_0}^{C}D_t^{0.5}\boldsymbol{\omega}(t) \tag{3.109}$$

where $\boldsymbol{\omega}(t) = [\omega_1 \quad \omega_2 \quad \omega_3]^T$ and $\mathbf{L}(t) = [L_1 \quad L_2 \quad L_3]^T$. By letting $x_i(t) = {}_{t_0}^{C}D_t^{0.5}\omega_i(t), i = 1,2,3$, we obtain the closed-loop dynamics as

$$ {}_{t_0}^{C}D_t^{0.5}\mathbf{p}(t) = A\mathbf{p}(t) + \frac{1}{2}\begin{bmatrix} 0 \\ \mathbf{p^T}D_1\mathbf{p} \\ 0 \\ \mathbf{p^T}D_2\mathbf{p} \\ 0 \\ 0 \end{bmatrix} \tag{3.110} $$

where

$$ \mathbf{p}(t) = [\omega_1 \quad x_1 \quad \omega_2 \quad x_2 \quad \omega_3 \quad x_3]^T \tag{3.111} $$

$$ A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_{21}}{J_1} & -\frac{k_{11}}{J_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{k_{22}}{J_2} & -\frac{k_{12}}{J_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{k_{23}}{J_3} & -\frac{k_{13}}{J_3} \end{bmatrix} \tag{3.112} $$

$$ D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(J_2 - J_3)}{J_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(J_2 - J_3)}{J_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.113} $$

$$ D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{(J_3 - J_1)}{J_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{(J_3 - J_1)}{J_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.114} $$

Then we can obtain the characteristic equations to obtain the eigenvalues of A.

$$s^2 + \frac{k_{1i}}{J_i}s + \frac{k_{2i}}{J_i} = 0, \quad i = 1,2,3 \tag{3.115}$$

Thus, we obtain six eigenvalues

$$\lambda_{2i-1,2i} = \frac{-\frac{k_{1i}}{J_i} \pm \sqrt{\left(\frac{k_{1i}}{J_i}\right)^2 - 4\frac{k_{2i}}{J_i}}}{2}, \quad i = 1,2,3 \tag{3.116}$$

All eigenvalues of matrix A should satisfy the stability condition.

$$\arg(\lambda_i) > \frac{\pi}{4}, \quad i = 1,2,\dots,6 \tag{3.117}$$

*Corollary 3.7*

The stability region of Eq. (3.110) around the origin is given by

$$(\omega_1^2 + \omega_2^2 + \omega_3^2 + x_1^2 + x_2^2 + x_3^2)^2 < \frac{\mu_1}{\left(\left(\frac{J_2 - J_3}{J_1}\right)^2 + \left(\frac{J_3 - J_1}{J_2}\right)^2\right)^{\frac{1}{2}}} \tag{3.118}$$

# CHAPTER IV

# ROBUST EIGENSTRUCTURE ASSIGNMENT FOR THE FRACTIONAL ORDER SYSTEM

## A. PROBLEM STATEMENT

Let us consider the following fractional order system.

$$ {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{4.1} $$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the time-invariant matrices with full rank. Input is given as the state feedback control form

$$ \mathbf{u}(t) = F\mathbf{x}(t) + \mathbf{v}(t) \tag{4.2} $$

where $F \in \mathbb{R}^{m\times n}$ is a constant gain matrix. Now, the system has the following closed-loop form.

$$ {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = (A + BF)\mathbf{x}(t) + B\mathbf{v}(t) \tag{4.3} $$

In this form, our objective is as follows.

1) Pole Assignment: Gain matrix F should consist of only real numbers and should be chosen to put the closed-loop eigenvalues of the fractional order system into the desired specific places.
2) Robustness: Assigned poles are as insensitive as possible to the perturbations in A+BF.

To achieve the robustness, we use some robust stability measures to represent the eigenvalue sensitivity such as the condition number of the closed loop eigenvector matrix mentioned previously, and try to minimize it. Although we have a fractional order

system, we can still use the following important facts obtained from previous research for the integer order system:

1) A gain solution F exists if and only if the A, B pair is completely controllable[6][34]. If the A, B pair is not completely controllable, a solution F exists if and only if the closed-loop eigenvalues contains the eigenvalue set of all uncontrollable modes of (A,B)[6].
2) F can be chosen to be a real matrix if complex eigenvalues are complex-conjugate pairs and the corresponding eigenvectors are also complex-conjugate pairs[34].
3) If A+BF is non-defective, it can be diagonalized. This means that it has n linearly independent eigenvectors.
4) According to the dimension of the control input, a solution F can be divided into three cases[6]:
    a) Single input $(m = 1)$: a solution F is unique, if it exists. The condition number cannot be adjusted.
    b) Multiple inputs $(1 < m < n)$: multiple solutions may exist and a specific solution can be obtained when we have enough additional specific conditions imposed.
    c) Multiple inputs $(m = n)$: A, B pair is always completely controllable so that solution F always exists. We can find the orthogonal(unitary) eigenvectors set which has the lower limit of the condition number.

From these observations, we consider the case of multiple inputs.

Now, let us consider the closed loop *eigenvalue problem* from Eq. (4.3).

$$(A + BF)\Phi = \Phi\Lambda \tag{4.4}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are the assigned eigenvalues. First, we want to find the explicit form of the feedback gain matrix F and the admissible space of a nonsingular eigenvector matrix $\Phi$ satisfying the above equation. Then, we will try to minimize the robust stability measure defined by the condition number. The following theorem and corollary will give us the explicit form of the gain matrix F and an admissible eigenvector matrix $\Phi$.

*Theorem 4.1*[6]

Let us assume that $\Phi$ and $\Lambda$ are given as in Eq. (4.4). Then there exists a feedback gain matrix F if and only if

$$U_1^T(A\Phi - \Phi\Lambda) = 0 \tag{4.5}$$

where

$$B = [U_0 \quad U_1]\begin{bmatrix} Z \\ 0 \end{bmatrix} \tag{4.6}$$

with $U = [U_0 \quad U_1]$ orthogonal and Z nonsingular. Then, matrix F is explicitly given by

$$F = Z^{-1}U_0^T(\Phi\Lambda\Phi^{-1} - A) \tag{4.7}$$

Proof

From Eq. (4.4), we have

$$BF = \Phi\Lambda\Phi^{-1} - A \tag{4.8}$$

From the assumption of full rank of a matrix B, B can be decomposed into Eq. (4.6). Factorization of matrix B can be obtained by singular value decomposition(SVD) or QR

decomposition[3][6]. By the pre-multiplication by $U^T$ on both sides, we obtain two equations

$$ZF = U_0^T(\Phi\Lambda\Phi^{-1} - A) \tag{4.9a}$$

$$0 = U_1^T(\Phi\Lambda\Phi^{-1} - A) \tag{4.9b}$$

Since Z is invertible, we can obtain Eq. (4.7) from Eq. (4.9a). Therefore, if we have the desired eigenvalue set and the corresponding eigenvector matrix $\Phi$, we can easily obtain the feedback gain matrix F. Now, we need to know how to find the eigenvector matrix $\Phi$. The following corollary will give us the corresponding admissible space of each eigenvector, which allows us to parameterize the eigenvector matrix $\Phi$.

*Corollary 4.1*[6]

From Eq. (4.9b), the eigenvector $\boldsymbol{\phi}_j$ of A+BF corresponding to $\lambda_j$ must belong to the following null space of $U_1^T(A - \lambda_j I)$.

$$\boldsymbol{\phi}_j \in S_j \equiv \mathcal{N}\left(U_1^T(A - \lambda_j I)\right), \quad j = 1,2,\dots,n \tag{4.10}$$

Proof

From Eq. (4.9b), we can directly obtain the following equation.

$$U_1^T(A - \lambda_j I)\boldsymbol{\phi}_j = 0, \quad \forall j \tag{4.11}$$

This means that $\boldsymbol{\phi}_j$ is the null space of $U_1^T(A - \lambda_j I)$. If it is completely controllable, the dimension of $S_j$ is m. This equation tells us what form eigenvector $\boldsymbol{\phi}_j$ should take. Null space of $U_1^T(A - \lambda_j I)$ can be obtained by singular value decomposition(SVD) or QR

decomposition[3][6]. Either decomposition gives us the orthonormal vectors which span the null space of $U_1^T(A - \lambda_j I)$. Hence, each eigenvector $\boldsymbol{\phi}_j$ can be described by

$$\boldsymbol{\phi}_j = \Sigma_j \Xi_j \tag{4.12}$$

where

$$\Sigma_j = [\mathbf{p}_1 \quad \cdots \quad \mathbf{p}_m] \in \mathbb{C}^{n\times m}(\text{or } \mathbb{R}^{n\times m}), \qquad \mathbf{p}_i \in \mathbb{C}^{n\times 1}(\text{or } \mathbb{R}^{n\times 1}) \tag{4.13}$$

$$\Xi_j = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} \in \mathbb{C}^{m\times 1}(\text{or } \mathbb{R}^{m\times 1}) \tag{4.14}$$

Each $\mathbf{p}_i, i = 1, \dots, m$ represents one of orthonormal basis vectors which span the null space of $U_1^T(A - \lambda_j I)$ in Eq. (4.11) and $\Xi_j$ describes the coordinate matrix with arbitrary numbers, $\alpha_i, i = 1, \dots, m$. If the eigenvalue $\lambda_j$ is complex(or real), the eigenvector is also complex(or real), $\boldsymbol{\phi}_j \in \mathbb{C}^{n\times 1}\left(\text{or } \boldsymbol{\phi}_j \in \mathbb{R}^{n\times 1}\right)$. Hence, the corresponding orthonormal basis vectors and the coordinate matrix also differ depending on whether the eigenvalue is real or complex. With this parameterization, eigenvector matrix $\Phi$ can be described by

$$\Phi = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_n] = \Sigma D \tag{4.15}$$

where

$$\Sigma = [\Sigma_1 \quad \Sigma_2 \quad \cdots \quad \Sigma_n] \tag{4.16}$$

$$D = \begin{bmatrix} \Xi_1 & 0 & 0 & 0 \\ 0 & \Xi_2 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \Xi_n \end{bmatrix} \tag{4.17}$$

Now, $\Phi$ can be parameterized with $\alpha_i, i = 1, \dots, n \times m$ as

$$\Phi = \Phi(\alpha_1, \alpha_2, \cdots, \alpha_{n\times m}) \tag{4.18}$$

Then, the cost function for the robust stability measure can be expressed in terms of $\alpha_i, i = 1, \dots, n \times m$ .

$$\min_{\alpha} \kappa_F \left( \Phi\left(\alpha_1, \alpha_{2,} \cdots, \alpha_{n\times m}\right)\right) \tag{4.19}$$

By using this parameterization and the above cost function, R. Byers and S. G. Nash found a numerical solution[34]. However, they mentioned that this parameterization has some problems. The condition number has the property of $\kappa_F(\Phi) = \kappa_F(\rho\Phi)$ for any $\rho \in \mathbb{R}, \rho \neq 0$ . This means that this optimization problem does not have a unique solution, which makes the Hessian of $\kappa_F^2(\Phi)$ singular at minimum, reducing the performance of the optimization algorithms[34]. Therefore, Byers and Nash propose the following ways to eliminate this property:

1) $\|\Phi\|_F = 1$ : This is a nonlinear constraint making it difficult to handle computationally.
2) Alternative objective function:

$$\kappa_F(\Phi) = \|\Phi\|_F \|\Phi^{-1}\|_F \leq \frac{1}{2}(\|\Phi\|_F^2 + \|\Phi^{-1}\|_F^2) \tag{4.20}$$

When the cost function in Eq. (4.20) is optimal, a solution Φ minimizes both cost functions.

To solve this scaling ambiguity, we use a normalized eigenvector by using $\|\boldsymbol{\Phi}_j\|_2 = 1, \text{for all } j, j = 1, \dots, n$ . In Reference[6], it is shown that by the assumption of normalizing the right eigenvectors, it is convenient to derive the measures of robustness as shown in the previous chapter.

In summary, if we use the normalized right eigenvectors, we can avoid the scaling ambiguity and obtain the convenient measures of robustness as a cost function. However, we are led to a problem with the nonlinear constraints. How can we accomplish our goals without an eigenvector norm nonlinear constraint? To answer this question, we propose the angle parameterization. To use an angle parameterization, we need to understand the concept of n-dimensional rigid rotation method developed by Mortari[5].

## B. ROTATION METHOD

### N-dimensional Rigid Rotation

We know that the rotation in 3-dimensional space is performed about an axis, 1-dimensional subspace. If we extend this concept to n-dimensions, the rigid rotation in the n-dimensional space is performed about an (n-2)-dimensional subspace. And the rotation is performed on the plane of rotation which has a dimension of 2. The rotation matrix in the n-dimensional space can be expressed as[5]

$$\mathrm{R}(\mathrm{P},\theta) = \mathrm{I}_\mathrm{n} + (\cos\theta - 1)\mathrm{P}\mathrm{P}^\mathrm{T} + \mathrm{P}\mathrm{J}_2\mathrm{P}^\mathrm{T}\sin\theta \tag{4.21}$$

where $\mathrm{P} = [\mathbf{p}_1 \quad \mathbf{p}_2] \in \mathbb{R}^{\mathrm{n}\times 2}$ is the plane of rotation described by two orthonormal vectors. These vectors are on the plane of rotation with the rotation angle, $\theta$. $\mathrm{J}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the symplectic matrix. And the rotation matrix is orthogonal.

$$\mathrm{R}(\mathrm{P},\theta)^\mathrm{T}\mathrm{R}(\mathrm{P},\theta) = \mathrm{R}(\mathrm{P},\theta)\mathrm{R}(\mathrm{P},\theta)^\mathrm{T} = \mathrm{I}_\mathrm{n} \tag{4.22}$$

This orthogonal property is easily proved by substitution of Eq. (4.21).

*Theorem 4.2*[5]

The rotation matrix defined in Eq. (4.21) performs a rigid rotation in the n-dimensional space.

Proof

See Reference [5].

Let us examine the n-dimensional rotation more. We assume that C is an $n \times n$ proper orthogonal matrix.

$$C = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_n] \tag{4.23}$$

Then we can perform any rotation on the plane defined as

$$P = [\hat{\mathbf{p}}_1 \quad \hat{\mathbf{p}}_2] \equiv [\mathbf{c}_i \quad \mathbf{c}_j] \in \mathbb{R}^{n\times 2} \quad \forall i, j \text{ and } i \neq j \tag{4.24}$$

Also, this rotation is performed about an $(n-2)$ dimension subspace defined as

$$\begin{aligned} A &= [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_{n-2}] \\ &\equiv [\mathbf{c}_1 \quad \cdots \quad \mathbf{c}_{i-1} \quad \mathbf{c}_{i+1} \quad \cdots \quad \mathbf{c}_{j-1} \quad \mathbf{c}_{j+1} \quad \cdots \quad \mathbf{c}_n] \end{aligned} \tag{4.25}$$

where i and j are defined in Eq. (4.24) and $A \in \mathbb{R}^{n\times(n-2)}$ is the matrix which removed the $\mathbf{c}_i$ and $\mathbf{c}_j$ columns from C matrix. Then we can define the rotation matrix R by using Eq. (4.21). From the proof of Theorem 4.2 with Eq. (4.22) to Eq. (4.24), we have the following equations[5].

$$\mathbf{a}_k^T\mathbf{v} = \mathbf{a}_k^T R(P,\theta)\mathbf{v}, \quad \forall k(k = 1, \ldots, n-2) \tag{4.26a}$$

$$\mathbf{v}^T\mathbf{w} = \mathbf{v}^T R(P,\theta)^T R(P,\theta)\mathbf{w} = \mathbf{v}^T\mathbf{w} \tag{4.26b}$$

where $\mathbf{v}$ and $\mathbf{w} \in \mathbb{R}^{n\times 1}$ are arbitrary vectors. First equation explains the rotation and the second one tells about the *rigidity*. From Eq. (4.26a) and Eq. (4.26b), we can have the following propositions.

*Proposition 4.1*

If an arbitrary vector $\mathbf{v}$ is defined in the subspace of matrix P in Eq. (4.24), $\mathbf{v}$ still does not have any column components defined in matrix A after performing the rotation defined by $\mathrm{R}(\mathrm{P}, \theta)$.

Proof

An arbitrary vector $\mathbf{v}$ can be expressed as

$$\mathbf{v} = \alpha\hat{\mathbf{p}}_1 + \beta\hat{\mathbf{p}}_2 \tag{4.27}$$

where $\alpha$ and $\beta$ are arbitrary real numbers. Since $\mathbf{a}_k^T\hat{\mathbf{p}}_1 = \mathbf{a}_k^T\hat{\mathbf{p}}_2 = 0, \mathbf{a}_k$ defined in Eq. (4.25), we have

$$\mathbf{a}_k^T\mathbf{v} = \mathbf{a}_k^T(\alpha\hat{\mathbf{p}}_1 + \beta\hat{\mathbf{p}}_2) = 0, \quad \forall k(k = 1, \dots, n-2) \tag{4.28}$$

After performing the rotation with $\mathrm{R}(\mathrm{P}, \theta)$, we have

$$\mathbf{a}_k^T\mathrm{R}(\mathrm{P}, \theta)\mathbf{v} = \mathbf{a}_k^T\mathrm{R}(\mathrm{P}, \theta)(\alpha\hat{\mathbf{p}}_1 + \beta\hat{\mathbf{p}}_2), \quad \forall k(k = 1, \dots, n-2) \tag{4.29}$$

From Eq. (4.26a), Eq. (4.28) and Eq. (4.29) should be the same. Therefore, Eq. (4.29) should be zero. This means that $\mathbf{v}$ still does not have any column vectors defined in matrix A after performing the rotation.

*Proposition 4.2*

The rotation matrix defined in Eq. (4.21) preserves the length of any vector $\mathbf{v}$ during the rotation.

Proof

An arbitrary vector is chosen as $\mathbf{v} \in \mathbb{R}^{n\times 1}$. This is the special case $\mathbf{w} = \mathbf{v}$ in Eq. (4.26b). After the rotation in Eq. (4.21), the vector $\mathbf{m} \in \mathbb{R}^{n\times 1}$ can be given by

$$\mathbf{m} = \mathrm{R}(\mathrm{P}, \theta)\mathbf{v} \tag{4.30}$$

By the definition of the norm and the property in Eq. (4.22),

$$\|\mathbf{m}\| = \mathbf{m}^{\mathrm{T}}\mathbf{m} = \mathbf{v}^{\mathrm{T}}\mathrm{R}(\mathrm{P}, \theta)^{\mathrm{T}}\mathrm{R}(\mathrm{P}, \theta)\mathbf{v} = \|\mathbf{v}\| \tag{4.31}$$

*Theorem 4.3*

Assume that we have an n-dimensional real Euclidean space. The column space of an orthogonal matrix C given by Eq. (4.23) can be described by

$$\mathrm{Col\ C} = \mathrm{Span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \tag{4.32}$$

Then, a typical vector in Col C can be written by either way

$$\mathbf{b} = \sum_{i=1}^{n} \alpha_i \mathbf{c}_i \tag{4.33}$$

$$\mathbf{b} = \gamma \prod_{k=1}^{n-1} \mathrm{R}(\mathrm{P}_k, \theta_k)\, \mathbf{c}_1 \tag{4.34}$$

where $\alpha_i (i = 1, \dots, n)$ is an arbitrary real coefficient and $\gamma$ is an arbitrary real coefficient. The rotation matrix $\mathrm{R}(\mathrm{P}_k, \theta_k)$ is given by Eq. (4.21). $\theta_k$ is the rotation angle and $\mathrm{P}_k$ is the plane of rotation which is defined by

$$P_1 \equiv [\mathbf{c}_1 \quad \mathbf{c}_2] \tag{4.35a}$$

$$P_2 \equiv [R(P_1, \theta_1)\mathbf{c}_1 \quad \mathbf{c}_3] \tag{4.35b}$$

$$\vdots$$

$$P_k \equiv \left[\prod_{i=1}^{k-1} R(P_i, \theta_i)\mathbf{c}_1 \quad \mathbf{c}_{k+1}\right] \tag{4.35c}$$

Proof

Let us consider the orthogonal matrix C given in Eq. (4.23). Then we can express an arbitrary vector in Col C formed by the linear combination of the columns of C, which leads us to have Eq. (4.33). Alternatively, we can use a rotation concept. First, we choose the first two columns from the matrix C. Then the plane of rotation $P_1$ can be given in Eq. (4.35a). If we have the rotation angle $\theta_1$ with respect to $\mathbf{c}_1$, the rotation matrix $R(P_1, \theta_1)$ can be given by Eq. (4.21). Thus, this rotation expression can be given by

$$\mathbf{v} = R(P_1, \theta_1)\mathbf{c}_1 \tag{4.36}$$

By Propositions 4.1 and 4.2, Eq. (4.36) does not have any components in $\mathbf{c}_k(\forall k, k = 3,4, \dots, n)$ and the length of $\mathbf{v}$ is unity. Therefore, if we put a scaling real number $\gamma$ on the right side of Eq. (4.36), the vector $\mathbf{v}$ can be an arbitrary vector in the whole space spanned by $\mathbf{c}_1$ and $\mathbf{c}_2$. Now, we can choose the second plane of rotation as Eq. (4.35b) since $\mathbf{v}^T\mathbf{c}_1 = 0$. The rotation angle $\theta_2$ is given with respect to $\mathbf{v}$. The rotation matrix $R(P_2, \theta_2)$ can be given by Eq. (4.21). Thus, this second rotation expression can be described by

$$\mathbf{w} = \mathrm{R}(\mathrm{P}_2, \theta_2)\mathbf{v} = \prod_{\mathrm{i}=1}^{2} \mathrm{R}(\mathrm{P}_\mathrm{i}, \theta_\mathrm{i})\mathbf{c}_1 \tag{4.37}$$

By Propositions 4.1 and 4.2, $\mathbf{w}$ does not have any components in $\mathbf{c}_\mathrm{k}(\forall \mathrm{k}, \mathrm{k} = 4, \dots, \mathrm{n})$ and the length of $\mathbf{w}$ has unity. Therefore, if we put a scaling real number $\gamma$ on the right side of Eq. (4.37), the vector $\mathbf{w}$ can be an arbitrary vector in the whole space spanned by $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$. From these observations, we can extend this procedure to the (n-1) step. Then we obtain the plane of rotation as Eq. (4.35c) for an arbitrary k and a typical vector in Col C can be written by Eq. (4.33).

If the matrix is not square but rectangular, the following corollary can be used.

*Corollary 4.2*

In an n-dimensional real Euclidean space, the column space of an $\mathrm{n} \times \mathrm{m}(\mathrm{n} > m)$ matrix C can be described by

$$\mathrm{Col\ C} = \mathrm{Span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_\mathrm{m}\} \tag{4.38}$$

where each $\mathbf{c}_\mathrm{i}, \mathrm{i} = 1, \dots, \mathrm{m}$ is an orthonormal vector. Then a typical vector in Col C can be written by either way

$$\mathbf{b} = \sum_{\mathrm{i}=1}^{\mathrm{m}} \alpha_\mathrm{i}\mathbf{c}_\mathrm{i} \tag{4.39}$$

$$\mathbf{b} = \gamma \prod_{\mathrm{k}=1}^{\mathrm{m}-1} \mathrm{R}(\mathrm{P}_\mathrm{k}, \theta_\mathrm{k})\, \mathbf{c}_1 \tag{4.40}$$

We use the same variables shown in Theorem 4.3. Equation (4.39) is completed in the similar manner of Theorem 4.3. To perform an n-dimensional rotation, we need n

orthonormal vectors, but we have only m orthonormal vectors. But we can add deficient (n-m) orthonormal vectors to $\mathrm{Col}\,\mathrm{C}$ by using the null space of $\mathrm{C}^{\mathrm{T}}$ to make $\mathrm{n} \times \mathrm{n}$ orthogonal matrix which columns are used for the rotation axes because total n dimensional axes are required to perform an n-dimensional rotation about (n-2) axes with the plane of rotation. Although this is required theoretically, the actual computation to obtain the null space of $\mathrm{C}^{\mathrm{T}}$ is not needed because we can perform the rotation if we choose any two axes in $\mathrm{Col}\,\mathrm{C}$ for the plane of rotation in Eq. (4.21). Note that two orthonormal axes for the plane of rotation should be selected only from $\mathrm{Col}\,\mathrm{C}$. Therefore, we need only (m-1) rotations to express a typical vector in $\mathrm{Col}\,\mathrm{C}$ described in Eq. (4.38). The proof is almost the same with one in Theorem 4.3.

With Theorem 4.3 and Corollary 4.2, we can find the angle parameterization for expressing the normalized eigenvector $\boldsymbol{\phi}_j$ in the admissible eigenvector space given by Eq. (4.10).

**Angle Representation of Eigenvector**

As the previous rotation matrix in Eq. (4.21) has shown, it contains only real numbers. As mentioned previously, if the eigenvalue is a complex number, the corresponding eigenvector has complex numbers. Thus, we have to consider how the rotation matrix can be defined in the complex space, which leads us to consider the real eigenvector space and the complex eigenvector space differently. First, let us examine the real eigenvector space.

**Angle Representation in the Real Space**

From Corollary 4.1, each eigenvector can be expressed by Eq. (4.12) and it becomes one column of the eigenvector matrix. This means that each eigenvector is a normalized vector in the column space of $\Sigma_j = [\hat{\mathbf{p}}_1 \quad \cdots \quad \hat{\mathbf{p}}_m] \in \mathbb{R}^{n\times m}$ given in Eq. (4.12). From Corollary 4.2, we have two methods to express each eigenvector. Equation (4.39) explains the concept of the previous parameterization given in Eq. (4.12), while Eq. (4.40) with a scaling coefficient $\gamma = 1$ gives us the fundamental idea of our proposed angle parameterization.

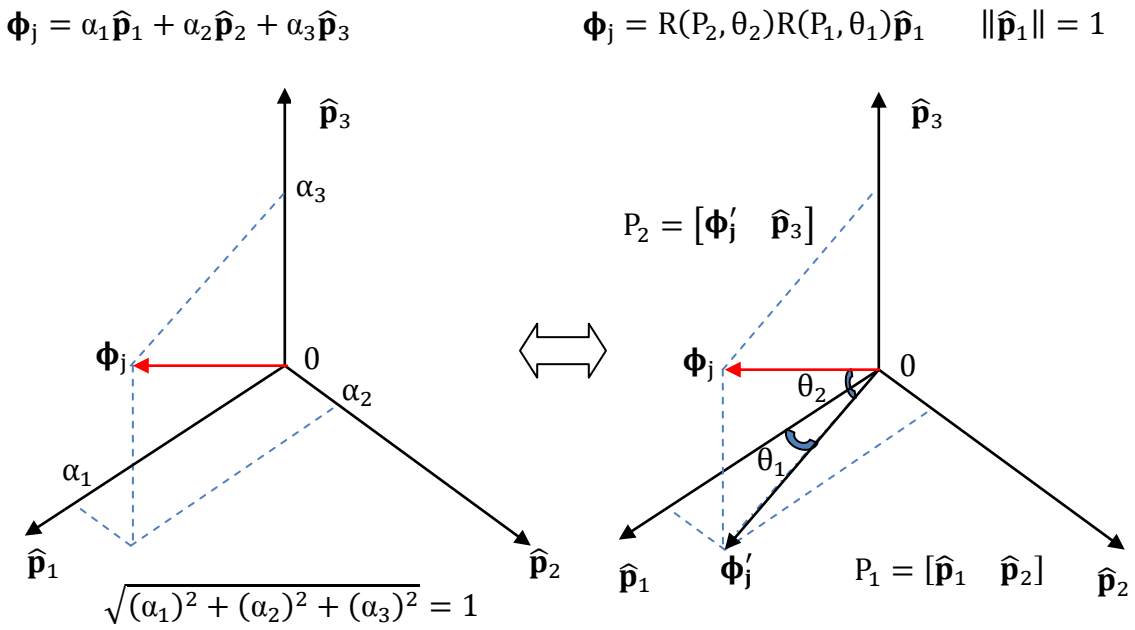


**Figure 4.1.** Equality between Two Parameterizations

Let us consider a 3-dimensional case to explain our concept of the angle representation easily. In order to express each eigenvector, previous parameterization in Eq. (4.12) requires three coordinates with three given orthonormal vectors. Since we need a normalized eigenvector, the summation of coordinates' squares should be one. So we can express each normalized eigenvector by using previous parameterization described in Eq. (4.12) as follows

$$\boldsymbol{\phi}_j = \alpha_1 \hat{\mathbf{p}}_1 + \alpha_2 \hat{\mathbf{p}}_2 + \alpha_3 \hat{\mathbf{p}}_3 \tag{4.41}$$

with

$$\sqrt{(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2} = 1 \tag{4.42}$$

This nonlinear constraint makes the numerical optimization to minimize the measure of robustness more difficult. However, the angle representation of eigenvector can avoid this undesired constraint. By using Eq. (4.34), the eigenvector can be given by

$$\boldsymbol{\phi}_j = R(P_2, \theta_2) R(P_1, \theta_1) \hat{\mathbf{p}}_1 \tag{4.43}$$

where

$$P_1 \equiv [\hat{\mathbf{p}}_1 \quad \hat{\mathbf{p}}_2] \tag{4.44a}$$

$$P_2 \equiv [\boldsymbol{\phi}'_j \quad \hat{\mathbf{p}}_3], \quad \boldsymbol{\phi}'_j = R(P_1, \theta_1) \hat{\mathbf{p}}_1 \tag{4.44b}$$

So it needs only two parameters(two angles) for a 3-dimensional real space case. Figure (4.1) explains this procedure well. If we consider an n-dimensional real space case, we need (n-1) angle parameters. On the other hand, the previous parameterization needs n parameters with one length constraint.

As shown in Eq. (4.44b), the plane of rotation $P_2$ has a vector $\boldsymbol{\phi}'_j$ which depends on the first rotation matrix $R(P_1, \theta_1)$. This means that the plane of rotation can be

variable, which complicates the parameterization. In order to make the plane of rotation invariant and simple, we want to find an alternative way to define the plane of rotation. In the following section, we will explore this idea.

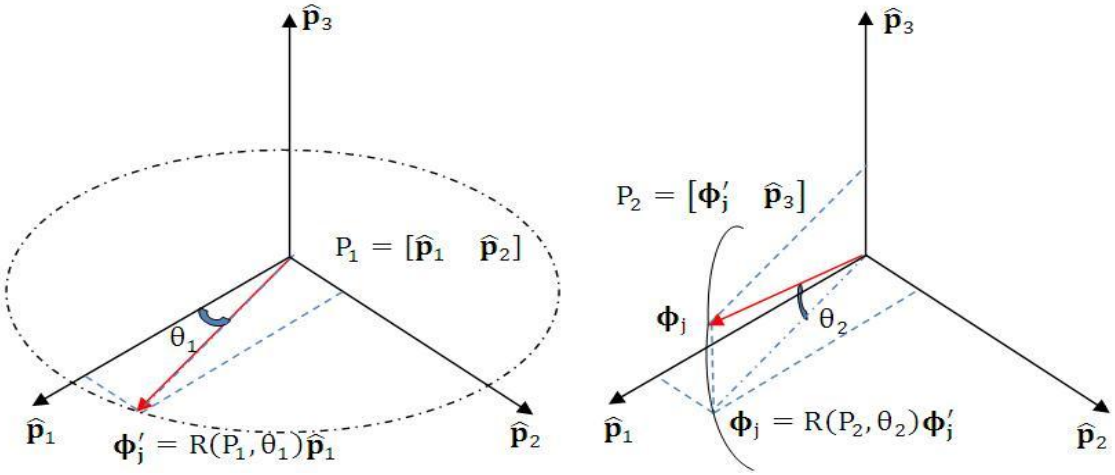


**Figure 4.2.** Rotation with the First Planes of Rotation

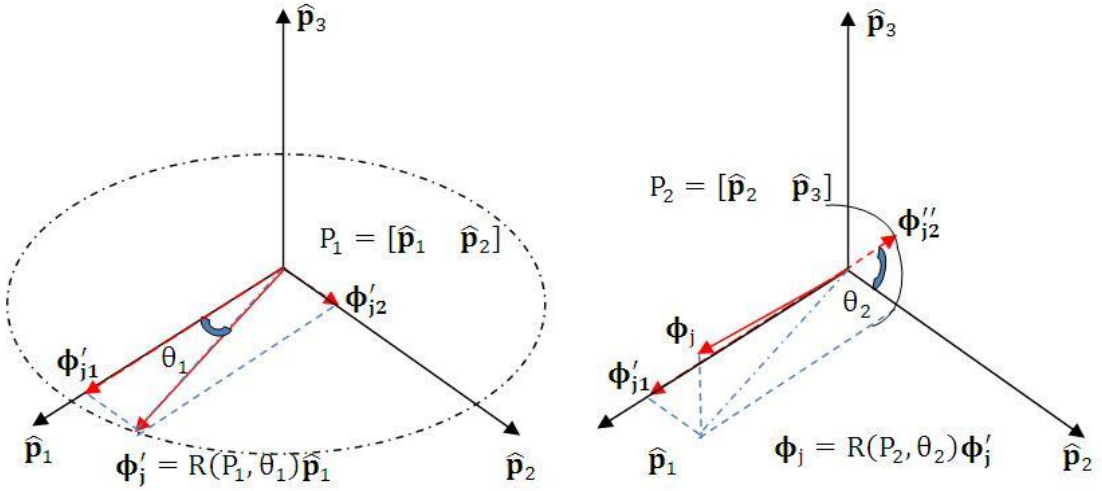


**Figure 4.3.** Rotation with the Alternative Planes of Rotation

**How to Choose a Plane of Rotation?**

From the orthonormal vector set which is a spanning set for the admissible space of eigenvector given in Eq. (4.10), we construct the plane of rotation by choosing any two of them. If the dimension of matrix B in Eq. (4.1) is m, we need (m-1) planes of rotation and (m-1) angles to represent a typical vector in the whole admissible space from Corollary 4.2. For example, let us consider a 3-dimensional space in Fig. (4.2). In this case, we have the eigenvector expression defined in Eq. (4.39). The second plane of rotation in an eigenvector expression includes the first rotation vector( $\boldsymbol{\phi}_j'$ ), which complicates the parameterization. Thus, we want a more simple expression, which leads us to consider the following corollary.

*Corollary 4.3*

From Theorem 4.3 and Corollary 4.2, we can have the alternative planes of rotation as follows.

$$P_1 = [\hat{\mathbf{p}}_1 \quad \hat{\mathbf{p}}_2] \equiv [\mathbf{c}_1 \quad \mathbf{c}_2] \tag{4.45a}$$

$$P_2 = [\hat{\mathbf{p}}_2 \quad \hat{\mathbf{p}}_3] \equiv [\mathbf{c}_2 \quad \mathbf{c}_3] \tag{4.45b}$$

$$\vdots$$

$$P_k = [\hat{\mathbf{p}}_k \quad \hat{\mathbf{p}}_{k+1}] \equiv [\mathbf{c}_k \quad \mathbf{c}_{k+1}] \tag{4.45c}$$

Then, a typical vector in Col C can still be written by Eq. (4.40).

Proof

First rotation is the same with that of Theorem 4.3. Thus, this rotation expression can be given by Eq. (4.36). The vector $\mathbf{v}$ can be an arbitrary vector in the whole space

spanned by $\mathbf{c}_1$ and $\mathbf{c}_2$. Now, we can choose the second plane of rotation as Eq. (4.45b). And we can project the vector onto $\hat{\mathbf{p}}_1$ axis and $\hat{\mathbf{p}}_2$ axis, which gives us two vectors as

$$\boldsymbol{\phi}'_{\mathbf{j1}} = \mathrm{Proj}_{\hat{\mathbf{p}}_1}\boldsymbol{\phi}'_{\mathbf{j}} \tag{4.46a}$$

$$\boldsymbol{\phi}'_{\mathbf{j2}} = \mathrm{Proj}_{\hat{\mathbf{p}}_2}\boldsymbol{\phi}'_{\mathbf{j}} \tag{4.46b}$$

where $\mathrm{Proj}_{\mathbf{a}}\mathbf{b}$ is the projection of $\mathbf{b}$ onto $\mathbf{a}$ and is given by

$$\mathrm{Proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} \tag{4.47}$$

The rotation angle $\theta_2$ is given with respect to $\mathbf{v}$. The rotation matrix $\mathrm{R}(\mathrm{P}_2, \theta_2)$ can be given by Eq. (4.21). Thus, this second rotation expression can be described by

$$\mathbf{w} = \mathrm{R}(\mathrm{P}_2, \theta_2)\boldsymbol{\phi}'_{\mathbf{j}} = \prod_{\mathrm{i}=1}^{2} \mathrm{R}(\mathrm{P}_\mathrm{i}, \theta_\mathrm{i})\mathbf{c}_1 \tag{4.48}$$

Geometrically, the second rotation can be given by

$$\mathbf{w} = \boldsymbol{\phi}'_{\mathbf{j1}} + \mathrm{R}(\mathrm{P}_2, \theta_2)\boldsymbol{\phi}'_{\mathbf{j2}} \tag{4.49}$$

This means that $\mathbf{w}$ can be a typical vector in the whole space spanned by $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ by putting a scaling real number $\gamma$ in the both sides of Eq. (4.49). Equation (4.48) is mathematically the same with Eq. (4.49). For a 3-dimensional real space case, Fig. (4.3) explains this procedure well. From these observations, we can extend this procedure to the (n-1) step. Therefore, a typical vector in Col C can be written by Eq. (4.34) with the following planes of rotation.

$$\mathrm{P}_1 = [\hat{\mathbf{p}}_1 \quad \hat{\mathbf{p}}_2]\,,\ \mathrm{P}_2 = [\hat{\mathbf{p}}_2 \quad \hat{\mathbf{p}}_3],\ \dots\ ,\ \mathrm{P}_{\mathrm{m}-1} = [\hat{\mathbf{p}}_{\mathrm{m}-1} \quad \hat{\mathbf{p}}_\mathrm{m}] \tag{4.50}$$

Each plane of rotation has the following rotation matrix.

$$\mathrm{R}(\mathrm{P}_\mathrm{i}, \theta_\mathrm{i}) = \mathrm{I}_\mathrm{n} + (\cos\theta_\mathrm{i} - 1)\mathrm{P}_\mathrm{i}\mathrm{P}_\mathrm{i}^\mathrm{T} + \mathrm{P}_\mathrm{i}\mathrm{J}_2\mathrm{P}_\mathrm{i}^\mathrm{T}\sin\theta_\mathrm{i}\,, \quad \mathrm{i} = 1, \dots, \mathrm{m}-1 \tag{4.51}$$

Finally, we have the angle representation as

$$\boldsymbol{\phi}_j = \prod_{k=1}^{m-1} R(P_k, \theta_k)\, \hat{\mathbf{p}}_1 \tag{4.52}$$

With this angle parameterization, eigenvector matrix $\Phi$ can be described by

$$\Phi = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_n] \tag{4.53}$$

where

$$\boldsymbol{\phi}_j = \prod_{k=1}^{m-1} R\left(P_{j,k}, \theta_{j,k}\right) \hat{\mathbf{p}}_{j,1} \tag{4.54}$$

Now, $\Phi$ can be parameterized with $\theta_i, i = 1, \ldots, n \times (m-1)$ as

$$\Phi = \Phi\left(\theta_{1,1}, \theta_{1,2}, \cdots, \theta_{n,(m-1)}\right) \tag{4.55}$$

From these parameters, our cost function can be expressed in terms of angles and we will find the angel set which minimizes the condition number.

$$\min_{\theta} \kappa_F \left(\Phi = \Phi\left(\theta_{1,1}, \theta_{1,2}, \cdots, \theta_{n,(m-1)}\right)\right) \tag{4.56}$$

**Angle Representation in the Complex Space**

In the real space, we can easily set up the angle parameterization, but we need to modify it for the complex space.

**Approach I**

From Reference [5], a rotation in n-dimensional complex spaces is given by

$$R(P, \theta) = I_n + (\cos\theta - 1)PP^H + PJ_2P^H \sin\theta \tag{4.57}$$

The difference between Eq. (4.21) and Eq. (4.57) is the change of the transpose of matrix P to the complex conjugate. This angle of rotation should be real. However, when we

choose two arbitrary complex vectors in complex Euclidean space, there can also be the imaginary part of angles between both of them. This means that the inner product of two arbitrary complex vectors can be complex. Therefore, a vector cannot reach the whole admissible space by using the above rotation equation. However, if we want to restrict a rotation angle as the real angle, this rotation transformation matrix can work. Also, we use the same procedure with the real space case, which simplifies the implementation of the numerical simulation. If we want to consider no restrictions on the complex values of angles between two complex vectors, we need another approach. We propose the second approach in the next section.

*Theorem 4.4*

Let us consider an n-dimensional complex Euclidean space and its complex conjugate space. Then we have two $\mathrm{n} \times \mathrm{m}(\mathrm{n} > m)$ matrice as

$$\mathrm{C} = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_\mathrm{m}] \quad \text{and} \quad \mathrm{C}^* = [\mathbf{c}_1^*, \mathbf{c}_2^*, \dots, \mathbf{c}_\mathrm{m}^*] \tag{4.58}$$

where each $\mathbf{c}_\mathrm{i}$ and $\mathbf{c}_\mathrm{i}^*, \mathrm{i} = 1, \dots, \mathrm{m}$ are orthonormal vectors. Each column space can be described by

$$\mathrm{Col}\ \mathrm{C} = \mathrm{Span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_\mathrm{m}\} \quad \text{and} \quad \mathrm{Col}\ \mathrm{C}^* = \mathrm{Span}\{\mathbf{c}_1^*, \mathbf{c}_2^*, \dots, \mathbf{c}_\mathrm{m}^*\} \tag{4.59}$$

Then typical vectors in Col C and $\mathrm{Col}\ \mathrm{C}^*$ can be written by

$$\mathbf{b} = \gamma \prod_{\mathrm{k}=1}^{\mathrm{m}-1} \mathrm{R}(\mathrm{P}_\mathrm{k}, \theta_\mathrm{k})\, \mathbf{c}_1 \quad \text{and} \quad \mathbf{r} = \gamma^* \prod_{\mathrm{k}=1}^{\mathrm{m}-1} \mathrm{R}(\mathrm{P}_\mathrm{k}^*, \psi_\mathrm{k})\, \mathbf{c}_1^* \tag{4.60}$$

where the complex rotation R can be performed by using Eq. (4.57), and the planes of rotation are given as follows.

$$P_{1,1} = [\hat{\mathbf{p}}_{1,1} \quad \hat{\mathbf{p}}_{1,2}] \equiv [\mathbf{c}_1 \quad \mathbf{c}_2] \tag{4.61a}$$

$$P_{1,2} = [\hat{\mathbf{p}}_{1,2} \quad \hat{\mathbf{p}}_{1,3}] \equiv [\mathbf{c}_2 \quad \mathbf{c}_3] \tag{4.61b}$$

$$\vdots$$

$$P_{1,k} = [\hat{\mathbf{p}}_{1,k} \quad \hat{\mathbf{p}}_{1,k+1}] \equiv [\mathbf{c}_k \quad \mathbf{c}_{k+1}] \tag{4.61c}$$

If vector $\mathbf{r}$ wants to be the complex conjugate of a vector $\mathbf{b}$, the rotation angle relationship should be the same.

$$\psi_k = \theta_k, \quad k = 1, \dots, m-1 \tag{4.62}$$

Proof

By taking the complex conjugate of $\mathbf{b}$ in Eq. (4.60), $\mathbf{b}^*$ should be $\mathbf{r}$ in Eq. (4.60).

$$\mathbf{b}^* = \gamma^* \prod_{k=1}^{m-1} R(P_k, \theta_k)^* \, \mathbf{c}_1^* = \gamma^* \prod_{k=1}^{m-1} R(P_k^*, \psi_k) \, \mathbf{c}_1^* \tag{4.63}$$

By examining $R(P_k, \theta_k)^*$, we have

$$\begin{aligned} R(P_k, \theta_k)^* &= \left(I_n + (\cos\theta - 1)P_k P_k^H + P_k J_2 P_k^H \sin\theta\right)^* \\ &= I_n + (\cos\theta - 1)P_k^* (P_k^*)^H + P_k^* J_2 (P_k^*)^H \sin\theta \\ &= R(P_k^*, \theta_k) \end{aligned} \tag{4.64}$$

Substitution Eq. (4.64) into Eq. (4.63) yields

$$R(P_k^*, \theta_k) = R(P_k^*, \psi_k) \tag{4.65}$$

Finally, we have Eq. (4.62).

To have the real gain matrix F in Eq. (4.2), the corresponding eigenvectors($\boldsymbol{\phi}_j$ and $\boldsymbol{\phi}_j^*$) which have the complex conjugate pair eigenvalues($\boldsymbol{\lambda}_j$ and $\boldsymbol{\lambda}_j^*$) should also be the complex conjugate pair. Theorem 4.4 tells how to choose the plane of rotation and

the rotation angle for the conjugate pair, which results in reducing the number of angle parameters.

**Approach II**

Here, we transform the complex space into the real space. Then we apply the rotation method developed previously to the transformed real space. Finally, we transform them back to the original complex space.

*Theorem 4.5*

Let us consider an n-dimensional complex Euclidean space, $\mathcal{S} \in \mathbb{C}^{n\times 1}$ spanned by a normalized complex vector $\mathbf{q}$. Then a typical vector in $\mathcal{S}$ can be described by

$$\mathbf{s} = \mathrm{p} \times \mathbf{q} \tag{4.66}$$

where $\mathrm{p} \in \mathbb{C}^{1\times 1}$ is the complex coordinate and $\mathbf{q} \in \mathbb{C}^{n\times 1}$ is the complex vector. Assume that p and $\mathbf{q}$ are given by

$$\mathrm{p} = \mathrm{a} + \mathrm{bj}, \qquad \mathrm{a}, \mathrm{b} \in \mathbb{R}^{1\times 1} \tag{4.67a}$$

$$\mathbf{q} = \mathbf{x} + \mathbf{y}\mathrm{j}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^{n\times 1} \tag{4.67b}$$

Then we can express it by real numbers as

$$\begin{cases} \text{Real part of vector } \mathbf{s} \rightarrow \\ \text{Imaginary part of vector } \mathbf{s} \rightarrow \end{cases} \quad \mathrm{a}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} + \mathrm{b}\begin{bmatrix}-\mathbf{y}\\ \mathbf{x}\end{bmatrix} = \begin{bmatrix}\mathbf{x} & -\mathbf{y}\\ \mathbf{y} & \mathbf{x}\end{bmatrix}\begin{bmatrix}\mathrm{a}\\ \mathrm{b}\end{bmatrix} \tag{4.68}$$

Proof

By using the assumption, a vector $\mathbf{s}$ can be given by

$$\mathbf{s} = (\mathrm{a} + \mathrm{bj})(\mathbf{x} + \mathbf{y}\mathrm{j}) = (\mathrm{a}\mathbf{x} - \mathrm{b}\mathbf{y}) + (\mathrm{a}\mathbf{y} + \mathrm{b}\mathbf{x})\mathrm{j} \tag{4.69}$$

This implies that $(a\mathbf{x} - b\mathbf{y})$ can express any real part of $\mathbf{s}$ and $(a\mathbf{y} + b\mathbf{x})$ can describe any imaginary part of $\mathbf{s}$. Therefore, if we have the form given in Eq. (4.68), it can express the same vector in Eq. (4.69).

As a result, if we want to have a complex space spanned by an n-dimensional complex vector, $2n \times 2$ real vectors in real space are needed from Theorem 4.5.

$$\mathbf{q} \in \mathbb{C}^{n\times 1} = (\mathbf{x} + \mathbf{y}j) \leftrightarrow \mathbf{q} \in \mathbb{R}^{2n\times 2} = \left\{\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}, \begin{bmatrix}-\mathbf{y}\\ \mathbf{x}\end{bmatrix}\right\} \tag{4.70}$$

*Proposition 4.3*

An $n \times n$ unitary complex matrix A transforms into $2n \times 2n$ orthogonal matrix B by Theorem 4.5.

Proof

Let a matrix A be given by

$$A = X + Yj, \quad X, Y \in \mathbb{R}^{n\times n} \tag{4.71}$$

Since the matrix A is unitary, we have

$$A^H A = I \tag{4.72}$$

By substituting Eq. (4.71) into Eq. (4.72), we have

$$\begin{aligned}(X + Yj)^H (X + Yj) &= (X - Yj)^T (X + Yj)\\ &= X^T X + Y^T Y + (-Y^T X + X^T Y)j = I\end{aligned} \tag{4.73}$$

From Eq. (4.72), we have

$$X^T X + Y^T Y = I \quad \text{and} \quad -Y^T X + X^T Y = 0 \tag{4.74}$$

By using Theorem 4.5, the matrix B can be given by

$$B = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \tag{4.75}$$

To show that the matrix B is orthogonal, we obtain $B^T B$ as

$$\begin{aligned} B^T B &= \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}^T \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} = \begin{bmatrix} X^T & Y^T \\ -Y^T & X^T \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \\ &= \begin{bmatrix} X^T X + Y^T Y & -X^T Y + Y^T X \\ -Y^T X + X^T Y & X^T X + Y^T Y \end{bmatrix} \end{aligned} \tag{4.76}$$

By using Eq. (4.72), we have $B^T B = I$.

From this proposition, the orthogonal matrix which represents the rotation axes becomes the orthogonal matrix after the transformation.

*Theorem 4.6*

Let us consider an n-dimensional complex Euclidean spaces and its complex conjugate space. Then, we have two $n \times m (n > m)$ matrice as

$$C = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m] \quad \text{and} \quad C^* = [\mathbf{c}_1^*, \mathbf{c}_2^*, \dots, \mathbf{c}_m^*] \tag{4.77}$$

where each $\mathbf{c}_i$ and $\mathbf{c}_i^*, i = 1, \dots, m$ are orthonormal vectors. Let the matrix C be given by

$$C = C_R + C_I j, \quad C_R, C_I \in \mathbb{R}^{n \times n} \tag{4.78}$$

By Proposition 4.3, matrix C and matrix $C^*$ can be transformed into $C_t$ and $C_t^*$, respectively.

$$C_t = \begin{bmatrix} C_R & -C_I \\ C_I & C_R \end{bmatrix} \quad \text{and} \quad C_t^* = \begin{bmatrix} C_R & C_I \\ -C_I & C_R \end{bmatrix} \tag{4.79}$$

Each column space can be described by

$$\text{Col } C_t = \text{Span}\left\{ \begin{bmatrix} \text{Re}\{\mathbf{c}_1\} \\ \text{Im}\{\mathbf{c}_1\} \end{bmatrix}, \begin{bmatrix} -\text{Im}\{\mathbf{c}_1\} \\ \text{Re}\{\mathbf{c}_1\} \end{bmatrix}, \dots, \begin{bmatrix} \text{Re}\{\mathbf{c}_m\} \\ \text{Im}\{\mathbf{c}_m\} \end{bmatrix}, \begin{bmatrix} -\text{Im}\{\mathbf{c}_m\} \\ \text{Re}\{\mathbf{c}_m\} \end{bmatrix} \right\} \tag{4.80a}$$

$$\text{Col } C_t^* = \text{Span}\left\{\begin{bmatrix}\text{Re}\{\mathbf{c}_1\}\\ -\text{Im}\{\mathbf{c}_1\}\end{bmatrix}, \begin{bmatrix}\text{Im}\{\mathbf{c}_1\}\\ \text{Re}\{\mathbf{c}_1\}\end{bmatrix}, \dots, \begin{bmatrix}\text{Re}\{\mathbf{c}_m\}\\ -\text{Im}\{\mathbf{c}_m\}\end{bmatrix}, \begin{bmatrix}\text{Im}\{\mathbf{c}_m\}\\ \text{Re}\{\mathbf{c}_m\}\end{bmatrix}\right\} \tag{4.80b}$$

Then typical vectors in Col $C_t$ and Col $C_t^*$ can be written by

$$\mathbf{b} = \gamma \prod_{k=1}^{2m-1} R(P_{1,k}, \theta_k) \begin{bmatrix}\text{Re}\{\mathbf{c}_1\}\\ \text{Im}\{\mathbf{c}_1\}\end{bmatrix} \quad \text{and} \quad \mathbf{r} = \gamma \prod_{k=1}^{2m-1} R(P_{2,k}, \psi_k) \begin{bmatrix}\text{Re}\{\mathbf{c}_1\}\\ -\text{Im}\{\mathbf{c}_1\}\end{bmatrix} \tag{4.81}$$

where the complex rotation R can be performed by using Eq. (4.57) and the planes of rotation can be defined as follows.

$$P_{1,1} = [\hat{\mathbf{p}}_{1,1} \quad \hat{\mathbf{p}}_{1,2}] \equiv \begin{bmatrix}\text{Re}\{\mathbf{c}_1\} & -\text{Im}\{\mathbf{c}_1\}\\ \text{Im}\{\mathbf{c}_1\} & \text{Re}\{\mathbf{c}_1\}\end{bmatrix} \tag{4.82a}$$

$$P_{1,2} = [\hat{\mathbf{p}}_{1,2} \quad \hat{\mathbf{p}}_{1,3}] \equiv \begin{bmatrix}-\text{Im}\{\mathbf{c}_1\} & \text{Re}\{\mathbf{c}_2\}\\ \text{Re}\{\mathbf{c}_1\} & \text{Im}\{\mathbf{c}_2\}\end{bmatrix} \tag{4.82b}$$

$$\vdots$$

$$P_{1,2m-1} = [\hat{\mathbf{p}}_{1,2m-1} \quad \hat{\mathbf{p}}_{1,2m-1}] \equiv \begin{bmatrix}\text{Re}\{\mathbf{c}_m\} & -\text{Im}\{\mathbf{c}_m\}\\ \text{Im}\{\mathbf{c}_m\} & \text{Re}\{\mathbf{c}_m\}\end{bmatrix} \tag{4.82c}$$

And,

$$P_{2,1} = [\hat{\mathbf{p}}_{2,1} \quad \hat{\mathbf{p}}_{2,2}] \equiv \begin{bmatrix}\text{Re}\{\mathbf{c}_1\} & \text{Im}\{\mathbf{c}_1\}\\ -\text{Im}\{\mathbf{c}_1\} & \text{Re}\{\mathbf{c}_1\}\end{bmatrix} \tag{4.83a}$$

$$P_{2,2} = [\hat{\mathbf{p}}_{2,2} \quad \hat{\mathbf{p}}_{2,3}] \equiv \begin{bmatrix}\text{Im}\{\mathbf{c}_1\} & \text{Re}\{\mathbf{c}_2\}\\ \text{Re}\{\mathbf{c}_1\} & -\text{Im}\{\mathbf{c}_2\}\end{bmatrix} \tag{4.83b}$$

$$\vdots$$

$$P_{2,2m-1} = [\hat{\mathbf{p}}_{2,2m-1} \quad \hat{\mathbf{p}}_{2,2m-1}] \equiv \begin{bmatrix}\text{Re}\{\mathbf{c}_m\} & \text{Im}\{\mathbf{c}_m\}\\ -\text{Im}\{\mathbf{c}_m\} & \text{Re}\{\mathbf{c}_m\}\end{bmatrix} \tag{4.83c}$$

If a vector $\mathbf{r}$ wants to be the complex conjugate of a vector $\mathbf{b}$, the rotation angle relationship should be the same.

$$\psi_k = -\theta_k, \quad k = 1, \dots, 2m-1 \tag{4.84}$$

Proof

From the plane of rotation given in Eq. (4.82a) and (4.82b), the real space expression of the plane of rotation can be divided by

1) Case I: $k = 2l - 1, l = 1, \dots, m$

$$P_{1,k} = \begin{bmatrix} \mathrm{Re}\{\mathbf{c}_k\} & -\mathrm{Im}\{\mathbf{c}_k\} \\ \mathrm{Im}\{\mathbf{c}_k\} & \mathrm{Re}\{\mathbf{c}_k\} \end{bmatrix} \in \mathbb{R}^{2n \times 4} \tag{4.85}$$

2) Case II: $k = 2l, l = 1, \dots, m - 1$

$$P_{1,k} = \begin{bmatrix} -\mathrm{Im}\{\mathbf{c}_k\} & \mathrm{Re}\{\mathbf{c}_{k+1}\} \\ \mathrm{Re}\{\mathbf{c}_k\} & \mathrm{Im}\{\mathbf{c}_{k+1}\} \end{bmatrix} \in \mathbb{R}^{2n \times 4} \tag{4.86}$$

For the simplicity, let $\mathbf{x} = \mathrm{Re}\{\mathbf{c}_k\}$, $\mathbf{y} = \mathrm{Im}\{\mathbf{c}_k\}$, $\mathbf{m} = \mathrm{Re}\{\mathbf{c}_{k+1}\}$ and $\mathbf{n} = \mathrm{Im}\{\mathbf{c}_{k+1}\}$.

Now, let us prove the first case in Eq. (4.85). Rotation matrix can be given by Eq. (4.21).

$$R(P_{1,k}, \theta_k) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tag{4.87}$$

where

$$\begin{aligned} R_{11} = R_{22} = I_{n \times n} + (\mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T)(1 - \cos\theta_k) \\ + (\mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T)\sin\theta_k \end{aligned} \tag{4.88a}$$

$$R_{12} = -R_{21} = (\mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T)(1 - \cos\theta_k) + (-\mathbf{x}\mathbf{x}^T - \mathbf{y}\mathbf{y}^T)\sin\theta_k \tag{4.88b}$$

Because

$$P_{1,k}P_{1,k}^T = \begin{bmatrix} \mathbf{x} & -\mathbf{y} \\ \mathbf{y} & \mathbf{x} \end{bmatrix} \begin{bmatrix} \mathbf{x}^T & \mathbf{y}^T \\ -\mathbf{y}^T & \mathbf{x}^T \end{bmatrix} = \begin{bmatrix} \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T & \mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T \\ -\mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T & \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T \end{bmatrix} \tag{4.89a}$$

$$P_{1,k}J_2P_{1,k}^T = \begin{bmatrix} \mathbf{x} & -\mathbf{y} \\ \mathbf{y} & \mathbf{x} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x}^T & \mathbf{y}^T \\ -\mathbf{y}^T & \mathbf{x}^T \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T & -\mathbf{x}\mathbf{x}^T - \mathbf{y}\mathbf{y}^T \\ \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T & \mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T \end{bmatrix} \tag{4.89b}$$

Let the rotation matrix about $P_{2,k}$ be given by Eq. (4.21).

$$R(P_{2,k}, \psi_k) = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \tag{4.90}$$

where

$$O_{11} = O_{22} = I_{n\times n} + (\mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T)(1 - \cos\psi_k) + (-\mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T)\sin\psi_k \tag{4.91a}$$

$$O_{12} = -O_{21} = (-\mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T)(1 - \cos\psi_k) + (-\mathbf{x}\mathbf{x}^T - \mathbf{y}\mathbf{y}^T)\sin\psi_k \tag{4.91b}$$

Because

$$P_{2,k}P_{2,k}^T = \begin{bmatrix} \mathbf{x} & \mathbf{y} \\ -\mathbf{y} & \mathbf{x} \end{bmatrix}\begin{bmatrix} \mathbf{x}^T & -\mathbf{y}^T \\ \mathbf{y}^T & \mathbf{x}^T \end{bmatrix} = \begin{bmatrix} \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T & -\mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T \\ \mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T & \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T \end{bmatrix} \tag{4.92a}$$

$$P_{2,k}J_2P_{2,k}^T = \begin{bmatrix} \mathbf{x} & \mathbf{y} \\ -\mathbf{y} & \mathbf{x} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x}^T & -\mathbf{y}^T \\ \mathbf{y}^T & \mathbf{x}^T \end{bmatrix}$$
$$= \begin{bmatrix} -\mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T & -\mathbf{x}\mathbf{x}^T - \mathbf{y}\mathbf{y}^T \\ \mathbf{x}\mathbf{x}^T + \mathbf{y}\mathbf{y}^T & -\mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T \end{bmatrix} \tag{4.92b}$$

From Eq. (4.81), we denote them as

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_R \\ \mathbf{b}_I \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} \mathbf{r}_R \\ \mathbf{r}_I \end{bmatrix} \tag{4.93}$$

Now, we want to have the vector **r** is the complex conjugate of the vector **b** so that we have

$$\mathbf{b}_R = \mathbf{r}_R \quad \text{and} \quad \mathbf{b}_I = -\mathbf{r}_I \tag{4.94}$$

For k=1, with Eq. (4.81) and Eq. (4.87), we have

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_R \\ \mathbf{b}_I \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \mathrm{Re}\{\mathbf{c}_1\} \\ \mathrm{Im}\{\mathbf{c}_1\} \end{bmatrix} = \begin{bmatrix} R_{11}\mathrm{Re}\{\mathbf{c}_1\} + R_{12}\mathrm{Im}\{\mathbf{c}_1\} \\ R_{21}\mathrm{Re}\{\mathbf{c}_1\} + R_{22}\mathrm{Im}\{\mathbf{c}_1\} \end{bmatrix} \tag{4.95a}$$

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_R \\ \mathbf{r}_I \end{bmatrix} = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} \mathrm{Re}\{\mathbf{c}_1\} \\ -\mathrm{Im}\{\mathbf{c}_1\} \end{bmatrix} = \begin{bmatrix} O_{11}\mathrm{Re}\{\mathbf{c}_1\} - O_{12}\mathrm{Im}\{\mathbf{c}_1\} \\ O_{21}\mathrm{Re}\{\mathbf{c}_1\} - O_{22}\mathrm{Im}\{\mathbf{c}_1\} \end{bmatrix} \tag{4.95b}$$

By using Eq. (4.94),

$$R_{11}\mathrm{Re}\{\mathbf{c}_1\} + R_{12}\mathrm{Im}\{\mathbf{c}_1\} = O_{11}\mathrm{Re}\{\mathbf{c}_1\} - O_{12}\mathrm{Im}\{\mathbf{c}_1\} \tag{4.96a}$$

$$R_{21}\mathrm{Re}\{\mathbf{c}_1\} + R_{22}\mathrm{Im}\{\mathbf{c}_1\} = -O_{21}\mathrm{Re}\{\mathbf{c}_1\} + O_{22}\mathrm{Im}\{\mathbf{c}_1\} \tag{4.96b}$$

Substitution of Eq. (4.88) and Eq. (4.91) into Eq. (4.96) gives us

$$\therefore \sin\theta_1 = -\sin\psi_1 \text{ and } \cos\theta_1 = \cos\psi_1 \tag{4.97}$$

Therefore, we have the following angle relationship.

$$\theta_1 = -\psi_1 \tag{4.98}$$

Now, let us prove the second case in Eq. (4.86). Rotation matrix can be given by Eq. (4.21).

$$R(P_{1,k}, \theta_k) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tag{4.99}$$

where

$$R_{11} = I_{n\times n} + (\mathbf{m}\mathbf{m}^T + \mathbf{y}\mathbf{y}^T)(1 - \cos\theta_k) + (-\mathbf{m}\mathbf{y}^T + \mathbf{y}\mathbf{m}^T)\sin\theta_k \tag{4.100a}$$

$$R_{22} = I_{n\times n} + (\mathbf{x}\mathbf{x}^T + \mathbf{n}\mathbf{n}^T)(1 - \cos\theta_k) + (\mathbf{n}\mathbf{x}^T - \mathbf{x}\mathbf{n}^T)\sin\theta_k \tag{4.100b}$$

$$R_{12} = (\mathbf{m}\mathbf{n}^T - \mathbf{y}\mathbf{x}^T)(1 - \cos\theta_k) + (\mathbf{m}\mathbf{x}^T + \mathbf{y}\mathbf{n}^T)\sin\theta_k \tag{4.100c}$$

$$R_{21} = (\mathbf{n}\mathbf{m}^T - \mathbf{x}\mathbf{y}^T)(1 - \cos\theta_k) + (-\mathbf{n}\mathbf{y}^T - \mathbf{x}\mathbf{m}^T)\sin\theta_k \tag{4.100d}$$

Because

$$P_{1,k}P_{1,k}^T = \begin{bmatrix} -\mathbf{y} & \mathbf{m} \\ \mathbf{x} & \mathbf{n} \end{bmatrix} \begin{bmatrix} -\mathbf{y}^T & \mathbf{x}^T \\ \mathbf{m}^T & \mathbf{n}^T \end{bmatrix} = \begin{bmatrix} \mathbf{m}\mathbf{m}^T + \mathbf{y}\mathbf{y}^T & \mathbf{m}\mathbf{n}^T - \mathbf{y}\mathbf{x}^T \\ \mathbf{n}\mathbf{m}^T - \mathbf{x}\mathbf{y}^T & \mathbf{x}\mathbf{x}^T + \mathbf{n}\mathbf{n}^T \end{bmatrix} \tag{4.101a}$$

$$\begin{aligned} P_{1,k}J_2P_{1,k}^T &= \begin{bmatrix} -\mathbf{y} & \mathbf{m} \\ \mathbf{x} & \mathbf{n} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\mathbf{y}^T & \mathbf{x}^T \\ \mathbf{m}^T & \mathbf{n}^T \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{m}\mathbf{y}^T + \mathbf{y}\mathbf{m}^T & \mathbf{m}\mathbf{x}^T + \mathbf{y}\mathbf{n}^T \\ -\mathbf{n}\mathbf{y}^T - \mathbf{x}\mathbf{m}^T & \mathbf{n}\mathbf{x}^T - \mathbf{x}\mathbf{n}^T \end{bmatrix} \end{aligned} \tag{4.101b}$$

Let the rotation matrix about $P_{2,k}$ be given by Eq. (4.21).

$$R(P_{2,k}, \psi_k) = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \tag{4.102}$$

where

$$O_{11} = I_{n\times n} + (\mathbf{m}\mathbf{m}^T + \mathbf{y}\mathbf{y}^T)(1 - \cos\psi_k) + (\mathbf{m}\mathbf{y}^T - \mathbf{y}\mathbf{m}^T)\sin\psi_k \tag{4.103a}$$

$$O_{22} = I_{n\times n} + (\mathbf{x}\mathbf{x}^T + \mathbf{n}\mathbf{n}^T)(1 - \cos\psi_k) + (-\mathbf{n}\mathbf{x}^T + \mathbf{x}\mathbf{n}^T)\sin\psi_k \tag{4.103b}$$

$$O_{12} = (-\mathbf{m}\mathbf{n}^T + \mathbf{y}\mathbf{x}^T)(1 - \cos\psi_k) + (\mathbf{m}\mathbf{x}^T + \mathbf{y}\mathbf{n}^T)\sin\psi_k \tag{4.103c}$$

$$O_{21} = (\mathbf{x}\mathbf{y}^T - \mathbf{n}\mathbf{m}^T)(1 - \cos\psi_k) + (-\mathbf{n}\mathbf{y}^T - \mathbf{x}\mathbf{m}^T)\sin\psi_k \tag{4.103d}$$

Because

$$P_{2,k}P_{2,k}^T = \begin{bmatrix} \mathbf{y} & \mathbf{m} \\ \mathbf{x} & -\mathbf{n} \end{bmatrix} \begin{bmatrix} \mathbf{y}^T & \mathbf{x}^T \\ \mathbf{m}^T & -\mathbf{n}^T \end{bmatrix} = \begin{bmatrix} \mathbf{m}\mathbf{m}^T + \mathbf{y}\mathbf{y}^T & -\mathbf{m}\mathbf{n}^T + \mathbf{y}\mathbf{x}^T \\ \mathbf{x}\mathbf{y}^T - \mathbf{n}\mathbf{m}^T & \mathbf{x}\mathbf{x}^T + \mathbf{n}\mathbf{n}^T \end{bmatrix} \tag{4.104a}$$

$$\begin{aligned} P_{2,k}J_2P_{2,k}^T &= \begin{bmatrix} \mathbf{y} & \mathbf{m} \\ \mathbf{x} & -\mathbf{n} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}^T & \mathbf{x}^T \\ \mathbf{m}^T & -\mathbf{n}^T \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{m}\mathbf{y}^T - \mathbf{y}\mathbf{m}^T & \mathbf{m}\mathbf{x}^T + \mathbf{y}\mathbf{n}^T \\ -\mathbf{n}\mathbf{y}^T - \mathbf{x}\mathbf{m}^T & -\mathbf{n}\mathbf{x}^T + \mathbf{x}\mathbf{n}^T \end{bmatrix} \end{aligned} \tag{4.104b}$$

For k=2, with Eq. (4.93), Eq. (4.94), Eq. (4.81) and Eq. (4.87), we have

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_R \\ \mathbf{b}_I \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \mathrm{Re}\{\mathbf{x}\} \\ \mathrm{Im}\{\mathbf{x}\} \end{bmatrix} = \begin{bmatrix} R_{11}\mathrm{Re}\{\mathbf{x}\} + R_{12}\mathrm{Im}\{\mathbf{x}\} \\ R_{21}\mathrm{Re}\{\mathbf{x}\} + R_{22}\mathrm{Im}\{\mathbf{x}\} \end{bmatrix} \tag{4.105a}$$

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_R \\ \mathbf{r}_I \end{bmatrix} = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \begin{bmatrix} \mathrm{Re}\{\mathbf{x}\} \\ -\mathrm{Im}\{\mathbf{x}\} \end{bmatrix} = \begin{bmatrix} O_{11}\mathrm{Re}\{\mathbf{x}\} - O_{12}\mathrm{Im}\{\mathbf{x}\} \\ O_{21}\mathrm{Re}\{\mathbf{x}\} - O_{22}\mathrm{Im}\{\mathbf{x}\} \end{bmatrix} \tag{4.105b}$$

where $\mathbf{x}$ is the vector after the first rotation with $P_{1,1}$ and $\theta_1$. By using Eq. (4.94),

$$R_{11}\mathrm{Re}\{\mathbf{x}\} + R_{12}\mathrm{Im}\{\mathbf{x}\} = O_{11}\mathrm{Re}\{\mathbf{x}\} - O_{12}\mathrm{Im}\{\mathbf{x}\} \tag{4.106a}$$

$$R_{21}\mathrm{Re}\{\mathbf{x}\} + R_{22}\mathrm{Im}\{\mathbf{x}\} = -O_{21}\mathrm{Re}\{\mathbf{x}\} + O_{22}\mathrm{Im}\{\mathbf{x}\} \tag{4.106b}$$

Substitution of Eq. (4.100) and Eq. (4.103) into Eq. (4.105) gives us

$$\therefore \sin\theta_2 = -\sin\psi_2 \text{ and } \cos\theta_2 = \cos\psi_2 \tag{4.107}$$

Therefore, we have the following angle relationship.

$$\theta_2 = -\psi_2 \tag{4.108}$$

By using these angle relationships, we can express the eigenvector complex conjugate pair easily, which results in reducing the number of angle parameters.

Finally, if we want to have the expression in the complex space, we use the following transformation matrix.

$$T_{R2C} = [I_{n\times n} \quad I_{n\times n}j] \tag{4.109}$$

By using this transformation matrix, we can get the complex vector.

$$\boldsymbol{\phi}_j = T_{R2C} \prod_{k=1}^{2m-1} R(P_k, \theta_k)\, \hat{\mathbf{p}}_1 \tag{4.110}$$

With this angle parameterization, eigenvector matrix $\Phi$ can be described by

$$\Phi = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_n] \tag{4.111}$$

where

$$\boldsymbol{\phi}_j = T_{R2C} \prod_{k=1}^{2m-1} R\left(P_{j,k}, \theta_{j,k}\right) \hat{\mathbf{p}}_{j,1} \tag{4.112}$$

Now, $\Phi$ can be parameterized with $\theta_i, i = 1, \ldots, n \times (2m-1)$ as

$$\Phi = \Phi\left(\theta_{1,1}, \theta_{1,2}, \cdots, \theta_{n,(2m-1)}\right) \tag{4.113}$$

**Properties of the Optimized Condition Number**

Minimization of the condition number has the following good properties which also minimize the upper bounds of the gain matrix F and the transient response of the fractional order system.

*Theorem 4.7*[6]

The gain matrix F obtained in Eq. (4.7) satisfy the inequalities

$$\|F\|_2 \leq \left( \|A\|_2 + \max_j \{|\lambda_j|\} \kappa_2(\Phi) \right) / \sigma_{min}\{B\} \tag{4.114}$$

where $\sigma_{min}(B)$ is the smallest singular value of B.

Proof

It can easily be shown by taking the norms on Eq. (4.7).

*Theorem 4.8*

The norm of the transient response $\mathbf{x}(t)$ of the closed-loop continuous fractional order system has the following inequality equation.

$$\|\mathbf{x}(t)\|_2 \leq \kappa_2(\Phi) \frac{C_{max}}{1 + \sigma t^{\alpha}} \|\mathbf{x_0}\|_2 \tag{4.115}$$

where $\mathbf{x}(0) = \mathbf{x_0}$ and $\mathbf{v} = 0$.

Proof

The transient response $\mathbf{x}(t)$ of the fractional order system is given by

$$\mathbf{x}(t) = E_{\alpha}\left((A + BF)t^{\alpha}\right)\mathbf{x_0} = \Phi E_{\alpha}(\Lambda t^{\alpha})\Phi^{-1}\mathbf{x_0} \tag{4.116}$$

By taking the norms on the above equation and using Corollary 3.2, we can obtain the following inequality condition.

$$\|\mathbf{x}(t)\|_2 \leq \|\Phi\|\|\Phi^{-1}\|\|E_\alpha(\Lambda t^\alpha)\|\|\mathbf{x_0}\|_2 \leq \kappa_2(\Phi)\frac{C_{max}}{1+\sigma t^\alpha}\|\mathbf{x_0}\|_2 \tag{4.117}$$

where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and $\sigma = \text{Min}[|\lambda_i|]$. And $C_{max} \equiv \text{Max}[C_i]$ where $C_i$ can be given by

$$\left\|E_{\alpha,\beta}(\lambda_i t^\alpha)\right\|_s \leq \frac{C_i}{1+|\lambda_i|t^\alpha}, \quad i = 1, \dots, n \tag{4.118}$$

## C. NUMERICAL SIMULATION

For robustness, we use our cost function as the weighted condition number mentioned in Eq. (3.38). For the convenience of calculating the derivative of a cost function, we use the squared form of the weighted robust stability measure.

$$J = \nu_3(D)^2 = \left\|D\Phi^{-1}\right\|_F^2 / \|D\|_F^2 \tag{4.119}$$

where $\Phi$ is the eigenvector matrix and $D = \text{diag}\{d_1, d_2, \dots, d_n\}$. $d_j = \delta^{-1}$ where $\delta$ can be obtained by using Eq. (3.45) for the fractional order system. To minimize the above cost function numerically, we need the gradient and the Hessian with respect to angles. Let $Y = \Phi^{-1}$ for the convenience. And the derivative of the inverse matrix can be done by

$$\frac{\partial Y}{\partial \theta_j} = -Y\frac{\partial \Phi}{\partial \theta_j}Y \tag{4.120}$$

### Gradient

The gradient can be obtained by

$$\begin{aligned}\frac{\partial\|\mathrm{D}\Phi^{-1}\|_{\mathrm{F}}^{2}}{\partial\theta_{\mathrm{j}}} &= \mathrm{tr}\left(\left(\frac{\partial\mathrm{Y}}{\partial\theta_{\mathrm{j}}}\right)^{\mathrm{H}}\mathrm{D}^{\mathrm{H}}\mathrm{D}\mathrm{Y}+\mathrm{Y}^{\mathrm{H}}\mathrm{D}^{\mathrm{H}}\mathrm{D}\left(\frac{\partial\mathrm{Y}}{\partial\theta_{\mathrm{j}}}\right)\right)\\ &= -\mathrm{tr}\left(\left(\mathrm{Y}\frac{\partial\Phi}{\partial\theta_{\mathrm{j}}}\mathrm{Y}\right)^{\mathrm{H}}\mathrm{D}^{\mathrm{H}}\mathrm{D}\mathrm{Y}+\mathrm{Y}^{\mathrm{H}}\mathrm{D}^{\mathrm{H}}\mathrm{D}\mathrm{Y}\frac{\partial\Phi}{\partial\theta_{\mathrm{j}}}\mathrm{Y}\right)\end{aligned} \tag{4.121}$$

For the real eigenvector matrix or the first case of the complex eigenvector matrix, the eigenvector matrix is given by Eq. (4.51) and its derivative is given by

$$\frac{\partial\Phi}{\partial\theta_{\mathrm{j}}}=\left[0\quad\cdots\quad 0\quad\frac{\partial\boldsymbol{\phi}_{\mathrm{r}}}{\partial\theta_{\mathrm{j}}}\quad 0\quad\cdots\quad 0\right],\quad \mathrm{r}=\left\lceil\frac{\mathrm{j}}{\mathrm{m}-1}\right\rceil \tag{4.122}$$

where

$$\begin{aligned}\frac{\partial\boldsymbol{\phi}_{\mathrm{r}}}{\partial\theta_{\mathrm{j}}} &= \mathrm{R}\left(\mathrm{P}_{\mathrm{r,m-1}},\theta_{\mathrm{r,m-1}}\right)\times\cdots\\ &\quad\times\mathrm{R}\left(\mathrm{P}_{\mathrm{r,k+1}},\theta_{\mathrm{r,k+1}}\right)\frac{\partial\mathrm{R}\left(\mathrm{P}_{\mathrm{r,k}},\theta_{\mathrm{r,k}}\right)}{\partial\theta_{\mathrm{j}}}\mathrm{R}\left(\mathrm{P}_{\mathrm{r,k-1}},\theta_{\mathrm{r,k-1}}\right)\\ &\quad\times\cdots\times\mathrm{R}\left(\mathrm{P}_{\mathrm{r,1}},\theta_{\mathrm{r,1}}\right)\widehat{\mathbf{p}}_{\mathrm{r,1}}\end{aligned} \tag{4.123a}$$

$$\frac{\partial\mathrm{R}\left(\mathrm{P}_{\mathrm{r,k}},\theta_{\mathrm{r,k}}\right)}{\partial\theta_{\mathrm{j}}}=-\sin\theta_{\mathrm{r,k}}\,\mathrm{P}_{\mathrm{r,k}}\mathrm{P}_{\mathrm{r,k}}^{\mathrm{T}}+\mathrm{P}_{\mathrm{r,k}}\mathrm{J}_{2}\mathrm{P}_{\mathrm{r,k}}^{\mathrm{T}}\cos\theta_{\mathrm{r,k}},\quad\theta_{\mathrm{j}}=\theta_{\mathrm{r,k}} \tag{4.123b}$$

with $\mathrm{j}=1,2,\ldots,\mathrm{n}\times(\mathrm{m}-1)$, $\mathrm{r}=1,2,\ldots,\mathrm{n}$ and $\mathrm{k}=1,2,\ldots,(\mathrm{m}-1)$. Because $\theta_{\mathrm{r,k}}$ denotes the $\mathrm{k}^{\mathrm{th}}$ angle of $\mathrm{r}^{\mathrm{th}}$ column eigenvector, we have the following index relationship.

$$\mathrm{j}=(\mathrm{r}-1)(\mathrm{m}-1)+\mathrm{k} \tag{4.124}$$

From Eq. (4.120), the derivative of the eigenvector matrix has the only $\mathrm{r}^{\mathrm{th}}$ column eigenvector.

For the second case of the complex eigenvector matrix, the eigenvector matrix is given by Eq. (4.111) and its derivative is given by

$$\frac{\partial \Phi}{\partial \theta_j} = \left[ 0 \quad \cdots \quad 0 \quad \frac{\partial \boldsymbol{\phi}_r}{\partial \theta_j} \quad 0 \quad \cdots \quad 0 \right], \quad r = \left\lfloor \frac{j}{2m-1} \right\rfloor \tag{4.125}$$

where

$$\frac{\partial \boldsymbol{\phi}_r}{\partial \theta_j} = T_{R2C} R\left(P_{r,m-1}, \theta_{r,m-1}\right) \times \cdots$$

$$\times R\left(P_{r,k+1}, \theta_{r,k+1}\right) \frac{\partial R\left(P_{r,k}, \theta_{r,k}\right)}{\partial \theta_j} R\left(P_{r,k-1}, \theta_{r,k-1}\right) \tag{4.126a}$$

$$\times \cdots \times R\left(P_{r,1}, \theta_{r,1}\right) \hat{\mathbf{p}}_{r,1}$$

$$\frac{\partial R\left(P_{r,k}, \theta_{r,k}\right)}{\partial \theta_j} = -\sin \theta_{r,k} P_{r,k} P_{r,k}^T + P_{r,k} J_2 P_{r,k}^T \cos \theta_{r,k}, \quad \theta_j = \theta_{r,k} \tag{4.126b}$$

with $j = (r-1)(2m-1) + k$, $r = 1,2,\ldots,n$ and $k = 1,2,\ldots,(2m-1)$. Because $\theta_{r,k}$ denotes the $k^{th}$ angle of $r^{th}$ column eigenvector, we have the following index relationship.

$$j = (r-1)(2m-1) + k \tag{4.127}$$

**Hessian**

The Hessian can be obtained by

$$\frac{\partial^2 \|Y\|_F^2}{\partial \theta_i \partial \theta_j} = -\frac{\partial}{\partial \theta_i} \left( \mathrm{tr} \left( \left( Y \frac{\partial \Phi}{\partial \theta_j} Y \right)^H D^H D Y + Y^H D^H D Y \frac{\partial \Phi}{\partial \theta_j} Y \right) \right)$$

$$
\begin{aligned}
= -\mathrm{tr}\Bigg( & \left(\frac{\partial \mathrm{Y}}{\partial \theta_i}\right)^{\mathrm{H}} \left(\frac{\partial \Phi}{\partial \theta_j}\right)^{\mathrm{H}} \mathrm{Y}^{\mathrm{H}}\mathrm{D}^{\mathrm{H}}\mathrm{D}\mathrm{Y} + \mathrm{Y}^{\mathrm{H}} \left(\frac{\partial^2 \Phi}{\partial \theta_i\, \partial \theta_j}\right)^{\mathrm{H}} \mathrm{Y}^{\mathrm{H}}\mathrm{D}^{\mathrm{H}}\mathrm{D}\mathrm{Y} \\
& + \mathrm{Y}^{\mathrm{H}} \left(\frac{\partial \Phi}{\partial \theta_j}\right)^{\mathrm{H}} \left(\frac{\partial \mathrm{Y}}{\partial \theta_i}\right)^{\mathrm{H}} \mathrm{D}^{\mathrm{H}}\mathrm{D}\mathrm{Y} + \left(\mathrm{Y}\frac{\partial \Phi}{\partial \theta_j}\mathrm{Y}\right)^{\mathrm{H}} \mathrm{D}^{\mathrm{H}}\mathrm{D}\frac{\partial \mathrm{Y}}{\partial \theta_i} \\
& + \left(\frac{\partial \mathrm{Y}}{\partial \theta_i}\right)^{\mathrm{H}} \mathrm{D}^{\mathrm{H}}\mathrm{D}\mathrm{Y}\frac{\partial \Phi}{\partial \theta_j}\mathrm{Y} + \mathrm{Y}^{\mathrm{H}}\mathrm{D}^{\mathrm{H}}\mathrm{D}\frac{\partial \mathrm{Y}}{\partial \theta_i}\frac{\partial \Phi}{\partial \theta_j}\mathrm{Y} \\
& + \mathrm{Y}^{\mathrm{H}}\mathrm{D}^{\mathrm{H}}\mathrm{D}\mathrm{Y}\frac{\partial^2 \Phi}{\partial \theta_i\, \partial \theta_j}\mathrm{Y} + \mathrm{Y}^{\mathrm{H}}\mathrm{D}^{\mathrm{H}}\mathrm{D}\mathrm{Y}\frac{\partial \Phi}{\partial \theta_j}\frac{\partial \mathrm{Y}}{\partial \theta_i}\Bigg)
\end{aligned}
\tag{4.128}
$$

For the real eigenvector matrix or the first case of the complex eigenvector matrix, we have

$$
\frac{\partial^2 \Phi}{\partial \theta_i\, \partial \theta_j} = \begin{cases} 0_{n\times n}, & \left\lceil \frac{i}{m-1} \right\rceil \neq \left\lceil \frac{j}{m-1} \right\rceil \\ \left[0 \quad \cdots \quad 0 \quad \frac{\partial^2 \boldsymbol{\phi}_r}{\partial \theta_i\, \partial \theta_j} \quad 0 \quad \cdots \quad 0\right], & r = \left\lceil \frac{i}{m-1} \right\rceil = \left\lceil \frac{j}{m-1} \right\rceil \end{cases}
\tag{4.129}
$$

where $i = (r-1)(m-1) + d$ and

$$
\begin{aligned}
\frac{\partial^2 \boldsymbol{\phi}_r}{\partial \theta_i\, \partial \theta_j} = {} & \mathrm{R}\left(\mathrm{P}_{r,m-1}, \theta_{r,m-1}\right) \times \cdots \\
& \times \mathrm{R}\left(\mathrm{P}_{r,d+1}, \theta_{r,d+1}\right) \frac{\partial \mathrm{R}\left(\mathrm{P}_{r,d}, \theta_{r,d}\right)}{\partial \theta_i} \mathrm{R}\left(\mathrm{P}_{r,d-1}, \theta_{r,d-1}\right) \\
& \times \cdots \\
& \times \mathrm{R}\left(\mathrm{P}_{r,k+1}, \theta_{r,k+1}\right) \frac{\partial \mathrm{R}\left(\mathrm{P}_{r,k}, \theta_{r,k}\right)}{\partial \theta_j} \mathrm{R}\left(\mathrm{P}_{r,k-1}, \theta_{r,k-1}\right) \\
& \times \cdots \times \mathrm{R}\left(\mathrm{P}_{r,1}, \theta_{r,1}\right) \hat{\mathbf{p}}_{r,1}
\end{aligned}
\tag{4.130a}
$$

$$
\frac{\partial \mathrm{R}\left(\mathrm{P}_{r,d}, \theta_{r,d}\right)}{\partial \theta_i} = -\sin\theta_{r,d}\, \mathrm{P}_{r,d}\mathrm{P}_{r,d}^{\mathrm{T}} + \mathrm{P}_{r,d}\mathrm{J}_2\mathrm{P}_{r,d}^{\mathrm{T}} \cos\theta_{r,d}\,, \quad \theta_i = \theta_{r,d}
\tag{4.130b}
$$

$$\frac{\partial^2 \mathrm{R}\left(\mathrm{P}_{\mathrm{r,d}}, \theta_{\mathrm{r,d}}\right)}{\partial^2 \theta_{\mathrm{i}}} = -\cos\theta_{\mathrm{r,d}}\, \mathrm{P}_{\mathrm{r,d}} \mathrm{P}_{\mathrm{r,d}}^{\mathrm{T}} - \mathrm{P}_{\mathrm{r,d}} \mathrm{J}_2 \mathrm{P}_{\mathrm{r,d}}^{\mathrm{T}} \sin\theta_{\mathrm{r,d}}, \quad \mathrm{i} = \mathrm{j} \tag{4.130c}$$

For the second case of the complex eigenvector matrix, we have

$$\frac{\partial^2 \Phi}{\partial \theta_{\mathrm{i}}\, \partial \theta_{\mathrm{j}}} = \begin{cases} 0_{\mathrm{n}\times\mathrm{n}}, & \left\lceil \frac{\mathrm{i}}{2\mathrm{m}-1} \right\rceil \neq \left\lceil \frac{\mathrm{j}}{2\mathrm{m}-1} \right\rceil \\ \begin{bmatrix} 0 & \cdots & 0 & \frac{\partial^2 \boldsymbol{\phi}_{\mathrm{r}}}{\partial \theta_{\mathrm{i}}\, \partial \theta_{\mathrm{j}}} & 0 & \cdots & 0 \end{bmatrix}, & \mathrm{r} = \left\lceil \frac{\mathrm{i}}{2\mathrm{m}-1} \right\rceil = \left\lceil \frac{\mathrm{j}}{2\mathrm{m}-1} \right\rceil \end{cases} \tag{4.131}$$

where $\mathrm{i} = (\mathrm{r}-1)(2\mathrm{m}-1) + \mathrm{d}$ and

$$\begin{aligned} \frac{\partial^2 \boldsymbol{\phi}_{\mathrm{r}}}{\partial \theta_{\mathrm{i}}\, \partial \theta_{\mathrm{j}}} = {} & \mathrm{T}_{\mathrm{R2C}} \mathrm{R}\left(\mathrm{P}_{\mathrm{r,m-1}}, \theta_{\mathrm{r,m-1}}\right) \times \cdots \\ & \times \mathrm{R}\left(\mathrm{P}_{\mathrm{r,d+1}}, \theta_{\mathrm{r,d+1}}\right) \frac{\partial \mathrm{R}\left(\mathrm{P}_{\mathrm{r,d}}, \theta_{\mathrm{r,d}}\right)}{\partial \theta_{\mathrm{i}}} \mathrm{R}\left(\mathrm{P}_{\mathrm{r,d-1}}, \theta_{\mathrm{r,d-1}}\right) \times \cdots \\ & \times \mathrm{R}\left(\mathrm{P}_{\mathrm{r,k+1}}, \theta_{\mathrm{r,k+1}}\right) \frac{\partial \mathrm{R}\left(\mathrm{P}_{\mathrm{r,k}}, \theta_{\mathrm{r,k}}\right)}{\partial \theta_{\mathrm{j}}} \mathrm{R}\left(\mathrm{P}_{\mathrm{r,k-1}}, \theta_{\mathrm{r,k-1}}\right) \times \cdots \\ & \times \mathrm{R}\left(\mathrm{P}_{\mathrm{r,1}}, \theta_{\mathrm{r,1}}\right) \widehat{\mathbf{p}}_{\mathrm{r,1}} \end{aligned} \tag{4.132a}$$

$$\frac{\partial \mathrm{R}\left(\mathrm{P}_{\mathrm{r,d}}, \theta_{\mathrm{r,d}}\right)}{\partial \theta_{\mathrm{i}}} = -\sin\theta_{\mathrm{r,d}}\, \mathrm{P}_{\mathrm{r,d}} \mathrm{P}_{\mathrm{r,d}}^{\mathrm{T}} + \mathrm{P}_{\mathrm{r,d}} \mathrm{J}_2 \mathrm{P}_{\mathrm{r,d}}^{\mathrm{T}} \cos\theta_{\mathrm{r,d}}, \quad \theta_{\mathrm{i}} = \theta_{\mathrm{r,d}} \tag{4.132b}$$

$$\frac{\partial^2 \mathrm{R}\left(\mathrm{P}_{\mathrm{r,d}}, \theta_{\mathrm{r,d}}\right)}{\partial^2 \theta_{\mathrm{i}}} = -\cos\theta_{\mathrm{r,d}}\, \mathrm{P}_{\mathrm{r,d}} \mathrm{P}_{\mathrm{r,d}}^{\mathrm{T}} - \mathrm{P}_{\mathrm{r,d}} \mathrm{J}_2 \mathrm{P}_{\mathrm{r,d}}^{\mathrm{T}} \sin\theta_{\mathrm{r,d}}, \quad \mathrm{i} = \mathrm{j} \tag{4.132c}$$

Note that the gradient and the Hessian for the complex conjugate eigenvectors should be considered differently by using the angle relationships given in Theorem 4.4 and Theorem 4.5.

By using the gradient and Hessian of our cost function, we can obtain the numerical results. For our numerical simulation, we use the BFGS(Broyden-Fletcher-Goldfarb-Shanno) algorithm considered as a quasi-Newton and conjugate gradient

method[35][36]. For one dimensional line search, the search with a fixed step size is used to bracket the minimum point and the golden section method is used to find the optimum point[35].

**Numerical Tests**

The following examples are chosen to test our rotation method. These examples are all integer order cases, which allows us to compare our results with the existing results. Initial angles are given by $\frac{\pi}{4}$. For the complex conjugate pair, the corresponding angles are given by $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ because of Theorem 4.5.

In each example, a) presents eigenvector matrix resulted from optimization process and b) shows the gain matrix and its norm obtained from the same process. From the observation of following results, we can see that all gain matrices contain all real numbers, which also shows that our angle relationships for the complex conjugate pair of the eigenvectors are valid numerically. c) shows the rotation angles in the each admissible space of each eigenvector. d) gives us the condition numbers in the final iteration. e) shows the optimization process with the figure. From these figures, we can see that the local optimal solution can be converged in a few iterations in the most cases. If the initial values are near singular, this might not be true.

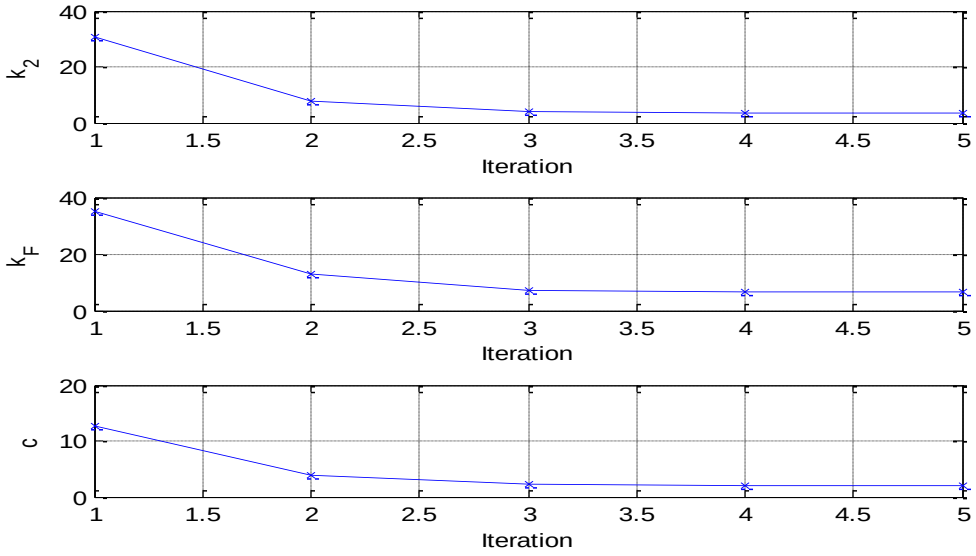


**Figure 4.4.** Optimization History of Condition Numbers for Example 4.1

Example 4.1. Chemical reactor, n=4, m=2

$$A = \begin{bmatrix} 1.3800 & -0.2077 & 6.7150 & -5.6760 \\ -0.5814 & -4.2900 & 0.0000 & 0.6750 \\ 1.0670 & 4.2730 & -6.6540 & 5.8930 \\ 0.0480 & 4.2730 & 1.3430 & -2.1040 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0.0000 & 5.6790 & 1.1360 & 1.1360 \\ 0.0000 & 0.0000 & -3.1460 & 0.0000 \end{bmatrix}$$

$$\Lambda = \mathrm{diag}(-0.2, \quad -0.5, \quad -5.0566, \quad -8.6659)$$

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} -0.7533 & 0.6821 & 0.4175 & -0.5670 \\ 0.0485 & 0.1040 & -0.5524 & -0.1924 \\ 0.5184 & 0.3485 & 0.1745 & 0.7993 \\ 0.4019 & 0.6344 & 0.7001 & -0.0508 \end{bmatrix}$$

b) Gain matrix and its norm

$$F = \begin{bmatrix} 0.1622 & -0.0909 & 0.2028 & -0.1703 \\ 1.1168 & -0.2355 & 0.7232 & -0.1372 \end{bmatrix} \quad \|F\|_2 = 1.3860$$

c) Rotation angles: $\theta = [-0.0073 \quad 1.6403 \quad 1.4805 \quad -0.0579]$

d) Condition numbers: $\kappa_2(\Phi) = 3.2332 \quad \kappa_F(\Phi) = 6.4660 \quad c = 1.9175$

e) Optimization process is given in Fig. (4.4).

Example 4.2. Distillation column, n=5, m=2.

$$A = \begin{bmatrix} -0.1094 & 0.0628 & 0.0000 & 0.0000 & 0.0000 \\ 1.3060 & -2.1320 & 0.9807 & 0.0000 & 0.0000 \\ 0.0000 & 1.5950 & -3.1490 & 1.5470 & 0.0000 \\ 0.0000 & 0.0355 & 2.6320 & -4.2570 & 1.8550 \\ 0.0000 & 0.0023 & 0.0000 & 0.1636 & -0.1625 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0.0000 & 0.0638 & 0.0838 & 0.1004 & 0.0063 \\ 0.0000 & 0.0000 & -1396 & -0.2060 & -0.0128 \end{bmatrix}$$

$$\Lambda = \text{diag}(-0.2, \quad -0.5, \quad -1.0, \quad -1.0 \pm 1.0i)$$

Approach I

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} 0.4696 & 0.1377 & 0.0155 & -0.0288 + 0.0073\text{i} & -0.0288 - 0.0073\text{i} \\ -0.6775 & -0.8565 & -0.2205 & 0.2920 - 0.5620\text{i} & 0.2920 + 0.5620\text{i} \\ -0.4213 & -0.4688 & 0.6171 & 0.5082 - 0.3447\text{i} & 0.5082 + 0.3447\text{i} \\ -0.2197 & -0.1394 & 0.7211 & 0.4522 - 0.0984\text{i} & 0.4522 + 0.0984\text{i} \\ -0.3079 & -0.0907 & -0.2245 & -0.0104 - 0.0814\text{i} & -0.0104 + 0.0814\text{i} \end{bmatrix}$$

b) Gain matrix and its norm

$$F = 10^2 \times \begin{bmatrix} -0.3645 & 1.1259 & -2.2547 & 1.9202 & -0.5500 \\ -0.0846 & 0.3794 & -0.5755 & 0.4433 & 0.0098 \end{bmatrix} \quad \|F\|_2 = 333.4916$$

c) Rotation angles: $\theta = [-0.5112 \quad 0.4934 \quad 0.1933 \quad 1.1721 \quad 1.1721]$

d) Condition numbers: $\kappa_2(\Phi) = 47.5828 \quad \kappa_F(\Phi) = 62.3486 \quad c = 17.3772$

e) Optimization process is given in Fig. (4.5).

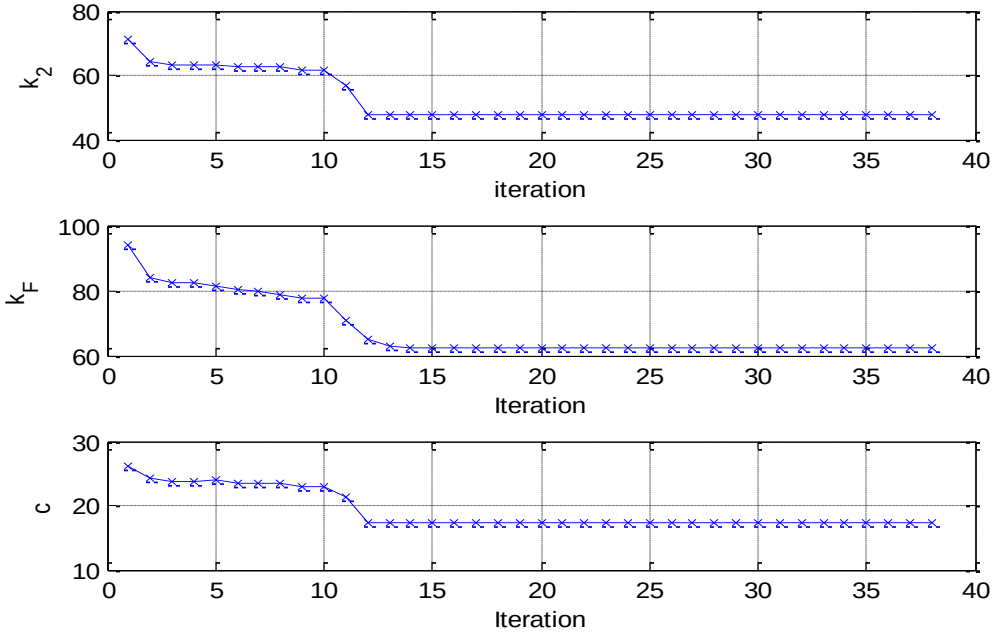


**Figure 4.5**. Optimization History of Condition Numbers for Example 4.2, Approach I

Approach II

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} 0.3983 & 0.1378 & -0.0353 & -0.0050 - 0.0062i & -0.0050 + 0.0062i \\ -0.5747 & -0.8571 & 0.5003 & 0.1695 + 0.0076i & 0.1695 - 0.0076i \\ -0.4676 & -0.0554 & 0.7009 & 0.6499 + 0.2467i & 0.6499 - 0.2467i \\ -0.0811 & 0.2264 & 0.4984 & 0.6292 + 0.2791i & 0.6292 - 0.2791i \\ 0.5347 & -0.4383 & -0.0938 & 0.0158 - 0.1177i & 0.0158 + 0.1177i \end{bmatrix}$$

b) Gain matrix and its norm

$$F = \begin{bmatrix} -49.6504 & 51.5669 & -93.7388 & 71.7377 & -13.0355 \\ -30.5997 & -9.4528 & 41.9827 & -47.3266 & 26.0617 \end{bmatrix} \quad \|F\|_2 = 148.1282$$

c) Rotation angles

$$\theta_{real} = [0.3825 \quad -0.1711 \quad 0.9858] \quad \theta_{complex} = \begin{bmatrix} 0.7923 & -0.7923 \\ 1.1165 & -1.1165 \\ 1.0674 & -1.0674 \end{bmatrix}$$

d) Condition numbers: $\kappa_2(\Phi) = 45.9794 \quad \kappa_F(\Phi) = 60.8576 \quad c = 17.5521$

e) Optimization process is given in Fig. (4.6).

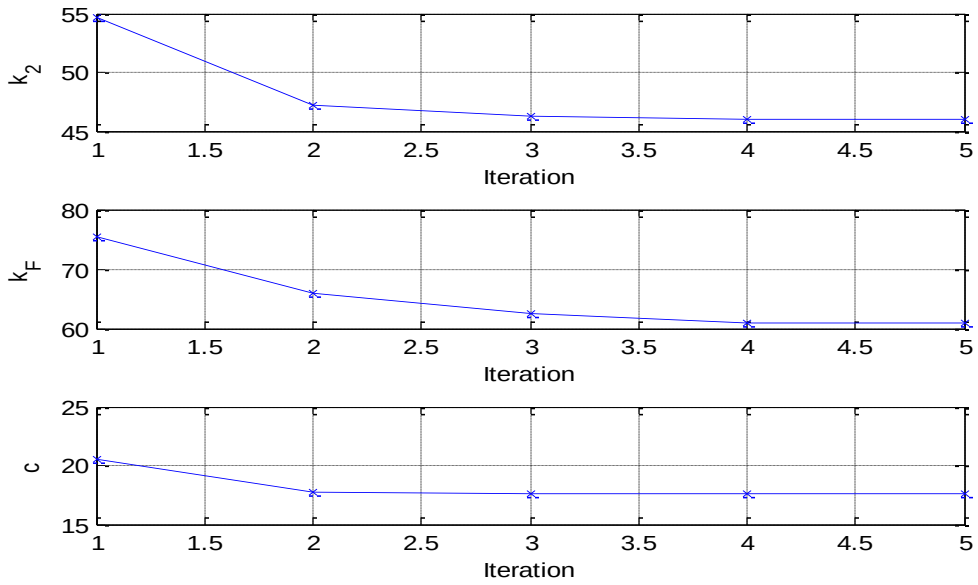


**Figure 4.6**. Optimization History of Condition Numbers for Example 4.2, Approach II

Example 4.3. Nuclear rocket engine, n=4, m=2.

$$A = \begin{bmatrix} -65.0 & 65.0 & -19.5 & 19.5 \\ 0.1 & -0.1 & 0.0 & 0.0 \\ 1.0 & 0.0 & -0.5 & -1.0 \\ 0.0 & 0.0 & 0.4 & -0.0 \end{bmatrix}, \; B^T = \begin{bmatrix} 65 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 \end{bmatrix} \text{ and}$$

$$\Lambda = \text{diag}(-1, \quad -2, \quad -3, \quad -4)$$

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} 0.6835 & -0.5753 & 0.3704 & -0.9029 \\ -0.0759 & 0.0303 & -0.0128 & 0.0232 \\ 0.0769 & 0.6829 & 0.2134 & 0.1422 \\ 0.7219 & 0.4491 & 0.9039 & -0.4050 \end{bmatrix}$$

b) Gain matrix and its norm

$$F = \begin{bmatrix} 0.9096 & -1.5234 & 0.2596 & -0.2797 \\ -1.6857 & -70.8605 & 3.2261 & -8.8083 \end{bmatrix} \quad \|F\|_2 = 71.5150$$

c) Rotation angles: $\theta = [1.6320 \quad 0.2639 \quad 1.5967 \quad -0.6362]$

d) Condition numbers: $\kappa_2(\Phi) = 35.7006 \quad \kappa_F(\Phi) = 45.7106 \quad c = 21.8205$

e) Optimization process is given in Fig. (4.7).

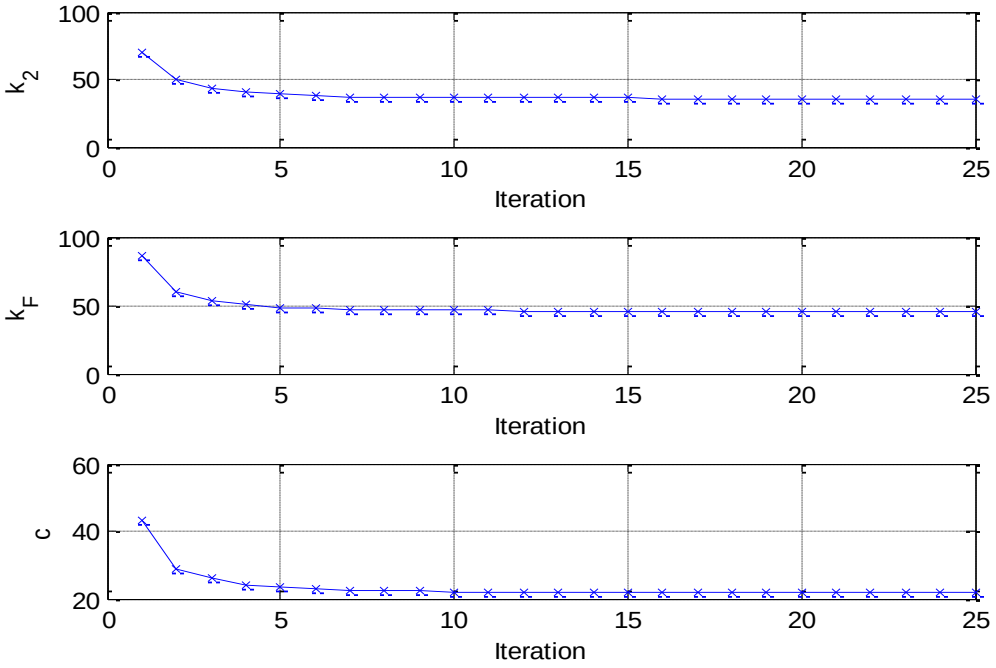


**Figure 4.7**. Optimization History of Condition Numbers for Example 4.3

Example 4.4. Multiple-input multiple-output system, n=4, m=2.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, B^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \Lambda = \text{diag}(-1, \quad -2, \quad -3)$$

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} -0.6648 & 0.5773 & -0.5661 \\ 0.0782 & -0.5773 & 0.2265 \\ 0.7430 & 0.5774 & 0.7926 \end{bmatrix}$$

b) Gain matrix and its norm

$$\mathrm{F} = \begin{bmatrix} 2.4167 & 7.6667 & 3.2500 \\ -0.8333 & -3.3333 & -1.5000 \end{bmatrix} \quad \|\mathrm{F}\|_2 = 9.4441$$

c) Rotation angles: $\theta = [1.1099 \quad 2.6445 \quad 1.0402]$

d) Condition numbers: $\kappa_2(\Phi) = 10.7738 \quad \kappa_F(\Phi) = 13.4211 \quad c = 6.1644$

e) Optimization process is given in Fig. (4.8).

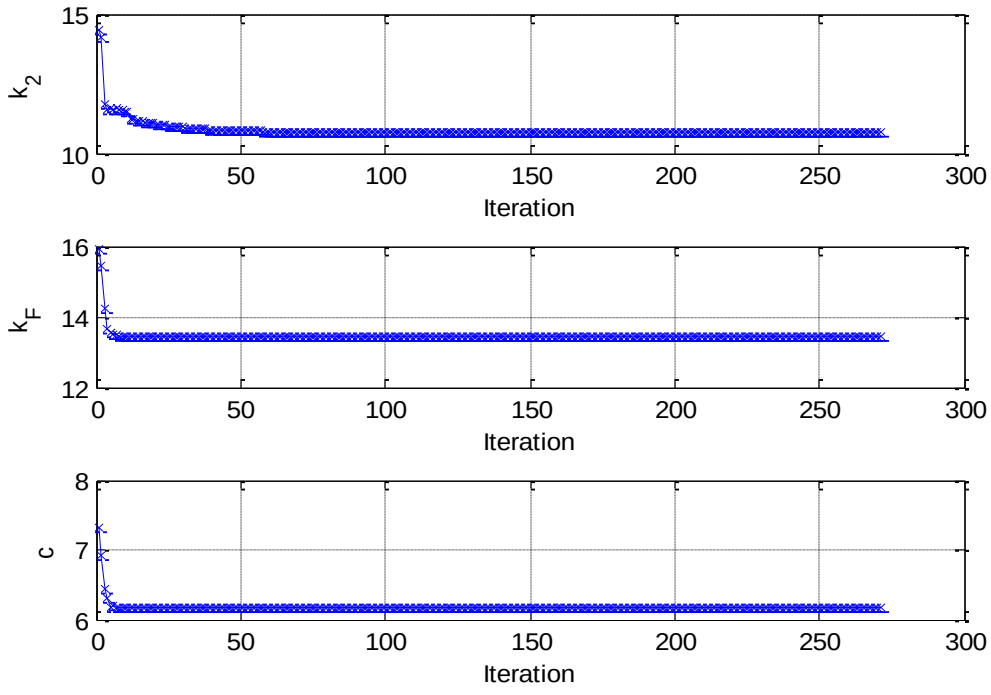


**Figure 4.8**. Optimization History of Condition Numbers for Example 4.4

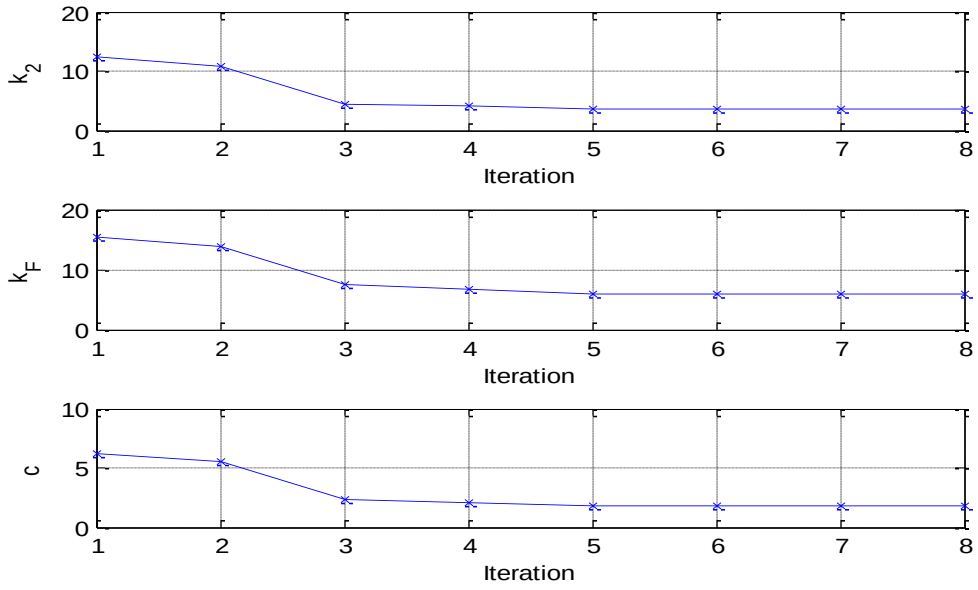


**Figure 4.9**. Optimization History of Condition Numbers for Example 4.5, Approach I

Example 4.5. Random problem, n=4, m=2.

$$A = \begin{bmatrix} 5.8765 & 9.3456 & 4.5634 & 9.3520 \\ 6.6526 & 0.5867 & 3.5829 & 0.6534 \\ 0.0000 & 9.6738 & 7.4876 & 4.7654 \\ 0.0000 & 0.0000 & 6.6784 & 2.5678 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 3.9878 & 0.0000 & 0.0000 & 0.0000 \\ 0.5432 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$\Lambda = \mathrm{diag}(-29.4986, \quad -10.0922, \quad 2.5201 \pm 6.8910i)$$

Approach I

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} -0.7227 & -0.3070 & 0.0606 + 0.2159\mathrm{i} & 0.0606 - 0.2159\mathrm{i} \\ 0.6665 & -0.7702 & -0.3713 - 0.5499\mathrm{i} & -0.3713 + 0.5499\mathrm{i} \\ -0.1791 & 0.4945 & -0.2773 + 0.4311\mathrm{i} & -0.2773 - 0.4311\mathrm{i} \\ 0.0373 & -0.2609 & 0.4196 + 0.2659\mathrm{i} & 0.4196 - 0.2659\mathrm{i} \end{bmatrix}$$

b) Gain matrix and its norm

$$F = \begin{bmatrix} -7.7856 & 0.9304 & 3.8716 & 3.5833 \\ 2.1570 & -7.6648 & -9.9235 & -10.7339 \end{bmatrix} \quad \|F\|_2 = 17.9154$$

c) Rotation angles: $\theta = [1.1242 \quad -0.6710 \quad 2.0131 \quad 2.0131]$

d) Condition numbers: $\kappa_2(\Phi) = 3.8827 \quad \kappa_F(\Phi) = 6.2410 \quad c = 1.7981$

e) Optimization process is given in Fig. (4.9).

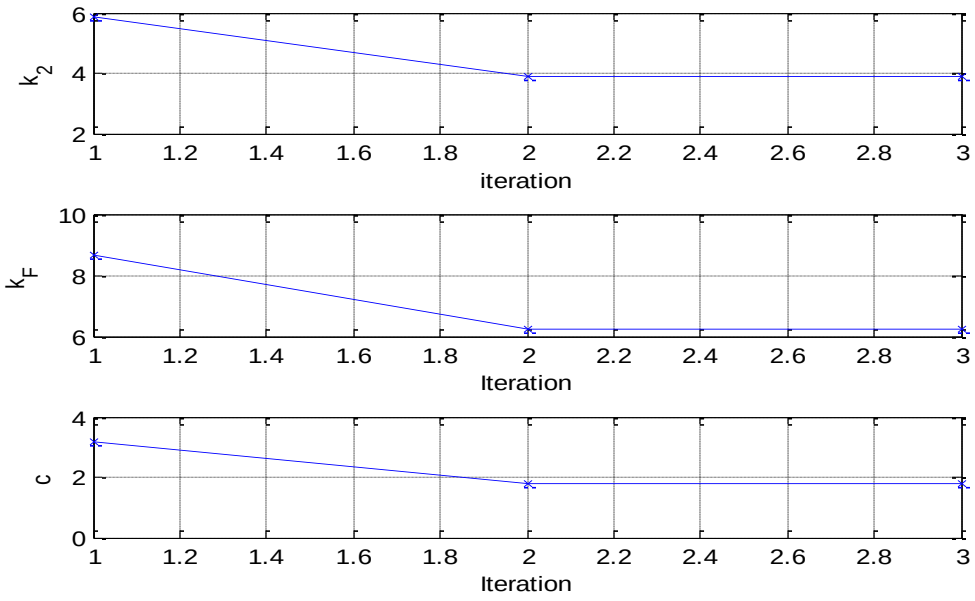


**Figure 4.10**. Optimization History of Condition Numbers for Example 4.5, Approach II

Approach II

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} -0.5046 & -0.6414 & 0.1076 - 0.0557i & 0.1076 + 0.0557i \\ 0.8326 & -0.6209 & 0.3033 + 0.6040i & 0.3033 - 0.6040i \\ -0.2238 & 0.3987 & 0.3365 - 0.3992i & 0.3365 + 0.3992i \\ 0.0466 & -0.2103 & -0.3891 - 0.3234i & -0.3891 + 0.3234i \end{bmatrix}$$

b) Gain matrix and its norm

$$\mathrm{F} = \begin{bmatrix} -5.9208 & 1.4377 & 2.4445 & 1.3895 \\ 1.7372 & -10.2719 & -10.8257 & -12.0135 \end{bmatrix} \quad \|\mathrm{F}\|_2 = 19.5904$$

c) Rotation angles

$$\theta_{\mathrm{real}} = [1.4029 \quad -0.2867] \quad \theta_{\mathrm{complex}} = \begin{bmatrix} 1.1455 & -1.1455 \\ -1.7140 & 1.7140 \\ 0.2560 & -0.2560 \end{bmatrix}$$

d) Condition numbers: $\kappa_2(\Phi) = 3.6473 \quad \kappa_{\mathrm{F}}(\Phi) = 5.9347 \quad \mathrm{c} = 1.7834$

e) Optimization process is given in Fig. (4.10).

Example 4.6. Random problem, n=8, m=3.

$$\mathrm{A} = \begin{bmatrix} 0.5257 & 0.8544 & 0.5591 & 0.5901 & 0.0259 & 0.6213 & 0.7227 & 0.5617 \\ 0.9931 & 0.0643 & 0.1249 & 0.3091 & 0.5174 & 0.3455 & 0.8977 & 0.4682 \\ 0.6489 & 0.8279 & 0.7279 & 0.2552 & 0.3917 & 0.7065 & 0.2428 & 0.7795 \\ 0.9923 & 0.9262 & 0.2678 & 0.6252 & 0.2414 & 0.5211 & 0.4338 & 0.9677 \\ 0.0000 & 0.5667 & 0.5465 & 0.1157 & 0.5064 & 0.2870 & 0.7901 & 0.9809 \\ 0.0000 & 0.0000 & 0.8672 & 0.6117 & 0.4236 & 0.6503 & 0.5069 & 0.8187 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.2894 & 0.0881 & 0.5233 & 0.4257 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4499 & 0.5597 & 0.2462 \end{bmatrix}$$

$$\mathrm{B}^{\mathrm{T}} = \begin{bmatrix} 0.9230 & 0.3950 & 0.8325 \\ 0.0000 & 0.0366 & 0.6105 \\ 0.0000 & 0.0000 & 0.1871 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$\Lambda$

$$= \mathrm{diag}(-0.1778, \quad 0.5628, \quad 1.2715 \pm 0.3923i, \quad 0.6297 \pm 0.5605i, \quad -0.5113 \pm 0.1013i)$$

Approach I

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} 0.4113 & 0.2173 & -0.0497 \mp 0.1127i & 0.2140 \pm 0.1386i & -0.6797 \pm 0.1509i \\ -0.2743 & -0.5515 & -0.5985 \pm 0.1998i & -0.6237 \mp 0.1759i & 0.1172 \mp 0.0729i \\ -0.1713 & 0.1103 & -0.3358 \pm 0.0097i & -0.1360 \pm 0.3184i & -0.3877 \mp 0.0043i \\ -0.4275 & -0.0127 & 0.0168 \pm 0.3455i & -0.0518 \pm 0.1190i & 0.5731 \mp 0.0439i \\ 0.5726 & -0.5594 & 0.1312 \pm 0.1946i & -0.2502 \pm 0.1873i & 0.0623 \pm 0.0473i \\ 0.3350 & 0.4543 & 0.3359 \pm 0.2426i & 0.4135 \pm 0.1716i & -0.0587 \mp 0.0061i \\ -0.3156 & -0.2145 & 0.2319 \mp 0.0232i & -0.0083 \mp 0.0812i & -0.0318 \mp 0.0291i \\ 0.0611 & 0.2664 & 0.2703 \mp 0.0096i & 0.1894 \mp 0.1940i & 0.0540 \pm 0.0324i \end{bmatrix}$$

b) Gain matrix and its norm

$$F = 10^2 \begin{bmatrix} -0.5193 & -0.5300 & 0.1699 & -0.1525 & 0.0572 & -0.3734 & -0.3245 & -0.0630 \\ 1.3224 & 1.3234 & -0.3897 & 0.3786 & -0.1042 & 0.9986 & 0.8917 & 0.1842 \\ -0.0732 & -0.0591 & -0.0079 & -0.0241 & -0.0165 & -0.0658 & -0.0792 & -0.0383 \end{bmatrix}$$

$$\|F\|_2 = 255.0325$$

c) Rotation angles

$$\theta = \begin{bmatrix} 0.6237 & 0.5506 & 1.2998 & 1.2998 & 0.7972 & 0.7972 & 1.2576 & 1.2576 \\ 0.6875 & 0.5865 & 0.6919 & 0.6919 & 0.7203 & 0.7203 & 0.2492 & 0.2492 \end{bmatrix}$$

d) Condition numbers: $\kappa_2(\Phi) = 17.4947 \quad \kappa_F(\Phi) = 34.7828 \quad c = 7.3554$

e) Optimization process is given in Fig. (4.11).

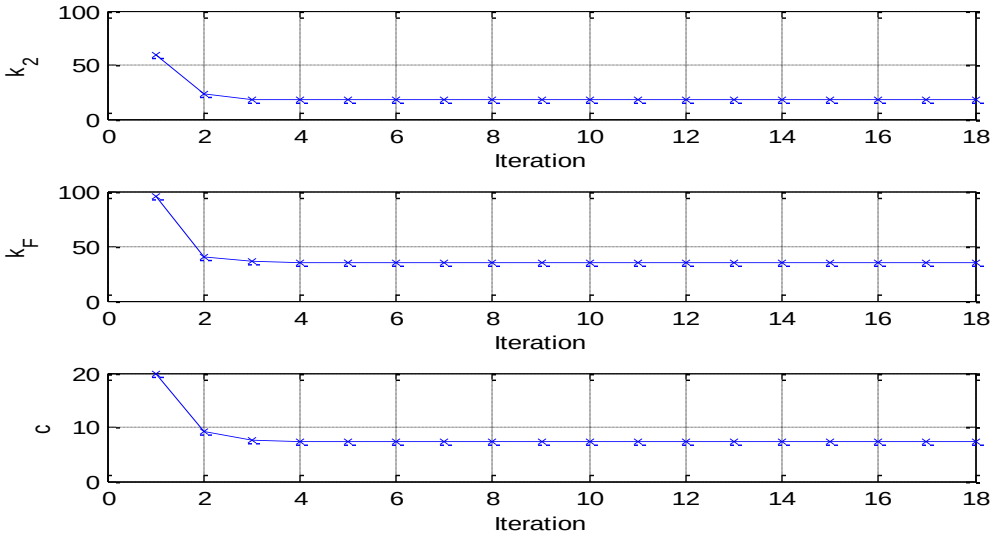


**Figure 4.11**. Optimization History of Condition Numbers for Example 4.6, Approach I

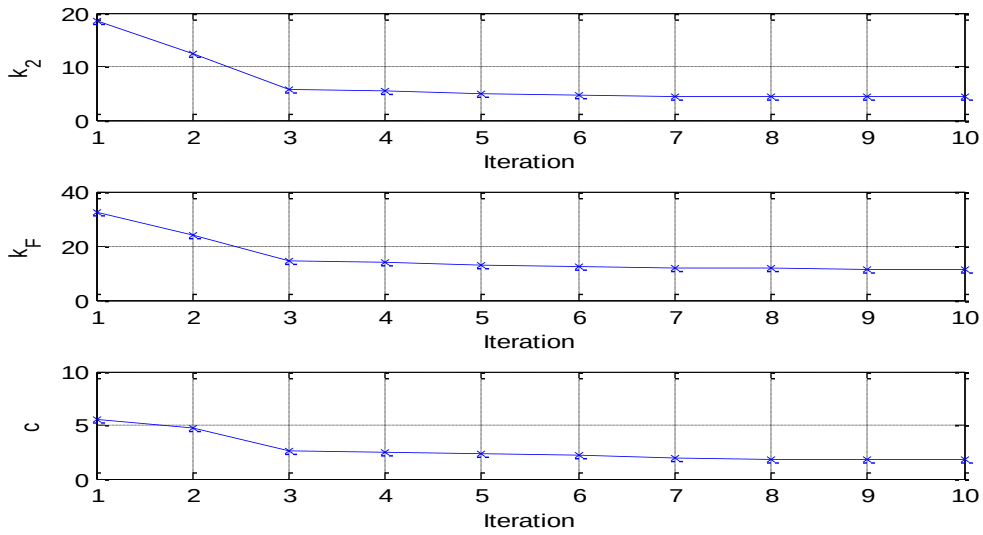


**Figure 4.12**. Optimization History of Condition Numbers for Example 4.6, Approach II

Approach II

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} 0.2421 & -0.0112 & 0.0616 \mp 0.1263i & -0.1532 \pm 0.2761i & -0.0014 \mp 0.3832i \\ -0.4125 & -0.1562 & -0.4786 \mp 0.1167i & -0.2791 \mp 0.2199i & -0.0043 \mp 0.1143i \\ -0.2306 & -0.4033 & -0.3540 \mp 0.3513i & 0.2165 \pm 0.4761i & -0.4560 \mp 0.2246i \\ -0.2970 & 0.6509 & -0.0935 \pm 0.3482i & -0.1181 \pm 0.2305i & 0.0191 \pm 0.4449i \\ 0.2702 & -0.4546 & -0.0282 \pm 0.2759i & -0.1815 \mp 0.0722i & 0.3562 \pm 0.1624i \\ 0.0359 & 0.3338 & 0.0418 \pm 0.3872i & 0.5521 \mp 0.0108i & 0.3790 \mp 0.1371i \\ -0.4682 & -0.1399 & 0.1860 \pm 0.1596i & -0.1871 \mp 0.0758i & -0.0354 \mp 0.0847i \\ 0.5799 & 0.2270 & 0.1904 \pm 0.1842i & 0.0621 \mp 0.2139i & -0.2144 \pm 0.1153i \end{bmatrix}$$

b) Gain matrix and its norm

$$F = 10^2 \begin{bmatrix} -0.0167 & -0.4888 & -0.2880 & -0.0305 & -0.0326 & -0.3794 & -0.3213 & -0.6785 \\ 0.0820 & 1.2410 & 0.7558 & 0.0885 & 0.1103 & 1.0005 & 0.9096 & 1.8052 \\ -0.0271 & -0.0617 & -0.0494 & -0.0139 & -0.0167 & -0.0679 & -0.0860 & -0.1169 \end{bmatrix}$$

$$\|F\|_2 = 287.8762$$

c) Rotation angles

$$\theta_{real} = \begin{bmatrix} 1.1714 & 1.1503 \\ 1.3817 & 1.7719 \end{bmatrix}$$

$$\theta_{\text{complex}} = \begin{bmatrix} 1.6034 & -1.6034 & -0.0729 & 0.0729 & 0.7872 & -0.7872 \\ 1.5142 & -1.5142 & 0.0900 & -0.0900 & 1.3554 & -1.3554 \\ 0.8441 & -0.8441 & 0.6983 & -0.6983 & 1.7461 & -1.7461 \\ 0.8837 & -0.8837 & 0.8946 & -0.8946 & 0.7734 & -0.7734 \\ 0.6940 & -0.6940 & 0.9734 & -0.9734 & 1.4082 & -1.4082 \end{bmatrix}$$

d) Condition numbers: $\kappa_2(\Phi) = 4.3664 \quad \kappa_F(\Phi) = 11.5549 \quad c = 1.8295$

e) Optimization process is given in Fig. (4.12).

Example 4.7. Aircraft control example I, n=4, m=3.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1.40 \times 10^{-4} & -2.04 & -1.950 & 1.33 \times 10^{-2} \\ -2.51 \times 10^{-4} & 1 & -1.320 & -2.38 \times 10^{-2} \\ -5.61 \times 10^{-1} & 0 & 0.358 & -2.79 \times 10^{-1} \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & -5.33 & -1.60 \times 10^{-1} & 0 \\ 0 & 6.45 \times 10^{-3} & -1.16 \times 10^{-2} & 1.06 \times 10^{-1} \\ 0 & -2.67 \times 10^{-1} & -2.51 \times 10^{-1} & 8.62 \times 10^{-2} \end{bmatrix}$$

$$\Lambda = \text{diag}(-1, \quad -2, \quad -3, \quad -4)$$

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} -0.7048 & -0.0614 & -0.0717 & -0.2423 \\ 0.7048 & 0.1228 & 0.2150 & 0.9691 \\ -0.0806 & -0.8974 & -0.3714 & -0.0298 \\ -0.0120 & 0.4193 & -0.9004 & 0.0368 \end{bmatrix}$$

b) Gain matrix and its norm

$$F = \begin{bmatrix} 0.8534 & 0.3837 & -0.5545 & -0.1409 \\ 7.1707 & -3.1747 & -10.5909 & -26.3029 \\ -1.9790 & 3.3335 & 4.1783 & 2.5514 \end{bmatrix} \quad \|F\|_2 = 29.7926$$

c) Rotation angles: $\theta = \begin{bmatrix} -0.0816 & 1.4330 & 1.3422 & 0.0474 \\ 0.1477 & 2.7045 & -1.9620 & 2.2514 \end{bmatrix}$

d) Condition numbers: $\kappa_2(\Phi) = 3.7576 \quad \kappa_F(\Phi) = 6.2246 \quad c = 2.3332$

e) Optimization process is given in Fig. (4.13).

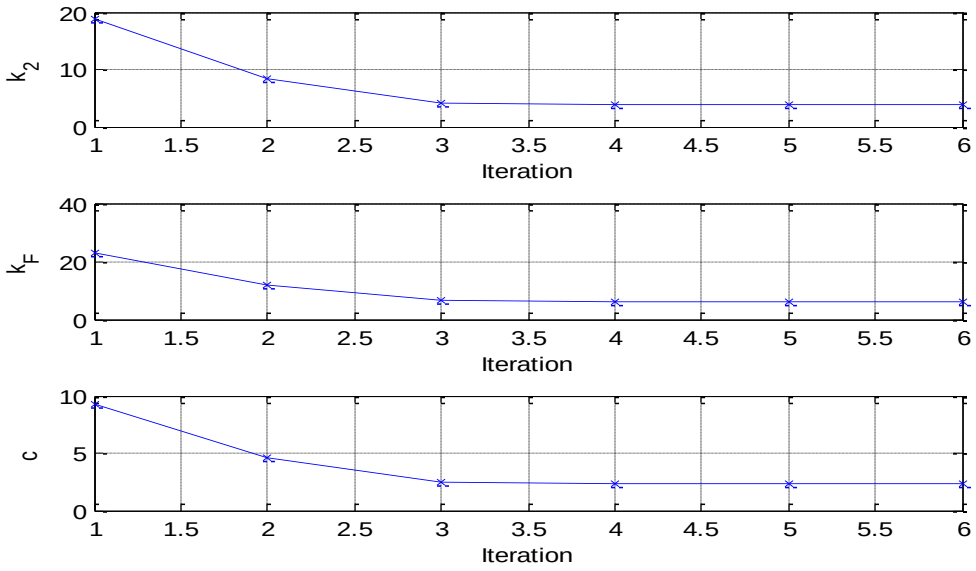


**Figure 4.13**. Optimization History of Condition Numbers for Example 4.7

Example 4.8. Aircraft control example II, n=4, m=2.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.32 \times 10^{-7} & -4.18 \times 10^{-1} & -0.120 & 2.32 \times 10^{-2} \\ -4.62 \times 10^{-9} & 1 & -0.752 & -2.39 \times 10^{-2} \\ -5.61 \times 10^{-1} & 0 & -0.300 & -1.74 \times 10^{-2} \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & -1.72 \times 10^{-1} & -2.38 \times 10^{-2} & 0 \\ 0 & 7.45 \times 10^{-6} & -7.78 \times 10^{-5} & 3.69 \times 10^{-3} \end{bmatrix}$$

$$\Lambda = \mathrm{diag}(-1, \quad -2, \quad -3, \quad -4)$$

a) Eigenvector matrix

$$\Phi = \begin{bmatrix} 0.2519 & 0.3902 & 0.0683 & -0.2395 \\ -0.2519 & -0.7805 & -0.2050 & 0.9580 \\ 0.9133 & 0.4880 & 0.0429 & -0.1500 \\ 0.1973 & 0.0206 & 0.9754 & 0.0485 \end{bmatrix}$$

b) Gain matrix and its norm

$$F = \begin{bmatrix} 62.3969 & 35.0790 & -9.2018 & 0.3208 \\ -264.7163 & -85.4444 & 289.2737 & -806.2011 \end{bmatrix} \quad \|F\|_2 = 900.9102$$

c) Rotation angles: $\theta = [0.1969 \quad 0.0268 \quad 1.3611 \quad 0.0690]$

d) Condition numbers: $\kappa_2(\Phi) = 18.7281 \quad \kappa_F(\Phi) = 23.8973 \quad c = 11.1326$

e) Optimization process is given in Fig. (4.14).

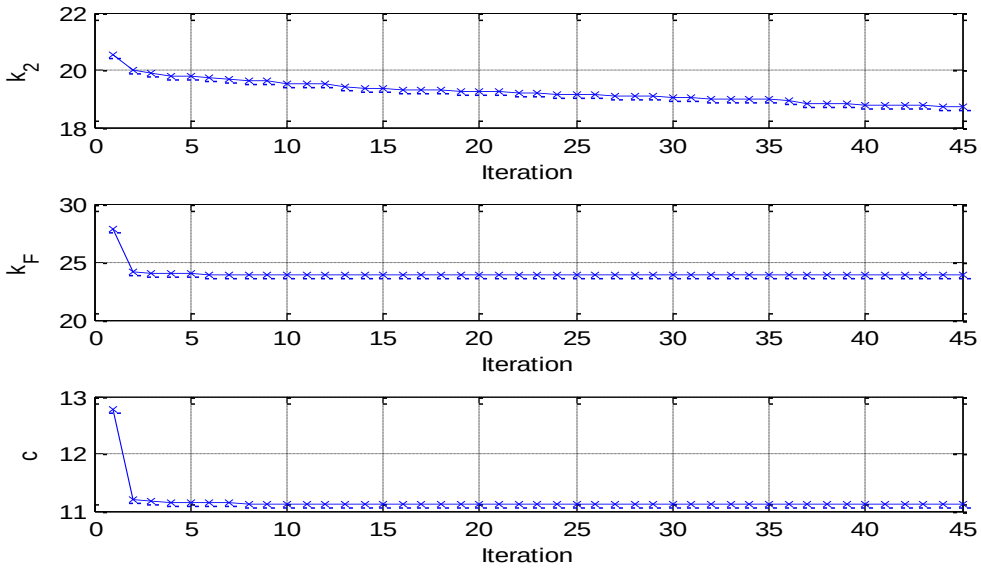


**Figure 4.14.** Optimization History of Condition Numbers for Example 4.8

**Comparison**

Table 4.1 compares numerical results obtained from previous examples with the ones obtained by existing methods. Some of the examples show that our methods produce better results although it is not always the case. We suspect that our methods would lead us to better results if we alternate initial values, but for the fair comparison we applied the same initial values for each example.

**Table 4.1.** Comparisons between Various Methods

| Ex. | Cond. No. | Rotation Method (Real case or Approach I) | Rotation Method (Approach II) | Beyers and Nash | J.Kautsky et al. | | |
|---|---|---|---|---|---|---|---|
| | | | | | Method 0 | Method 1 | Method 2/3 |
| 1 | $\kappa_2(\Phi)$ | 3.2332 | | 3.35 | 1.82 | 3.32 | 4.54 |
| | $\kappa_F(\Phi)$ | 6.4660 | | 6.40 | | | |
| | $\|F\|_2$ | 1.3860 | | 1.46 | 1.47 | 1.40 | 1.17 |
| 2 | $\kappa_2(\Phi)$ | 47.5828 | 45.9794 | 33.07 | | 39.4 | 66.1 |
| | $\kappa_F(\Phi)$ | 62.3486 | 60.8576 | 39.11 | | | |
| | $\|F\|_2$ | 333.4916 | 148.1282 | 354.85 | | 311.5 | 283.1 |
| 3 | $\kappa_2(\Phi)$ | 35.7006 | | 33.85 | | | |
| | $\kappa_F(\Phi)$ | 45.7106 | | 41.23 | | | |
| | $\|F\|_2$ | 71.5150 | | 77.15 | | | |
| 4 | $\kappa_2(\Phi)$ | 10.7738 | | 10.77 | | | |
| | $\kappa_F(\Phi)$ | 13.4211 | | 11.87 | | | |
| | $\|F\|_2$ | 9.4441 | | 9.44 | | | |
| 5 | $\kappa_2(\Phi)$ | 3.8827 | 3.6473 | 3.55 | | | |
| | $\kappa_F(\Phi)$ | 6.2410 | 5.9347 | 5.84 | | | |
| | $\|F\|_2$ | 17.9154 | 19.5904 | 23.00 | | | |
| 6 | $\kappa_2(\Phi)$ | 17.4947 | 4.3664 | 4.74 | | | |
| | $\kappa_F(\Phi)$ | 34.7828 | 11.5549 | 11.91 | | | |
| | $\|F\|_2$ | 255.0325 | 287.8762 | 305.50 | | | |
| 7 | $\kappa_2(\Phi)$ | 3.7576 | | 3.61 | | | |
| | $\kappa_F(\Phi)$ | 6.2246 | | 5.89 | | | |
| | $\|F\|_2$ | 29.7926 | | 28.25 | | | |
| 8 | $\kappa_2(\Phi)$ | 18.7281 | | 18.59 | | | |
| | $\kappa_F(\Phi)$ | 23.8973 | | 21.24 | | | |
| | $\|F\|_2$ | 900.9102 | | 807.57 | | | |

**Graphic Visualization of the Cost Function**

Each angle parameter in the rotation method has the finite angle space$[-\pi, \pi]$. With the angle space being finite, we can now visualize our cost function by sweeping each angle parameter from the obtained local minimum. This means that one angle parameter is sweeping from $-\pi$ to $\pi$ while the other angle parameters are fixed with the

corresponding local minimum values, which leads us to see if our local minimum is the good solution and another good local minimum candidate can be found. This is also one of advantages of our angle parameterization.

Figures (4.15) to (4.18) give us the visualization of our cost function by sweeping one angle parameter for each case. Because rotating the vector with π changes only the sign of it, figures show us the symmetry characteristics with π.

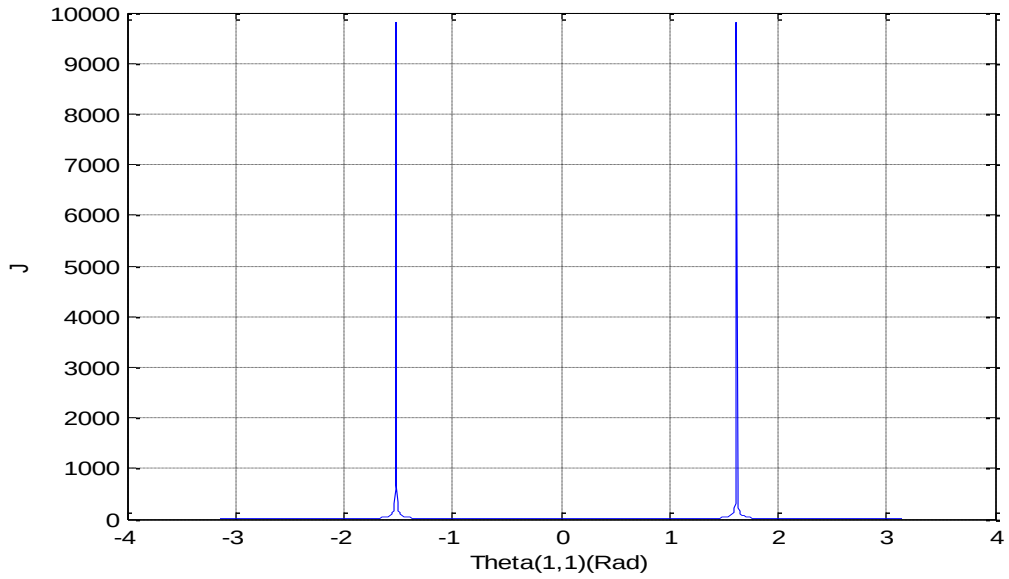


**Figure 4.15.** Angle Sweep$(-\pi$ to $\pi)$ of First Rotation Angle in Example 4.1

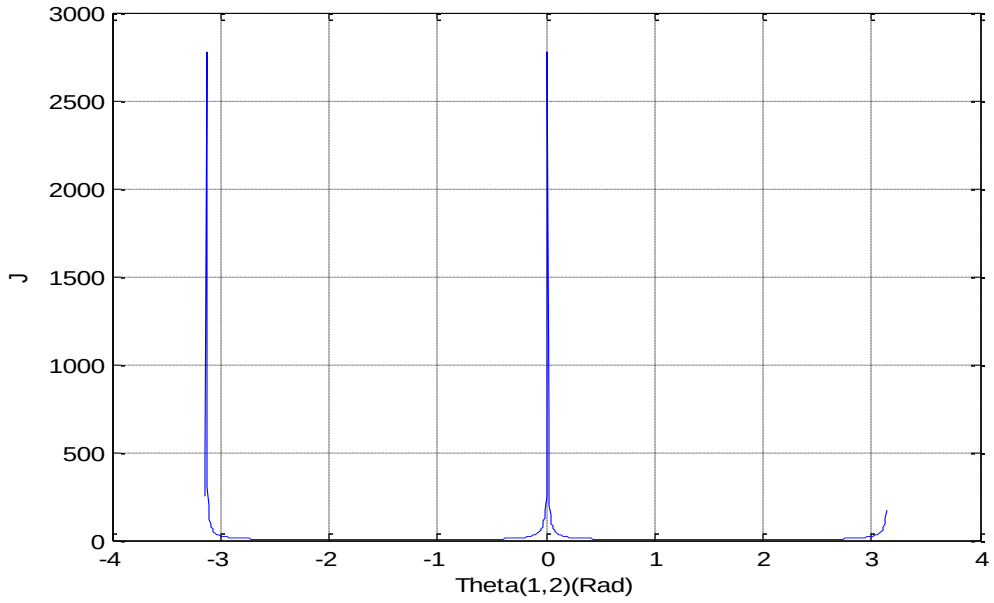


**Figure 4.16.** Angle Sweep($-\pi$ to $\pi$) of Second Rotation Angle in Example 4.1

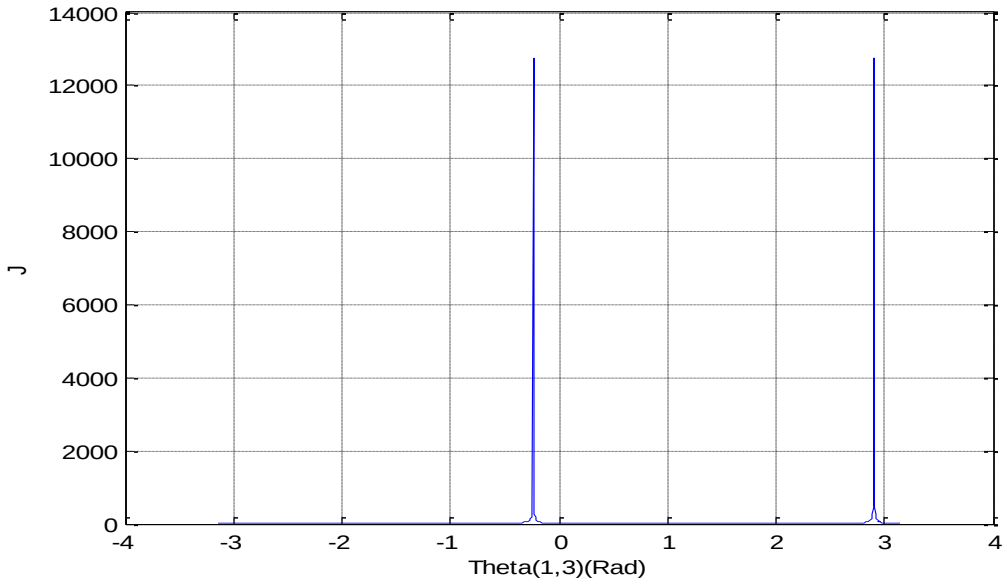


**Figure 4.17.** Angle Sweep($-\pi$ to $\pi$) of Third Rotation Angle in Example 4.1

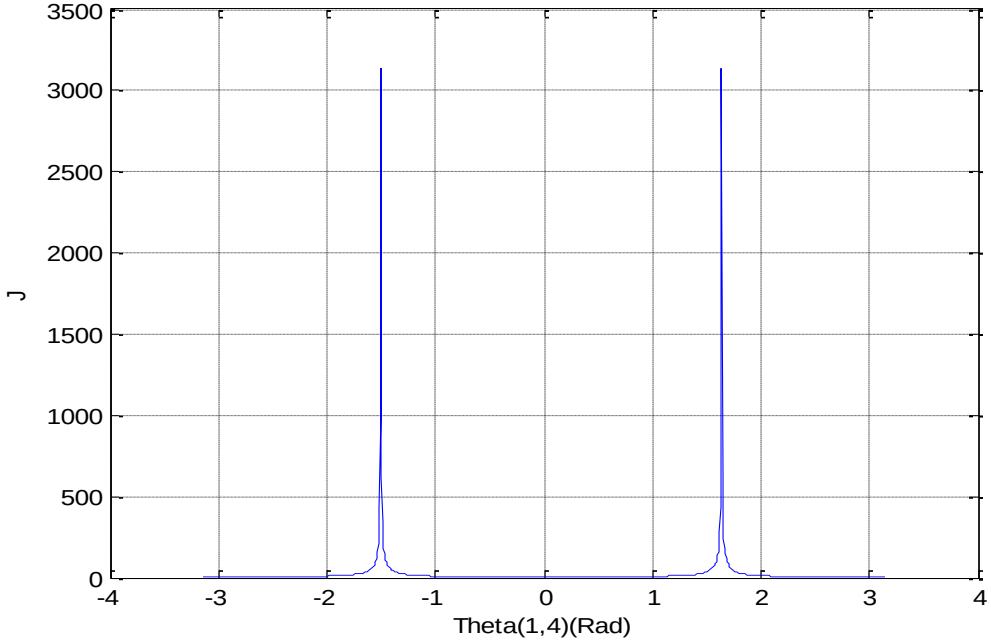


**Figure 4.18.** Angle Sweep$(-\pi$ to $\pi)$ of Fourth Rotation Angle in Example 4.1

**Weighted Cost Function**

Let us consider the example 2 with the fractional order $\alpha = 0.25$. Then the diagonal weight matrix can be obtained as $D = \text{diag}\{5,2,1,0.7654,0.7654\}$ by using Eq. (3.45).

Approach I

f) Eigenvector matrix

$$\Phi = \begin{bmatrix} 0.2925 & 0.1339 & -0.0214 & -0.0029 + 0.0127i & -0.0029 - 0.0127i \\ -0.4220 & -0.8329 & 0.3041 & -0.1613 - 0.2268i & -0.1613 + 0.2268i \\ -0.3945 & -0.5042 & 0.7225 & 0.5856 - 0.1580i & 0.5856 + 0.1580i \\ -0.0107 & -0.1782 & 0.6041 & 0.7266 - 0.0727i & 0.7266 + 0.0727i \\ 0.7620 & -0.0477 & -0.1421 & -0.0537 - 0.1354i & -0.0537 + 0.1354i \end{bmatrix}$$

g) Gain matrix and its norm

$$F = 10^2 \times \begin{bmatrix} -0.9490 & 0.0929 & -0.1004 & -0.0012 & 0.1970 \\ -0.6698 & -0.4588 & 1.1584 & -1.1213 & 0.5314 \end{bmatrix} \quad \|F\|_2 = 191.4713$$

h) Rotation angles: $\theta = [0.6944 \quad 0.5653 \quad 0.7558 \quad 0.4945 \quad 0.4945]$

i) Condition numbers: $\upsilon_3(\Phi) = 9.6761$

j) Optimization process is given in Fig. (4.19).

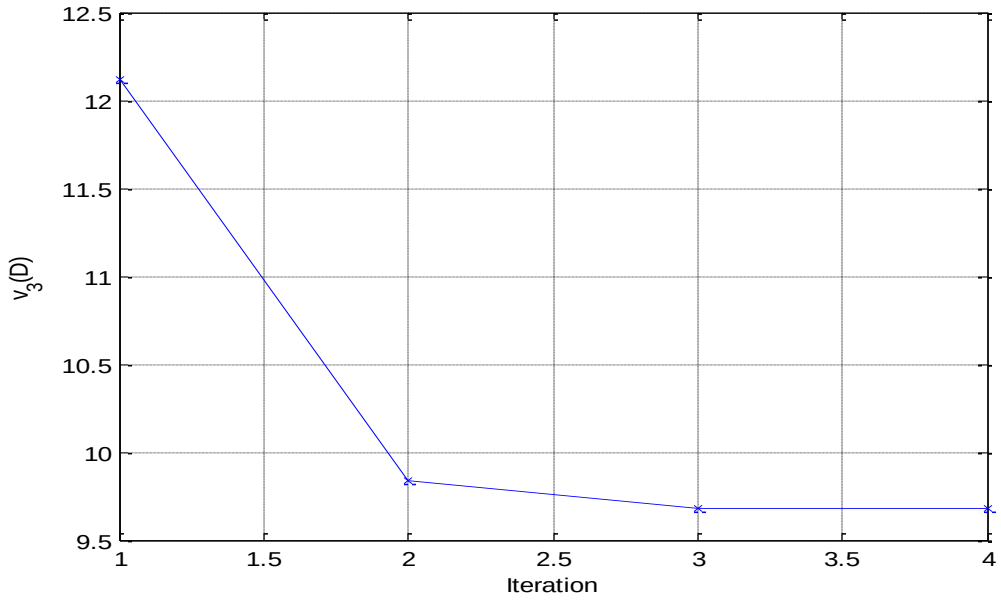


**Figure 4.19**. Optimization Process for Example with Weighted Cost Function, Approach I

Approach II

f) Eigenvector matrix

$$\Phi = \begin{bmatrix} 0.2515 & 0.1330 & -0.0188 & -0.0032 + 0.0043i & -0.0032 - 0.0043i \\ -0.3628 & -0.8274 & 0.2660 & -0.0235 - 0.1116i & -0.0235 + 0.1116i \\ -0.3621 & -0.5114 & 0.7223 & 0.6013 + 0.2377i & 0.6013 - 0.2377i \\ 0.0127 & -0.1864 & 0.6202 & 0.6714 + 0.3136i & 0.6714 - 0.3136i \\ 0.8208 & -0.0386 & -0.1502 & 0.0166 - 0.1394i & 0.0166 + 0.1394i \end{bmatrix}$$

g) Gain matrix and its norm

$$F = \begin{bmatrix} -78.2063 & 21.0097 & -28.7506 & 14.2437 & 7.4686 \\ -55.8073 & -33.4609 & 92.8076 & -93.1437 & 41.4066 \end{bmatrix} \quad \|F\|_2 = 152.4390$$

h) Rotation angles

$$\theta_{real} = [0.7956 \quad 0.5806 \quad 0.7135] \quad \theta_{complex} = \begin{bmatrix} 0.6616 & -0.6616 \\ 0.8388 & -0.8388 \\ 0.9136 & -0.9136 \end{bmatrix}$$

i) Condition numbers: $\upsilon_3(\Phi) = 7.6279$

j) Optimization process is given in Fig. (4.20).

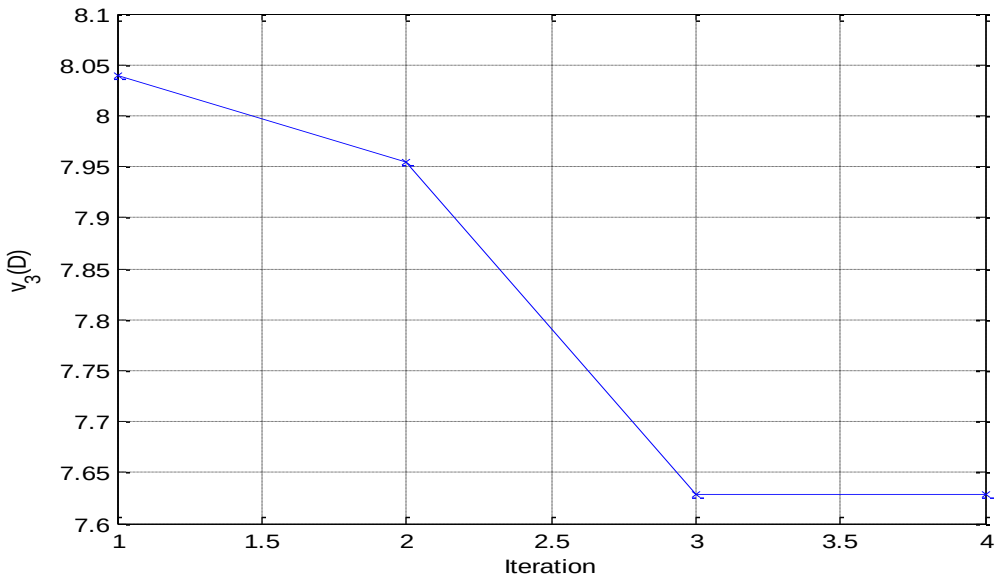


**Figure 4.20**. Optimization Process for Example with Weighted Cost Function, Approach II

# CHAPTER V

# KALMAN FILTER FOR FRACTIONAL ORDER SYSTEMS

In this chapter, we derive the discrete-time Kalman filter for fractional order systems. In order to implement it, we first derive a discrete-time model for fractional order systems. Then we obtain the propagation equation and "update" equation. Because the solution to the fractional order differential equation has the history-dependent terms, the propagation equation of the error covariance matrix is not simple and contains all of the history-dependent terms. By using the derived Kalman filter, we show the numerical examples and investigate some important properties of this filter.

## A. KALMAN FILTER DEVIATION

Let us consider a continuous fractional order system as

$$ {}_{t_0}^{C}D_t^{\alpha}\mathbf{x}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + G(t)\mathbf{w}(t) \tag{5.1} $$

where $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ and $G(t) \in \mathbb{R}^{n\times n}$. Assuming that $A(t), B(t)$ and $G(t)$ are constant and the input $\mathbf{u}(t)$ and the system noise $\mathbf{w}(t)$ are constant during a sample interval $\Delta t$, we can derive the discrete-time model of the above continuous fractional order system by using the second approach shown in the Chapter II. Then we want to formulate the discrete-time Kalman filter for the discrete-time model of the fractional order system. By using Eq. (2.86), we can obtain the following discrete-time model with the initial time $t_0 = 0$.

$$\underline{\mathbf{x}}(t_{n+1}) = \Phi(t_{n+1}, t_n)\underline{\mathbf{x}}(t_n) + \sum_{k=0}^{n-1} N(t_{n+1}, t_n, t_{k+1}, t_k)\big(B\mathbf{u}(t_k) + \Upsilon\mathbf{w}(t_k)\big) + \Gamma(\Delta t)\big(Bu(t_n) + \Upsilon\mathbf{w}(t_n)\big) \tag{5.2}$$

where $\Phi(t_{n+1}, t_n)$ $N(t_{n+1}, t_n, t_{k+1}, t_k)$ and $\Gamma(\Delta t)$ are given in Eqs. (2.87) to (2.89). For the simplicity and the convenience in later use, we express it with the simple notations. Then the "truth" discrete-time model can be described as

$$\mathbf{x}_{k+1} = \Phi_k\mathbf{x}_k + \Gamma B\mathbf{u}_k + \Gamma\Upsilon\mathbf{w}_k + \sum_{m=0}^{k-1}\big(N_{k,m}B\mathbf{u}_m + N_{k,m}\Upsilon\mathbf{w}_m\big) \tag{5.3}$$

where $\mathbf{x}_k = \underline{\mathbf{x}}(t_n)$, $\Phi_k = \Phi(t_{n+1}, t_n)$, $\Gamma = \Gamma(\Delta t)$, $N_{k,m} = N(t_{n+1}, t_n, t_{k+1}, t_k)$, $\mathbf{u}_m = \mathbf{u}(t_k)$ and $\mathbf{w}_m = \mathbf{w}(t_k)$. Note that even if system matrices A and B are time-invariant, $\Phi_k$ is a time-variant function. Summation terms in Eq. (5.3) characterize the difference between the integer order systems and the fractional order systems by showing the memory property of the fractional order systems. The measurement model is given by

$$\tilde{\mathbf{y}}_k = H_k\mathbf{x}_k + \boldsymbol{\upsilon}_k \tag{5.4}$$

where $\boldsymbol{\upsilon}_k$ is the measurement noise. The system noise and the measurement noise are considered to be zero-mean Gaussian white-noise processes. The properties of them are given by

$$E\{\mathbf{w}_i\mathbf{w}_j^T\} = Q_i\delta_{ij} \text{ and } E\{\boldsymbol{\upsilon}_i\boldsymbol{\upsilon}_j^T\} = R_i\delta_{ij} \tag{5.5}$$

where $\delta_{ij}$ is the Kronecker delta. $Q_i$ and $R_i$ present the corresponding covariance matrix of each noise. And we assume that both noises are uncorrelated to each other.

Now, we can obtain the propagation of the estimated state by taking the expectation of the true dynamic model. Then the propagation equation of the estimated state can be described by

$$\hat{\mathbf{x}}_{k+1}^{-} = \Phi_k \hat{\mathbf{x}}_k^{+} + \Gamma B \mathbf{u}_k + \sum_{m=0}^{k-1} N_{k,m} B \mathbf{u}_m \tag{5.6}$$

"+" and "-" notation means the times after and before the measurement, respectively. Also, once we have the measurement, we can update our state estimate. Therefore, we can think that "+" and "-" notation presents the times after and before the update, respectively. In order to obtain an error covariance matrix, we need the error dynamics of the state. By subtracting Eq. (5.6) from Eq. (5.3), the estimated state error can be obtained as

$$\tilde{\mathbf{x}}_{k+1}^{-} = \hat{\mathbf{x}}_{k+1}^{-} - \mathbf{x}_{k+1} = \Phi_k \tilde{\mathbf{x}}_k^{+} - \Gamma \Upsilon \mathbf{w}_k - \sum_{m=0}^{k-1} N_{k,m} \Upsilon \mathbf{w}_m \tag{5.7}$$

Then the propagation equation of the error covariance matrix can be defined by

$$\begin{aligned} P_{k+1}^{-} &\equiv E\{\tilde{\mathbf{x}}_{k+1}^{-} \tilde{\mathbf{x}}_{k+1}^{-T}\} \\ &= \Phi_k E\{\tilde{\mathbf{x}}_k^{+} \tilde{\mathbf{x}}_k^{+T}\} \Phi_k^{T} - \sum_{m=0}^{k-1} \Phi_k E\{\tilde{\mathbf{x}}_k^{+} \mathbf{w}_m^{T}\} \Upsilon^{T} N_{k,m}^{T} \\ &\quad + \Gamma \Upsilon E\{\mathbf{w}_k \mathbf{w}_k^{T}\} \Upsilon^{T} \Gamma^{T} - \sum_{m=0}^{k-1} N_{k,m} \Upsilon E\{\mathbf{w}_m \tilde{\mathbf{x}}_k^{+T}\} \Phi_k^{T} \\ &\quad + \sum_{m=0}^{k-1} N_{k,m} \Upsilon E\{\mathbf{w}_m \mathbf{w}_m^{T}\} \Upsilon^{T} N_{k,m}^{T} \end{aligned} \tag{5.8}$$

By substituting $P_k^{+} = E\{\tilde{\mathbf{x}}_k^{+} \tilde{\mathbf{x}}_k^{+T}\}$, $Q_m = E\{\mathbf{w}_m \mathbf{w}_m^{T}\}$ into Eq. (5.8), we have

$$P_{k+1}^{-} = \Phi_k P_k^{+} \Phi_k^{T} + \Gamma \Upsilon_k Q_k \Upsilon^{T} \Gamma^{T} + \sum_{m=0}^{k-1} N_{k,m} \Upsilon Q_m \Upsilon^{T} N_{k,m}^{T} - \sum_{m=0}^{k-1} \Phi_k E\{\tilde{\mathbf{x}}_k^{+} \mathbf{w}_m^{T}\} \Upsilon^{T} N_{k,m}^{T} - \sum_{m=0}^{k-1} N_{k,m} \Upsilon E\{\mathbf{w}_m \tilde{\mathbf{x}}_k^{+T}\} \Phi_k^{T} \tag{5.9}$$

Now, we need to examine $E\{\tilde{\mathbf{x}}_k^{+} \mathbf{w}_m^{T}\}$ and $E\{\mathbf{w}_m \tilde{\mathbf{x}}_k^{+T}\}$. Taking the transpose of the former becomes the latter. So we only need one of them. To obtain $E\{\mathbf{w}_m \tilde{\mathbf{x}}_k^{+T}\}$, we have to know the state updated form. This "update" equation can be given as[4][37].

$$\hat{\mathbf{x}}_k^{+} = \hat{\mathbf{x}}_k^{-} + K_k(\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_k^{-}) \tag{5.10}$$

where $K_k$ is the Kalman gain. We will obtain it later. The state error can be defined as the difference between the estimated state and the truth state. By using the above equation, the updated state error can be obtained as

$$\begin{aligned} \tilde{\mathbf{x}}_k^{+} = \hat{\mathbf{x}}_k^{+} - \mathbf{x}_k &= \hat{\mathbf{x}}_k^{-} - \mathbf{x}_k + K_k(\tilde{\mathbf{y}}_k - H_k(\hat{\mathbf{x}}_k^{-} - \mathbf{x}_k) - H_k \mathbf{x}_k) \\ &= \tilde{\mathbf{x}}_k^{-} + K_k(\boldsymbol{\upsilon}_k - H_k \tilde{\mathbf{x}}_k^{-}) = (I - K_k H_k)\tilde{\mathbf{x}}_k^{-} + K_k \boldsymbol{\upsilon}_k \end{aligned} \tag{5.11}$$

where $\tilde{\mathbf{x}}_k^{-} = \hat{\mathbf{x}}_k^{-} - \mathbf{x}_k$. By using Eq. (5.11) and Eq. (5.7), we can obtain

$$\tilde{\mathbf{x}}_k^{+} = (I - K_k H_k)\left( \Phi_{k-1} \tilde{\mathbf{x}}_{k-1}^{+} - \Gamma \Upsilon \mathbf{w}_{k-1} - \sum_{m=0}^{k-2} N_{k-1,m} \Upsilon \mathbf{w}_m \right) + K_k \boldsymbol{\upsilon}_k \tag{5.12}$$

Then $E\{\tilde{\mathbf{x}}_k^{+} \mathbf{w}_m^{T}\}$ and $E\{\mathbf{w}_m \tilde{\mathbf{x}}_k^{+T}\}$ can be expressed as

$$E\{\tilde{\mathbf{x}}_k^{+} \mathbf{w}_m^{T}\} = (I - K_k H_k)\Phi_{k-1} E\{\tilde{\mathbf{x}}_{k-1}^{+} \mathbf{w}_m^{T}\} - (I - K_k H_k)\Gamma \Upsilon E\{\mathbf{w}_{k-1} \mathbf{w}_m^{T}\} - (I - K_k H_k) \sum_{j=0}^{k-2} N_{k-1,j} \Upsilon E\{\mathbf{w}_j \mathbf{w}_m^{T}\} \tag{5.13}$$

Because of the property of $E\{\mathbf{w}_m\tilde{\mathbf{x}}_k^{+T}\} = (E\{\tilde{\mathbf{x}}_k^{+}\mathbf{w}_m^T\})^T$, we only need one calculation between both of them.

$$
\begin{aligned}
&-\sum_{m=0}^{k-1}\Phi_k E\{\tilde{\mathbf{x}}_k^{+}\mathbf{w}_m^T\}\Upsilon^T N_{k,m}^T \\
&\quad = -\sum_{m=0}^{k-1}\Phi_k(I-K_kH_k)\Phi_{k-1}E\{\tilde{\mathbf{x}}_{k-1}^{+}\mathbf{w}_m^T\}\Upsilon^T N_{k,m}^T \\
&\quad +\sum_{m=0}^{k-1}\Phi_k(I-K_kH_k)\Gamma\Upsilon E\{\mathbf{w}_{k-1}\mathbf{w}_m^T\}\Upsilon^T N_{k,m}^T \\
&\quad +\sum_{m=0}^{k-1}\Phi_k(I-K_kH_k)\sum_{j=0}^{k-2}N_{k-1,j}\Upsilon E\{\mathbf{w}_j\mathbf{w}_m^T\}\Upsilon^T N_{k,m}^T
\end{aligned}
\tag{5.14}
$$

By using Eq. (5.5), the second and third terms on the right hand side of Eq. (5.14) can be changed to

$$
\begin{aligned}
&\sum_{m=0}^{k-1}\Phi_k(I-K_kH_k)\Gamma\Upsilon E\{\mathbf{w}_{k-1}\mathbf{w}_m^T\}\Upsilon^T N_{k,m}^T \\
&\quad = \Phi_k(I-K_kH_k)\Gamma\Upsilon Q_{k-1}\Upsilon^T N_{k,k-1}^T
\end{aligned}
\tag{5.15a}
$$

$$
\begin{aligned}
&\sum_{m=0}^{k-1}\Phi_k(I-K_kH_k)\sum_{j=0}^{k-2}N_{k-1,j}\Upsilon E\{\mathbf{w}_j\mathbf{w}_m^T\}\Upsilon^T N_{k,m}^T \\
&\quad = \Phi_k(I-K_kH_k)\sum_{m=0}^{k-2}N_{k-1,m}\Upsilon Q_m\Upsilon^T N_{k,m}^T
\end{aligned}
\tag{5.15b}
$$

By the induction of $E\{\tilde{\mathbf{x}}_{k-1}^{+}\mathbf{w}_m^T\}$, we can finally obtain

$$\begin{aligned} Z \equiv & -\sum_{m=0}^{k-1} \Phi_k E\{\tilde{\mathbf{x}}_k^+ \mathbf{w}_m^T\} \Upsilon^T N_{k,m}^T - \sum_{m=0}^{k-1} N_{k,m} \Upsilon E\{\mathbf{w}_m \tilde{\mathbf{x}}_k^{+T}\} \Phi_k^T \\ & = Z_1 + Z_2 + Z_1^T + Z_2^T \end{aligned} \tag{5.16}$$

where

$$\begin{aligned} Z_1 = & \Phi_k (I - K_k H_k) \Gamma \Upsilon Q_{k-1} \Upsilon^T N_{k,k-1}^T \\ & + \Phi_k (I - K_k H_k) \Phi_{k-1} (I - K_{k-1} H_{k-1}) \Gamma \Upsilon Q_{k-2} \Upsilon^T N_{k,k-2}^T + \cdots \\ & + \Phi_k (I - K_k H_k) \Phi_{k-1} (I - K_{k-1} H_{k-1}) \times \cdots \\ & \times \Phi_1 (I - K_1 H_1) \Gamma \Upsilon Q_0 \Upsilon^T N_{k,0}^T \end{aligned} \tag{5.17a}$$

$$\begin{aligned} Z_2 = & \Phi_k (I - K_k H_k) \sum_{m=0}^{k-2} N_{k-1,m} \Upsilon Q_m \Upsilon^T N_{k,m}^T \\ & + \Phi_k (I - K_k H_k) \Phi_{k-1} (I - K_{k-1} H_{k-1}) \\ & \times \sum_{m=0}^{k-3} N_{k-2,m} \Upsilon Q_m \Upsilon^T N_{k,m}^T + \cdots + \Phi_k (I - K_k H_k) \Phi_{k-1} \times \cdots \\ & \times \Phi_2 (I - K_2 H_2) N_{1,0} \Upsilon Q_0 \Upsilon^T N_{k,0}^T \end{aligned} \tag{5.17b}$$

Therefore, the propagation equation of the error covariance matrix is finally given by

$$P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Gamma \Upsilon Q_k \Upsilon^T \Gamma^T + \sum_{m=0}^{k-1} N_{k,m} \Upsilon Q_m \Upsilon^T N_{k,m}^T + Z \tag{5.18}$$

where Z is given in Eq. (5.16).

The “update” equation of the error covariance matrix can be described as

$$P_k^+ = E\{\tilde{\mathbf{x}}_k^+ \tilde{\mathbf{x}}_k^{+T}\}$$

$$= (I - K_k H_k) E\{\tilde{\mathbf{x}}_k^- \tilde{\mathbf{x}}_k^{-T}\}(I - K_k H_k)^T + K_k E\{\boldsymbol{\upsilon}_k \boldsymbol{\upsilon}_k^T\} K_k^T \quad (5.19)$$

$$= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T$$

In order to find the gain $K_k$, we use the minimum variance measure of optimality by defining our cost function as the trace of the updated error covariance matrix, $J(K_k) = Tr(P_k^+)$. Using the trace identities in Reference [1], we can obtain

$$\frac{\partial J}{\partial K_k} = 0 = -2(I - K_k H_k) P_k^- H_k^T + 2K_k R_k \quad (5.20)$$

From this equation, we can obtain the gain $K_k$.

$$K_k = P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k\right)^{-1} \quad (5.21)$$

Note that this solution is the same with that of the conventional Kalman filter because we assume the "update" equation is given in Eq. (5.10) which is the same "update" solution in the conventional Kalman filter. Substituting Eq. (5.21) into Eq. (5.19) gives us

$$P_k^+ = (I - K_k H_k) P_k^- \quad (5.22)$$

Therefore, we obtain the all propagation and "update" equations for the discrete-time Kalman filter for fractional order systems. Table 5.1 shows the summary of it.

**Table 5.1**. Discrete-time Linear Kalman Filter for Fractional Order Systems

| | |
|---|---|
| Model | $\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma B \mathbf{u}_k + \Gamma \Upsilon \mathbf{w}_k + \sum_{m=0}^{k-1} \left( N_{k,m} B \mathbf{u}_m + N_{k,m} \Upsilon \mathbf{w}_m \right)$ <br> $\tilde{\mathbf{y}}_k = H_k \mathbf{x}_k + \boldsymbol{\upsilon}_k$ |
| Initialize | $\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ <br> $P_0 = E\{\tilde{\mathbf{x}}(t_0)\tilde{\mathbf{x}}(t_0)^T\}$ |
| Gain | $K_k = P_k^- H_k^T \left( H_k P_k^- H_k^T + R_k \right)^{-1}$ |
| Update | $\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k(\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_k^-)$ <br> $P_k^+ = (I - K_k H_k) P_k^-$ |
| Propagation | $\hat{\mathbf{x}}_{k+1}^- = \Phi_k \hat{\mathbf{x}}_k^+ + \Gamma B \mathbf{u}_k + \sum_{m=0}^{k-1} N_{k,m} B \mathbf{u}_m$ <br> $P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Gamma \Upsilon_k Q_k \Upsilon_k^T \Gamma^T + \sum_{m=0}^{k-1} N_{k,m} \Upsilon_m Q_m \Upsilon_m^T N_{k,m}^T + Z$ |

**Moving Window**

The solution for the fractional order differential equation has the *short memory principle* introduced in Chapter I, which allows us to have an approximate solution because the solution depends on mainly the "recent past" values. So it is attractable and useful for dealing with the practical engineering problem because we have finite memory. By using this *short memory principle*, an approximate truth model can be changed into

$$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma B \mathbf{u}_k + \Gamma \Upsilon \mathbf{w}_k + \sum_{m=k-1-L}^{k-1} \left( N_{k,m} B \mathbf{u}_m + N_{k,m} \Upsilon \mathbf{w}_m \right) \tag{5.23}$$

where L is a fixed length of the moving window. Then the propagation equation of the estimated state can be expressed as

$$\hat{\mathbf{x}}_{k+1}^{-} = \Phi_k \hat{\mathbf{x}}_k^{+} + \Gamma B \mathbf{u}_k + \sum_{m=k-1-L}^{k-1} N_{k,m} B \mathbf{u}_m \tag{5.24}$$

By using this equation, the propagation equation of the error covariance matrix in Eq. (5.18) can also be changed with fewer terms.

$$P_{k+1}^{-} = \Phi_k P_k^{+} \Phi_k^{T} + \Gamma \Upsilon_k Q_k \Upsilon_k^{T} \Gamma^{T} + \sum_{m=k-1-L}^{k-1} N_{k,m} \Upsilon_m Q_m \Upsilon_m^{T} N_{k,m}^{T} + Z \tag{5.25}$$

where

$$\begin{aligned} Z \equiv & - \sum_{m=k-1-L}^{k-1} \Phi_k E\{\tilde{\mathbf{x}}_k^{+} \mathbf{w}_m^{T}\} \Upsilon^{T} N_{k,m}^{T} - \sum_{m=k-1-L}^{k-1} N_{k,m} \Upsilon E\{\mathbf{w}_m \tilde{\mathbf{x}}_k^{+T}\} \Phi_k^{T} \\ & = Z_1 + Z_2 + Z_1^{T} + Z_2^{T} \end{aligned} \tag{5.26}$$

with

$$\begin{aligned} Z_1 = & \Phi_k (I - K_k H_k) \Gamma \Upsilon Q_{k-1} \Upsilon^{T} N_{k,k-1}^{T} \\ & + \Phi_k (I - K_k H_k) \Phi_{k-1} (I - K_{k-1} H_{k-1}) \Gamma \Upsilon Q_{k-2} \Upsilon^{T} N_{k,k-2}^{T} + \cdots \\ & + \Phi_k (I - K_k H_k) \Phi_{k-1} (I - K_{k-1} H_{k-1}) \times \cdots \\ & \times \Phi_{k-L} (I - K_{k-L} H_{k-L}) \Gamma \Upsilon Q_{k-1-L} \Upsilon^{T} N_{k,k-1-L}^{T} \end{aligned} \tag{5.27a}$$

$$
\begin{aligned}
Z_2 = \Phi_k(I - K_k H_k) \sum_{m=k-1-L}^{k-2} N_{k-1,m} \Upsilon Q_m \Upsilon^T N_{k,m}^T \\
+ \Phi_k(I - K_k H_k)\Phi_{k-1}(I - K_{k-1} H_{k-1}) \\
\times \sum_{m=k-1-L}^{k-3} N_{k-2,m} \Upsilon Q_m \Upsilon^T N_{k,m}^T + \cdots + \Phi_k(I - K_k H_k)\Phi_{k-1} \\
\times \cdots \times \Phi_{k-L}(I - K_{k-L} H_{k-L}) \sum_{m=k-1-L}^{k-1-L} N_{k-1-L,m} \Upsilon Q_m \Upsilon^T N_{k,m}^T
\end{aligned}
\tag{5.27b}
$$

Therefore, this property of the fractional order system leads to the reduction of the memory for storing the previous data. But note that this is an approximate solution, which gives us larger errors. We will compare this approximate solution with exact solution in numerical examples.

**B. Numerical Example**

Let us assume that we have the following fractional order system with the fractional order, $\alpha = 0.5$.

$$
A = \begin{bmatrix} 0 & 1 \\ -1.1 & -1.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad H = [1 \quad 0] \tag{5.28}
$$

The noise parameters are given by $\sigma_w = 0.04 \times [1 \quad 1]^T$ and $\sigma_v = 0.1$. And, the initial error covariance matrix is set to $P_0 = \text{diag}\{10^3 \quad 10^3\}$. The design parameters for the Kalman filter are chosen as $Q = 0.16 \times \text{diag}\{10^{-3} \quad 10^{-3}\}$ and $R = 0.01$. The input is given by $u(t) = \sin(\frac{12}{\pi}t)$. For examining the property of the filter, we run 100 numerical simulations. For each simulation, the total time is set to be 3.9sec with $\Delta t = 0.1\text{sec}$.

Figures 5.1 to 5.4 show only one simulation result among the 100 numerical simulations. In this example, the moving window of the filter with $L = 5$ gives us the appropriate result. If we increase the length of the moving window, we can expect a more precise result. Therefore, for some examples, it is required to check what length can be a numerically appropriate candidate for the filter. From Fig. (5.4), we can see that the state estimates are in 3σ bound.

Figure 5.5 shows that the 100 ensembles of the estimation errors and 3σ bounds obtained from $P_k^+$ in the last simulation and from the statistical analysis. Most ensembles are within in 3σ bound. And, 3σ bounds almost coincide each other. This means that the error covariance matrix obtained in Eq. (5.19) predicts the real error covariance fairly accurately.

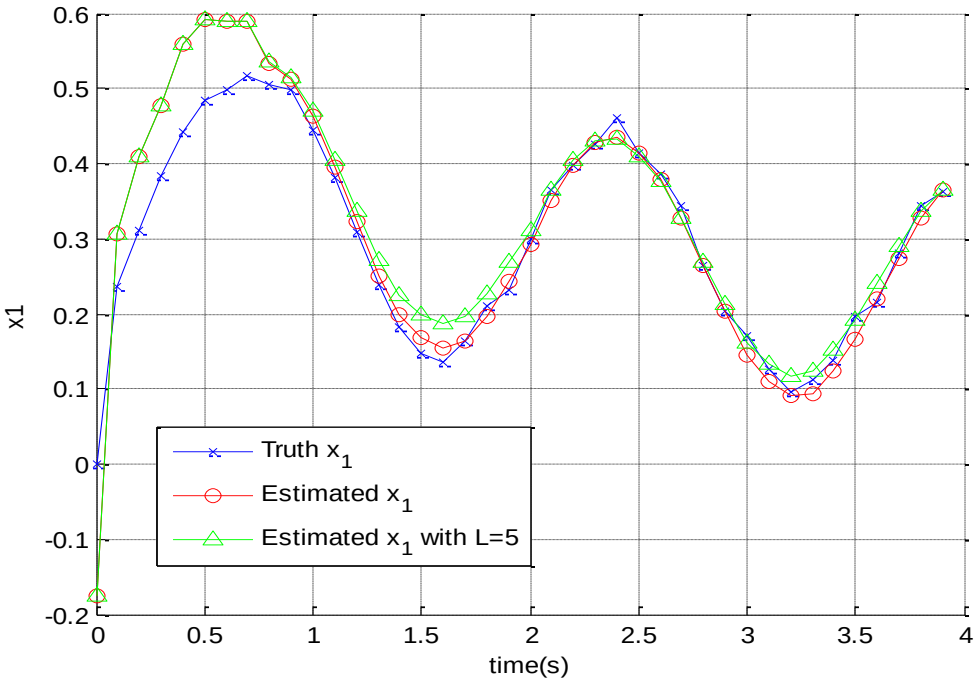


**Figure 5.1**. Estimate of the State $\hat{\mathbf{x}}(1)$

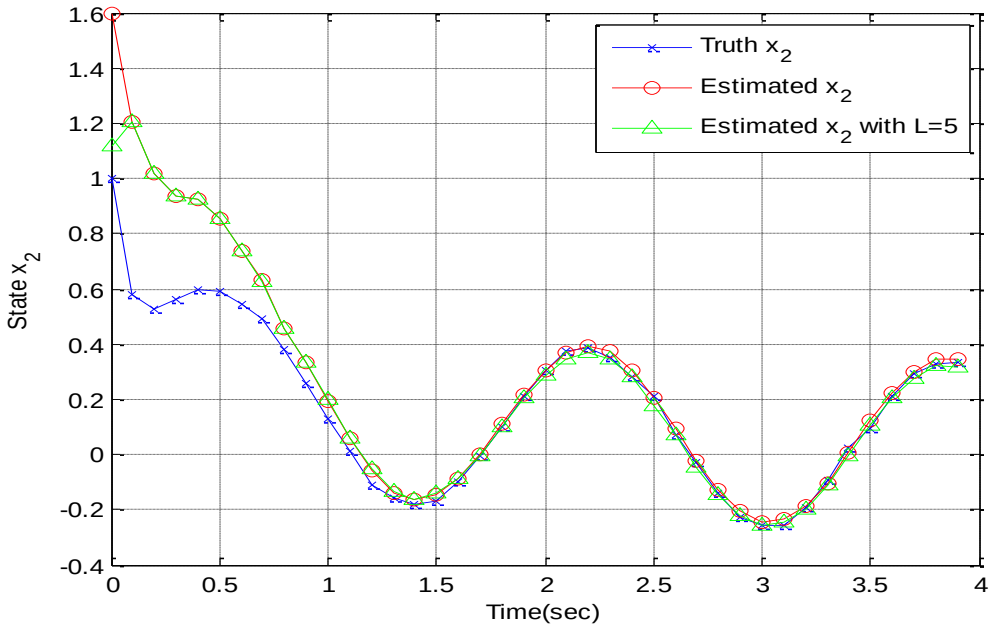


**Figure 5.2**. Estimate of the State $\hat{\mathbf{x}}(2)$

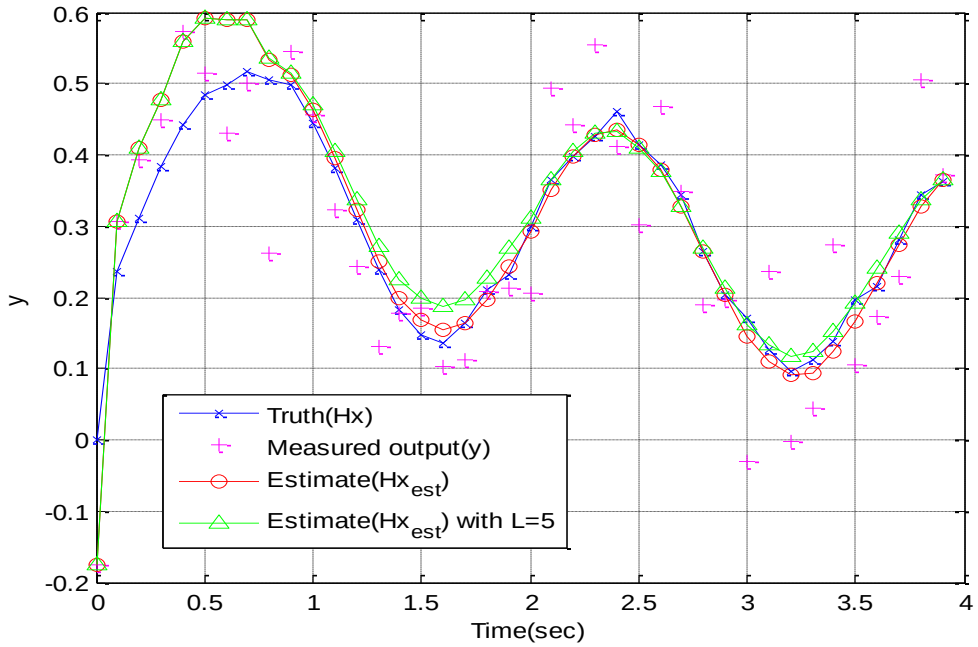


**Figure 5.3**. Measured Output and Outputs with Estimated States

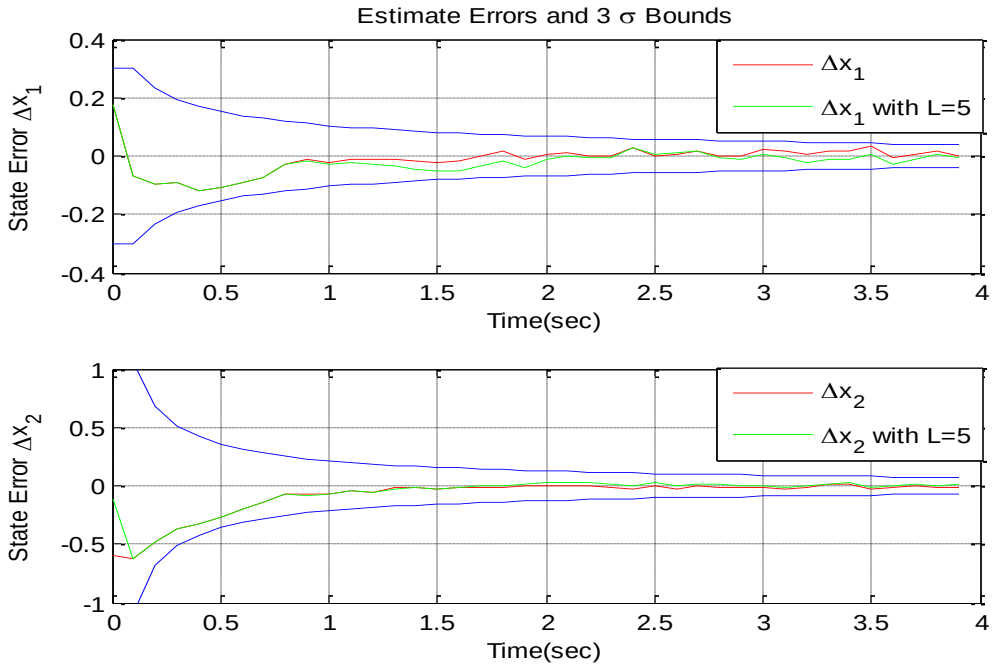


**Figure 5.4**. Estimation Errors and 3σ Bounds

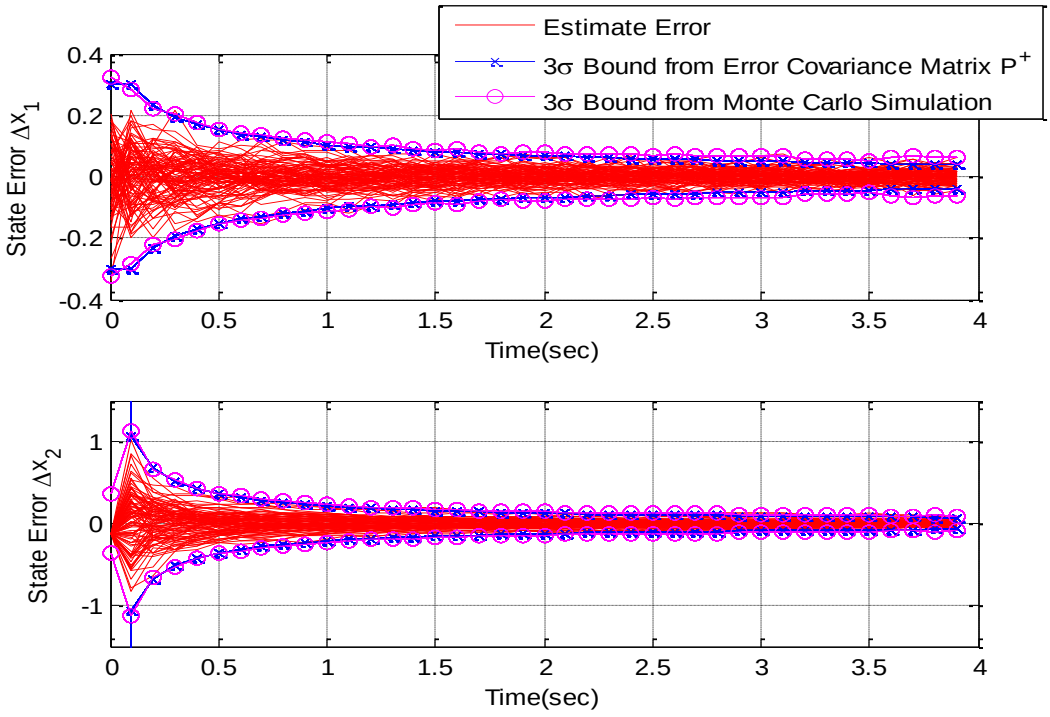


**Figure 5.5**. Ensembles of the Estimation Errors and 3σ Bounds

**Study of the Error Covariance Matrix**

From Eq. (5.18), the error covariance matrix has many previous data information, which leads us to have a very large computational cost and a memory problem. We want to rearrange and regroup the terms of $Q_m, m = k, k-1, \dots, 0$ in order to know the effect of each term of $Q_m$ on $P_{k+1}^-$. The following propagation equation of the error covariance shows this rearrangement.

$$P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + \Gamma\Upsilon Q_k \Upsilon^T \Gamma^T + Q_{k-1}^* + Q_{k-2}^* + \cdots \tag{5.29}$$

where

$$\begin{aligned} Q_{k-1}^* = {} & N_{k,k-1}\Upsilon Q_{k-1}\Upsilon^T N_{k,k-1}^T + \Phi_k(I - K_k H_k)\Gamma\Upsilon Q_{k-1}\Upsilon^T N_{k,k-1}^T \\ & + N_{k,k-1}\Upsilon Q_{k-1}(I - K_k H_k)^T \Phi_k^T \end{aligned} \tag{5.30a}$$

$$\begin{aligned} Q_{k-2}^* = {} & N_{k,k-2}\Upsilon Q_{k-2}\Upsilon^T N_{k,k-2}^T \\ & + \Phi_k(I - K_k H_k)\Phi_{k-1}(I - K_{k-1}H_{k-1}) \\ & \times \Gamma\Upsilon Q_{k-2}\Upsilon^T N_{k,k-2}^T + N_{k,k-2}\Upsilon Q_{k-2} \\ & \times (I - K_{k-1}H_{k-1})^T \Phi_{k-1}^T (I - K_k H_k)^T \Phi_k^T \\ & + \Phi_k(I - K_k H_k) N_{k-1,k-2}\Upsilon Q_{k-2}\Upsilon^T N_{k,k-2}^T \\ & + N_{k,k-2}\Upsilon Q_{k-2}\Upsilon^T N_{k-1,k-2}^T (I - K_k H_k)^T \Phi_k^T \end{aligned} \tag{5.30b}$$

If the effect of the $Q_{k-p-1}^*, Q_{k-p-2}^*, \cdots, Q_0^*$ terms is negligible, we only have to account for a finite number of $Q_m^*, m = k, k-1, \dots, k-p$. This enables us to calculate the propagation equation of the error covariance matrix with finite memory.

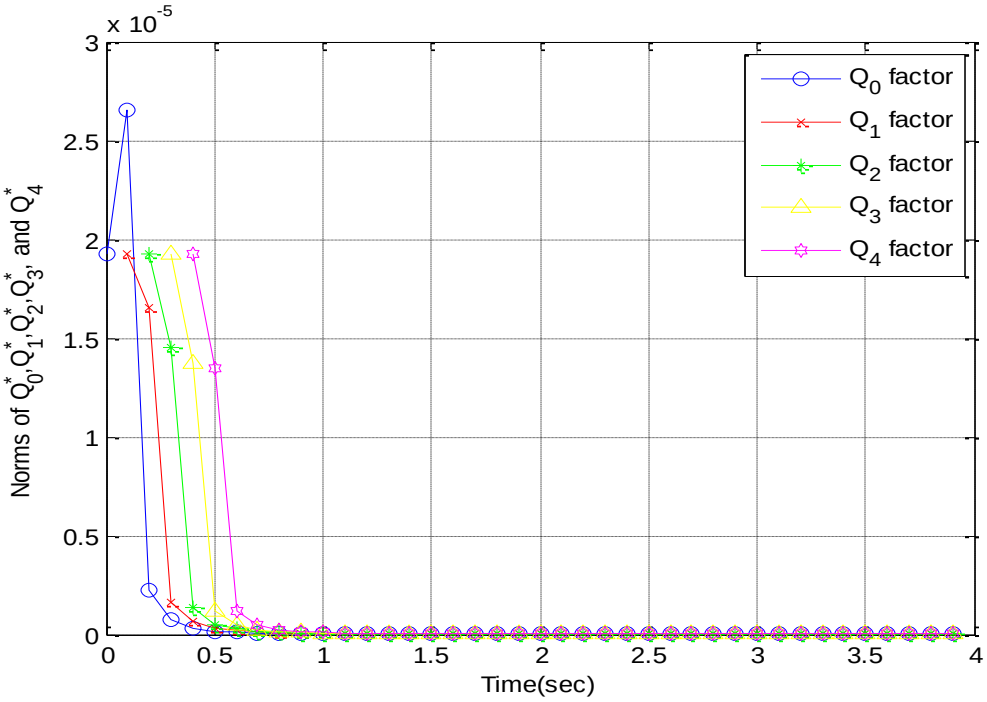


**Figure 5.6**. Effect of $Q_m$ Terms

For the previous numerical example, Fig. (5.6) shows that the effect of $Q_m^*$ after it is populated decreases abruptly in a finite time. For example, $Q_0^*$ is populated at t=0. Then it has the maximum at t=0.1sec and it is decreasing to almost zero after 0.5sec. Like $Q_0^*, Q_1^*$ is populated at t=0.1sec. Then it is decreasing to almost zero from 0.6sec. This means that the effect of $Q_m^*$ 0.5sec later after it is populated is negligible in this example.

**Influence of the Initial Error**

Fractional order systems have memory terms. So, we want to investigate how much the initial error can be influenced into the current time estimate by examining

$E\{\tilde{\mathbf{x}}_k^+\tilde{\mathbf{x}}_0^{+T}\}$. In other words, we want to know how long the initial errors "dominate" the time-varying estimation errors by checking to see if $E\{\tilde{\mathbf{x}}_k^+\tilde{\mathbf{x}}_0^{+T}\}$ is going to be zero for a sufficiently large k. By using Eq. (5.12) we can obtain the relationship between $E\{\tilde{\mathbf{x}}_k^+\tilde{\mathbf{x}}_0^{+T}\}$ and $E\{\tilde{\mathbf{x}}_0^+\tilde{\mathbf{x}}_0^{+T}\}$. Equation (5.31) and (5.32) show it.

$$E\{\tilde{\mathbf{x}}_k^+\tilde{\mathbf{x}}_0^{+T}\} = E\left\{\left((I - K_kH_k)\left(\Phi_{k-1}\tilde{\mathbf{x}}_{k-1}^+ - \Gamma\Upsilon\mathbf{w}_{k-1} - \sum_{m=0}^{k-2} N_{k-1,m}\Upsilon\mathbf{w}_m\right)\right.\right.$$
$$\left.\left. + K_k\boldsymbol{\upsilon}_k\right)\tilde{\mathbf{x}}_0^{+T}\right\} = (I - K_kH_k)\Phi_{k-1}E\{\tilde{\mathbf{x}}_{k-1}^+\tilde{\mathbf{x}}_0^{+T}\} \tag{5.31}$$

By substituting $\tilde{\mathbf{x}}_j^+$ until j is 0 into Eq. (5.31), we can finally obtain the following equation.

$$E\{\tilde{\mathbf{x}}_k^+\tilde{\mathbf{x}}_0^{+T}\} = (I - K_kH_k)\Phi_{k-1}(I - K_{k-1}H_{k-1})\Phi_{k-2} \times \cdots$$
$$\times (I - K_1H_1)\Phi_0E\{\tilde{\mathbf{x}}_0^+\tilde{\mathbf{x}}_0^{+T}\} \tag{5.32}$$

By checking the $(I - K_kH_k)\Phi_{k-1}(I - K_{k-1}H_{k-1})\Phi_{k-2} \times \cdots \times (I - K_1H_1)\Phi_0$ term in Eq. (5.32), we can examine the influence of the initial error. Also, for the previous numerical example, we can obtain $E\{\tilde{\mathbf{x}}_k^+\tilde{\mathbf{x}}_0^{+T}\}$ numericallly. Figure 5.7 shows that if $k > 5$, the norm of $E\{\tilde{\mathbf{x}}_k^+\tilde{\mathbf{x}}_0^{+T}\}$ is quite small.

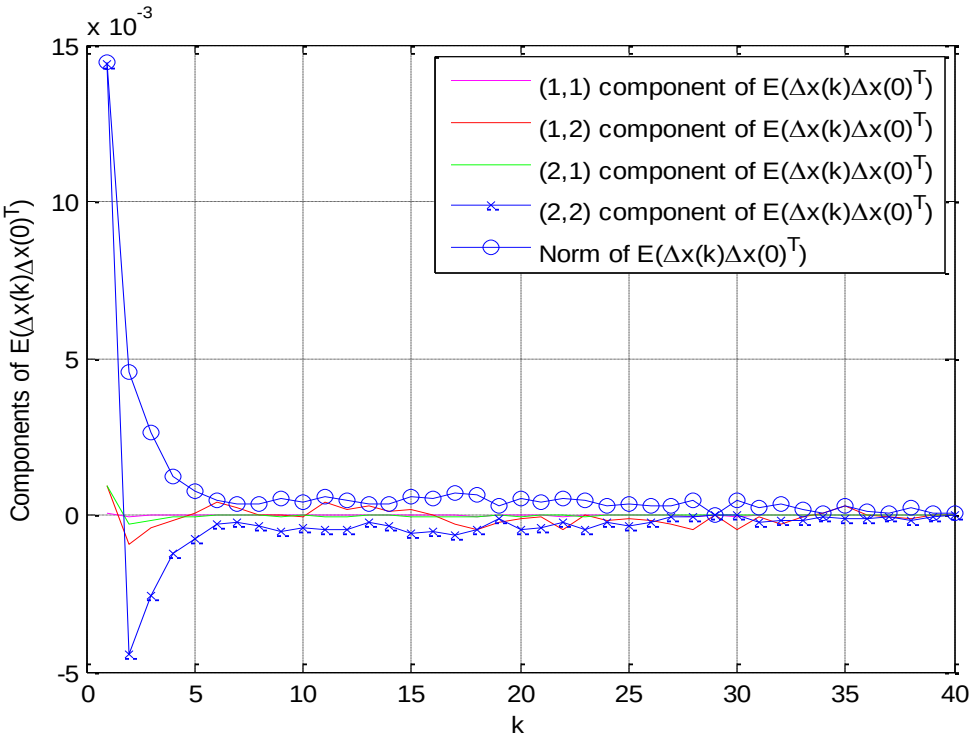


**Figure 5.7**. Norm History of Components of $E\{\tilde{\mathbf{x}}_k^+ \tilde{\mathbf{x}}_0^{+T}\}$

**Unbiased Estimator**

It is important to check to see if this filter is an unbiased estimator. By checking the expectation of the state error, we can investigate the "unbiased" property. From Eq. (5.7), we have

$$E\{\tilde{\mathbf{x}}_{k+1}^-\} = \Phi_k \, E\{\tilde{\mathbf{x}}_k^+\} \tag{5.33}$$

Substitution of Eq. (5.11) into Eq. (5.33) gives us

$$E\{\tilde{\mathbf{x}}_{k+1}^-\} = \Phi_k \, (I - K_k H_k) E\{\tilde{\mathbf{x}}_k^-\} \tag{5.34}$$

By repeating the same procedure, we finally have

$$E\{\tilde{\mathbf{x}}_{k+1}^-\} = \Phi_k \, (I - K_k H_k) \Phi_{k-1} \, (I - K_{k-1} H_{k-1}) \, \ldots \, E\{\tilde{\mathbf{x}}_0^-\} \tag{5.35}$$

This equation means that the estimate error at time k is not biased if the initial error is not biased. For the previous numerical example, we can obtain $E\{\tilde{\mathbf{x}}_k^+\}$ numerically. Figure 5.8 shows that the filter is the "unbiased" estimator.

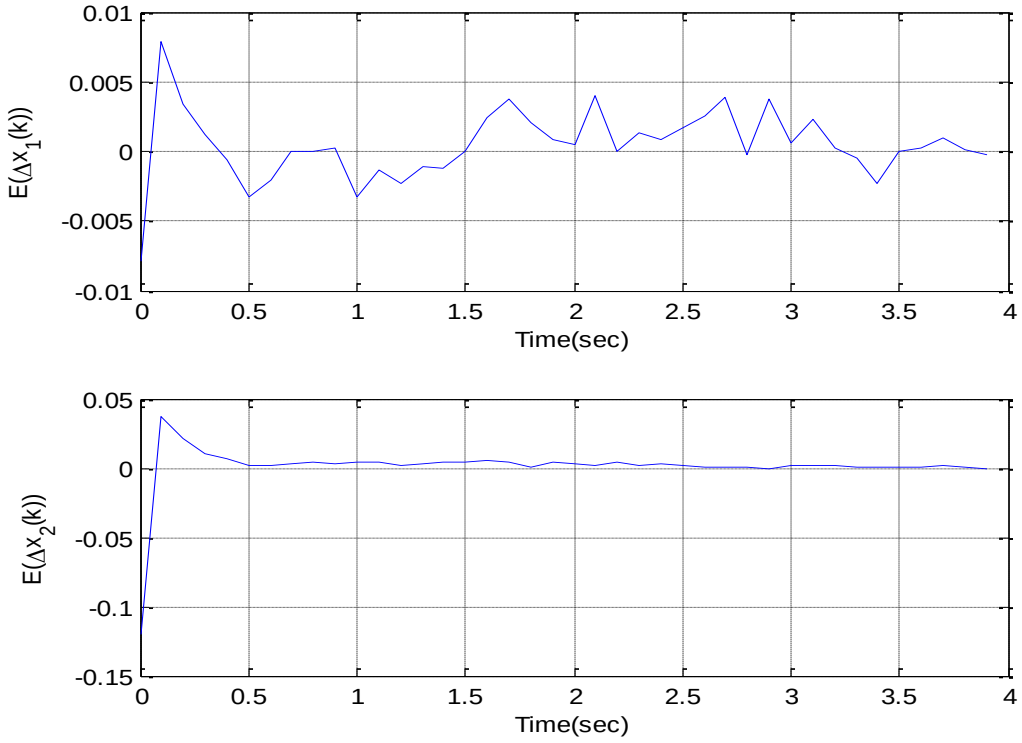


**Figure 5.8**. Norm History of $E\{\tilde{\mathbf{x}}_k^+\}$

**Orthogonality Property**

The standard Kalman filter for the integer order system has the orthogonality of the state estimate and its error[4][37]. It can be shown by

$$E\{\hat{\mathbf{x}}_k^+\tilde{\mathbf{x}}_k^{+T}\} = 0, \quad \forall k \tag{5.36}$$

This means that the state estimate is not correlated with its error. Now, let us examine the Kalman filter for the fractional order system. First, we will show theoretically that the Kalman filter for the fractional order system does not have the orthogonality property, but we expect $\mathrm{E}\{\hat{\mathbf{x}}_k^+\tilde{\mathbf{x}}_k^{+\mathrm{T}}\}$ has very small values although we cannot show it theoretically. This will be shown by the numerical result for the previous numerical example. Therefore, by checking $\mathrm{E}\{\hat{\mathbf{x}}_k^+\tilde{\mathbf{x}}_k^{+\mathrm{T}}\}$ numerically from the Monte Carlo simulation, we can verify that $\mathrm{E}\{\hat{\mathbf{x}}_k^+\tilde{\mathbf{x}}_k^{+\mathrm{T}}\}$ has very small values.

Let us assume that the initial estimate and its error are uncorrelated.

$$\mathrm{E}\{\hat{\mathbf{x}}_0^+\tilde{\mathbf{x}}_0^{+\mathrm{T}}\} = 0 \tag{5.37}$$

In order to examine the "orthogonality" property, we obtain $\mathrm{E}\{\hat{\mathbf{x}}_k^+\tilde{\mathbf{x}}_k^{+\mathrm{T}}\}$ directly. At first, we obtain $\hat{\mathbf{x}}_k^+$ by substituting Eq. (5.3), Eq. (5.4) and Eq. (5.6) into Eq. (5.9).

$$\begin{aligned}\hat{\mathbf{x}}_k^+ = \Phi_{k-1}\hat{\mathbf{x}}_{k-1}^+ + \Gamma B\mathbf{u}_{k-1} + \sum_{m=0}^{k-2} N_{k-1,m}B\mathbf{u}_m \\ + K_k\left(H_k\Gamma\Upsilon\mathbf{w}_{k-1} + H_k\sum_{m=0}^{k-2} N_{k-1,m}\Upsilon\mathbf{w}_m + \boldsymbol{\upsilon}_k\right. \\ \left. - H_k\Phi_{k-1}\tilde{\mathbf{x}}_{k-1}^+\right)\end{aligned} \tag{5.38}$$

And, $\tilde{\mathbf{x}}_k^+$ can be obtained from Eq. (5.12). Then, with $\mathrm{E}\{\hat{\mathbf{x}}_{k-1}^+\mathbf{w}_{k-1}^{\mathrm{T}}\} = \mathrm{E}\{\mathbf{w}_{k-1}\tilde{\mathbf{x}}_{k-1}^{+\mathrm{T}}\} = 0$, $\mathrm{E}\{\hat{\mathbf{x}}_k^+\tilde{\mathbf{x}}_k^{+\mathrm{T}}\}$ can be given by

$$
\begin{aligned}
\mathrm{E}\{\hat{\mathbf{x}}_k^+\tilde{\mathbf{x}}_k^{+\mathrm{T}}\} &= \Phi_{k-1}\mathrm{E}\{\hat{\mathbf{x}}_{k-1}^+\tilde{\mathbf{x}}_{k-1}^{+\mathrm{T}}\}\Phi_{k-1}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_k\mathrm{H}_k)^{\mathrm{T}} \\
&\quad - \Phi_{k-1}\sum_{m=0}^{k-2}\mathrm{E}\{\hat{\mathbf{x}}_{k-1}^+\mathbf{w}_m^{\mathrm{T}}\}\Upsilon^{\mathrm{T}}\mathrm{N}_{k-1,m}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_k\mathrm{H}_k)^{\mathrm{T}} \\
&\quad - \mathrm{K}_k\mathrm{H}_k\Gamma\Upsilon\mathrm{Q}_{k-1}\Upsilon^{\mathrm{T}}\Gamma^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_k\mathrm{H}_k)^{\mathrm{T}} \\
&\quad + \mathrm{K}_k\mathrm{H}_k\sum_{m=0}^{k-2}\mathrm{N}_{k-1,m}\Upsilon\mathrm{E}\{\mathbf{w}_m\tilde{\mathbf{x}}_{k-1}^{+\mathrm{T}}\}\Phi_{k-1}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_k\mathrm{H}_k)^{\mathrm{T}} \\
&\quad - \mathrm{K}_k\mathrm{H}_k\sum_{m=0}^{k-2}\mathrm{N}_{k-1,m}\Upsilon\mathrm{Q}_m\Upsilon^{\mathrm{T}}\mathrm{N}_{k-1,m}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_k\mathrm{H}_k)^{\mathrm{T}} + \mathrm{K}_k\mathrm{R}_k\mathrm{K}_k^{\mathrm{T}} \\
&\quad - \mathrm{K}_k\mathrm{H}_k\Phi_{k-1}\mathrm{E}\{\tilde{\mathbf{x}}_{k-1}^+\tilde{\mathbf{x}}_{k-1}^{+\mathrm{T}}\}\Phi_{k-1}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_k\mathrm{H}_k)^{\mathrm{T}} \\
&\quad + \mathrm{K}_k\mathrm{H}_k\Phi_{k-1}\sum_{m=0}^{k-2}\mathrm{E}\{\tilde{\mathbf{x}}_{k-1}^+\mathbf{w}_m^{\mathrm{T}}\}\Upsilon^{\mathrm{T}}\mathrm{N}_{k-1,m}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_k\mathrm{H}_k)^{\mathrm{T}}
\end{aligned}
\tag{5.39}
$$

To know if this filter has the orthogonality principle, we check Eq. (5.39) from the time step k=1. By substitution of k=1, we can obtain

$$
\begin{aligned}
\mathrm{E}\{\hat{\mathbf{x}}_1^+\tilde{\mathbf{x}}_1^{+\mathrm{T}}\} &= \Phi_0\mathrm{E}\{\hat{\mathbf{x}}_0^+\tilde{\mathbf{x}}_0^{+\mathrm{T}}\}\Phi_0^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_1\mathrm{H}_1)^{\mathrm{T}} - \mathrm{K}_1\mathrm{H}_1\Gamma\Upsilon\mathrm{Q}_0\Upsilon^{\mathrm{T}}\Gamma^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_1\mathrm{H}_1)^{\mathrm{T}} \\
&\quad + \mathrm{K}_1\mathrm{R}_1\mathrm{K}_1^{\mathrm{T}} - \mathrm{K}_1\mathrm{H}_1\Phi_0\mathrm{E}\{\tilde{\mathbf{x}}_0^+\tilde{\mathbf{x}}_0^{+\mathrm{T}}\}\Phi_0^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_1\mathrm{H}_1)^{\mathrm{T}} \\
&= -\mathrm{K}_1\mathrm{H}_1(\Gamma\Upsilon\mathrm{Q}_0\Upsilon^{\mathrm{T}}\Gamma^{\mathrm{T}} + \Phi_0\mathrm{P}_0^+\Phi_0^{\mathrm{T}})(\mathrm{I}-\mathrm{K}_1\mathrm{H}_1)^{\mathrm{T}} + \mathrm{K}_1\mathrm{R}_1\mathrm{K}_1^{\mathrm{T}}
\end{aligned}
\tag{5.40}
$$

By using Eq. (5.18), we can obtain the following relationship

$$
\mathrm{P}_1^- = \Gamma\Upsilon\mathrm{Q}_0\Upsilon^{\mathrm{T}}\Gamma^{\mathrm{T}} + \Phi_0\mathrm{P}_0^+\Phi_0^{\mathrm{T}} \tag{5.41}
$$

Replacing $\Gamma\Upsilon\mathrm{Q}_0\Upsilon^{\mathrm{T}}\Gamma^{\mathrm{T}} + \Phi_0\mathrm{P}_0^+\Phi_0^{\mathrm{T}}$ with $\mathrm{P}_1^-$ yields

$$
\mathrm{E}\{\hat{\mathbf{x}}_1^+\tilde{\mathbf{x}}_1^{+\mathrm{T}}\} = -\mathrm{K}_1\mathrm{H}_1\mathrm{P}_1^-(\mathrm{I}-\mathrm{K}_1\mathrm{H}_1)^{\mathrm{T}} + \mathrm{K}_1\mathrm{R}_1\mathrm{K}_1^{\mathrm{T}} \tag{5.42}
$$

By using Eq. (5.22), $\mathrm{P}_1^-(\mathrm{I}-\mathrm{K}_1\mathrm{H}_1)^{\mathrm{T}}$ can be replaced with $\mathrm{P}_1^+$.

$$\mathrm{E}\{\hat{\mathbf{x}}_1^+\tilde{\mathbf{x}}_1^{+\mathrm{T}}\} = -\mathrm{K}_1\mathrm{H}_1\mathrm{P}_1^+ + \mathrm{K}_1\mathrm{R}_1\mathrm{K}_1^{\mathrm{T}} \tag{5.43}$$

From Eq. (5.21), the gain can be given by

$$\mathrm{K}_1 = \mathrm{P}_1^+\mathrm{H}_1^{\mathrm{T}}\mathrm{R}_1^{-1} \tag{5.44}$$

By the substitution of Eq. (5.43) into Eq. (5.44), we finally obtain $\mathrm{E}\{\hat{\mathbf{x}}_1^+\tilde{\mathbf{x}}_1^{+\mathrm{T}}\} = 0$.

Now, we check the orthogonality at k=2. From Eq. (5.39),

$$\begin{aligned}\mathrm{E}\{\hat{\mathbf{x}}_2^+\tilde{\mathbf{x}}_2^{+\mathrm{T}}\} = {} & \Phi_1\mathrm{E}\{\hat{\mathbf{x}}_1^+\tilde{\mathbf{x}}_1^{+\mathrm{T}}\}\Phi_1^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_2\mathrm{H}_2)^{\mathrm{T}} \\ & - \Phi_1\mathrm{E}\{\hat{\mathbf{x}}_1^+\mathbf{w}_0^{\mathrm{T}}\}\Upsilon^{\mathrm{T}}\mathrm{N}_{1,0}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_2\mathrm{H}_2)^{\mathrm{T}} \\ & - \mathrm{K}_2\mathrm{H}_2\Gamma\Upsilon\mathrm{Q}_1\Upsilon^{\mathrm{T}}\Gamma^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_2\mathrm{H}_2)^{\mathrm{T}} \\ & + \mathrm{K}_2\mathrm{H}_2\mathrm{N}_{1,0}\Upsilon\mathrm{E}\{\mathbf{w}_0\tilde{\mathbf{x}}_1^{+\mathrm{T}}\}\Phi_1^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_2\mathrm{H}_2)^{\mathrm{T}} \\ & - \mathrm{K}_2\mathrm{H}_2\mathrm{N}_{1,0}\Upsilon\mathrm{Q}_0\Upsilon^{\mathrm{T}}\mathrm{N}_{1,0}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_2\mathrm{H}_2)^{\mathrm{T}} + \mathrm{K}_2\mathrm{R}_2\mathrm{K}_2^{\mathrm{T}} \\ & - \mathrm{K}_2\mathrm{H}_2\Phi_1\mathrm{P}_1^+\Phi_1^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_2\mathrm{H}_2)^{\mathrm{T}} \\ & + \mathrm{K}_2\mathrm{H}_2\Phi_1\mathrm{E}\{\tilde{\mathbf{x}}_1^+\mathbf{w}_0^{\mathrm{T}}\}\Upsilon^{\mathrm{T}}\mathrm{N}_{1,0}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_2\mathrm{H}_2)^{\mathrm{T}}\end{aligned} \tag{5.45}$$

By $\hat{\mathbf{x}}_1^+$ and $\tilde{\mathbf{x}}_1^+$ obtained in Eq. (5.38) and Eq. (5.14) respectively, we can obtain the following equations.

$$\mathrm{E}\{\hat{\mathbf{x}}_1^+\mathbf{w}_0^{\mathrm{T}}\} = \mathrm{K}_1\mathrm{H}_1\Gamma\Upsilon\mathrm{Q}_0 \quad \text{and} \quad \mathrm{E}\{\tilde{\mathbf{x}}_1^+\mathbf{w}_0^{\mathrm{T}}\} = (\mathrm{K}_1\mathrm{H}_1-\mathrm{I})\Gamma\Upsilon\mathrm{Q}_0 \tag{5.46}$$

Substituting Eq. (5.46) into Eq. (5.45) yields

$$\begin{aligned}\mathrm{E}\{\hat{\mathbf{x}}_2^+\tilde{\mathbf{x}}_2^{+\mathrm{T}}\} = {} & -\Phi_1\mathrm{K}_1\mathrm{H}_1\Gamma\Upsilon\mathrm{Q}_0\Upsilon^{\mathrm{T}}\mathrm{N}_{1,0}^{\mathrm{T}}(\mathrm{I}-\mathrm{K}_2\mathrm{H}_2)^{\mathrm{T}} + \mathrm{K}_2\mathrm{H}_2 \\ & \times\big(-\Gamma\Upsilon\mathrm{Q}_1\Upsilon^{\mathrm{T}}\Gamma^{\mathrm{T}} + \mathrm{N}_{1,0}\Upsilon\mathrm{Q}_0\Upsilon^{\mathrm{T}}\Gamma^{\mathrm{T}}(\mathrm{K}_1\mathrm{H}_1-\mathrm{I})^{\mathrm{T}}\Phi_1^{\mathrm{T}} \\ & - \mathrm{N}_{1,0}\Upsilon\mathrm{Q}_0\Upsilon^{\mathrm{T}}\mathrm{N}_{1,0}^{\mathrm{T}} - \Phi_1\mathrm{P}_1^+\Phi_1^{\mathrm{T}} + \Phi_1(\mathrm{K}_1\mathrm{H}_1-\mathrm{I})\Gamma\Upsilon\mathrm{Q}_0\Upsilon^{\mathrm{T}}\mathrm{N}_{1,0}^{\mathrm{T}}\big) \\ & \times(\mathrm{I}-\mathrm{K}_2\mathrm{H}_2)^{\mathrm{T}} + \mathrm{K}_2\mathrm{R}_2\mathrm{K}_2^{\mathrm{T}}\end{aligned} \tag{5.47}$$

By using Eq. (5.18), we can obtain the relationship

$$P_2^- = \Gamma\Upsilon Q_1 \Upsilon^T \Gamma^T - N_{1,0}\Upsilon Q_0 \Upsilon^T \Gamma^T (K_1 H_1 - I)^T \Phi_1^T + N_{1,0}\Upsilon Q_0 \Upsilon^T N_{1,0}^T + \Phi_1 P_1^+ \Phi_1^T - \Phi_1 (K_1 H_1 - I)\Gamma\Upsilon Q_0 \Upsilon^T N_{1,0}^T \quad (5.48)$$

Then,

$$\begin{aligned} E\{\hat{\mathbf{x}}_2^+ \tilde{\mathbf{x}}_2^{+T}\} &= -\Phi_1 K_1 H_1 \Gamma\Upsilon Q_0 \Upsilon^T N_{1,0}^T (I - K_2 H_2)^T \\ &\quad + K_2 H_2 P_2^- (I - K_2 H_2)^T + K_2 R_2 K_2^T \\ &= -\Phi_1 K_1 H_1 \Gamma\Upsilon Q_0 \Upsilon^T N_{1,0}^T (I - K_2 H_2)^T \end{aligned} \quad (5.49)$$

Since $K_2 H_2 P_2^- (I - K_2 H_2)^T + K_2 R_2 K_2^T = 0$, we finally have

$$E\{\hat{\mathbf{x}}_2^+ \tilde{\mathbf{x}}_2^{+T}\} = -\Phi_1 K_1 H_1 \Gamma\Upsilon Q_0 \Upsilon^T N_{1,0}^T (I - K_2 H_2)^T \quad (5.50)$$

Therefore, the filter does not have the orthogonality property between the state estimate and its error. However, we can check its correlation numerically. Figure 5.9 shows that $E\{\hat{\mathbf{x}}_k^+ \tilde{\mathbf{x}}_k^{+T}\}$ is very small after k is greater than 5.

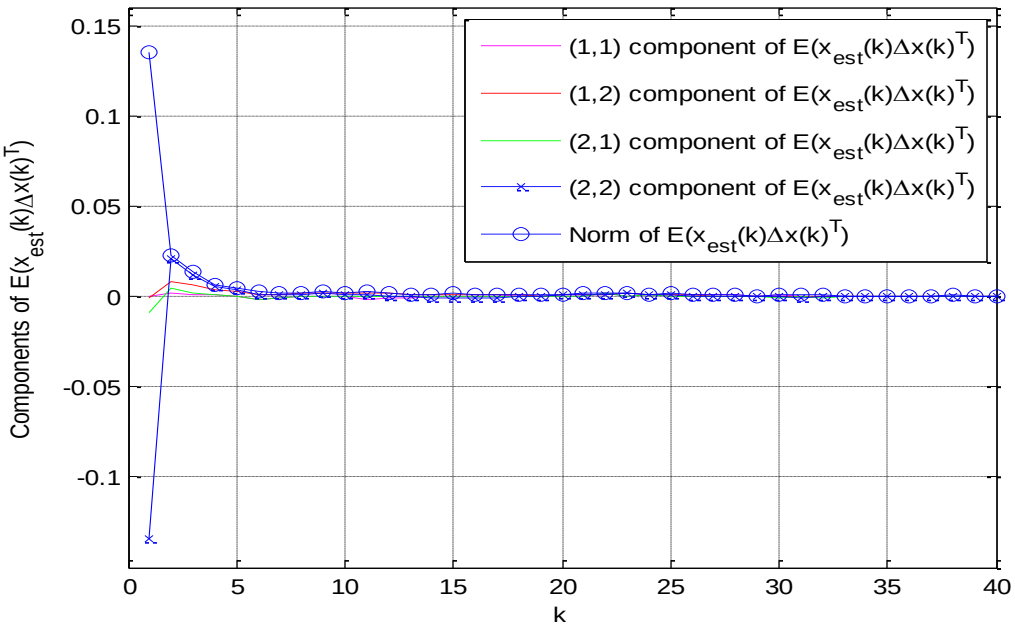


**Figure 5.9**. Norm History of Components of $E\{\hat{\mathbf{x}}_k^+ \tilde{\mathbf{x}}_k^{+T}\}$

**Stability of the Discrete-Time Fractional Kalman Filter**

The stability of the Kalman filter for the fractional order system can be checked by the Lyapunov's direct method[4]. We want to examine if the state error is stable. Let us have the following Lyapunov function.

$$V(\tilde{\mathbf{x}}) = \tilde{\mathbf{x}}_k^T P_k^{-1} \tilde{\mathbf{x}}_k \tag{5.51}$$

This Lyapunov function should be positive, which means that $P_k$ and $P_k^{-1}$ should be positive definite. The Lyapunov's direct method tells that if the increment of the Lyapunov function is negative, the state error is stable. The increment of the Lyapunov function is given by

$$\Delta V(\tilde{\mathbf{x}}) = V_{k+1} - V_k = \tilde{\mathbf{x}}_{k+1}^T P_{k+1}^{-1} \tilde{\mathbf{x}}_{k+1} - \tilde{\mathbf{x}}_k^T P_k^{-1} \tilde{\mathbf{x}}_k \tag{5.52}$$

Let $\tilde{\mathbf{x}}_{k+1} = \tilde{\mathbf{x}}_{k+1}^-$ and $\tilde{\mathbf{x}}_k = \tilde{\mathbf{x}}_k^-$. Then the substitution of Eq. (5.10) into Eq. (5.7) gives us the following equation.

$$\tilde{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1} - \mathbf{x}_{k+1} = \Phi_k (I - K_k H_k) \tilde{\mathbf{x}}_k + \Phi_k K_k \boldsymbol{\upsilon}_k - \Gamma \Upsilon \mathbf{w}_k - \sum_{m=0}^{k-1} N_m \Upsilon \mathbf{w}_m \tag{5.53}$$

Now, we only consider the homogeneous part, $\Phi_k (I - K_k H_k) \tilde{\mathbf{x}}_k$, of Eq. (5.52). The increment of the Lyapunov function can be given by

$$\begin{aligned} \Delta V(\tilde{\mathbf{x}}) &= \tilde{\mathbf{x}}_{k+1}^T P_{k+1}^{-1} \tilde{\mathbf{x}}_{k+1} - \tilde{\mathbf{x}}_k^T P_k^{-1} \tilde{\mathbf{x}}_k \\ &= \tilde{\mathbf{x}}_k^T \left( (I - K_k H_k)^T \Phi_k^T P_{k+1}^{-1} \Phi_k (I - K_k H_k) - P_k^{-1} \right) \tilde{\mathbf{x}}_k \end{aligned} \tag{5.54}$$

This increment should be negative for a stable solution. So we have

$$(I - K_k H_k)^T \Phi_k^T P_{k+1}^{-1} \Phi_k (I - K_k H_k) - P_k^{-1} < 0 \tag{5.55}$$

By pre-multiplying $\Phi_k^{-T} (I - K_k H_k)^{-T}$ and post-multiplying $(I - K_k H_k)^{-1} \Phi_k^{-1}$, we have

$$P_{k+1}^{-1} - \Phi_k^{-T}(I - K_k H_k)^{-T} P_k^{-1}(I - K_k H_k)^{-1}\Phi_k^{-1} < 0 \tag{5.56}$$

Then the premultiplication of $P_{k+1}$ gives us

$$I - P_{k+1}\Phi_k^{-T}(I - K_k H_k)^{-T} P_k^{-1}(I - K_k H_k)^{-1}\Phi_k^{-1} < 0 \tag{5.57}$$

By the substitution of Eq. (5.19) into Eq. (5.33), we have

$$\begin{aligned} P_{k+1} = {} & \Phi_k(I - K_k H_k)P_k(I - K_k H_k)^T\Phi_k^T + \Phi_k K_k R_k K_k^T \Phi_k^T \\ & + \Gamma\Upsilon Q_k \Upsilon^T \Gamma^T + Q_{k-1}^* + Q_{k-2}^* + \cdots \end{aligned} \tag{5.58}$$

Substitution of Eq. (5.58) into Eq. (5.57) gives

$$\begin{aligned} I - \big(& \Phi_k(I - K_k H_k)P_k(I - K_k H_k)^T\Phi_k^T + \Phi_k K_k R_k K_k^T \Phi_k^T \\ & + \Gamma\Upsilon Q_k \Upsilon^T \Gamma^T + Q_{k-1}^* + Q_{k-2}^* + \cdots \big) \\ & \times \Phi_k^{-T}(I - K_k H_k)^{-T} P_k^{-1}(I - K_k H_k)^{-1}\Phi_k^{-1} < 0 \end{aligned} \tag{5.59}$$

Then we have

$$\begin{aligned} -\big(& \Phi_k K_k R_k K_k^T \Phi_k^T + \Gamma\Upsilon Q_k \Upsilon^T \Gamma^T + Q_{k-1}^* + Q_{k-2}^* + \cdots \big) \\ & \times \Phi_k^{-T}(I - K_k H_k)^{-T} P_k^{-1}(I - K_k H_k)^{-1}\Phi_k^{-1} < 0 \end{aligned} \tag{5.60}$$

Since $\Phi_k^{-T}(I - K_k H_k)^{-T} P_k^{-1}(I - K_k H_k)^{-1}\Phi_k^{-1}$ is positive definite, the following condition should be satisfied.

$$\big(\Phi_k K_k R_k K_k^T \Phi_k^T + \Gamma\Upsilon Q_k \Upsilon^T \Gamma^T + Q_{k-1}^* + Q_{k-2}^* + \cdots\big) > 0 \tag{5.61}$$

Although $R_k$ and $Q_k$ is positive definite, we still need each $Q_m^*, m = k-1, \ldots, 0$ is positive semi-definite in order to guarantee the stability of the filter. Or the either following conditions should be satisfied.

$$\big(\Phi_k K_k R_k K_k^T \Phi_k^T + \Gamma\Upsilon Q_k \Upsilon^T \Gamma^T\big) > -(Q_{k-1}^* + Q_{k-2}^* + \cdots) \tag{5.62a}$$

$$\Gamma\Upsilon Q_k \Upsilon^T \Gamma^T > -(Q_{k-1}^* + Q_{k-2}^* + \cdots) \tag{5.62b}$$

Also, $P_k$ must remain positive definite. Now, we want to show if $P_k$ is positive definite $P_{k+1}$is also positive definite. By substitution of Eq. (5.22) into Eq. (5.51), we have

$$P_{k+1} = \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \Gamma \Upsilon Q_k \Upsilon^T \Gamma^T + Q_{k-1}^* + Q_{k-2}^* + \cdots > 0 \tag{5.63}$$

By using Eq. (5.62a), we can have

$$\Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T - \Phi_k K_k R_k K_k^T \Phi_k^T > 0 \tag{5.64}$$

And, if $Q_m^*, m = k-1, \ldots, 0$ is positive semi-definite or Eq. (5.62b) is satisfied, we have

$$\Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T > 0 \tag{5.65}$$

Then,

$$P_k > K_k H_k P_k \tag{5.66}$$

By substituting Eq. (5.21) into Eq. (5.66), we have

$$P_k > P_k H_k^T \left(H_k P_k H_k^T + R_k\right)^{-1} H_k P_k \tag{5.67}$$

By pre-multiplying $H_k$ and post-multiplying $H_k^T$, we can have

$$H_k P_k H_k^T > H_k P_k H_k^T \left(H_k P_k H_k^T + R_k\right)^{-1} H_k P_k H_k^T \tag{5.68}$$

From the above equation, we can obtain

$$H_k P_k H_k^T + R_k > H_k P_k H_k^T \tag{5.69}$$

Since $R_k$ is positive definite, this condition is satisfied. Note that this condition is satisfied only if $Q_m^*, m = k-1, \ldots, 0$ or Eq. (5.62b) is valid. Otherwise, we have to check if Eq. (5.62a) and Eq. (5.65) is satisfied for the stability of the filter. Therefore, if $Q_m^*, m = k-1, \ldots, 0$ is positive definite or Eq. (5.62b) is satisfied, the stability of the filter is guaranteed.

# CHAPTER VI

# CONCLUSIONS

In this dissertation, we built the base methodology with a focus on robust feedback control and state estimation for fractional order systems described by Caputo's definition. This includes systems where fractional order models do not arise "naturally" but can be an option in the quest for more general feedback control laws. As mentioned before, Caputo definition was chosen because it requires *only integer-order initial conditions* to obtain the solution of the fractional order differential equation and it enables us to define the fractional-order initial conditions physically.

To achieve this objective, we first built up the foundations of the needed mathematical properties and concepts such as the linearity, composition rules, Leibniz rules and short memory principle and so on for Caputo's fractional derivatives.

With these foundations, we could transform any fractional-order differential equations described by Caputo definition into the state-space representation. If we have a very small common fractional order, we have to worry about the *curse of dimensionality*. In order to avoid this problem, one needs to use an incommensurate order, but, for the convenience of analysis we examined only a commensurate order case, opening up possibilities for further research into an incommensurate order.

From the state-space representation, we showed how to obtain the eigenvalue and eigenvector of the Caputo fractional order system and check the stability by using the argument of eigenvalues. And we constructed the linear system theory for the Caputo

fractional order system. In doing so, we examined the controllability and the observability for fractional order systems. Two kinds of discrete-time approximate solutions were also developed by utilizing the general solution forms of the fractional differential equation. They gave us the good numerical results when compared to the existing method and an analytical solution. All these works become the useful mathematical tools for the robust eigenstructure assignment and the Kalman filter for fractional order systems.

We draw some useful results about the eigenvalue and eigenvector sensitivity under the perturbations in fractional order systems from the analogy between the integer order system and the fractional order system about. Condition number used in integer order case for measuring the robust stability still plays an important role in the fractional order case analogous to the integer derivative case. Also we found the weighted robust stability measure in order to weigh the factor to maximize the stability margin by calculating the inverse of the distance between the locations of the eigenvalues and the stability lines. We found the relationship between the perturbation to the closed-loop system matrix and the condition number of the closed loop systems modal(eigenvector) matrix. And we found the upper bound of perturbation to make the fractional order system stay stable. For the applications, it can be used for the controller parameter and it also gave us the stability region of the bilinear systems.

With the robust stability measures obtained previously, we addressed the robust eigenstructure assignment problem for the fractional order system which has already been widely and successfully used in the integer order cases. We developed the new

algorithm based on the n-dimensitional rotation for the robust eigenstructure assignment problem in a novel way. This algorithm can be applied to the integer order system as well as the fractional order system. When compared to the existing methods numerically, our new algorithm also gave us good results. Our algorithm was made even more effective when we used weighted robust stability since we could yield a numerical result that the algorithm developed in Reference [34] could not provide.

For the state estimation of the fractional order system defined by Caputo definition, we derived the discrete-time fractional Kalman filter. In contradistinction to the Fractional Kalman filter developed by using Gründwald-Letnikov definition, the new Fractional Kalman filter that we established by utilizing Caputo definition provides us with new and powerful means for solving practical state estimation problems for fractional order systems. Also, we investigated the properties of the filter such as the influence of the initial error, the "unbiased" property, the "near" orthogonality property and the stability of the discrete-time fractional Kalman filter. All these properties were examined numerically by a Monte Carlo simulation. For the continuous fractional Kalman filter, it is difficult to derive the propagation equation of the error covariance matrix. If we want to obtain it approximately, we can use the discrete-time approximate model for the propagation equation. Further research into a continuous fractional Kalman filter is required.

## REFERENCES

[1] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999

[2] Podlubny, I., "Fractional order systems and $PI^{\lambda}D^{\mu}$ controller," *IEEE Transaction on Automatic Control*, Vol. 44, No. 2, 1999, pp. 208-214.

[3] Junkins, J. L. and Kim, Y., *Introduction to Dynamics and Control of Flexible Structures*, AIAA, Wahshington, DC, 1993

[4] Crassidis, J. L. and Junkins, J. L., *Optimal Estimation of Dynamic Systems*, Chapman & Hall/CRC, Boca Raton, FL, 2004

[5] Mortari, D., "On the rigid rotation concept in n-dimensional spaces," *The Journal of the Astronautical Sciences*, Vol. 49, No. 3, 2001, pp. 401-429.

[6] Kautsky, J., Nichols, N. K., and Van Dooren, P., "Robust pole assignment in linear state feedback," *Int. J. Control,* Vol. 41, No. 5, 1985, pp. 1129-1155.

[7] Diethelm, K., and Ford, N. J., "Numerical solution of the Bagley Torvik equation," *BIT Numerical Mathematics*, Vol. 42, No. 3, 2002, pp. 490-507.

[8] Hartley, T. T. and Lorenzo, C. F., "Initialized fractional calculus," NASA TP-2000-209943.

[9] Bagley, R. L. and Calico, R. A., "Fractional order state equations for the control of viscoelastically damped structures," *Journal of Guidance, Control and Dynamcis*, Vol. 14, No. 2, 1991, pp. 304-311.

[10] Leszczynski, J. and Ciesielski, M., "A numerical method for solution of ordinary differential equations of fractional order," *In Parallel Processing and Applied*

*Mathematics (PPAM2001) Conference, under publishing in Lecture Notes in Computer Science*, Springer Verlag, 2001, pp. 695-702.

[11] Ford, N. J. and Simpson, A. C., "The numerical solution of fractional differential equations: Speed versus accuracy," *Numer. Algorithms*, Vol. 26, 2001, pp. 336-346.

[12] Matignon, D., "Stability properties for generalized fractional differential systems," in *Proc. Of Fractional Differential Systems: Models, Methods and Applications*, 1998, pp. 145-158.

[13] Sierociuk, D. and Dzielinski, A., "Fractional Kalman filter algorithm for the states, parameters and order of fractional system estimation," *Int. J. Appl. Math. Comput. Sci*. Vol. 16, No. 1, 2006, pp. 129-140.

[14] Diethelm, K. and Ford, N. J., "Multi-order fractional differential equations and their numerical solution," *Applied Mathematics and Computation*, Vol. 154, 2004, pp. 621-640.

[15] Deng, W., Li, C. and Lü J., "Stability analysis of linear fractional differential system with multiple time delays," *Nonlinear Dynamics*, Vol. 48, 2007, pp. 409-416.

[16] Hartley, T. T. and Lorenzo, C. F., "A solution to the fundamental linear fractional order differential equation," NASA TP-1998-208693.

[17] Hartley, T. T. and Lorenzo, C. F., "Dynamics and control of initialized fractional-order systems," *Nonlinear Dynamics*, Vol. 29, 2002, pp. 201-233.

[18] Aoun, M., Malti, R., Levron, F., and Oustaloup, A., "Numerical simulations of fractional systems: An overview of existing methods and improvements," *Nonlinear Dynamics*, Vol. 38, 2004, pp. 117-131.

[19] Chen, Y., Ahn, H., and Xue, D., "Robust controllability of interval fractional order linear time invariant systems," *Signal Processing*, Vol. 86, 2006, pp. 2794-2802.

[20] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006

[21] Diethelm, K., Ford, N., and Freedm, A., "A predictor-corrector approach for the numerical solution of fractional differential equations," *Nonlinear Dynamics*, Vol. 29, 2002, pp. 3-22.

[22] Agrawal, O. P. and Kumar, P., "Comparison of five numerical schemes for fractional differential equations," *J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, pp. 43-60, Springer, Dordrecht, Netherlands, 2007

[23] El-Khazali, R. and Momani, S., "Stability analysis of composite fractional systems," *International Journal of Applied Mathematics*, Vol. 12, No. 1, 2003, pp. 73-85.

[24] Moze, M., Sabatier, J. and Oustaloup, A., "LMI characterization of fractional systems stability," *J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, pp. 419-434, Springer, Dordrecht, Netherlands, 2007

[25] Lim, K. B. and Junkins, J. L., "Probability of stability: New measures of stablity robustness for linear dynamical systems," *The Journal of the Astronautical Sciences*, Vol. 35, No. 4, October-December, 1987, pp. 383-397.

[26] Bauer, F. L. and Fike, C. T., "Norms and exclusion theorems," *Numerische Mathematik*, Vol. 2, 1960, pp. 137-141.

[27] Wilkinson, J. H., The Algebraic Eigenvalue Problem, Oxford University Press, New York, 1965.

[28] Datta, B. N., Numerical Methods for Linear Control Systems, Elsevier, Academic Press, Amsterdam, 2004.

[29] Patel, R. V. and Toda, M., "Quantitative measures of robustness for multivariable systems," *in Proc. Joint Automat. Contr. Conf*., San Francisco, CA, 1980, TP8-A.

[30] Wen, X., Wu, Z., and Lu, J., "Stability analysis of a class of nonlinear fractional-order systems," *IEEE Trans. On Circuits and Systems II:Express Briefs*, Vol. 55, No. 11, November, 2008, pp. 1178-1182.

[31] Lin, W., "Global existence theory and chaos control of fractional differential equations," *J. Math. Anal. Appl.,* Vol. 332, 2007, pp. 709-726.

[32] Pachpatte, B. G., Inequalities for Differential and Integral Equations, Academic Press, San Diego, CA, 1998.

[33] Mahmudov, A. P. and Musaev, V. M., "On the theory of solutions of nonlinear Volterra Urysohn integral equations," Akad. Nauk. Azerbaidzan SSR Dokl. Vol. 25, pp. 3-6, 1969.

[34] Byers, R. and Nash, S. G., "Approaches to robust pole assignment," *Int. J. Control,* Vol. 49, No. 1, 1989, pp. 97-117.

[35] Rao, S. S., Engineering Optimization, John Wiley & Sons, Inc., New York, 1996.

[36] Fletcher, R., Practical Methods of Optimization, John Wiley & Sons, Inc., New York, 1987.

[37] Gelb, A., Applied Optimal Estimation, The Analytic Sciences Corporation, Cambridge, MA, 1974.

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