EXTREMAL FIELDS AND NEIGHBORING OPTIMAL CONTROL OF CONSTRAINED SYSTEMS

A Thesis

by

MATTHEW WADE HARRIS

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 2010

Major Subject: Aerospace Engineering

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ABSTRACT

Extremal Fields and Neighboring Optimal Control of Constrained Systems.

(December 2010)

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Chair of Advisory Committee: Dr. John Valasek

This work provides first and second-order expressions to approximate neighboring solutions to the m-point boundary value problem. Multi-point problems arise in optimal control because of interior constraints or switching dynamics. Many problems have this form, and so this work fills a void in the study of extremal fields and neighboring optimal control of constrained systems. Only first and second-order terms are written down, but the approach is systematic and higher order expressions can be found similarly. The constraints and their parameters define an extremal field because any solution to the problem must satisfy the constraints. The approach is to build a Taylor series using constraint differentials, state differentials, and state variations. The differential is key to these developments, and it is a unifying element in the optimization of points, optimal control, and neighboring optimal control. The method is demonstrated on several types of problems including lunar descent, which has nonlinear dynamics, bounded thrust, and free final time. The control structure is bang-off-bang, and the method successfully approximates the unknown initial conditions, switch times, and final time. Compared to indirect shooting, computation time decreases by about three orders of magnitude.

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CHAPTER I
INTRODUCTION

Optimal control theory provides a systematic way to determine the controls so that a process satisfies all constraints in the most efficient manner. The theory is applicable to many problems in mathematics, physics, and engineering. Application of optimal control to linear systems is mostly understood because problems that result have analytical solutions. Application of optimal control to nonlinear systems is less understood because many problems that result do not have analytical solutions. In this case, one must resort to numerical methods [1]. Indirect shooting [2] is used in this work.

Optimal control problems that do not have interior constraints are called unconstrained, and they can be solved as two-point boundary value problems. In spite of their name, these problems still have constraints at the initial and final time. Optimal control problems that do have interior constraints are called constrained, and they can be solved as multi-point boundary value problems. In either case, some combination of analytical and numerical methods can be used to solve the problem. However, one may also be interested in generating families of solutions or solving neighboring optimal control problems.

The meaning of an extremal field depends upon the context. Commonly, one refers to an extremal field as a family of extremal solutions, and it is generally assumed that the solutions are similar and local to each other. Another definition considers an extremal field as a family of constraints that define a problem. This work is motivated

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This thesis follows the style of *Journal of Guidance, Control, and Dynamics.*

by some mixture of the definitions. The goal is to find neighboring solutions to constrained optimal control problems, but the approach is to compute differentials of constraints. The differentials form a Taylor series approximation local to a reference solution. Such an approach is known as neighboring optimal control.

The study of extremal fields and neighboring optimal control is not new. These topics originated with calculus of variations in the 1600s [3], and modern extensions followed the development of optimal control theory in the 1960s. One approach to neighboring optimal control is to solve the accessory optimization problem [4] whereby one minimizes the second variation subject to constraints linearized about a reference solution. The result is a first-order approximation. A second approach is to consider the Euler-Lagrange equations and compute variations about a reference solution [5]. First-order variations result in a first-order approximation. Second-order variations result in a second-order approximation, and so on. The two approaches are fundamentally the same. Choosing one is mostly a matter of personal preference, however, the second approach more easily permits higher order approximations. Work in the 1960s focused on unconstrained problems that can be solved as two-point boundary value problems.

Every real system, however, is subject to constraints. Temperature cannot be less than absolute zero, mass cannot be negative, thrust cannot be infinite, and so on. Optimal solutions may or may not encounter the constraints, and the solution can be made of any number of constrained and unconstrained arcs. Predicting the number of arcs and switch times is difficult and an area of ongoing research [6–9]. Neighboring optimal control of constrained systems is still possible under certain restrictions discussed later. In the late 1960s and 1970s, the accessory optimization approach was extended to constrained, multi-point boundary value problems [10–12]. In the 1980s, the approach was successfully applied as a guidance scheme for a constrained

Space Shuttle reentry [13–15]. The work addressed several issues regarding prediction of switching times, trajectory tracking, and numerical implementation. Neighboring optimal control as guidance requires storage of the reference solution and state transition matrices. Large storage requirements and frequent table look-up have made real-time implementation infeasible for fast systems. Higher order approximations further increase the burden.

Nonetheless, technology is always improving and higher approximations reduce truncation error. Research in the early 2000s focused on derivative computation for two-point boundary value problems. There are at least three methods for computing derivatives: automatic differentiation [16], differential algebra [17], and complex differentiation [18]. Algebraic equations, feedback control, and unconstrained optimization are a few applications for these methods [19–21].

To summarize, previous work focused on first and higher order approximations for unconstrained problems and first-order approximations for constrained problems. This research develops first and second-order expressions to approximate solutions to neighboring optimal control problems with constraints. Only first and second-order terms are written down, but the approach is systematic and higher order terms can be found similarly. Constraints naturally occur in every real system, and this work helps fill a void in the study of extremal fields and neighboring optimal control of constrained systems.

The approach is to find a reference solution, compute differentials of constraints, and obtain approximate solutions using a Taylor series. This work is based on two assumptions. First, the reference and neighboring solutions satisfy the strengthened Legendre-Clebsch condition. Second, the neighboring solutions are optimal in the same sense as the reference. Extensive use of the differential is made, and it is noted that the differential serves a unifying role in the optimization of points, optimal control, and neighboring optimal control.

Four types of problems are investigated: 1) fixed final time, 2) free final time, 3) control constrained, and 4) state constrained. These problems include characteristics common to many optimal control problems such as unknown initial conditions, switch times, interior jumps, and final time. The method is then demonstrated with the lunar descent problem, which has nonlinear dynamics, bounded thrust, and free final time. The problem is one of historical significance and current interest. As stated by Apollo engineers [22]:

> The powered descent and landing on the lunar surface from lunar orbit is perhaps the most critical phase of the lunar-landing mission. Because of the large effect of weight upon the booster requirements of the earth launch and upon the payload delivered to the lunar surface, the weight of the fuel expended during powered descent and landing must be minimized.

Chapter II introduces fundamental concepts relevant to the research. Chapter III develops the method to solve neighboring optimal control problems with constraints. Chapter IV solves the four example problems. Chapter V introduces and solves the lunar descent problem. Finally, Chapter VI summarizes the results and draws conclusions.

CHAPTER II

PRELIMINARIES

A. Differentials

Differentials are critical elements in this research. They provide a convenient way to write derivatives and build Taylor series for algebraic, differential, and integral equations [23]. Naturally, they play an important role in the optimization of points, optimal control, and neighboring optimal control. Necessary conditions are derived by setting the first differential to zero, and sufficiency is verified by checking the positiveness of the second differential [24]. Appendix A provides additional information regarding functions, their derivatives, and the Kronecker product.

1. Differentials of Algebraic Equations

Consider the algebraic equation $y = f(x)$. Let x be the independent variable and let y be the dependent variable. The total change in y is

$$\Delta y = \mathrm{d}y + \tfrac{1}{2}\mathrm{d}^2 y + \cdots \,. \tag{2.1}$$

The differentials of y are

$$\mathrm{d}y = f_x \mathrm{d}x \tag{2.2}$$

$$\mathrm{d}^2 y = f_{xx}(\mathrm{d}x \cdot \mathrm{d}x) \tag{2.3}$$

and so on. Note that differentials of independent differentials are zero, i.e., $\mathrm{d}(\mathrm{d}x) = \mathrm{d}^2 x = 0$. It follows that

$$\Delta y = f_x \mathrm{d}x + \tfrac{1}{2} f_{xx}(\mathrm{d}x \cdot \mathrm{d}x) + \cdots \,. \tag{2.4}$$

2. Differentials of Differential Equations

Consider the first-order, ordinary differential equation

$$\dot{x} = f(t, x). \tag{2.5}$$

The total change in x at a given time is the sum of differentials.

$$\Delta x = \mathrm{d}x + \tfrac{1}{2}\mathrm{d}^2 x + \cdots \tag{2.6}$$

The time-fixed change in x at a given time is the sum of variations.

$$\tilde{\Delta} x = \delta x + \tfrac{1}{2}\delta^2 x + \cdots \tag{2.7}$$

The differential of a time varying function consists of a variation and a time part.

$$\mathrm{d}(\cdot) = \delta(\cdot) + \frac{\mathrm{d}}{\mathrm{d}t}(\cdot)\mathrm{d}t \tag{2.8}$$

Applying the formula gives the differentials of x.

$$\mathrm{d}x = \delta x + \dot{x}\mathrm{d}t \tag{2.9}$$

$$\mathrm{d}^2 x = \delta^2 x + \dot{x}\mathrm{d}^2 t + 2\delta\dot{x}\mathrm{d}t + \ddot{x}\mathrm{d}t^2 \tag{2.10}$$

Variations and differentials are interchangeable with time derivatives. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x = \delta\dot{x}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}x = \mathrm{d}\dot{x}. \tag{2.11}$$

One can then write expressions for the total and time-fixed changes in $\dot{x}$.

$$\Delta\dot{x} = \mathrm{d}\dot{x} + \tfrac{1}{2}\mathrm{d}^2\dot{x} + \cdots \tag{2.12}$$

$$\tilde{\Delta}\dot{x} = \delta\dot{x} + \tfrac{1}{2}\delta^2\dot{x} + \cdots \tag{2.13}$$

The same goes for higher order time derivatives.

3. Differentials of Integral Equations

Consider the integral equation

$$J = \int_{t_0}^{t_f} F(t, x)\ \mathrm{d}t. \tag{2.14}$$

The total change in J is the sum of differentials.

$$\Delta J = \mathrm{d}J + \tfrac{1}{2}\mathrm{d}^2 J + \cdots \tag{2.15}$$

The formula for a differential of an integral is given by Leibniz's rule.

$$\mathrm{d}J = [F\mathrm{d}t]_{t_0}^{t_f} + \int_{t_0}^{t_f} \delta F\ \mathrm{d}t \tag{2.16}$$

Leibniz's rule is important in optimal control because the performance index is an integral equation. Fixing the initial time and applying the formula gives the differentials of J.

$$\mathrm{d}J = F_f \mathrm{d}t_f + \int_{t_0}^{t_f} \delta F\ \mathrm{d}t \tag{2.17}$$

$$\mathrm{d}^2 J = F_f \delta^2 \mathrm{d}t_f + 2(F_x)_f \delta x_f \delta t_f + \dot{F}_f \delta t_f^2 + \int_{t_0}^{t_f} [F_x \delta^2 x + F_{xx}(\delta x \cdot \delta x)]\ \mathrm{d}t \tag{2.18}$$

B. State Transition Matrix

The state transition matrix relates variational changes at one time to variational changes at another time. To see this, consider the first-order, ordinary differential equation

$$\dot{x} = f(t, x), \quad x(t_0) = x_0. \tag{2.19}$$

Differentiation with respect to the initial states gives

$$\dot{\Phi}^{(1)} = f_x \Phi^{(1)}, \quad \Phi^{(1)}(t_0, t_0) = I \tag{2.20}$$

$$\dot{\Phi}^{(2)} = f_{xx}(\Phi^{(1)} \cdot \Phi^{(1)}) + f_x \Phi^{(2)}, \quad \Phi^{(2)}(t_0, t_0) = 0 \tag{2.21}$$

where

$$\Phi^{(1)}(t, t_0) = \frac{\partial x(t)}{\partial x(t_0)} \quad \text{and} \quad \Phi^{(2)}(t, t_0) = \frac{\partial^2 x(t)}{\partial x(t_0)^2} \tag{2.22}$$

are the first and second-order state transition matrices. For convenience, let $\Phi_{ji} = \Phi(t_j, t_i)$. Integrating the first-order state transition matrix forward in time then gives

$$\Phi_{00}^{(1)}, \quad \Phi_{10}^{(1)}, \quad \Phi_{20}^{(1)} \tag{2.23}$$

and so on. These matrices have the following first-order group property.

$$\Phi_{21}^{(1)} = \Phi_{20}^{(1)} \Phi_{01}^{(1)} \tag{2.24}$$

$$= \Phi_{20}^{(1)} [\Phi_{10}^{(1)}]^{-1}. \tag{2.25}$$

Thus, intermediate state transition matrices can be found with a single forward integration of the state and state transition matrix differential equations. Analogous group properties do not exist for the higher order state transition matrices. One can of course append the first-order matrix to the state vector, integrate the augmented vector, and obtain some group property. This is not a higher-order property, but it is satisfactory from a computational perspective.

If the initial condition is considered independent, then a time-fixed change at the initial time has only a first-order part, i.e., $\tilde{\Delta} x_0 = \delta x_0$. Variations at a later time t_1 are

$$\delta x_1 = \Phi_{10}^{(1)} \tilde{\Delta} x_0 \tag{2.26}$$

$$\delta^2 x_1 = \Phi_{10}^{(2)} (\tilde{\Delta} x_0 \cdot \tilde{\Delta} x_0) \tag{2.27}$$

so that the time-fixed change at t_1 is

$$\tilde{\Delta}x_1 = \Phi_{10}^{(1)}\tilde{\Delta}x_0 + \tfrac{1}{2}\Phi_{10}^{(2)}(\tilde{\Delta}x_0 \cdot \tilde{\Delta}x_0) + \cdots . \tag{2.28}$$

Series reversion gives the inverse mapping

$$\tilde{\Delta}x_0 = [\Phi_{10}^{(1)}]^{-1}\tilde{\Delta}x_1 - \tfrac{1}{2}\Phi_{10}^{(2)}([\Phi_{10}^{(1)}]^{-1} \cdot [\Phi_{10}^{(1)}]^{-1})(\tilde{\Delta}x_1 \cdot \tilde{\Delta}x_1) - \cdots . \tag{2.29}$$

C. Optimal Control Theory

Optimal control theory provides a systematic way to determine the controls so that a process satisfies all constraints in the most efficient manner. Optimal control problems that do not have interior constraints are called unconstrained, and they can be solved as two-point boundary value problems. In spite of their name, these problems still have constraints at the initial and final time. Optimal control problems that do have interior constraints are called constrained, and they can be solved as multi-point boundary value problems. To begin, consider the following optimal control problem with constraints. Find the control history $u(t)$ that minimizes the performance index

$$J = \phi(t_f, x_f) + \int_{t_0}^{t_f} L(t, x, u)\,\mathrm{d}t, \quad t_f = \text{free} \tag{2.30}$$

subject to the differential constraints

$$\dot{x} = f(t, x, u) \tag{2.31}$$

interior state and control constraints

$$S(t, x) \leq 0 \tag{2.32}$$

$$C(t, u) \leq 0 \tag{2.33}$$

and prescribed initial and final conditions

$$\theta(t_0, x_0) = 0 \tag{2.34}$$

$$\psi(t_f, x_f) = 0. \tag{2.35}$$

The state vector is $x \in \mathbb{R}^n$; the control vector is $u \in \mathbb{R}^m$; the interior state constraint is $S \in \mathbb{R}^s$; the interior control constraint is $C \in \mathbb{R}^r$ where $r \leq m$; the initial point constraint is $\theta \in \mathbb{R}^q$ where $q \leq n$; the final point constraint is $\psi \in \mathbb{R}^{p+1}$ where $p \leq n$; and ϕ and L are scalars. Note that the initial time is fixed and the final time is free. Because the states are differentiated variables, they are required to be continuous. Because the controls are not differentiated variables, they can be discontinuous. Here, they are assumed to be piecewise continuous, i.e., single-valued over intervals and possibly double-valued at interior points.

Necessary conditions for optimal control can be found using differentials. It is convenient to first define the Hamiltonian

$$H = L(t, x, u) + \lambda^\top f(t, x, u) \tag{2.36}$$

with $\lambda \in \mathbb{R}^n$ being a Lagrange multiplier. First-order necessary conditions for the unconstrained problem are below [24].

$$\dot{x} - H_\lambda^\top = 0 \tag{2.37}$$

$$\dot{\lambda} + H_x^\top = 0 \tag{2.38}$$

$$H_u^\top = 0 \tag{2.39}$$

$$\theta(t_0, x_0) = 0 \tag{2.40}$$

$$\theta_x^\top \mu + \lambda(t_0) = 0 \tag{2.41}$$

$$\theta_t^\top \mu - H(t_0) = 0 \tag{2.42}$$

$$\psi(t_f, x_f) = 0 \tag{2.43}$$

$$\phi_x^\top + \psi_x^\top \nu - \lambda(t_f) = 0 \tag{2.44}$$

$$\phi_t^\top + \psi_t^\top \nu + H(t_f) = 0 \tag{2.45}$$

The initial condition Lagrange multiplier is $\mu \in \mathbb{R}^q$, and the final condition Lagrange multiplier is $\nu \in \mathbb{R}^{p+1}$. Differentiating the Hamiltonian with respect to time along the optimal path gives

$$\dot{H} = H_t + H_u \dot{u} + H_x \dot{x} + H_\lambda \dot{\lambda} \tag{2.46}$$

$$= H_t. \tag{2.47}$$

Consequently, the Hamiltonian is constant for time invariant systems. This is especially useful for single state systems and free time problems. Further, the optimal control problem is nonsingular if it satisfies the strengthened Legendre-Clebsch condition.

$$H_{uu} > 0 \tag{2.48}$$

When interior control constraints are present and active, additional corner conditions exist [24].

$$H(t_1^+) = H(t_1^-) \tag{2.49}$$

$$\lambda(t_1^+) = \lambda(t_1^-) \tag{2.50}$$

The corner time is t_1, and the same conditions hold at entry and exit. When interior state constraints are present and active, additional point and corner conditions exist.

For a q^{th}-order state constraint, q point constraints result [24].

$$\Omega_1 = S^{(q-1)}(t_1, x_1) = 0 \tag{2.51}$$

$$\Omega_2 = S^{(q-2)}(t_1, x_1) = 0 \tag{2.52}$$

$$\vdots$$

$$\Omega_q = S^{(0)}(t_1, x_1) = 0 \tag{2.53}$$

The corner conditions at an entry time t_1 are

$$H(t_1^+) = H(t_1^-) + \Omega_t^\top \zeta \tag{2.54}$$

$$\lambda(t_1^+) = \lambda(t_1^-) - \Omega_x^\top \zeta \tag{2.55}$$

where $\zeta \in \mathbb{R}^q$ is a Lagrange multiplier. The corner conditions at an exit time t_2 are

$$H(t_2^+) = H(t_2^-) \tag{2.56}$$

$$\lambda(t_2^+) = \lambda(t_2^-). \tag{2.57}$$

D. Boundary Value Problem

The necessary conditions for optimal control result in a multi-point boundary value problem. To motivate the development, consider the m-point problem: determine the initial conditions z_0, times t_i, and constants ξ_i so that the ordinary differential equations

$$\dot{z} = f_i(t, z), \quad i = 0, \ldots, m-1 \tag{2.58}$$

satisfy all point constraints

$$\theta^i(t_i, z_i^-, p_i) = 0 \tag{2.59}$$

and corner conditions

$$z_i^+ = z_i^- + \xi_i \tag{2.60}$$

for a given set of parameters p_i. The shorthand z_i^- means $z(t_i^-)$ and z_i^+ means $z(t_i^+)$. The state vector is $z \in \mathbb{R}^n$ and it is piecewise continuous. The constants ξ_i are $\mathbb{R}^n$. The dimension of each constraint vector θ^i is free, but all constraints and corner conditions must sum to $mn+m+n-1$. The parameters p_i may have any dimension, and they serve the convenient role of resizing or reshaping the constraints.

CHAPTER III

NEIGHBORING SOLUTIONS

The boundary value problem described by Equations 2.58 through 2.60 can be solved when the parameters p_i are set. If the parameters change, the solution changes. The goal is to find this relationship. A well known fact in calculus and perturbation theory is that such a relationship can be approximated using differentials [3, 24–26].

The constraints and their parameters define an extremal field because any solution to the problem must satisfy the constraints. Computing differentials of constraints gives the first and second-order relationship between the parameters and the solution. One can then compute state differentials and variations, substitute into the constraint differentials, and solve two linear equations for the first and second-order terms of a Taylor series.

The work is based on two assumptions. First, the reference and neighboring solutions satisfy the strengthened Legendre-Clebsch condition so that the problem is nonsingular. Solving the linear equations requires matrix inversion, and this assumption guarantees that the inverse exists. The second assumption is that the neighboring solutions are optimal in the same sense as the reference. In other words, neighboring problems must have as many or fewer interior constraints as the reference, and the form of the interior constraints cannot change. The following sections derive the first and second-order expressions to approximate neighboring solutions to the m-point boundary value problem.

A. Taylor Series

Recall that a solution to the original problem is given by the initial conditions z_0, corner times t_i, and constants ξ_i. A solution to the neighboring problem is given by z_0^*, t_i^*, and ξ_i^* where

$$z_0^* = z_0 + \Delta z_0 \tag{3.1}$$

$$t_i^* = t_i + \Delta t_i \tag{3.2}$$

$$\xi_i^* = \xi_i + \Delta \xi_i. \tag{3.3}$$

The initial time is fixed so that

$$\Delta z_0 = \mathrm{d}z_0 + \tfrac{1}{2}\mathrm{d}^2 z_0 + \cdots \tag{3.4}$$

$$= \delta z_0 + \tfrac{1}{2}\delta^2 z_0 + \cdots . \tag{3.5}$$

Similarly,

$$\Delta t_i = \mathrm{d}t_i + \tfrac{1}{2}\mathrm{d}^2 t_i + \cdots \tag{3.6}$$

$$\Delta \xi_i = \mathrm{d}\xi_i + \tfrac{1}{2}\mathrm{d}^2 \xi_i + \cdots . \tag{3.7}$$

A first-order neighboring solution needs δz_0, $\mathrm{d}t_i$, and $\mathrm{d}\xi_i$. A second-order neighboring solution needs the first-order terms along with $\delta^2 z_0$, $\mathrm{d}^2 t_i$, and $\mathrm{d}^2 \xi_i$. The goal of upcoming sections is to find these terms as functions of the parameters.

B. Constraint Differentials

A solution satisfies the constraints exactly. This means that the total change in the constraints must be zero.

$$\Delta \theta^i = \mathrm{d}\theta^i + \tfrac{1}{2}\mathrm{d}^2 \theta^i + \cdots = 0 \tag{3.8}$$

Setting the differentials to zero gives

$$\mathrm{d}\theta^i = 0, \quad \mathrm{d}^2\theta^i = 0. \tag{3.9}$$

Computing the differentials gives the first and second-order relationship between the solution and the parameters.

$$\mathrm{d}\theta^i = \theta^i_z \mathrm{d}z_i^- + \theta^i_p \mathrm{d}p_i + \theta^i_t \mathrm{d}t_i \tag{3.10}$$

$$\begin{aligned} \mathrm{d}^2\theta^i = \theta^i_{zz}(\mathrm{d}z_i^- \cdot \mathrm{d}z_i^-) + \theta^i_z \mathrm{d}^2 z_i^- + \theta^i_{tt}(\mathrm{d}t_i \cdot \mathrm{d}t_i) + \theta^i_t \mathrm{d}^2 t_i + \theta^i_{pp}(\mathrm{d}p_i \cdot \mathrm{d}p_i) \\ +2\theta^i_{zp}(\mathrm{d}z_i^- \cdot \mathrm{d}p_i) + 2\theta^i_{zt}(\mathrm{d}z_i^- \cdot \mathrm{d}t_i) + 2\theta^i_{pt}(\mathrm{d}p_i \cdot \mathrm{d}t_i) \end{aligned} \tag{3.11}$$

C. State Differentials

The state differentials represent changes in state. Because of discontinuities, one must distinguish between the "just before" and "just after" state differentials. First-order differentials at the "just after" time depend on first-order jumps.

$$\mathrm{d}z_i^- = \delta z_i^- + \dot{z}_i^- \mathrm{d}t_i \tag{3.12}$$

$$\mathrm{d}z_i^+ = \delta z_i^+ + \dot{z}_i^+ \mathrm{d}t_i = \mathrm{d}z_i^- + \mathrm{d}\xi_i \tag{3.13}$$

Second-order differentials at the "just after" time depend on second-order jumps.

$$\mathrm{d}^2 z_i^- = \delta^2 z_i^- + \dot{z}_i^- \mathrm{d}^2 t_i + 2\delta \dot{z}_i^- \mathrm{d}t_i + \ddot{z}_i^- \mathrm{d}t_i^2 \tag{3.14}$$

$$\mathrm{d}^2 z_i^+ = \delta^2 z_i^+ + \dot{z}_i^+ \mathrm{d}^2 t_i + 2\delta \dot{z}_i^+ \mathrm{d}t_i + \ddot{z}_i^+ \mathrm{d}t_i^2 = \mathrm{d}^2 z_i^- + \mathrm{d}^2 \xi_i \tag{3.15}$$

D. State Variations

Similarly, one must make the time distinction for state variations. The "just before" variations come by definition of the state transition matrix. The "just after" variations come by rearranging the state differential equations above. To illustrate the procedure, first-order variations at the first two interior times are provided.

$$\delta z_1^- = \Phi_{10}^{(1)} \delta z_0 \tag{3.16}$$

$$\delta z_1^+ = \Phi_{10}^{(1)} \delta z_0 + (\dot{z}_1^- - \dot{z}_1^+)\mathrm{d}t_1 + \mathrm{d}\xi_1 \tag{3.17}$$

$$\delta z_2^- = \Phi_{21}^{(1)} \Phi_{10}^{(1)} \delta z_0 + \Phi_{21}^{(1)} (\dot{z}_1^- - \dot{z}_1^+)\mathrm{d}t_1 + \Phi_{21}^{(1)} \mathrm{d}\xi_1 \tag{3.18}$$

$$\delta z_2^+ = \Phi_{21}^{(1)} \Phi_{10}^{(1)} \delta z_0 + \Phi_{21}^{(1)} (\dot{z}_1^- - \dot{z}_1^+) + (\dot{z}_2^- - \dot{z}_2^+)\mathrm{d}t_1 + \Phi_{21}^{(1)} \mathrm{d}\xi_1 + \mathrm{d}\xi_2 \tag{3.19}$$

The general series expressions for first-order state variations are

$$\delta z_i^- = \Phi_{i0}^{(1)} \delta z_0 + \sum_{j=1}^{i-1} \Phi_{ij}^{(1)} (\dot{z}_j^- - \dot{z}_j^+)\mathrm{d}t_j + \sum_{j=1}^{i-1} \Phi_{ij}^{(1)} \mathrm{d}\xi_j \tag{3.20}$$

$$\begin{aligned} \delta z_i^+ &= \Phi_{i0}^{(1)} \delta z_0 + \sum_{j=1}^{i} \Phi_{ij}^{(1)} (\dot{z}_j^- - \dot{z}_j^+)\mathrm{d}t_j + \sum_{j=1}^{i} \Phi_{ij}^{(1)} \mathrm{d}\xi_j &(3.21)\\ &= \delta z_i^- + (\dot{z}_i^- - \dot{z}_i^+)\mathrm{d}t_i + \mathrm{d}\xi_i. &(3.22)\end{aligned}$$

The general series expressions for second-order state variations are

$$\begin{aligned} \delta^2 z_i^- = \Phi_{i0}^{(1)} \delta^2 z_0 &+ \sum_{j=1}^{i-1} \Phi_{ij}^{(1)} (\dot{z}_j^- - \dot{z}_j^+)\mathrm{d}^2 t_j + \sum_{j=1}^{i-1} \Phi_{ij}^{(1)} \mathrm{d}^2\xi_j \\ &+ \sum_{j=0}^{i-1} \Phi_{i(j+1)}^{(1)} \Phi_{(j+1)j}^{(2)} (\delta z_j^+ \cdot \delta z_j^+) + 2\sum_{j=1}^{i-1} \Phi_{ij}^{(1)} (\delta\dot{z}_j^- - \delta\dot{z}_j^+)\mathrm{d}t_j \\ &+ \sum_{j=1}^{i-1} \Phi_{ij}^{(1)} (\ddot{z}_j^- - \ddot{z}_j^+)\mathrm{d}t_j^2 \end{aligned} \tag{3.23}$$

$$\begin{aligned} \delta^2 z_i^+ = \Phi_{i0}^{(1)} \delta^2 z_0 &+ \sum_{j=1}^{i} \Phi_{ij}^{(1)} (\dot{z}_j^- - \dot{z}_j^+)\mathrm{d}^2 t_j + \sum_{j=1}^{i} \Phi_{ij}^{(1)} \mathrm{d}^2\xi_j \\ &+ \sum_{j=0}^{i-1} \Phi_{i(j+1)}^{(1)} \Phi_{(j+1)j}^{(2)} (\delta z_j^+ \cdot \delta z_j^+) + 2\sum_{j=1}^{i} \Phi_{ij}^{(1)} (\delta\dot{z}_j^- - \delta\dot{z}_j^+)\mathrm{d}t_j &(3.24)\\ &+ \sum_{j=1}^{i} \Phi_{ij}^{(1)} (\ddot{z}_j^- - \ddot{z}_j^+)\mathrm{d}t_j^2 \\ = \delta^2 z_i^- + (\dot{z}_i^- - \dot{z}_i^+)&\mathrm{d}^2 t_i + \mathrm{d}^2\xi_i + 2(\delta\dot{z}_i^- - \delta\dot{z}_i^+)\mathrm{d}t_i + (\ddot{z}_i^- - \ddot{z}_i^+)\mathrm{d}t_i^2 &(3.25)\end{aligned}$$

Summations that have upper limits less than lower limits are defined to be zero.

E. Solution Procedure

The neighboring solution can now be found. 1) Substitute the state variations into the state differentials. 2) Substitute the state differentials into the constraint differentials. 3) Solve the first-order constraint differential for δz_0, $\mathrm{d}t_i$, and $\mathrm{d}\xi_i$. 4) Solve the second-order constraint differential for $\delta^2 z_0$, $\mathrm{d}^2 t_i$, and $\mathrm{d}^2\xi_i$. 5) Substitute these values into the Taylor series. This approach provides first and second-order expressions to approximate neighboring solutions to the m-point boundary value problem.

CHAPTER IV
NUMERICAL EXAMPLES

Four types of problems are investigated: 1) fixed final time, 2) free final time, 3) control constrained, and 4) state constrained. These problems include characteristics common to many optimal control problems such as unknown initial conditions, switch times, interior jumps, and final time. The problems are presented in order of increasing difficulty, however, difficulties arise only in finding a reference solution. Once a solution is known, computing neighboring solutions is systematic.

In each problem, four neighboring solutions are found. The first two, denoted a and b, lie on one side of the reference. The third and fourth, denoted c and d, lie on the other side of the reference. In each case, an exact solution is calculated using an indirect method for illustration purposes and error analysis. Exact solutions are denoted by solid curves, and approximate solutions are denoted by circle markers. The approximation has “graphing accuracy” when the circles lie on the solid curves, and this qualitatively suggests that the approach is successful. Figures show exact solutions and second-order approximations. Tables provide errors for first and second-order approximations.

Neighboring solutions arise because of changes in parameters, and so parameters are embedded in the problem wherever a change is desired. Parameters, p, are zero in the original problem and take on different values for each neighboring solution. Over-barred symbols are set for the original problem and do not change for neighboring problems. As an example, let the family of initial conditions be $x_0 = \{3, 4, 5, 6, 7\}$. Then $x_0 = \bar{x}_0 + p_{x_0}$ with $\bar{x}_0 = 5$ and $p_{x_0} = \{-2, -1, 0, 1, 2\}$. Similar constructs are made for final conditions, control bounds, and state bounds.

A. Fixed Final Time Problem

Consider the following optimal control problem with fixed final time [21]. Minimize

$$J[u] = \frac{1}{2} \int_0^5 \left(x^2 + v^2 + u^2\right)\ \mathrm{d}t \tag{4.1}$$

subject to

$$\dot{x} = v, \quad x_0 = \bar{x}_0 + p_{x_0} \tag{4.2}$$

$$\dot{v} = u, \quad v_0 = \bar{v}_0 + p_{v_0} \tag{4.3}$$

$$x_f^2 + v_f^2 = (\bar{\alpha} + p_{\alpha_f})^2. \tag{4.4}$$

The initial conditions require that the states begin at specified points. The final condition requires that the states terminate on a circle. There are no interior control or state constraints, and so the controls and states are unbounded. The Hamiltonian is

$$H = \tfrac{1}{2}(x^2 + v^2 + u^2) + \lambda_x v + \lambda_v u. \tag{4.5}$$

Along with the state equations, the Euler-Lagrange equations are

$$\dot{\lambda}_x = 0, \quad \dot{\lambda}_v = -\lambda_x, \quad u = -\lambda_v. \tag{4.6}$$

The constraints that define the extremal field are

$$\theta^0 = \begin{bmatrix} x_0 - \bar{x}_0 - p_{x_0} \\ v_0 - \bar{v}_0 - p_{v_0} \end{bmatrix}, \quad \theta^f = \begin{bmatrix} x_f^2 + v_f^2 - (\bar{\alpha} + p_{\alpha_f})^2 \\ \lambda_{x_f} v_f - \lambda_{v_f} x_f \end{bmatrix}. \tag{4.7}$$

Conditions for the original problem and parameters for the neighboring problems are in Tables 1 and 2.

Table 1. Reference conditions for the fixed final time problem.

$\bar{x}_0$	$\bar{v}_0$	$\bar{\alpha}$
4	4	2

Table 2. Parameters for the fixed final time problem.

Case	p_{x_0}	p_{v_0}	p_{α_f}
a	1.0	1.0	0.5
b	0.5	0.5	0.25
c	-0.5	-0.5	-0.25
d	-1.0	-1.0	-0.5

Figures 1 and 2 show the state trajectories and control histories.

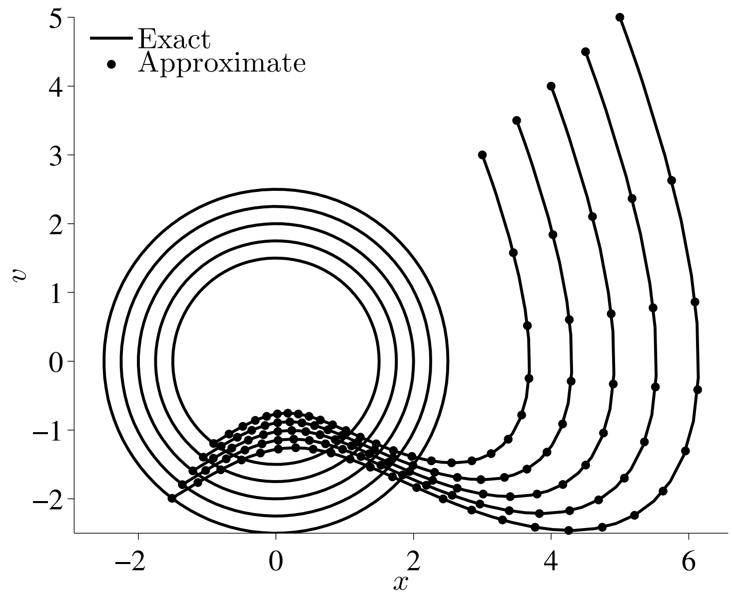

Figure 1. State trajectories for the fixed final time problem.

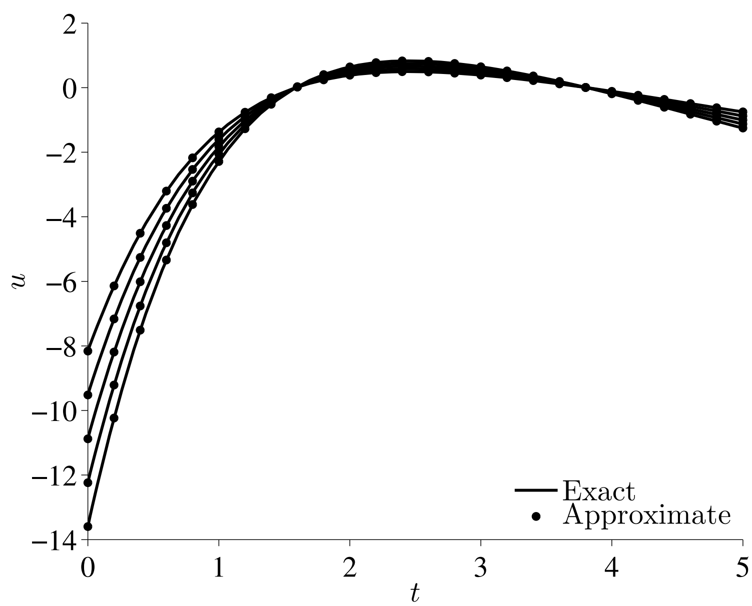

Figure 2. Control histories for the fixed final time problem.

Tables 3 and 4 show absolute errors in the unknown initial conditions.

Table 3. First-order errors for the fixed final time problem.

Case	λ_{x_0}	λ_{v_0}
a	2.7×10^{-8}	4.6×10^{-9}
b	1.3×10^{-8}	2.3×10^{-9}
c	1.3×10^{-8}	2.3×10^{-9}
d	2.7×10^{-8}	4.6×10^{-9}

Table 4. Second-order errors for the fixed final time problem.

Case	λ_{x_0}	λ_{v_0}
a	2.7×10^{-8}	4.6×10^{-9}
b	1.3×10^{-8}	2.3×10^{-9}
c	1.3×10^{-8}	2.3×10^{-9}
d	2.7×10^{-8}	4.6×10^{-9}

Although not apparent from the tables, the second-order terms do affect the approximation. Their contribution does not appear within the two decimals shown.

B. Free Final Time Problem

Consider the following free-time optimal control problem. Minimize

$$J[u] = \frac{1}{2}\int_0^{t_f} \left(x^2 + v^2 + u^2\right)\ \mathrm{d}t, \quad t_f = \text{free} \tag{4.8}$$

subject to

$$\dot{x} = v, \quad x_0 = \bar{x}_0 + p_{x_0} \tag{4.9}$$

$$\dot{v} = u, \quad v_0 = \bar{v}_0 + p_{v_0} \tag{4.10}$$

$$x_f^2 + v_f^2 = (\bar{\alpha} + p_{\alpha_f})^2. \tag{4.11}$$

The initial conditions require that the states begin at specified points. The final condition requires that the states terminate on a circle in an optimal time. As evident in the last example, there is more than one way for the states to contact the final constraint. In fact, there are infinitely many solutions that terminate on the circle. The cost function monotonically increases with time, and so the optimal solution is the one that takes the least time. There are no interior control or state constraints, and so the controls and states are unbounded. The Hamiltonian is

$$H = \tfrac{1}{2}(x^2 + v^2 + u^2) + \lambda_x v + \lambda_v u. \tag{4.12}$$

Along with the state equations, the Euler-Lagrange equations are

$$\dot{\lambda}_x = 0, \quad \dot{\lambda}_v = -\lambda_x, \quad \dot{h} = H, \quad u = -\lambda_v. \tag{4.13}$$

The constraints that define the extremal field are

$$\theta^0 = \begin{bmatrix} x_0 - \bar{x}_0 - p_{x_0} \\ v_0 - \bar{v}_0 - p_{v_0} \\ h_0 \end{bmatrix}, \quad \theta^f = \begin{bmatrix} x_f^2 + v_f^2 - (\bar{\alpha} + p_{\alpha_f})^2 \\ \lambda_{x_f} v_f - \lambda_{v_f} x_f \\ h_f \end{bmatrix}. \tag{4.14}$$

Conditions for the original problem and parameters for the neighboring problems are in Tables 5 and 6.

Table 5. Reference conditions for the free final time problem.

$\bar{x}_0$	$\bar{v}_0$	$\bar{\alpha}$
4	4	2

Table 6. Parameters for the free final time problem.

Case	p_{x_0}	p_{v_0}	p_{α_f}
a	0.5	0.5	-0.3
b	0.25	0.25	-0.15
c	-0.25	-0.25	0.15
d	-0.5	-0.5	0.3

Figures 3 and 4 show the state trajectories and control histories.

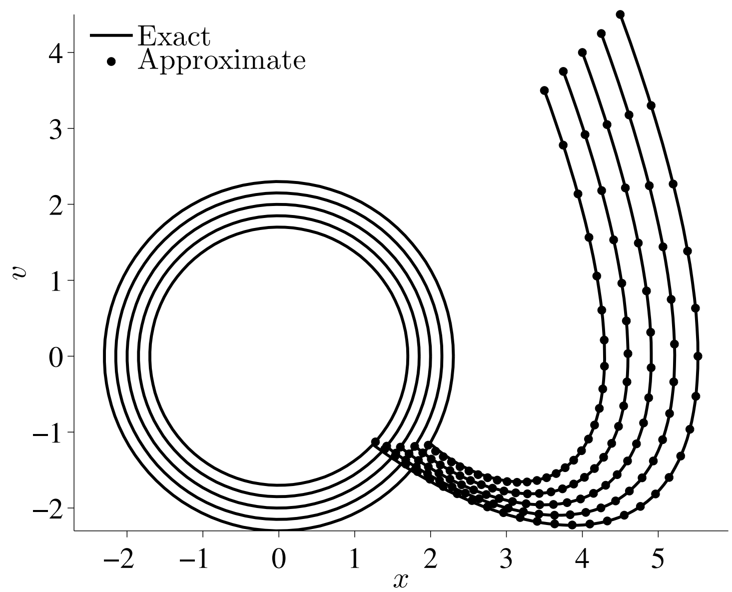

Figure 3. State trajectories for the free final time problem.

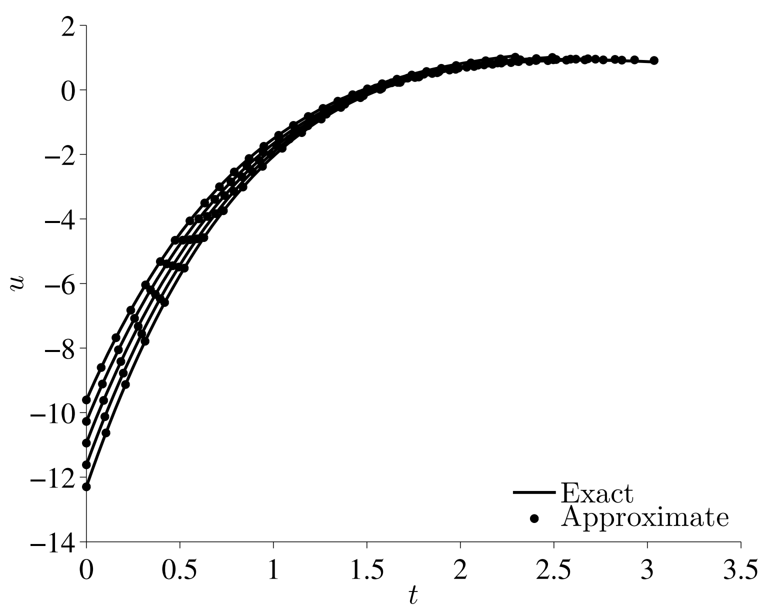

Figure 4. Control histories for the free final time problem.

Tables 7 and 8 show absolute errors in the unknown initial conditions and final time.

Table 7. First-order errors for the free final time problem.

Case	λ_{x_0}	λ_{v_0}	t_f
a	2.0×10^{-2}	7.3×10^{-3}	1.2×10^{-2}
b	5.8×10^{-3}	2.1×10^{-3}	3.6×10^{-3}
c	8.3×10^{-3}	3.0×10^{-3}	4.9×10^{-3}
d	4.2×10^{-2}	1.5×10^{-2}	2.3×10^{-2}

Table 8. Second-order errors for the free final time problem.

Case	λ_{x_0}	λ_{v_0}	t_f
a	7.3×10^{-3}	2.7×10^{-3}	4.7×10^{-3}
b	1.0×10^{-3}	4.0×10^{-4}	6.0×10^{-4}
c	1.5×10^{-3}	5.0×10^{-4}	7.0×10^{-4}
d	1.4×10^{-2}	5.2×10^{-3}	6.5×10^{-3}

C. Control Constrained Problem

Consider the following optimal control problem with control constraints [27]. Minimize

$$J[u] = \int_0^{35} |u| \, \mathrm{d}t \tag{4.15}$$

subject to

$$\dot{x} = v, \quad x_0 = \bar{x}_0 + p_{x_0}, \quad x_f = \bar{x}_f + p_{x_f} \tag{4.16}$$

$$\dot{v} = u, \quad v_0 = \bar{v}_0 + p_{v_0}, \quad v_f = \bar{v}_f + p_{v_f} \tag{4.17}$$

$$|u| \leq \bar{c} + p_c. \tag{4.18}$$

The initial conditions require that the states begin at specified points, and the final conditions require that the states terminate at specified points. There is an interior control constraint, and so the control magnitude is bounded while the states are unbounded. One might expect a solution with any number of constrained and unconstrained arcs, but because the differential constraints are linear, one can show that there are at most three arcs [27]. In other words, the solution is bang-off-bang. The Hamiltonian is

$$H = |u| + \lambda_x v + \lambda_v u. \tag{4.19}$$

Along with the state equations, the Euler-Lagrange equations are

$$\dot{\lambda}_x = 0, \quad \dot{\lambda}_v = -\lambda_x, \quad u = \{u_{\min},\ 0,\ u_{\max}\}. \tag{4.20}$$

The constraints that define the extremal field at the initial and final times are

$$\theta^0 = \begin{bmatrix} x_0 - \bar{x}_0 - p_{x_0} \\ v_0 - \bar{v}_0 - p_{v_0} \end{bmatrix}, \quad \theta^f = \begin{bmatrix} x_f - \bar{x}_f - p_{x_f} \\ v_f - \bar{v}_f - p_{v_f} \end{bmatrix}. \tag{4.21}$$

The constraints at interior times are

$$\theta^1 = \begin{bmatrix} \lambda_{v_1} \end{bmatrix}, \quad \theta^2 = \begin{bmatrix} \lambda_{v_2} \end{bmatrix}. \tag{4.22}$$

Conditions for the original problem and parameters for the neighboring problems are in Tables 9 and 10.

Table 9. Reference conditions for the control constrained problem.

$\bar{x}_0$	$\bar{v}_0$	$\bar{x}_f$	$\bar{v}_f$	$\bar{c}$
10	10	2	2	1

Table 10. Parameters for the control constrained problem.

Case	p_{x_0}	p_{v_0}	p_{x_f}	p_{v_f}	p_c
a	1.0	-1.0	0.1	0.1	-0.12
b	0.5	-0.5	0.05	0.05	-0.06
c	-0.5	0.5	-0.05	-0.05	0.06
d	-1.0	1.0	-0.1	-0.1	0.12

Figures 5 and 6 show the state trajectories and control histories.

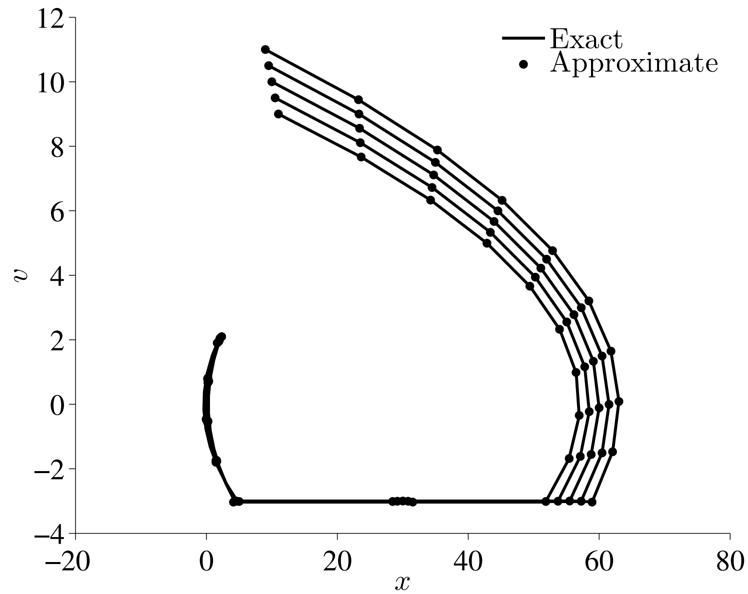

Figure 5. State trajectories for the control constrained problem.

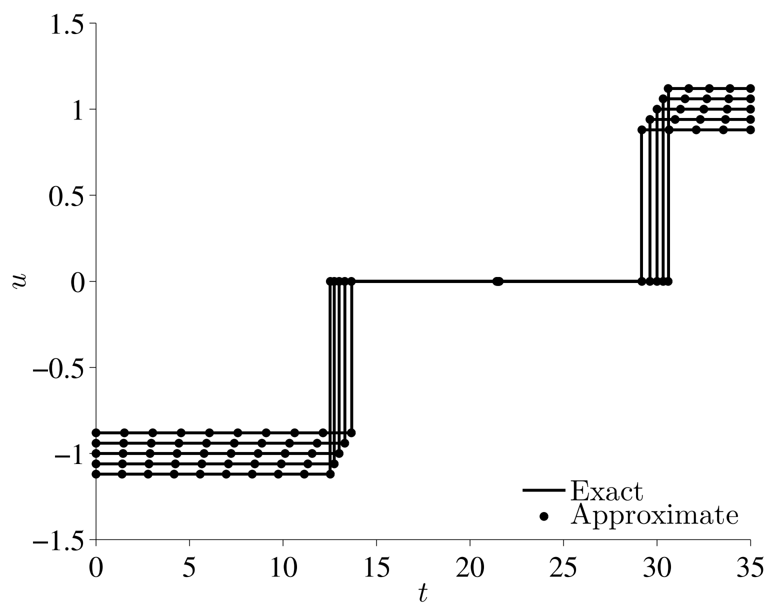

Figure 6. Control histories for the control constrained problem.

Tables 11 and 12 show absolute errors in the unknown initial conditions and switch times.

Table 11. First-order errors for the control constrained problem.

Case	λ_{x0}	λ_{v0}	t_1	t_2
a	1.3×10^{-3}	2.7×10^{-2}	1.1×10^{-1}	1.3×10^{-1}
b	2.8×10^{-3}	5.8×10^{-3}	2.5×10^{-2}	3.0×10^{-2}
c	2.2×10^{-3}	4.6×10^{-3}	2.1×10^{-2}	2.5×10^{-2}
d	7.9×10^{-4}	1.6×10^{-2}	7.8×10^{-2}	9.3×10^{-2}

Table 12. Second-order errors for the control constrained problem.

Case	λ_{x0}	λ_{v0}	t_1	t_2
a	3.2×10^{-4}	6.7×10^{-3}	1.9×10^{-2}	2.1×10^{-2}
b	3.4×10^{-5}	7.2×10^{-4}	2.1×10^{-3}	2.4×10^{-3}
c	2.7×10^{-5}	5.6×10^{-4}	1.8×10^{-3}	2.0×10^{-3}
d	1.9×10^{-4}	4.0×10^{-3}	1.3×10^{-2}	1.5×10^{-2}

D. State Constrained Problem

Consider the following optimal control problem with state constraints [28]. Minimize

$$J[u] = \frac{1}{2}\int_0^1 u^2 \,\mathrm{d}t \tag{4.23}$$

subject to

$$\dot{x} = v, \quad x_0 = \bar{x}_0, \quad x_f = \bar{x}_f \tag{4.24}$$

$$\dot{v} = u, \quad v_0 = \bar{v}_0, \quad v_f = \bar{v}_f \tag{4.25}$$

$$|x_1| \leq \bar{c} + p_c. \tag{4.26}$$

The initial conditions require that the states begin at specified points, and the final conditions require that the states terminate at specified points. There is an interior state constraint. This reduces to a control constraint and additional point constraints at entrance times. The Hamiltonian and Lagrange multipliers can jump at any entrance time. The Hamiltonian is

$$H = \tfrac{1}{2}u^2 + \lambda_x v + \lambda_v u. \tag{4.27}$$

Along with the state equations, the Euler-Lagrange equations are

$$\dot{\lambda}_x = 0, \quad \dot{\lambda}_v = -\lambda_x, \quad u = \{-\lambda_v,\ 0,\ -\lambda_v\}. \tag{4.28}$$

The constraints that define the extremal field at the initial and final times are

$$\theta^0 = \begin{bmatrix} x_0 - \bar{x}_0 \\ v_0 - \bar{v}_0 \end{bmatrix}, \quad \theta^f = \begin{bmatrix} x_f - \bar{x}_f \\ v_f - \bar{v}_f \end{bmatrix}. \tag{4.29}$$

The constraints at interior times are

$$\theta^1 = \begin{bmatrix} x_1 - \bar{c} - p_c \\ v_1 - 0 \\ \lambda_{v_1} - 0 \end{bmatrix}, \quad \theta^2 = \begin{bmatrix} \lambda_{v_2} \end{bmatrix}. \tag{4.30}$$

Conditions for the original problem and parameters for the neighboring problems are in Tables 13 and 14. Note that only the state constraint varies.

Table 13. Reference conditions for the state constrained problem.

$\bar{x}_0$	$\bar{v}_0$	$\bar{x}_f$	$\bar{v}_f$	$\bar{c}$
0	1	0	-1	.1

Table 14. Parameters for the state constrained problem.

Case	p_c
a	-0.01
b	-0.005
c	0.005
d	0.01

Figures 7 and 8 show the state trajectories and control histories.

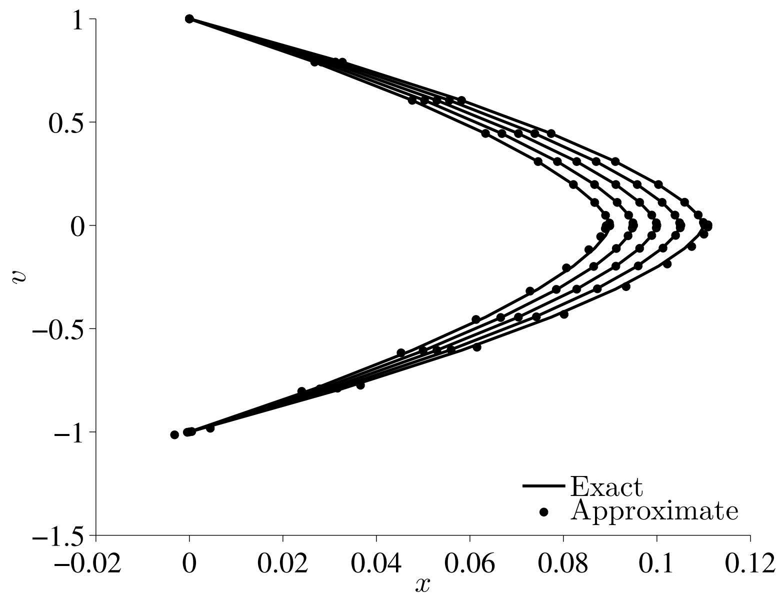

Figure 7. State trajectories for the state constrained problem.

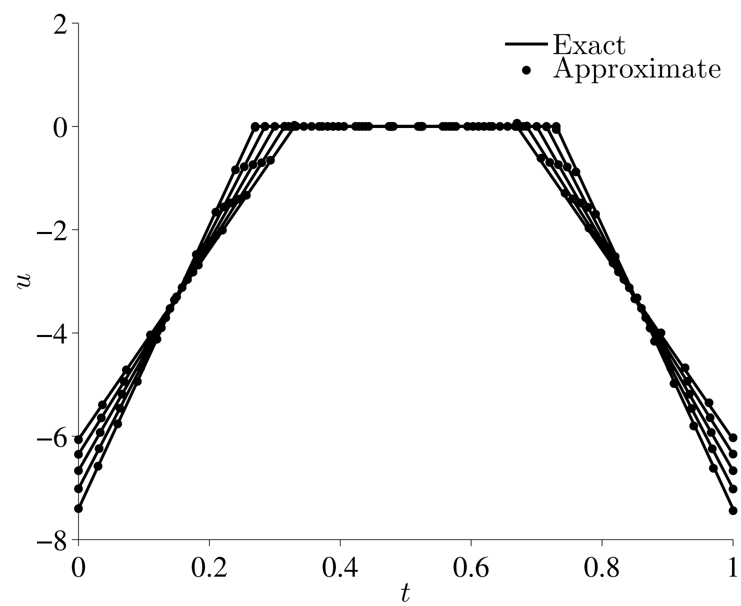

Figure 8. Control histories for the state constrained problem.

Tables 15 and 16 show absolute errors in the unknown initial conditions, switch times, and jumps.

Table 15. First-order errors for the state constrained problem.

Case	λ_{x_0}	λ_{v_0}	ξ_{λ_x}	ξ_{λ_v}	t_1	t_2
a	7.7×10^{-1}	7.4×10^{-2}	1.5×10^{0}	6.2×10^{-1}	0	0
b	1.8×10^{-1}	1.8×10^{-2}	3.6×10^{-1}	1.4×10^{-1}	0	0
c	1.6×10^{-1}	1.6×10^{-2}	3.1×10^{-1}	1.2×10^{-1}	0	0
d	5.9×10^{-1}	6.1×10^{-2}	1.2×10^{0}	4.7×10^{-1}	0	0

Table 16. Second-order errors for the state constrained problem.

Case	λ_{x_0}	λ_{v_0}	ξ_{λ_x}	ξ_{λ_v}	t_1	t_2
a	1.0×10^{-1}	7.4×10^{-3}	2.0×10^{-1}	8.7×10^{-2}	0	0
b	1.2×10^{-2}	8.7×10^{-4}	2.4×10^{-2}	1.0×10^{-2}	0	0
c	1.0×10^{-2}	7.9×10^{-4}	2.1×10^{-2}	8.9×10^{-3}	0	0
d	7.9×10^{-2}	6.1×10^{-3}	1.6×10^{-1}	6.7×10^{-2}	0	0

E. Discussion

Four types of problems are solved, and neighboring solutions are computed for each. Figures show “graphing accuracy”, but a conclusive statement about errors cannot be made. The fixed time problem has errors around 1×10^{-8}, and second-order terms are negligible. Letting the final time be free increases the error to about 1×10^{-3}, and second-order terms improve the approximation by an order of magnitude. The significant increase in error is caused by additional nonlinear equations needed to solve for the final time. The presence of interior constraints increases the error to about 1×10^{-1}. Again, second-order terms improve the approximation by an order of magnitude.

CHAPTER V
LUNAR DESCENT

The lunar descent problem is well studied [22, 29–36]. In short, the descent mission begins in a circular parking orbit around the moon. An impulsive burn lowers the orbit and subsequent phases null velocity to prepare for landing. Within a few kilometers of the landing site, the vehicle can retarget to avoid craters and uneven surfaces. A successful landing is defined as one where the vehicle arrives at the landing site with zero velocity and vertical pitch angle.

Mass is an important factor and so the goal is to land successfully and minimize fuel usage. Unfortunately, fuel minimal solutions have at least a few undesirable properties. Mainly, they pass through the moon and do not have vertical pitch angles at the final time. An alternative is to penalize fuel usage and flight time. The time penalty keeps the vehicle away from the surface until the final moments, and the final pitch angle is near vertical. The thrust magnitude is bounded, and consequently, the optimal solution is bang-off-bang.

This work focuses on the last few kilometers where the vehicle retargets to new landing sites. A reference solution is calculated using indirect shooting, and neighboring solutions are found for a variety of initial states, landing sites, and thrust magnitudes. The lunar gravitational parameter is $\mu = 4,902.8$ km^3/s^2 and the lunar radius is $R = 1,737.4$ km. Normalized units are used for computations.

Table 17. Normalized units for lunar descent.

DU	TU	MU
1,737.4 km	500 s	9,000 kg

A. Problem Statement

Consider the following optimal control problem. Minimize

$$J[T, \psi] = -m_f + \int_0^{t_f} 1 \text{ dt}, \quad t_f = \text{free} \tag{5.1}$$

subject to the differential constraints

$$\dot{r} = v, \quad \dot{v} = \frac{T}{m}\sin\psi - \frac{\mu}{r^2} + r\omega^2 \tag{5.2}$$

$$\dot{\phi} = \omega, \quad \dot{\omega} = -\frac{T}{mr}\cos\psi - 2\frac{v\omega}{r} \tag{5.3}$$

$$\dot{m} = -T/C \tag{5.4}$$

interior state and control constraints

$$r \geq R \tag{5.5}$$

$$0 \leq T \leq \bar{T} + p_T \tag{5.6}$$

and initial and final conditions

$$r_0 = \bar{r}_0 + p_{r_0}, \qquad r_f = R \tag{5.7}$$

$$v_0 = \bar{v}_0 + p_{v_0}, \qquad v_f = 0 \tag{5.8}$$

$$\phi_0 = 0, \qquad \phi_f = \bar{\phi}_f + p_{\phi_f} \tag{5.9}$$

$$\omega_0 = \bar{\omega}_0, \qquad \omega_f = 0 \tag{5.10}$$

$$m_0 = \bar{m}_0, \qquad m_f = \text{free}. \tag{5.11}$$

The radius is r such that the altitude is $h = r - R$. The radial velocity is v. The central angle is ϕ such that the projected range is $d = R\phi$. The central angle rate is ω such that the range rate is $w = r\omega$. The two control variables are the thrust T and the thrust angle ψ. All of the states are constrained at the initial and final times except mass which is to be optimized. Time is also free for optimization. The interior control constraint is active and causes the thrust to alternate between its maximum and minimum value, i.e., the thrust is bang-off-bang. The Hamiltonian is

$$H = 1 + \lambda_r v + \lambda_v \left(\frac{T}{m} \sin\psi - \frac{\mu}{r^2} + r\omega^2 \right) + \lambda_\phi \omega - \lambda_\omega \left(\frac{T}{mr} \cos\psi + 2\frac{v\omega}{r} \right) - \lambda_m \frac{T}{C}. \tag{5.12}$$

Along with the state equations, the Euler-Lagrange equations are

$$\dot{\lambda}_r = -2\lambda_v \frac{\mu}{r^3} - \lambda_v \omega^2 - \lambda_\omega \frac{T}{mr^2} \cos\psi - 2\lambda_\omega \frac{v\omega}{r^2}, \quad \dot{\lambda}_v = -\lambda_r + 2\lambda_\omega \frac{\omega}{r} \tag{5.13}$$

$$\dot{\lambda}_\phi = 0, \quad \dot{\lambda}_\omega = -2\lambda_v r\omega - \lambda_\phi + 2\lambda_\omega \frac{v}{r} \tag{5.14}$$

$$\dot{\lambda}_m = \lambda_v \frac{T}{m^2} \sin\psi - \lambda_\omega \frac{T}{rm^2} \cos\psi, \quad \dot{h} = H \tag{5.15}$$

where

$$\sin\psi = -\frac{r\lambda_v}{\sqrt{r^2\lambda_v^2 + \lambda_\omega^2}}, \quad \cos\psi = \frac{\lambda_\omega}{\sqrt{r^2\lambda_v^2 + \lambda_\omega^2}}. \tag{5.16}$$

The constraints that define the extremal field at the initial and final times are

$$\theta^0 = \begin{bmatrix} r_0 - \bar{r}_0 - p_{r_0} \\ v_0 - \bar{v}_0 - p_{v_0} \\ \phi \\ \omega_0 - \bar{\omega}_0 \\ m_0 - \bar{m}_0 \\ h_0 \end{bmatrix}, \quad \theta^f = \begin{bmatrix} r_f - R \\ v_f \\ \phi_f - \bar{\phi}_f - p_{\phi_f} \\ \omega_f \\ \lambda_{m_f} + 1 \\ h_f \end{bmatrix}. \tag{5.17}$$

The constraints at interior times are

$$\theta^1 = \begin{bmatrix} h_1 \end{bmatrix}, \quad \theta^2 = \begin{bmatrix} h_2 \end{bmatrix}. \tag{5.18}$$

Conditions for the original problem and parameters for the neighboring problems are in Tables 18 and 19.

Table 18. Reference conditions for lunar descent.

$\bar{T}$	C	$\bar{h}_0$	$\bar{v}_0$	$\bar{w}_0$	$\bar{m}_0$	$\bar{d}_f$
45 kN	3.5 km/s	2 km	-45 m/s	45 m/s	9,000 kg	1.5 km

Table 19. Parameters for lunar descent.

Case	p_T (N)	p_{r_0} (m)	p_{v_0} (m/s)	p_{d_f} (m)
a	-500	-250	5	-250
b	-250	-125	2.5	-125
c	250	125	-2.5	125
d	500	250	-5	250

The parameters change the thrust magnitude, initial altitude, initial altitude rate, and final downrange distance.

B. Results

Figure 9 shows altitude as a function of range. Each trajectory begins at a different altitude and ends at a different range.

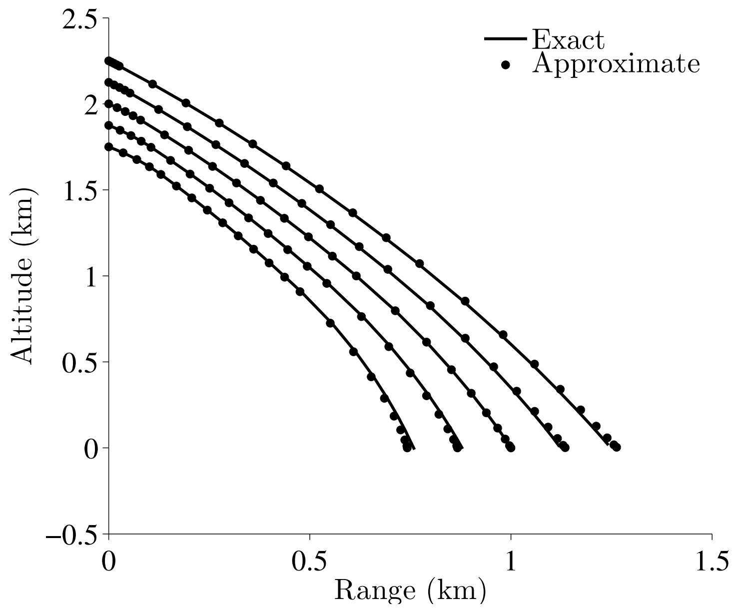

Figure 9. Altitude for lunar descent.

Figure 10 shows thrust as a function of time. Each thrust profile has the same bang-off-bang sequence, but the maximum allowable thrust is different for each case. The switch times and final times are approximated. Figure 11 shows thrust angle as a function of time. The thrust angle is continuous and ends so that the vehicle is nearly vertical.

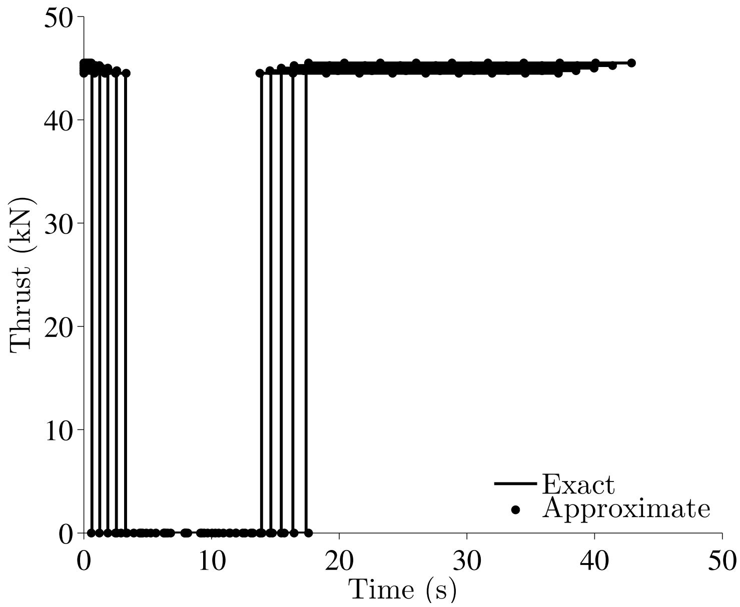

Figure 10. Thrust for lunar descent.

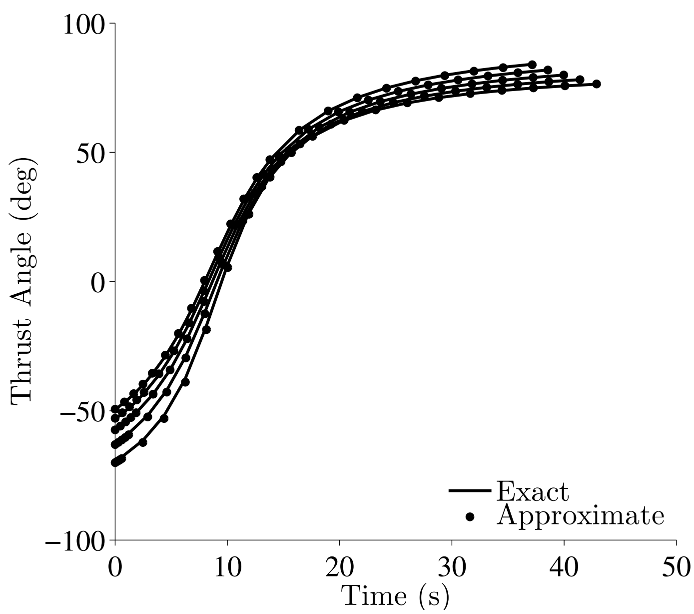

Figure 11. Thrust angle for lunar descent.

Tables 20 and 21 show the first-order, absolute errors in the unknown initial costates, switching times, and final time.

Table 20. First-order Lagrange multiplier errors for lunar descent.

Case	λ_{r_0}	λ_{v_0}	λ_{ϕ_0}	λ_{ω_0}	λ_{m_0}
a	1.9×10^{-1}	1.3×10^{-2}	5.1×10^{-1}	2.1×10^{-2}	1.1×10^{-3}
b	5.2×10^{-3}	4.5×10^{-3}	4.0×10^{-2}	1.1×10^{-2}	5.0×10^{-4}
c	1.4×10^{-1}	1.7×10^{-3}	1.8×10^{-1}	2.6×10^{-3}	3.8×10^{-4}
d	3.6×10^{-1}	2.7×10^{-3}	3.2×10^{-1}	1.6×10^{-2}	5.9×10^{-4}

Table 21. First-order time errors for lunar descent.

Case	t_1	t_2	t_f
a	3.3×10^{-2}	3.6×10^{-1}	1.6×10^{-1}
b	1.3×10^{-2}	1.4×10^{-1}	8.8×10^{-2}
c	5.0×10^{-2}	4.4×10^{-2}	1.2×10^{-1}
d	1.2×10^{-1}	3.8×10^{-2}	2.8×10^{-1}

Tables 22 and 23 show the second-order, absolute errors in the unknown initial costates, switching times, and final time.

Table 22. Second-order Lagrange multiplier errors for lunar descent.

Case	λ_{r_0}	λ_{v_0}	λ_{ϕ_0}	λ_{ω_0}	λ_{m_0}
a	9.7×10^{-2}	6.4×10^{-2}	2.0×10^{-1}	2.6×10^{-3}	7.4×10^{-4}
b	6.8×10^{-2}	2.9×10^{-3}	3.8×10^{-2}	4.6×10^{-3}	4.1×10^{-4}
c	7.1×10^{-2}	3.6×10^{-3}	1.0×10^{-1}	8.6×10^{-3}	4.6×10^{-4}
d	7.1×10^{-2}	9.2×10^{-3}	6.6×10^{-3}	7.6×10^{-3}	9.1×10^{-4}

Table 23. Second-order time errors for lunar descent.

Case	t_1	t_2	t_f
a	3.9×10^{-2}	1.3×10^{-1}	2.2×10^{-1}
b	3.1×10^{-2}	8.1×10^{-2}	1.0×10^{-1}
c	3.3×10^{-2}	1.0×10^{-1}	1.0×10^{-1}
d	4.5×10^{-2}	1.9×10^{-1}	2.2×10^{-1}

Table 24 shows computation times for exact and approximate solutions. All calculations are done in MATLAB on an eMachines ET1810-01 running Ubuntu Linux 10.04. Exact solutions are calculated using indirect shooting. Computation times for the approximate solutions are significantly less because the method is non-iterative and requires only algebraic operations.

Table 24. Computation times for lunar descent.

Case	Exact (s)	Approximate (s)
a	2.7	2.0×10^{-2}
b	2.0	8.0×10^{-3}
c	2.0	2.5×10^{-4}
d	2.4	2.4×10^{-4}

C. Discussion

Neighboring solutions are computed for the lunar descent problem, which has nonlinear dynamics, bounded thrust, and free final time. A reference solution is calculated using indirect shooting and neighboring solutions are found for a variety of initial states, landing sites, and thrust magnitudes. Second-order terms do not significantly reduce errors. If first-order errors are satisfactory, higher order terms are unnecessary. If ample computing power is available, higher order terms can reduce error. Compared to indirect shooting, computation time decreases by about three orders of magnitude.

CHAPTER VI
CONCLUSIONS

This work provides first and second-order expressions to approximate neighboring solutions to the m-point boundary value problem. Multi-point problems arise in optimal control because of interior constraints or switching dynamics. Many problems have this form, and so this work fills a void in the study of extremal fields and neighboring optimal control of constrained systems. Only first and second-order terms are written down, but the approach is systematic and higher order expressions can be found similarly.

The constraints and their parameters define an extremal field because any solution to the problem must satisfy the constraints. Computing differentials of constraints gives the first and second-order relationship between the parameters and the solution. One can then compute state differentials and variations, substitute into the constraint differentials, and solve two linear equations for the first and second-order terms of a Taylor series. The differential is key to these developments, and it is a unifying element in the optimization of points, optimal control, and neighboring optimal control.

Four types of problems are worked to illustrate the method: 1) fixed final time, 2) free final time, 3) control constrained, and 4) state constrained. These problems include characteristics common to many optimal control problems such as unknown initial conditions, switch times, interior jumps, and final time. Figures show "graphing accuracy", but a conclusive statement about errors cannot be made. Results are problem specific and depend on the nonlinearity of the system, magnitude of perturbations, and order of approximation.

The method is demonstrated on the lunar descent problem, which has nonlinear dynamics, bounded thrust, and free final time. The control structure is bang-off-bang, and the method approximates the unknown initial conditions, switch times, and final time. Compared to indirect shooting, computation time decreases by about three orders of magnitude. Computational improvements are expected because the strengthened Legendre-Clebsch condition guarantees that the Jacobian is full rank, and the method is non-iterative.

Applications outside of extremal fields and neighboring optimal control include feedback control and guidance. Both give control commands in real-time based on state feedback. If a reference solution is known this method is feasible because of the small computation time.

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APPENDIX A

DERIVATIVES OF FUNCTIONS

Consider the multi-valued vector function $f : \mathbb{R}^n \to \mathbb{R}^m$. In matrix notation,

$$f(x) = \begin{bmatrix} f^1 \\ f^2 \\ \vdots \\ f^m \end{bmatrix}. \tag{A.1}$$

The first derivative is

$$f_x(x) = \begin{bmatrix} f^1_{x_1} & \cdots & f^1_{x_n} \\ \vdots & \ddots & \vdots \\ f^m_{x_1} & \cdots & f^m_{x_n} \end{bmatrix}. \tag{A.2}$$

The second derivative is

$$f_{xx}(x) = \begin{bmatrix} f^1_{x_1x_1} & f^1_{x_1x_2} & \cdots & f^1_{x_2x_1} & f^1_{x_2x_2} & \cdots & f^1_{x_nx_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f^m_{x_1x_1} & f^m_{x_1x_2} & \cdots & f^m_{x_2x_1} & f^m_{x_2x_2} & \cdots & f^m_{x_nx_n} \end{bmatrix}. \tag{A.3}$$

Higher derivatives follow similarly. A Taylor series for the change in f is then

$$\Delta f = f_x \mathrm{d}x + \tfrac{1}{2} f_{xx}(\mathrm{d}x \cdot \mathrm{d}x) + \cdots \tag{A.4}$$

where $\mathrm{d}x$ is a differential and the dot operator is a Kronecker product.

APPENDIX B

ADDITIONAL FIGURES

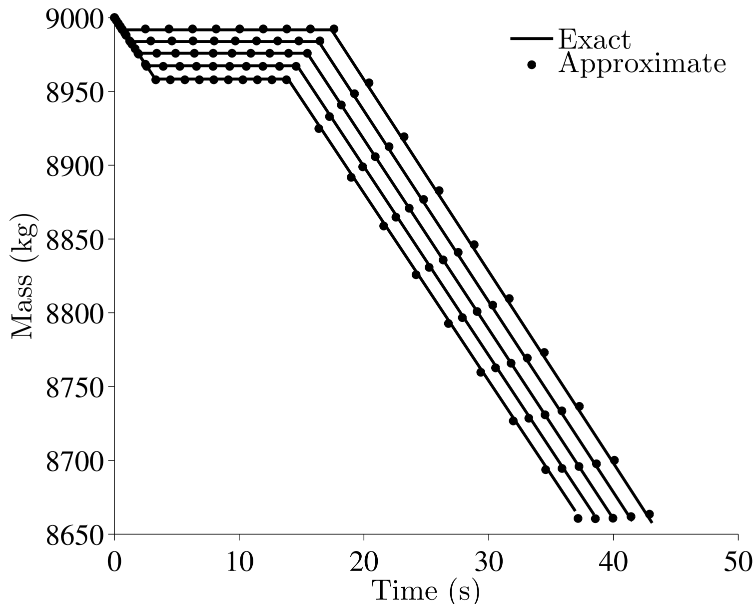

Figure 12. Mass for lunar descent.

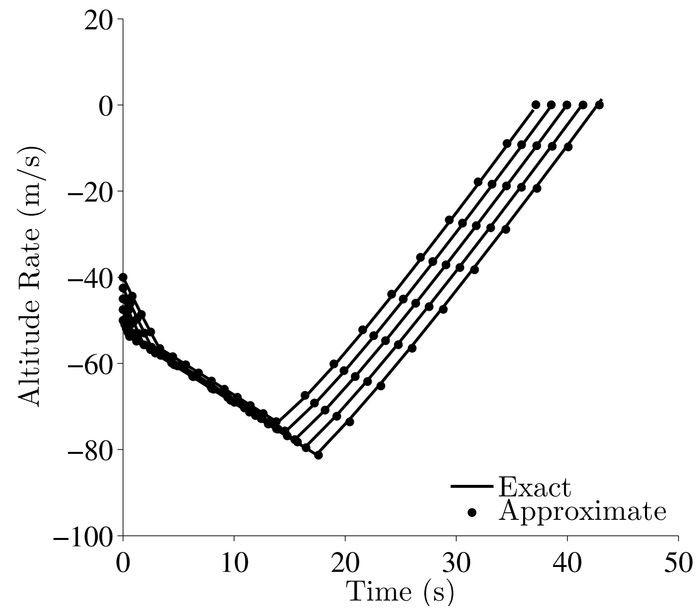

Figure 13. Altitude rate for lunar descent.

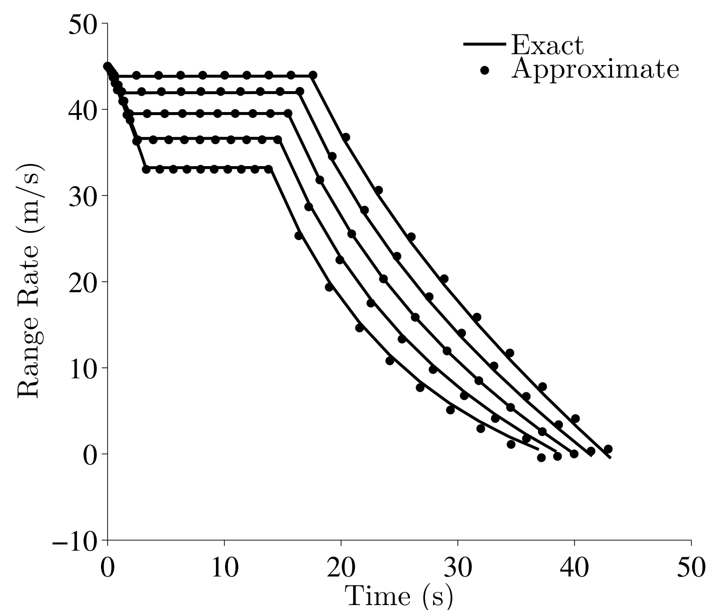

Figure 14. Range rate for lunar descent.

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