REDUCED–SENSING CONTROL METHODS FOR INFINITE–DIMENSIONAL SYSTEMS

A Thesis

by

KRISTEN HOLMSTROM JOHNSON

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2010

Major Subject: Aerospace Engineering

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Approved by:

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ABSTRACT

Reduced–sensing Control Methods

for Infinite–dimensional Systems. (August 2010)

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Dr. Thomas W. Strganac

Infinite dimensional systems such as flexible airplane wings and Vertical Axis Wind Turbine (VAWT) blades may require control to improve performance. Traditional control techniques use position and velocity information feedback, but velocity information for infinite dimensional systems is not easily attained. This research investigates the use of reduced-sensing control for these applications.

Reduced-sensing control uses feedback of position measurements and an associated filter state to stabilize the system dynamics. A filter state is a nonphysical entity that appends an additional ordinary differential equation to the system dynamics. Asymptotic stability of a system using this control approach is confirmed through a sequence of existing mathematical tools. These tools include equilibrium point solutions, Lyapunov functions for stability and control, and Mukherjee and Chen's Asymptotic Stability Theorem. This thesis research investigates the stability of a beam representing an airplane wing or a VAWT blade controlled using feedback of position and filter state terms only. Both of these infinite dimensional systems exhibit asymptotic stability with the proposed reduced-sensing control design. Additionally, the analytical stability response of the VAWT is verified through numerical simulation.

To Adam and Maxwell

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There are no words to describe how grateful I am to my best friend and husband, Adam Johnson. I choose to unconditionally love you every day through the love of Jesus Christ. You always support and encourage me through all of our adventures together.

"You have turned for me my mourning into dancing; You have loosed my sackcloth and girded me with gladness, That my soul may sing praise to You and not be silent O LORD my God, I will give thanks to You forever." Psalm 30:11-12

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CHAPTER I

INTRODUCTION

A finite dimensional system can move at a finite number of points; unlike an infinite dimensional system that can move at an infinite number of points on the structure. Consider an infinite dimensional system such as a wind turbine blade or an airplane wing. These systems need control for damping vibrations that can cause structural damage. For example, wind turbine blades continue to increase in length and consequently experience very large forces at the blade tip. When a large wind gust occurs, these forces may cause the blade tip to strike the turbine tower causing catastrophic failure. In aircraft wing design, finding new lighter, simpler wing configurations, while still achieving stability, is desired. To this end, one can design a flexible structure coupled with a controller as a viable alternative to a very stiff structure. Using feedback control, the blades and wings can be flexible structures that are commanded to a desired shape. However, gathering the necessary measurement data along the structure for position and velocity feedback can be a challenge.

Position measurement data can be attained using strain gauges and then translated into structural displacement using calibration information. This technique has been validated for measuring displacements in a vibrating body [1]. Velocity measurement data can be obtained using velocity transducers or by manipulating position measurement data coupled with measurement time stamps: a numerical time derivative of the position data gives an estimate of the velocity. In many cases, however, the velocity transducer and the time derivative of the position sensor do not provide an adequate estimation for the velocity of the structure. One solution to this problem

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The journal model is *IEEE Transactions on Automatic Control.*

is the concept of reduced-sensing control.

A. Reduced-sensing Control

Reduced-sensing control uses both position measurements and a software filter to provide feedback control information. The software filter is constructed by adding dynamic compensation to the system. After formulating the transfer function equation between the control and the error, one can construct a control equation that includes both the position and filter terms. The governing differential equation for the filter state is defined using terms in the transfer function.

The primary control design goals include stabilizing the system and successfully designing a controller that will drive the system to equilibrium. More specifically, we desire to drive the system to an equilibrium configuration asymptotically. This means that over a period of time, the system will asymptotically approach the equilibrium position [2].

There are several mathematical tools useful for achieving these objectives. Combining the tools in an overall control approach allows one to mathematically confirm asymptotic stability for a reduced-sensing control method. These tools include equilibrium points, Lyapunov functions for stability and control, and Mukherjee and Chen's Asymptotic Stability Theorem.

The concepts described above have been proven for finite dimensional systems, and the challenge of this research is to apply these techniques to an infinite dimensional system [3].

B. Objectives

The objective of this research is to show that reduced-sensing control is mathematically viable for infinite dimensional systems. Two different application problems will be considered in order to illustrate the process and demonstrate asymptotic stability. The application problems include both a cantilevered beam and a Vertical Axis Wind Turbine (VAWT) blade. After developing the control design and presenting the relevant example mathematical models, simulation results that validate the filter state controller for the VAWT blade will be presented and discussed.

CHAPTER II

TOOLS

In this thesis, several existing engineering tools are applied to achieve reduced-sensing control for infinite-dimensional systems. This chapter describes the following tools in detail: equilibrium points, Lyapunov functions, Mukherjee and Chen's theorem, and filter states for control.

A. Equilibrium Points

The equilibrium point of the system is the configuration where a system will stay at rest [2]. When the system is perturbed from rest, it is desired for the control to drive the system back to this equilibrium point. The equilibrium point of a second order system can be found by setting the velocity and acceleration terms to zero. When the equilibrium point found is nonzero, a simple change of variables can create a zero equilibrium point [2].

To illustrate this tool, consider a simple pendulum with the dynamical model displayed below.

$$\ddot{\theta} + \sin\theta = 0 \tag{2.1}$$

Setting the velocity and acceleration terms to zero, one can find an infinite number of equilibrium points, as shown in Eq. (2.2) where $\boldsymbol{\theta}$ is defined in Eq. (2.3).

$$\boldsymbol{\theta}_* = n\pi \quad n = 0, 1, 2, \ldots \tag{2.2}$$

$$\boldsymbol{\theta} = \left\{ \begin{array}{c} \theta \\ \dot{\theta} \end{array} \right\} \tag{2.3}$$

A system may have a unique equilibrium point, an infinite number of equilibrium points, or no equilibrium point.

B. Lyapunov Functions

A Lyapunov function is a locally positive definite function associated with a system that is used to analyze the stability of an equilibrium point [4]. There is not a straightforward method for creating a Lyapunov function; therefore, designing Lyapunov functions is more of an art. The Lyapunov function must meet specific requirements outlined below, in order to prove that system stability. Furthermore, these requirements can be used to design a stabilizing control for a system of interest.

1. Stability Requirements

The following conditions for a Lyapunov function can prove either stability or asymptotic stability. Here, stability is defined as system behavior such that motion near the equilibrium point will remain bounded [2]. That is, the states of a stable system remain within some bounds, unlike those of an unstable system, which become unbounded. The stable Lyapunov function requirements are displayed in Eq. (2.4) where W is the region in which the function is evaluated.

$$\begin{aligned} &\text{Stable:} \\ &V(\mathbf{0}) = \mathbf{0} \\ &V(\boldsymbol{x}) > \mathbf{0} \quad \forall\, \boldsymbol{x} \neq \mathbf{0},\ \boldsymbol{x} \in W \\ &\dot{V}(\mathbf{0}) = \mathbf{0} \\ &\dot{V}(\boldsymbol{x}) \leq \mathbf{0} \quad \forall\, \boldsymbol{x} \neq \mathbf{0},\ \boldsymbol{x} \in W \end{aligned} \tag{2.4}$$

An asymptotically stable system is defined as a system whose states converge to zero over time [2]. This behavior occurs when the $\dot{V}$ equation is less than zero for all

$\boldsymbol{x}$ not equal to zero, rather than less than or equal to zero for this set, as in the stable case. The asymptotically stable requirements are shown in Eq. (2.5).

$$\begin{aligned} &\text{Asymptotically Stable:} \\ &V(\mathbf{0}) = \mathbf{0} \\ &V(\boldsymbol{x}) > \mathbf{0} \quad \forall\, \boldsymbol{x} \neq \mathbf{0},\ \boldsymbol{x} \in W \\ &\dot{V}(\mathbf{0}) = \mathbf{0} \\ &\dot{V}(\boldsymbol{x}) < \mathbf{0} \quad \forall\, \boldsymbol{x} \neq \mathbf{0},\ \boldsymbol{x} \in W \end{aligned} \tag{2.5}$$

2. Control Design

Stability analysis is the primary use of the Lyapunov function technique, however, the requirements can be used for control design [4]. Here, the goal is to design a controller such that the Lyapunov function is stable or asymptotically stable. More specifically, the controller design is constructed using an analysis of $\dot{V}$. To illustrate this technique, once again consider the simple pendulum. Equation (2.6) is the equation of motion (Eq. (2.1)), modified with a controller on the right hand side.

$$\ddot{\theta} + \sin\theta = u \tag{2.6}$$

Here, a Lyapunov function is chosen based on the kinetic and potential energy of the system, as shown in Eq. (2.7).

$$V(\boldsymbol{\theta}) = \frac{1}{2}\dot{\theta}^2 + (1 - \cos\theta) \tag{2.7}$$

Evaluating Eq. (2.7) using the stability conditions proves that the function satisfies the first two stability requirements: $V(\mathbf{0}) = \mathbf{0}$ and $V(\boldsymbol{\theta}) > \mathbf{0}$. Taking the time derivative of this function, one can design the controller such that $\dot{V}(\boldsymbol{\theta}) \leq \mathbf{0}$. Equation (2.8) shows both the time derivative and the control chosen to satisfy the requirement,

$u = -\dot{\theta}$.

$$\dot{V}(\boldsymbol{\theta}) = \dot{\theta}u = -\dot{\theta}^2 \tag{2.8}$$

This controller choice stabilizes the system, but does not provide asymptotic stability. Note that the system is not considered asymptotically stable because θ does not appear in the $\dot{V}$ equation. The value of θ could be any value, including zero, which violates the conditions necessary for asymptotic stability.

C. Mukherjee and Chen

Because there is no straightforward way of creating a Lyapunov function, they are not unique for each system. In fact, a single system may have many Lyapunov functions that capture the system behavior. Thus, a system might be asymptotically stable, but may only be proved asymptotically stable if the correct Lyapunov function is found. To create and evaluate Lyapunov functions until asymptotic stability is proved could be quite tedious. Even if such an approach is taken, a Lyapunov function may never be found that proves asymptotic stability [4].

Mukherjee and Chen created a theorem that takes the analysis of a Lyapunov function further in order to prove asymptotic stability [5]. Their theorem involves taking additional derivatives and evaluating them on a particular set Z. Equation (2.9) is the list of requirements for Mukherjee and Chen's Asymptotic Stability Theorem where integer k is an odd number.

$$
\begin{aligned}
&V(\mathbf{0}) = \mathbf{0} \\
&V(\boldsymbol{x}) > \mathbf{0} \quad \forall\, \boldsymbol{x} \neq \mathbf{0},\ \boldsymbol{x} \in W \\
&\dot{V}(\mathbf{0}) = \mathbf{0} \\
&\dot{V}(\boldsymbol{x}) < \mathbf{0} \quad \forall\, \boldsymbol{x} \neq \mathbf{0},\ \boldsymbol{x} \in Z \subset W \\
&V^{(i)} \equiv \mathrm{d^iV/d}t^i(\boldsymbol{x}) = 0 \quad \forall\, \boldsymbol{x} \in Z \quad i = 1, 2, \ldots, k-1 \\
&V^{(k)} \equiv \mathrm{d^kV/d}t^k(\mathbf{0}) = 0, \quad k \text{ odd} \\
&V^{(k)} \equiv \mathrm{d^kV/d}t^k(\mathbf{0}) < 0, \quad \forall\, \boldsymbol{x} \neq \mathbf{0},\ \boldsymbol{x} \in Z,\ k \text{ odd}
\end{aligned}
\tag{2.9}
$$

To illustrate this theorem, again consider the simple pendulum problem. In the previous section, the controller $u = -\dot{\theta}$ resulted in a stable system, but asymptotic stability was not yet proved. Using Mukherjee and Chen's Asymptotic Stability Theorem, the set Z is defined as shown in Eq. (2.10).

$$
Z = \{\boldsymbol{\theta} \mid \dot{\theta} = 0\} \tag{2.10}
$$

Here, Z includes θ evaluated at all possible values and $\dot{\theta}$ evaluated at zero. Equations (2.11) and (2.12) show the second and third time derivatives of the Lyapunov function evaluated on the set Z.

$$
\ddot{V}\,\big|_{Z} = -2\dot{\theta}\ddot{\theta}\,\big|_{Z} = 0 \tag{2.11}
$$

$$
V^{(3)}\,\big|_{Z} = -2\ddot{\theta}^2 - 2\dot{\theta}\theta^{(3)}\,\big|_{Z} = -2\sin^2\theta < 0 \tag{2.12}
$$

Because Eq. (2.12) is nonzero and negative definite on the set Z with $k = 3$ odd, the simple pendulum system is asymptotically stable with this controller.

Consider a second example of a rolling cart with a rocket attached, a simple linear system as shown in Fig. (1). Equation (2.13) gives a simplified model of the

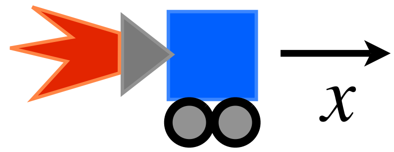

Fig. 1. Rolling cart rocket with the coordinate definition.

system dynamics and Eq. (2.14) is an associated Lyapunov function.

$$\ddot{x} = u \tag{2.13}$$

$$V(\boldsymbol{x}) = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2 \tag{2.14}$$

Equation (2.14) successfully meets the first two Lyapunov requirements: $V(\mathbf{0}) = \mathbf{0}$ and $V(\boldsymbol{x}) > \mathbf{0}$. The first time derivative of V and substitution of the controller design selected, $u = -x - \dot{x}$, are shown in Eq. (2.15) .

$$\dot{V}(\boldsymbol{x}) = x\dot{x} + \dot{x}u = -\dot{x}^2 \tag{2.15}$$

Equation (2.15) satisfies the next two Lyapunov requirements for stability: $\dot{V}(\mathbf{0}) = \mathbf{0}$ and $\dot{V}(\boldsymbol{x}) \leq \mathbf{0}$. As before, the system is not proven to be asymptotically stable because x does not appear in the $\dot{V}$ equation and therefore could be anything; including zero. To investigate asymptotic stability, further analysis can be performed using Mukherjee and Chen's theorem. Equation (2.16) is the set Z on which the second and third time derivatives are evaluated.

$$Z = \{\boldsymbol{x} \mid \dot{x} = 0\} \tag{2.16}$$

Below, Eqs. (2.17) and (2.18) show the second and third time derivatives of the Lyapunov function evaluated on the set Z.

$$\ddot{V}\mid_z \; = -2\dot{x}\ddot{x}\mid_z \; = 0 \tag{2.17}$$

$$V^{(3)}\ |_{z}\ = -2\ddot{x}^2 - 2\dot{x}x^{(3)}\ |_{z}\ = -2x^2 < 0 \tag{2.18}$$

Because Eq. (2.18) is nonzero and negative definite on the set Z, the rolling cart rocket system is asymptotically stable with the chosen controller.

In this rolling cart rocket example problem, the controller design includes both position and velocity terms. However, the velocity information is not always available In the next section, a filter state is introduced that can be included in the control design to provide asymptotic stability for a system lacking velocity information feedback.

D. Filter State

The goal of adding a filter state to the control design is to achieve velocity-free control and still have an asymptotically stable system. In the previous section, the simple pendulum and the rolling cart rocket examples had control designs that used velocity state information. Both of those controllers achieved asymptotic stability with static compensation. Static compensation is the addition of damping or stiffness to a structure to make a degree of freedom easier to control [6]. In some cases, proportional gain alone does not provide a satisfactory response and the dynamics of the system need to be modified. To begin the investigation of the filter state, we have to look at dynamic compensation.

The term dynamic compensation means that the controller behaves like a dynamical system. The controller can either have lead or lag compensation. For a lead controller, the output leads the input in the frequency domain, and for a lag controller, the output lags the input in the frequency domain [7]. Figure (2) shows a block diagram of the system.

A dynamic controller is of the form shown in Eq. (2.19). If $z < a$, the controller

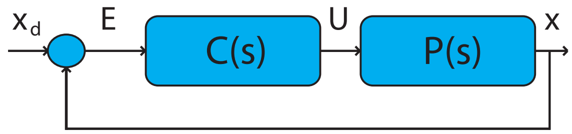

Fig. 2. Block diagram of a system with dynamic compensation.

is a lead compensator, and if $z > a$, the controller is a lag compensator.

$$C(s) = k\frac{s+z}{s+a} \tag{2.19}$$

To determine if the controller needs to be a lead or a lag controller for this system, consider a second order, linear time invariant system without damping, shown in Eq. (2.20).

$$\ddot{x} + {\omega_n}^2 x = 0 \tag{2.20}$$

The characteristic equation of the system is derived from the transfer function of the block diagram shown in Fig. (2).

$$x(s) = P(s)C(s)E(s) \tag{2.21}$$

After substituting for the definition of the error, $E(s) = x_d(s) - x(s)$, algebraic manipulation creates a relationship between x and x_d.

$$x(s) = P(s)C(s)(x_d(s) - x(s)) \tag{2.22}$$

$$x(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} x_d(s) \tag{2.23}$$

Setting the denominator of the right hand side of Eq. (2.23) equal to zero and substituting terms for $P(s)$ and $C(s)$ produces the following characteristic equation.

$$s^3 + as^2 + ({\omega_n}^2 + k)s + (a{\omega_n}^2 + kz) = 0 \tag{2.24}$$

Performing a stability analysis using a Routh Array determines that the stabilizing criterion are $a > 0$, $z > 0$, and $a > z$. Thus, the dynamic compensation must be a lead controller. Further investigation requires the evaluation of the transfer function between the controller and the error. Equations (2.25) through (2.27) show the transfer function and two definitions.

$$U(s) = kE(s) + k_2 x_0(s) \tag{2.25}$$

$$k_2 \equiv k(z - a) \tag{2.26}$$

$$x_0(s) \equiv \frac{1}{s + a} E(s) \tag{2.27}$$

For stability, the constant k_2 has to be less than zero, and x_0 is defined as the filter state. After taking the inverse Laplace Transform of $U(s)$ and $x_0(s)$, we find the equations for the controller and the filter state, shown in Eqs. (2.28) and (2.29) respectively [2].

$$u(t) = -kx + k_2 x_0 \tag{2.28}$$

$$\dot{x}_0(t) = -a x_0 - x \tag{2.29}$$

The controller contains position and filter information only, velocity information is not included. The filter state has an additional differential equation that is appended to the system. This state has no particular physical meaning; it is just an entity that is numerically integrated in real time. This controller-filter combination can achieve asymptotic stability when velocity level information is not available as shown through example problems in subsequent chapters.

CHAPTER III

FINITE DIMENSIONAL SYSTEMS

The filter state technique has been verified to effectively control finite dimensional systems [3]. This thesis not only applies the filter state control techniques to infinite dimensional systems for the first time, it also investigates rheonomic motion for the first time. This chapter describes the difference between a scleronomic system and a rheonomic system, and then presents an example problem of concepts for applying reduced-sensing control to a finite dimension rheonomic system. The following chapter extends these concepts to infinite dimensional scleronomic and rheonomic systems and develops an asymptotic stability verification for a reduced-sensing control technique that includes a filter state.

A. Scleronomic v. Rheonomic

This section describes the difference between a scleronomic and a rheonomic system[1]. A scleronomic system is defined as a system that does not explicitly depend on time [8]. Some examples of this type of system include a simple pendulum, a rolling cart with a rocket attached, and a vibrating airplane wing. In the previous chapter, the two finite dimensional example problems described were both scleronomic systems. A rheonomic system is defined as a system that explicitly depends on time [8]. Some examples of this kind of system include a ball with a spring attached in a rotating tube and a Vertical Axis Wind Turbine blade. These two types of systems are analyzed in examples throughout the remainder of the thesis.

[1] Here a scleronomic/rheonomic system is truthfully a system with a scleronomic/rheonomic constraint [8].

B. Finite Dimension Rheonomic Example Problem

This section presents an example problem in order to demonstrate the filter state technique for a finite dimension rheonomic system. Consider a ball connected to a spring in a rotating tube. Figure (3) shows the system and the coordinates used to define the system.

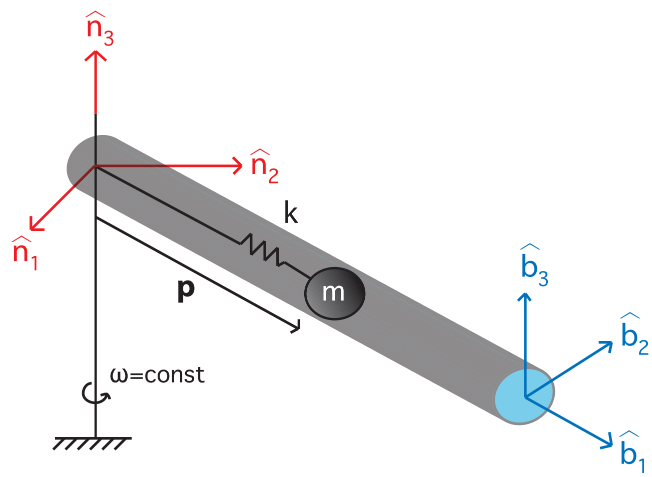

Fig. 3. Ball in rotating tube with coordinate definitions.

The position vector, $\boldsymbol{p}$ shown in Eq. (3.1), is the sum of r', the equilibrium position of the spring, and r, the distance the ball is perturbed from the equilibrium.

$$\boldsymbol{p} = (r' + r)\hat{\boldsymbol{b}}_{\mathbf{1}} \tag{3.1}$$

Note that the equilibrium position of the spring is not the same as the equilibrium configuration of the entire system. Equation (3.2) describes the angular velocity of the tube, which is constant. Equation (3.3) gives the translational velocity vector found by using the transport theorem [9].

$$\boldsymbol{\omega} = \omega\hat{\boldsymbol{b}}_{\mathbf{3}} \tag{3.2}$$

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{b_1}} + \omega(r' + r)\hat{\boldsymbol{b_2}} \tag{3.3}$$

Next, the equation of motion is derived using Lagrangian dynamics. The Lagrangian is defined as the difference between the the kinetic and potential energy, shown in Eq. (3.4) for this example.

$$L = \frac{1}{2}m[\dot{r}^2 + \omega^2(r' + r)^2] - \frac{1}{2}kr^2 \tag{3.4}$$

Equation (3.5) is the formula for obtaining the equations of motion using a Lagrangian function, where the subscript (np) denotes general forces that are not acquired from potential functions and $k = 1, \ldots, n$ [10].

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_{k(np)} \tag{3.5}$$

The partial derivatives of the Lagrangian function for this problem are as follows.

$$\frac{\partial L}{\partial r} = (m\omega^2 - k)r + m\omega^2 r' \tag{3.6}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} \tag{3.7}$$

Below is the equation of motion after subtracting Eq. (3.6) from Eq. (3.7) for the right hand side and finding that there are no generalized forces for the left hand side.

$$m\ddot{r} - (m\omega^2 - k)r - m\omega^2 r' = 0 \tag{3.8}$$

Using this equation of motion, the equilibrium point can be found as previously described in Chapter II. Setting the acceleration and velocity terms of the governing equation to zero results in a nonzero equilibrium point given by Eq. (3.9).

$$r_* = \frac{m\omega^2 r'}{k - m\omega^2} \neq 0 \tag{3.9}$$

Note that the equilibrium point is only nonzero if the constant r' is nonzero. As previously mentioned, when the equilibrium point produces a nonzero result it is desireable to perform a change in variables. Equation (3.10) presents a new variable, x, that is equal to the difference between r and r_*. Because r_* is a constant, it does not appear in the first and second time derivatives of x.

$$x = r - r_* \tag{3.10}$$

$$\dot{x} = \dot{r} \tag{3.11}$$

$$\ddot{x} = \ddot{r} \tag{3.12}$$

After substituting Eqs. (3.10) through (3.12) into Eq. (3.8), Eq. (3.13) gives the new equation of motion with the new variable x and control u. To include a filter state in the development, simply include Eq. (2.29) with the equation of motion to get an equation set that describes the full system dynamics. Equation (3.15) is the new equilibrium point for the system.

$$m\ddot{x} - (m\omega^2 - k)x = u \tag{3.13}$$

$$\dot{x}_0 = -ax_0 - x \tag{3.14}$$

$$x_* = 0 \tag{3.15}$$

For the newly defined system, the next step is to create a Lyapunov function. In the previous example, the Lyapunov function was created using the energy of the system. In this example, a more appropriate choice is a function that appears similar to an energy function but is not derived explicitly from the energy, as shown in Eq. (3.16). This function meets the first two stability requirements, $V(\mathbf{0}) = \mathbf{0}$ and $V(\boldsymbol{x}) > \mathbf{0}$.

$$V = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) + \frac{1}{2}kx^2 + \frac{1}{2}(ax_0 + x)^2 \tag{3.16}$$

Equation (3.17) shows the first time derivative of the Lyapunov equation. Setting the terms in brackets equal to zero and solving for the control results in a controller with position and filter terms only, given by Eq. (3.18).

$$\dot{V} = \dot{x}[u + (2m\omega^2 + 1)x + ax_0] - \frac{1}{2}(ax_0 + x)^2 \tag{3.17}$$

$$u = -x(2m\omega^2 + 1) - ax_0 \tag{3.18}$$

The first time derivative passes the last two requirements for stability, $\dot{V}(\mathbf{0}) = 0$ and $\dot{V}(\boldsymbol{x}) \leq 0$, after substitution of the selected control for u. The Lyapunov function passes all of the requirements for stability. However, asymptotic stability is not proven because $\dot{x}$ does not appear in the resulting equation for $\dot{V}$, Eq. (3.19), and therefore could be any value, including zero.

$$\dot{V} = -\frac{1}{2}(ax_0 + x)^2 \tag{3.19}$$

Recall that Mukherjee and Chen's Asymptotic Stability Theorem tests for asymptotic stability by analyzing the Lyapunov function further. To simplify the algebra, first define $l = ax_0 + x$. Using this definition, Eq. (3.20) gives the set Z against which the second and third time derivatives are evaluated.

$$Z = \{\boldsymbol{x} \mid l = 0\} \tag{3.20}$$

Equations (3.21) and (3.22), respectively, are the second and third time derivatives of the Lyapunov function evaluated on the set Z. The second time derivative evaluated on the set Z is zero, which correctly corresponds to the requirements stated in Chapter II. The third time derivative evaluated on the set Z is less than or equal to zero, or negative semidefinite. This result requires additional analysis of the Lyapunov function to prove asymptotic stability.

$$\ddot{V} \mid_z = -2al\dot{l} \mid_z = 0 \tag{3.21}$$

$$V^{(3)} \mid_{z} = -2a\dot{l}^2 - 2al\ddot{l} \mid_{z} = -2a\dot{x}^2 \leq 0 \tag{3.22}$$

To continue, the next step is to define a new set Z' to evaluate the fourth and fifth time derivatives of the Lyapunov equation. Equation (3.23) gives the new set and Eqs. (3.24) and (3.25) then show the fourth and fifth time derivatives evaluated on the new set Z'.

$$Z' = \{\boldsymbol{x} \mid l = 0 \ \& \ \dot{x} = 0\} \tag{3.23}$$

$$V^{(4)} \mid_{z'} = -6a\dot{l}\ddot{l} - 2all^{(3)} \mid_{z'} = 0 \tag{3.24}$$

$$\begin{aligned} V^{(5)} \mid_{z'} &= -6a\ddot{l}^2 - 8a\dot{l}l^{(3)} - 2all^{(4)} \mid_{z'} \\ &= -6a\ddot{l}^2 \mid_{z'} \\ &= -6\frac{a}{m^2}(m\omega^2 + k)^2 x^2 < 0 \end{aligned} \tag{3.25}$$

The fifth time derivative is always less than zero, or negative definite, and therefore the system is asymptotically stable with a control composed of position and filter terms only. This proves that asymptotic stability can be achieved for a finite dimensional rheonomic system using a control with no velocity feedback terms.

CHAPTER IV

CONTINUOUS SYSTEMS

A continuous structure is defined as such because deflections can occur at any point along the structure. Most long slender objects are modeled as continuous systems, also known as infinite dimensional systems. In this chapter, the primary goal is to apply reduced-sensing control to an infinite dimensional system. Similar techniques are applied to two example problems considered in this chapter as were implemented in the "ball in a tube" example from Ch. III. The first section describes an example problem of a continuous scleronomic system, a cantilevered beam, and the second section presents an example problem of a continuous rheonomic system, a Vertical Axis Wind Turbine (VAWT) blade.

The primary challenge of this research was overcome while working on the problems presented in this chapter. Finite dimensional systems are described by ordinary differential equations and employ an ordinary differential equation filter state. Infinite dimensional systems are described by partial differential equations and the challenge was to determine whether the filter state should be an ordinary differential equation or a partial differential equation.

A. Cantilevered Beam

Consider a cantilevered beam with a load distributed along the upper surface. Recall that this system is a scleronomic system because its dynamics do not depend explicitly on time. Figure (4) shows the beam and the coordinate frame definition for this problem. Displacement in the $\hat{\boldsymbol{b}}_2$ direction is described by the coordinate y, which is a function of both space and time.

Equation (4.1) gives the equation of motion for a cantilevered beam with a dis-

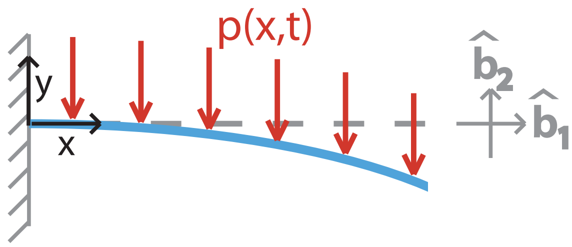

Fig. 4. Cantilevered beam with coordinate definitions.

tributed load. This equation is a partial differential equation because the variable y is differentiated in both time (as indicated with over dots) and space (as indicated with primes).

$$\rho\ddot{y} + EIy'''' = p(x,t) \tag{4.1}$$

Because the equation of motion is a linear equation, the first step to solving this problem is to perform a separation of variables. The separation of variables technique relies on the assumption that y can be written as the product of two separate functions of space and time. The variables of the distributed load are also separated, but using a different function of time coupled with the same function of space as y. Equations (4.2) and (4.3) give the separation of variables definitions [11].

$$y = f(t)g(x) \tag{4.2}$$

$$p = h(t)g(x) \tag{4.3}$$

After substituting Eqs. (4.2) and (4.3) into Eq. (4.1), one has an equation where each variable is a function of one parameter.

$$\rho\ddot{f}g + EIfg'''' = hg \tag{4.4}$$

The next step is to divide this equation by f, g, and ρ. Variables explicitly a function of time are placed on the left hand side of the equation, where as the variables in space are moved to the right hand side of the equation.

$$\frac{\ddot{f}}{f} - \frac{h}{\rho f} = -\frac{EI}{\rho}\frac{g''''}{g} \tag{4.5}$$

Because we are able to separate the variables on either side of the equation, we can set them both equal to an arbitrary constant known as the separation constant, $-\omega^2$ [11]. Equation (4.6) is the resulting equation where $h' = \frac{h}{\rho}$, and h' represents the controller to be designed.

$$\frac{\ddot{f}}{f} - \frac{h'}{f} = -\frac{EI}{\rho}\frac{g''''}{g} = -\omega^2 \tag{4.6}$$

First consider the variables that are a function of x. Equation (4.7) is the space equation after algebraically manipulating the right side of Eq. (4.6), where $\beta^4 \equiv \frac{\omega^2\rho}{EI}$. Equation (4.8) is the list of boundary conditions [4].

$$g'''' - \beta^4 g = 0 \tag{4.7}$$

$$\begin{aligned} g(0) &= 0 \\ g'(0) &= 0 \\ g''(L) &= 0 \\ g'''(L) &= 0 \end{aligned} \tag{4.8}$$

The general solution of Eq. (4.7) is given by the following function.

$$g(x) = C_1 \sin(\beta x) + C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x) \tag{4.9}$$

After applying the boundary conditions, the result is a set of linear equations. The trivial solution of $C_i = 0$ is not of interest for this application. To find the non-trivial solution, the determinant of the coefficients must vanish. This condition is shown in Eq. (4.10).

$$\cos(\beta L)\cosh(\beta L) = -1 \tag{4.10}$$

Equation (4.10) results in an infinite number of values for β and therefore an infinite number of values for ω, as shown in the relationship below.

$$\omega_i = (\beta_i L)^2 \sqrt{\frac{EI}{\rho L^4}} \tag{4.11}$$

An infinite number of ω values results in an infinite number of f and g solutions, which in turn result in an infinite number of solutions for the displacement, y. Now that the ω values are known, we can proceed to determining the solutions for the variables that are explicit functions of time.

A solution exists for the space part of Eq. (4.6); therefore, we can now focus on the time part of this equation. Because this equation is an ordinary differential equation, the same tools and approach can be used as for the finite dimensional system presented in Ch. III. Equation (4.12) restates the time equation. Setting the acceleration term to zero gives the equilibrium point of $\boldsymbol{f}_* = 0$.

$$\ddot{f} + \omega^2 f = h' \tag{4.12}$$

Here, the Lyapunov function is derived from the kinetic and potential energy. Because there are an infinite number of f and g, a new definition of y is now appropriate.

$$y = \sum_{i=1}^{\infty} f_i(t) g_i(x) \tag{4.13}$$

The equation for the kinetic energy of the system, below, is integrated over the length of the beam. Because the integral is over space, the time variables can be removed from the integrand, which results in an integrand constant with respect to time. This property allows us to define a constant, M_{ij}, that represents the integral of the space functions over the length of the beam.

$$T = \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} M_{ij}\dot{f}_i\dot{f}_j \tag{4.14}$$

$$M_{ij} = \int_0^L \rho g_i(x) g_j(x)\mathrm{d}x \tag{4.15}$$

In a similar way, the potential energy can be integrated over the length of the beam. This integral of the space variables can be used to form a constant, K_{ij}, that is then substituted into the potential energy equation.

$$V = \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K_{ij} f_i f_j \tag{4.16}$$

$$K_{ij} = \int_0^L EI g_i''(x) g_j''(x)\mathrm{d}x \tag{4.17}$$

Equation (4.18) gives the Lyapunov function, here denoted as J, for the system including the kinetic energy, potential energy, and filter state terms. Note that, because there are infinite possible f, there are also infinite possible filters.

$$J = \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} M_{ij}\dot{f}_i\dot{f}_j + \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K_{ij} f_i f_j + \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(a_i f_{0_i} + f_i)(a_j f_{0_j} + f_j) \tag{4.18}$$

For simplicity all of the states, including the filter states, can be written as infinite dimensional vectors. The constants, $\boldsymbol{M}$ and $\boldsymbol{K}$, then become infinite dimensional matrices. Our new Lyapunov equation is as follows, where $\boldsymbol{A}$ is now defined as a diagonal matrix of the a_i values.

$$J = \frac{1}{2}\dot{\boldsymbol{f}}^T\boldsymbol{M}\dot{\boldsymbol{f}} + \frac{1}{2}\boldsymbol{f}^T\boldsymbol{K}\boldsymbol{f} + \frac{1}{2}(\boldsymbol{A}\boldsymbol{f_0} + \boldsymbol{f})^T(\boldsymbol{A}\boldsymbol{f_0} + \boldsymbol{f}) \tag{4.19}$$

This Lyapunov function satisfies the first two stability requirements. After taking the first time derivative, the controller can be designed such that it includes position and filter terms only. The first time derivative of the Lyapunov function and the controller are below where $\boldsymbol{\Omega}$ is a diagonal matrix of ${\omega_i}^2$ values.

$$\dot{J} = \dot{\boldsymbol{f}}^T[\boldsymbol{M}\ddot{\boldsymbol{f}} + (\boldsymbol{K}+\boldsymbol{I})\boldsymbol{f} + \boldsymbol{A}\boldsymbol{f_0}] - (\boldsymbol{A}\boldsymbol{f_0}+\boldsymbol{f})^T\boldsymbol{A}(\boldsymbol{A}\boldsymbol{f_0}+\boldsymbol{f}) \tag{4.20}$$

$$\boldsymbol{h'} = [\boldsymbol{\Omega} - \boldsymbol{M}^{-1}(\boldsymbol{K}+\boldsymbol{I})]\boldsymbol{f} - \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{f_0} \tag{4.21}$$

After substitution of the controller, Eq. (4.22) gives the resulting first time derivative of the Lyapunov equation. This equation is negative semidefinite and it is therefore necessary to use Mukherjee and Chen's Asymptotic Stability Theorem to verify if the system is actually asymptotically stable.

$$\dot{J} = -(\boldsymbol{A}\boldsymbol{f_0}+\boldsymbol{f})^T\boldsymbol{A}(\boldsymbol{A}\boldsymbol{f_0}+\boldsymbol{f}) \tag{4.22}$$

The set Z is defined as all $\boldsymbol{f}$ such that $\boldsymbol{l} = 0$, where $\boldsymbol{l} = \boldsymbol{A}\boldsymbol{f}_0 + \boldsymbol{f}$. As before, the second and third time derivatives of the Lyapunov function are evaluated on this set. The second time derivative is zero on the set Z, and the third time derivative is less than or equal to zero on the set Z.

$$Z = \{\boldsymbol{f} \mid \boldsymbol{l} = 0\} \tag{4.23}$$

$$\ddot{J}\mid_Z = -\dot{\boldsymbol{l}}^T\boldsymbol{A}\boldsymbol{l} - \boldsymbol{l}^T\boldsymbol{A}\dot{\boldsymbol{l}}\mid_Z = 0 \tag{4.24}$$

$$J^{(3)}\mid_Z = -\ddot{\boldsymbol{l}}^T\boldsymbol{A}\boldsymbol{l} - 2\dot{\boldsymbol{l}}^T\boldsymbol{A}\dot{\boldsymbol{l}} - \boldsymbol{l}^T\boldsymbol{A}\ddot{\boldsymbol{l}}\mid_Z = -2\dot{\boldsymbol{f}}^T\boldsymbol{A}\dot{\boldsymbol{f}}\mid_Z \leq 0 \tag{4.25}$$

Because the third derivative is negative semidefinite, a new set is defined to evaluate the next two time derivatives. The new set Z' is defined for all $\boldsymbol{f}$ such that $\boldsymbol{l} = 0$ and $\dot{\boldsymbol{f}} = 0$. After evaluating the fourth and fifth derivatives on the new set, the fourth derivative equals zero and the fifth derivative is always less than zero.

$$Z' = \{\boldsymbol{f} \mid \boldsymbol{l} = 0 \ \& \ \dot{\boldsymbol{f}} = 0\} \tag{4.26}$$

$$J^{(4)}\mid_{Z'} = -\boldsymbol{l}^{(3)^T}\boldsymbol{A}\boldsymbol{l} - 3\ddot{\boldsymbol{l}}^T\boldsymbol{A}\dot{\boldsymbol{l}} - 3\dot{\boldsymbol{l}}^T\boldsymbol{A}\ddot{\boldsymbol{l}} - \boldsymbol{l}^T\boldsymbol{A}\boldsymbol{l}^{(3)}\mid_{Z'} = 0 \tag{4.27}$$

$$\begin{aligned} J^{(5)}\mid_{Z'} &= -\boldsymbol{l}^{(4)^T}\boldsymbol{A}\boldsymbol{l} - 4\boldsymbol{l}^{(3)^T}\boldsymbol{A}\dot{\boldsymbol{l}} - 6\ddot{\boldsymbol{l}}^T\boldsymbol{A}\ddot{\boldsymbol{l}} - 4\dot{\boldsymbol{l}}^T\boldsymbol{A}\boldsymbol{l}^{(3)} - \boldsymbol{l}^T\boldsymbol{A}\boldsymbol{l}^{(4)}\mid_{Z'} \\ &= -6\ddot{\boldsymbol{l}}^T\boldsymbol{A}\ddot{\boldsymbol{l}}\mid_{Z'} \\ &= -6\boldsymbol{f}^T\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{f} < 0 \end{aligned} \tag{4.28}$$

Because the fifth derivative (k odd) is negative definite on the set Z', the system is asymptotically stable with a controller composed of only position and filter state terms. This example shows that asymptotic stability is achievable for an infinite dimensional system using a controller with position and filter state terms only. The cantilevered beam could easily be replaced by many other scleronomic systems, such as an airplane wing. In this application, strain gauges could be placed on the beam to find position information, and this information could be integrated to construct the filter states for the controller.

B. Vertical Axis Wind Turbine Blade

The Vertical Axis Wind Turbine (VAWT) blade was chosen as the example problem for a rheonomic infinite dimensional system because many companies, such as We-Power and Windspire, are making VAWTs for commercial and residential properties to provide a renewable energy source to offset the cost of traditional energy from the city's power grid [12, 13].

1. Asymtptic Stability Investigation

The model chosen for this development is a VAWT blade with a pinned-pinned boundary condition. The blade is spinning about an axis parallel to the length of the blade at a distance, R, away. Figures (5) and (6) depict the system and the coordinate definitions used in this example. This model is similar to the WePower Falcon Series VAWT [12].

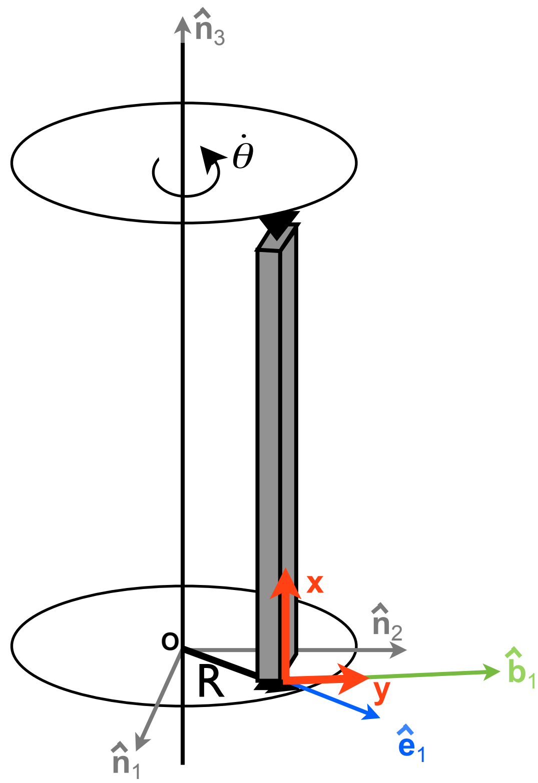

Fig. 5. VAWT blade side view with coordinate definitions.

For this development, the assumptions made are that the VAWT is spinning at a constant rate ($\dot{\theta}$), and the blade is positioned at a constant angle (ϕ). The position vector to any point on the blade is a function of x and y, while all the other quantities are constants.

$$\boldsymbol{p} = (R + y\cos\phi)\hat{\boldsymbol{e}_1} + y\sin\phi\hat{\boldsymbol{e}_2} + x\hat{\boldsymbol{e}_3} \tag{4.29}$$

Equation (4.30) is the angular velocity vector, which is constant with respect to time.

$$\boldsymbol{\omega} = \dot{\theta}\hat{\boldsymbol{e}_3} \tag{4.30}$$

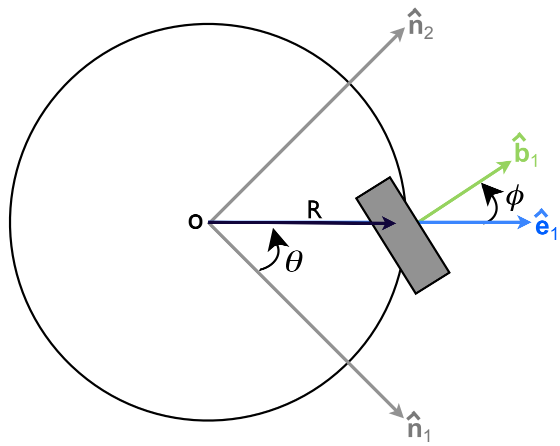

Fig. 6. VAWT blade top view with coordinate definitions.

The velocity vector comes from applying the Transport Theorem using the position and angular velocity vectors [9].

$$\boldsymbol{v} = (\dot{y}\cos\phi - \dot{\theta} y \sin\phi)\hat{\boldsymbol{e}}_1 + [\dot{y}\sin\phi + \dot{\theta}(R + y\cos\phi)]\hat{\boldsymbol{e}}_2 \tag{4.31}$$

Hamilton's principle is a valuable method for deriving equations of motion for continuous systems [10]. To begin with Hamilton's principle, consider the kinetic and potential energy of the system.

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \rho \boldsymbol{v}\cdot\boldsymbol{v}\mathrm{d}x \\ &= \frac{1}{2}\int_0^L \rho[(\dot{y}\cos\phi - \dot{\theta} y\sin\phi)^2 + (\dot{y}\sin\phi + \dot{\theta}(R + y\cos\phi))^2]\mathrm{d}x \end{aligned} \tag{4.32}$$

$$V = \frac{1}{2}\int_0^L EI{y''}^2\mathrm{d}x \tag{4.33}$$

Lagrange's equation is formed by taking the difference of the kinetic and potential energy.

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}\int_0^L \{\rho[(\dot{y}\cos\phi - \dot{\theta}y\sin\phi)^2 + (\dot{y}\sin\phi + \dot{\theta}(R + y\cos\phi))^2] - EIy''^2\}\mathrm{d}x \end{aligned} \tag{4.34}$$

The variation of the Lagrangian is integrated over time and set equal to zero.

$$\int_0^t \delta L = \int_0^t \int_0^L \{\rho[(\dot{y}+\dot{\theta}R\sin\phi)\delta\dot{y}+(\dot{\theta}^2 y+\dot{\theta}^2 R\cos\phi)\delta y]-EIy''\delta y''\}\mathrm{d}x\mathrm{d}t = 0 \tag{4.35}$$

The equation can be divided into three sections after integrating by parts.

$$\int_0^t \int_0^L \{\rho(\ddot{y}-\dot{\theta}^2 y-\dot{\theta}^2 R\cos\phi)+EIy''''\}\delta y\mathrm{d}x\mathrm{d}t+\int_0^t \{EIy''\delta y' \mid_L \}\mathrm{d}t-\int_0^t \{EIy'''\delta y \mid_L \}\mathrm{d}t = 0 \tag{4.36}$$

The first section, after applying the Mean Value Theorem, results in the equation of motion for the system [14].

$$\rho(\ddot{y} - \dot{\theta}^2 y - \dot{\theta}^2 R\cos\phi) + EIy'''' = 0 \tag{4.37}$$

The last two sections, set equal to zero, produce the four boundary conditions associated with the pinned-pinned assumption.

$$\begin{aligned} y(0,t) &= 0 \\ y(L,t) &= 0 \\ y''(0,t) &= 0 \\ y''(L,t) &= 0 \end{aligned} \tag{4.38}$$

Now we assume there is a distributed load on the blade, which appears on the right hand side of the equation of motion as $p(x,t)$.

$$\rho(\ddot{y} - \dot{\theta}^2 y - \dot{\theta}^2 R\cos\phi) + EIy'''' = p(x,t) \tag{4.39}$$

Dividing everything by ρ and moving the constants to the right hand side allows one to define a new variable, $\widetilde{p}$, that is the pseudo distributed load.

$$\begin{aligned}\ddot{y} - \dot{\theta}^2 y + \frac{EI}{\rho} y'''' &= \frac{p(x,t)}{\rho} + \dot{\theta}^2 R \cos\phi \\ &= \widetilde{p}(x,t) \end{aligned} \tag{4.40}$$

Because the equation of motion is linear, one can perform separation of variables as before in the cantilevered beam problem. The variable y is separated into a function of time, f, and a function of space, g. The distributed load p would have been separated into a function of time, h and a function of space, g; the pseudo distributed load $\widetilde{p}$, is separated into a function of time, $\widetilde{h}$, and a function of space, g. Note that y and $\widetilde{p}$ include the same function of space, g.

$$y = f(t)g(x) \tag{4.41}$$

$$\widetilde{p} = \widetilde{h}(t)g(x) \tag{4.42}$$

The equation of motion is in terms of separate functions of space and time after substituting Eqs. (4.41) and (4.42) into Eq. (4.40).

$$\ddot{f}g - \dot{\theta}^2 fg + \frac{EI}{\rho} fg'''' = \widetilde{h}g \tag{4.43}$$

As previously stated, we are able to separate the variables on either side of the equation, and therefore we can set them both equal to an arbitrary constant known as the separation constant, $-\omega^2$ [11].

$$\frac{\ddot{f}}{f} - \dot{\theta}^2 - \frac{\widetilde{h}}{f} = -\frac{EI}{\rho}\frac{g''''}{g} = -\omega^2 \tag{4.44}$$

Consider the variables of space in the previous equation. The variables present an ordinary differential equation where $\beta^4 \equiv \frac{\omega^2 \rho}{EI}$. Equation (4.46) is the list of boundary conditions for the pinned-pinned blade assumption.

$$g'''' - \beta^4 g = 0 \tag{4.45}$$

$$\begin{aligned} g(0) &= 0 \\ g(L) &= 0 \\ g''(0) &= 0 \\ g''(L) &= 0 \end{aligned} \tag{4.46}$$

For this set of boundary conditions, the solution to Eq. (4.45) is shown below [15].

$$g(x) = \sin\left(\frac{i\pi x}{L}\right) \tag{4.47}$$

This brings forth the value for the constant ω where $i = 0 \ldots \infty$ [15].

$$\omega_i = \frac{(i\pi)^2}{2\pi}\sqrt{\frac{EI}{mL^3}} \tag{4.48}$$

An infinite number of ω values results in an infinite number of f and g solutions, which in turn result in an infinite number of solutions for the displacement, y. Now that the ω values are known, we can proceed to determining the solutions for the variables that are explicit functions of time.

A solution exists for the space portion of Eq. (4.44); therefore, we can look at the time portion of this equation. Because this equation is an ordinary differential equation, the same tools and same approach can be used as for a finite dimensional system as presented in Ch. III. Equation (4.49) restates the time equation shown in matrix form where the states are infinite dimensional vectors. The quantity $\widetilde{\boldsymbol{h}}$ is the controller to be designed, and the filter state differential equation is included with the equation of motion to produce a set that describes the dynamics of the system.

The matrix quantity $\boldsymbol{K}$ is a diagonal matrix of the $({\omega_i}^2 - \dot{\theta}^2)$ values, and the matrix quantity $\boldsymbol{A}$ is a diagonal matrix of the a_i values.

$$\ddot{\boldsymbol{f}} + \boldsymbol{K}\boldsymbol{f} = \widetilde{\boldsymbol{h}}$$
$$\dot{\boldsymbol{f}}_0 = -\boldsymbol{A}\boldsymbol{f}_0 - \boldsymbol{f} \tag{4.49}$$

The equilibrium configuration is all states equal zero; found after setting the velocity and acceleration terms to zero.

$$\boldsymbol{f}_* = 0 \tag{4.50}$$

To investigate asymptotic stability, the next step is to compose the Lyapunov equation. The equation formulated is not an explicit function of the energy of the system, but it bears some resemblance to the energy and meets the first two stability requirements.

$$J = \frac{1}{2}\dot{\boldsymbol{f}}^T\dot{\boldsymbol{f}} + \frac{1}{2}\boldsymbol{f}^T\boldsymbol{K}\boldsymbol{f} + \frac{1}{2}(\boldsymbol{A}\boldsymbol{f_0} + \boldsymbol{f})^T(\boldsymbol{A}\boldsymbol{f_0} + \boldsymbol{f}) \tag{4.51}$$

The first time derivative of this equation is shown below. The control is hidden in the $\ddot{\boldsymbol{f}}$ term, and therefore setting the quantity in brackets to zero produces an equation for the control as a function of position and filter state only.

$$\dot{J} = \dot{\boldsymbol{f}}^T[\ddot{\boldsymbol{f}} + (\boldsymbol{K} + \boldsymbol{I})\boldsymbol{f} + \boldsymbol{A}\boldsymbol{f_0}] - (\boldsymbol{A}\boldsymbol{f_0} + \boldsymbol{f})^T\boldsymbol{A}(\boldsymbol{A}\boldsymbol{f_0} + \boldsymbol{f}) \tag{4.52}$$

$$\widetilde{\boldsymbol{h}} = -\boldsymbol{A}\boldsymbol{f}_0 - \boldsymbol{f} \tag{4.53}$$

Substituting Eq. (4.53) into Eq. (4.52) produces a negative semidefinite time derivative of the Lyapunov equation. The equation is negative semidefinite because one of the states is missing from the equation and therefore could be anything including zero. This trait exhibits stability but not asymptotic stability.

$$\dot{J} = -(\boldsymbol{A}\boldsymbol{f_0} + \boldsymbol{f})^T\boldsymbol{A}(\boldsymbol{A}\boldsymbol{f_0} + \boldsymbol{f}) \tag{4.54}$$

To test if the system is actually asymptotically stable, we continue with Mukherjee and Chen's Asymptotic Stability Theorem. Equation (4.55) is the defined set Z that the second and third time derivatives of the Lyapunov function are evaluated, where $\boldsymbol{l} = \boldsymbol{A}\boldsymbol{f}_0 + \boldsymbol{f}$.

$$Z = \{\boldsymbol{f} \mid \boldsymbol{l} = 0\} \tag{4.55}$$

The second time derivative evaluated on the set Z is zero, and the third time derivative evaluated on the set Z is less than or equal to zero.

$$\ddot{J} \mid_Z = -\dot{\boldsymbol{l}}^T \boldsymbol{A}\boldsymbol{l} - \boldsymbol{l}^T \boldsymbol{A}\dot{\boldsymbol{l}} \mid_Z = 0 \tag{4.56}$$

$$J^{(3)} \mid_Z = -\ddot{\boldsymbol{l}}^T \boldsymbol{A}\boldsymbol{l} - 2\dot{\boldsymbol{l}}^T \boldsymbol{A}\dot{\boldsymbol{l}} - \boldsymbol{l}^T \boldsymbol{A}\ddot{\boldsymbol{l}} \mid_Z = -2\dot{\boldsymbol{f}}^T \boldsymbol{A}\dot{\boldsymbol{f}} \mid_Z \leq 0 \tag{4.57}$$

The third time derivative equation is still negative semidefinite because we do not have information about all of the states. A new set Z' is defined to evaluate the next two time derivatives.

$$Z' = \{\boldsymbol{f} \mid \boldsymbol{l} = 0 \;\&\; \dot{\boldsymbol{f}} = 0\} \tag{4.58}$$

The fourth time derivative evaluated on the set Z' is zero, and the fifth time derivative evaluated on the set Z' is less than zero, therefore confirming asymptotic stability.

$$J^{(4)} \mid_{Z'} = -{\boldsymbol{l}^{(3)}}^T \boldsymbol{A}\boldsymbol{l} - 3\ddot{\boldsymbol{l}}^T \boldsymbol{A}\dot{\boldsymbol{l}} - 3\dot{\boldsymbol{l}}^T \boldsymbol{A}\ddot{\boldsymbol{l}} - \boldsymbol{l}^T \boldsymbol{A}\boldsymbol{l}^{(3)} \mid_{Z'} = 0 \tag{4.59}$$

$$\begin{aligned} J^{(5)} \mid_{Z'} &= -{\boldsymbol{l}^{(4)}}^T \boldsymbol{A}\boldsymbol{l} - 4{\boldsymbol{l}^{(3)}}^T \boldsymbol{A}\dot{\boldsymbol{l}} - 6\ddot{\boldsymbol{l}}^T \boldsymbol{A}\ddot{\boldsymbol{l}} - 4\dot{\boldsymbol{l}}^T \boldsymbol{A}\boldsymbol{l}^{(3)} - \boldsymbol{l}^T \boldsymbol{A}\boldsymbol{l}^{(4)} \mid_{Z'} \\ &= -6\ddot{\boldsymbol{l}}^T \boldsymbol{A}\ddot{\boldsymbol{l}} \mid_{Z'} \\ &= -6\boldsymbol{f}^T \boldsymbol{K}\boldsymbol{A}\boldsymbol{K}\boldsymbol{f} < 0 \end{aligned} \tag{4.60}$$

Now that the fifth time derivative is negative definite, we can claim that the system will be asymptotically stable with the defined control of position and filter state terms only.

2. Simulation Results

Simulations of the VAWT blade verify the analytical demonstration of asymptotic stability for the control based on position and filter state terms only, shown in Eq. (4.53). Matlab is used to numerically integrate the equations of motion and to plot the time history of the blade. Each simulation was initialized using a slight perturbation from the equilibrium configuration in terms of the mode shapes. For the simulations, an approximation was made to only include the first three modes. Here, the x-axis is time, the y-axis is space, and the z-axis is beam displacement. The figures show the deformation of the blade starting at time equal to zero for the initial conditions. As time continues the blade motion dampens to the equilibrium configuration.

In the first case considered, the number of simulation modes equals the number of modes controlled. Figure (7) shows the time history of the blade with all three modes excited.

Although Fig. (7) presents favorable system behavior, these are infinite dimensional systems and there are always more modes than one can control. Figure (8) shows the time history of the blade when the second mode is uncontrolled. The oscillating second mode is not damped, which is consistent with the theory presented in this thesis.

One reason that the second mode continues to oscillate is because the system model does not include any structural damping. However, real world structures have natural damping inherent to the system. For the next figure, proportional damping has been added to the system model to represent structural damping. Even though the second mode is uncontrolled, the blade is still driven to the equilibrium configuration due to the effect of the structural damping.

Proportional damping was chosen to represent structural damping for this prob-

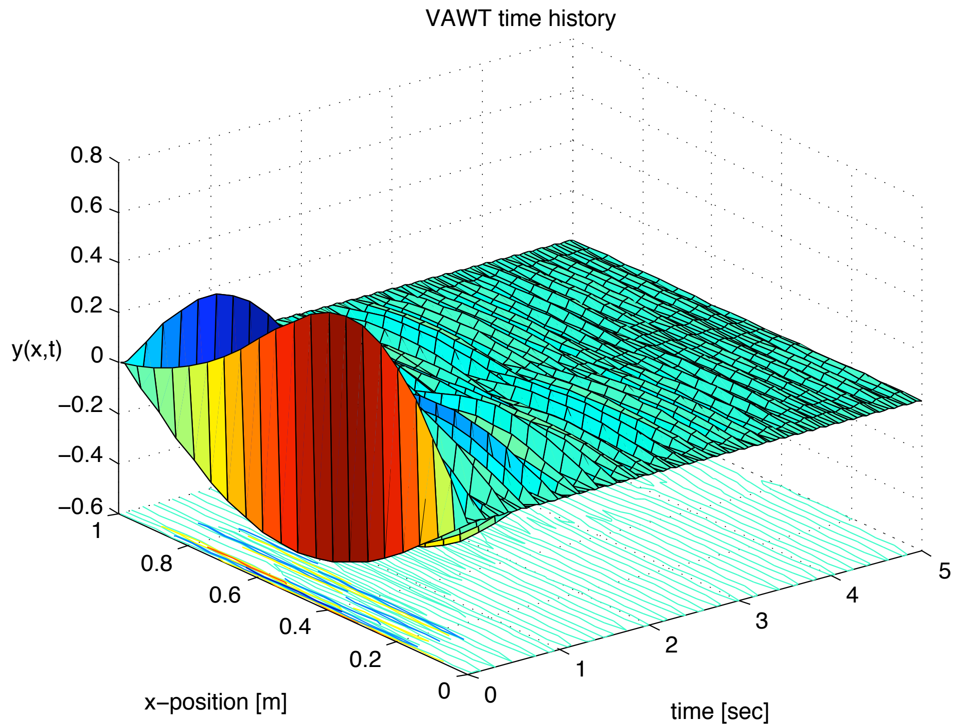

Fig. 7. VAWT blade simulation with the first three modes excited and controlled.

lem because the system dynamics are represented by diagonal matrices, and the damping ratio was calculated based on Eq. (4.61) below [16].

$$\xi_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \tag{4.61}$$

Here, ξ is the damping ratio, and α and β are constants. Figure (9) shows the VAWT blade time history with proportional damping included in the system dynamics.

These three figures verify the mathematical statements presented earlier in this chapter, and show that the VAWT system is asymptotically stable with a controller of position and filter state terms only.

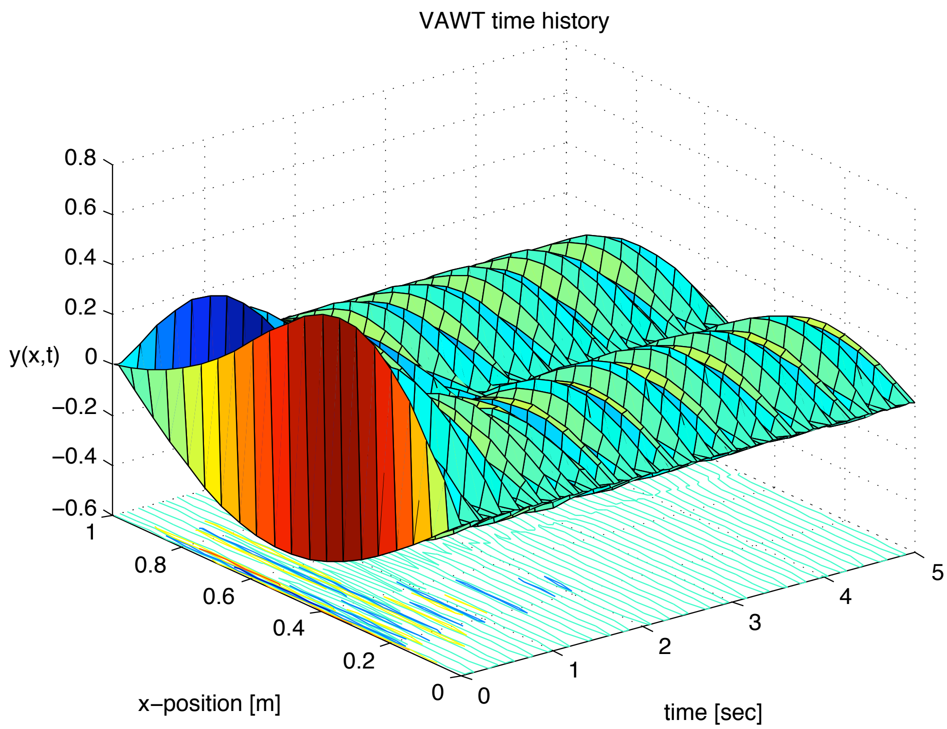

Fig. 8. VAWT blade simulation with the second mode uncontrolled.

C. Challenges

This chapter details two example problems of infinite dimensional systems with controllers that feedback on position and filter state terms only. In both problems, asymptotic stability is confirmed for the system, but during the development challenges were presented.

The infinite dimensional problems considered were both linear systems. Because of this quality, separation of variables could be applied to manipulate their governing equations. This approach created known variables in space and ordinary differential equations in time, and therefore ordinary differential equations for the filter states. If the problem considered was a nonlinear system, one would not be able to use the technique of separation of variables. For these cases, one might be able to use an

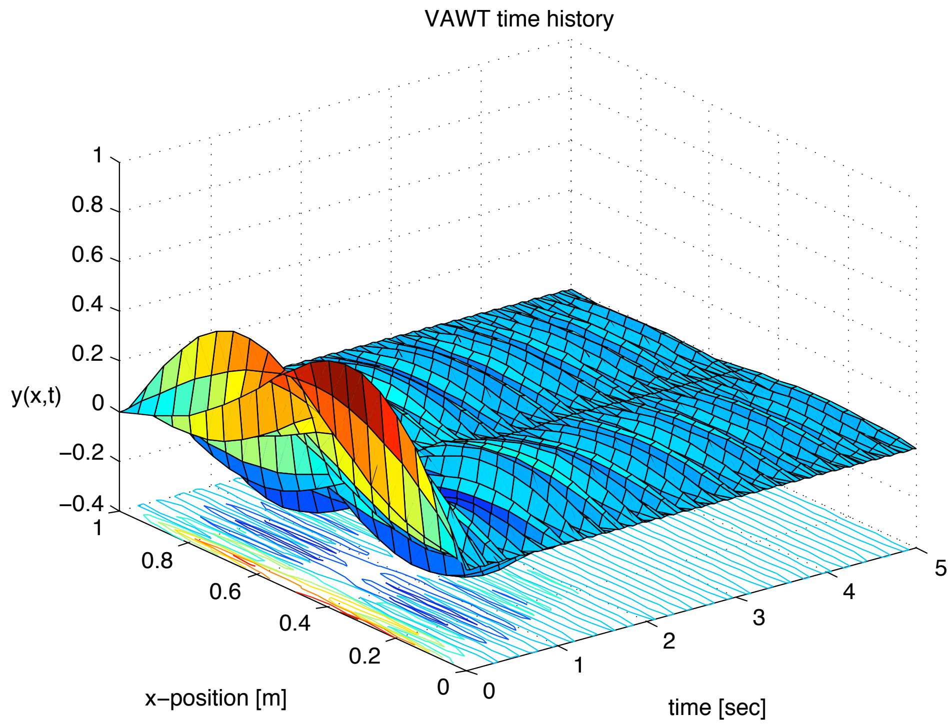

Fig. 9. VAWT blade simulation with proportional damping and the second mode uncontrolled.

assumed modes method to apply the reduced-sensing approach in spite of nonlinearity. The assumed modes method approximates a solution for the variables in space; this method produces a similar result as separation of variables for the problems presented in this thesis [10]. Further investigation would be necessary if the variables in space cannot be satisfactorily approximated.

Another challenge to be investigated in the future is the addition of aerodynamic loading to the VAWT model. In this thesis, the distributed load on the blade consists of control terms only. To include aerodynamic loading, the distributed load could be modeled as the addition of control and aerodynamic loads. The control is an explicit function of f, f_0, and x, and an implicit function of time. The aerodynamic loads would be an explicit function of y and time, and an implicit function of x and time.

CHAPTER V

CONCLUSION

The research described in this thesis is motivated by flexible structures such as wind turbines and airplane wings. The blades of these systems are modeled as infinite dimensional structures because they can be displaced at any point along the blade. To control the position of the blade, typical controllers require position and velocity information. Infinite dimensional systems often do not have velocity measurements available, but one still desires to control the system to achieve asymptotic stability. This research proposes the use of a filter state to control an infinite dimensional system.

A filter state is a nonphysical entity that can add damping to a system. In order to implement a filter state, a differential equation is added to the system and the controller feedback includes position and filter terms only. The filter state is simply numerically integrated in real time with the system. It can potentially provide asymptotic stability to a system that does not have access to velocity measurements.

The tools used to meet the research objective include the following: equilibrium points, Lyapunov functions for stability and control, and Mukherjee and Chen's Asymptotic Stability Theorem. The equilibrium points give the control a goal configuration to which the system should be driven. Lyapunov functions have specific stability requirements, and these requirements can be used to design a controller. Mukherjee and Chen's Asymptotic Stability Theorem analyzes the Lyapunov function further to prove asymptotic stability. All of these tools can be used together to prove asymptotic stability of a system utilizing control with a filter state.

Although the filter state technique has already been applied to finite dimensional systems in previous work, it was extended to infinite dimensional systems in this thesis. To illustrate the process and to demonstrate mathematically that a system can be asymptotically stable with a control of position and filter state only, finite dimensional example problems were presented. In the investigation of these techniques for infinite dimensional system, two types of systems were considered: scleronomic and rheonomic. The cantilevered beam represents a scleronomic system and the Vertical Axis Wind Turbine (VAWT) blade represents a rheonomic system. Asymptotic stability was exhibited in both cases with controllers based on position and filter state information only.

To implement the controller for an infinite-dimensional system, one method employs a piezoelectric patch. The patch is an actuator that can impart forces onto a surface without any support structure, and is shown to to be successful for a cantilever rectangular aluminum plate [17].

Simulations were presented that validate the asymptotic stability of the VAWT using the reduced-sensing control technique. First, the case considered had an equivalent number of simulation modes as control modes. Next, a case was considered where the number of simulation modes was greater than the number of control modes. The final case considered introduced structural damping to the model of the system dynamics.

Future work includes hardware testing and incorporating aerodynamic loading into the VAWT system model. The hardware testing could include a finite dimensional filter state implemented on a two degree of freedom wing. This test would provide data to compare with simulation work, and would be useful in determining the real world efficiency of the control. Another hardware test could include the implementation of the piezoelectric patch. If the test setup included an infinite dimensional system such as a beam, the patch could be placed on the beam in such a way so that it could control one of the natural modes. This test would determine the usefulness of the piezoelectric patch for the application and explore the effect of structural damping. In addition to hardware testing, the effect of aerodynamic loads could be investigated for the VAWT system model in order to determine the possible change in the control equation.

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VITA

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