SPECIAL VALUES OF THE GOSS L-FUNCTION AND SPECIAL POLYNOMIALS

A Dissertation

by

BRAD AUBREY LUTES

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2010

Major Subject: Mathematics

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ABSTRACT

Special Values of the Goss L-function

and Special Polynomials. (August 2010)

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Let K be the function field of an irreducible, smooth projective curve X defined over $\mathbb{F}_q$. Let ∞ be a fixed point on X and let $A \subseteq K$ be the Dedekind domain of functions which are regular away from ∞. Following the work of Greg Anderson, we define special polynomials and explain how they are used to define an A-module (in the case where the class number of A and the degree of ∞ are both one) known as the module of special points associated to the Drinfeld A-module ρ. We show that this module is finitely generated and explicitly compute its rank. We also show that if K is a function field such that the degree of ∞ is one, then the Goss L-function, evaluated at 1, is a finite linear combination of logarithms evaluated at algebraic points. We conclude with examples showing how to use special polynomials to compute special values of both the Goss L-function and the Goss zeta function.

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CHAPTER I

INTRODUCTION

Let X be an irreducible, smooth projective curve defined over $\mathbb{F}_q$ and let K be its function field. One should view K as a positive characteristic analogue of $\mathbb{Q}$. With this in mind, the study of function fields closely mirrors that of number fields, especially cyclotomic fields. The primary motivation for this dissertation is the paper of Anderson [2] in which he considers the following classical results from the theory of cyclotomic fields:

(1) Let p be an odd prime and set $\mathbf{e}(x) := \exp(2\pi i x)$. Consider the cyclotomic field $\mathbb{Q}(\mathbf{e}(1/p))$. Let $\mathcal{C}$ denote the circular units of this cyclotomic field. Then $\mathcal{C}$ is of rank $(p-3)/2$.

(2) Let χ be a Dirichlet character on $\mathbb{Z}/n\mathbb{Z}$ for some $n > 1$. If χ is a nontrivial character such that $\chi(-1) = 1$, then

$$L(1,\chi) = -\frac{\tau(\chi)}{f}\sum_{a=1}^{f}\bar{\chi}(a)\log|1-\mathbf{e}(a/f)|$$

where $L(s,\chi)$ is the Dirichlet L-function associated to χ, f is the conductor of χ, and $\tau(\chi)$ is a Gauss sum.

Note that the previous special value is a finite $\bar{\mathbb{Q}}$-linear combination of logarithms evaluated at algebraic points. In [2] Anderson proved a formula which relates the value of the Goss L-function at 1 to logarithms of so called special points of the Carlitz module. Also, a result analogous to (1) is shown in which these special points play the role of the circular units.

In Chapter II, we first begin with a discussion of the relevant notions from the theory of function fields. If X is the curve associated to K, then fix a point $\infty \in X$. We set $A \subseteq K$

This dissertation follows the style of Journal of Number Theory.

to be the Dedekind domain of functions whose only poles are at ∞. In §B, we discuss the primary object with which to study the arithmetic of function fields, namely Drinfeld A-modules. For more information about Drinfeld A-modules, see [7], [8], [9], [10], [13], [19]. In this dissertation, we only consider Drinfeld-Hayes A-modules which are just Drinfeld A-modules of rank 1 with some additional assumptions. In §C, we briefly discuss the Carlitz module. It is the simplest example of a Drinfeld-Hayes A-module. In §D, we discuss the notion of torsion points of a Drinfeld-Hayes A-module. It is here that the connection between cyclotomic fields and function fields begins to come into focus. In §E, we give a brief discussion of ramification in function fields.

In Sections F and G we discuss the connections between class field theory and characters in function fields. We begin with a homomorphism $\chi\colon A \to \mathbb{C}_\infty$ which we call a Dirichlet character. (Here $\mathbb{C}_\infty$ is the function field analogue of $\mathbb{C}$.) Let $\mathfrak{m} := \ker\chi$. Using the class field theory of §F, we extend χ to a character $\psi\colon \mathfrak{F}_\mathfrak{m}(A) \to \mathbb{C}_\infty$, where $\mathfrak{F}_\mathfrak{m}(A)$ consists of the fractional ideals of A relatively prime to $\mathfrak{m}$, such that $\psi|_A = \chi$. The Goss L-function for ψ is defined to be

$$L(j,\psi) = \sum_I \frac{\psi(I)}{I^{[j]}}$$

where the sum ranges over all integral ideals of A relatively prime to $\mathfrak{m}$. Here j denotes a positive integer and $I^{[j]}$ denotes a type of ideal exponentiation. For more background on the Goss L-function, see [4], [5], [6].

In Chapter III, we discuss the results of Anderson. In [2], a function is said to be log-algebraic if it is formally the logarithm of a power series algebraic over the field of rational functions in z where z is an indeterminate. In §B, a function denoted $\ell(b;z)$ is defined for $b \in H[t]$ where t is another indeterminate and H is the Hilbert class field of K. The following fundamental result is proven in [2] and restated in this dissertation as Theorem III.2:

Theorem I.1. *Let B be the integral closure of A in H. Let ρ be a Drinfeld-Hayes A-module and let $\exp_\rho$ be the exponential associated to ρ (cf., §II.B). For all $b \in B[t]$, the power series $\exp_\rho \ell(b;z)$ is in fact a polynomial in $B[t,z]$.*

The polynomials

$$\{\exp_\rho \ell(b;z) \mid b \in B[t]\}$$

are called special polynomials. In §C, we discuss the results of Anderson concerning special polynomials and special points in the case of the Carlitz module C.

Let h_A be the class number of A and let $d_\infty = \deg \infty$. In Chapter IV, we discuss our extensions of special points and special polynomials when $h_A = 1$ and $d_\infty = 1$. In §A, we note that there are only four such function fields ([11], [18]). The curves associated to these function fields are

- $X_1 \colon y^2 = t^3 - t - 1$ over $\mathbb{F}_3$;
- $X_2 \colon y^2 + y = t^3 + \alpha$ over $\mathbb{F}_4$ where $\alpha \in \mathbb{F}_4$ satisfies $\alpha^2 + \alpha + 1 = 0$;
- $X_3 \colon y^2 + y = t^3 + t + 1$ over $\mathbb{F}_2$;
- $X_4 \colon y^2 + y = t^5 + t^3 + 1$ over $\mathbb{F}_2$.

Let $K_1, \ldots, K_4$ be the function field of $X_1, \ldots, X_4$, respectively. Each K_j has exactly one Drinfeld-Hayes A_j-module, which we denote by ρ^j. In §B, we discuss shtuka functions which are rational functions which allow us to (1) recover the Drinfeld-Hayes A_j-module ρ^j and (2) explicitly compute the exponential associated to ρ^j (cf., Theorem IV.9). In §C–F, we explicitly compute the invariants $i_0(\rho^j)$ and $j_0(t^m;\rho^j)$ (cf., Propositions IV.10 and IV.15). In §G, we state as Proposition IV.17 our main result concerning the extension of special polynomials:

Proposition I.2. *For $1 \le j \le 4$, let K_j be the function field associated to the curve X_j and let ρ^j be the unique Drinfeld-Hayes A_j-module associated to K_j.*

(1) The power series

$$S(t^m;z) := \exp_{\rho^j} \ell(t^m;z) = \sum_{i\geq 0} e_i(\rho^j) \sum_{a\in(A_j)_+} \left(\frac{(\rho_a^j(t))^m}{a}\right)^{q^i} z^{q^{i+\deg a}}$$

lies in $A_j[t,z]$.

(2) One has $\exp_{\rho^j} l_m(x) = S(t^m;1)|_{t=\mathbf{e}_j(x)}$.

(3) For all $c \in \mathbb{F}_q^\times$*, one has* $S(t^m;z)|_{t=ct} = c^m S(t^m;z)$*, and moreover* $S(t^m;z)$ *is divisible by* t^m.

(4) One has

$$\left.\frac{S(t^m;z)}{t^m}\right|_{t=0} = \sum_{a\in(A_j)_+} a^{m-1} z^{q^{\deg a}}$$

for $m > 0$.

(5) Let $i_0(\rho^j)$ *and* $j_0(t^m;\rho^j)$ *be as in Propositions IV.10 and IV.15. The degree of* $S(t^m;z)$ *in z (respectively t) does not exceed* $q^{\lfloor i_0(\rho^j)+j_0(t^m;\rho^j)\rfloor}$ *(resp.* $mq^{\lfloor i_0(\rho^j)+j_0(t^m;\rho^j)\rfloor}$*).*

(6) The specialization $S(t^m;1) \in A_j[t]$ *vanishes identically if* $m > 1$ *and* $m \equiv 1 \bmod q-1$.

In §H, we discuss our extension of special points when $h_A = 1$ and $d_\infty = 1$. Let d be a positive integer and fix an irreducible $\mathbf{p} \in (A_j)_+$ of degree d. For $b \in (A_j/\mathbf{p})^\times$, set

$$s_m^j(b) := \exp_{\rho^j} l_m(x) = S(t^m;1)|_{t=\mathbf{e}_j(x)}$$

where $x = \tilde{b}/\mathbf{p}$ and $\tilde{b} \equiv b \bmod \mathbf{p}$. The A_j-module generated by points of the form $s_m^j(b)$ is called the module of special points of ρ^j. In §H and I, we show that this module is finitely generated by the special points of the form

$$\{s_m^j(1) \mid 0 \leq m \leq q^d - 1\}.$$

In Chapter V, we are concerned with expressing the Goss L-function evaluated at 1 as a linear combination of logarithms. In this chapter, we only assume that $d_\infty = 1$. We make no assumption about h_A. We first explicitly compute $L(1,\psi)$ where ψ is as in §II.G. The following result is restated as Proposition V.13 in Section E:

Proposition I.3. *Let χ be a Dirichlet character on A with kernel $\mathfrak{m}$. Let $\psi\colon \mathfrak{F}_{\mathfrak{m}}(A) \to \mathbb{C}_\infty$ be the character (as in §II.G) such that $\psi|_A = \chi$. Let $\{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ be a set of representatives of the equivalence classes of $Cl(A)$ that are relatively prime to $\mathfrak{m}$ where $h = \#Cl(A)$. Let μ be as in* (5.6) *in §V.C and let $d = \deg \mathfrak{m}$. Then*

$$L(1,\psi) = \sum_{j=1}^{h} L_{\mathfrak{a}_j}(1,\psi)$$
$$= \sum_{j=1}^{h} \left[-\psi(\mathfrak{a}_j) \sum_{m=1}^{q^d-1} \left(\frac{1}{\mathfrak{a}_j^{[1]}} \sum_{a \in (A/\mathfrak{m})^\times} \psi(a) \mathbf{e}^*_{m,\mathfrak{a}_j}(a/\nu_j) \right) \left(\sum_{b \in (A/\mathfrak{m})^\times} \psi(b)^{-1} l_{m,\mathfrak{a}_j}(b\mu) \right) \right]$$

where $\nu_j \in \mathfrak{a}_j \setminus \mathfrak{a}_j\mathfrak{m}$ are chosen as in (5.14) *in §V.E.*

The numbers $\mathbf{e}^*_{m,\mathfrak{a}_j}(a/\nu_j)$ are called the generalized dual coefficients. Their properties are investigated in §D. The function $l_{m,\mathfrak{a}_j}(x)$ is defined in §C.

We conclude Chapter V by showing that the expression $l_{m,\mathfrak{a}_j}(a\mu)$ is a finite linear combination of logarithms evaluated at algebraic points (cf., (5.20)). More precisely, we prove the following (which is restated as Theorem V.14):

Theorem I.4. *Let ψ be as in the previous Proposition. Then there exist $u_1, \dots, u_s \in \mathbb{C}_\infty$ with $\exp_\rho(u_i) \in \bar{K}$ and $\alpha_1, \dots, \alpha_s \in \bar{K}$ such that*

$$L(1,\psi) = \sum_{i=1}^{s} \alpha_i u_i.$$

In Chapter VI, we return to the case when $h_A = 1$ and $d_\infty = 1$. Our main result is the following (restated as Theorem VI.4):

Theorem I.5. *Let K be a function field over $\mathbb{F}_q$ (other than the rational function field $\mathbb{F}_q(T)$) satisfying $h_A = 1$ and $d_\infty = 1$. Let ρ be the unique Drinfeld-Hayes A-module with* *respect to a fixed sign function sgn. Let $\mathcal{S}$ be the A-module of special points of ρ as defined in §IV.H. Then the A-rank of $\mathcal{S}$ equals $(q^d-1)(q-2)/(q-1)$.*

In Chapter VII, we explicitly compute special polynomials for the function fields $K_1,\ldots,K_4$. In Chapter VIII, we compute special values of the Goss L-function and the Goss zeta function for these function fields. The following example is Example VIII.4: Consider the curve X_1. Consider the Dirichlet character

$$\begin{aligned}\chi\colon A_1 &\to \bar{\mathbb{F}}_9\\ a=a(t,y) &\mapsto a(0,\sqrt{-1}).\end{aligned}$$

Then

$$\begin{aligned}L(1,\chi) &= \frac{\log_\rho(\xi')+\sqrt{-1}\log_\rho(\xi)}{\xi'+\sqrt{-1}\xi}\\ L(1,\chi^3) &= \frac{\log_\rho(\xi')-\sqrt{-1}\log_\rho(\xi)}{\xi'-\sqrt{-1}\xi}\end{aligned}$$

where ξ is a generator of the t-torsion $\rho^1[t]$ and $\xi' := \rho^1_\eta(\xi)$. We also consider (in Examples VIII.5 and VIII.6) a function field for which $h_A=2$ to illustrate the computational complexity involved in going from class number one to class number two. Finally we conclude in Chapter IX with some remarks about special points and special polynomials when $h_A>1$. As a final note, it would be interesting to compare the results contained in this dissertation on special values of L-functions to recent work of Taelman [16], [17], [15] on Birch and Swinnerton-Dyer type formulas for Drinfeld modules.

CHAPTER II

PRELIMINARIES

A. Function Fields

Let X be an irreducible, smooth projective curve defined over the finite field $\mathbb{F}_q$, where $q = p^r$ for some prime p, and $r > 0$. Let ∞ denote a fixed point on X. If $X \subset \mathbb{P}^n$, then the *ideal of* X is

$$I(X) := \langle F \in \mathbb{F}_q[x_1, \ldots, x_{n+1}], F \text{ homogeneous} \mid F(P) = 0 \; \forall \; P \in X \rangle.$$

Because X is irreducible, $I(X)$ is a prime ideal; hence

$$\Gamma(X) := \mathbb{F}_q[x_1, \ldots, x_{n+1}]/I(X)$$

is an integral domain. Let $\mathrm{Frac}(\Gamma(X))$ denote the fraction field of $\Gamma(X)$. The *function field of* X is

$$K := \{z \in \mathrm{Frac}(\Gamma(X)) \mid \exists \; f, g \in \Gamma(X) \text{ of the same degree with } z = f/g\}.$$

The elements of K are called *rational functions on* X.

For $P \in X$, define

$$\mathcal{O}_P(X) := \{z \in K \mid z = f/g, f \text{ and } g \text{ of the same degree, } g(P) \neq 0\}.$$

Then $\mathcal{O}_P(X)$ is a *discrete valuation ring*, i.e. it is a Noetherian local ring whose unique maximal ideal is principal, and whose fraction field is K.

Given a discrete valuation ring R, another way to characterize it is that there exists an irreducible element $t \in R$ such that every nonzero element of R may be written uniquely as ut^n, where u is a unit in R and n is a non-negative integer ([3], §2.4, Proposition 4). The element t is called a *uniformizing parameter for R*. For fixed t, every nonzero $z \in K$ can be written uniquely as ut^n, where u is a unit in $\mathcal{O}_P(X)$ and $n \in \mathbb{Z}$. We define $\operatorname{ord}_P(z) := n$ and say that n is the order of vanishing of z at P.

The function $\operatorname{ord}_P\colon K \to \mathbb{Z} \cup \{\infty\}$ is a *valuation* satisfying:

1. $\operatorname{ord}_P(k) = +\infty$ if and only if $k = 0$;

2. $\operatorname{ord}_P(k_1 k_2) = \operatorname{ord}_P(k_1) + \operatorname{ord}_P(k_2)$;

3. $\operatorname{ord}_P(k_1 + k_2) \geq \min\{\operatorname{ord}_P(k_1), \operatorname{ord}_P(k_2)\}$

for all $k_1, k_2 \in K$. With this notation, we have that

$$\mathcal{O}_P(X) = \{z \in K \mid \operatorname{ord}_P(z) \geq 0\}$$

and its (unique) maximal ideal is

$$\mathcal{M}_P(X) = \{z \in K \mid \operatorname{ord}_P(z) > 0\}.$$

We thus have two natural objects to study: points on X and discrete valuation rings of K. The following result tells us that these objects are essentially the same.

Theorem II.1 ([3], §7.1, Corollary 4)**.** *Let X be an irreducible non-singular projective curve over a field F and let K be its function field. Then there is a natural one-to-one correspondence between the closed points of X and the discrete valuation rings of K. If $P \in X$, then $\mathcal{O}_P(X)$ is the corresponding discrete valuation ring.*

Let P be a point on X. Define the degree of P by

$$\deg P := \dim_{\mathbb{F}_q} \mathcal{O}_P(X)/\mathcal{M}_P(X).$$

We denote the degree of ∞ by d_∞.

Let A be the elements of K whose only poles are at ∞. Then A is a subring of K which is also a *Dedekind domain*, i.e. an integral domain for which each nonzero ideal can be factored uniquely into a product of prime ideals. For $a \neq 0$, let (a) be the ideal generated by a in A. Then $A/(a)$ is a finite dimensional $\mathbb{F}_q$-vector space. Define the degree of a by

$$\deg a := \dim_{\mathbb{F}_q} A/(a).$$

We set the degree of 0 to be $-\infty$. Equivalently, we have that

$$q^{\deg a} = \#(A/(a)).$$

Note that if A is a polynomial ring, then this notion of degree corresponds to the usual notion of degree. Similarly, if I is an ideal of A, the degree of I is defined by

$$q^{\deg I} = \#(A/I).$$

Given a point P on X, we may consider its associated valuation ord_P. Let $0 < \alpha < 1$ be a real number and consider the map $|\cdot|_P \colon K \to \mathbb{R}_{\geq 0}$ given by $|x|_P = \alpha^{\mathrm{ord}_P(x)}$. It is easy to see that $|\cdot|_P$ defines a *non-archimedean* absolute value on K, i.e. it satisfies

1. $|x|_P = 0$ if and only if $x = 0$;
2. $|xy|_P = |x|_P |y|_P$ for all $x, y \in K$;
3. $|x+y|_P \leq \max\{|x|_P, |y|_P\}$.

Note that in the case of function fields, *all* absolute values are nonarchimedean.

Let K_∞ be the completion of K with respect to the valuation ord_∞. For $x \in K_\infty$, set

$$\deg x := -d_\infty \mathrm{ord}_\infty(x).$$

Note that this definition of degree restricted to elements of A reduces to our previous definition of degree. We denote by $|\cdot|$ the normalized absolute value associated to the point ∞, i.e.

$$|x| := q^{-d_\infty \operatorname{ord}_\infty(x)}$$

for all $x \in K_\infty$. Let $\bar{K}_\infty$ be the algebraic closure of K_∞. We extend $|\cdot|$ to $\bar{K}_\infty$ via the formula

$$|z| = \left|N_{K_\infty}^{E}(z)\right|^{1/[E\colon K_\infty]}$$

where E is any intermediate field containing z of finite degree over K_∞ and $N_{K_\infty}^{E}$ denotes the norm from K_∞ to E ([13], Chapter 13).

Define $\mathbb{C}_\infty$ to be the completion of $\bar{K}_\infty$ with respect to $|\cdot|$. The absolute value $|\cdot|$ extends uniquely to $\mathbb{C}_\infty$ and so $\mathbb{C}_\infty$ is a nonarchimedean field. It is also complete and algebraically closed. Thus, in the context of function fields, $\mathbb{C}_\infty$ behaves like $\mathbb{C}$.

As for the other objects we have defined, one should think of A as an analogue of $\mathbb{Z}$. With this in mind, K and K_∞ correspond to $\mathbb{Q}$ and $\mathbb{R}$, respectively. Also, ∞ corresponds to the unique archimedean place of $\mathbb{Q}$, i.e. the Euclidean absolute value.

B. Drinfeld A-modules

Assuming that A is the function field analogue of $\mathbb{Z}$, we want to generalize the notion of the sign of a number. For nonzero elements of $\mathbb{R}$, there are only two such possibilities, namely positive or negative. This corresponds to the fact that $\mathbb{Z}^\times = \{\pm 1\}$. Note that $K^\times = \mathbb{F}_q^\times$ and we refer to $\mathbb{F}_q$ as the *constant field* of K. So the signs of elements of the function field are elements of the corresponding constant field. Since K_∞ is isomorphic to the field of Laurent series $\mathbb{F}_{q^{d_\infty}}((\pi))$, where π is any uniformizer for $\mathcal{O}_\infty(X)$, we have that the constant field of K_∞ is $\mathbb{F}_{q^{d_\infty}}$. (In this case we also say that π is a uniformizer at ∞.) Hence the signs of K_∞ should lie in $\mathbb{F}_{q^{d_\infty}}^\times$.

Let $\hat{\mathcal{M}}_\infty(X) := \{z \in K_\infty \mid \operatorname{ord}_\infty(z) > 0\}$. We say that x is a *1-unit* of K_∞ if $\operatorname{ord}_\infty(x) = 0$ and $x - 1 \in \hat{\mathcal{M}}_\infty(X)$. We denote the 1-units by $U^{(1)}$. Every element of K_∞ can be written as

$c\pi^n u$, where $c \in \mathbb{F}_{q^{d_\infty}}$, π is a fixed uniformizer at ∞, $n \in \mathbb{Z}$, and $u \in U^{(1)}$. A *sign function* $\mathrm{sgn} : K_\infty^\times \to (\mathbb{F}_{q^{d_\infty}})^\times$ is a homomorphism which is the identity on $(\mathbb{F}_{q^{d_\infty}})^\times$ and is trivial on $U^{(1)}$. We use the convention that sgn(0)=0. An element of sign one is called *monic*. For $\sigma \in \mathrm{Gal}(\mathbb{F}_{q^{d_\infty}}/\mathbb{F}_q)$, the composite $\sigma \circ \mathrm{sgn}$ is called *twisting of the sign function* sgn. The number of possible sign functions equals $\#\left(\mathbb{F}_{q^{d_\infty}}\right)^\times = q^{d_\infty} - 1$.

Define $\tau \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $\tau(z) = z^q$ for all $z \in \mathbb{C}_\infty$. We call τ the *q-th power Frobenius map*. Define $\mathbb{C}_\infty\langle\tau\rangle$ to be the ring of polynomials in τ with coefficients in $\mathbb{C}_\infty$ with "twisted" multiplication, i.e.

$$\tau z = z^q \tau$$

for all $z \in \mathbb{C}_\infty$. Note that $\mathbb{C}_\infty\langle\tau\rangle$ is a noncommutative ring. Let $\mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty)$ denote the $\mathbb{F}_q$-algebra of $\mathbb{F}_q$-endomorphisms of $\mathbb{C}_\infty$ viewed as an additive group. The endomorphism $\phi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is in $\mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty)$ if

(i) $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2)$ for all $z_1, z_2 \in \mathbb{C}_\infty$ and

(ii) $\phi(\alpha z) = \alpha\phi(z)$ for all $\alpha \in \mathbb{F}_q$ and for all $z \in \mathbb{C}_\infty$.

Then $\mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty) \subseteq \mathbb{C}_\infty\langle\tau\rangle$.

We define the homomorphism $D \colon \mathbb{C}_\infty\langle\tau\rangle \to \mathbb{C}_\infty$ by $D(\sum c_i \tau^i) = c_0$. A *Drinfeld A-module* over $\mathbb{C}_\infty$ consists of an $\mathbb{F}_q$-algebra homomorphism $\rho \colon A \to \mathbb{C}_\infty\langle\tau\rangle$ such that for all $a \in A$

$$D(\rho_a) = a$$

where $\rho_a := \rho(a)$. We also require that the image of ρ not be contained in $\mathbb{C}_\infty$. Denote by $\mathrm{Drin}_A(\mathbb{C}_\infty)$ the set of all Drinfeld A-modules over $\mathbb{C}_\infty$. The *rank* of the Drinfeld A-module ρ is defined to be the unique positive integer r such that

$$\deg_\tau(\rho_a) = r \, \deg a$$

for all $a \in A$, where $\deg_\tau(\rho_a)$ denotes the degree of ρ_a as a polynomial in τ. We next explain why Drinfeld A-modules exist.

A *lattice* Λ is a discrete, finitely generated, A-submodule of $\mathbb{C}_\infty$. The dimension of the vector space $K_\infty\Lambda$ over K_∞ is called the *rank* of the lattice. Given Λ, we define

$$e_\Lambda(z) := z \prod_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \left(1 - \frac{z}{\lambda}\right),$$

where $z \in \mathbb{C}_\infty$. Because Λ is discrete, e_Λ is entire on $\mathbb{C}_\infty$, and it is $\mathbb{F}_q$-linear. By construction, it is the unique entire function on $\mathbb{C}_\infty$ with simple zeros on the elements of Λ and with leading term z.

Let $\Lambda \subseteq \Lambda'$ be two lattices of the same rank. Then Λ'/Λ is a finite A-module. Consider the surjective map $e_\Lambda \colon \Lambda' \to e_\Lambda(\Lambda')$. The kernel of this map is clearly Λ, and thus $\Lambda'/\Lambda \cong e_\Lambda(\Lambda')$. So $e_\Lambda(\Lambda')$ is a finite set.

Consider the polynomial

$$P(z;\Lambda'/\Lambda) := z \prod_{\substack{\gamma\in e_\Lambda(\Lambda') \\ \gamma\neq 0}} \left(1 - \frac{z}{\gamma}\right).$$

The polynomial $P(z;\Lambda'/\Lambda)$ satisfies

$$e_{\Lambda'}(x) = P(e_\Lambda(x);\Lambda'/\Lambda)$$

([13], Proposition 13.22). Now fix a lattice Λ of rank r and for each nonzero $a \in A$ define $\rho_a^\Lambda \in \mathbb{C}_\infty\langle\tau\rangle$ by

$$\rho_a^\Lambda(z) := aP(z;a^{-1}\Lambda/\Lambda).$$

The map induced by sending a to ρ_a^Λ and 0 to 0 is a Drinfeld A-module of rank r ([13], Theorem 13.23). The existence of Drinfeld A-modules is thus guaranteed by the following result.

Theorem II.2 ([13], Theorem 13.24)**.** *Let* $\mathrm{Lat}_A(\mathbb{C}_\infty)$ *denote the set of lattices in* $\mathbb{C}_\infty$*. The map*

$$\mathrm{Lat}_A(\mathbb{C}_\infty) \to \mathrm{Drin}_A(\mathbb{C}_\infty)$$
$$\Lambda \mapsto \rho^\Lambda$$

is a rank-preserving bijection.

Therefore, if Λ is a lattice of rank r, then there is a unique corresponding Drinfeld A-module of rank r which we call ρ. The function which we previously denoted by e_Λ will now be denoted by $\exp_\rho$. We say that $\exp_\rho$ is the *exponential belonging to* ρ. The function $\exp_\rho$ satisfies the following functional equation:

$$\exp_\rho(az) = \rho_a(\exp_\rho(z)) \tag{2.1}$$

for all $a \in A$ and for all $z \in \mathbb{C}_\infty$. We denote the (multivalued) inverse of $\exp_\rho$ by $\log_\rho$, and we call this the *logarithm belonging to* ρ.

A Drinfeld A-module $\rho \in \mathrm{Drin}_A(\mathbb{C}_\infty)$ is called *normalized* if the leading coefficient $\mu_\rho(a)$ of ρ_a belongs to $\mathbb{F}_{q^{d_\infty}}$ for all $a \in A$. If for some sign function sgn, the map $a \mapsto \mu_\rho(a)$ is a twisting of sgn, then ρ is called sgn-*normalized.*

Fix a sign function sgn. A *Drinfeld-Hayes A-module with respect to* sgn is an injective $\mathbb{F}_q$-algebra homomorphism $\rho : A \to \mathbb{C}_\infty\langle\tau\rangle$ such that for all nonzero $a \in A$ the following properties hold:

1. the degree of ρ_a as a polynomial in τ is deg a;
2. the coefficient of τ^0 in ρ_a is a;
3. the leading coefficient of ρ_a is sgn(a).

Note that a Drinfeld-Hayes A-module is a rank one sgn-normalized Drinfeld A-module.

Let $\rho, \rho' \in \mathrm{Drin}_A(\mathbb{C}_\infty)$. A *morphism from* ρ *to* ρ' is an element $m \in \mathbb{C}_\infty\langle\tau\rangle$ such that $m\rho_a = \rho'_a m$ for all $a \in A$. The set of all such morphisms is denoted $\mathrm{Hom}_{\mathbb{C}_\infty}(\rho, \rho')$. We say that ρ, ρ' are *isogenous over* $\mathbb{C}_\infty$ if

$$\mathrm{Hom}_{\mathbb{C}_\infty}(\rho, \rho') \neq (0).$$

Fix a nonzero ideal $I \subset A$ and recall that ρ_I is defined to be the unique monic generator of the left ideal in $\mathbb{C}_\infty\langle\tau\rangle$ generated by $\{\rho_i \mid i \in I\}$. Then there is a uniquely determined Drinfeld A-module $I * \rho$ such that ρ_I is an isogeny from ρ to $I * \rho$, i.e. we have

$$\rho_I \rho_a = (I * \rho)_a \rho_I$$

for all $a \in A$ ([13], Proposition 13.13). Now suppose that ρ is a Drinfeld-Hayes A-module with respect to a fixed sign function sgn. Since $\deg_\tau \rho_I = \deg I$ ([13], Proposition 13.17), it follows that $I * \rho$ has rank one. Since ρ_I is a monic polynomial in $\mathbb{C}_\infty\langle\tau\rangle$ and since the leading coefficient of ρ_a is $\mathrm{sgn}(a)$, we have that the leading coefficient of $(I * \rho)_a$ is also $\mathrm{sgn}(a)$. Therefore, $I * \rho$ is a Drinfeld-Hayes A-module.

Two Drinfeld A-modules ρ and ρ' are isomorphic if there exists $c \in \mathbb{C}_\infty^\times$ such that $c\rho_a = \rho'_a c$ for all $a \in A$. Thus the set $\mathrm{Drin}_A(\mathbb{C}_\infty)$ may be partitioned into isomorphism classes. It is enough to study sgn-normalized Drinfeld A-modules because every Drinfeld A-module $\rho \in \mathrm{Drin}_A(\mathbb{C}_\infty)$ is isomorphic to a sgn-normalized Drinfeld A-module ([14], Theorem 13.5.14). So we are interested in the number of Drinfeld-Hayes A-modules in each isomorphism class.

For this dissertation, we consider only rank one Drinfeld A-modules. In this case, we have the following results.

Proposition II.3 ([14], Proposition 13.5.16)**.** *If* ρ *and* $\rho' = c\rho c^{-1}$ *are* sgn-*normalized rank one Drinfeld A-modules, then* $c \in \mathbb{F}_{q^{d_\infty}}^\times$ *and* $\mu_\rho(a) = \mu_{\rho'}(a)$ *for all* $a \in A$.

Theorem II.4 ([14], Corollary 13.5.17)**.** *Each isomorphism class of Drinfeld A-modules of rank one over $\mathbb{C}_\infty$ contains exactly $(q^{d_\infty}-1)/(q-1)$ Drinfeld-Hayes A-modules.*

Note that the number of Drinfeld-Hayes A-modules equals the number of sgn functions divided by $\#\mathbb{F}_q^\times$.

Let ρ be a Drinfeld-Hayes A-module with respect to sgn. Since ρ is a Drinfeld A-module of rank one, there exists a unique lattice Λ of rank one associated to ρ. The lattice Λ has the form $\tilde{\pi}_\rho I$, where $\tilde{\pi}_\rho \in \mathbb{C}_\infty$ and I is an integral ideal of A ([13], Chapter 13). The element $\tilde{\pi}_\rho$ is called the *period of* ρ.

C. The Carlitz Module

The simplest example of a Drinfeld-Hayes A-module is called the *Carlitz module*. Let $X = \mathbb{P}^1(\mathbb{F}_q)$ and denote the unique point at infinity by ∞. Then $K = \mathbb{F}_q(T)$ and $A = \mathbb{F}_q[T]$. Choose sgn such that sgn(T)=1. The Carlitz module $C\colon \mathbb{F}_q[T] \to \mathbb{C}_\infty\langle\tau\rangle$ is defined by

$$C(T) := C_T := T\tau^0 + \tau$$

and extended to all of A by $\mathbb{F}_q$-linearity and the "twisted" multiplication rule. The lattice associated to C is of the form $\tilde{\pi}_C\, A$ with

$$\tilde{\pi}_C = \sqrt[q-1]{T-T^q}\prod_{i=0}^{\infty}\left(1-\frac{T^{q^i}-T}{T^{q^{i+1}}-T}\right)$$

([13], Chapter 13). Notice that $\exp_C$, the exponential associated with C, has simple zeros on the elements of $\tilde{\pi}_C\, A$. So if A is viewed as the function field analogue of $\mathbb{Z}$, then the element $\tilde{\pi}_C$ is analogous to $2\pi i$.

D. Torsion Points

Returning now to the case of a general ring A, ρ is defined on the elements of A but can be extended to integral ideals of A as follows. One knows that $\mathbb{C}_\infty\langle\tau\rangle$ has a right division algorithm and every left ideal in $\mathbb{C}_\infty\langle\tau\rangle$ is principal ([13], Lemma 13.11). So if I is an ideal of A and J is the left ideal in $\mathbb{C}_\infty\langle\tau\rangle$ generated by $\{\rho_i|\ i \in I\}$, then ρ_I is defined to be the unique monic generator of J.

Let P be a prime ideal of A and let $\rho[P]$ denote the roots of the polynomial $\rho_P(x)$. These are the *P-torsion points of* ρ. If the lattice associated to ρ is $\tilde{\pi}_\rho I$, then

$$\rho[P] = \{\exp_\rho(\tilde{\pi}_\rho t) \mid t \in P^{-1}I\}$$

where $P^{-1} = \{x \in K \mid xP \subseteq A\}$.

E. Ramification

Let L be a finite separable extension of K. Let P be a point on the curve X and consider the associated discrete valuation ring $\mathcal{O}_P(X)$. Let R be the integral closure of $\mathcal{O}_P(X)$ in L. Since L is a separable extension of K, R is a Dedekind domain. We may view the ideal $\mathcal{M}_P(X)R$ as an ideal of R. Hence this ideal has a unique prime ideal decomposition in R:

$$\mathcal{M}_P(X)R = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$$

where each $\mathfrak{p}_i$ is a prime ideal of R. Let $R_{\mathfrak{p}_i}$ be the localization of R at $\mathfrak{p}_i$ which is a discrete valuation ring whose maximal ideal is $\mathfrak{P}_i := \mathfrak{p}_i R_{\mathfrak{p}_i}$. Set $\mathcal{O}_{\mathfrak{P}_i} := R_{\mathfrak{p}_i}$. We say that the ideal $\mathfrak{P}_i$ *lies above* P. The set $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_g\}$ consists of the all prime ideals of L which lie above P ([13], Chapter 7).

The *ramification index of* $\mathfrak{P}_i$ *with respect to* P, denoted $e(\mathfrak{P}_i/P)$, is the unique nonnegative integer such that

$$P\mathcal{O}_{\mathfrak{P}_i} = \mathfrak{P}_i^{e(\mathfrak{P}_i/P)}.$$

The *relative degree of* $\mathfrak{P}_i$ *with respect to* P, denoted $f(\mathfrak{P}_i/P)$, is the dimension of $\mathcal{O}_{\mathfrak{P}_i}/\mathfrak{P}_i$ over $\mathcal{O}_P(X)/\mathcal{M}_P(X)$. The point P is said to be *unramified in* L if all prime ideals above P in L are unramified. The point P is said to *split completely in* L if there are $[L\colon K]$ prime ideals above P in L.

F. Some Class Field Theory

Let H denote the *Hilbert class field* of A. The Hilbert class field H is the maximal abelian extension of K such that ∞ splits completely and every finite point is unramified. One knows that H is a separable extension of K ([10], §14). Let B denote the integral closure of A in H, i.e. the set of all elements of H which are integral over A.

Let $\rho \in \mathrm{Drin}_A(\mathbb{C}_\infty)$ and let E be a subfield of $\mathbb{C}_\infty$ containing K. Then E is a *field of definition for* ρ if ρ is isomorphic to some $\rho' \in \mathrm{Drin}_A(\mathbb{C}_\infty)$ such that $\rho'_a \in E\langle\tau\rangle$ for all $a \in A$. There exists a field of definition, denoted K_ρ, which is contained in every field of definition for ρ ([14], Theorem 13.5.9). The common field of definition of the rank one Drinfeld A-modules is precisely H ([10], §15). In fact, we can say more.

Theorem II.5 (Takahashi, [10], Theorem 15.8)**.** *Every Drinfeld A-module ρ is isomorphic to a Drinfeld A-module ρ' which is defined over B.*

The A-module $M \subseteq K$ is a *fractional ideal of* A if there exists a nonzero element $a \in A$ such that $aM \subseteq A$. Let $\mathfrak{F}(A)$ be the group of fractional ideals of A and let $\mathfrak{P}(A)$ be the subgroup of principal fractional ideals of A. The *class group of* A is

$$Cl(A) := \mathfrak{F}(A)/\mathfrak{P}(A).$$

Explicitly, two nonzero fractional ideals M, M' are *equivalent* if there exists nonzero α,

$\beta \in A$ such that $\alpha M = \beta M'$. We denote the equivalence of M and M' in $Cl(A)$ by $M \sim M'$. The relation defined by $\sim$ is an equivalence relation and $\mathfrak{P}(A)$ consists of the fractional ideals of A equivalent to the ideal $(1) = A$. The *class number of A* is defined by

$$h_A := \#Cl(A).$$

We have that $h_A = [H\colon K]$ ([10], Theorem 15.6).

We have a map

$$\mathfrak{F}(A) \times \mathrm{Drin}_A(\mathbb{C}_\infty) \to \mathrm{Drin}_A(\mathbb{C}_\infty)$$
$$(I, \rho) \mapsto I * \rho.$$

This gives an action of $\mathfrak{F}(A)$ on $\mathrm{Drin}_A(\mathbb{C}_\infty)$. If I is a principal ideal, then $I * \rho$ is isomorphic to ρ ([13], Proposition 13.14), so the above action descends to an action of $Cl(A)$ on the set of Drinfeld-Hayes A-modules. Furthermore, this action is one-to-one and transitive ([14], Theorem 13.5.18). We will return to this later.

If $\sigma \in \mathrm{Aut}(\mathbb{C}_\infty/K)$ and I is an ideal of A, then

$$I * \sigma\rho = \sigma(I * \rho)$$

where $\sigma\rho$ is the map defined by $a \mapsto \rho_a$ followed by the action of ρ([14], §13.7). If ρ is a Drinfeld-Hayes A-module, then so is $\sigma\rho$. In particular, we have that $\mathrm{Gal}(H/K)$ acts on the set of Drinfeld-Hayes A-modules.

Let L be a finite, Galois extension of K whose constant field is E and set $G := \mathrm{Gal}(L/K)$. Let P be a point on X and let $\mathfrak{P}$ be a prime ideal of L lying above P. Let

$$E_{\mathfrak{P}} := \mathcal{O}_{\mathfrak{P}}/\mathfrak{P}, \quad F_P := \mathcal{O}_P(X)/\mathcal{M}_P(X).$$

These (finite) fields are the *residue class fields* of $\mathfrak{P}$ and P, respectively. It is well known that $\mathrm{Gal}(E_{\mathfrak{P}}/F_P)$ is cyclic and generated by the Frobenius automorphism ϕ_P, which is defined by $\phi_P(x) = x^{q^{\deg P}}$ for all $x \in E_{\mathfrak{P}}$. Consider the following subgroup of G:

$$Z(\mathfrak{P}/P) := \{\sigma \in G \mid \sigma\mathfrak{P} = \mathfrak{P}\}.$$

This is the *decomposition group of* $\mathfrak{P}$ *over* P. If $\mathfrak{P}$ is unramified, then $Z(\mathfrak{P}/P) \cong \mathrm{Gal}(E_{\mathfrak{P}}/F_P)$ ([13], Corollary to Theorem 9.6). Denote by $(\mathfrak{P}, L/K) \in Z(\mathfrak{P}/P)$ the element which corresponds to ϕ_P under this isomorphism. The element $(\mathfrak{P}, L/K)$ is called the *Frobenius automorphism of* $\mathfrak{P}$ *for* L/K. Explicitly, the Frobenius automorphism satisfies

$$(\mathfrak{P}, L/K)\omega \equiv \omega^{q^{\deg P}} \pmod{\mathfrak{P}}$$

for all $\omega \in \mathcal{O}_{\mathfrak{P}}$. Furthermore, we have

$$(\sigma\mathfrak{P}, L/K) = \sigma(\mathfrak{P}, L/K)\sigma^{-1}$$

for all $\sigma \in G$ ([13], Proposition 9.10). Thus, as $\mathfrak{P}$ varies over the prime ideals lying above P, the associated Frobenius automorphisms fill out a conjugacy class in G which we call the *Artin conjugacy class of* P and denote by $(P, L/K)$.

Let L and G be as above but also assume that L is an abelian extension of K. Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be two unramified prime ideals of L lying above P. The Frobenius automorphisms $(\mathfrak{P}_1, L/K)$ and $(\mathfrak{P}_2, L/K)$ are thus conjugate in G, and since G is abelian, these automorphisms are equal. The Artin conjugacy class $(P, L/K)$ hence contains only one element of G. This element, which we also denote by $(P, L/K)$, is the *Artin automorphism associated to* P.

We extend this definition multiplicatively to all ideals of K which are not divisible by a ramified prime. Namely, let

$$I = P_1^{e_1} \cdots P_g^{e_g}$$

be an ideal of A where $P_1, \ldots, P_g$ are all unramified primes in L. Then the Artin automorphism associated to I is

$$(I, L/K) := (P_1, L/K)^{e_1} \cdots (P_g, L/K)^{e_g}.$$

If P is ramified in L, then set $(P, L/K) := \mathrm{id}_{\mathrm{Gal}(L/K)}$. Thus, we may consider the Artin automorphism associated to an arbitrary ideal of K.

Thus, $\mathrm{Gal}(H/K)$ and $Cl(A)$ act on the set of Drinfeld-Hayes A-modules and the equation $I * \sigma\rho = \sigma(I * \rho)$ shows that these actions commute with one another. Define a map

$$\kappa \colon \mathrm{Gal}(H/K) \to Cl(A)$$

as follows. Let $(I, H/K)$ be the Artin automorphism associated to I. Write $\rho_a = \sum_{i=0}^{\deg a} \rho_{a,i}\tau^i$ where $\rho_{a,i} \in H$ for all i. Define $(I, H/K)\rho$ by

$$((I, H/K)\rho)_a := \sum_{i=0}^{\deg a} \rho_{a,i}^{(I,H/K)}\tau^i$$

for all $a \in A$. Then

$$\kappa((I, H/K)) = [I_\rho]$$

where I_ρ satisfies $(I, H/K)\rho = I_\rho * \rho$. The map κ is an isomorphism ([14], Proposition 13.5.22).

Consider

$$\mathfrak{P}^+(A) = \{xA \mid x \in K, \mathrm{sgn}(x) = 1\}$$

and define $\mathrm{Pic}^+(A) := \mathfrak{F}(A)/\mathfrak{P}^+(A)$. Fix a Drinfeld-Hayes A-module ρ. Let H^+ be the field generated over K by the coefficients of ρ_y for some nonconstant $y \in A$. This field H^+ is independent of the choice of ρ ([10], §14). The field H^+ is an extension of H of degree

$(q^{d_\infty}-1)/(q-1)$ ([10], Theorems 14.7 and 15.6). The relation

$$(I,H^+/K)\rho = I*\rho$$

holds and thus, $\mathrm{Gal}(H^+/K)\cong \mathrm{Pic}^+(A)$ and $[H^+ : K] = \frac{q^{d_\infty}-1}{q-1}h_A$ ([14], Theorem 13.5.30). Note that H^+ is an extension of H which is nontrivial precisely when $d_\infty > 1$.

Fix a prime ideal $\mathfrak{m}$ of A and let $\mathfrak{F}_\mathfrak{m}(A)$ be the subgroup of $\mathfrak{F}(A)$ consisting of fractional ideals that are prime to $\mathfrak{m}$. Let

$$\mathfrak{P}^+_\mathfrak{m}(A) := \{xA \mid x\in K^\times, \mathrm{sgn}(x)=1, x\equiv 1 \pmod{\mathfrak{m}}\}$$

and define

$$\mathrm{Pic}^+_\mathfrak{m}(A) := \mathfrak{F}_\mathfrak{m}(A)/\mathfrak{P}^+_\mathfrak{m}(A).$$

Let ρ be a Drinfeld-Hayes A-module and let $\rho[\mathfrak{m}]$ denote the set of all $\mathfrak{m}$-torsion points of ρ. The cyclic A-module $\rho[\mathfrak{m}]$ is isomorphic to $A/\mathfrak{m}$ ([10], §16). Let $\lambda\in\rho[\mathfrak{m}]$ and let $K_\mathfrak{m} := H^+(\rho[\mathfrak{m}])$. If I is an ideal of A prime to $\mathfrak{m}$, then

$$(I,K_\mathfrak{m}/K)\lambda = \rho_I(\lambda)$$

for $(I,K_\mathfrak{m}/K)\in\mathrm{Gal}(K_\mathfrak{m}/K)$ ([14], Theorem 13.5.43). This also gives an action of $\mathrm{Pic}^+_\mathfrak{m}(A)$ on $K_\mathfrak{m}$. Furthermore, we have that $\mathrm{Pic}^+_\mathfrak{m}(A)\cong\mathrm{Gal}(K_\mathfrak{m}/K)$ ([10], §16). We also have that $(A/\mathfrak{m})^\times\cong\mathrm{Gal}(K_\mathfrak{m}/H^+)$ via the map $a\mapsto((a),K_\mathfrak{m}/H^+)$ ([10], §16).

G. Character Groups and the Goss L-function

Consider the following groups

$$G := \mathrm{Gal}(K_{\mathfrak{m}}/K),$$

$$G' := \mathrm{Gal}(H^+/K) \cong \mathrm{Pic}^+(A),$$

$$G'' := \mathrm{Gal}(K_{\mathfrak{m}}/H^+) \cong (A/\mathfrak{m})^\times.$$

Thus, since $G/G'' \cong \mathrm{Pic}^+(A)$, the following sequence is exact

$$0 \to G'' \to G \to G' \to 0.$$

Define $\hat{G} := \mathrm{Hom}(G, \mathbb{C}_\infty^\times)$ and similarly define $\hat{G}'$ and $\hat{G}''$. From these character groups, we get another exact sequence

$$0 \to \hat{G}' \to \hat{G} \to \hat{G}'' \to 0.$$

Let χ be a Dirichlet character on A, i.e. $\chi\colon A \to \mathbb{C}_\infty$ is a homomorphism. Assume the kernel of χ is $\mathfrak{m}$ and that the order of χ is relatively prime to p. Then $\chi \in \hat{G}''$. The map $\hat{G} \to \hat{G}''$ defined by $\psi \mapsto \psi|_A$ is surjective. Fix $\psi \in \hat{G}$ such that $\psi|_A = \chi$. Thus, ψ is defined on $G \cong \mathrm{Pic}_{\mathfrak{m}}^+(A)$. So we have extended the character

$$\chi\colon A \to \mathbb{C}_\infty$$

to the multiplicative character

$$\psi\colon \mathfrak{F}_{\mathfrak{m}}(A) \to \mathbb{C}_\infty.$$

Let I be a fractional ideal which is not prime to $\mathfrak{m}$. By setting $\psi(I) = 0$, we now have a multiplicative character defined on $\mathfrak{F}(A)$.

Fix a sign function sgn and let $\pi \in K_\infty$ be a monic uniformizer at ∞. Every $x \in K_\infty$ may be written uniquely as

$$x = \operatorname{sgn}(x)\pi^{\operatorname{ord}_\infty(x)}\langle x\rangle.$$

The element $\langle x\rangle$ is the 1-unit part of x. Recall that $U^{(1)}$ denotes the group of 1-units in K_∞. Let $\widehat{U^{(1)}}$ be the group of 1-units in $\mathbb{C}_\infty$. Consider the map $\mathfrak{P}^+(A) \to \widehat{U^{(1)}}$, $(\alpha) \mapsto \langle\alpha\rangle$. Because $\mathfrak{P}^+(A)$ has finite index in $\mathfrak{F}(A)$, and because $\widehat{U^{(1)}}$ is uniquely divisible, this map uniquely extends to a map $\mathfrak{F}(A) \to \widehat{U^{(1)}}$ (which we also denote by $\langle\cdot\rangle$) ([7], Corollary 8.2.4).

Let $\pi_* \in \mathbb{C}_\infty$ be a fixed d_∞-th root of π. Let I be a fractional ideal of A. For $j \in \mathbb{Z}$, set

$$I^{[j]} := (\pi_*^{-j})^{\deg I}\langle I\rangle^j.$$

Now suppose $\varpi \in K_\infty$ is another monic uniformizer at ∞. Let $\langle I\rangle_\varpi$ be the 1-unit part of I defined with respect to ϖ. Let $\varpi_* \in \mathbb{C}_\infty$ be a fixed d_∞-th root of ϖ. And let $I_\varpi^{[j]}$ be the exponentiation of I defined with respect to ϖ. Using ([7], Proposition 8.2.15) and the definitions, one can show that there exists a d_∞-th root of unity ζ such that

$$I^{[j]} = \zeta^{j\deg I} I_\varpi^{[j]}.$$

We now list some additional properties of this ideal exponentiation:

1. If $I = (i)$, where i is monic, then $I^{[j]} = i^j$ ([7], Proposition 8.1.4).

2. If $I = (i)$, where $i = \operatorname{sgn}(i)\pi^{\operatorname{ord}_\infty(i)}\langle i\rangle$, then $I^{[j]} = (\pi_*^{-j})^{\deg i}\langle i\rangle^j$. ([7], Proposition 8.2.6)

3. Let e be the order of I in $\mathfrak{F}(A)/\mathfrak{P}^+(A)$. If $I^e = (\lambda)$ for some monic $\lambda \in K$, then $I^{[j]} = (\pi_*^{-j})^{\deg I}\langle\lambda\rangle^{j/e}$ ([7], §8.2).

Let χ be a Dirichlet character on A whose kernel is $\mathfrak{m}$ and whose order is relatively prime to p. As before, we extend $\chi\colon A \to \mathbb{C}_\infty$ to the multiplicative character

$$\psi\colon \mathfrak{F}_\mathfrak{m}(A) \to \mathbb{C}_\infty.$$

For all fractional ideals I which are not prime to $\mathfrak{m}$, we set $\psi(I) = 0$ and so the multiplicative character ψ is now defined on $\mathfrak{F}(A)$. Note that $\psi \in \hat{G}$ and $\psi|_A = \chi$. The *Goss L-function for* ψ is defined to be

$$L(j, \psi) = \sum_I \frac{\psi(I)}{I^{[j]}}$$

where the sum ranges over all integral ideals I of A which are relatively prime to $\mathfrak{m}$.

We claim that $L(j, \psi)$ converges in $\mathbb{C}_\infty$ for integers $j > 0$. Write

$$L(j, \psi) = \sum_{d \geq 0} \sum_{\deg I = d} \frac{\psi(I)}{I^{[j]}} =: \sum_{d \geq 0} a_d.$$

The coefficients a_d are well-defined since there are only finitely many ideals of a given degree. Since $\mathbb{C}_\infty$ is a complete nonarchimedean field, it is enough to show that $|a_d| \to 0$ as $d \to \infty$. Now either $\psi(I) = 0$ or $|\psi(I)| = 1$ since $\psi(I)$ is in a finite extension of $\mathbb{F}_q$. To compute $|I^{[j]}|$, we use the expression for $I^{[j]}$ as given in (3) above. First, note that any 1-unit has degree 0. This follows since if $\langle i \rangle$ denotes a 1-unit, then $\langle i \rangle = 1 \cdot \pi^0 \cdot \langle i \rangle$ and this expression is unique. Hence, $\mathrm{ord}_\infty(\langle i \rangle) = 0$, which implies that the degree is 0. Second, the degree of π is -1. Again, this follows from the uniqueness of expansion of elements in K_∞. Since $\pi_*^{d_\infty} = \pi$, it follows that π_* has degree $-1/d_\infty$ and so $|\pi_*| = q^{-1/d_\infty}$. Therefore,

$$|I^{[j]}| = |\pi_*|^{-j \deg I} |\langle \lambda \rangle|^{j/e} = q^{(j \deg I)/d_\infty}$$

and so

$$|a_d| = \left| \sum_{\deg I = d} \frac{\psi(I)}{I^{[j]}} \right| \leq \max_{\deg I = d} \left(\left| \frac{\psi(I)}{I^{[j]}} \right| \right) = q^{(-jd)/d_\infty}.$$

This proves the claim.

CHAPTER III

RESULTS OF ANDERSON

In this chapter, we will review the techniques and results from [2].

A. Characteristic Zero and Log-algebraicity

We first recall some results from characteristic 0. Let $\mathbf{e}(z) := \exp(2\pi i z)$. Fix an odd prime p and consider the ring of integers $\mathbb{Z}[\mathbf{e}(1/p)]$ of $\mathbb{Q}(\mathbf{e}(1/p))$.

Let $\mathcal{C}$ be the subgroup of $\mathbb{Z}[\mathbf{e}(1/p)]^\times$ generated by

$$\frac{1-\mathbf{e}(a/p)}{1-\mathbf{e}(1/p)}$$

for $a = 1, \ldots, p-1$. The elements of $\mathcal{C}$ are called *circular units*.

Theorem III.1 ([12], §19.1)**.** (1) *The group $\mathcal{C}$ of circular units is of rank $(p-3)/2$.* (2) *Moreover, $\mathcal{C}$ is of finite index in $\mathbb{Z}[\mathbf{e}(1/p)]^\times$.*

Let $\chi\colon (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Choose $d > 0$ such that $d|n$. Then d is called an induced modulus for χ if

$$\chi(a) = 1 \text{ whenever } (a,n) = 1 \text{ and } a \equiv 1 \pmod{d}.$$

The smallest induced modulus for χ is called the conductor of χ which we denote by f.

The L-series attached to χ is

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$. If χ is not the trivial character and if $\chi(-1) = 1$, then

$$L(1,\chi) = -\frac{\tau(\chi)}{f} \sum_{a=1}^{f} \bar{\chi}(a) \log|1-\mathbf{e}(a/f)| \tag{3.1}$$

where

$$\tau(\chi) = \sum_{a=1}^{f} \chi(a)\mathbf{e}(a/f)$$

is a Gauss sum ([20], Theorem 4.9).

In [2], a formula analogous to (3.1) is proven which relates the value of the Goss L-function at 1 to logarithms of so called *special points* of the Carlitz module. Also, a result analogous to Theorem III.1(1) is proven in which the special points of the Carlitz module play the role of the circular units.

Anderson says that (3.1) is proved by analyzing the formal power series identity

$$\exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\right) = 1-z \tag{3.2}$$

over $\mathbb{C}$. Hence, one of the goals of [2] is to prove a function field analogue of (3.2). In [2], a function is said to be *log-algebraic* if it is formally the logarithm of a power series algebraic over the field of rational functions in z. For example, the series $\sum_{n=1}^{\infty}\frac{z^n}{n}$ is log-algebraic.

B. Special Polynomials

We return now to the case of function fields. Recall that H is the Hilbert class field of A and B is the integral closure of A in H. Fix a Drinfeld-Hayes A-module ρ relative to a fixed sign function sgn. Recall that $\exp_\rho(z)$, for $z \in \mathbb{C}_\infty$, is the exponential belonging to ρ. Since $\exp_\rho(z)$ is $\mathbb{F}_q$-linear and has leading term z, it has a power series expansion of the form

$$\exp_\rho(z) = z + \sum_{i=1}^{\infty} e_i(\rho) z^{q^i}$$

for $e_i(\rho) \in \mathbb{C}_\infty$. Set $e_0(\rho) = 1$ and $e_i(\rho) = 0$ for $i < 0$.

Let I be an integral ideal of A and let $b \in H[t]$ where t is a variable. Set

$$I * b := \sum_i b_i^{(I,H/K)} \rho_I(t)^i$$

where $b = \sum_i b_i t^i$, $(I, H/K)$ is the Artin automorphism of I and we write $b_i^{(I,H/K)}$ to denote $(I, H/K)$ acting on b_i. Note that the map $H[t] \to H[t]$ defined by $b \mapsto I * b$ is an A-algebra endomorphism that stabilizes $B[t]$.

For $b \in H[t]$, define

$$\ell(b;z) := \sum_I \frac{I * b}{D(\rho_I)} z^{q^{\deg I}} \in H[t][[z]]$$

where the sum is over all nonzero integral ideals I of A and $D(\rho_I)$ is the constant term of ρ_I. Anderson calls this function a *twisted A-harmonic series* and views it as the analogue of the harmonic series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Now expand $\ell(b;z)$ as a power series:

$$\ell(b;z) := \sum_{i=0}^{\infty} \ell_i(b) z^{q^i}.$$

Note that the coefficients $\ell_i(b) \in H[t]$ for $i \geq 0$. Set $\ell_i(b) = 0$ for $i < 0$. For all integers i, define

$$Z_i(b) := \sum_{j=0}^{\infty} e_j(\rho) \ell_{i-j}(b)^{q^j} \in H[t].$$

The sum defining $Z_i(b)$ is actually finite and clearly $Z_i(b) = 0$ for $i < 0$. Define the formal power series

$$Z(b;z) := \sum_{i=0}^{\infty} Z_i(b) z^{q^i} \in H[t][[z]].$$

As formal power series, we have

$$Z(b;z) = \exp_\rho \ell(b;z).$$

The following fundamental result is proven in [2].

Theorem III.2. *([2], Theorem 3) For all $b \in B[t]$, the power series $\exp_\rho \ell(b;z)$ is in fact a polynomial in $B[t,z]$.*

Theorem III.2 is viewed by Anderson as the analogue of

$$\exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\right) = 1 - z$$

since it implies that the power series $Z(b;z)$ is in $B[t,z]$. Hence, the twisted A-harmonic series $\ell(b;z)$ is log-algebraic. The polynomials

$$\{Z(b;z) = \exp_\rho \ell(b;z) \mid b \in B[t]\}$$

are called *special polynomials*.

Anderson's strategy for proving Theorem III.2 is as follows:

1. Equip $H[t]$ with a norm $\|\cdot\|$ for which $B[t]$ is discrete.

2. Prove that the coefficients $Z_i(b)$ belong to $B[t]$ if $b \in B[t]$.

3. Prove that $\|Z_i(b)\| \to 0$ as $i \to \infty$.

From this, Anderson concludes that $Z_i(b)$ vanishes for $i \gg 0$ if $b \in B[t]$. In fact, Anderson proves that if i is beyond a certain explicit index, then the coefficients $Z_i(b)$ are identically zero. We will be concerned with computing this index in a later chapter.

Let π be a fixed monic uniformizer at ∞. The *imaginary axis* is defined to be the one-dimensional K_∞-subspace of $\bar{K}_\infty$ spanned by the $(q-1)$st roots of $-\pi^{-1}$. We denote this subspace by $K_\infty \cdot \sqrt[q-1]{-\pi^{-1}}$.

For $b = \sum_i b_i t^i \in H[t]$ and for an integral ideal I of A, consider

$$(I * b)(\exp_\rho(x)) = \sum_i b_i^{(I,H/K)} \rho_I(\exp_\rho(x))^i,$$

where x is in the imaginary axis. Recall that $|\cdot|$ denotes the normalized absolute value associated to ∞. Let

$$\|b\|_I := \sup_x |(I * b)(\exp_\rho(x))|$$

where x ranges over the imaginary axis.

The following result shows that this supremum is always finite.

Lemma III.3 ([2], Lemma 1)**.** *For all b and I, the supremum $\|b\|_I$ is finite and depends only on the Artin automorphism $(I,H/K)$ via the formula*

$$\|b\|_I = \sup_x \left| \sum_i b_i^{(I,H/K)} (\exp_{I*\rho}(x))^i \right|. \tag{3.3}$$

Now set

$$\|b\| := \sup_I \|b\|_I \tag{3.4}$$

where I ranges over all integral ideals of A. By the previous Lemma, $\|b\|$ is finite for all b. The function $\|\cdot\|$ is an ultrametric norm for $H[t]$. The properties of this norm are spelled out in the next result.

Proposition III.4 ([2], Proposition 2)**.** (1) *There exists a positive integer n such that for all $b \in B[t]$ such that $\|b\| \leq 1$ one has $b^{q^n} = b$ (and hence b is constant).*

(2) *If $b \in B[t]$ satisfies $\|b\| < 1$, then $b = 0$.*

(3) *The ring $B[t]$ is discretely embedded in $H[t]$ with respect to the topology defined by the norm $\|\cdot\|$.*

Next Anderson shows that the coefficients $Z_i(b)$, which a priori are in $H[t]$, lie in $B[t]$. Let $b = \sum_i b_i t^i \in H[t]$. Let I be an integral ideal of A and let v be a finite valuation on H. Set

$$v(b) := \min_i v(b_i).$$

Proposition III.5 ([2], Proposition 6)**.** *For all $b \in H[t]$ such that $v(b) \geq 0$, and for all integers i, we have $v(Z_i(b)) \geq 0$.*

The third part of Anderson's strategy for proving his Theorem III.2 is to give upper bounds on the coefficients $e_i(\rho)$ and $\ell_i(b)$ in terms of the norm $\|\cdot\|$. First, the upper bound for $e_i(\rho)$.

Proposition III.6. *([2], Proposition 4) There exists a real number* $i_0 \geq 0$ *such that*

$$\|e_i(\rho)\| \leq q^{(i_0-i)q^i}$$

for all integers i.

And now the upper bound for $\ell_i(b)$.

Proposition III.7. *([2], Proposition 7) For all* $b \in H[t]$*, there exists a real number* $j_0(b) \geq 0$*, such that*

$$\|\ell_i(b)\| \leq q^{j_0(b)-i}$$

for all integers i.

The number $j_0(b)$ can be explicitly computed as follows. Set

$$\gamma := \max_I \; q^{\deg I} \left\| \frac{1}{D(\rho_I)} \right\|$$

where I ranges over all integral ideals of A. Note that the maximum exists because if $I \sim J$ in $Cl(A)$, then

$$q^{\deg I} \left\| \frac{1}{D(\rho_I)} \right\| = q^{\deg J} \left\| \frac{1}{D(\rho_J)} \right\|.$$

Thus, the maximum depends only on the ideal class of I. Then $j_0(b)$ is defined by

$$q^{j_0(b)} = \max(1, \gamma \, \|b\|).$$

The proof of Theorem III.2 can now be completed as follows. From the definition of $Z_i(b)$ and by Propositions III.6 and III.7, we conclude

$$\|Z_i(b)\| \leq \sup_{j \geq 0} q^{(i_0+j_0(b)-i)q^j}. \tag{3.5}$$

Proposition III.5 implies that $Z_i(b) \in B[t]$ for all $b \in B[t]$. Thus, by Proposition III.4, we have that if $i > i_0 + j_0(b)$ then $Z_i(b) = 0$. This completes the proof. The numbers i_0 and $j_0(b)$ will be computed explicitly, first in the case of the Carlitz module, and later for a more general function field.

C. Special Polynomials for the Carlitz Module

For the rest of this chapter (and for the rest of Anderson's paper), the Drinfeld-Hayes A-module in question is the Carlitz module C. Recall that the curve $X = \mathbb{P}^1(\mathbb{F}_q)$ and that the distinguished point ∞ is the unique point at infinity on X. The degree of ∞, denoted d_∞, is 1. The function field $K = \mathbb{F}_q(T)$ and the functions regular away from ∞ are $A = \mathbb{F}_q[T]$. The completion of K at ∞ is $K_\infty = \mathbb{F}_q((1/T))$. Elements of K_∞ are of the form

$$\sum_{i=m}^{\infty} a_i \left(\frac{1}{T}\right)^i$$

for some $m \in \mathbb{Z}$, $a_i \in \mathbb{F}_q$ for all i, and $a_m \neq 0$. The field $\mathbb{C}_\infty$ is the completion of $\bar{K}_\infty$ with respect to $|\cdot|$. Fix a sign function sgn such that $\operatorname{sgn}(T) = 1$. The Carlitz module $C\colon A \to \mathbb{C}_\infty\langle\tau\rangle$ is determined by

$$C(T) := C_T := T\tau^0 + \tau,$$

where τ is the q-th power Frobenius map, and is extended to all of A by the "twisted" multiplication rule. It is easy to see (using the "twisted" multiplication rule and the definition of C_T) that the image of A under the Carlitz module actually lies in $A\langle\tau\rangle$.

Let $\exp_C(z)$ be the exponential belonging to C. We also refer to this function as the *Carlitz exponential*. It is known ([14], §13.4) that

$$\exp_C(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i}$$

where $D_0 = 1$ and, for $i \geq 1$,

$$D_i := \prod_{j=0}^{i-1} (T^{q^i} - T^{q^j}).$$

In terms of our previous notation, $e_i(C) = 1/D_i$. The Carlitz exponential satisfies the following functional equation

$$\exp_C(az) = C_a(\exp_C(z))$$

for all $a \in A$ and for all $z \in \mathbb{C}_\infty$.

The lattice associated to C is of the form $\tilde{\pi}_C\, A$ where

$$\tilde{\pi}_C = \sqrt[q-1]{T - T^q} \prod_{i=0}^{\infty} \left(1 - \frac{T^{q^i} - T}{T^{q^{i+1}} - T}\right).$$

The element $\tilde{\pi}_C$ is called the Carlitz period.

Let $\log_C(z)$ be the logarithm belonging to C. This function, which by definition is the formal power series inverse of $\exp_C$, is also called the *Carlitz logarithm*. It is known ([14], §13.4) that

$$\log_C(z) = \sum_{k=0}^{\infty} \frac{z^{q^k}}{L_k}$$

where $L_0 = 1$ and, for $k \geq 1$,

$$L_k := \prod_{j=1}^{k} (T - T^{q^j}).$$

Define a function $\mathbf{e}\colon K_\infty \to \bar{K}_\infty$ by

$$\mathbf{e}(x) := \exp_C(\tilde{\pi}_C x).$$

For each nonnegative integer m, define the function $l_m\colon K_\infty \to \bar{K}_\infty$ by

$$l_m(x) := \sum_{a \in A_+} \frac{\mathbf{e}(ax)^m}{a}$$

where A_+ denotes the set of monic elements of A. Define $l_0\colon K_\infty \to \bar{K}_\infty$ by

$$l_0(x) := \sum_{a\in A_+} \frac{1}{a}.$$

The following is an explicit version of Theorem III.2 in the case of the Carlitz module. We will be concerned with proving a generalization of this result in a later chapter.

Proposition III.8 ([2], Proposition 8)**.** *Let m be a nonnegative integer.*

(1) The power series

$$S_m(t,z) := \exp_C \ell(t^m;z) = \sum_{i=0}^{\infty}\sum_{a\in A_+} \frac{1}{D_i}\left(\frac{(C_a(t))^m}{a}\right)^{q^i} z^{q^{i+\deg a}}$$

lies in $A[t,z]$.

(2) If $m<q$, then $S_m(t,z) = t^m z$.

(3) One has $\exp_C l_m(x) = S_m(\mathbf{e}(x),1)$ for all $x\in K_\infty$.

(4) For all $\theta\in\mathbb{F}_q^\times$ one has $S_m(\theta t,z) = \theta^m S_m(t,z)$, and moreover $S_m(t,z)$ is divisible by t^m.

(5) One has

$$\left.\frac{S_m(t,z)}{t^m}\right|_{t=0} = \sum_{a\in A_+} a^{m-1} z^{q^{\deg a}}$$

for $m>0$.

(6) The degree of $S_m(t,z)$ in z (respectively t and T) does not exceed $q^{\lfloor (m-1)/(q-1)\rfloor}$ (resp. $mq^{\lfloor (m-1)/(q-1)\rfloor}$ and $(m/q)q^{\lfloor (m-1)/(q-1)\rfloor}$).

(7) The specialization $S_m(t,1)\in A[t]$ vanishes identically if $m>1$ and $m\equiv 1 \bmod q-1$.

We conclude our discussion of this Proposition with some remarks.

Remark III.9. The polynomial $S_m(t,z)$ is called the *m-th special polynomial* for the Carlitz module.

Remark III.10. The choice of $b \in B[t]$ in this case is $b = t^m$. Note that since $h_A = 1$, we have that $A = B$.

Remark III.11. Recall that we have $Z(t^m; z) = \exp_C \ell(t^m; z)$ and that $Z(t^m; z)$ may be expanded as a power series:

$$Z(t^m; z) = \sum_{i=0}^{\infty} Z_i(t^m) z^{q^i}.$$

Proposition III.8(1) implies that

$$Z_i(t^m) = \frac{1}{D_i} \sum_{a \in A_+} \left(\frac{(C_a(t))^m}{a} \right)^{q^i} z^{q^{\deg a}}.$$

From the remarks following Proposition III.7, we have that if $i > i_0 + j_0(t^m)$, then $Z_i(t^m) = 0$. Hence, $q^{\lfloor i_0 + j_0(t^m) \rfloor}$ is an upper bound for the degree of $S_m(t,z)$ in z. Thus, to prove Proposition III.8(6), it is necessary to compute the numbers i_0 and $j_0(t^m)$. These numbers are:

(1) $i_0 = 0$;

(2) $j_0(t^m) = \frac{m}{q-1}$.

Also, to compute $j_0(t^m)$, it is necessary to compute the value $\|t^m\|$. In [2], it is shown that

$$\|t^m\| = \sup_{x \in K_\infty} \{ |\exp_C(\tilde{\pi}_C x)|^m \}.$$

Since

$$|\exp_C(\tilde{\pi}_C x)| \le \max_{i \ge 0} (|e_i(C)| |\tilde{\pi}_C x|^{q^i}),$$

it is necessary to compute $|\tilde{\pi}_C|$. This value is $|\tilde{\pi}_C| = q^{q/(q-1)}$. We will return to these types of calculations during the proof of our analogue of Proposition III.8.

D. Special Points for the Carlitz Module

Fix a positive integer d. Let

$$\mathcal{M} := \{m \in \mathbb{Z} \mid 1 \leq m \leq q^d - 1, m \not\equiv 1 \bmod q-1\}.$$

Fix an irreducible $\mathbf{p} \in A_+$ of degree d and let $(\mathbf{p})$ denote the ideal generated by $\mathbf{p}$ in A. Consider the $(\mathbf{p})$-torsion of C:

$$\begin{aligned} C[(\mathbf{p})] = C[\mathbf{p}] &= \{\exp_C(\tilde{\pi}_C t) \mid t \in (\mathbf{p})^{-1}A\} \\ &= \{\exp_C(\tilde{\pi}_C a/\mathbf{p}) \mid a \in A\} \\ &= \{\mathbf{e}(a/\mathbf{p}) \mid a \in A\}. \end{aligned}$$

Suppose $a \equiv a' \bmod \mathbf{p}$. Then $a = a' + \mathbf{p}a''$ for some $a'' \in A$. Therefore,

$$\mathbf{e}(a/\mathbf{p}) = \mathbf{e}(a'/\mathbf{p}) + \mathbf{e}(a'') = \mathbf{e}(a'/\mathbf{p})$$

since the $\mathbb{F}_q$-linear function $\mathbf{e}(x)$ vanishes when $x \in A$. So the function $\mathbf{e}(x/\mathbf{p})$ for $x \in A$ depends only the residue of x mod $\mathbf{p}$. Let $\mathbf{F_p} := A/\mathbf{p}$. For $y \in \mathbf{F_p}$, choose $y' \in A$ such that $y \equiv y' \bmod \mathbf{p}$. Set

$$\mathbf{e}(y/\mathbf{p}) := \mathbf{e}(y'/\mathbf{p}). \tag{3.6}$$

This definition is independent of the choice of y' as we have previously shown. Thus, we consider $\mathbf{e}(x/\mathbf{p})$ as a function on $\mathbf{F_p}$.

It is known that $C[\mathbf{p}] \cong \mathbf{F_p}$ as A-modules ([13], Proposition 12.4). Let $\lambda := \mathbf{e}(1/\mathbf{p})$ be a generator of this module. Since $d_\infty = 1$, we have that $H^+ = H$. Since $h_A = 1$, we have $H = K$. Set

$$\begin{aligned} K_{\mathbf{p}} &:= K(C[\mathbf{p}]) = K(\lambda); \\ \mathcal{R} &:= A[\lambda]. \end{aligned}$$

The fraction field of $\mathcal{R}$ is $K(\lambda)$ and, moreover, the integral closure of A in $K_{\mathbf{p}}$ is $\mathcal{R}$ ([13], Proposition 12.9).

Let $G := \mathrm{Gal}(K_{\mathbf{p}}/K)$. Let

$$a \mapsto \sigma_a \colon \mathbf{F}_{\mathbf{p}}^{\times} \to G$$

be the unique isomorphism such that

$$\sigma_a(\mathbf{e}(b/\mathbf{p})) = \mathbf{e}(ab/\mathbf{p})$$

for all $a, b \in \mathbf{F}_{\mathbf{p}}$. Note that by the functional equation for the Carlitz module, it follows that $\mathbf{e}(ab/\mathbf{p}) = C_a(\mathbf{e}(b/\mathbf{p}))$.

For any A-algebra R, let R^C be a copy of R equipped with the A-module structure

$$(a, r) \mapsto a * r := C_a(r) \colon A \times R \to R$$

The Carlitz exponential $\exp_C(z)$ becomes A-linear if viewed as a map $\bar{K}_\infty \to \bar{K}_\infty^C$. If $r \in R$ is viewed as an element of R^C, i.e. via $(1, r) \mapsto C_1(r) = r$, then r is the coordinate of an *R-valued point* of the Carlitz module.

Let m be a nonnegative integer. For each $b \in \mathbf{F}_{\mathbf{p}}^{\times}$, let $s_m(b)$ be the $\mathcal{R}$-valued point of the Carlitz module with coordinate

$$s_m(b) := \exp_C \ell_m(x) = S_m(\mathbf{e}(x), 1),$$

where $x = \tilde{b}/\mathbf{p}$ and $\tilde{b} \equiv b \pmod{\mathbf{p}}$.

We claim that this definition does not depend upon the choice of $\tilde{b}$. Suppose $b' \in \mathbf{F}_{\mathbf{p}}^{\times}$ such that $b' \equiv b \pmod{\mathbf{p}}$. Set $x' = b'/\mathbf{p}$. Now $b' \equiv \tilde{b} \pmod{\mathbf{p}}$, so we may write $b' = \tilde{b} + \mathbf{p}a$ for some $a \in A$. Then

$$\mathbf{e}(x') = \mathbf{e}(b'/\mathbf{p}) = \mathbf{e}(\tilde{b}/\mathbf{p} + a) = \mathbf{e}(\tilde{b}/\mathbf{p}) = \mathbf{e}(x)$$

since the $\mathbb{F}_q$-linear function $\mathbf{e}(x) = \exp_C(\tilde{\pi}_C x)$ vanishes if $x \in A$. This proves the claim.

Let $\mathcal{S}$ be the A-submodule of $\mathcal{R}^C$ generated by the points of the form $s_m(b)$. Since

$$\sigma_a s_m(b) = s_m(ab)$$

for all $a, b \in \mathbf{F}_{\mathbf{p}}^{\times}$, the A-module $\mathcal{S}$ is G-stable. The elements of $\mathcal{S}$ are *special points* of the Carlitz module. A special point is considered by Anderson to be the analogue of a circular unit.

Since $S_0(t,z) = z$ by Proposition III.8(2), it follows that $1 \in A$ is the coordinate of the special point $s_0(1)$. Note that $\lambda \in \mathcal{R}$ is the coordinate of the special point $s_1(1)$. To see this, the coordinate of $s_1(1)$ is $S_1(\mathbf{e}(1/\mathbf{p})) = \mathbf{e}(1/\mathbf{p})$ by Proposition III.8(2). Also, $s_1(1)$ is annihilated by $\mathbf{p}$. In this case, observe that $\mathbf{p} * s_1(1) = C_{\mathbf{p}}(\mathbf{e}(1/\mathbf{p})) = \mathbf{e}(\mathbf{p} \cdot 1/\mathbf{p}) = 0$.

Since the module of special points is viewed as the analogue of the circular units, Anderson proved a result analogous to Theorem III.1(1). First, Anderson showed that the $\mathcal{R}$-valued points of the Carlitz module of the form

$$\{s_m(1) \mid 0 \leq m \leq q^d - 1\}$$

generate $\mathcal{S}$ ([2], Proposition 9). Therefore, $\mathcal{S}$ is a finitely generated A-module. Then the following result was proved.

Theorem III.12 ([2], Theorem 4)**.** *The A-rank of $\mathcal{S}$ is*

$$(q^d - 1)((q-2)/(q-1)).$$

Remark III.13. We will refer to this result as *Anderson's A-rank theorem.*

The rest of this chapter will be concerned with Anderson's analogue of (3.1). Consider $\omega\colon A \to \mathbb{C}_\infty^{\times}$ where $\omega(a) \equiv a \bmod \mathbf{p}$ for all $a \in A$ relatively prime to $\mathbf{p}$. This is the Teichmüller character on A. Notice that if $a' \equiv a'' \bmod \mathbf{p}$, then $\omega(a') \equiv \omega(a'') \bmod \mathbf{p}$. Thus, for all $a \in A$ relatively prime to $\mathbf{p}$, $\omega(a)$ depends upon the residue of a mod $\mathbf{p}$. Hence, we consider ω as a character on $\mathbf{F}_{\mathbf{p}}^{\times}$.

Proposition III.14 ([2], Proposition 10)**.** *Let m be an integer such that $1 \leq m \leq q^d - 1$.*

(1) For every m and for every $a \in \mathbf{F}_{\mathbf{p}}^{\times}$, there exists a unique element $\mathbf{e}_m^*(a) \in K_{\mathbf{p}}$ such that

$$\sum_{m=1}^{q^d-1} \mathbf{e}(b/\mathbf{p})^m \mathbf{e}_m^*(a) = \mathbf{p} \cdot \delta_{ba}$$

for all $b \in \mathbf{F}_{\mathbf{p}}^{\times}$. Here,

$$\delta_{ba} = \begin{cases} 1 & \text{if } b = a; \\ 0 & \text{if } b \neq a. \end{cases}$$

(2) For all $c \in \mathbb{F}_q^{\times}$ and for all $a \in \mathbf{F}_{\mathbf{p}}^{\times}$, we have

$$\mathbf{e}_m^*(ca) = c^{-m} \mathbf{e}_m^*(a).$$

(3) For all $a, b \in \mathbf{F}_{\mathbf{p}}^{\times}$, we have

$$\sigma_a \mathbf{e}_m^*(b) = \mathbf{e}_m^*(ab).$$

(4) For all $a \in \mathbf{F}_{\mathbf{p}}^{\times}$ and for every m, we have

$$\mathbf{e}_m^*(a) - \mathbf{e}(a/\mathbf{p})^{q^d-1-m} \in \mathbf{p} \cdot \mathcal{R}.$$

The numbers $\mathbf{e}_m^*(a)$ are called dual coefficients.

Now let us recall our construction of the Goss L-function from the previous chapter. Fix an integer $1 \leq i \leq q^d - 1$. We have that $G = \mathrm{Gal}(K_{\mathbf{p}}/K) \cong \mathbf{F}_{\mathbf{p}}^{\times}$. The character $\omega^i \in \hat{G}$ and note that the kernel of ω^i is $\mathbf{p}$. The Goss L-function associated to ω^i is

$$L(j, \omega^i) = \sum_{\substack{I \subseteq A \\ (I, \mathbf{p}) = 1}} \frac{\omega^i(I)}{I^{[j]}}.$$

Since $h_A = 1$, the ideals of A are in bijection with A_+. So, setting $j = 1$, we have

$$L(1,\omega^i) = \sum_{\substack{a\in A_+\\(a,\mathbf{p})=1}} \frac{\omega^i((a))}{(a)^{[1]}} = \sum_{\substack{a\in A_+\\(a,\mathbf{p})=1}} \frac{\omega^i(a)}{a}.$$

Anderson then proved

$$L(1,\omega^i) = -\sum_{m=1}^{q^d-1} \left(\frac{1}{\mathbf{p}} \sum_{a\in\mathbf{F}_{\mathbf{p}}^{\times}} \omega^i(a)\mathbf{e}_m^*(a)\right) \left(\sum_{b\in\mathbf{F}_{\mathbf{p}}^{\times}} \omega^{-i}(b) l_m(b/\mathbf{p})\right). \tag{3.7}$$

We briefly explain the proof of this result. First, using Proposition III.14(1) and the definition of the function $l_m(x)$, one shows

$$\frac{1}{\mathbf{p}}\sum_{m=1}^{q^d-1} \mathbf{e}_m^*(a) l_m(b/\mathbf{p}) = \sum_{\substack{n\in A_+\\(n,\mathbf{p})=1\\ bn\equiv a \bmod \mathbf{p}}} \frac{1}{n}$$

for all $a,b \in \mathbf{F}_{\mathbf{p}}^{\times}$. Now multiply both sides of the preceding equation by $\omega^i(a)\omega^{-1}(b)$ and sum over a and b. And (3.7) follows.

By Proposition III.8(3) it follows that $L(1,\omega^i)$ is an algebraic linear combination of logarithms of special points. According to Anderson, this formula is the main reason for viewing special points as analogues of circular units.

CHAPTER IV

EXTENSIONS OF SPECIAL POINTS AND SPECIAL POLYNOMIALS WHEN $h_A = 1$ AND $d_\infty = 1$

A. Function Fields with $h_A = 1$ and $d_\infty = 1$

Our goals in this chapter are to prove an analogue of Proposition III.8 in the case of function fields (other than $\mathbb{F}_q(T)$) satisfying $h_A = 1$ and $d_\infty = 1$ and to define the module of special points for such function fields. To do this, we will use the fact that there are only four such function fields ([11], Theorem 2). The curves associated to these function fields are as follows (cf., [18], Examples A–D):

- $X_1 : y^2 = t^3 - t - 1$ over $\mathbb{F}_3$;
- $X_2 : y^2 + y = t^3 + \alpha$ over $\mathbb{F}_4$ where $\alpha \in \mathbb{F}_4$ satisfies $\alpha^2 + \alpha + 1 = 0$;
- $X_3 : y^2 + y = t^3 + t + 1$ over $\mathbb{F}_2$;
- $X_4 : y^2 + y = t^5 + t^3 + 1$ over $\mathbb{F}_2$.

Note that we are writing these curves in the affine coordinates (t, y), but we also view X_1, X_2, and X_3 as projective plane curves as follows. Introduce a third variable z and write coordinates in $\mathbb{P}^2$ as $[t, y, z]$. The equation defining each curve in $\mathbb{P}^2$ is given by homogenizing the corresponding affine equation.

Consider X_1. It is easy to check that the corresponding affine equation has no solutions over $\mathbb{F}_3$. So the only possible points on this curve over $\mathbb{F}_3$ are at infinity. Homogenizing the affine equation, we get

$$zy^2 = t^3 - z^2 t - z^3.$$

Setting $z = 0$, we conclude $t = 0$. Thus $y \neq 0$ (since at least on projective coordinate must be nonzero), and since we are dealing with homogeneous coordinates, we conclude that the only point on X_1 over $\mathbb{F}_3$ is $[0,1,0]$. A similar analysis shows that the only point on all four curves (over $\mathbb{F}_3$, $\mathbb{F}_4$, $\mathbb{F}_2$, and $\mathbb{F}_2$ respectively) is the point $[0,1,0]$. Therefore, for $i = 1,\dots,4$, set

$$\infty_i := \infty := [0,1,0].$$

We do not view X_4 as a plane curve. It is easy to see that X_4 is not smooth at ∞_4. But there exists a smooth projective curve X_4' and a birational morphism ϕ from X_4' onto X_4. The curve X_4' is called the nonsingular model of X_4. And the function field of X_4' is identified, using the map ϕ, with the function field of X_4 ([3], §7.5).

Set

- $A_1 := \mathbb{F}_3[t,y]/(y^2 - t^3 + t + 1)$;
- $A_2 := \mathbb{F}_4[t,y]/(y^2 - y - t^3 - \alpha)$;
- $A_3 := \mathbb{F}_2[t,y]/(y^2 - y - t^3 - t - 1)$;
- $A_4 := \mathbb{F}_2[t,y]/(y^2 - y - t^5 - t^3 - 1)$.

We have that A_1, A_2 and A_3 are Dedekind domains since X_1, X_2 and X_3 are smooth curves. Since A_4 consists of those rational functions which are regular away from ∞_4, and ∞_4 is the only singular point on X_4, we have that A_4 is also a Dedekind domain. For $i = 1,\dots,4$, the function field K_i is defined to be the quotient field of A_i. For each K_i, we have $h_{A_i} = 1$ and $d_\infty = 1$ ([18], §2).

For each curve, we claim that $\deg t = 2$. We will prove this for X_1. The proof for the remaining curves is similar. By definition, the number $\deg t$ satisfies

$$3^{\deg t} = \#(A/(t)).$$

Every element of A may be written as

$$f_1(t)+yf_2(t)$$

for $f_1(t), f_2(t) \in \mathbb{F}_3[t]$. Thus,

$$A/(t) \cong \{c+dy \mid c,d \in \mathbb{F}_3\},$$

and this proves that $\deg t = 2$.

For X_1, X_2, and X_3, we claim that $\deg y = 3$. Again, we will prove this for X_1 since the proof for the remaining curves is similar. Observe that $t^3 = y^2+t+1 \equiv (t+1) \bmod y$ and an easy induction argument shows that if $j \geq 3$, then $t^j \bmod y$ is a polynomial in t of degree ≤ 2. Hence,

$$A/(y) \cong \{F_1(t) \mid F_1(t) \in \mathbb{F}_3[t], \deg F_1 \leq 2\},$$

and so $\#(A/(y)) = 27$. Therefore, $\deg y = 3$. A similar analysis shows that $\deg y = 5$ in the case of X_4.

B. Shtuka Functions

Let X be a smooth, irreducible projective curve defined over $\mathbb{F}_q$ and let K be its associated function field. The *divisor group of* X, denoted $\mathrm{Div}(X)$, is the free abelian group generated by the points of X. So a divisor D is a formal sum

$$D = \sum_{P \in X} n_P(P)$$

where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many $P \in X$. The group $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acts on $\mathrm{Div}(X)$:

$$D^\sigma := \sum_{P \in X} n_P(P^\sigma)$$

for $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. Here P^σ is the point obtained by applying σ to the coordinates of P. The divisor D is *defined over* K if $D^\sigma = D$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. We denote the *group of divisors defined over* K by $\mathrm{Div}_K(X)$.

Let $k \in K^\times$. We associate to k a divisor by

$$\mathrm{div}(k) := \sum_{P\in X} \mathrm{ord}_P(k)(P).$$

(For a proof that $\mathrm{div}(k)$ is actually a divisor, see [13], Proposition 5.1.) Since

$$\mathrm{div}(k^\sigma) = \mathrm{div}(k)^\sigma$$

for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, we conclude that $\mathrm{div}(k) \in \mathrm{Div}_K(X)$.

Let ρ be a Drinfeld-Hayes A-module with respect to a fixed sign function sgn. For $i \geq 0$, let $\tau^i \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ be the q^i-th power Frobenius, i.e. $\tau^i(z) = z^{q^i}$ for all $z \in \mathbb{C}_\infty$. If $P \in X$, then set $P^{(i)}$ to be the point obtained by raising the coordinates of P to the q^i-th power. If $D = \sum_{P\in X} n_P(P)$ is a divisor, set

$$D^{(i)} := \sum_{P\in X} n_P(P^{(i)}).$$

If f is a function on X, set $f^{(i)}$ to be the function obtained by applying τ^i to the coefficients of f. Note that

$$\mathrm{div}(f^{(i)}) = (\mathrm{div}(f))^{(i)}$$

for all $i \geq 0$.

Let $\Xi := (\theta, \eta)$. We view $\theta \in \mathbb{C}_\infty$ as a constant which is a copy of the variable t and $\eta \in \mathbb{C}_\infty$ as a constant which is a copy of the variable y. In other words, if (t, y) is a point on $X(\mathbb{F}_q)$, then we think of (θ, η) as a point on $X(\mathbb{C}_\infty)$.

Let $\pi \in K_\infty$ be a monic uniformizer at ∞. Consider $\bar{X} := \mathbb{C}_\infty \otimes_{\mathbb{F}_q} X$. Let F be a function on $\bar{X}$. Then F can be written

$$\sum_{j=-m}^{\infty} a_j \pi^j,$$

with $a_j \in \mathbb{C}_\infty$ and $a_{-m} \neq 0$. We set $\widetilde{\operatorname{sgn}}\, F = a_{-m}$.

Proposition IV.1 ([7], §7.11)**.** *Let $\bar{X} := \mathbb{C}_\infty \otimes_{\mathbb{F}_q} X$. There exists a divisor $V \in \operatorname{Div}_{\mathbb{C}_\infty}(\bar{X})$ and a function f on $\bar{X}$ such that*

$$V^{(1)} - V + (\Xi) - (\infty^{(1)}) = div(f) \tag{4.1}$$

holds in $\operatorname{Div}_{\mathbb{C}_\infty}(\bar{X})$. If f is normalized so that $\widetilde{\operatorname{sgn}}\, f = 1$, then f is unique.

Definition IV.2. The function f is the *shtuka function* associated to ρ.

Remark IV.3. Suppose the curve X is defined by the vanishing of a polynomial $g(t,y) \in \mathbb{F}_q[t,y]$. Then $A = \mathbb{F}_q[t,y]/(g)$. Define

$$\bar{A} := \mathbb{C}_\infty \otimes_{\mathbb{F}_q} A = \mathbb{C}_\infty[t,y]/(g).$$

The function field of $\bar{X}$ is the fraction field of $\bar{A}$. Therefore, the shtuka function f is an element of the fraction field of $\bar{A}$ such that (4.1) holds. And if we assume that $\widetilde{\operatorname{sgn}}\, f = 1$, then f is unique.

Now let us return to the case of our four curves. Let $1 \leq i \leq 4$. Since $h_{A_i} = 1$ and $d_\infty = 1$, it follows that for each K_i, there is only one corresponding Drinfeld A_i-module, and furthermore, it is a Drinfeld-Hayes A_i-module. Let ρ^i denote this Drinfeld-Hayes A_i-module. We list them now (cf., [18] §2 and [7] §7.11).

- $\rho^1_t = \theta + \eta(\theta^3 - \theta)\tau + \tau^2$,
  $\rho^1_y = \eta + \eta(\eta^3 - \eta)\tau + (\eta^9 + \eta^3 + \eta)\tau^2 + \tau^3$;
- $\rho^2_t = \theta + (\theta^8 + \theta^2)\tau + \tau^2$,
  $\rho^2_y = \eta + (\theta^{10} + \theta)\tau + (\theta^{32} + \theta^8 + \theta^2)\tau^2 + \tau^3$;

- $\rho_t^3 = \theta + (\theta^2+\theta)\tau + \tau^2$,
  $\rho_y^3 = \eta + (\eta^2+\eta)\tau + \theta(\eta^2+\eta)\tau^2 + \tau^3$;

- $\rho_t^4 = \theta + (\theta^2+\theta)^2\tau + \tau^2$,
  $\rho_y^4 = \eta + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^4 + \tau^5$, where
  $b_1 = (\eta^2+\eta)(\theta^2+\theta)$
  $b_2 = \theta^2(\theta+1)(\eta^2+\eta)(\theta^3+\eta)(\theta^3+\eta+1)$
  $b_3 = \eta(\eta+1)(\theta^5+\theta^3+\theta^2+\theta+1)((1+\theta^2+\theta^3)\eta+\theta^2+\theta^4+\theta^7)$
  $\times((1+\theta^2+\theta^3)\eta+1+\theta^3+\theta^4+\theta^7)$
  $b_4 = (\theta(\eta^2+\eta)(\theta^5+\theta^2+1)(\theta+\eta)(\theta+1+\eta))^2$.

Let $f_1(t,y),\ldots,f_4(t,y)$ denote the shtuka function associated to $\rho^1,\ldots,\rho^4$, respectively. These functions are given by

- $f_1(t,y) = \dfrac{-\eta(t-\theta)+y-\eta}{t-\theta-1}$;
- $f_2(t,y) = \dfrac{\theta^2(t+\theta)+y+\eta}{t+\theta}$;
- $f_3(t,y) = \dfrac{\theta(t+\theta)+y+\eta}{t+\theta+1}$;
- $f_4(t,y) = \dfrac{(\theta+t)(\theta^4+\theta^3+(1+t)\theta^2)+y+\eta}{\theta^3+t\theta^2+(1+t)\theta+t^2+t}$;

(cf., [7], §7.11).

Given the shtuka function $f_i(t,y)$, one can recover the Drinfeld-Hayes A_i-module ρ^i. But first we need some definitions.

Definition IV.4. Let $D = \sum_{P\in X} n_P(P)$ be a divisor of X_j. The *degree* of D is defined as

$$\deg D := \sum_{P\in X_j} n_P \deg P.$$

Definition IV.5. Let $D = \sum_{P\in X} n_P(P)$ be a divisor of X_j. We say that D is an *effective divisor* if $n_P \geq 0$ for all P. We denote this by $D \geq 0$.

Definition IV.6. Let D be a divisor of X_j. Define

$$L(D) := \{k \in K_j^{\times} \mid \operatorname{div}(k) + D \geq 0\} \cup \{0\}.$$

The space $L(D)$ is a finite dimensional $\mathbb{F}_q$-vector space ([13], Chapter 5). We define

$$l(D) := \dim_{\mathbb{F}_q} L(D).$$

Equivalently,

$$q^{l(D)} = \#L(D).$$

Definition IV.7. The *genus* of K_j, denoted g_{K_j}, is defined to be the genus of X_j.

The following is a corollary of the Riemann-Roch Theorem. It will be sufficient for our applications.

Theorem IV.8 ([13], Corollary 4 to Theorem 5.4)**.** *Let D be a divisor of X_j. If* $\deg D > 2g_{K_j} - 2$*, then* $l(D) = \deg D - g_{K_j} + 1$.

We are now ready to explain how to use the shtuka function $f_i(t,y)$ to recover the Drinfeld-Hayes A_i-module ρ^i. We give the details only for $i = 1$ as the remaining cases are similar. Let $\mathcal{L} := \bigcup_{m \geq 0} L(V + m\infty)$. We claim that

$$1, f_1, f_1 f_1^{(1)}, f_1 f_1^{(1)} f_1^{(2)}, \ldots$$

is a basis for $\mathcal{L}$. First since $\infty = [0,1,0]$ for A_1, it follows that $\infty^{(i)} = \infty$ for all $i \geq 0$. Second, the observation that

$$\operatorname{div}(f_1^{(i)}) = (\operatorname{div}(f))^{(i)}$$

for all $i \geq 0$ and (4.1) imply

$$\operatorname{div}(f_1^{(i)}) = V^{(i+1)} - V^{(i)} + (\Xi^{(i)}) - (\infty)$$

for all $i \geq 0$. Hence,

$$\begin{aligned}\operatorname{div}(f_1 f_1^{(1)} \cdots f_1^{(i)}) &= \operatorname{div}(f_1) + \operatorname{div}(f_1^{(1)}) + \cdots + \operatorname{div}(f_1^{(i)}) \\ &= V^{(i+1)} - V + \Xi + \Xi^{(1)} + \cdots + \Xi^{(i)} - (i+1)\infty\end{aligned}$$

for all $i \geq 0$. It follows that $f_1 f_1^{(1)} \cdots f_1^{(i)} \in L(V + (i+1)\infty)$ since V is an effective divisor (§7.11, [7]). And $1 \in \mathcal{L}$ since $\operatorname{div}(1) = 0$. Now $\deg V = g_{K_1}$ (§7.11, [7]) and so $\deg(V + m\infty) = g_{K_1} + m = 1 + m$. Theorem IV.8 implies that

$$l(V + m\infty) = 1 + m - 1 + 1 = m + 1.$$

We have

$$L(V + m\infty) \subsetneq L(V + (m+1)\infty)$$

since $f_1 f_1^{(1)} \cdots f_1^{(m)} \in L(V + (m+1)\infty)$ and $f_1 f_1^{(1)} \cdots f_1^{(m)} \notin L(V + m\infty)$. It follows that $\mathcal{L}$ is infinite-dimensional and this concludes the proof of the claim since

$$\{1, f_1, f_1 f_1^{(1)}, f_1 f_1^{(1)} f_1^{(2)}, \ldots\}$$

is clearly a linearly independent subset of $\mathcal{L}$ over $\mathbb{C}_\infty$.

Now $t \in \mathcal{L}$ since $\operatorname{div}(t) = 2(0) - 2(\infty)$. Write

$$t = a_0 + a_1 f_1(t,y) + a_2 f_1(t,y) \cdot f_1^{(1)}(t,y) + \sum_{n \geq 3} a_n f_1(t,y) f_1^{(1)}(t,y) \cdots f_1^{(n-1)}(t,y) \tag{4.2}$$

for some constants $a_0, a_1, \ldots$. From (4.1), we conclude that $f_1(\theta, \eta) = 0$ which implies that $a_0 = \theta$. Also, (4.1) implies

$$\operatorname{div}(f_1^{(1)}) = V^{(2)} - V^{(1)} + \Xi^{(1)} - (\infty^{(2)})$$

so that $f_1^{(1)}(t,y)$ vanishes at $\Xi^{(1)} = (\theta^3, \eta^3)$ and does not vanish at $\Xi^{(2)} = (\theta^9, \eta^9)$. Therefore,

$$a_1 = \left.\frac{t-\theta}{f_1(t,y)}\right|_{(t,y)=(\theta^3,\eta^3)} = \eta(\theta^3-\theta)$$

and

$$a_2 = \left.\frac{t-\theta-a_1f_1(t,y)}{f_1(t,y)f_1^{(1)}(t,y)}\right|_{(t,y)=(\theta^9,\eta^9)} = 1$$

after some algebra because $f_1^{(2)}(t,y)$ vanishes at $\Xi^{(2)}$.

We now claim that $a_n = 0$ for $n \geq 3$. We have

$$\deg(f_1f_1^{(1)}\cdots f_1^{(n-1)}) = -\mathrm{ord}_\infty(f_1f_1^{(1)}\cdots f_1^{(n-1)}) = n$$

for all $n \geq 1$. Taking the degree of both sides of (4.2) and using the fact that $\deg t = 2$, we see that $a_n = 0$ for $n \geq 3$. Hence, we have recovered ρ_t^1. Since ρ^1 is uniquely determined by ρ_t^1, we have thus recovered ρ^1 from the shtuka function $f_1(t,y)$.

The following result is the main reason why we consider the shtuka function.

Theorem IV.9 ([7], Proposition 7.11.4)**.** *Let $1 \leq i \leq 4$. Let K_i be as above. Let $f_i(t,y)$ be the shtuka function associated to the Drinfeld-Hayes A_i-module ρ^i. Then*

$$\exp_{\rho^i}(z) = z + \sum_{n\geq 1}\frac{z^{q^n}}{(f_i^{(0)}\cdots f_i^{(n-1)})\mid_{(t,y)=(\theta^{q^n},\eta^{q^n})}},$$

where $(f_i^{(0)}\cdots f_i^{(n-1)})\mid_{(t,y)=(\theta^{q^n},\eta^{q^n})} = f_i^{(0)}(\theta^{q^n},\eta^{q^n})\cdots f_i^{(n-1)}(\theta^{q^n},\eta^{q^n})$.

To see why this is true, consider the case $i = 1$ (and we will stop writing the index 1 for the moment) and let $E(z)$ be the function on the right hand side of the previous displayed equation. We now explain how to use (4.2) to verify that $E(z)$ satisfies

$$\exp_\rho(\theta z) = \rho_t(E(z)). \tag{4.3}$$

First,

$$\exp_\rho(\theta z) = \theta z + \sum_{n\geq 1}\frac{\theta^{3^n}z^{3^n}}{f\cdots f^{(n-1)}|_{\Xi^{(n)}}}.$$

We compute

$$\begin{aligned}
\rho_t(\exp_\rho(z)) &= \theta\exp_\rho(z) + a_1(\exp_\rho(z))^3 + (\exp_\rho(z))^9 \quad (a_1 = \eta(\theta^3-\theta) \text{ as before}) \\
&= \theta\exp_\rho(z) + a_1\left(z^3 + \sum_{m\geq 1}\frac{z^{3^{m+1}}}{[f\cdots f^{(m-1)}|_{\Xi^{(m)}}]^3}\right) + z^9 + \sum_{n\geq 1}\frac{z^{3^{n+2}}}{[f\cdots f^{(n-1)}|_{\Xi^{(n)}}]^9} \\
&= \theta z + \sum_{l\geq 1}\frac{\theta z^{3^l}}{f\cdots f^{(l-1)}|_{\Xi^{(l)}}} + a_1\left(z^3 + \sum_{m\geq 1}\frac{z^{3^{m+1}}}{f^{(1)}\cdots f^{(m)}|_{\Xi^{(m+1)}}}\right) \\
&+ z^9 + \sum_{n\geq 1}\frac{z^{3^{n+2}}}{f^{(2)}\cdots f^{(n+1)}|_{\Xi^{(n+2)}}} \\
&= \theta z + \sum_{l\geq 1}\frac{\theta z^{3^l}}{f\cdots f^{(l-1)}|_{\Xi^{(l)}}} + a_1\left(z^3 + \sum_{m\geq 1}\frac{z^{3^{m+1}} f|_{\Xi^{(m+1)}}}{f\cdots f^{(m)}|_{\Xi^{(m+1)}}}\right) \\
&+ z^9 + \sum_{n\geq 1}\frac{z^{3^{n+2}} ff^{(1)}|_{\Xi^{(n+2)}}}{f\cdots f^{(n+1)}|_{\Xi^{(n+2)}}}.
\end{aligned}$$

The coefficient of z in $\rho_t(\exp_\rho(z))$ is θ. The coefficient of z^3 in $\rho_t(\exp_\rho(z))$ is

$$\frac{\theta}{f|_{\Xi^{(1)}}} + a_1 = \frac{\theta}{f|_{\Xi^{(1)}}} + \frac{\theta^3-\theta}{f|_{\Xi^{(1)}}} = \frac{\theta^3}{f|_{\Xi^{(1)}}}$$

as expected. The coefficient of z^9 in $\rho_t(\exp_\rho(z))$ is

$$\frac{\theta}{ff^{(1)}|_{\Xi^{(2)}}} + \frac{a_1 f|_{\Xi^{(2)}}}{ff^{(1)}|_{\Xi^{(2)}}} + 1 = \frac{\theta^9 - ff^{(1)}|_{\Xi^{(2)}}}{ff^{(1)}|_{\Xi^{(2)}}} + 1 = \frac{\theta^9}{ff^{(1)}|_{\Xi^{(2)}}}$$

where the first equality follows from

$$\theta^9 - \theta - a_1 f|_{\Xi^{(2)}} = ff^{(1)}|_{\Xi^{(2)}}$$

which is how the coefficient a_2 was defined. Finally, if $N \geq 3$, the coefficient of z^{3^N} in $\rho_t(\exp_\rho(z))$ is

$$\frac{\theta}{f\cdots f^{(N-1)}|_{\Xi^{(N)}}} + \frac{a_1 f|_{\Xi^{(N)}}}{f\cdots f^{(N-1)}|_{\Xi^{(N)}}} + \frac{ff^{(1)}|_{\Xi^{(N)}}}{f\cdots f^{(N-1)}|_{\Xi^{(N)}}} = \frac{\theta^{3^N}}{f\cdots f^{(N-1)}|_{\Xi^{(N)}}}$$

which follows by evaluating both sides of Equation (4.2) at $\Xi^{(N)}$. This concludes the proof of (4.3).

C. Computation of $i_0(\rho)$

Let $1 \le j \le 4$. We set $i_0(\rho^j)$ to be the real number (from Proposition III.6) satisfying

$$\|e_i(\rho^j)\| \le q^{(i_0(\rho^j)-i)q^i}$$

where $e_0(\rho^j) = 1$ and the coefficients $e_i(\rho^j)$ for $i \ge 1$ are determined by

$$\exp_{\rho^j}(z) = z + \sum_{i \ge 1} e_i(\rho^j) z^{q^i}.$$

Theorem IV.9 implies that

$$e_i(\rho^j) = \frac{1}{f_j^{(0)} \cdots f_j^{(i-1)} \, |_{(t,y)=(\theta^{q^i},\eta^{q^i})}} \tag{4.4}$$

for $i \ge 1$. Our explicit formulas for the shtuka functions imply that $e_i(\rho^j) \in K_j$ for all $i \ge 0$. Hence,

$$\|e_i(\rho^j)\| = |e_i(\rho^j)|.$$

Let $D_i(\rho^j)$ denote the denominator of the right hand side of (4.4). It follows that $\deg e_i(\rho^j) = -\deg D_i(\rho^j)$ and so

$$|e_i(\rho^j)| = q^{-\deg D_i(\rho^j)} \tag{4.5}$$

for $i \ge 1$. We will also use the observation that for $i \ge l$,

$$f_j^{(l)}(\theta^{q^i}, \eta^{q^i}) = f_j(\theta^{q^{i-l}}, \eta^{q^{i-l}})^{q^l}. \tag{4.6}$$

Set

- $\mathcal{A}_1 := \mathbb{F}_3[\theta,\eta]/(\eta^2 - \theta^3 + \theta + 1)$;

- $\mathcal{A}_2 := \mathbb{F}_4[\theta,\eta]/(\eta^2-\eta-\theta^3-\alpha)$;
- $\mathcal{A}_3 := \mathbb{F}_2[\theta,\eta]/(\eta^2-\eta-\theta^3-\theta-1)$;
- $\mathcal{A}_4 := \mathbb{F}_2[\theta,\eta]/(\eta^2-\eta-\theta^5-\theta^3-1)$.

Note that each $\mathcal{A}_i$ is just A_i where we have replaced t with θ and y with η. We may define the degree of an element in $\mathcal{A}_i$, which we also denote by deg, exactly as we have done with A_i. With this in mind, we have that $\deg t = \deg\theta$ and $\deg y = \deg\eta$ where the first deg is the degree in A_i and the second deg is the degree in $\mathcal{A}_i$.

First, consider K_1. We have

$$f_1(\theta^{3^i},\eta^{3^i}) = \frac{-\eta(\theta^{3^i}-\theta)+\eta^{3^i}-\eta}{\theta^{3^i}-\theta-1}.$$

Note that $\deg\eta^{3^i} = 3\cdot 3^i \geq \deg(\eta\cdot\theta^{3^i}) = 3+2\cdot 3^i$ with equality if $i=1$. Since

$$f_1(\theta^3,\eta^3) = \frac{-\eta(\theta^3-\theta)+\eta^3-\eta}{\theta^3-\theta-1} = \frac{\eta(\eta^2+1)+\eta^3-\eta}{\eta^2} = \frac{1}{\eta},$$

we conclude that $\deg f_1(\theta^3,\eta^3) = -3$. For $i \geq 2$, we have

$$\begin{aligned}\deg f_1(\theta^{3^i},\eta^{3^i}) &= 3\cdot 3^i - \deg\theta^{3^i}\\ &= 3\cdot 3^i - 2\cdot 3^i\\ &= 3^i.\end{aligned}$$

If follows that

$$\begin{aligned}\deg f_1 f_1^{(1)}\cdots f_1^{(i-1)}(\theta^{3^i},\eta^{3^i}) &= \deg f_1(\theta^{3^i},\eta^{3^i}) f_1(\theta^{3^{i-1}},\eta^{3^{i-1}})^3\cdots f_1(\theta^3,\eta^3)^{3^{i-1}}\\ &= 3^i + 3\cdot 3^{i-1} + \cdots 3^{i-2}\cdot 3^2 + 3^{i-1}(-3)\\ &= (i-2)3^i\end{aligned}$$

for $i \geq 1$ and so $|e_i(\rho^1)| = 3^{(2-i)3^i}$, for $i \geq 1$. Since $|e_0(\rho^1)| = 1 \leq 3^{(2-0)3^0}$, we may take

$\boxed{i_0(\rho^1) = 2.}$

For K_2, we have

$$f_2(\theta^{4^i}, \eta^{4^i}) = \frac{\theta^2(\theta^{4^i} + \theta) + \eta^{4^i} + \eta}{\theta^{4^i} + \theta},$$

and $\deg \eta^{4^i} = 3 \cdot 4^i \geq \deg(\theta^2\ \theta^{4^i}) = 4 + 2 \cdot 4^i$ with equality if $i = 1$. Since

$$\begin{aligned} f_2(\theta^4, \eta^4) = \frac{\theta^2(\theta^4 + \theta) + \eta^4 + \eta}{\theta^4 + \theta} &= \frac{\theta^6 + \theta^3 + (\theta^6 + \alpha^2 + \eta^2) + \eta}{\theta^4 + \theta} \\ &= \frac{\theta^3 + \alpha^2 + \theta^3 + \alpha}{\theta^4 + \theta} \\ &= \frac{1}{\theta^4 + \theta}, \end{aligned}$$

we have that $\deg f_2(\theta^4, \eta^4) = -8$. If $i \geq 2$, then

$$\begin{aligned} \deg f_2(\theta^{4^i}, \eta^{4^i}) &= 3 \cdot 4^i - \deg \theta^{4^i} \\ &= 3 \cdot 4^i - 2 \cdot 4^i \\ &= 4^i. \end{aligned}$$

It follows that

$$\begin{aligned} \deg f_2 f_2^{(1)} \cdots f_2^{(i-1)}(\theta^{4^i}, \eta^{4^i}) &= \deg f_2(\theta^{4^i}, \eta^{4^i}) f_2(\theta^{4^{i-1}}, \eta^{4^{i-1}})^4 \cdots f_2(\theta^4, \eta^4)^{4^{i-1}} \\ &= 4^i + 4 \cdot 4^{i-1} + \cdots 4^{i-2} \cdot 4^2 + 4^{i-1}(-8) \\ &= (i-3)4^i \end{aligned}$$

for $i \geq 1$ and so $|e_i(\rho^2)| = 4^{(3-i)4^i}$ for $i \geq 1$. Hence, we may take $\boxed{i_0(\rho^2) = 3.}$

For K_3, we have

$$f_3(\theta^{2^i}, \eta^{2^i}) = \frac{\theta(\theta^{2^i} + \theta) + \eta^{2^i} + \eta}{\theta^{2^i} + \theta + 1}.$$

Note that $\deg \eta^{2^i} = 3 \cdot 2^i \geq \deg(\theta\ \theta^{2^i}) = 2 + 2 \cdot 2^i$ with equality if $i = 1$. Since

$$f_3(\theta^2, \eta^2) = \frac{\theta(\theta^2 + \theta) + \eta^2 + \eta}{\theta^2 + \theta + 1} = \frac{\theta^3 + \theta^2 + \theta^3 + \theta + 1}{\theta^2 + \theta + 1} = 1,$$

we conclude that $\deg f_3(\theta^2,\eta^2)=0$. For $i\geq 2$, we have

$$\begin{aligned}\deg f_3(\theta^{2^i},\eta^{2^i}) &= 3\cdot 2^i-\deg\theta^{2^i}\\ &= 3\cdot 2^i-2\cdot 2^i\\ &= 2^i.\end{aligned}$$

It follows that

$$\begin{aligned}\deg f_3 f_3^{(1)}\cdots f_3^{(i-1)}(\theta^{2^i},\eta^{2^i}) &= \deg f_3(\theta^{2^i},\eta^{2^i})f_3(\theta^{2^{i-1}},\eta^{2^{i-1}})^2\cdots f_3(\theta^2,\eta^2)^{2^{i-1}}\\ &= 2^i+2\cdot 2^{i-1}+\cdots 2^{i-2}\cdot 2^2+0\\ &= (i-1)2^i\end{aligned}$$

for $i\geq 1$ and so $|e_i(\rho^3)|=2^{(1-i)2^i}$ for $i\geq 1$. Hence, we may take $\boxed{i_0(\rho^3)=1.}$

The analysis for K_4 is similar but more involved. Recall that for this curve, $\deg\eta=5$. We first begin by simplifying the shtuka function:

$$\begin{aligned}f_4(t,y) &= \frac{(\theta+t)(\theta^4+\theta^3+(1+t)\theta^2)+y+\eta}{\theta^3+t\theta^2+(1+t)\theta+t^2+t}\\ &= \frac{\theta^2t^2+(\theta^4+\theta^2)t+y+\theta^5+\theta^4+\theta^3+\eta}{t^2+(\theta^2+\theta+1)t+\theta^3+\theta}\end{aligned}$$

Then

$$f_4(\theta^{2^i},\eta^{2^i})=\frac{\theta^{2^{i+1}+2}+\theta^{2^i+4}+\theta^{2^i+2}+\eta^{2^i}+\theta^5+\theta^4+\theta^3+\eta}{\theta^{2^{i+1}}+\theta^{2^i+2}+\theta^{2^i+1}+\theta^{2^i}+\theta^3+\theta}.$$

First,

$$\begin{aligned}f_4(\theta^2,\eta^2) &= \frac{\theta^6+\theta^6+\theta^4+\theta^5+\theta^4+\theta^3+\eta^2+\eta}{\theta^4+\theta^4+\theta^3+\theta^2+\theta^3+\theta}\\ &= \frac{\theta^5+\theta^3+\theta^5+\theta^3+1}{\theta^2+\theta}\\ &= \frac{1}{\theta^2+\theta},\end{aligned}$$

and so $\deg f_4(\theta^2,\eta^2) = -4$. Next,

$$f_4(\theta^4,\eta^4) = \frac{\theta^{10}+\theta^8+\theta^6+\theta^5+\theta^4+\theta^3+\eta^4+\eta}{\theta^8+\theta^6+\theta^5+\theta^4+\theta^3+\theta}.$$

Since $\eta^4 = \theta^{10}+\theta^6+\theta^5+\theta^3+\eta$, it follows that the term of highest degree in both the numerator and denominator is θ^8. Therefore, $\deg f_4(\theta^4,\eta^4) = 0$. For $i \geq 3$, the term of highest degree in the numerator and denominator of $f_4(\theta^{2^i},\eta^{2^i})$ is η^{2^i} and $\theta^{2^{i+1}}$, respectively. Thus, $\deg f_4(\theta^{2^i},\eta^{2^i}) = 5\cdot 2^i - 2\cdot 2^{i+1} = 2^i$ for $i \geq 3$.

We have that $|e_1(\rho^4)| = 2^4$. Since

$$\begin{aligned}\deg f_4 f_4^{(1)}(\theta^4,\eta^4) &= \deg f_4(\theta^4,\eta^4) + \deg f_4(\theta^2,\eta^2)^2 \\ &= -8,\end{aligned}$$

we conclude that $|e_2(\rho^4)| = 2^8$. For $i \geq 3$, we have

$$\begin{aligned}\deg f_4 f_4^{(1)}\cdots f_4^{(i-1)}(\theta^{2^i},\eta^{2^i}) &= \deg f_4(\theta^{2^i},\eta^{2^i}) f_4(\theta^{2^{i-1}},\eta^{2^{i-1}})^2 \cdots f_4(\theta^4,\eta^4)^{2^{i-2}} f_4(\theta^2,\eta^2)^{2^{i-1}} \\ &= 2^i + 2\cdot 2^{i-1} + \cdots + 8\cdot 2^{i-3} + 0 + (-4)2^{i-1} \\ &= (i-4)2^i,\end{aligned}$$

and so $|e_i(\rho^4)| = 2^{(4-i)2^i}$. It follows that we can take $\boxed{i_0(\rho^4) = 4.}$

We record our results for future reference.

Proposition IV.10. *Let $1 \leq j \leq 4$. Let K_j and $i_0(\rho^j)$ be as above. Then*

(1) $i_0(\rho^1) = 2$.

(2) $i_0(\rho^2) = 3$.

(3) $i_0(\rho^3) = 1$.

(4) $i_0(\rho^4) = 4$.

D. Computation of $j_0(t^m;\rho^j)$

We continue with the notation of the previous sections.

Let π_j be a fixed monic uniformizer at ∞_j. Recall that the imaginary axis is the one-dimensional $(K_j)_\infty$-subspace of $(\bar{K_j})_\infty$ spanned by the $(q-1)$st roots of $-\pi_j^{-1}$. We denote this subspace by $(K_j)_\infty \cdot \sqrt[q-1]{-\pi_j^{-1}}$.

For each K_j, we set $b = t^m$. In this section we will compute $j_0(t^m;\rho^j)$ which is determined by

$$q^{j_0(t^m;\rho^j)} = \max(1, \gamma_j \|t^m\|)$$

where

$$\gamma_j = \max_I q^{\deg I} \left\| \frac{1}{D(\rho_I^j)} \right\|.$$

The maximum is taken over all integral ideals I of A_j. We first compute γ_j since it will be the same for each of our four function fields.

Since $h_{A_j} = 1$, it follows that A_j is a principal ideal domain. Furthermore, the generator of every ideal may be chosen to be a monic element of A_j. Hence, there is a bijection between $(A_j)_+$ and the integral ideals of A_j. This implies that

$$\gamma = \max_{a \in (A_j)_+} q^{\deg a} \left\| \frac{1}{D(\rho_{(a)}^j)} \right\|.$$

Since $a \in (A_j)_+$, we have $\rho_{(a)}^j = \rho_a^j$ ([13], Proposition 13.14) and since ρ^j is a Drinfeld-Hayes A_j-module, it follows that $D(\rho_a^j) = a$. Thus we are reduced to computing $\left\| \frac{1}{a} \right\|$. By Lemma III.3 and (3.4), it follows that

$$\left\| \frac{1}{a} \right\| = \left| \frac{1}{a} \right| = q^{-\deg a},$$

and so $\gamma_j = 1$ for all $1 \le j \le 4$. The computation of $j_0(t^m;\rho^j)$ is thus reduced to computing $\|t^m\|$.

Again by Lemma III.3 and (3.4), and since $h_{A_j} = 1$, we have

$$\begin{aligned}\|t^m\| &= \sup_{a \in (A_j)_+} \sup_x |(\exp_{(a)*\rho^j}(x))^m| \\ &= \sup_x |(\exp_{\rho^j}(x))^m|,\end{aligned}$$

where the second supremum is taken over all x in the imaginary axis. The second equality holds since $(a) * \rho^j = \rho^j$.

Suppose that the period lattice of ρ^j equals $\tilde{\pi}_{\rho^j} A$. We claim that

$$(K_j)_\infty \cdot \tilde{\pi}_{\rho^j} = (K_j)_\infty \cdot \sqrt[q-1]{-\pi_j^{-1}}, \tag{4.7}$$

where $(K_j)_\infty \cdot \tilde{\pi}_{\rho^j}$ denotes the $(K_j)_\infty$-subspace of $\bar{K}_{j\infty}$ spanned by $\tilde{\pi}_{\rho^j}$. Since the period lattice is contained in the imaginary axis ([2], §2.6), this proves $\subseteq$. As for the other inclusion, since $\sqrt[q-1]{-\pi_j^{-1}} = \pi_j \sqrt[q-1]{-\pi_j^{-q}}$ and $\pi_j \in (K_j)_\infty$, it is enough to show that $(K_j)_\infty \cdot \sqrt[q-1]{-\pi_j^{-q}} \subseteq (K_j)_\infty \cdot \tilde{\pi}_{\rho^j}$. Thus, we need only show that

$$\sqrt[q-1]{-\pi_j^{-q}} \in (K_j)_\infty \cdot \tilde{\pi}_{\rho^j}. \tag{4.8}$$

We know that $\tilde{\pi}_{\rho^j}^{q-1} \in (K_j)_\infty$ ([7], Theorem 7.10.10), so $\tilde{\pi}_{\rho^j}^{q-1}$ may be expressed in terms of π_j as

$$\tilde{\pi}_{\rho^j}^{q-1} = c\pi_j^{(1-q)\deg(\tilde{\pi}_{\rho^j})} u,$$

where $c := \operatorname{sgn}(\tilde{\pi}_{\rho^j}^{q-1})$. Thus

$$\pi_j^{-q} = c^{-1} u^{-1} \pi_j^{-q+(q-1)\deg(\tilde{\pi}_{\rho^j})} \cdot \tilde{\pi}_{\rho^j}^{q-1}$$

which, upon taking $(q-1)$st roots, proves (4.8) and thus (4.7).

Consider K_1. Set $\pi_1 = t/y$. Consider the following inclusions:

$$\begin{array}{ccccc} A_1 = \mathbb{F}_3[t,y] & \subseteq & K_1 = \mathbb{F}_3(t,y) & \subseteq & (K_1)_\infty = \mathbb{F}_3((\frac{t}{y})) \\ | & & | & & | \\ \mathbb{F}_3[t] & \subseteq & \mathbb{F}_3(t) & \subseteq & \mathbb{F}_3((\frac{1}{t})). \end{array}$$

We have that $[K_1 : \mathbb{F}_3(t)] = 2$ and we claim that

$$[(K_1)_\infty \colon \mathbb{F}_3((1/t))] = 2. \tag{4.9}$$

We will use the following two results.

Theorem IV.11 ([14], Theorem 5.4.8)**.** *Let K be a function field and let X be the associated curve. Let L/K be a finite extension of function fields. Let $P \in X$ and let $\mathfrak{P}_i$ be an ideal of L lying above P. Denote by K_P (respectively $L_{\mathfrak{P}_i}$) the completion of K (resp. L) with respect to P (resp. $\mathfrak{P}_i$). Then*

$$[L_{\mathfrak{P}_i} \colon K_P] = e(\mathfrak{P}_i/P) f(\mathfrak{P}_i/P).$$

Theorem IV.12 ([13], Proposition 9.3)**.** *Let K, P and L be as before but assume now that L/K is a Galois extension. Let $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_{g(P)}\}$ be the prime ideals of L lying above P. Then $f(\mathfrak{P}_i/P) = f(\mathfrak{P}_j/P)$ and $e(\mathfrak{P}_i/P) = e(\mathfrak{P}_j/P)$ for all $1 \le i, j \le g(P)$. If we denote by $f(P)$ the common relative degree and by $e(P)$ the common ramification index, then*

$$e(P) f(P) g(P) = [L \colon K].$$

Since $K_1/\mathbb{F}_3(t)$ is Galois of degree 2, it follows that $[(K_1)_\infty \colon \mathbb{F}_3((1/t))]$ is either 1 or 2. If it is 1, then $(K_1)_\infty = \mathbb{F}_3((1/t))$. Therefore, $y \in \mathbb{F}_3((1/t))$ and so

$$y = \sum_{i \ge m} \alpha_i \left(\frac{1}{t}\right)^i$$

for some $m \in \mathbb{Z}, \alpha_i \in \mathbb{F}_3$ and $\alpha_m \neq 0$. Taking degrees we conclude that $3 = -2m$, which is a contradiction. This proves (4.9). We conclude that

$$(K_1)_\infty = \mathbb{F}_3((1/t))(y).$$

Now if $y \in (K_1)_\infty$, then $y = \alpha + \beta y$ for some $\alpha, \beta \in \mathbb{F}_3((1/t))$. We have that $y = \alpha + \beta y \in A_1$ if and only if $\alpha, \beta \in \mathbb{F}_3[t]$. Putting everything together, we conclude that if x is in the imaginary axis, then

$$\begin{aligned} x &= y\tilde{\pi}_{\rho^1} \quad \text{for some } y \in (K_1)_\infty \\ &= (a+z)\tilde{\pi}_{\rho^1} \end{aligned}$$

for some $a \in A_1$ and for

$$z \in \frac{1}{t}\mathbb{F}_3[[1/t]] + \frac{1}{t}\mathbb{F}_3[[1/t]]y.$$

Since $\exp_{\rho^1}(x) = \exp_{\rho^1}((a+z)\tilde{\pi}_{\rho^1}) = \exp_{\rho^1}(z\tilde{\pi}_{\rho^1})$, we conclude that

$$\|t^m\| = \sup\left\{ |\exp_{\rho^1}(z\tilde{\pi}_{\rho^1})|^m : z \in \frac{1}{t}\mathbb{F}_3[[1/t]] + \frac{1}{t}\mathbb{F}_3[[1/t]]y \right\}.$$

Exactly as above, we may determine $\|t^m\|$ for the cases of K_2 and K_3. For K_2, we have

$$\|t^m\| = \sup\left\{ |\exp_{\rho^2}(z\tilde{\pi}_{\rho^2})|^m : z \in \frac{1}{t}\mathbb{F}_4[[1/t]] + \frac{1}{t}\mathbb{F}_4[[1/t]]y \right\}.$$

For K_3, we have

$$\|t^m\| = \sup\left\{ |\exp_{\rho^3}(z\tilde{\pi}_{\rho^3})|^m : z \in \frac{1}{t}\mathbb{F}_2[[1/t]] + \frac{1}{t}\mathbb{F}_2[[1/t]]y \right\}.$$

For K_4, set $\pi_4 = t^2/y$. Again, using the same argument as above, we get

$$(K_4)_\infty = \mathbb{F}_2((t^2/y)) = \mathbb{F}_2((1/t))(y)$$

and so

$$\|t^m\| = \sup\left\{ |\exp_{\rho^4}(z\tilde{\pi}_{\rho^4})|^m : z \in \frac{1}{t}\mathbb{F}_2[[1/t]] + \frac{1}{t}\mathbb{F}_2[[1/t]]y \right\}.$$

Now, for $1 \leq j \leq 4$,

$$|\exp_{\rho^j}(z\tilde{\pi}_{\rho^j})| = \left|\sum_{i=0}^{\infty} e_i(\rho^j)(z\tilde{\pi}_{\rho^j})^{q^i}\right| \leq \max_{i\geq 0}(|e_i(\rho^j)|\,|z\tilde{\pi}_{\rho^j}|^{q^i})$$

with equality holding if the values

$$\{|e_i(\rho^j)|\cdot|z\tilde{\pi}_{\rho^j}|^{q^i}\ :\ i\geq 0\}$$

are distinct. Since we have already computed the values $|e_i(\rho^j)|$, our computation of $\|t^m\|$ will be concluded upon computing $|\tilde{\pi}_{\rho^j}|$.

E. Computing the Absolute Value of the Period

We again continue with the notations of the previous sections. Let $1 \leq j \leq 4$.

Definition IV.13. Let $I_j \subseteq A_j$ be an ideal and $a \in A_j$. Set

$$Z_{a,I_j}(u) := \sum_{\substack{b\in A_j \\ b\equiv a \bmod I_j}} u^{\deg b}.$$

The absolute value of the period $\tilde{\pi}_{\rho^j}$ and the function $Z_{a,I_j}(u)$ are related to one another via the following result.

Proposition IV.14 ([7], Corollary 7.10.11)**.** *Let* $Z'_{a,I_j}(u) = \frac{d}{du}Z_{a,I_j}(u)$. *Then* $|\tilde{\pi}_{\rho^j}| = q^{Z'_{0,A_j}(1)}$.

So we need to compute

$$Z_{0,A_j}(u) = \sum_{b\in A_j} u^{\deg b} = \sum_{i\geq 0} \#(A_j)_i u^i$$

where $(A_j)_i$ denotes the set of elements of A_j of degree i. To compute $\#(A_j)_i$, we will appeal to Theorem IV.8 from Section B.

We first consider K_1, K_2, and K_3 together since they each have genus 1. In what follows, $1 \leq j \leq 3$.

Clearly, $\#(A_j)_0 = \#\mathbb{F}_{q_j}^{\times} = q_j - 1$ and, for $i \geq 1$, we have

$$\begin{aligned}
\#(A_j)_i &= \#\{b \in A_j \colon \mathrm{ord}_{\infty_j}(b) = -i\} \\
&= \#\{b \in A_j \colon \mathrm{ord}_{\infty}(b) \geq -i\} - \#\{b \in A_j \colon \mathrm{ord}_{\infty_j}(b) \geq -i+1\} \\
&= \#L(i\infty_j) - \#L((i-1)\infty_j) \\
&= q_j^{l(i\infty_j)} - q_j^{l((i-1)\infty_j)}.
\end{aligned}$$

Now # $(A_j)_1 = q_j^{l(\infty_j)} - q_j^{l(0)} = q^{l(\infty_j)} - q_j$ since $l(0) = 1$. By Theorem IV.8, since $d_{\infty_j} = 1$, we have $l(i\infty_j) = i$ for $i \geq 1$. Hence,

$$\#(A_j)_i = \begin{cases} q_j - 1 & \text{if } i = 0; \\ 0 & \text{if } i = 1; \\ q_j^i - q_j^{i-1} & \text{if } i \geq 2. \end{cases}$$

Hence,

$$Z_{0,A_j}(u) = q_j - 1 + (q_j - 1)\frac{q_j u^2}{1 - q_j u} \tag{4.10}$$

and so

$$Z'_{0,A_j}(1) = \frac{q_j(q_j - 2)}{1 - q_j}. \tag{4.11}$$

For K_4, $g_{K_4} = 2$, and so our computation will be similar. As opposed to our previous analysis, we will use the fact that we know A_4 explicitly. Recall that for K_4, $A_4 = \mathbb{F}_2[t,y]/(y^2 - y - t^5 - t^3 - 1)$ with $\deg t = 2$, $\deg y = 5$ and $d_{\infty_4} = 1$.

Note that

$$\begin{aligned}
&(A_4)_0 = \mathbb{F}_2^{\times}; \\
&(A_4)_1 = \varnothing; \\
&(A_4)_2 = \{t + a \mid a \in \mathbb{F}_2\}; \\
&(A_4)_3 = \varnothing;
\end{aligned}$$

$$(A_4)_4 = \{t^2 + at + b \mid a, b \in \mathbb{F}_2\};$$

$$(A_4)_5 = \{y + at^2 + bt + c \mid a, b, c \in \mathbb{F}_2\}.$$

Theorem IV.8 implies $\ell(j\infty_4) = j - 1$ for $j \geq 5$. Thus, for $i \geq 6$,

$$\begin{aligned}
\#(A_4)_i &= 2^{\ell(i\infty_4)} - 2^{\ell((i-1)\infty_4)} \\
&= 2^{i-1} - 2^{i-2} \\
&= 2^{i-2}
\end{aligned}$$

exactly as above. We conclude that

$$\begin{aligned}
Z_{0,A_4}(u) &= 1 + 2u^2 + 4u^4 + 8u^5 + \sum_{i \geq 6} 2^{i-2} u^i \\
&= 1 + 2u^2 + 4u^4 + 8u^5 + \frac{16u^6}{1 - 2u},
\end{aligned}$$

and so

$$Z'_{0,A_4}(1) = -4. \tag{4.12}$$

F. Computing $\|t^m\|$

For K_1, we have that $|\tilde{\pi}_{\rho^1}| = 3^{-3/2}$ by (4.11) and

$$|e_i(\rho^1)| = \begin{cases} 1 & \text{if } i = 0 \\ 3^{-(i-2)3^i} & \text{if } i \geq 1 \end{cases}$$

from Section C. Thus, for

$$z \in \frac{1}{t}\mathbb{F}_3[[1/t]] + \frac{1}{t}\mathbb{F}_3[[1/t]]y,$$

we have

$$|e_i(\rho^1)(z\tilde{\pi}_{\rho^1})^{3^i}|^m = \begin{cases} (3^{\deg z}3^{-3/2})^m & \text{if } i=0 \\ (3^{-(i-2)3^i}3^{3^i\deg z}3^{(-3/2)3^i})^m & \text{if } i\geq 1 \end{cases}$$
$$= \begin{cases} 3^{m(\deg z-3/2)} & \text{if } i=0 \\ 3^{m3^i(1/2+\deg z-i)} & \text{if } i\geq 1. \end{cases}$$

Let $F_1(i) := 3^i(1/2+\deg z-i)$. Then F_1 is maximized when $i=\deg z+1/2-1/\log 3$. Since i must be an integer, we conclude that F_1 is maximized at either $i=\deg z$ or at $i=\deg z-1$. Since $\deg z\leq 1$, the maximum value of F_1 on $[1,\infty)\cap\mathbb{Z}$ is

$$F_1(\deg z) = \frac{1}{2}3^{\deg z}.$$

So

$$|\exp_{\rho^1}(z\tilde{\pi}_{\rho^1})|^m = \max_{i\geq 1}(3^{m(\deg z-3/2)}, 3^{m3^i(1/2+\deg z-i)})$$
$$= \max(3^{m(\deg z-3/2)}, 3^{m(1/2)\deg z})$$
$$= 3^{(m/2)3^{\deg z}}.$$

Therefore

$$\|t^m\| = \sup\left\{3^{(m/2)3^{\deg z}} : z\in\frac{1}{t}\mathbb{F}_3[[1/t]]+\frac{1}{t}\mathbb{F}_3[[1/t]]y\right\}$$
$$= 3^{3(m/2)}$$

since the supremum is attained for $z=y/t$ which is of degree 1. From the expression

$$3^{j_0(t^m;\rho^1)} = \max(1,\|t^m\|),$$

we conclude that

$$j_0(t^m;\rho^1) = \frac{3}{2}m. \tag{4.13}$$

For K_2, we have $|\tilde{\pi}_{\rho^2}| = 4^{-8/3}$ by (4.11) and

$$|e_i(\rho^2)| = \begin{cases} 1 & \text{if } i = 0 \\ 4^8 & \text{if } i = 1 \\ 4^{-(i-3)4^i} & \text{if } i \geq 2 \end{cases}$$

from Section C. Thus, for

$$z \in \frac{1}{t}\mathbb{F}_4[[1/t]] + \frac{1}{t}\mathbb{F}_4[[1/t]]y,$$

we have

$$\begin{aligned} |e_i(\rho^2)(z\tilde{\pi}_{\rho^2})^{4^i}|^m &= \begin{cases} (4^{\deg z}4^{-8/3})^m & \text{if } i = 0 \\ (4^8 4^{4\deg z}4^{-32/3})^m & \text{if } i = 1 \\ (4^{-(i-3)4^i}4^{4^i \deg z}4^{(-8/3)4^i})^m & \text{if } i \geq 2 \end{cases} \\ &= \begin{cases} 4^{m(\deg z - 8/3)} & \text{if } i = 0 \\ 4^{m(\deg z - 8/3)} & \text{if } i = 1 \\ 4^{m4^i(1/3 + \deg z - i)} & \text{if } i \geq 2. \end{cases} \end{aligned}$$

Let $F_2(i) := 4^i(1/3 + \deg z - i)$. Then F_2 is maximized when $i = \deg z + 1/3 - 1/\log 4$. Using the same argument as for K_1, we conclude that the maximum value of F_2 on $[1,\infty) \cap \mathbb{Z}$ is

$$F_2(\deg z) = \frac{1}{3}4^{\deg z}.$$

So

$$
\begin{aligned}
|\exp_{\rho^2}(z\tilde{\pi}_{\rho^2})|^m &= \max_{i\geq 2}(4^{m(\deg z-8/3)}, 4^{m(4\deg z-8/3)}, 4^{m4^i(1/3+\deg z-i)}) \\
&= \max(4^{m(\deg z-8/3)}, 4^{m(4\deg z-8/3)}, 4^{m(1/3)\deg z}) \\
&= 4^{(m/3)4^{\deg z}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|t^m\| &= \sup\left\{4^{(m/3)4^{\deg z}} : z \in \frac{1}{t}\mathbb{F}_4[[1/t]] + \frac{1}{t}\mathbb{F}_4[[1/t]]y\right\} \\
&= 4^{4(m/3)}
\end{aligned}
$$

since the supremum is attained for $z = y/t$ which is of degree 1. From the expression

$$
4^{j_0(t^m;\rho^2)} = \max(1, \|t^m\|),
$$

we conclude that

$$
j_0(t^m;\rho^2) = \frac{4}{3}m. \tag{4.14}
$$

For K_3, we have $|\tilde{\pi}_{\rho^3}| = 1$ by (4.11) and

$$
|e_i(\rho^3)| = \begin{cases} 1 & \text{if } i = 0, 1 \\ 2^{-(i-1)2^i} & \text{if } i \geq 2 \end{cases}
$$

from Section C. Thus, for

$$
z \in \frac{1}{t}\mathbb{F}_2[[1/t]] + \frac{1}{t}\mathbb{F}_2[[1/t]]y,
$$

we have

$$|e_i(\rho^3)(z\tilde{\pi}_{\rho^3})^{2^i}|^m = \begin{cases} (2^{\deg z})^m & \text{if } i=0 \\ (2^{2\deg z})^m & \text{if } i=1 \\ (2^{-(i-1)2^i}2^{2^i \deg z})^m & \text{if } i \geq 2 \end{cases}$$

$$= \begin{cases} 2^{m\deg z} & \text{if } i=0 \\ 2^{2m\deg z} & \text{if } i=1 \\ 2^{m2^i(1+\deg z-i)} & \text{if } i \geq 2 \end{cases}$$

Let $F_3(i) := 2^i(1+\deg z - i)$. Then F_3 is maximized when $i = \deg z + 1 - 1/\log 2$. Using the same argument as for K_1 and K_2, we conclude that the maximum value of F_3 on $[1,\infty) \cap \mathbb{Z}$ is

$$F_3(\deg z) = 2^{\deg z}.$$

So

$$\begin{aligned} |\exp_{\rho^3}(z\tilde{\pi}_{\rho^3})|^m &= \max_{i\geq 2}(2^{m\deg z}, 2^{2m\deg z}, 2^{m2^i(1+\deg z-i)}) \\ &= \max(2^{m\deg z}, 2^{2m\deg z}, 2^{m2^{\deg z}}) \\ &= 2^{m2^{\deg z}}. \end{aligned}$$

Therefore

$$\begin{aligned} \|t^m\| &= \sup\left\{2^{m2^{\deg z}} : z \in \frac{1}{t}\mathbb{F}_2[[1/t]] + \frac{1}{t}\mathbb{F}_2[[1/t]]y\right\} \\ &= 2^{2m} \end{aligned}$$

since the supremum is attained for $z = y/t$ which is of degree 1. From the expression

$$2^{j_0(t^m;\rho^3)} = \max(1, \|t^m\|),$$

we conclude that

$$j_0(t^m;\rho^3) = 2m. \tag{4.15}$$

The analysis for K_4 is almost the same as our previous cases. The main difference is that now $\deg\eta = 5$. We have $|\tilde{\pi}_{\rho^4}| = 2^{-4}$ by (4.12) and

$$|e_i(\rho^4)| = \begin{cases} 1 & \text{if } i = 0 \\ 2^4 & \text{if } i = 1 \\ 2^8 & \text{if } i = 2 \\ 2^{-(i-4)2^i} & \text{if } i \geq 3 \end{cases}$$

from Section C. Thus, for

$$z \in \frac{1}{t}\mathbb{F}_2[[1/t]] + \frac{1}{t}\mathbb{F}_2[[1/t]]y,$$

we have

$$\begin{aligned} |e_i(\rho^4)(z\tilde{\pi}_{\rho^4})^{2^i}|^m &= \begin{cases} (2^{\deg z}2^{-4})^m & \text{if } i = 0 \\ (2^4 2^{2\deg z}2^{-8})^m & \text{if } i = 1 \\ (2^8 2^{4\deg z}2^{-16})^m & \text{if } i = 2 \\ (2^{-(i-4)2^i}2^{2^i\deg z}2^{2^i(-4)})^m & \text{if } i \geq 3 \end{cases} \\ &= \begin{cases} 2^{m(\deg z-4)} & \text{if } i = 0 \\ 2^{m(2\deg z-4)} & \text{if } i = 1 \\ 2^{m(4\deg z-8)} & \text{if } i = 2 \\ 2^{m2^i(\deg z-i)} & \text{if } i \geq 3 \end{cases} \end{aligned}$$

$$= \begin{cases} 2^{m(\deg z - 4)} & \text{if } i = 0 \\ 2^{m(2\deg z - 4)} & \text{if } i = 1 \\ 2^{m2^i(\deg z - i)} & \text{if } i \geq 2. \end{cases}$$

Note that now $\deg z \leq 3$.

Let $F_4(i) := 2^i(\deg z - i)$. Then F_4 is maximized when $i = \deg z - 1/\log 2$. Since i must be an integer, F_4 is maximized at either $i = \deg z - 1$ or at $i = \deg z - 2$. Note that in this case,

$$F_4(\deg z - 1) = F_4(\deg z - 2) = 2^{\deg z - 1}.$$

So

$$\begin{aligned} |\exp_{\rho^4}(z\tilde{\pi}_{\rho^4})|^m &= \max_{i \geq 2}(2^{m(\deg z - 4)}, 2^{m(2\deg z - 4)}, 2^{m2^i(\deg z - i)}) \\ &= \max(2^{m(\deg z - 4)}, 2^{m(2\deg z - 4)}, 2^{m2^{\deg z - 1}}) \\ &= 2^{m2^{\deg z - 1}}. \end{aligned}$$

Therefore

$$\begin{aligned} \|t^m\| &= \sup\left\{2^{m2^{\deg z - 1}} : z \in \frac{1}{t}\mathbb{F}_2[[1/t]] + \frac{1}{t}\mathbb{F}_2[[1/t]]y\right\} \\ &= 2^{4m} \end{aligned}$$

since the supremum is attained for $z = y/t$ which is of degree 3. From the expression

$$2^{j_0(t^m;\rho^4)} = \max(1, \|t^m\|),$$

we conclude that

$$j_0(t^m;\rho^4) = 4m. \tag{4.16}$$

Let us recap our results.

Proposition IV.15. *For $1 \leq j \leq 4$, let K_j be the function field associated to the curve X_j and let ρ^j be the unique Drinfeld-Hayes A_j-module associated to K_j. Let $j_0(t^m;\rho^j)$ be as in Proposition III.7. Then*

(1) $j_0(t^m;\rho^1) = (3/2)m$.

(2) $j_0(t^m;\rho^2) = (4/3)m$.

(3) $j_0(t^m;\rho^3) = 2m$.

(4) $j_0(t^m;\rho^4) = 4m$.

Remark IV.16. We note the following pattern evident in our previous result. For all $1 \leq j \leq 4$,

$$j_0(t^m;\rho^j) = g_{K_j}\frac{q_j}{q_j - 1}m.$$

G. Special Polynomials

We still continue with the notations of the previous sections. Let $1 \leq j \leq 4$.

Consider the function $\mathbf{e}_j\colon (K_j)_\infty \to (\bar{K}_j)_\infty$ defined by

$$\mathbf{e}_j(x) := \exp_{\rho^j}(\tilde{\pi}_{\rho^j}x)$$

and, for each nonnegative integer m, the function $l_m\colon (K_j)_\infty \to (\bar{K}_j)_\infty$ defined by

$$l_m(x) := \sum_{a\in(A_j)_+}\frac{\mathbf{e}_j(ax)^m}{a}.$$

Define $l_0\colon (K_j)_\infty \to (\bar{K}_j)_\infty$ by

$$l_0(x) := \sum_{a\in(A_j)_+}\frac{1}{a}.$$

Proposition IV.17. *For $1 \leq j \leq 4$, let K_j be the function field associated to the curve X_j and let ρ^j be the unique Drinfeld-Hayes A_j-module associated to K_j.*

(1) The power series

$$S(t^m;z) := \exp_{\rho^j} \ell(t^m;z) = \sum_{i\geq 0} e_i(\rho^j) \sum_{a\in(A_j)_+} \left(\frac{(\rho_a^j(t))^m}{a}\right)^{q^i} z^{q^{i+\deg a}}$$

lies in $A_j[t,z]$.

(2) One has $\exp_{\rho^j} l_m(x) = S(t^m;1)|_{t=\mathbf{e}_j(x)}$.

(3) For all $c \in \mathbb{F}_q^\times$, *one has* $S(t^m;z)|_{t=ct} = c^m S(t^m;z)$, *and moreover* $S(t^m;z)$ *is divisible by* t^m.

(4) One has

$$\left.\frac{S(t^m;z)}{t^m}\right|_{t=0} = \sum_{a\in(A_j)_+} a^{m-1} z^{q^{\deg a}}$$

for $m > 0$.

(5) Let $i_0(\rho^j)$ *and* $j_0(t^m;\rho^j)$ *be as in Propositions IV.10 and IV.15. The degree of* $S(t^m;z)$ *in* z *(respectively* t*) does not exceed* $q^{\lfloor i_0(\rho^j)+j_0(t^m;\rho^j)\rfloor}$ *(resp.* $mq^{\lfloor i_0(\rho^j)+j_0(t^m;\rho^j)\rfloor}$*).*

(6) The specialization $S(t^m;1) \in A_j[t]$ *vanishes identically if* $m > 1$ *and* $m \equiv 1 \bmod q-1$.

Proof. (1): Since $h_{A_j} = 1$, we have that K_j is its own Hilbert class field. Hence, $\exp_{\rho^j} \ell(t^m;z) \in A_j[t,z]$ by Theorem III.2.

(2): For all $x \in (K_j)_\infty$, we have

$$\rho_a^j(\mathbf{e}_j(x)) = \rho_a^j(\exp_{\rho^j}(\tilde{\pi}_{\rho^j} x)) = \exp_{\rho^j}(a\tilde{\pi}_{\rho^j} x) = \mathbf{e}(ax).$$

Hence,

$$S(t^m;1)|_{t=\mathbf{e}_j(x)} = \sum_{i\geq 0} e_i(\rho^j) \sum_{a\in(A_j)_+} \left(\frac{\mathbf{e}_j(ax)^m}{a}\right)^{q^i} = \exp_{\rho^j} l_m(x).$$

(3): This follows from $(\rho_a^j(ct))^m = (c\rho_a^j(t))^m$ and that $(c^m)^{q^i} = c^m$ for all $c \in \mathbb{F}_q^\times$.

(4): Since $\rho_a^j = a + (\text{higher order terms in } \tau)$, it follows that

$$\rho_a^j(t) = at + (\text{higher order terms in } t).$$

Therefore, if $i \geq 1$, then

$$\left(\frac{(\rho_a^j(t))^m}{t^m}\right)^{q^i}\Bigg|_{t=0} = 0$$

and

$$\left(\frac{(\rho_a^j(t))^m}{t^m}\right)\Bigg|_{t=0} = a^m.$$

Hence,

$$\frac{S(t^m;z)}{t^m}\Bigg|_{t=0} = \sum_{a \in (A_j)_+} \frac{1}{a}\left(\frac{(\rho_a^j(t))^m}{t^m}\right)\Bigg|_{t=0} z^{q^{\deg a}},$$

which gives the desired result. (Recall that $e_0(\rho^j) = 1$.)

(5): The bound in the degree in z follows exactly as in Remark III.11. As for the other bound, the general term in the sum defining $S(t^m;z)$ is

$$e_i(\rho^j)\left(\frac{(\rho_a^j(t))^m}{a}\right)^{q^i} z^{q^{i+\deg a}}.$$

It satisfies

$$(\text{degree in } t) \leq m \cdot (\text{degree in } z)$$

since the degree in t is $q^{\deg a} m q^i$ and the degree in z is $q^{i+\deg a}$. Since all terms of degree in z exceeding $q^{\lfloor i_0(\rho^j)+j_0(t^m;\rho^j)\rfloor}$ may be ignored in computing $S(t^m;z)$, the bound for the degree in t follows.

(6): We will stop writing the index j for this part of the proof. Let $\mathbf{p} \in A_+$ be irreducible of degree d and fix $m > 1$ such that $m \equiv 1 \bmod q-1$. It is enough to show that $S(t^m;1)|_{t=\mathbf{e}(1/\mathbf{p})} = 0$ since there are infinitely many $\mathbf{p}$ to choose from. By (2), this is equivalent to showing that $\exp_\rho l_m(1/\mathbf{p}) = 0$ which is equivalent to showing that

$$\frac{l_m(1/\mathbf{p})}{\tilde{\pi}_\rho} \in A \tag{4.17}$$

since $\exp_\rho$ vanishes on $\tilde{\pi}_\rho A$. Recall that $\exp_\rho$ may be written as an infinite product:

$$\exp_\rho(z) = z \prod_{\substack{a\in A \\ a\neq 0}} \left(1 - \frac{z}{a\tilde{\pi}_\rho}\right)$$

for all $z \in \mathbb{C}_\infty$. Taking the logarithmic derivative with respect to z of the preceding expression, and using the fact that $\exp_\rho$ has derivative 1, we get

$$\frac{1}{\exp_\rho(z)} = \frac{1}{z} + \sum_{\substack{a\in A \\ a\neq 0}} \frac{-1/(a\tilde{\pi}_\rho)}{1 - z/(a\tilde{\pi}_\rho)} = \frac{1}{z} + \sum_{\substack{a\in A \\ a\neq 0}} \frac{1}{z - a\tilde{\pi}_\rho} = \sum_{a\in A} \frac{1}{z - a\tilde{\pi}_\rho}.$$

Choose $a_0 \in A$ such that $a_0 \not\equiv 0 \bmod \mathbf{p}$ and set $z = a_0\tilde{\pi}_\rho/\mathbf{p}$ in the previous expression. We conclude

$$\begin{aligned}
\mathbf{e}(a_0/\mathbf{p})^{-1} &= \mathbf{p}\sum_{a\in A} \frac{1}{\tilde{\pi}_\rho(a_0 - \mathbf{p}a)} \\
&= \mathbf{p}\sum_{b\in A} \frac{1}{\tilde{\pi}_\rho(a_0 + \mathbf{p}b)} \qquad \text{where } b = -a \\
&= \mathbf{p} \sum_{\substack{a\in A \\ a\equiv a_0 \bmod \mathbf{p}}} \frac{1}{\tilde{\pi}_\rho a}. \qquad (4.18)
\end{aligned}$$

Also note

$$\frac{l_m(1/\mathbf{p})}{\tilde{\pi}_\rho} = \sum_{a\in A_+} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho} = \sum_{\substack{a\in A_+ \\ (a,\mathbf{p})=1}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho}. \qquad (4.19)$$

Since $d_\infty = 1$, $\mathrm{sgn}(a) \in \mathbb{F}_q^\times$. Consider the equation

$$\sum_{d\in\mathbb{F}_q^\times} \sum_{\substack{a\in A \\ (a,\mathbf{p})=1 \\ \mathrm{sgn}(a)=d}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho} = \sum_{\substack{a\in A \\ (a,\mathbf{p})=1}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho}. \qquad (4.20)$$

If $\mathrm{sgn}(a) = d \in \mathbb{F}_q^\times$, then

$$\begin{aligned}
\frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho} &= \frac{\mathbf{e}(da'/\mathbf{p})^m}{da'\tilde{\pi}_\rho} \text{ for some } a' \in A_+ \\
&= \frac{d^m\mathbf{e}(a'/\mathbf{p})^m}{da'\tilde{\pi}_\rho}
\end{aligned}$$

$$= \frac{d\mathbf{e}(a'/\mathbf{p})^m}{da'\tilde{\pi}_\rho} \text{ since } m \equiv 1 \bmod q-1 \text{ and } d \in \mathbb{F}_q^\times$$
$$= \frac{\mathbf{e}(a'/\mathbf{p})^m}{a'\tilde{\pi}_\rho}. \tag{4.21}$$

Hence, (4.20) becomes

$$\begin{aligned}\sum_{\substack{a\in A\\(a,\mathbf{p})=1}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho} &= (q-1)\sum_{\substack{a\in A_+\\(a,\mathbf{p})=1}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho}\\ &= -\sum_{\substack{a\in A_+\\(a,\mathbf{p})=1}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho}. \end{aligned}\tag{4.22}$$

Therefore (4.19) becomes

$$\begin{aligned}\frac{l_m(1/\mathbf{p})}{\tilde{\pi}_\rho} &= \sum_{\substack{a\in A_+\\(a,\mathbf{p})=1}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho}\\
&= -\sum_{\substack{a\in A\\(a,\mathbf{p})=1}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho} \quad \text{(by (4.22))}\\
&= -\sum_{\substack{a\in A\\a\not\equiv 0 \bmod \mathbf{p}}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho} \quad ((a,\mathbf{p})=1 \Leftrightarrow a\not\equiv 0 \bmod \mathbf{p})\\
&= -\sum_{b\in(A/\mathbf{p})^\times}\ \sum_{\substack{a\in A\\a\equiv b \bmod \mathbf{p}}} \frac{\mathbf{e}(a/\mathbf{p})^m}{a\tilde{\pi}_\rho}\\
&\quad (a\not\equiv 0 \bmod \mathbf{p} \Leftrightarrow a\equiv b \bmod \mathbf{p} \text{ for some } b\in(A/\mathbf{p})^\times)\\
&= -\sum_{b\in(A/\mathbf{p})^\times} \mathbf{e}(b/\mathbf{p})^m \sum_{\substack{a\in A\\a\equiv b \bmod \mathbf{p}}} \frac{1}{a\tilde{\pi}_\rho}\\
&= -\sum_{b\in(A/\mathbf{p})^\times} \mathbf{e}(b/\mathbf{p})^m \frac{1}{\mathbf{p}\mathbf{e}(b/\mathbf{p})} \quad \text{(by (4.18))}\\
&= -\sum_{b\in(A/\mathbf{p})^\times} \frac{\mathbf{e}(b/\mathbf{p})^{m-1}}{\mathbf{p}}.\end{aligned}$$

(Refer to §III.D and (3.6) for how we view $\mathbf{e}(x/\mathbf{p})$ as a function on $A/\mathbf{p}$.) We claim that this last element belongs to A. To do this, we will use the *Newton formulas*.

Lemma IV.18 ([12], pp. 26-27)**.** *(a) Let Y_1, Y_2, ..., Y_n be indeterminates and consider the symmetric polynomials*

$$s_1 := s_1(Y_1, Y_2, \dots, Y_n) := Y_1 + Y_2 + \cdots + Y_n,$$

$$s_2 := s_2(Y_1, Y_2, \dots, Y_n) := Y_1Y_2 + Y_1Y_3 + \cdots Y_2Y_3 + \cdots Y_{n-1}Y_n;$$

$$\dots$$

$$s_k := s_k(Y_1, Y_2, \dots, Y_n) := \sum_{i_1 < i_2 < \cdots < i_k} Y_{i_1}Y_{i_2} \cdots Y_{i_k} \quad (\textit{where } k \leq n),$$

$$\dots$$

$$s_n := s_n(Y_1, Y_2, \dots, Y_n) := Y_1Y_2 \cdots Y_n.$$

If Z is any indeterminate, then

$$Z^n - s_1Z^{n-1} + s_2Z^{n-2} - \cdots + (-1)^k s_k Z^{n-k} + \cdots + (-1)^n s_n = \prod_{k=1}^{n} (Z - Y_k).$$

(b) Let $p_0 = n$ and $p_k := p_k(Y_1, Y_2, \dots, Y_n) := Y_1^k + Y_2^k + \cdots + Y_n^k$ where $k \geq 1$. The Newton formulas are as follows.

(i) If $k \leq n$, then

$$p_k - p_{k-1}s_1 + p_{k-2}s_2 - \cdots + (-1)^{k-1} p_1 s_{k-1} + (-1)^k k s_k = 0.$$

(ii) If $k > n$, then

$$p_k - p_{k-1}s_1 + \cdots + (-1)^n p_{k-n} s_n = 0.$$

Recall that the $\mathbf{p}$-torsion points of ρ are, by definition, the roots of the polynomial $\rho_{\mathbf{p}}(z)$. Explicitly, we have

$$\rho[\mathbf{p}] = \rho[\mathbf{p}A] = \{\mathbf{e}(x/\mathbf{p}) \mid x \in A/\mathbf{p}A\}.$$

Therefore,

$$\begin{aligned}\rho_{\mathbf{p}}(z) &= \prod_{x\in A/\mathbf{p}A}(z-\mathbf{e}(x/\mathbf{p}))\\ &= \sum_{j=1}^{q^d}(-1)^{j+1}\hat{s}_{j-1}z^{q^d+1-j}\end{aligned}$$

by Lemma IV.18(a) with $\{Y_1,\ldots,Y_{q^d}\}:=\{\mathbf{e}(x/\mathbf{p})\mid x\in A/\mathbf{p}A\}$, $\hat{s}_{j-1}:=s_{j-1}(Y_1,\ldots,Y_{q^d})$ and $\hat{s}_0:=1$.

Since $h_A=1$ and since, for each of our function fields, there is only one Drinfeld A-module, we have that $\rho_{\mathbf{p}}(z)\in A[z]$ by Theorem II.5. This implies that $\hat{s}_j\in A$ for all $0\le j\le q^d-1$. Also, $\rho_{\mathbf{p}}(z)$ is Eisenstein at $\mathbf{p}$ ([10], Proposition 11.4). This means that

$$\mathbf{p}\nmid\hat{s}_0,\quad \frac{\hat{s}_j}{\mathbf{p}}\in A \text{ for } 1\le j\le q^d-1,\quad \mathbf{p}^2\nmid\hat{s}_{q^d-1}. \tag{4.23}$$

Set $\hat{p}_k:=p_k(Y_1,\ldots,Y_{q^d})$. Using part (i) of (b), it follows that $\hat{p}_1\in A$. A straightforward induction argument then implies that $\hat{p}_k\in A$ for all k. Now using either (i) or (ii) of part (b) (depending on whether $m-1\le q^d$ or $m-1>q^d$) and (4.23), we conclude $\hat{p}_{m-1}/\mathbf{p}\in A$. Therefore,

$$-\frac{\hat{p}_{m-1}}{\mathbf{p}}=-\sum_{b\in(A/\mathbf{p})^\times}\frac{\mathbf{e}(b/\mathbf{p})^{m-1}}{\mathbf{p}}\in A.$$

This completes the proof.

□

H. The Module of Special Points

Our setup of the module of special points will be similar to the case of the Carlitz module. Let d be a positive integer. Let

$$\mathcal{M}:=\{m\in\mathbb{Z}\mid 1\le m\le q^d-1, m\not\equiv 1 \pmod{q-1}\}.$$

Fix a sign function sgn and let $\mathbf{p} \in (A_j)_+$ be irreducible of degree d_j. As in the case of the Carlitz module, the $\mathbf{p}$-torsion of ρ^j is

$$\rho^j[\mathbf{p}] = \{\exp_{\rho^j}(\tilde{\pi}_{\rho^j} a/\mathbf{p}) \mid a \in A_j\}.$$

As before, $\rho^j[\mathbf{p}] \cong A_j/\mathbf{p}$ as A_j-modules, so let $\lambda_j := \exp_{\rho^j}(\tilde{\pi}_{\rho^j}/\mathbf{p})$ be a generator. Since $d_{\infty_j} = 1$ and $h_{A_j} = 1$, we have that $H_j = K_j$ where H_j is the Hilbert class field of K_j.

Set

$$(K_j)_{\mathbf{p}} := K_j(\rho^j[\mathbf{p}]) = K_j(\lambda_j);$$
$$\mathcal{R}_j := A_j[\lambda_j].$$

The fraction field of $\mathcal{R}_j$ is $K_j(\lambda_j)$ and, moreover, the integral closure of A_j in $(K_j)_{\mathbf{p}}$ is $\mathcal{R}_j$.

Let $G_j := \mathrm{Gal}((K_j)_{\mathbf{p}_j}/K_j)$. Let

$$a \mapsto \sigma_a \colon (A_j/\mathbf{p})^\times \to G_j$$

be the unique isomorphism such that

$$\sigma_a(\mathbf{e}_j(b/\mathbf{p})) = \mathbf{e}_j(ab/\mathbf{p})$$

for all $a, b \in (A_j/\mathbf{p})^\times$. By the functional equation (2.1), we have

$$\mathbf{e}_j(ab/\mathbf{p}) = \rho^j_a(\mathbf{e}_j(b/\mathbf{p})).$$

Exactly as in the case of the Carlitz module, for any A_j-algebra R, we let R^{ρ^j} be a copy of R equipped with the A_j-module structure

$$(a, r) \mapsto a * r := \rho^j_a(r) \colon A_j \times R \to R.$$

The exponential function $\exp_{\rho^j}$ is A_j-linear if viewed as a map $(\bar{K}_j)_\infty \to (\bar{K}_j)_\infty^{\rho^j}$. If $r \in R$ is viewed as an element of R^{ρ^j} via $(1,r) \mapsto \rho_1^j(r) = r$, then r is the coordinate of an R-valued point of ρ^j.

Let m be a nonnegative integer. For each $b \in (A_j/\mathbf{p})^\times$, let $s_m^j(b)$ be the $\mathcal{R}_j$-valued point of ρ^j with coordinate

$$s_m^j(b) := \exp_{\rho^j} l_m(x) = S(t^m;1)|_{t=\mathbf{e}_j(x)}$$

where $x = \tilde{b}/\mathbf{p}$ and $\tilde{b} \equiv b \pmod{\mathbf{p}}$. This definition does not depend upon the choice of $\tilde{b}$; the proof is exactly the same as in the case of the Carlitz module.

Definition IV.19. Let $\mathcal{S}^j$ denote the A_j-submodule of $\mathcal{R}^{\rho^j}$ generated by points of the form $s_m^j(b)$. The elements of $\mathcal{S}^j$ are called *special points* of ρ^j.

For the rest of this chapter, we will forgo writing the letter j since our results hold for all $1 \leq j \leq 4$. We begin our analysis of $\mathcal{S}$ by first examining dependence relations among the functions

$$\{l_m(x) \mid m \geq 0\}.$$

Lemma IV.20. *Let A be one of $A_1, \ldots, A_4$. Let $a \in A$, $a \neq 0$, $x \in K_\infty$, and m be a nonnegative integer. Define $a_j \in A$ for $j = m, \ldots, mq^{\deg a}$ by*

$$\rho_a(t)^m = \sum_{j=m}^{mq^{\deg a}} a_j t^j.$$

Then

$$l_m(ax) = \sum_{j=m}^{mq^{\deg a}} a_j l_j(x).$$

Proof. First note that since $h_A = 1$, we have $\rho_a(t) \in A[t]$. So the coefficients a_j do lie in A.

We have

$$\sum_{j=m}^{mq^{\deg a}} a_j \mathbf{e}(x)^j = \rho_a(\mathbf{e}(x))^m = \rho_a(\exp_\rho(\tilde{\pi}_\rho x))^m = \exp_\rho(a\tilde{\pi}_\rho x)^m = \mathbf{e}(ax)^m,$$

where the penultimate equality follows from the functional equation (2.1).

Therefore,

$$\begin{aligned}\sum_{j=m}^{mq^{\deg a}} a_j l_j(x) &= \sum_{j=m}^{mq^{\deg a}} a_j \left(\sum_{a\in A_+} \frac{\mathbf{e}(ax)^j}{a}\right) = \sum_{a\in A_+} \frac{1}{a}\left(\sum_{j=m}^{mq^{\deg a}} a_j \mathbf{e}(ax)^j\right)\\ &= \sum_{a\in A_+} \frac{1}{a}\mathbf{e}(a^2x)^m\\ &= l_m(ax).\end{aligned}$$

□

Before we state our next lemma, we need to make a definition. Let A be one of $A_1,\dots,A_4$. For $a\in A$, define $\rho_{a,i}\in A$ for $i=0,\dots,\deg a$ by

$$\rho_a(t) = \sum_{i=0}^{\deg a} \rho_{a,i}\, t^{q^i}.$$

Note that since ρ is a Drinfeld-Hayes A-module, we have $\rho_{a,0}=a$ and $\rho_{a,\deg a}=\operatorname{sgn}(a)$.

Lemma IV.21. *Let A be one of $A_1,\dots,A_4$. Let m be a nonnegative integer. Then*

$$l_{m+q^d}(1/\mathbf{p}) = -\sum_{i=0}^{d-1} \rho_{\mathbf{p},i} l_{m+q^i}(1/\mathbf{p}).$$

Proof. We calculate

$$\begin{aligned}\sum_{i=0}^{d-1} \rho_{\mathbf{p},i} l_{m+q^i}(1/\mathbf{p}) + l_{m+q^d}(1/\mathbf{p}) &= \sum_{i=0}^{d} \rho_{\mathbf{p},i} l_{m+q^i}(1/\mathbf{p})\\ &= \sum_{i=0}^{d} \rho_{\mathbf{p},i} \sum_{a\in A_+} \frac{\mathbf{e}(a/\mathbf{p})^{m+q^i}}{a}\\ &= \sum_{a\in A_+} \frac{1}{a}\mathbf{e}(a/\mathbf{p})^m \sum_{i=0}^{d} \rho_{\mathbf{p},i} \exp_\rho(a\tilde{\pi}_\rho/\mathbf{p})^{q^i}.\end{aligned}$$

The inner sum is

$$\begin{aligned}\rho_{\mathbf{p}}(\exp_\rho(a\tilde{\pi}_\rho/\mathbf{p})) &= \exp_\rho(a\tilde{\pi}_\rho) \quad \text{by the functional equation (2.1)}\\ &= 0 \quad \text{since } \exp_\rho \text{ vanishes on } A\tilde{\pi}_\rho.\end{aligned}$$

This completes the proof. □

Lemma IV.22. *Let A be one of $A_1,\dots,A_4$. We have*

$$\sum_{i=1}^{d}\rho_{\mathbf{p},i}l_{q^i-1}(1/\mathbf{p}) = (1-\mathbf{p})l_0(1/\mathbf{p}).$$

Proof. Let

$$\phi(z) := \frac{\rho_{\mathbf{p}}(z)}{z} = \mathbf{p} + \sum_{i=1}^{d}\rho_{\mathbf{p},i}z^{q^i-1}.$$

Then

$$\begin{aligned}\mathbf{p}l_0(1/\mathbf{p}) + \sum_{i=1}^{d}\rho_{\mathbf{p},i}l_{q^i-1}(1/\mathbf{p}) &= \sum_{i=0}^{d}\rho_{\mathbf{p},i}l_{q^i-1}(1/\mathbf{p})\\ &= \sum_{i=0}^{d}\rho_{\mathbf{p},i}\sum_{a\in A_+}\frac{\mathbf{e}(a/\mathbf{p})^{q^i-1}}{a}\\ &= \sum_{a\in A_+}\frac{1}{a}\sum_{i=0}^{d}\rho_{\mathbf{p},i}\mathbf{e}(a/\mathbf{p})^{q^i-1}\\ &= \sum_{a\in A_+}\frac{\phi(\mathbf{e}(a/\mathbf{p}))}{a}\end{aligned}$$

since $\rho_{\mathbf{p},0} = \mathbf{p}$.

If $\mathbf{p} \mid a$, then $a/\mathbf{p} \in A$, and so $\mathbf{e}(a/\mathbf{p}) = 0$. Therefore,

$$\phi(\mathbf{e}(a/\mathbf{p})) = \mathbf{p} + \sum_{i=1}^{d}\rho_{\mathbf{p},i}(\mathbf{e}(a/\mathbf{p}))^{q^i-1} = \mathbf{p}.$$

If $\mathbf{p} \nmid a$, then $\mathbf{e}(a/\mathbf{p}) \neq 0$ and $\rho_{\mathbf{p}}(\mathbf{e}(a/\mathbf{p})) = 0$ exactly as in the previous lemma. Therefore,

$$\phi(\mathbf{e}(a/\mathbf{p})) = \begin{cases} \mathbf{p} & \text{if } \mathbf{p} \mid a, \\ 0 & \text{if } \mathbf{p} \nmid a. \end{cases}$$

Continuing, we have

$$\begin{aligned} \sum_{a \in A_+} \frac{\phi(\mathbf{e}(a/\mathbf{p}))}{a} &= \sum_{\substack{a \in A_+ \\ a \equiv 0 \pmod{\mathbf{p}}}} \frac{\mathbf{p}}{a} = \sum_{a' \in A_+} \frac{\mathbf{p}}{\mathbf{p}a'} \quad \text{by setting } a = \mathbf{p}a' \\ &= l_0(1/\mathbf{p}). \end{aligned}$$

This completes the proof. □

I. The Finite Generation of $\mathcal{S}$

In this section, A is one of $A_1, \ldots, A_4$. Let S be a subset of an A-module M. We write $\langle S \rangle_A$ to denote the A-submodule of M generated by the elements of S. We recall that

$$\mathcal{M} := \{m \in \mathbb{Z} \mid 1 \leq m \leq q^d - 1, m \not\equiv 1 \pmod{q-1}\}.$$

Proposition IV.23. *The quotient of $\mathcal{S}$ by $\langle s_m(1) \mid m \in \mathcal{M} \cup \{1\} \rangle_A$ is generated, as an A-module, by $s_0(1)$ and is annihilated by $\mathbf{p} - 1$.*

Proof. Set

$$\mathcal{L} := \langle l_m(a/\mathbf{p}) \mid m \geq 0, a \in A \rangle_A.$$

We claim

$$\mathcal{L} = \langle l_m(1/\mathbf{p}) \mid 0 \leq m \leq q^d - 1 \rangle_A. \tag{4.24}$$

Since every element of the form $l_m(1/\mathbf{p})$ for $0 \leq m \leq q^d - 1$ is clearly in $\mathcal{L}$, this proves

$\supseteq$. As for the other inclusion, Lemma IV.20 implies

$$l_m(a/\mathbf{p}) = \sum_{j=m}^{mq^{\deg a}} a_j l_j(1/\mathbf{p}) \in \langle l_r(1/\mathbf{p}) \mid m \le r \le mq^{\deg a} \rangle_A.$$

Suppose $r \in \{m, \ldots, mq^{\deg a}\}$ such that $r \ge q^d$. Then $r = q^d + n$ for some $n \ge 0$. Therefore,

$$l_r(1/\mathbf{p}) = l_{q^d+n}(1/\mathbf{p}) = -\sum_{i=0}^{d-1} \rho_{\mathbf{p},i} l_{n+q^i}(1/\mathbf{p})$$

by Lemma IV.21. If $n + q^i < q^d$ for all $0 \le i \le d-1$, then this proves $\subseteq$. Otherwise, there exists an index j such that $n + q^j \ge q^d$, and so $n + q^j = q^d + k$ for some $k \ge 0$. Thus,

$$l_{n+q^j}(1/\mathbf{p}) = l_{k+q^d}(1/\mathbf{p}),$$

and we may repeat the above argument using Lemma IV.21. This proves (4.24).

Given the original definition of $\mathcal{L}$, note that exponentiating the elements of $\mathcal{L}$ gives the coordinates of the special points of ρ. Therefore, by (4.24), we conclude

$$\mathcal{S} = \langle s_m(1) \mid 0 \le m \le q^d - 1 \rangle_A. \tag{4.25}$$

By Proposition III.8(6), the special point $s_m(1)$ vanishes for all $m > 1$ such that $m \equiv 1 \pmod{q-1}$. Hence,

$$\mathcal{S} = \langle s_m(1) \mid m \in \mathcal{M} \cup \{0, 1\} \rangle_A.$$

This proves the first assertion of the Proposition.

As for the second assertion, it is enough to show

$$(\mathbf{p} - 1) * s_0(1) \in \langle s_m(1) \mid m \in \mathcal{M} \cup \{1\} \rangle_A.$$

Note that the coordinate of the special point $s_0(1)$ is $\exp_\rho l_0(1/\mathbf{p})$. We calculate

$$(\mathbf{p} - 1) * s_0(1) = \rho_{\mathbf{p}-1}(s_0(1))$$

$$
\begin{aligned}
&= \rho_{\mathbf{p}-1}(\exp_\rho l_0(1/\mathbf{p})) \\
&= \exp_\rho((\mathbf{p}-1)l_0(1/\mathbf{p})) \\
&= \exp_\rho\left(-\sum_{i=1}^{d} \rho_{\mathbf{p},i} l_{q^i-1}(1/\mathbf{p})\right)
\end{aligned}
$$

where the last two equalities follow from the functional equation (2.1) and Lemma IV.22, respectively. We view the sum inside the parentheses as an element of $\bar{K}_\infty^\rho$. Since $\rho_{\mathbf{p},i} \in A$ for all i and since $\exp_\rho$ is A-linear on $\bar{K}_\infty^\rho$, we conclude

$$
(\mathbf{p}-1) * s_0(1) = -\sum_{i=1}^{d} \rho_{\mathbf{p},i} * \exp_\rho(l_{q^i-1}(1/\mathbf{p})) = -\sum_{i=1}^{d} \rho_{\mathbf{p},i} * s_{q^i-1}(1).
$$

This last expression clearly belongs to $\langle s_m(1) \mid m \in \mathcal{M} \cup \{1\} \rangle_A$. This concludes the proof.

□

Remark IV.24. We record here the content of (4.25): The module of special points $\mathcal{S}$ is finitely generated as an A-module by the special points of the form

$$
\{s_m(1) \mid 0 \leq m \leq q^d - 1\}.
$$

CHAPTER V

EXPRESSING SPECIAL VALUES AS LINEAR COMBINATIONS OF LOGARITHMS

Recall that in the case of the Carlitz module, if $\omega\colon \mathbf{F}_{\mathbf{p}}^{\times} \to \mathbb{C}_{\infty}^{\times}$ is the Teichmüller character and $\mathbf{e}_m^*(a)$ are the dual coefficients of Proposition III.14, then we have

$$L(1,\omega^i) = -\sum_{m=1}^{q^d-1}\left(\frac{1}{\mathbf{p}}\sum_{a\in\mathbf{F}_{\mathbf{p}}^{\times}}\omega^i(a)\mathbf{e}_m^*(a)\right)\left(\sum_{b\in\mathbf{F}_{\mathbf{p}}^{\times}}\omega^{-i}(b)l_m(b/\mathbf{p})\right).$$

Our goal in this chapter is to prove an analogue of this formula for an *arbitrary* function field.

A. Preliminary Functions–Part 1

Let K be an arbitrary function field over the finite field $\mathbb{F}_q$. We assume that $d_\infty = 1$. Let $\mathfrak{m}$ be a prime ideal of $A \subseteq K$ and set $d := \deg \mathfrak{m}$. Recall from §II.F that H is the Hilbert class field of K and B be the integral closure of A in H. Fix a sign function sgn and let ρ be a Drinfeld-Hayes A-module with respect to sgn (cf., §II.B). Set $K_{\mathfrak{m}} := H(\rho[\mathfrak{m}])$, $G := \mathrm{Gal}(K_{\mathfrak{m}}/K)$ (§II.F, II.G) and let C be the integral closure of B in $K_{\mathfrak{m}}$.

Let χ be a Dirichlet character on A whose kernel is $\mathfrak{m}$. Since $\mathfrak{m}$ is a prime ideal, $A/\mathfrak{m}$ is a field with $\#(A/\mathfrak{m})^{\times} = q^{\deg\mathfrak{m}} - 1$. Thus χ has order relatively prime to p. Fix $\psi \in \hat{G}$ such that $\psi|_A = \chi$ as in §II.G. Recall from §II.G that the Goss L-function for ψ is

$$L(s,\psi) = \sum_I \frac{\psi(I)}{I^{[s]}}$$

where the sum ranges over all integral ideals I of A which are relatively prime to $\mathfrak{m}$. Here s is a positive integer.

Recall that two nonzero fractional ideals M and M' are equivalent in $Cl(A)$, and we write $M \sim M'$, if there exist nonzero $\alpha, \beta \in A$ such that $\alpha M = \beta M'$. Write $h = h_A$ and let

$\{\mathfrak{a}_1, \ldots, \mathfrak{a}_h\}$ be a set of representatives of the equivalence classes of $Cl(A)$ that are relatively prime to $\mathfrak{m}$. Then

$$L(s,\psi) = \sum_{i=1}^{h} \sum_{I \sim \mathfrak{a}_i} \frac{\psi(I)}{I^{[s]}}.$$

Let

$$A_{\mathfrak{m}} := \{x \in K \mid \mathrm{ord}_{\mathfrak{m}}(x) \geq 0\}.$$

We extend $\chi \colon A \to \mathbb{C}_\infty$ to a map (which we also call χ) defined on $A_{\mathfrak{m}}$ as follows:

$$\begin{aligned} \chi \colon A_{\mathfrak{m}} &\to \mathbb{C}_\infty \\ \frac{a}{b} &\mapsto \frac{\chi(a)}{\chi(b)} \end{aligned}$$

for $b \notin \mathfrak{m}$. Since $b \notin \mathfrak{m}$, it follows that $\chi(b) \neq 0$ and so the above map is well-defined.

Let $I \sim \mathfrak{a}_i$. Pick $\beta, \alpha \in A_+$ such that $\beta I = \alpha \mathfrak{a}_i$. Then

$$\mathrm{ord}_{\mathfrak{m}}(\beta I) = \mathrm{ord}_{\mathfrak{m}}(\alpha \mathfrak{a}_i)$$

which implies

$$\mathrm{ord}_{\mathfrak{m}}(\beta) + \mathrm{ord}_{\mathfrak{m}}(I) = \mathrm{ord}_{\mathfrak{m}}(\alpha) + \mathrm{ord}_{\mathfrak{m}}(\mathfrak{a}_i).$$

We have that $\mathrm{ord}_{\mathfrak{m}}(I) \geq 0$. Since $\mathfrak{a}_i$ and $\mathfrak{m}$ are relatively prime, $\mathrm{ord}_{\mathfrak{m}}(\mathfrak{a}_i) = 0$. We conclude

$$\mathrm{ord}_{\mathfrak{m}}(\alpha/\beta) \geq 0,$$

i.e. $\alpha/\beta \in A_{\mathfrak{m}}$. Therefore,

$$\psi(I) = \psi(\alpha/\beta)\psi(\mathfrak{a}_i)$$

with $\psi(\alpha/\beta) = 0$ if α/β is not a unit in $A_{\mathfrak{m}}$. Thus,

$$L(s,\psi) = \sum_{i=1}^{h} \sum_{\gamma \in (\mathfrak{a}_i^{-1})_+} \frac{\psi(\gamma \mathfrak{a}_i)}{(\gamma \mathfrak{a}_i)^s}$$

$$= \sum_{i=1}^{h} \frac{\psi(\mathfrak{a}_i)}{\mathfrak{a}_i^s} \sum_{\gamma \in (\mathfrak{a}_i^{-1})_+} \frac{\psi(\gamma)}{\gamma^s}$$

where if $\gamma = \alpha/\beta \in A_{\mathfrak{m}}$, then $\psi(\gamma) = \psi(\alpha)/\psi(\beta)$.

Definition V.1. Let I be an integral ideal of A. Define

$$L_I(s,\psi) := \frac{\psi(I)}{I^{[s]}} \sum_{\omega \in (I^{-1})_+} \frac{\psi(\omega)}{\omega^s};$$

$$\zeta_{I,c}(s) := \frac{1}{I^{[s]}} \sum_{\substack{n \in (I^{-1})_+ \\ n \equiv c \pmod{I^{-1}\mathfrak{m}}}} \frac{1}{n^s}$$

for $c \in I^{-1}/I^{-1}\mathfrak{m}$.

Remark V.2. These two functions are related to each other via

$$L_I(s,\psi) = \psi(I) \sum_{c \in I^{-1}/I^{-1}\mathfrak{m}} \psi(c)\zeta_{I,c}(s).$$

Lemma V.3. *Suppose $I \sim J$ in $Cl(A)$. Then $L_I(s,\psi) = L_J(s,\psi)$.*

Proof. If $I \sim J$, then $\alpha I = \beta J$ for some $\alpha, \beta \in A_+$ relatively prime to $\mathfrak{m}$. The maps

$$I^{-1} \to J^{-1} : \gamma \mapsto \gamma\frac{\beta}{\alpha} \quad \text{and} \quad J^{-1} \to I^{-1} : \varepsilon \mapsto \varepsilon\frac{\alpha}{\beta}$$

give a one-to-one correspondence between I^{-1} and J^{-1}. Hence,

$$\begin{aligned} L_I(s,\psi) &= \frac{\psi(I)}{I^{[s]}} \sum_{\omega \in (I^{-1})_+} \frac{\psi(\omega)}{\omega^s} \\ &= \frac{\psi(\beta)\psi(J)}{\psi(\alpha)\left(\frac{\beta J}{\alpha}\right)^{[s]}} \sum_{\omega \in (I^{-1})_+} \frac{\psi(\omega)}{\omega^s} \\ &= \frac{\psi(J)}{J^{[s]}} \sum_{\omega \in (I^{-1})_+} \frac{\psi(\beta)\psi(\omega)}{\psi(\alpha)} \left(\frac{1}{\frac{\beta}{\alpha}\omega}\right)^s \\ &= \frac{\psi(J)}{J^{[s]}} \sum_{\omega' \in (J^{-1})_+} \frac{\psi(\omega')}{(\omega')^s} \text{ where } \omega' := \omega\beta/\alpha \end{aligned}$$

$$= L_J(s, \psi).$$

□

This Lemma implies that the function $L_I(s,\psi)$ depends only on the ideal class of I in $Cl(A)$. Therefore,

$$L(s,\psi) = \sum_{i=1}^{h} L_{\mathfrak{a}_i}(s,\psi), \tag{5.1}$$

and so the study of $L(s,\psi)$ reduces to the study of the functions $L_{\mathfrak{a}_i}(s,\psi)$ for $1 \le i \le h$.

B. Preliminary Functions–Part 2

We want to do a similar analysis in this section on the function $\ell(b;z)$ which was originally defined in Chapter III. We begin by recalling the definition of this function.

Let I be an integral ideal of A and let $b(t) = \sum_i b_i t^i \in B[t]$ where t is a variable. Set

$$I * b(t) := \sum_i b_i^{(I,H/K)} \rho_I(t)^i$$

where $(I,H/K)$ is the Artin automorphism of I and we write $b_i^{(I,H/K)}$ to denote $(I,H/K)$ acting on b_i. The map

$$H[t] \to H[t] \colon b \mapsto I * b \tag{5.2}$$

is an A-algebra endomorphism that stabilizes $B[t]$.

Define

$$\ell(b(t);z) := \sum_I \frac{I * b(t)}{D(\rho_I)} z^{q^{\deg I}}$$

where the sum is over all integral ideals of A and $D(\rho_I)$ is the constant term of ρ_I. Note that

$$\ell(b(t);z) = \sum_{i=1}^{h} \sum_{I \sim \mathfrak{a}_i} \frac{I * b(t)}{D(\rho_I)} z^{q^{\deg I}}.$$

We now consider a general term in the second sum. If $I \sim \mathfrak{a}_i$, then there exist $\beta_i, \gamma_i \in A_+$ such that $\beta_i I = \gamma_i \mathfrak{a}_i$. This implies that $I = (\gamma_i/\beta_i)\mathfrak{a}_i$ and so $\gamma_i/\beta_i \in (\mathfrak{a}_i^{-1})_+$. Also, note that $(I, H/K) = ((\gamma_i/\beta_i)\mathfrak{a}_i, H/K)$. We would like to determine the relationship between $D(\rho_I)$ and $D(\rho_{\mathfrak{a}_i})$. To do this, we will need the following result.

Lemma V.4 ([13], Propositions 13.14 and 13.15)**.** *Let ρ be a Drinfeld A-module over $\mathbb{C}_\infty$.*

1. *Let I, J be nonzero ideals of A. Then*

$$\rho_{IJ} = (J * \rho)_I \rho_J$$

where $J * \rho$ is the Drinfeld A-module satisfying

$$\rho_J \rho_a = (J * \rho)_a \rho_J$$

for all $a \in A$.

2. *If $0 \neq b \in A$, then write (b) for the ideal of A generated by b. Then $\rho_{(b)} = c^{-1}\rho_b$, where c is the leading coefficient of ρ_b. Moreover, $c[(b) * \rho]_a = \rho_a c$ for all $a \in A$; i.e., $(b) * \rho$ is isomorphic to ρ over $\mathbb{C}_\infty$.*

Continuing, we have

$$\rho_{\beta_i I} = \rho_{I(\beta_i)} = ((\beta_i) * \rho)_I \rho_{(\beta_i)}.$$

Since $\beta_i \in A_+$ and since ρ is a Drinfeld-Hayes A-module, we conclude that

$$D(\rho_{(\beta_i)}) = D(\rho_{\beta_i}) = \beta_i.$$

Since $((\beta_i) * \rho)$ is isomorphic to ρ over $\mathbb{C}_\infty$, it follows that

$$D(((\beta_i) * \rho)_I) = D(\rho_I),$$

and we get that

$$D(\rho_{\beta_i I}) = \beta_i D(\rho_I). \tag{5.3}$$

Therefore,

$$D(\rho_I) = \frac{\gamma_i}{\beta_i} D(\rho_{\mathfrak{a}_i}). \tag{5.4}$$

Putting all of this together, we may write

$$\begin{aligned} \ell(b(t);z) &= \sum_{i=1}^{h} \sum_{I\sim\mathfrak{a}_i} \frac{I * b(t)}{D(\rho_I)} z^{q^{\deg I}} \\ &= \sum_{i=1}^{h} \sum_{\omega\in(\mathfrak{a}_i^{-1})_+} \frac{(\omega\mathfrak{a}_i) * b(t)}{\omega D(\rho_{\mathfrak{a}_i})} z^{q^{\deg \mathfrak{a}_i + \deg\omega}}, \end{aligned}$$

and this motivates our next definition.

Definition V.5. Let I be an integral ideal of A and let $b(t) \in B[t]$. Define

$$\ell_I(b(t);z) := \frac{1}{D(\rho_I)} \sum_{\omega\in(I^{-1})_+} \frac{(\omega I) * b(t)}{\omega} z^{q^{\deg I + \deg\omega}}.$$

Lemma V.6. *Suppose $I \sim J$ in $Cl(A)$. Then $\ell_I(b(t);z) = \ell_J(b(t);z)$.*

Proof. If $I \sim J$ in $Cl(A)$, then $\alpha I = \beta J$ for some $\alpha, \beta \in A_+$. Since the $*$ map from (5.2) is an A-algebra endomorphism, we get

$$\frac{(\omega I) * b(t)}{\omega} = \frac{(\frac{\beta}{\alpha}\omega \cdot \alpha I) * b(t)}{\beta\omega} = \frac{(\frac{\beta}{\alpha}\omega \cdot \beta J) * b(t)}{\beta\omega} = \frac{(\frac{\beta}{\alpha}\omega \cdot J) * b(t)}{\omega}.$$

Now proceed as in the proof of Lemma V.3. □

The function $\ell_I(b(t);z)$ therefore only depends on the class of I in $Cl(A)$. Exactly as in the case of $L(s,\psi)$, we have

$$\ell(b(t);z) = \sum_{i=1}^{h} \ell_{\mathfrak{a}_i}(b(t);z), \tag{5.5}$$

and so the study of $\ell(b(t);z)$ reduces to studying the functions $\ell_{\mathfrak{a}_i}(b(t);z)$ for $1 \le i \le h$.

C. Preliminary Functions–Part 3

Let ρ be a Drinfeld-Hayes A-module with respect to a fixed sign function sgn. Since ρ is a Drinfeld A-module of rank one, then the period lattice of ρ has the form $\tilde{\pi}_\rho \mathfrak{I}$ where $\mathfrak{I}$ is an integral ideal of A and $\tilde{\pi}_\rho \in \mathbb{C}_\infty$. Recall that χ is our Dirichlet character on A with kernel $\mathfrak{m}$.

The map

$$\begin{aligned} A/\mathfrak{m} &\to \mathfrak{m}^{-1}\mathfrak{I}/\mathfrak{I} \\ a &\mapsto a\mu \end{aligned} \tag{5.6}$$

for $\mu \in \mathfrak{m}^{-1}\mathfrak{I} \setminus \mathfrak{I}$ is an isomorphism. We fix a choice of μ.

Definition V.7. Let I be an integral ideal of A. We define the function $\mathbf{e}_I \colon K_\infty \to \bar{K}_\infty$ by

$$\mathbf{e}_I(x) := \exp_{I*\rho}(D(\rho_I)\tilde{\pi}_\rho x).$$

Remark V.8. If $I = A = (1)$, then

$$\mathbf{e}_I(x) = \exp_{(1)*\rho}(D(\rho_{(1)})\tilde{\pi}_\rho x) = \exp_\rho(\tilde{\pi}_\rho x) = \mathbf{e}(x).$$

Let us investigate how the function $\mathbf{e}_I$ depends on the ideal I. If $I \sim J$ in $Cl(A)$, then $\alpha I = \beta J$ for some $\alpha, \beta \in A_+$ relatively prime to $\mathfrak{m}$. Set $\gamma := \beta/\alpha \in (J^{-1})_+$. Then

$$\mathbf{e}_I(x) = \mathbf{e}_{\gamma J}(x) = \exp_{\gamma J*\rho}(D(\rho_{\gamma J})\tilde{\pi}_\rho x)$$

Recall that $Cl(A)$ acts on $\mathrm{Drin}_A(\mathbb{C}_\infty)$ via the $*$ operation. Therefore, since $\gamma J \sim J$ in $Cl(A)$, it follows that $\gamma J * \rho$ and $J * \rho$ are isomorphic over $\mathbb{C}_\infty$. Hence, $\exp_{\gamma J*\rho}(z) = \exp_{J*\rho}(z)$ for all $z \in \mathbb{C}_\infty$. We have already seen that $D(\rho_{\gamma J}) = \gamma D(\rho_J)$. We conclude that

$$\mathbf{e}_I(x) = \mathbf{e}_J(\gamma x).$$

Recall that we have a function $l_m \colon K_\infty \to \bar{K}_\infty$ defined by

$$l_m(x) = \sum_{a \in A_+} \frac{\mathbf{e}(ax)^m}{a}$$

if $m > 0$ and

$$l_0(x) = \sum_{a \in A_+} \frac{1}{a}.$$

Suppose that $h = 1$ and consider

$$\ell(t^m; z) = \sum_{a \in A_+} \frac{(\rho_a(t))^m}{a} z^{q^{\deg a}}.$$

If we set $t = \mathbf{e}(x)$ and $z = 1$, we get

$$\begin{aligned} \ell(t^m; z)|_{t=\mathbf{e}(x), z=1} &= \sum_{a \in A_+} \frac{(\rho_a(\mathbf{e}(x)))^m}{a} \\ &= \sum_{a \in A_+} \frac{\mathbf{e}(ax)^m}{a} \\ &= l_m(x). \end{aligned} \tag{5.7}$$

We want to investigate the value of

$$\ell_I(t^m; z)|_{t=\mathbf{e}_A(a\mu), z=1} \tag{5.8}$$

for $a \in A$ and μ as in (5.6).

Lemma V.9 ([7], pp. 68-69)**.** *If I is an integral ideal of A, then*

$$\rho_I(\exp_\rho(z)) = \exp_{I*\rho}(D(\rho_I)z)$$

for all $z \in \mathbb{C}_\infty$.

In order to compute

$$\ell_I(t^m; z)|_{t=\mathbf{e}_A(a\mu), z=1},$$

we need to compute $(\omega I * t^m)|_{t=\mathbf{e}_A(a\mu)}$ for $\omega \in (I^{-1})_+$. We calculate

$$\begin{aligned}(\omega I * t^m)|_{t=\mathbf{e}_A(a\mu)} &= \rho_{\omega I}(\mathbf{e}_A(a\mu))^m \\ &= \exp_{\omega I*\rho}(D(\rho_{\omega I})\tilde{\pi}_\rho a\mu)^m \text{ by Lemma V.9.}\end{aligned}$$

We have already seen that $D(\rho_{\omega I}) = \omega D(\rho_I)$ and $\exp_{\omega I*\rho}(z) = \exp_{I*\rho}(z)$. Hence,

$$(\omega I * t^m)|_{t=\mathbf{e}_A(a\mu)} = \exp_{I*\rho}(D(\rho_I)\omega\tilde{\pi}_\rho a\mu)^m = \mathbf{e}_I(\omega a\mu)^m,$$

and so

$$\ell_I(t^m;z)|_{t=\mathbf{e}_A(a\mu),z=1} = \frac{1}{D(\rho_I)} \sum_{\omega\in(I^{-1})_+} \frac{\mathbf{e}_I(\omega a\mu)^m}{\omega}. \tag{5.9}$$

The similarities between the preceding expression and the analogous relation (5.7) in the case of $h = 1$ inspire our next definition.

Definition V.10. For $m \geq 0$, we set

$$l_{m,I}(x) := \frac{1}{D(\rho_I)} \sum_{\omega\in(I^{-1})_+} \frac{\mathbf{e}_I(\omega x)^m}{\omega}$$

for $x \in K_\infty$.

We note that $l_{m,A}(x) = l_m(x)$ since $D(\rho_A) = 1$, $A^{-1} = A$, and $\mathbf{e}_A(x) = \mathbf{e}(x)$.

D. Generalized Dual Coefficients

In this section, we will prove the following generalization of Proposition III.14. We first review the concept of $\mathfrak{m}$-torsion with respect to some of our preliminary functions. Let ρ be a Drinfeld-Hayes A-module with respect to a fixed sign function sgn. Recall that the period lattice of ρ is $\tilde{\pi}_\rho \mathfrak{I}$. The $\mathfrak{m}$-torsion points of ρ are (cf., [19], §2.4)

$$\begin{aligned}\rho[\mathfrak{m}] = \{\exp_\rho(\tilde{\pi}_\rho r) \mid r \in \mathfrak{m}^{-1}\mathfrak{I}/\mathfrak{I}\} &= \{\exp_\rho(\tilde{\pi}_\rho a\mu) \mid a \in A/\mathfrak{m}\} \\ &= \{\mathbf{e}_A(a\mu) \mid a \in A/\mathfrak{m}\}.\end{aligned}$$

Recall from §III.D and (3.6) how we considered the function $\mathbf{e}(x/\mathbf{p})$ as a function on $A/\mathbf{p}$. Similarly, we now consider $\mathbf{e}_A(x\mu)$ for $x \in A$. We claim that we may view $\mathbf{e}_A(x\mu)$ as a function on $A/\mathfrak{m}$. Let $a \in A$ and suppose $a \equiv a'$ mod $\mathfrak{m}$. Then $a = a' + \mathfrak{m}a''$ and so

$$\begin{aligned}\mathbf{e}_A(a\mu) = \exp_\rho(\tilde{\pi}_\rho \mu a' + \tilde{\pi}_\rho \mu \mathfrak{m} a'') &= \exp_\rho(\tilde{\pi}_\rho \mu a') + \exp_\rho(\tilde{\pi}_\rho \mu \mathfrak{m} a'') \\ &= \exp_\rho(\tilde{\pi}_\rho \mu a') \\ &= \mathbf{e}_A(a'\mu)\end{aligned}$$

since $\mu\mathfrak{m}a'' \in \mathfrak{I}$ and $\exp_\rho$ vanishes on $\tilde{\pi}_\rho\mathfrak{I}$. For $y \in A/\mathfrak{m}$, choose $y' \in A$ such that $y \equiv y'$ mod $\mathfrak{m}$. Set

$$\mathbf{e}_A(y\mu) := \mathbf{e}_A(y'\mu). \tag{5.10}$$

This definition is independent of the choice of y' as we have previously shown. Thus, we consider $\mathbf{e}_A(y\mu)$ as a function on $A/\mathfrak{m}$.

Now given an integral ideal I, we may consider the Drinfeld-Hayes A-module $I * \rho$ and consider its $\mathfrak{m}$-torsion. The period lattice of $I * \rho$ is $D(\rho_I)I^{-1}\tilde{\pi}_\rho\mathfrak{I}$ ([13], Chapter 13). We will compute the $\mathfrak{m}$-torsion points of $I * \rho$, denoted $I * \rho[\mathfrak{m}]$, by mimicking the above computation. We have

$$\begin{aligned}I * \rho[\mathfrak{m}] &= \{\exp_\rho(D(\rho_I)\tilde{\pi}_\rho r) \mid r \in \mathfrak{m}^{-1}I^{-1}\mathfrak{I}/I^{-1}\mathfrak{I}\} \\ &= \{\exp_\rho(\tilde{\pi}_\rho D(\rho_I)\omega\mu) \mid \omega \in I^{-1}/I^{-1}\mathfrak{m}\} \\ &= \{\mathbf{e}_I(\omega\mu) \mid \omega \in I^{-1}/I^{-1}\mathfrak{m}\}.\end{aligned}$$

As above, we consider $\mathbf{e}_I(x\mu)$ as a function on $I^{-1}/I^{-1}\mathfrak{m}$.

Recall that $K_{\mathfrak{m}} := H(\rho[\mathfrak{m}])$. It is known that $K_{\mathfrak{m}}$ is independent of the choice of Drinfeld-Hayes A-module ([10], §16). Since $I * \rho$ is also a Drinfeld-Hayes A-module, it follows that $I * \rho[\mathfrak{m}] \subseteq K_{\mathfrak{m}}$. We are now ready to state our generalization of Proposition III.14. Set $d := \deg \mathfrak{m}$.

Proposition V.11. *Let m be an integer such that $1 \leq m \leq q^d - 1$. Then for every m and for every nonzero $\alpha \in I^{-1}/I^{-1}\mathfrak{m}$, there exists a unique element $\mathbf{e}^*_{m,I}(\alpha) \in K_{\mathfrak{m}}$ such that*

$$\sum_{m=1}^{q^d-1} \mathbf{e}_I(\beta\mu)^m \mathbf{e}^*_{m,I}(\alpha) = D(\rho_I) \cdot \delta_{\beta,\alpha}$$

for all nonzero $\beta \in I^{-1}/I^{-1}\mathfrak{m}$.

Proof. The idea of the proof is to apply Lagrange interpolation to $I^{-1}/I^{-1}\mathfrak{m}$. For nonzero $\gamma \in I^{-1}/I^{-1}\mathfrak{m}$, consider the Lagrange basis polynomial

$$\mathcal{P}_\gamma(x) := \prod_{\substack{0 \neq \lambda \in I^{-1}/I^{-1}\mathfrak{m} \\ \lambda \neq \gamma}} \frac{x - \mathbf{e}_I(\lambda\mu)}{\mathbf{e}_I(\gamma\mu) - \mathbf{e}_I(\lambda\mu)}.$$

By the previous discussion, $\mathcal{P}_\gamma(x)$ is a well-defined polynomial in $K_{\mathfrak{m}}[x]$. The degree of $\mathcal{P}_\gamma(x)$ is $q^d - 1$ since $I^{-1}/I^{-1}\mathfrak{m} \cong A/\mathfrak{m}$. Also,

$$\mathcal{P}_\gamma(\mathbf{e}_I(\omega\mu)) = \delta_{\gamma,\omega}$$

for all $\omega \in I^{-1}/I^{-1}\mathfrak{m}$ by construction. For nonzero $\beta \in I^{-1}/I^{-1}\mathfrak{m}$, set

$$F_\beta(x) := D(\rho_I)\mathcal{P}_\beta(x).$$

Then

$$F_\beta(\mathbf{e}_I(\omega\mu)) = D(\rho_I)\delta_{\gamma,\omega}$$

for all $\omega \in I^{-1}/I^{-1}\mathfrak{m}$. Let

$$\mathcal{D}_\gamma := \prod_{\substack{0\neq\lambda\in I^{-1}/I^{-1}\mathfrak{m}\\ \lambda\neq\gamma}} (\mathbf{e}_I(\gamma\mu) - \mathbf{e}_I(\lambda\mu)).$$

Then $\mathcal{D}_\gamma^{-1}$ is the leading coefficient of $\mathcal{P}_\gamma(x)$. Now $\dfrac{\mathcal{D}_\beta}{D(\rho_I)} F_\beta(x) = \mathcal{D}_\beta \mathcal{P}_\beta(x)$ is a *monic* polynomial of degree $q^d - 1$ satisfying

(i) it vanishes on $Z_\beta := \{\mathbf{e}_I(\alpha\mu) \mid 0 \neq \alpha \in I^{-1}/I^{-1}\mathfrak{m}, \alpha \neq \beta\}$;

(ii) it equals $\mathcal{D}_\beta$ when $x = \mathbf{e}_I(\beta\mu)$.

Suppose $G_\beta(x)$ is another monic polynomial of degree $q^d - 1$ satisfying (i) and (ii). Then $\mathcal{D}_\beta\mathcal{P}_\beta(x) - G_\beta(x)$ vanishes on $Z_\beta \cup \{\mathbf{e}_I(\beta\mu)\}$ and has degree $< q^d - 1$. Since $\#(Z_\beta \cup \{\mathbf{e}_I(\beta\mu)\}) = q^d - 1$, it follows that $\mathcal{D}_\beta\mathcal{P}_\beta(x) - G_\beta(x)$ is identically zero. Hence, $\mathcal{D}_\beta\mathcal{P}_\beta(x)$ is the *unique* monic polynomial of degree $q^d - 1$ in $K_\mathfrak{m}[x]$ satisfying (i) and (ii). Equivalently, $F_\beta(x)$ is the unique polynomial of degree $q^d - 1$ in $K_\mathfrak{m}[x]$ that vanishes on Z_β and equals $D(\rho_I)$ when $x = \mathbf{e}_I(\beta\mu)$.

Write

$$F_\beta(x) =: \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,I}(\beta) x^m.$$

Then

$$F_\beta(\mathbf{e}_I(\alpha\mu)) = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ D(\rho_I) & \text{if } \alpha = \beta. \end{cases}$$

This completes the proof. $\square$

Corollary V.12. *Let m be an integer such that $1 \leq m \leq q^d - 1$ and let $0 \neq \alpha \in I^{-1}/I^{-1}\mathfrak{m}$. Suppose that J is an integral ideal of A such that $I \sim J$ in $Cl(A)$. Then*

$$\mathbf{e}^*_{m,I}(\alpha) = \gamma^{-1}\mathbf{e}^*_{m,J}(\alpha\gamma^{-1})$$

where $\gamma \in K_+$ is such that $I = \gamma^{-1}J$.

Proof. First note that $0 \neq \alpha\gamma^{-1} \in J^{-1}/J^{-1}\mathfrak{m}$ and so the expression $\mathbf{e}^*_{m,J}(\alpha\gamma^{-1})$ makes sense. Since $I^{-1} = \gamma J^{-1}$, $\alpha = \gamma\omega$ for some $\omega \in J^{-1}$. By the Proposition, we have

$$\begin{aligned} D(\rho_I) &= \sum_{m=1}^{q^d-1} \mathbf{e}_I(\alpha\mu)^m \mathbf{e}^*_{m,I}(\alpha) \\ &= \sum_{m=1}^{q^d-1} \mathbf{e}_I(\gamma\omega\mu)^m \mathbf{e}^*_{m,I}(\gamma\omega) \\ &= \sum_{m=1}^{q^d-1} \mathbf{e}_J(\omega\mu)^m \mathbf{e}^*_{m,I}(\gamma\omega) \end{aligned} \tag{5.11}$$

where the last equality follows from Remark V.8.

Applying the Proposition to $0 \neq \omega \in J^{-1}/J^{-1}\mathfrak{m}$, we get

$$\sum_{m=1}^{q^d-1} \mathbf{e}_J(\omega\mu)^m \mathbf{e}^*_{m,J}(\omega) = D(\rho_J) = \gamma D(\rho_I).$$

Therefore,

$$\sum_{m=1}^{q^d-1} \mathbf{e}_J(\omega\mu)^m \frac{\mathbf{e}^*_{m,J}(\omega)}{\gamma} = \sum_{m=1}^{q^d-1} \mathbf{e}_J(\omega\mu)^m \mathbf{e}^*_{m,I}(\gamma\omega).$$

By the uniqueness of the dual coefficients, we conclude

$$\frac{1}{\gamma}\mathbf{e}^*_{m,J}(\omega) = \mathbf{e}^*_{m,I}(\gamma\omega)$$

which is equivalent to

$$\frac{1}{\gamma}\mathbf{e}^*_{m,J}(\alpha\gamma^{-1}) = \mathbf{e}^*_{m,I}(\alpha).$$

□

E. Main Result

We are now ready to compute our analogue of (3.7). Let $0 \neq \alpha \in I^{-1}/I^{-1}\mathfrak{m}$ and $b \in (A/\mathfrak{m})^{\times}$. We calculate

$$\begin{aligned}
\sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,I}(\alpha) l_{m,I}(b\mu) &= \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,I}(\alpha) \frac{1}{D(\rho_I)} \sum_{\omega \in (I^{-1})_+} \frac{\mathbf{e}_I(\omega b\mu)^m}{\omega} \\
&= \sum_{\omega \in (I^{-1})_+} \frac{1}{D(\rho_I)} \frac{1}{\omega} \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,I}(\alpha) \mathbf{e}_I(\omega b\mu)^m \\
&= \sum_{\substack{\omega \in (I^{-1})_+ \\ \omega \notin I^{-1}\mathfrak{m}}} \frac{1}{D(\rho_I)} \frac{1}{\omega} D(\rho_I) \delta_{\alpha,\omega b} \quad \text{by Proposition V.11} \\
&= \sum_{\substack{\omega \in (I^{-1})_+ \\ \omega \notin I^{-1}\mathfrak{m} \\ \alpha \equiv \omega b \pmod{I^{-1}\mathfrak{m}}}} \frac{1}{\omega} \\
&= I^{[1]} \zeta_{I,\alpha b^{-1}}(1), \qquad (5.12)
\end{aligned}$$

and so

$$\frac{1}{I^{[1]}} \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,I}(\alpha) l_{m,I}(b\mu) = \zeta_{I,\alpha b^{-1}}(1). \qquad (5.13)$$

The map

$$\begin{aligned}
I^{-1}/I^{-1}\mathfrak{m} &\to A/\mathfrak{m} \qquad (5.14) \\
\alpha &\mapsto \alpha\nu
\end{aligned}$$

for $\nu \in I \setminus I\mathfrak{m}$ is an isomorphism. We fix a choice of ν with the added condition that $\psi(\nu) = 1$.

We multiply both sides of (5.13) by $\psi(\alpha\nu)\psi(b)^{-1}$ to get

$$\psi(\alpha\nu)\psi(b)^{-1} \frac{1}{I^{[1]}} \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,I}(\alpha) l_{m,I}(b\mu) = \psi(\alpha\nu)\psi(b)^{-1} \zeta_{I,\alpha b^{-1}}(1). \qquad (5.15)$$

Set $a := \alpha\nu$ and notice that $a \in (A/\mathfrak{m})^\times$. We have

$$\psi(I) \sum_{a,b \in (A/\mathfrak{m})^\times} \psi(a)\psi(b)^{-1} \frac{1}{I^{[1]}} \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,I}(\alpha) l_{m,I}(b\mu)$$
$$= \psi(I) \sum_{a,b \in (A/\mathfrak{m})^\times} \psi(a)\psi(b)^{-1} \zeta_{I,\alpha b^{-1}}(1). \quad (5.16)$$

The righthand side of (5.16), after some rearranging and appealing to the definition of ζ, is

$$\frac{\psi(I)}{I^{[1]}} \sum_{b \in (A/\mathfrak{m})^\times} \psi(b)^{-1} \sum_{a \in (A/\mathfrak{m})^\times} \sum_{\substack{n \in (I^{-1})_+ \\ \alpha \equiv bn \bmod I^{-1}\mathfrak{m}}} \frac{\psi(a)}{n}. \quad (5.17)$$

Now $\alpha \equiv bn \bmod I^{-1}\mathfrak{m}$ implies $a \equiv bn\nu \bmod I^{-1}\mathfrak{m}$. These congruences are equivalent provided $\nu \notin I^{-1}\mathfrak{m}$, which follows from the fact that $\nu \in I \setminus I\mathfrak{m}$. And the congruence $a \equiv bn\nu \bmod I^{-1}\mathfrak{m}$ is equivalent to $a \equiv bn\nu \bmod \mathfrak{m}$ since $n\nu \in A$. Therefore, (5.17) becomes

$$\begin{aligned}
&\frac{\psi(I)}{I^{[1]}} \sum_{b \in (A/\mathfrak{m})^\times} \psi(b)^{-1} \sum_{a \in (A/\mathfrak{m})^\times} \sum_{\substack{n \in (I^{-1})_+ \\ a \equiv bn\nu \bmod \mathfrak{m}}} \frac{\psi(a)}{n} \\
&\qquad = \frac{\psi(I)}{I^{[1]}} \sum_{b \in (A/\mathfrak{m})^\times} \psi(b)^{-1} \sum_{n \in (I^{-1})_+} \frac{\psi(bn\nu)}{n} \\
&\qquad = \frac{-\psi(I)}{I^{[1]}} \psi(\nu) \sum_{n \in (I^{-1})_+} \frac{\psi(n)}{n} \\
&\qquad = -L_I(1, \psi)
\end{aligned}$$

where we used the facts that (i) the kernel of ψ is $\mathfrak{m}$, (ii) $\#(A/\mathfrak{m})^\times = q^d - 1 = -1 \bmod p$, and (iii) $\psi(\nu) = 1$ respectively. Plugging this back into (5.16) and rearranging the left hand side yields

$$L_I(1,\psi) = -\psi(I) \sum_{m=1}^{q^d-1} \left(\frac{1}{I^{[1]}} \sum_{a \in (A/\mathfrak{m})^\times} \psi(a) \mathbf{e}^*_{m,I}(\alpha) \right) \left(\sum_{b \in (A/\mathfrak{m})^\times} \psi(b)^{-1} l_{m,I}(b\mu) \right).$$

We immediately deduce our desired analogue of Anderson's Equation (3.7).

Proposition V.13. *Let χ be a Dirichlet character on A with kernel $\mathfrak{m}$. Let $\psi\colon \mathfrak{F}_{\mathfrak{m}}(A) \to \mathbb{C}_\infty$ be the character (as in §II.G) such that $\psi|_A = \chi$. Let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_h\}$ be a set of representatives of the equivalence classes of $Cl(A)$ that are relatively prime to $\mathfrak{m}$ where $h = \#Cl(A)$. Let μ be as in* (5.6) *and let $d = \deg \mathfrak{m}$. Then*

$$\begin{aligned} L(1,\psi) &= \sum_{j=1}^{h} L_{\mathfrak{a}_j}(1,\psi) \\ &= \sum_{j=1}^{h} \left[-\psi(\mathfrak{a}_j) \sum_{m=1}^{q^d-1} \left(\frac{1}{\mathfrak{a}_j^{[1]}} \sum_{a\in(A/\mathfrak{m})^\times} \psi(a)\mathbf{e}^*_{m,\mathfrak{a}_j}(a/\nu_j) \right) \left(\sum_{b\in(A/\mathfrak{m})^\times} \psi(b)^{-1} l_{m,\mathfrak{a}_j}(b\mu) \right) \right] \end{aligned}$$

where $\nu_j \in \mathfrak{a}_j \setminus \mathfrak{a}_j\mathfrak{m}$ are chosen as in (5.14).

F. The Log-algebraicity of $L(1,\psi)$

We continue with the notations of the previous sections. By Theorem III.2, we have that

$$\exp_\rho \ell(b(t);z) = \exp_\rho \left(\sum_{j=1}^{h} \ell_{\mathfrak{a}_j}(b(t);z) \right) = \sum_{j=1}^{h} \exp_\rho \ell_{\mathfrak{a}_j}(b(t);z)$$

is a polynomial in $B[t,z]$ for all $b(t) \in B[t]$. But this does not imply that $\exp_\rho \ell_{\mathfrak{a}_j}(b(t);z)$ is a polynomial in $B[t,z]$ for any j. As in the class number one case, to study the log-algebraicity of $L(1,\psi)$, we must look at $l_{m,\mathfrak{a}_j}(x)$ because of Proposition V.13.

Let $S(t^m,z) := \exp_\rho \ell(t^m;z)$ which is a polynomial in $B[t,z]$ by Theorem III.2. For $a \in A$, we get

$$S(t^m,z)|_{t=\mathbf{e}_A(a\mu),z=1} = \exp_\rho \left(\sum_{j=1}^{h} \ell_{\mathfrak{a}_j}(t^m;z)|_{t=\mathbf{e}_A(a\mu),z=1} \right) = \exp_\rho \left(\sum_{j=1}^{h} l_{m,\mathfrak{a}_j}(a\mu) \right)$$

so that

$$\sum_{j=1}^{h} l_{m,\mathfrak{a}_j}(a\mu) = \log_\rho \left(S(t^m,z)|_{t=\mathbf{e}_A(a\mu),z=1} \right). \tag{5.18}$$

We would like to determine whether *each* $l_{m,\mathfrak{a}_j}(a\mu)$ can be expressed in a way similar to (5.18).

For every $\beta \in B$ and for every integral ideal I of A, we have

$$\ell_I(\beta t^m; z)|_{t=\mathbf{e}_A(a\mu), z=1} = \frac{1}{D(\rho_I)} \sum_{\omega \in (I^{-1})_+} \frac{\beta^{(\omega I, H/K)} \mathbf{e}_I(\omega a\mu)^m}{\omega}$$

since $(\omega I * \beta t^m) = \beta^{(\omega I, H/K)} \rho_{\omega I}(t)^m$. Now $\omega I \sim I$ in $Cl(A)$. Since $Cl(A) \cong \mathrm{Gal}(H/K)$, it follows that $(\omega I, H/K) = (I, H/K)$. Therefore,

$$\ell_I(\beta t^m; z)|_{t=\mathbf{e}_A(a\mu), z=1} = \beta^{(I, H/K)} l_{m,I}(a\mu).$$

Hence,

$$\sum_{j=1}^{h} \beta^{(\mathfrak{a}_j, H/K)} l_{m,\mathfrak{a}_j}(a\mu) = \log_\rho(\text{Polynomial}|_{t=\mathbf{e}_A(a\mu), z=1}). \tag{5.19}$$

So we now investigate sums which look like the lefthand side of the preceding expression.

For $\beta \in B$, set

$$L_m(\beta, x) := \sum_{j=1}^{h} \beta^{(\mathfrak{a}_j, H/K)} l_{m,\mathfrak{a}_j}(x)$$

where x is a variable. Let $\{\beta_1, \ldots, \beta_h\}$ be a subset of B which is linearly independent over A. Consider the system

$$\begin{pmatrix} L_m(\beta_1, x) \\ L_m(\beta_2, x) \\ \vdots \\ L_m(\beta_h, x) \end{pmatrix} = \begin{pmatrix} \beta_1^{(\mathfrak{a}_1, H/K)} & \beta_1^{(\mathfrak{a}_2, H/K)} & \ldots & \beta_1^{(\mathfrak{a}_h, H/K)} \\ \beta_2^{(\mathfrak{a}_1, H/K)} & \beta_2^{(\mathfrak{a}_2, H/K)} & \ldots & \beta_2^{(\mathfrak{a}_h, H/K)} \\ & & \vdots & \\ \beta_h^{(\mathfrak{a}_1, H/K)} & \beta_h^{(\mathfrak{a}_2, H/K)} & \ldots & \beta_h^{(\mathfrak{a}_h, H/K)} \end{pmatrix} \begin{pmatrix} l_{m,\mathfrak{a}_1}(x) \\ l_{m,\mathfrak{a}_2}(x) \\ \vdots \\ l_{m,\mathfrak{a}_h}(x) \end{pmatrix}.$$

Denote by $\mathcal{D}$ the h by h matrix on the right. Since $\{\beta_1, \ldots, \beta_h\}$ are linearly independent over A, the determinant of $\mathcal{D}$ is nonzero. By Cramer's Rule, we conclude

$$l_{m,\mathfrak{a}_j}(x) = (\det \mathcal{D})^{-1} \sum_{i=1}^{h} (-1)^{i+j} L_m(\beta_i, x) \det \mathcal{D}_{ij}$$

for all $1 \leq j \leq h$ where $\mathcal{D}_{ij}$ denotes the ij-minor of $\mathcal{D}$. Hence,

$$l_{m,\mathfrak{a}_j}(a\mu) = (\det \mathcal{D})^{-1} \sum_{i=1}^{h} (-1)^{i+j} L_m(\beta_i, a\mu) \det \mathcal{D}_{ij}. \tag{5.20}$$

This, coupled with (5.19), implies the desired log-algebraic expression for $l_{m,\mathfrak{a}_j}(a\mu)$. We record our result in the form of the following Theorem.

Theorem V.14. *Let ψ be as in Proposition V.13. Then there exist $u_1, \ldots, u_s \in \mathbb{C}_\infty$ with $\exp_\rho(u_i) \in \bar{K}$ and $\alpha_1, \ldots, \alpha_s \in \bar{K}$ such that*

$$L(1, \psi) = \sum_{i=1}^{s} \alpha_i u_i.$$

In determining the log-algebraicity of $L(1, \psi)$, we used the expression given in Proposition V.13 as a starting point. We now briefly describe another way to prove Theorem V.14 by determining a simpler expression for $L(1, \psi)$.

Recall (5.13):

$$\frac{1}{I^{[1]}} \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,I}(\alpha) l_{m,I}(b\mu) = \zeta_{I,\alpha b^{-1}}(1)$$

where $0 \neq \alpha \in I^{-1}/I^{-1}\mathfrak{m}$ and $b \in (A/\mathfrak{m})^\times$. We compute

$$\begin{aligned}
L(1, \psi) &= \sum_{j=1}^{h} L_{\mathfrak{a}_j}(1, \psi) \\
&= \sum_{j=1}^{h} \psi(\mathfrak{a}_j) \sum_{0 \neq c \in \mathfrak{a}_j^{-1}/\mathfrak{a}_j^{-1}\mathfrak{m}} \psi(c) \zeta_{\mathfrak{a}_j,c}(1) \quad \text{(by Remark V.2)} \\
&= \sum_{j=1}^{h} \psi(\mathfrak{a}_j) \sum_{0 \neq c \in \mathfrak{a}_j^{-1}/\mathfrak{a}_j^{-1}\mathfrak{m}} \psi(c) \left(\frac{1}{\mathfrak{a}_j^{[1]}} \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,\mathfrak{a}_j}(c) l_{m,\mathfrak{a}_j}(\mu) \right) \\
&\quad \text{((5.13) with } I = \mathfrak{a}_j \text{ and } b = 1) \\
&= \sum_{m=1}^{q^d-1} \left(\sum_{j=1}^{h} \frac{\psi(\mathfrak{a}_j)}{\mathfrak{a}_j^{[1]}} \sum_{0 \neq c \in \mathfrak{a}_j^{-1}/\mathfrak{a}_j^{-1}\mathfrak{m}} \psi(c) \mathbf{e}^*_{m,\mathfrak{a}_j}(c) \right) l_{m,\mathfrak{a}_j}(\mu).
\end{aligned}$$

This expression and (5.20) immediately imply Theorem V.14.

CHAPTER VI

THE A-RANK THEOREM WHEN $h_A = 1$ AND $d_\infty = 1$

In this chapter, we will prove our analogue of the A-rank Theorem in the case of a function field satisfying $h_A = 1$ and $d_\infty = 1$. Recall from §IV.A that there are only four such function fields which we called $K_1, \ldots, K_4$. We refer the reader to §IV.A for the remaining notation. The reader should assume that all objects A, ρ, etc., written without the index j, correspond to one of A_j, ρ^j, etc. for $1 \leq j \leq 4$. We follow the argument of §4.7 and §4.8 in [2].

Let $\mathfrak{m}$ be an ideal of A. Since $h_A = 1$ and $d_\infty = 1$, we have $H^+ = H = K$. Let ρ be a Drinfeld-Hayes A-module with respect to a fixed sign function sgn. Set $K_\mathfrak{m} := K(\rho[\mathfrak{m}]), G := \mathrm{Gal}(K_\mathfrak{m}/K)$, and $G'' := \mathrm{Gal}(K_\mathfrak{m}/K)$. Then $G = G'' \cong (A/\mathfrak{m})^\times$ and so $\hat{G} = \hat{G}''$. Since A is a principal ideal domain, there exists an irreducible $\mathbf{p} \in A_+$ such that $\mathfrak{m} = (\mathbf{p})$. Then $A/\mathfrak{m} = A/\mathbf{p} =: \mathbf{F_p}$ is a field. Set $d := \deg \mathbf{p}$.

Recall from (5.6) that the element $\mu \in (\mathbf{p})^{-1} \setminus A$ is chosen such that

$$A/(\mathbf{p}) \to (\mathbf{p})^{-1}/A$$

$$a \mapsto a\mu$$

is an isomorphism. We choose $\mu = 1/\mathbf{p}$.

Let $a \in (\mathbf{p})^{-1}/A$, $b \in \mathbf{F}_\mathbf{p}^\times$ and consider Equation (5.12) with $I = (\mathbf{p})$:

$$\frac{1}{\mathbf{p}} \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,(\mathbf{p})}(\alpha) l_{m,(\mathbf{p})}(b/\mathbf{p}) = \zeta_{(\mathbf{p}),\alpha b^{-1}}(1). \tag{6.1}$$

Since $(\mathbf{p}) = \mathbf{p}A$, Corollary V.12 implies

$$\mathbf{e}^*_{m,(\mathbf{p})}(\alpha) = \mathbf{p}\mathbf{e}^*_{m,A}(\alpha\mathbf{p}).$$

Remark V.8 implies that

$$\mathbf{e}_{(\mathbf{p})}(x) = \mathbf{e}_A(x)$$

and so

$$\begin{aligned}
l_{m,(\mathbf{p})}(b/\mathbf{p}) &= \frac{1}{\mathbf{p}} \sum_{\omega \in (\mathbf{p})_+^{-1}} \frac{\mathbf{e}_{(\mathbf{p})}(\omega b/\mathbf{p})^m}{\omega} \\
&= \frac{1}{\mathbf{p}} \sum_{\omega \in (\mathbf{p})_+^{-1}} \frac{\mathbf{e}_A(\omega b)^m}{\omega} \\
&= \frac{1}{\mathbf{p}} \sum_{n \in A_+} \frac{\mathbf{e}_A(nb/\mathbf{p})^m}{\frac{1}{\mathbf{p}} n} \quad \text{change variables } \omega \to (1/\mathbf{p})n \\
&= \sum_{n \in A_+} \frac{\mathbf{e}_A(nb/\mathbf{p})^m}{n} \\
&= l_{m,A}(b/\mathbf{p}).
\end{aligned}$$

We also have

$$\begin{aligned}
\zeta_{(\mathbf{p}),\alpha b^{-1}}(1) &= \frac{1}{\mathbf{p}} \sum_{\substack{n \in (\mathbf{p})_+^{-1} \\ n \equiv \alpha b^{-1} \bmod A}} \frac{1}{n} \quad \text{by definition of } \zeta \\
&= \frac{1}{\mathbf{p}} \sum_{\substack{c \in A_+ \\ (c,\mathbf{p})=1 \\ bc \equiv \alpha \mathbf{p} \bmod \mathbf{p}}} \frac{1}{\frac{1}{\mathbf{p}} c} \quad n \to (1/\mathbf{p})c.
\end{aligned}$$

First note that the congruence

$$nc = \frac{1}{\mathbf{p}} c \equiv \alpha b^{-1} \bmod A$$

is equivalent to

$$bc \equiv \alpha \mathbf{p} \bmod \mathbf{p}.$$

The constraint $(c,\mathbf{p})=1$ arises since $\alpha\mathbf{p} \bmod \mathbf{p} \neq 0$. Therefore,

$$\zeta_{(\mathbf{p}),\alpha b^{-1}}(1) = \sum_{\substack{c\in A_+ \\ (c,\mathbf{p})=1 \\ bc\equiv \alpha\mathbf{p} \bmod \mathbf{p}}} \frac{1}{c} = \zeta_{A,\alpha\mathbf{p}b^{-1}}(1).$$

Putting everything together, (6.1) becomes

$$\sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,A}(\alpha\mathbf{p}) l_{m,A}(b/\mathbf{p}) = \zeta_{A,\alpha\mathbf{p}b^{-1}}(1). \tag{6.2}$$

Set $a := \alpha\mathbf{p} \in \mathbf{F}_{\mathbf{p}}^{\times}$. The group $G \cong \mathbf{F}_{\mathbf{p}}^{\times}$ is cyclic implies that $\hat{G}$ is cyclic. Let ϕ be a generator of $\hat{G}$. Choose $1 \leq i, j \leq q^d - 1$. Multiply both sides of (6.2) by $\phi^i(a)\phi^{-j}(b)$ and sum over a and b. We get

$$\sum_{a,b\in\mathbf{F}_{\mathbf{p}}^{\times}} \phi^i(a)\phi^{-j}(b) \sum_{m=1}^{q^d-1} \mathbf{e}^*_{m,A}(a) l_{m,A}(b/\mathbf{p}) = \sum_{a,b\in\mathbf{F}_{\mathbf{p}}^{\times}} \phi^i(a)\phi^{-j}(b)\zeta_{A,ab^{-1}}(1). \tag{6.3}$$

The left hand side of (6.3) becomes

$$\sum_{m=1}^{q^d-1} \left(\sum_{a\in\mathbf{F}_{\mathbf{p}}^{\times}} \phi^i(a)\mathbf{e}^*_{m,A}(a) \right) \left(\sum_{b\in\mathbf{F}_{\mathbf{p}}^{\times}} \phi^{-j}(b) l_{m,A}(b/\mathbf{p}) \right). \tag{6.4}$$

The right hand side of (6.3) is

$$\begin{aligned}
\sum_{a,b\in\mathbf{F}_{\mathbf{p}}^{\times}} \phi^i(a)\phi^{-j}(b)\zeta_{A,ab^{-1}}(1) &= \sum_{a,b\in\mathbf{F}_{\mathbf{p}}^{\times}} \phi^i(a)\phi^{-j}(b) \sum_{\substack{n\in A_+ \\ bn\equiv a \bmod \mathbf{p}}} \frac{1}{n} \\
&= \sum_{a,b\in\mathbf{F}_{\mathbf{p}}^{\times}} \phi^{-j}(b) \sum_{\substack{n\in A_+ \\ bn\equiv a \bmod \mathbf{p}}} \frac{\phi^i(a)}{n} \\
&= \sum_{a,b\in\mathbf{F}_{\mathbf{p}}^{\times}} \phi^{-j}(b) \sum_{n\in A_+} \frac{\phi^i(bn)}{n} \\
&\quad \text{(the value of } \phi(a) \text{ is determined by } a \bmod \mathbf{p}) \\
&= \left(\sum_{a\in\mathbf{F}_{\mathbf{p}}^{\times}} 1 \right) \sum_{b\in\mathbf{F}_{\mathbf{p}}^{\times}} \phi^{-j}(b)\phi^i(b) \sum_{n\in A_+} \frac{\phi^i(n)}{n}
\end{aligned}$$

$$= -\sum_{b \in \mathbf{F}_\mathbf{p}^\times} \phi^{-j}(b)\phi^i(b) \sum_{n \in A_+} \frac{\phi^i(n)}{n} \quad (\text{since } \# \mathbf{F}_\mathbf{p}^\times = q^d - 1)$$

$$= -L_A(1, \phi^i) \sum_{b \in \mathbf{F}_\mathbf{p}^\times} \phi^{-j}(b)\phi^i(b)$$

$$= -L_A(1, \phi^i)(q^d - 1)\delta_{ij}$$

(by the orthogonality relations, [13], Proposition 4.2)

$$= L_A(1, \phi^i)\delta_{ij}.$$

Again, $\delta_{ij} = 0$ if $i \neq j$ and 1 if $i = j$. Hence, we have

$$L_A(1, \phi^i)\delta_{ij} = \sum_{m=1}^{q^d-1} \left(\sum_{a \in \mathbf{F}_\mathbf{p}^\times} \phi^i(a)\mathbf{e}^*_{m,A}(a) \right) \left(\sum_{b \in \mathbf{F}_\mathbf{p}^\times} \phi^{-j}(b) l_{m,A}(b/\mathbf{p}) \right) \tag{6.5}$$

for all $1 \leq i, j \leq q^d - 1$.

Lemma VI.1. *Let ω be any generator of $\mathbf{F}_\mathbf{p}^\times$. Let Y be the $(q^d - 1)$ by $(q^d - 1)$ matrix whose i, j entry is $l_{i,A}(\omega^j/\mathbf{p})$. Then* $\det Y \neq 0$.

Proof. Let R be the $(q^d - 1)$ by $(q^d - 1)$ matrix whose i, j entry is the right hand side of (6.5). Let R_1 be the $(q^d - 1)$ by $(q^d - 1)$ matrix whose r, s entry is

$$\sum_{a \in \mathbf{F}_\mathbf{p}^\times} \phi^r(a)\mathbf{e}^*_{s,A}(a)$$

and let R_2 be the $(q^d - 1)$ by $(q^d - 1)$ matrix whose t, u entry is

$$\sum_{b \in \mathbf{F}_\mathbf{p}^\times} \phi^{-u}(b) l_{t,A}(b/\mathbf{p}).$$

Notice that $R = R_1 R_2$. If we write

$$\sum_{b \in \mathbf{F}_\mathbf{p}^\times} \phi^{-u}(b) l_{t,A}(b/\mathbf{p}) = \sum_{k=1}^{q^d-1} \phi^{-u}(\omega^k) l_{t,A}(\omega^k/\mathbf{p}),$$

then

$$R_2 = Y \begin{pmatrix} \phi^{-1}(\omega) & \dots & \phi^{-(q^d-1)}(\omega) \\ & \vdots & \\ \phi^{-1}(\omega^{q^d-1}) & \dots & \phi^{-(q^d-1)}(\omega^{q^d-1}) \end{pmatrix}.$$

If $\det Y = 0$, then

$$0 = \det R = \prod_{i=1}^{q^d-1} L_A(1,\phi^i).$$

But $L_A(1,\phi^i)$ does not vanish for all $1 \leq i \leq q^d - 1$. This contradiction completes the proof.

□

Let π be a fixed monic uniformizer at ∞.

Lemma VI.2. *Let m be a nonnegative integer such that $m \not\equiv 1 \bmod q-1$ and let $x \in K_\infty$. Let Tr denote the trace from $K_\infty \cdot \sqrt[q-1]{-\pi^{-1}}$ to K_∞. Then $\mathrm{Tr}(l_{m,A}(x)/\tilde{\pi}_\rho) = 0$.*

Remark VI.3. If $q = 2$, then no value of m satisfies the hypothesis of the Lemma. Also, if $q = 2$, then $K_\infty \cdot \sqrt[q-1]{-\pi^{-1}} = K_\infty$ since $\tilde{\pi}_\rho^{q-1} \in K_\infty$ for all q.

Proof. We claim that if n is a nonnegative integer such that $n \not\equiv 0 \bmod q-1$, then $\mathrm{Tr}(\tilde{\pi}_\rho^{-n}) = 0$. Write $n = n_1 + r(q-1)$ where $0 < n_1 < q-1$. Since $\tilde{\pi}_\rho^{q-1} \in K_\infty$, we have

$$\mathrm{Tr}(\tilde{\pi}_\rho^{-n}) = \mathrm{Tr}(\tilde{\pi}_\rho^{-n_1}\tilde{\pi}_\rho^{-r(q-1)}) = \tilde{\pi}_\rho^{-r(q-1)}\mathrm{Tr}(\tilde{\pi}_\rho^{-n_1}).$$

Recall from (4.7) that

$$K_\infty \cdot \tilde{\pi}_\rho = K_\infty \cdot \sqrt[q-1]{-\pi^{-1}}.$$

So $\tilde{\pi}_\rho = k \sqrt[q-1]{-\pi^{-1}}$ for some $k \in K_\infty$. We have

$$\mathrm{Tr}(\tilde{\pi}_\rho^{-n_1}) = k^{-n_1}\mathrm{Tr}((\sqrt[q-1]{-\pi^{-1}})^{-n_1}) = k^{-n_1}\sum_\sigma \sigma\left((\sqrt[q-1]{-\pi^{-1}})^{-n_1}\right)$$

where the sum ranges over all $\sigma \in \mathrm{Gal}(K_\infty \cdot \sqrt[q-1]{-\pi^{-1}}/K_\infty)$.

An element of $\mathrm{Gal}(K_\infty \cdot \sqrt[q-1]{-\pi^{-1}}/K_\infty)$ permutes the roots of the equation $z^{q-1} +$

$\pi^{-1} = 0$. And the roots of this equation are

$$\{c \sqrt[q-1]{-\pi^{-1}} \mid c \in \mathbb{F}_q^\times \}.$$

Thus,

$$\mathrm{Tr}(\tilde{\pi}_\rho^{-n_1}) = k^{-n_1} \sum_{c \in \mathbb{F}_q^\times} (c \sqrt[q-1]{-\pi^{-1}})^{-n_1} = \tilde{\pi}_\rho^{-n_1} \sum_{c \in \mathbb{F}_q^\times} c^{-n_1}.$$

Note that inversion is an automorphism of $\mathbb{F}_q^\times$ and that $\mathbb{F}_q^\times$ is a cyclic group. Let $g \neq 1$ be a generator. Then

$$\begin{aligned}
\sum_{c \in \mathbb{F}_q^\times} c^{-n_1} = \sum_{c \in \mathbb{F}_q^\times} c^{n_1} = \sum_{i=1}^{q-1} (g^i)^{n_1} &= \frac{g^{qn_1} - g^{n_1}}{g^{n_1} - 1} \\
&= \frac{g^{n_1}(g^{(q-1)n_1} - g^{n_1})}{g^{n_1} - 1} \\
&= \frac{g^{n_1}(g^{n_1} - g^{n_1})}{g^{n_1} - 1} \\
&= 0.
\end{aligned}$$

This proves the claim.

We may now finish the proof. Since $l_{m,A}(x)/\tilde{\pi}_\rho \in K_\infty$, we conclude

$$\mathrm{Tr}\left(\frac{l_{m,A}(x)}{\tilde{\pi}_\rho^m}\right) = \mathrm{Tr}\left(\frac{l_{m,A}(x)}{\tilde{\pi}_\rho} \frac{1}{\tilde{\pi}_\rho^{m-1}}\right) = \frac{l_{m,A}(x)}{\tilde{\pi}_\rho} \mathrm{Tr}\left(\frac{1}{\tilde{\pi}_\rho^{m-1}}\right) = 0$$

if $m - 1 \not\equiv 0 \bmod q - 1$ by our earlier claim. This concludes the proof. □

Recall from §IV.H the set

$$\mathcal{M} = \{m \in \mathbb{Z} \mid 1 \leq m \leq q^d - 1, m \not\equiv 1 \bmod q - 1\}.$$

We compute $\#\mathcal{M}$. Note that

$$\#\mathcal{M} = q^d - 1 - \#\{1 \leq m \leq q^d - 1 \mid m \equiv 1 \bmod q - 1\}.$$

If $1 \le m \le q^d - 1$ and $m \equiv 1 \mod q-1$, then $m = 1 + k(q-1)$ with $0 \le k \le (q^d-2)/(q-1)$. Therefore,

$$\#\mathcal{M} = q^d - 1 - \left(\left\lfloor \frac{q^d-1}{q-1} \right\rfloor + 1 \right) = q^d - 2 - \left\lfloor \frac{q^d-1}{q-1} \right\rfloor .$$

Since

$$\frac{q^d-2}{q-1} = \frac{q^d-1}{q-1} - \frac{1}{q-1},$$

we have

$$\left\lfloor \frac{q^d-1}{q-1} \right\rfloor = \begin{cases} 2^d - 2 & \text{if } q = 2 \\ \frac{q^d-1}{q-1} - 1 & \text{if } q > 2. \end{cases}$$

Therefore,

$$\#\mathcal{M} = \begin{cases} 0 & \text{if } q = 2 \\ (q^d-1)\frac{q-2}{q-1} & \text{if } q > 2. \end{cases}$$

We are now ready to state our analogue of the A-rank theorem.

Theorem VI.4. *Let K be a function field over $\mathbb{F}_q$ (other than the rational function field) satisfying $h_A = 1$ and $d_\infty = 1$. Let ρ be the unique Drinfeld-Hayes A-module with respect to a fixed sign function* sgn*. Let $\mathcal{S}$ be the A-module of special points of ρ as defined in §IV.H. Then the A-rank of $\mathcal{S}$ equals $(q^d-1)(q-2)/(q-1)$.*

Proof. We begin by showing that the special point

$$s_1(1) := \exp_\rho l_{m,A}(1/\mathbf{p})$$

is annihilated by $\mathbf{p}$. That is, we must show

$$\mathbf{p} * s_1(1) := \rho_{\mathbf{p}}(s_1(1)) = 0.$$

From the proof of Proposition III.8(6), we have

$$\frac{l_{1,A}(1/\mathbf{p})}{\tilde{\pi}_\rho} = -\sum_{b \in (A/\mathbf{p})^\times} \frac{1}{\mathbf{p}} = -\#(A/\mathbf{p})^\times \frac{1}{\mathbf{p}} = -(q^d-1)\frac{1}{\mathbf{p}} = \frac{1}{\mathbf{p}}.$$

Then

$$s_1(1) = \exp_\rho(\tilde{\pi}_\rho/\mathbf{p}) = \mathbf{e}(1/\mathbf{p})$$

and so

$$\rho_{\mathbf{p}}(s_1(1)) = \rho_{\mathbf{p}}(\mathbf{e}(1/\mathbf{p}) = \mathbf{e}(1) = 0.$$

Thus, $s_1(1)$ is annihilated by $\mathbf{p}$. Therefore, to prove the theorem, by Proposition IV.23, it is enough to show that the set of special points $\{s_m(1) \mid m \in \mathcal{M}\}$ is linearly independent over A.

Let $\{a_m \mid m \in \mathcal{M}\} \subset A$ be such that

$$\sum_{m\in\mathcal{M}} a_m * s_m(1) = 0. \tag{6.6}$$

Recall that

$$a \mapsto \sigma_a \colon \mathbf{F}_{\mathbf{p}}^\times \to \mathrm{Gal}(K_{\mathbf{p}}/K)$$

is the unique isomorphism such that

$$\sigma_a(\mathbf{e}(b/\mathbf{p})) = \mathbf{e}(ab/\mathbf{p})$$

for all $a, b \in \mathbf{F}_{\mathbf{p}}^\times$. Also, we have that

$$\sigma_a(s_m(b)) = s_m(ab).$$

For any $b \in \mathbf{F}_{\mathbf{p}}^\times$, apply σ_b to both sides of (6.6):

$$\sum_{m\in\mathcal{M}} a_m * s_m(b) = 0.$$

By the definition of $s_m(b)$ (cf., §IV.H), we have

$$\sum_{m\in\mathcal{M}} a_m * \exp_\rho(l_{m,A}(b/\mathbf{p})) = 0$$

which implies

$$\sum_{m\in\mathcal{M}} \rho_{a_m}(\exp_\rho(l_{m,A}(b/\mathbf{p}))) = 0.$$

Therefore,

$$0 = \sum_{m\in\mathcal{M}} \exp_\rho(a_m l_{m,A}(b/\mathbf{p})) = \exp_\rho\left(\sum_{m\in\mathcal{M}} a_m l_{m,A}(b/\mathbf{p})\right).$$

Hence,

$$\sum_{m\in\mathcal{M}} a_m l_{m,A}(b/\mathbf{p}) \in \tilde{\pi}_\rho A$$

so that

$$\sum_{m\in\mathcal{M}} a_m l_{m,A}(b/\mathbf{p})/\tilde{\pi}_\rho = a'.$$

for some $a' \in A$.

Let Tr denote the trace as in Lemma VI.2. Then

$$a' = \mathrm{Tr}(a') = \sum_{m\in\mathcal{M}} a_m \mathrm{Tr}\left(\frac{l_{m,A}(b/\mathbf{p})}{\tilde{\pi}_\rho}\right) = 0$$

by Lemma VI.2. Therefore,

$$\sum_{m\in\mathcal{M}} a_m l_{m,A}(b/\mathbf{p})/\tilde{\pi}_\rho = 0$$

which implies

$$\sum_{m\in\mathcal{M}} a_m l_{m,A}(b/\mathbf{p}) = 0. \tag{6.7}$$

Define a column vector $[\alpha_n]_{n=1,\ldots,q^d-1}$ by

$$\alpha_n = \begin{cases} a_n & \text{if } n \in \mathcal{M} \\ 0 & \text{if } n \notin \mathcal{M}. \end{cases}$$

Let Y be the matrix from Lemma VI.1. Then

$$Y^T[\alpha_n] = \begin{pmatrix} l_{1,A}(\omega/\mathbf{p}) & \cdots & l_{q^d-1,A}(\omega/\mathbf{p}) \\ & \vdots & \\ l_{1,A}(\omega^{q^d-1}/\mathbf{p}) & \cdots & l_{q^d-1,A}(\omega^{q^d-1}/\mathbf{p}) \end{pmatrix} [\alpha_n].$$

For $1 \leq k \leq q^d - 1$, the k-th row of $Y^T[\alpha_n]$ is

$$\begin{aligned} \alpha_1 l_{1,A}(\omega^k/\mathbf{p}) + \alpha_2 l_{2,A}(\omega^k/\mathbf{p}) + \cdots + \alpha_{q^d-1} l_{q^d-1,A}(\omega^k/\mathbf{p}) \\ = \sum_{m\in\mathcal{M}} \alpha_m l_{m,A}(\omega^k/\mathbf{p}) = 0 \end{aligned}$$

by (6.7). Therefore, $[\alpha_n] \in \ker(Y^T) = \{0\}$ by Lemma VI.1. Hence, the coefficients $\{a_m \mid m \in \mathcal{M}\}$ are all zero. This completes the proof of the theorem. $\square$

CHAPTER VII

EXAMPLES OF SPECIAL POLYNOMIALS

The following special polynomials were computed using Maple 12.

For K_1, we have:

- $S(t^0;z) = z + \eta z^3 + z^9$
- $S(t;z) = tz + \eta t^3 z^3 + \left(-t^3(\eta - t^6)\right) z^9 - t^9 z^{27}$
- $S(t^2;z) = t^2 z + \eta t^6 z^3 + \left[-\theta(\theta+1)(\theta-1)(\theta^3-\theta-1)t^6 + \eta t^{12} + t^{18}\right] z^9 + \left[-\eta(\theta^{12}-\theta^{10}-\theta^9-\theta^6+\theta^4-\theta^3-\theta+1)t^{18} + t^{36}\right] z^{27} + t^{54} z^{81}$

For K_2, we have:

- $S(t^0;z) = z + (\theta^4+\theta)z^4 + (\theta^8+\theta^2)z^{16} + z^{64}$
- $S(t;z) = tz + (\theta^4+\theta)t^4 z^4 + \left[(\theta^4+\theta)t^4 + (\theta^8+\theta^2)t^{16}\right] z^{16} + \left[(\theta^8+\theta^2)t^{16} + t^{64}\right] z^{64} + t^{64} z^{256}$

For $S(t;z)$, this is assuming that $Z_5(t) = 0$ (cf., §III.B). For K_3, we have:

- $S(t^0;z) = z + z^2$
- $S(t;z) = tz + t^2 z^2 + t^2 z^4$
- $S(t^2;z) = t^2 z + t^4 z^2 + \left[t^2 + \theta(\theta+1)t^4\right] z^4 + \left[(\theta^2+\theta+1)t^4 + t^8\right] z^8 + t^8 z^{16}$

We note that each of the above special polynomials $S(t^0;z)$ appears in §5.2 of [1] as well.

CHAPTER VIII

EXAMPLES OF SPECIAL VALUES OF GOSS L-FUNCTIONS AND GOSS ZETA FUNCTIONS

Definition VIII.1. Let K be the function field of an irreducible, smooth projective curve X defined over $\mathbb{F}_q$. Let ∞ be a fixed point on X and let A be the Dedekind domain of functions which are regular away from ∞. Fix a sign function sgn. For a nonnegative integer n, the Goss zeta function for K is

$$\zeta_K(n) := \sum_I \frac{1}{I^{[n]}}$$

where the sum is over all ideals I of A.

Note that if $d_\infty = 1$ and $h_A = 1$, then

$$\zeta_K(n) = \sum_{a \in A_+} \frac{1}{a^n}.$$

Example VIII.2 ([18], Equation (31))**.** Consider K_1. Let ρ^1 denote the unique Drinfeld-Hayes A_1-module as defined in §IV.B. By Proposition III.8, we have

$$\begin{aligned} S(t^0;z) &= \exp_{\rho^1} \ell(t^0;z) \\ &= \exp_{\rho^1}\left(\sum_{a \in (A_1)_+} \frac{(a) * 1}{a} z^{3^{\deg a}}\right) \\ &= \exp_{\rho^1}\left(\sum_{a \in (A_1)_+} \frac{1}{a} z^{3^{\deg a}}\right). \end{aligned}$$

Therefore,

$$S(t^0;1) = \exp_{\rho^1}(\zeta_{A_1}(1))$$

and so

$$\zeta_{K_1}(1) = \log_{\rho^1}(S(t^0;1)) = \log_{\rho^1}(\eta - 1).$$

Example VIII.3 ([18], Theorems VIII and X)**.** Let $j = 2,3$ and consider K_j. Exactly as in the previous example, we have

$$\zeta_{K_2}(1) = \log_{\rho^2}\left(1 + (\theta^4+\theta) + (\theta^8+\theta^2) + 1\right) = \log_{\rho^2}(\theta^8+\theta^4+\theta^2+\theta)$$

and

$$\zeta_{K_3}(1) = \log_{\rho^3}(0).$$

Example VIII.4. Consider the curve X_1. In this example, we will stop writing the index 1. Consider the Dirichlet character

$$\begin{aligned} \chi \colon A &\to \bar{\mathbb{F}}_9 \\ a = a(t,y) &\mapsto a(0,\sqrt{-1}). \end{aligned}$$

Recall that

$$A = \{F(t) + yG(t) \mid F(t), G(t) \in \mathbb{F}_3[t]\}.$$

Then $\ker\chi = (t)$. We know

$$A/(t) \cong \rho[t].$$

Let ξ be a generator of $\rho[t]$. Consider $a = F(t) + yG(t) \in A$. Since $F(t) - F(0) \in (t)$, we conclude

$$\begin{aligned} \rho_{F(t)}(\xi) = \rho_{F(t)-F(0)+F(0)}(\xi) &= \rho_{F(t)-F(0)}(\xi) + \rho_{F(0)}(\xi) \\ &= \rho_{F(0)}(\xi) \\ &= F(0)\xi. \end{aligned}$$

Since $G(t) - G(0) \in (t)$, we conclude

$$\begin{aligned} \rho_{yG(t)}(\xi) = \rho_{yG(0)}(\xi) + \rho_{yG(t)-yG(0)}(\xi) &= \rho_{yG(0)}(\xi) \\ &= G(0)\rho_y(\xi) \\ &=: G(0)\xi'. \end{aligned}$$

Note that $\xi' \in \rho[t]$ since

$$\rho_t(\xi') = \rho_t(\rho_y(\xi)) = \rho_{ty}(\xi) = 0.$$

Therefore,

$$\rho_a(\xi) = F(0)\xi + G(0)\xi'$$

for all $a = F(t) + yG(t) \in A$.

By Proposition III.8(1), we have

$$S(t;z) = \exp_\rho\left(\sum_{a\in A_+} \frac{\rho_a(t)}{a} z^{3^{\deg a}}\right) \tag{8.1}$$

which implies

$$\xi = S(\xi;1) = \exp_\rho\left(\sum_{\substack{a\in A_+ \\ a=F(t)+yG(t)}} \frac{F(0)\xi + G(0)\xi'}{a}\right). \tag{8.2}$$

Let $\mathrm{Gal}(K/\mathbb{F}_3(t)) = \{\mathrm{id}, \sigma\}$. Since $[K\colon \mathbb{F}_3(t)] = 2$, we have $\sigma(t) = t$ and $\sigma(y) = -y$. Then

$$\begin{aligned}
\chi(a) &= F(0) + G(0)\sqrt{-1} \\
a - \sigma(a) &= F(t) + yG(t) - (F(t) - yG(t)) = -yG(t) \\
a + \sigma(a) &= -F(t).
\end{aligned}$$

Therefore,

$$F(0) = F(0,\sqrt{-1}) = -a(0,\sqrt{-1}) - \sigma(a)(0,\sqrt{-1})$$

and

$$
\begin{aligned}
G(0) = G(0,\sqrt{-1}) &= \frac{\sigma(a)(0,\sqrt{-1}) - a(0,\sqrt{-1})}{\sqrt{-1}} \\
&= -\sqrt{-1}\left(\sigma(a)(0,\sqrt{-1}) - a(0,\sqrt{-1})\right) \\
&= \sqrt{-1}\left(a(0,\sqrt{-1}) - \sigma(a)(0,\sqrt{-1})\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\chi^3(a) &= \chi(F(t)^3 + y^3 G(t)^3) \\
&= F(0)^3 + (-\sqrt{-1})G(0)^3 \quad \text{since } y^3 = y(t^3 - t - 1) \\
&= F(0) - \sqrt{-1}G(0) \quad \text{since } F(0), G(0) \in \mathbb{F}_3 \\
&= \chi(\sigma(a)).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{F(0)\xi + G(0)\xi'}{a} &= \frac{F(0,\sqrt{-1})\xi + G(0,\sqrt{-1})\xi'}{a} \\
&= \frac{\left(-a(0,\sqrt{-1}) - \sigma(a)(0,\sqrt{-1})\right)\xi}{a} \\
&\quad + \frac{\sqrt{-1}\left(a(0,\sqrt{-1}) - \sigma(a)(0,\sqrt{-1})\right)\xi'}{a} \\
&= \frac{\left(\sqrt{-1}\xi' - \xi\right)\chi(a)}{a} + \frac{\left(-\sqrt{-1}\xi' - \xi\right)\chi(\sigma(a))}{a} \\
&= \frac{\left(\sqrt{-1}\xi' - \xi\right)\chi(a)}{a} + \frac{\left(-\sqrt{-1}\xi' - \xi\right)\chi^3(a)}{a}.
\end{aligned}
$$

(8.2) implies that

$$
\begin{aligned}
\log_\rho(\xi) &= (\sqrt{-1}\xi' - \xi)\sum_{a\in A_+}\frac{\chi(a)}{a} + (-\sqrt{-1}\xi' - \xi)\sum_{a\in A_+}\frac{\chi^3(a)}{a} \\
&= (\sqrt{-1}\xi' - \xi)L(1,\chi) + (-\sqrt{-1}\xi' - \xi)L(1,\chi^3). \qquad (8.3)
\end{aligned}
$$

Now suppose we had taken ξ' as our generator of $\rho[t]$. If $a = F(t) + yG(t)$, then as before, we have

$$\begin{aligned}\rho_a(\xi') &= F(0)\xi' + G(0)\rho_y(\xi') \\ &= F(0)\xi' + G(0)\rho_y(\rho_y(\xi)) \\ &= F(0)\xi' + G(0)\rho_{y^2}(\xi) \\ &= F(0)\xi' - G(0)\xi\end{aligned}$$

since

$$\rho_{y^2}(\xi) = \rho_{t^3}(\xi) - \rho_t(\xi) - \rho_1(\xi) = -\rho_1(\xi) = -\xi.$$

Setting $t = \xi'$ and $z = 1$ in (8.1) yields

$$\log_\rho(\xi') = \sum_{\substack{a \in A_+ \\ a = F(t) + yG(t)}} \frac{F(0)\xi' - G(0)\xi}{a}. \tag{8.4}$$

Exactly as above, we compute

$$\begin{aligned}\frac{F(0)\xi' - G(0)\xi}{a} &= \frac{\left(-\chi(a) - \chi^3(a)\right)\xi'}{a} - \frac{\sqrt{-1}\left(\chi(a) - \chi^3(a)\right)\xi}{a} \\ &= \frac{\left(-\xi' - \sqrt{-1}\xi\right)\chi(a)}{a} + \frac{\left(-\xi' + \sqrt{-1}\xi\right)\chi^3(a)}{a}.\end{aligned}$$

Thus, (8.4) implies

$$\log_\rho(\xi') = \left(-\xi' - \sqrt{-1}\xi\right) L(1,\chi) + \left(-\xi' + \sqrt{-1}\xi\right) L(1,\chi^3). \tag{8.5}$$

(8.3) and (8.5) yield the following linear system:

$$\begin{bmatrix} \sqrt{-1}\xi' - \xi & -\sqrt{-1}\xi' - \xi \\ -\xi' - \sqrt{-1}\xi & -\xi' + \sqrt{-1}\xi \end{bmatrix} \begin{bmatrix} L(1,\chi) \\ L(1,\chi^3) \end{bmatrix} = \begin{bmatrix} \log_\rho(\xi) \\ \log_\rho(\xi') \end{bmatrix}.$$

Let $\mathcal{A}$ denote the 2 by 2 matrix on the left hand side of the preceding expression. Since

$$\det \mathcal{A} = \sqrt{-1}\left((\xi')^2 + \xi^2\right) \neq 0,$$

our system has a unique solution. We conclude

$$L(1,\chi) = \frac{\log_\rho(\xi') + \sqrt{-1}\log_\rho(\xi)}{\xi' + \sqrt{-1}\xi}$$
$$L(1,\chi^3) = \frac{\log_\rho(\xi') - \sqrt{-1}\log_\rho(\xi)}{\xi' - \sqrt{-1}\xi}.$$

Example VIII.5. Consider the elliptic curve $E\colon y^2 = t^3 - t^2 - t$ defined over $\mathbb{F}_3$. Note that $E(\mathbb{F}_3) = \{(0,0),[0,1,0]\}$. Let $\infty := [0,1,0]$. We have $d_\infty = 1$. Set

$$A := \mathbb{F}_3[t,y]/(y^2 - t^3 + t^2 + t)$$

and let K be the fraction field of A. Here $h_A = 2$.

Consider the Dirichlet character

$$\chi\colon A \to \bar{\mathbb{F}}_3$$
$$a = a(t,y) \mapsto a(0,0).$$

Choose a sign function sgn and fix a Drinfeld-Hayes A-module ρ with respect to sgn. Set $I_2 := (t,y)$. Note that I_2 is a nonprincipal ideal of A such that $\ker\chi = I_2$.

Let ξ be a generator of $\rho[I_2] \cong (A/I_2)^\times$. Since $G'' := \mathrm{Gal}(H(\rho[I_2])/H) \cong (A/I_2)^\times$, it follows that $\chi \in \hat{G}''$. Let $G := \mathrm{Gal}(H(\rho[I_2])/K)$. As described in §II.G, we let $\psi \in \hat{G}$ be such that $\psi|_A = \chi$.

By Theorem III.2,

$$\exp_\rho \ell(t;z) =: S(t;z)$$

is a polynomial in $B[t,z]$ where B is the integral closure of A in the Hilbert class field H. Let $\{A, I_2\}$ be a set of representatives of the equivalence classes of $Cl(A)$. We compute (cf., §V.D)

$$\ell(t;z) = \sum_{\omega_1 \in (A^{-1})_+} \frac{(\omega_1 A) * t}{\omega_1 D(\rho_A)} z^{3^{\deg A + \deg \omega_1}} + \sum_{\omega_2 \in (I_2^{-1})_+} \frac{(\omega_2 I_2) * t}{\omega_2 D(\rho_{I_2})} z^{3^{\deg I_2 + \deg \omega_2}}$$

$$= \sum_{\omega_1 \in A_+} \frac{\rho_{\omega_1}(t)}{\omega_1} z^{3^{\deg \omega_1}} + \frac{1}{D(\rho_{I_2})} \sum_{\omega_2 \in (I_2^{-1})_+} \frac{\rho_{\omega_2 I_2}(t)}{\omega_2} z^{3^{1+\deg \omega_2}} \tag{8.6}$$

where we have used that $A^{-1} = A$, $D(\rho_A) = 1$, $\deg A = 0$ and $\deg I_2 = 1$. This last fact follows from the observations that $I_2^2 = (\theta)$ and $\deg t = 2$.

The maps

$$I_2 \to I_2^{-1} \colon a \mapsto \frac{a}{\theta} \quad \text{and} \quad I_2^{-1} \to I_2 \colon \omega_2 \mapsto \omega_2 \theta$$

give a one-to-one correspondence between I_2^{-1} and I_2. Setting $\omega_2' := \omega_2\theta$ in (8.6) yields

$$\ell(t;z) = \sum_{\omega_1 \in A_+} \frac{\rho_{\omega_1}(t)}{\omega_1} z^{3^{\deg \omega_1}} + \frac{\theta}{D(\rho_{I_2})} \sum_{\omega_2' \in (I_2)_+} \frac{\rho_{(\omega_2'/\theta) I_2}(t)}{\omega_2'} z^{3^{\deg \omega_2' - 1}}. \tag{8.7}$$

We now investigate the values $\rho_a(\xi)$ for $a \in A_+$ and $\rho_{(\omega_2'/\theta)I_2}(\xi)$ for $\omega_2' \in I_2$. First note that if $a \in A$, then $a = F(t) + yG(t)$ for some $F(t), G(t) \in \mathbb{F}_3[t]$. We calculate

$$\begin{aligned}
\rho_a(\xi) = \rho_{F(t)}(\xi) + \rho_{yG(t)}(\xi) &= \rho_{F(t)}(\xi) \quad \text{since } yG(t) \in I_2 \\
&= \rho_{F(0)}(\xi) + \rho_{F(t)-F(0)}(\xi) \\
&= \rho_{F(0)}(\xi) \quad \text{since } F(t) - F(0) \in I_2 \\
&= F(0)\xi \\
&= \chi(a)\xi.
\end{aligned}$$

Now let $\omega_2' \in I_2$. Then $\omega_2' = f(t) + yg(t)$ for some $f(t), g(t) \in \mathbb{F}_3[t]$. Furthermore, $\omega_2' \in I_2$ implies $\theta \mid f(t)$. We have

$$I_2 \left(\frac{\omega_2'}{\theta} \right) \subseteq I_2$$

if and only if

$$(\omega_2') \subseteq (\theta)$$

if and only if $\theta \mid \omega_2'$. Hence, $\theta \mid \omega_2'$ implies $\rho_{I_2(\omega_2'/\theta)}(\xi) = 0$.

Now suppose $\theta \nmid \omega_2'$. Then

$$\frac{\eta\omega_2'}{\theta} = \eta\frac{f(t)}{\theta} + (\theta^2 - \theta - 1)g(t).$$

Since $I_2(\omega_2'/\theta) = (\omega_2', \eta\omega_2'/\theta)$, it follows that $\rho_{I_2(\omega_2'/\theta)}$ is the right gcd of $\rho_{\omega_2'}$ and $\rho_{\eta\omega_2'/\theta}$, i.e. there exists $h(\tau) \in \mathbb{C}\langle\tau\rangle$ such that

$$\rho_{\eta\omega_2'/\theta} = h(\tau)\rho_{\omega_2'} + \rho_{I_2(\omega_2'/\theta)}.$$

This implies

$$\rho_{I_2(\omega_2'/\theta)}(\xi) = \rho_{\eta\omega_2'/\theta}(\xi) = \rho_{(\theta^2-\theta-1)g(t)}(\xi).$$

Therefore, in computing $\rho_{I_2(\omega_2'/\theta)}(\xi)$ we need only consider elements of $(I_2)_+$ of the form $\eta g(t)$ where $\theta \nmid g(t)$. In other words, we are reduced to looking at $\rho_{I_2(\alpha\eta/\theta)}(\xi)$ where $\alpha \in A_+$ such that $(\alpha, I_2) = 1$.

Setting $t = \xi$ and $z = 1$ in (8.7) yields

$$\ell(\xi;1) = \sum_{\omega_1 \in A_+} \frac{\rho_{\omega_1}(\xi)}{\omega_1} + \frac{\theta}{D(\rho_{I_2})} \sum_{\substack{a \in A_+ \\ (a,I_2)=1}} \frac{\rho_{I_2(a\eta/\theta)}(\xi)}{a\eta}. \tag{8.8}$$

We claim that $\rho_{I_2(a\eta/\theta)}(\xi) \in \rho[I_2]$. Since $I_2^2 = (\theta)$, we have

$$\rho_{I_2}(\rho_{I_2(a\eta/\theta)}(\xi)) = \rho_{I_2^2(a\eta/\theta)}(\xi) = \rho_{a\eta}(\xi) = 0$$

since $a\eta \in I_2$. Since $\deg I_2 = 1$, $\rho_{I_2}(t) = D(\rho_{I_2})t + t^3 = t(t-\xi)(t+\xi)$. Therefore, $\rho[I_2] = \{0, \pm\xi\}$. Furthermore, we have that

$$\xi = \pm\sqrt{-D(\rho_{I_2})}.$$

For specificity, we fix $\xi = \sqrt{-D(\rho_{I_2})}$. Now

$$\rho_{I_2(a\eta/\theta)}(\xi) = 0 \iff I_2\left(\frac{a\eta}{\theta}\right) \subseteq I_2 \iff (a\eta) \subseteq I_2^2 = (\theta) \iff \mathrm{ord}_{I_2}(a\eta) \geq 2.$$

Since $\eta^2 \in I_2^2$, it follows that $2\mathrm{ord}_{I_2}(\eta) \geq 2$. And $(a, I_2) = 1$ implies $\mathrm{ord}_{I_2}(a) = 0$. So

$$\mathrm{ord}_{I_2}(a\eta) = \mathrm{ord}_{I_2}(a) + \mathrm{ord}_{I_2}(\eta) \geq 1.$$

Therefore, $\rho_{I_2(a\eta/\theta)}(\xi) \neq 0$.

Recall from §II.G that we have the exact sequence

$$0 \to G'' \to G \to G' \to 0.$$

From §II.G, we know

$$G \cong \mathfrak{F}_{I_2}(A)/\mathfrak{P}_{I_2}(A)$$

and

$$G' \cong Cl(A) \cong \mathfrak{F}(A)/\mathfrak{P}(A).$$

It is easy to see that G is a cyclic group of order 4. Define

$$J := \frac{\eta}{\theta} I_2 = (\eta, \theta^2 - \theta - 1).$$

Consider

$$\frac{\mathfrak{F}_{I_2}(A)}{\mathfrak{P}_{I_2}(A)} \to \frac{\mathfrak{F}(A)}{\mathfrak{P}(A)}$$
$$J \bmod \mathfrak{P}_{I_2}(A) \mapsto J \bmod \mathfrak{P}(A).$$

Since $\mathrm{ord}_{I_2}(\theta) = 2$ and $\mathrm{ord}_{I_2}(\eta) = 1$, it follows that $\mathrm{ord}_{I_2}(J) = 0$ and so $J \in \mathfrak{F}_{I_2}(A)$. Since $\mathfrak{F}_{I_2}(A)/\mathfrak{P}_{I_2}(A)$ is cyclic of order 4, the above map is uniquely determined by J provided that $J \bmod \mathfrak{P}_{I_2}(A)$ generates $\mathfrak{F}_{I_2}(A)/\mathfrak{P}_{I_2}(A)$. This follows at once since

$$J^2 = (\theta^2 - \theta - 1), \quad J^4 = (\theta^4 + \theta^3 - \theta^2 - \theta + 1)$$

and

$$\theta^4 + \theta^3 - \theta^2 - \theta + 1 \equiv 1 \bmod I_2.$$

(8.8) now becomes

$$\ell(\xi;1)=\sum_{\omega_1\in A_+}\frac{\rho_{\omega_1}(\xi)}{\omega_1}+\frac{\theta}{D(\rho_{I_2})}\sum_{\substack{a\in A_+\\(a,I_2)=1}}\frac{\rho_{aJ}(\xi)}{a\eta}. \tag{8.9}$$

If $a\in A_+$ and $(a,I_2)=1$, we have

$$\rho_{aJ}=\rho_{Ja}=((a)*\rho)_J\rho_a=\rho_J\rho_a$$

hence

$$\rho_{aJ}(\xi)=\rho_J\rho_a(\xi)=\rho_J(\chi(a)\xi)=\chi(a)\rho_J(\xi).$$

By the right division algorithm in $\mathbb{C}_\infty\langle\tau\rangle$, there exists $h(\tau)\in\mathbb{C}_\infty\langle\tau\rangle$ such that

$$\rho_{\theta^2-\theta-1}=h(\tau)\rho_\eta+\rho_J.$$

Therefore,

$$\rho_J(\xi)=\rho_J(\xi)+h(\tau)\rho_\eta(\xi)=\rho_{\theta^2-\theta-1}(\xi)=\rho_{-1}(\xi)=-\xi.$$

Hence,

$$\rho_{aJ}(\xi)=-\chi(a)\xi$$

and so (8.9) now becomes

$$\ell(\xi;1)=\sum_{\omega_1\in A_+}\frac{\rho_{\omega_1}(\xi)}{\omega_1}+\frac{\theta}{D(\rho_{I_2})}\sum_{\substack{a\in A_+\\(a,I_2)=1}}\frac{-\chi(a)\xi}{a\eta}. \tag{8.10}$$

Fix $\psi\in\hat{G}$ such that $\psi|_A=\chi$ as in §II.G. By our previous analysis, ψ is uniquely determined by its value at J and $\psi(J)$ is a fourth root of unity. So set $\psi(J)=\sqrt{-1}$. If $\mathfrak{a}\in\mathfrak{F}_{I_2}(A)/\mathfrak{P}_{I_2}(A)$ then $\mathfrak{a}=\beta J^i$ for some $\beta\in\mathfrak{P}_{I_2}(A)$, $1\le i\le 4$. Since

$$\psi(\beta)=\psi((\beta))=\chi(\beta)=1,$$

we conclude

$$\psi(\mathfrak{a}) = \psi(J^i) = (\sqrt{-1})^i.$$

Now let I be an ideal of A such that $I \sim I_2$ in $Cl(A)$. By our previous work,

$$I = a(\eta/\theta)I_2 = aJ$$

for some $a \in A_+$ such that $(a, I_2) = 1$. Note that

$$I^{[1]} = \frac{a\eta}{\theta} I_2^{[1]}.$$

We now compute $I_2^{[1]}$. Let π be a monic uniformizer at ∞. Since $d_\infty = 1$, $\deg I_2 = 1$ and $I_2^2 = (\theta)$, we conclude from §II.G that

$$I_2^{[1]} = \pi^{-1}\langle\theta\rangle^{1/2}.$$

Since θ is monic of degree 2, we have

$$\theta = \pi^{-2}\langle\theta\rangle$$

which implies that

$$I_2^{[1]} = \sqrt{\theta}.$$

Hence,

$$I^{[1]} = \frac{a\eta}{\theta}\sqrt{\theta}.$$

Also,

$$\psi(I) = \psi(aJ) = \psi(a)\psi(J) = \chi(a)\sqrt{-1}.$$

And we have

$$\rho_{aJ}(\xi) = -\chi(a)\xi = -\chi(a)\sqrt{-1}\sqrt{D(\rho_{I_2})} = -\sqrt{D(\rho_{I_2})}\psi(I).$$

Putting everything together, we conclude

$$\begin{aligned}\ell(\xi;1) &= \sum_{\omega_1\in A_+}\frac{\rho_{\omega_1}(\xi)}{\omega_1}+\frac{\theta}{D(\rho_{I_2})}\sum_{\substack{a\in A_+\\(a,I_2)=1}}\frac{-\chi(a)\xi}{a\eta}\\ &= \xi\sum_{\omega_1\in A_+}\frac{\chi(\omega_1)}{\omega_1}+\frac{-\theta}{\sqrt{D(\rho_{I_2})}}\sum_{I\sim I_2}\frac{\psi(I)}{I^{[1]}}\\ &= \xi\sum_{(\omega_1)\sim A}\frac{\psi((\omega_1))}{(\omega_1)^{[1]}}-\frac{\sqrt{\theta}}{\sqrt{D(\rho_{I_2})}}\sum_{I\sim I_2}\frac{\psi(I)}{I^{[1]}}\\ &= \log_\rho S(t;1)|_{t=\xi}.\end{aligned}\tag{8.11}$$

Now another application of Theorem III.2 using $\sqrt{\theta}t$ instead of t yields

$$\exp_\rho \ell(\sqrt{\theta}t;z) =: S(\sqrt{\theta}t;z)$$

is a polynomial in $B[t,z]$. Exactly as above, one shows

$$\begin{aligned}\ell(\sqrt{\theta}\xi;1) &= \sqrt{\theta}\xi\sum_{(\omega_1)\sim A}\frac{\psi((\omega_1))}{(\omega_1)^{[1]}}+\frac{\theta}{\sqrt{D(\rho_{I_2})}}\sum_{I\sim I_2}\frac{\psi(I)}{I^{[1]}}\\ &= \log_\rho S(\sqrt{\theta}t;1)|_{t=\xi}.\end{aligned}\tag{8.12}$$

Let $\Sigma_1 := \sum_{(\omega_1)\sim A}\frac{\psi((\omega_1))}{(\omega_1)^{[1]}}$ and $\Sigma_2 := \sum_{I\sim I_2}\frac{\psi(I)}{I^{[1]}}$. Then (8.11) and (8.12) yield the following linear system

$$\begin{aligned}\xi\Sigma_1-\frac{\sqrt{\theta}}{\sqrt{D(\rho_{I_2})}}\Sigma_2 &= \log_\rho S(t;1)|_{t=\xi}\\ \sqrt{\theta}\xi\Sigma_1+\frac{\theta}{\sqrt{D(\rho_{I_2})}}\Sigma_2 &= \log_\rho S(\sqrt{\theta}t;1)|_{t=\xi}\end{aligned}$$

which we solve to obtain

$$\begin{aligned}\Sigma_1 &= -\frac{1}{\xi}\log_\rho S(t;1)|_{t=\xi}-\frac{1}{\xi\sqrt{\theta}}\log_\rho S(\sqrt{\theta}t;1)|_{t=\xi}\\ \Sigma_2 &= \frac{\sqrt{D(\rho_{I_2})}}{\sqrt{\theta}}\log_\rho S(t;1)|_{t=\xi}-\frac{\sqrt{D(\rho_{I_2})}}{\theta}\log_\rho S(\sqrt{\theta}t;1)|_{t=\xi}.\end{aligned}$$

Hence,

$$\begin{aligned} L(1,\psi) &= \Sigma_1 + \Sigma_2 \\ &= \left(\frac{\sqrt{D(\rho_{I_2})}}{\sqrt{\theta}} - \frac{1}{\xi} \right) \log_\rho S(t;1)|_{t=\xi} + \left(\frac{-\sqrt{D(\rho_{I_2})}}{\theta} - \frac{1}{\xi\sqrt{\theta}} \right) \log_\rho S(\sqrt{\theta}t;1)|_{t=\xi}. \end{aligned}$$

Using Maple 12, we were able to compute $S(t;z)$:

$$S(t,z) = tz + \left((-\eta - \sqrt{\theta})(\theta - 1)t^3 + t \right) z^3 + \left(t^9 + (\eta + \sqrt{\theta}(\theta - 1))t^3 \right) z^9 - t^9 z^{27}.$$

But we were unable to compute $S(\sqrt{\theta}t;z)$.

Example VIII.6. We continue with the previous example. Consider

$$\zeta_K(1) = \sum_I \frac{1}{I^{[1]}} = \sum_{a \in A_+} \frac{1}{a} + \sum_{I \sim I_2} \frac{1}{I^{[1]}}.$$

Claim 1:

$$I \sim I_2 \iff II_2 = (\alpha) \text{ for some } \alpha \in A_+.$$

The implication $\Leftarrow$ follows immediately since $I_2^2 = (\theta)$. As for $\Rightarrow$, if $I \sim I_2$ then $aI = bI_2$ for some $a, b \in A_+$. This implies that $aII_2 = b\theta$ which implies

$$II_2 = \frac{b\theta}{a} \in A.$$

This proves the claim.

Claim 2:

$$II_2 = (\alpha) \text{ for some } \alpha \in A_+ \iff \alpha^2/\theta \in A.$$

If $II_2 = (\alpha)$, then

$$I^2 = \frac{(\alpha^2)}{I_2^2} = \frac{(\alpha^2)}{(\theta)} \subseteq A$$

which proves $\Rightarrow$. Conversely, if $\alpha \in A_+$ and $\alpha^2/\theta \in A$, then $(\alpha^2) = (\theta)I$ for some ideal I. This implies $(\alpha^2) = I_2^2 I$ and so $(\alpha) = I_2 I'$ for some ideal I'.

Claim 3:

$$\{\alpha \in A_+ \mid \alpha^2/\theta \in A\} = (I_2)_+.$$

If $\alpha^2/\theta \in A$, then Claim 2 implies that

$$\mathrm{ord}_{I_2}(\alpha) = \mathrm{ord}_{I_2}(II_2) = \mathrm{ord}_{I_2}(I) + 1 \geq 1$$

which proves $\subseteq$. Now if $\beta \in (I_2)_+$, then $\beta = f(t) + yg(t)$ for $f(t), g(t) \in \mathbb{F}_3[t]$ and $\theta \mid f(t)$. It follows that $\theta \mid \beta^2$ which completes the proof of the claim.

By Claim 1, we have

$$\alpha = (\alpha)^{[1]} = I^{[1]} I_2^{[1]} = I^{[1]}\sqrt{\theta}.$$

Hence, Claims 2 and 3 imply

$$\zeta_K(1) = \sum_{a\ A_+} \frac{1}{a} + \sqrt{\theta} \sum_{\alpha \in (I_2)_+} \frac{1}{\alpha}.$$

As in the previous example, we compute

$$\ell(t^0;1) = \sum_{a \in A_+} \frac{1}{a} + \frac{\theta}{D(\rho_{I_2})} \sum_{\alpha \in (I_2)_+} \frac{1}{\alpha};$$

$$\ell(\sqrt{\theta}t^0;1) = \sqrt{\theta} \sum_{a \in A_+} \frac{1}{a} - \sqrt{\theta} \frac{\theta}{D(\rho_{I_2})} \sum_{\alpha \in (I_2)_+} \frac{1}{\alpha}.$$

We set

$$S(t^0;1) := \exp_\rho \ell(t^0;1);$$

$$S(\sqrt{\theta}t^0;1) := \exp_\rho \ell(\sqrt{\theta}t^0;1).$$

Let $\Sigma_1 := \sum_{a \in A_+} \frac{1}{a}$ and $\Sigma_2 := \sum_{\alpha \in (I_2)_+} \frac{1}{\alpha}$. Then

$$\zeta_K(1) = \Sigma_1 + \sqrt{\theta}\Sigma_2$$

and we have the linear system

$$
\begin{aligned}
\Sigma_1 + \frac{\theta}{D(\rho_{I_2})}\Sigma_2 &= \log_\rho S(t^0;1) \\
\sqrt{\theta}\Sigma_1 - \frac{\theta^{3/2}}{D(\rho_{I_2})}\Sigma_2 &= \log_\rho S(\sqrt{\theta}t^0;1).
\end{aligned}
$$

Solving the system we obtain

$$
\begin{aligned}
\Sigma_1 &= -\log_\rho S(t^0;1) - \frac{1}{\sqrt{\theta}}\log_\rho S(\sqrt{\theta}t^0;1); \\
\Sigma_2 &= -\frac{D(\rho_{I_2})}{\theta}\log_\rho S(t^0;1) + \frac{D(\rho_{I_2})}{\theta^{3/2}}\log_\rho S(\sqrt{\theta}t^0;1).
\end{aligned}
$$

Using Maple 12, one can compute

$$
\begin{aligned}
S(t^0,z) &= z - (\theta^{3/2} - \theta^{1/2} + \eta)z^3 + z^9; \\
S(\sqrt{\theta}t^0,z) &= \sqrt{\theta}z - \left(1 - \theta - \frac{\eta}{\sqrt{\theta}} + \theta^3 + \eta\theta^{3/2} - \eta\sqrt{\theta}\right)z^3 \\
&\quad + \left(-\eta\theta^3 + \eta\theta^2 + \eta\theta - \eta - \theta^{9/2} + \sqrt{\theta}\right)z^9 + z^{27}.
\end{aligned}
$$

Hence, the value $\zeta_K(1)$ can be explicitly computed as opposed to our previous example.

CHAPTER IX

CONCLUSIONS

We conclude with some remarks about special points and special polynomials in the case where $h_A > 1$.

The special polynomials $\{\exp_\rho \ell(b;z) \mid b \in B[t]\}$ of Theorem III.2 exist for any function field. The only concern we have about special polynomials for a general function field is with respect to computation. If $h_A > 1$, then the special polynomials are now polynomials with coefficients in B. Examples VIII.5 and VIII.6 suggest that since the coefficients lie in B, this adds to the difficulty in computing special polynomials explicitly. We think that this suspicion is well founded due to our calculations which we presented in the aforementioned examples. We are unsure if our choice of computational package is to blame for our difficulties or if the special polynomials are truly difficult to compute for higher class numbers.

Another issue concerning computation of special polynomials, which was addressed in [2] but was not addressed in this dissertation, is the case of recursive formulas for the special polynomials. In the case of the Carlitz module, the special polynomials satisfy certain recursions which can reduce the amount of computational work ([2], Equations 27 and 28). We were able to derive a formula similar to [2] Equation 27 in the case of the function fields $K_1, \ldots, K_4$, but we did not see how it would help in computing special polynomials. This is why we did not include it in this dissertation. It remains to see if there is a more efficient way of computing special polynomials.

A third issue is that of the number of terms to expect in a general special polynomial. We suspect that the degree bounds of the special polynomials as given in Proposition III.8(5) depend upon h_A. Furthermore, we suspect that these degree bounds increase as h_A increases. The special polynomials listed in Examples VIII.5 and VIII.6 seem to support our claim. Concerning the function fields $K_1,\dots,K_4$, we have $g_{K_j} = 1$ for $1 \le j \le 3$ and $g_{K_4} = 2$. It seems to us that special polynomials in general should depend upon the genus of the function field. For $K_1,\dots,K_4$, we only saw this dependence in the computation of $j_0(t^m;\rho^i)$ for Proposition IV.15. We are not sure of the exact dependence between the genus and special polynomials.

In computing special polynomials, we arrived at difficulties concerning the function $\ell(b;z)$ for $b \in B[t]$. If $h_A = 2$, as we have already seen, the majority of the work that goes into manipulating $\ell(b;z)$ is directed towards the sum over the nonprincipal ideals of A. If $h_A > 1$, then in order to compute $\ell(b;z)$, one must deal with $h_A - 1$ such sums. And each sum is over a class of nonprincipal ideals of A. It remains to be seen if there is a more efficient way of explicitly dealing with these sums than the methods we employed.

As for special points when $h_A > 1$, we are unable to make any predictions since it is not clear to us how to define the module of special points in this case. Recall that our definition of a special point when $h_A = 1$ arises from Proposition III.8(2):

$$\exp_\rho l_m(x) = S(t^m;1)|_{t=\mathbf{e}(x)}.$$

The proof of this identity follows from the functional equation (2.1) and the fact that $\ell(t^m;z)$ can be written as a sum over A_+. If $h_A > 1$, then as we have already noted, $\ell(b;z)$ for $b \in B[t]$ is a sum over the ideal classes of A. It is not clear to us what the analogue of Proposition III.8(2) should be for a general function field. Thus, we are not sure how one should define a special point when $h_A > 1$.

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