A STUDY OF PROSPECTIVE MATHEMATICS TEACHERS' KNOWLEDGE DEVELOPMENT AND BELIEFS CHANGES FOR TEACHING FRACTION DIVISION

A Dissertation

by

XI CHEN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2010

Major Subject: Curriculum and Instruction

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Approved by:

Co-Chairs of Committee,	Yeping Li
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	Donald Allen
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ABSTRACT

A Study of Prospective Mathematics Teachers' Knowledge Development and Beliefs Changes for Teaching Fraction Division. (May 2010)

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Co-Chairs of Advisory Committee: Dr. Yeping Li
Dr. Gerald Kulm

The purpose of this study is to examine prospective mathematics teachers' knowledge development and beliefs changes for teaching fraction division through the undergraduate mathematics method course to the field practice. Further, it reveals the correlation between the knowledge development and beliefs changes. Therefore, this study uses a qualitative methodology. I analyze the data from two time periods using three steps. In the method course period, interviews concerning knowledge and beliefs are triangulated with the tests, surveys, concept mapping and the writing assignment. There are two steps in this time period. First, I focus on a total of 27 prospective teachers' subject matter knowledge (SMK), including common content knowledge (CCK) and special content knowledge (SCK), and its development. Further, I examine their beliefs changes towards fraction division and mathematics teaching and learning during the method course. Next, I choose six participants from the total 27, based on different mathematics achievement. I do this to identify 1) whether CCK differences impact SCK development and 2) whether SCK development influence beliefs changes in the method course. In the field practice period, classroom observation of fraction division is triangulated with the interviews. I follow up one prospective teacher in his field practice and focus on the way his beliefs influence his teaching behavior and the development of the pedagogical content knowledge (PCK), through the teaching.

The results indicate that the prospective teachers developed both CCK and SCK in their method course. Their beliefs towards to teaching and learning fraction division progress from procedural-oriented to conceptual-oriented. The knowledge development and beliefs changes derived from the different learning experiences from their past school experiences and method course. Moreover, prospective teachers who had high CCK developed his/her SCK significantly. Thus, his/her beliefs changes became more significant. Further, the prospective teacher's beliefs changes in the method course influenced the way of teaching behavior in the field practice and SCK impacts PCK in teaching. On the other hand, field practice changed prospective teacher's beliefs and the development of PCK. Therefore, further attention is called for in the prospective teachers' knowledge transition and beliefs changes from a student to a future teacher.

DEDICATION

To my grandparents, Xiyi Chen and Yaoying Wang

For their care, and love

To my parents, Zhizhong Chen and Weijun Rong

For their support, and encouragement

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CHAPTER I

INTRODUCTION

Does teacher education matter in the U.S? For decades, researchers have debated about the effectiveness of undergraduate mathematics teacher preparation programs. For example, undergraduate mathematics teacher preparation programs have been criticized as deficient compared with academic mathematics department because prospective teachers' mathematics knowledge is limited. On the other hand, since the late of 1980s, researchers (e.g., Ball, 1990a, 1990b; Borko et al., 1992; Graeber, Tirosh, & Glover, 1989; Ma, 1999; McDiarmid & Wilson, 1991; Simon, 1993) have concerned with prospective teachers and in-service teachers' mathematics knowledge for teaching in U.S. , which is "the knowledge that a teacher needs to have or uses in the course of teaching a particular school-level curriculum in mathematics" (Leihardt, Rutnam, Stein, & Baxter, 1991, p.96), rather than "the knowledge of advanced topic that a mathematician might have" (Leinhardt, Rutnam, Stein, & Baxter, 1991, p. 97). Simply taking more mathematics courses alone is not helpful in improving mathematics knowledge for teaching (e.g., Jonker, 2008). Prospective teachers have to learn how to carry out their job and foster students' mathematics learning in their undergraduate mathematics teacher preparation program. Because they need to learn how to link "knowing" with enable others to learn, prospective teachers require different experiences in mathematics education from what they received before they enrolled the teacher

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This dissertation follows the style of *Journal for Research in Mathematics Education*.

preparation program.

If knowledge is the "cognitive outcome of teacher education" (Ernest, 1989, p.17), then "beliefs and attitudes represent the affective outcome" (Ernest, 1989, p. 18). The current mathematics reform movement indicates that new forms of mathematics teaching need to "create a culture of mathematical inquiry aimed at developing deep and flexible understanding of the domain" (Goldsmith & Shifter, 1997, p.20). Some research has found that many prospective teachers enter teacher education programs with preconceived conception and beliefs about teaching and learning mathematics based on their previous school experience (e.g., Ball, 1988). These preconceived attitude and beliefs also consistently influence prospective teachers' teaching in their further teaching career. Teacher preparation programs provide an opportunity to prospective teachers to reflect their beliefs and perception about teaching, learning and abut mathematics itself. If teacher knowledge develops, how do perceptions and beliefs change? Further, if the prospective teachers' perceptions and beliefs change, what kind of impact may this have in their knowledge development? In other words, how knowledge and beliefs interact to each other.

A better understanding of the correlation between prospective teachers' development of knowledge and beliefs changes thus becomes very important to mathematics educators, when we consider the impact that mathematics teacher education has on prospective teachers. These questions call for further research.

1.1 Rationale for This Study

These is a common consensus that mathematics teacher education should "aim at improving teachers' *beliefs*, their *knowledge* and their practice, at increasing their *motivation*, their *self-confidence* and their *identity* as mathematics teachers, and, most importantly, at contributing to their students *affective and cognitive growth*" (Krainer, 2008, p. 225). Here, it can be assume that teachers' mathematics knowledge and their beliefs are two important factors that may influence their effective teaching. According to Schoenfeld (1998), teachers' knowledge and beliefs are "critically important determinants of what teachers do and why they do it" (p.2). In other words, the knowledge, beliefs, decisions, and actions of teachers affect what is taught and ultimately learned in the classroom instruction.

Shulman (1986) proposed a new model of teacher knowledge, which he called the "subject matter *for teaching"* (p.9). That means, what a good teacher knows is distinguished from what a knowledgeable person knows. In other words, knowledge of a content domain does not necessarily include knowledge of how to assist other to acquire the knowledge. Harel (1993) explored Shulman's ideas and indicated that there are three components, which are mathematics content, epistemology, and pedagogy. Here, mathematics content refers to the depth and breadth of the mathematics knowledge; epistemology refers to the teachers' understanding of how students learn mathematics; and pedagogy refers to the teachers have ability to teach.

Ball (1988, 1991) identified teachers' understanding of mathematics as interweaving ideas *of* and *about* the subject (1990b, 1991). By knowledge *of* mathematics she meant substantive knowledge of subject: comprehension of particular topics, procedures, and concepts, and the relationship among these topics, procedures, and concepts. By knowledge *about* mathematics she meant syntactic knowledge, say, comprehension of the nature and discourse of mathematics. Ma (1999) provided the idea of *Profound Understanding of Fundamental Mathematics* (PUFM) that includes breadth, depth, and thoroughness. Further, Ball, Hill and Bass (2005) conceptualized and measured “mathematical knowledge for teaching.” They indicated that teacher content knowledge demands mathematical reasoning and insight into common teaching patterns (i.e., teaching a topic, responding to a student’s mistake, generating a representation of a certain topic, and responding to a naïve idea raised by students).

When prospective teachers enroll in the undergraduate mathematics teacher preparation program, they are trained to develop this “mathematics knowledge for teaching.” This knowledge development may help them to reflect on their beliefs, attitudes, and perceptions of mathematics teaching and learning, which is another factor that directly or indirectly influences teachers’ instructional behavior (e.g., Thompson, 1984, 1992). Mathematics teacher preparation should have some influence on prospective teachers’ changing their preconceived beliefs about teaching and learning based on their previous school experience, which is “the mastery of symbols and procedures, ignoring the processes of mathematics and the fact that mathematical knowledge often emerges from dealing with problem situations” (Thompson, 1992, p. 128). Cooney’s (1999) study indicates that this change did happen in the undergraduate teacher preparation program but not consistently. It is necessary for mathematics educators to understand how prospective teachers' prior beliefs about teaching and learning mathematics in order to change them.

Until several decades ago, researchers had noted that the distinctions between knowledge and beliefs are unclear because of the close connection (e.g., Sheffler, 1965). Thompson (1992) provides the distinction between beliefs and knowledge and thus, research on teachers' knowledge and beliefs becomes two major genres. If Shulman's (1986) PCK refers to the complex knowledge connected to the content and pedagogy and teachers' must possess to make the curriculum accessible to their students, the beliefs, on the other hand, provokes the relationship between what a teacher thinks about mathematics and how the teacher teaches. Ma (1999) argues that professional development occurs in three main stages of a teacher's career, which are schooling, teacher preparation, and teaching. In the teacher preparation stage, prospective teachers learn about learning and teaching mathematics that departs from their previous experience, "both directly in a specific designated context and indirectly through their practice" (Zaslavsky, Chapman, & Leikin, 2003, p.878).

Most mathematics educators share the view that teaching is strongly influenced by a teacher's personal experiences as a learner (e.g., Zaslavsky, 1995; Stigler & Hiebert, 1999), the current study investigates the issue of knowledge and beliefs in depth, examining teacher knowledge as it develops in the context of the classroom (both in the undergraduate teacher preparation program and the field practice) in an attempt to examine the interactive and dynamic nature of teacher knowledge. A constructivist perspective is use to examine the development of knowledge and beliefs of prospective teachers through a case study of teaching division of fractions. The focus of the study is on the prospective teachers' last year of undergraduate mathematics method course and the first year of field practice.

In this study, I focus on two time periods using three steps. In the method course period, first, I focus on a total of 27 prospective teachers' subject matter knowledge and its development and their beliefs changes in the method course. Next, I choose six participants from the total 27, based on different mathematics achievement, to identify the correlation between knowledge development and beliefs changes. In the field practice period, I follow up one prospective teacher in his field practice and focus on the way his beliefs influence his teaching and the development of the pedagogical content knowledge (PCK), through the teaching.

Through the descriptions of the prospective teachers' understanding of division of fractions (DoF) and the student teaching, the analysis is done by how teachers' knowledge for teaching DoF related to their changes in beliefs and how this change in beliefs might influence their classroom instruction and decision making of teaching DoF. Finally, the effects of the student teaching experiences in enhancing their understanding of DoF and their beliefs of mathematics learning and teaching was examined.

The content topic of DoF is chosen for this study because it is considered as one of the most mechanical and least understood topics in middle school (e.g., Fendel, 1987). Many students and many prospective teachers' knowledge of DoF are limited to the invert-and-multiply algorithm. However, teaching DoF requires a deep conceptual understanding of both division and fraction concepts (e.g., Armstrong & Bezuk, 1995;

Kieren, 1993; Sinincrop et al., 2002). Thus, this topic can provide more information about how prospective teachers develop their understanding of DoF, their learning and teaching of DoF, and their students' difficulties and misconception of this topic.

1.2 Purpose of This Study

Many studies have focused on in-service teachers' knowledge and beliefs for teaching mathematics (e.g., An, 2000; Ma, 1999; Thompson, 1984), but few of them focus on the way prospective teachers develop their knowledge and beliefs changes in their mathematics teacher education program. Further, the study how prospective teachers' field practice is impacted by their knowledge and beliefs they experienced in their method course and how the filed practice influence their further development also needs to be explored.

Thus, the current study focuses on the process of prospective teachers' knowledge development and changing beliefs from their mathematics teacher education to the field practice. The study investigates 1) what knowledge develops and beliefs changes happen though method course to the filed practice and 2) whether there is a relationship or connection between prospective teachers' knowledge and beliefs for teaching mathematics during their teacher education programs and student teaching. Is there a correlation between prospective teachers' knowledge development and possible changes in their beliefs for teaching fraction division? If there is a relationship, what is it and how do they interact to each other? Put another way, how does prospective teachers' mathematics knowledge and beliefs connect and influence to each other? How does knowledge development and beliefs changes impact the teaching behavior in the field practice? What does the field practice experiences influence their understanding for the effective mathematics teaching?

The study attempts to characterize prospective teachers' content knowledge (including common content knowledge and special content knowledge) (Ball, 2006) development or subject matter knowledge (SMK), especially their special content knowledge (SCK) (Ball, 2006; Ball, Thames & Phelps, 2008) for teaching and pedagogical content knowledge (PCK) during the last year in teacher education program. In particular, I focus on prospective teachers' SCK (Ball, 2006) and PCK (including knowledge of content and students and knowledge of content and teaching) (Ball, 2006; Ball, Thames & Phelps, 2008). The study also examines how these prospective teachers' beliefs changes both about DoF and about learning and teaching. Further, I intend to reveal the correlation between the knowledge development and beliefs changes.

Therefore, what follows are the research questions for this study in specific:

1) What and how does these prospective teachers' knowledge for teaching and understanding of DoF develop in their undergraduate mathematics methods course? In other words, what kind of SMK about DoF do they acquire through this time period? Here, I specific SCK in SMK to see what knowledge that prospective teachers develop for their teaching DoF. Moreover, What kinds of beliefs about DoF, about teaching and learning DoF does these prospective teachers hold and how did it change based on their learning experiences in the method course?

2) How do knowledge development and beliefs changes relate to each other? In particular, I intend to reveal whether the differences of the prospective teachers' common content knowledge (CCK) influences their SCK development. Further, I study whether and how the differences of CCK and SCK influence their beliefs changes.
3) Through reflection about the field practice, how does his knowledge for teaching DoF develop by his teaching experience? Especially, what kind of knowledge about students and teaching related to teaching content DoF does he develop? What does his beliefs and attitude about DoF as a content topic, teaching and learning, and the interaction of instruction do prospective teachers changes through their teaching experiences in field practice?
4) How does the beliefs impact his teaching behavior in the classroom instruction? How do the beliefs he holds impact his PCK (especial knowledge of content and students and knowledge of content and teaching) development? Further, how the field practice influence their knowledge (both content knowledge and PCK) development and beliefs changes?

1.3 Methodology of This Study

Therefore, this study uses a qualitative methodology. I analyze the data from two time periods by three steps. In the method course period, interviews concerning knowledge and beliefs are triangulated with the tests, surveys, concept mapping and the writing assignment. There are two steps in this time period. First, I focus on a total of 27 prospective teachers' SMK, including CCK and SCK, and its development and their beliefs changes in the method course. Next, I choose six participants from the total 27 based on different mathematics achievement to identify whether CCK differences impact the SCK development and beliefs changes and the correlation between SCK development and beliefs changes in the method course. In the field practice period, classroom observation of fraction division is triangulated with the interview. I follow up one prospective teacher in his field practice and focus on the way his beliefs influence his teaching and the development of PCK, through the teaching.

1.4 Limitation of This Study

Mention previously, I analyze the data from three steps through two time periods. In the second time period, which is in the time of the field practice, only one participant was followed up. He was teaching in seventh grade in the field practice and was willing to participate the study from the middle school mathematics methods course in 2007 to examine their knowledge reconstruction and development. The limitation of the number of the participants in the case study may only restrain the results as one aspect of perception of their teaching. Nevertheless, other multiple data can help me to have better understanding. Thus, the study mainly focuses on prospective teachers' documents from the methods course (including their concept mapping, lesson plan, and homework) and the interview and observation.

1.5 Significance of This Study

This study is significant in several ways. First, many researchers focus on middle grade prospective teachers' knowledge (e.g., Borko et al., 1992; Even & Tirosh, 2008) and indicate that prospective teachers' are not well-prepared in both conceptual and procedural knowledge. Most of these studies examine what kind of knowledge that prospective teachers have (e.g., Ball, 1990a; Borko et al., 1992). However, few studies directly focus on what knowledge specific prospective teachers learn and acquire in their teacher preparation program. Further, research on prospective teachers' beliefs change is also limited (e.g., Cooney, 2001). The case study provides some perspective on how prospective teacher's knowledge development and change in beliefs. Moreover, it examines how these two aspects are intertwined and interrelated with each other. Teaching is a complicated activity, and through teaching, prospective teachers' knowledge continuously develops and the beliefs continuously changes and becomes stable. This study may provide an opportunity to see in depth the possibility that teacher education program provided.

The case study considers the prospective teachers' last year of the teacher preparation program and field practice as a continuous, ongoing process. This approach can provide a complete portrait of prospective teacher's development in one of their important stages. It is different from the stage of their schooling, which they may hold some preconception of teaching and learning, and mathematics. It is also different from the stage that they are in their teaching career, in which their knowledge and beliefs are comparatively stable. Therefore, this is a transition stage for our future teachers and the better understanding is needed.

Finally, this study is not limited to content topic of DoF. Through the portrait of how prospective teachers developed and reconstructed their understanding and beliefs in certain ways, it is expected to show how teachers' knowledge and beliefs affects their instructional decision making, which directly connected with students' achievement.

CHAPTER Ⅱ

LITERATURE REVEIW

In this chapter prior research related to the study is reviewed from three aspects: (1) Teachers' SCK, PCK and MKT; (2) Teachers' beliefs and perceptions; and (3) Knowledge and beliefs about teaching and learning of division of fractions (DoF). These three aspects are not separated because knowledge and beliefs are closely connected (e.g., Thompson, 1984, 1992). Since teacher education is an important stage for prospective teachers to reconstruct and develop their knowledge and challenge their beliefs, I also address both knowledge and beliefs in this stage about the context of teaching and learning DoF as example.

2.1 Teachers' Knowledge for Teaching Mathematics

In *Foundation for Success: The Final Report of the National Mathematics Advisory Panel* (2008), the researchers argued that students' substantial differences in mathematics achievement are attributable to differences in teachers. Thus, teachers are crucial to students' learning mathematics.

According to Ball, Lubienski, and Mewborn (2001), teachers and teacher knowledge have been a significant focus of research since the publication of the third edition of *Handbook of Research on Teaching* (Wittrock, 1986). In the same handbook, Shulman (1986) argues that there has "missing program" in educational research of the study of "teachers' cognitive understanding of subject matter content and the relationships between such understanding and the instruction teachers provide for students" (1986a, p.25) and indicates that teacher knowledge is one important fact that influences classroom instruction. He proposes three kinds of knowledge are important for teaching school mathematics, subject matter knowledge, pedagogical knowledge, and curricular knowledge. Further, he proposes the idea of PCK and refined the model of teacher's knowledge in his paper. According to Shulman, teacher should have (1) knowledge of subject matter, which is mathematics for mathematics teacher; (2) knowledge of pedagogy, the knowledge of student; and (3) pedagogical content knowledge, which is the knowledge of instructional practice (e.g., Shulman, 1986). Shulman (1986) extended subject-matter knowledge (SMK) for teaching more in-depth and proposed PCK as "a knowledge of subject matter topics and issues appropriate to the diverse abilities and interest of learners" (Shulman, 1986, p.9).

In another paper, Shulman (1987) explores to seven knowledge bases needed for teaching, which is in detail from the previous three parts. It is the way that teachers need to hold and use mathematics in order to teach mathematics. Harel (1993) explores Shulman's ideas and indicates that there are three components, which are mathematics content, epistemology, and pedagogy. Here, mathematics content refers to the depth and breadth of the mathematics knowledge; epistemology refers to the teachers' understanding of how students learn mathematics; and pedagogy refers to the teachers have ability to teach. It is the way of knowing and using mathematics that differs from the way mathematicians hold and use mathematics (Ball & Bass, 2000).

In mathematics teaching, Ball & Bass (2000, 2003) explore PCK to teachers' mathematical knowledge for teaching (MTC). Thus, teacher knowledge is "a large, integrated, functioning system with each part difficult to isolate" (Fennema & Frank, 1992, p.152). It includes the several components that mentioned earlier.

2.1.1 Prior research on teachers' mathematics knowledge

Research on mathematics SMK. The teacher's primary role is to help students achieve understanding of the subject matter. In order to do so, teachers need to have solid knowledge of the subject matter. "A teacher who has solid mathematical knowledge for teaching is more capable of helping his/her students to achieve a meaningful understanding of the subject matter" (Even, 1990, p.521). Teachers' mathematics understanding and knowledge have been the focus of attention of researchers. Various terminologies are used to describe teacher's mathematics or subject matter knowledge (SMK).

SMK is one of the important knowledge components as well prepared teachers. But, what is a "solid mathematics knowledge"? Initially, teacher's SMK was defined by the number of courses taken in college or teachers' scores on superficial standardized tests (e.g., Wilson, Shulman & Richert, 1987), which was using a quantitative measurement. Leinhard and Smith (1985) describe mathematics SMK in detail. It includes "conceptual understanding, the particular algorithmic operations, the connection between different algorithmic procedures, the subset of the number system being drawn upon, understanding of classes of students errors and curriculum presentation" (p.248). Shulman (1986) makes a distinction between of SMK and PCK.

Teachers' SMK, also defined as content knowledge in Shulman's paper, is not defined by the number of courses they have taken or their success on standardized test, but by analyzed what it means to know mathematics.

What does it mean to know mathematics? According to Shulman (1986):

> The teacher needs to not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied. Moreover, we expect the teacher to understand why a given topic is particularly central to a discipline whereas another may be somewhat peripheral. (Shulman, 1986, p.9)

Shulman (1986) distinguishes between two kinds of understanding of the subject matter that teachers need to have- knowing "that" and knowing "why". They need to not only define the mathematical fact or claim to students, but also be able to explain "why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice" (p.9). Therefore, according to Shulman, to think subject content knowledge properly should go beyond knowledge of the facts or concepts of a domain. It also requires understanding the structure of the subject matter, which Schwab (1978) defined decades ago.

Further, Shulman and his colleagues (1987) reframe the definition of subject matter understanding to include the "nature, form, organization, and content of teacher knowledge" (Grossamn, 1989, pp.25-26). Thus, SMK includes knowledge of key facts, concepts, principles, and explanatory frameworks of a discipline, as well as the rules of evidence used to guide inquiry in the field.

Based on the previous definition, Ball (1988, 1991) identifies teachers' understanding of mathematics as interweaving ideas *of* and *about* the subject (1990b, 1991). According to Ball, teachers need have *substantive knowledge of mathematics,* which is the knowledge particular concepts and procedures. By knowledge *of* mathematics she means substantive knowledge of subject: comprehension of particular topics, procedures, and concepts, and the relationship among these topics, procedures, and concepts. By knowledge *about* mathematics she meant syntactic knowledge, which is comprehension of the nature and discourse of mathematics. Teachers also need to understand *about* mathematics, which means to "understand where the knowledge comes from and how it is justified, what is means to *do* mathematics, what the connections are between mathematics and other domain" (p.21). Also, the analysis of teachers' knowledge *about* mathematics includes knowledge about the nature of mathematics (Even, 1990). This is more general knowledge about a discipline, which guides the use of conceptual and procedural knowledge. Ball (1988) indicates that individual teacher must have knowledge of mathematics characterized by an explicit conceptual understanding of the principles and meaning underlying mathematical procedures and by connectedness of mathematics, rules and definitions.

According to Ball (1988), the substantive knowledge of mathematics includes "understandings about the nature of knowledge in the discipline — where it comes from, how it changes, and how truth is established" (p.44). It also included what it means to "know" and "do" mathematics, the relative centrality of different ideas" (p.44).

Ball (1988) finds that prospective teachers do not think about mathematics from a disciplinary perspective but instead think of a body of rules and procedures. They think that "doing mathematics" means following standard procedures to arrive at the right answers. They do algorithms without understanding the concept or principle back of the algorithm. Moreover, for substantive knowledge, Ball (1988) indicates the prospective teachers have difficulties to explain why a procedure works, to connect one mathematical concept to another, and to links between mathematics and other domains.

Fennema and Franke (1992) also use Shulman's model as a base to discuss five components of teachers' knowledge: the knowledge f the content of mathematics, knowledge of pedagogy, knowledge of students' cognitions, context specific knowledge, and teachers' beliefs. The content of mathematics includes teachers' knowledge of the concepts, procedures, and problem-solving processes within the domain in which they teach. Ma (1999) further provides the idea of *Profound Understanding of Fundamental Mathematics* (PUFM) that has 1) depth that referred to large and powerful basic ideas; 2) breath that had to do with multiple perspectives, and 3) thoroughness that was essential to weave ideas in a coherent whole. According to Ma, Chinese teachers presented their own "knowledge package" through making explicit the connections between and among mathematical topics to facilitate learning. Ma provides the examples of teachers' understanding of subtraction with regrouping. In the context of subtraction regrouping, "proficiency in composing and decomposing a 10 is such a procedural topic" (p.23).

Moreover, conceptual topics include "mainly for a thorough understanding of the rationale underlying the algorithm" (p.23). Therefore, "a comprehensive understanding of the concept of regrouping" is the conceptual topic, which is supporting the learning of mathematics. For example, "the concept of *the rate of composing a higher value unit*" and the concept of *"inverse operations"* are basic principles. In order to enable to students learn this content topic, Ma indicated that teachers should hold PUMF for teaching.

According to Kilpatrick et al. (2001), teachers should have not only the knowledge of mathematics concepts and procedures, their relationship, representation, and mathematics as a discipline, but also a "consideration of the goals of mathematics instruction and provides a basis for discriminating and prioritizing those goals" (p.371). Further, Ball, Hill and Bass (2005) conceptualize and measure "mathematical knowledge for teaching"(MKT) based on four common teaching patterns (i.e., teaching a topic, responding to a student's mistake, generating a representation of a certain topic, and responding to a naïve idea raised by students). Thus, high quality mathematics teaching requires teachers to have sufficient mathematics knowledge that mentioned above.

Research on PCK. According to Shulman (1986), pedagogical content knowledge (PCK) is "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (p.9). PCK includes 1) how to use the way of the representation to make the subject matter comprehensible to other" (p.9); 2) "an understanding of what makes the learning specific topics easy or difficult" (p.9); moreover, teachers should have 3) an understanding of the misconception that students may hold and how to help students to overcome. It means to represent "specific topics and issues in ways that are appropriate to the diverse abilities and interests of learners" (Borko et al., P.196). Sowder (1995) uses *pedagogy* and *epistemology* to identify PCK. *Pedagogy* refers to "the ability to teach in accordance with the nature of how students learn mathematics" (p.1), while *epistemology* refers to "the teachers' understanding of how students learn mathematics" (p.1).

Although PCK is commonly believed to be a transformation of at least two constituent knowledge domains: general pedagogical knowledge and subject matter knowledge, the interpretation of PCK has ambiguities (Marks, 1990). Since the ambiguities, many researchers developed their study of PCK by focusing on different aspects. Marks (1990) indicated the ambiguities of PCK from three reasons. First, PCK contains elements of both SMK and general pedagogical knowledge. It included that "teachers' anticipates students' error and designs instruction to avoid it" (p.8) and that teachers aware to identify significant subject matter concepts ad highlight them in instruction. Therefore, "each level of interpretation is valid" (p.8). Another reason of ambiguity involves the questions whether there is a difference of PCK about mathematics teaching or learning to apply to any other mathematics content topics.

Ma (1999) focuses on teachers' knowledge of student' misconception and representation. Ma indicates that Chinese teachers used multiple representation and alternative computational approaches to make sense of the "invert and multiply" algorithm. For example, some teachers proved by converting the operation with fractions into one with whole numbers, which is

$$1\frac{3}{4} \div \frac{1}{2} = 1\frac{3}{4} \div (1 \div 2)$$

$$= 1\frac{3}{4} \div 1 \times 2$$

$$= 1\frac{3}{4} \times 2 \div 1$$

$$= 1\frac{3}{4} \times 2$$

According to this Chinese teacher in Ma's study, since students have the prior knowledge of commutative law, knowledge of taking of f and add parentheses, and the knowledge a fraction equivalent to the result of a division, students can easily understand the proof and even more, they can do it by themselves. Another example that Ma (1999) examines Chinese teachers used verbal explanation to justify the algorithm through drawing on the meaning of expression $1\frac{3}{4} \div \frac{1}{2}$:

> Why is it equal to multiplying by the reciprocal of the divisor? $1\frac{3}{4} \div \frac{1}{2}$ means that $\frac{1}{2}$ of a number is $1\frac{3}{4}$. The answer, as one can imagine, will be $3\frac{1}{2}$, which is exactly the same as the answer of $1\frac{3}{4} \times 2$. 2 is the reciprocal of $\frac{1}{2}$. This is how I would explain to my students. (p.60)

Chinese teachers' PUMK provided them with a solid knowledge base to use multiple representations in their classroom instruction.

Ma's study provides an important aspect of teachers' mathematical knowledge for teaching. "Ma's notion of knowledge packages represents a particularly generative form of and structure for pedagogical content knowledge" (Ball, Lubienske, & Mewborn, 2001, p.449). However, "(t)o understanding the mathematical work of teaching would require a closer look at practice, with an eye to the mathematical understanding that is needed to carry out the work" (p.449).

Feenema and Franke's (1992) model also indicates that pedagogical knowledge is teachers' knowledge of teaching procedures, including planning, organization, management, and motivation. Learners' cognitions include knowledge of how students think and learn.

Kilpatrick et al (2001) argues that teachers should have knowledge of instructional practice, which "includes knowledge of curriculum, knowledge of tasks and tools for teaching important mathematical idea, knowledge of how to design and manage classroom discourse, and knowledge of classroom norms that support the development of mathematical proficiency" (Kilpatrick et al., p.372). This requires teachers not only have a deep understanding of mathematics as a discipline, but also have an understanding of students' cognitive development in order to design and plan the classroom instruction.

In summary, due to the unclear definition and boundary of PCK and SMK, researchers define and explore the idea of PCK for understanding and developing teachers' classroom instruction in different ways. For Shulman (1987), SMK will be converted or transformed to PCK, which is a form appropriate for teaching. According to Kinach (2002), the obvious interface between SMK and PCK is central to the transformation process. Thus, Ball et al. (2001) explore Shulman's study and develop the idea of MKT to identify teacher knowledge.

Research on mathematics knowledge for teaching (MKT). Ball and her colleagues (Ball, 2006; Ball &Bass 2006) explores Shulman's idea of SMK and PCK, developing the idea of teachers' Mathematical Knowledge for Teaching (MKT). She provides a clear relationship between Shulman's (1986) SMK and PCK (see Figure 1). There are three components in MKT related to SMT. They are common content knowledge (CCK), specialized content knowledge (SCK), knowledge at mathematical horizon. CCK is the mathematical knowledge and skill expected of any well-educated adult. For example, a teacher needs to recognize wrong answers, spot inaccurate definitions in textbooks, and use notation correctly for CCK. SCK is a knowledge base mainly needed by teachers in their work and beyond that expected of any well-educated adult. Thus, although is not "contained in pedagogical content knowledge, but …is essential to effective teaching" (Ball et al., 2008 p.390). As a teacher, s/he should be able to analyze errors and evaluate alternative ideas, give mathematical explanations and use mathematical representation, and be explicit about mathematical language and practice.

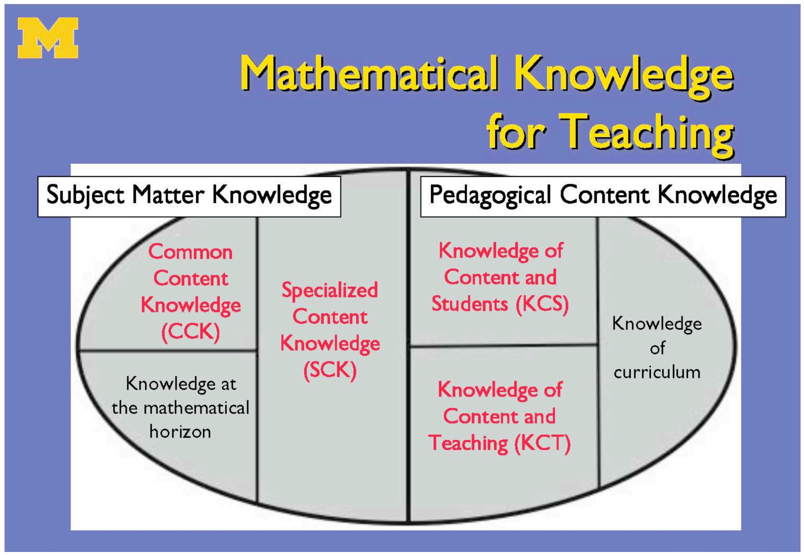


Figure 1. MKT and Shulman's SMK.

Ball (2006) further provides examples to show how the different knowledge components play different roles concerning students' error (see Figure 2).

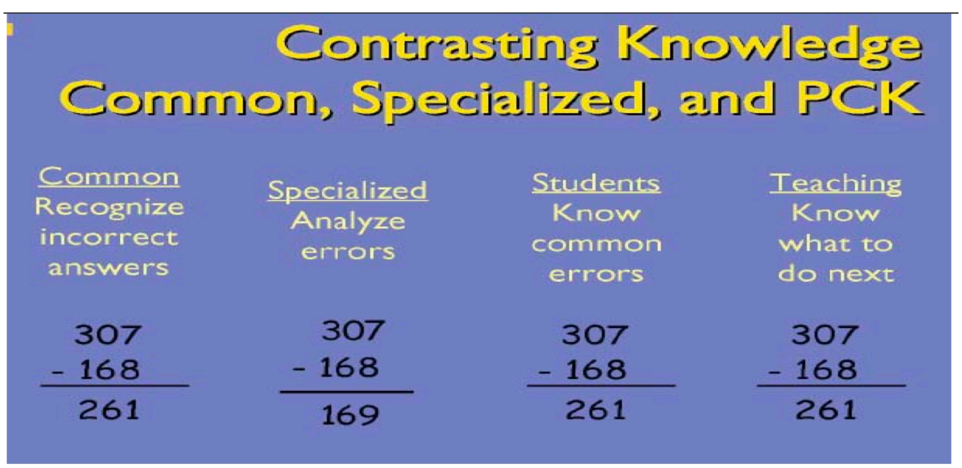


Figure 2. Example for the components of MKT.

As a teacher, s/he must be able to point that 261 is incorrect. This “does not require any special knowledge to do” (Ball et al, p.397). However, “teaching involves more than identifying an incorrect answer” (p.397). Teachers need to be able to “perform this kind of mathematical error analysis efficiently and fluently” (p.397). Ball argues that teachers’ should know the possible difficulties with the algorithm for subtracting multi-digit numbers that cause the errors presented here. Thus, teachers require a kind of mathematical reasoning “that most adults do not need on a regular basis” (p.397). Ball et al. (2008) provide SCK required for teaching (see Figure 3).

Presenting mathematical ideas
Responding to students' “why” questions
Finding an example to make a specific mathematical point
Recognizing what is involved in using a particular representation
Linking representations to underlying ideas and to other representations
Connecting a topic being taught to topics from prior or future years
Explaining mathematical goals and purposes to parents
Appraising and adapting the mathematical content of textbooks
Modifying tasks to be either easier or harder
Evaluating the plausibility of students’ claims (often quickly)
Giving or evaluating mathematical explanations
Choosing and developing useable definitions
Using mathematical notation and language and critiquing its use
Asking productive mathematical questions
Selecting representations for particular purposes
Inspecting equivalencies

Figure 3. SCK for mathematics teaching.

In general, "teaching involves the use of decompressed mathematical knowledge that might be taught directly to students as they develop understanding" (p.400).

Moreover, Ball (2006) argues that teachers should combine knowledge of mathematics with knowledge of students (KCS) and knowledge of teaching (KCT). According to Ball and her colleagues (2006, 2008), KCS is defined as content knowledge intertwined with knowledge of how students think about, know, or learn the particular content:

> KCS is used in tasks of teaching that involve attending to both the specific content and something particular about learners, for instance, how students typically learn to add fractions and the mistakes or misconceptions that commonly arise during this process. In teaching students to add fractions, a teacher might be aware that students, who often have difficulty with the multiplicative nature of fractions, are likely to add the numerators and denominators of two fractions. (Ball et al., 2008, p.375)

Such knowledge might help the teacher design instruction to address this likely issue. Thus, for knowledge of students, teachers need to be able to 1) anticipate student errors and common misconceptions; 2) interpret students' thinking; and 3) predict what students are likely to do with specific tasks and what they will find interesting or challenging. Further, teachers need to be able to sequence content for instruction and using different representation.

According to Hill et al. (2005), MKT means "the mathematical knowledge used to carry out *the work of teaching mathematics*" (p. 373). According to Ball et al., (2005):

> (K)nowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the algorithm reliably. Further, it indicates that there are predictable and recurrent tasks that teachers face that are deeply entwined with mathematics and mathematical reasoning—figuring where a student has gone wrong (error analysis), explaining the basis for an algorithm in words that children can understand and showing why it works (principled knowledge of algorithms and mathematical reasoning), and using mathematical representations. Important to note is that each of these common tasks of teaching involves *mathematical* reasoning as much as it does pedagogical thinking. (p.21)

MKT requires teachers not only explain why the algorithm works, but also use "an effective way to represent the *meaning* of the algorithm" (Ball, Hill, & Bass, 2005, p.20). That is, teachers need to think from students' perspective and to consider what it helps to understand a mathematical idea for someone first seeing it. Moreover, it requires teachers to recognize students' errors and to analyze the source of the error.

Overall, in order to be an effective teacher, teachers should have knowledge of mathematics and the ability to use it in practice.

2.1.2 Research on the evaluation and assessment of SMK, PCK, and MKT

After identifying and conceptualizing SMK, PCK and MKT, the next focal point for researchers is to determine the extent of SMK, PCK and MKT that teachers hold, including in-service teachers or prospective teachers. Most of assessment efforts attempt to appraise the adequacy of individual teachers' knowledge or the quality of their performance (Hill, Sleep, Lewis & Ball, 2007). Others try to assess teachers' knowledge for mathematics classroom teaching, connecting classroom behavior with their knowledge (e.g., Borko et al., 1992; Leinhardt & Smith, 1985). Researchers use both quantitative and qualitative methodology to examine SMK and PCK. This section focuses on the major methodologies used by researchers, including interviews, pictorial representations (concept mapping and card sorting), and classroom observation.

Interviews and questionnaires are widely used to examine SMK, PCK and MKT, especially PCK and MKT (e.g., Ball, 1988, 1990; Even, 1990; Ma, 1999). In most studies (e.g., Ball, 1988, 1990; Even, 1990; Ma, 1999), questionnaires are "grounded in scenarios of classroom teaching and woven with particular subject matter topics" (p.452).

Conceptual mapping has been defined as a "metalearning strategy" (Wandersee, 1990, p.923). Cognitive psychologists seem to agree that the internal representation of knowledge resembles networks of ideas that are organized and structured (e.g., Hiebert & Carpenter, 1992). Thus, concept mapping is a direct method of looking at the organization and structure of an individual's knowledge within a particular domain and at the fluency and efficiency with which the knowledge can be used (Williams, 1998). Concept mapping can be a helpful meta-cognitive tool, promoting understanding in which new material interacts with the students' existing cognitive structure. The interaction of new and existing knowledge is made easier if the existing knowledge is made explicit to both teacher and student. This is described as 'meaningful learning' (Kinchin et al., 2000). The construction of a concept map is intended to reveal the perceptions of the map's author, rather than a reproduction of memorized facts (Jonassen et al., 1991). Researchers have argued that the concept maps can represent graphically the development of knowledge constructed and reconstructed (Kinchin et al., 2000).

An alternative way of concept mapping is card sorting. A set of cards with each card containing a particular concept, idea, and principle are provided. Teachers are asked to place the cards in an arrangement that best illustrates the relationship among the idea or principle containing on the cards.

Classroom observation is another way to assess teacher knowledge (e.g., Borko et al., 1992; Leinhardt & Smith, 1985; TIMSS, 1999). Hill et al. (2007) indicated "analysis that appeared in print was primarily qualitative, with researchers using methods and coding systems tailored specifically to the mathematical topics and questions at hand" (p.125). Also, researchers mainly combined observational data with other data to gain insight into teacher mathematical knowledge for teaching (Hill et al., 2007).

Ball (1988, 1990a, 1990b) mainly studies teacher education students' subject knowledge (mainly substantive knowledge) in their beginning of undergraduate teacher preparation program. In the study (Ball, 1990a), Ball uses questionnaires and interviews to assess teachers' understanding of DoF.

She first gave the questionnaire item with the statement of "$4\frac{1}{4} \div \frac{1}{2}$" with five story problems representing a given division problem, one of which corresponded to the expression. The results showed that although 30.3% of elementary prospective teachers and 40% students who chose an appropriate representation for story, 30% also marked one or more of the inappropriate representations. A majority of prospective teachers

(69.7% of elementary and 60% of secondary) chose an inappropriate representation or could not generate a representation.

Further, Ball asked teacher to do the problem of $1\frac{3}{4} \div \frac{1}{2}$ and asked them to "think of something to represent the statement $1\frac{3}{4} \div \frac{1}{2}$" (Ball, 1990a, p.453). She found that only four of 35 secondary teacher candidates of and none of the elementary candidates were able to generate a representation for the division. For the inappropriate representations, the most frequent error was to represent division by 2 instead of division by ½. Ball indicates that prospective teachers made this kind of error they were familiar with round model (i.e., pizzas or pies) representation when they learned division.   Ball (1990a) concluded that teacher education students' substantive mathematics knowledge was both rule-bound and compartmentalized. They lack explicit understanding of concepts and principles. Thus, it is difficult for them to discuss, justify and explain their calculation procedure.

Ball (1990b) further interviewed the prospective teachers to examine mathematical and pedagogical knowledge in a particular content topic context, division. The prospective teachers were in the same stage with the previous study. Explored from the previous study, Ball provides another two scenarios about division by zero and solving algebraic equations involving division. Prospective teachers in this study need to response students' question that what 7 divided by 0 is. Ball revealed that instead of view case to case on the concept of division, teacher candidates viewed each question as a specific, single piece of knowledge, which means they viewed each piece of knowledge separated. Two scenarios asked these teacher candidates how they respond to the question that students may raise. Since prospective teachers' knowledge of division seemed founded more on memorization than on conceptual understanding (Ball, 1988, 1990a, 1990b), they could not provide an answer to articulate underlying meanings and principle (Ball, 1990b).

Even (1990, 1993) used the questionnaire and interview to address prospective teachers' SMK and PCK in their last stage. He chose 162 students to build a theoretical framework of SMK for teaching mathematical concepts in general and the function concept in particular, addressing teachers' SMK about conceptual understanding of function: arbitrariness and ambivalence. The questionnaire included nine nonstandard problems addressing the different aspects of teachers' SMK about functions and six items referring to "students" mistaken solutions or misunderstandings to be analyzed. In addition, an interview was conducted, asking students to reflect their thinking and to explain and clarify their answers to the questionnaire. Many prospective teachers held the concepts that function are equations and all functions can be represented by formulas. From the results, Even and Torish (2008) also indicates that students lack subject matter understanding of arbitrariness. Moreover, he finds that many prospective teachers only presented procedural knowledge without concern for meaning, since they did not know why univalence is needed.

Borko et al. (1992) examines prospective teachers' knowledge through another perspective by observing the prospective teacher's student teaching. The study focused on a few minutes of a lesson that was taught by a student teacher in which for an explanation of the division of fractions algorithm. The results show that this teacher used a concrete model to represent multiplication rather than division. Based on the classroom observation, the researchers used the interview data and open-ended mathematical problems to explain her classroom instruction.

2.1.3 Research on prospective teachers' knowledge development

Grossman (1990) argued that prospective teachers' subject matter and pedagogical understanding would interact during the process of learning to teach a subject. Thus, according to Grossman (1990), the subject-specific method course provided them to develop PCK. Grossman indicated the role of subject-specific method courses is to shape prospective teachers' conceptions of the school subject, the purposes for teaching the subject, and desirable learning outcomes for each subject.

Recent studies have indicated that teacher preparation may be one predictor of students' mathematics achievement (e.g., Schmidt et al., 2007). Although academic mathematics preparation may positively connect with a higher student achievement, some researchers have also emphasized positive effects of mathematics education courses, with courses in education contributing more to student achievement gains than undergraduate mathematics courses (e.g., Monk, 1994). In other words, education coursework, including subject-specific method courses, is useful and have a higher correlation with student achievement than mathematics study alone (e.g., Wilson, Floden, &Ferrini-Mundy, 2001; 2002).

In Ball’s (1988) study about prospective teachers’ knowledge and beliefs before they enrolled in a math methods course, concludes that the prospective teachers thought mathematics as school subject, which is a body of rules and procedures. Therefore, for them, "doing mathematics" meant to follow standard procedures to get the right answers and few of them can explicitly articulate underlying meanings or principles of mathematical concept and procedures (Ball, 1988). On the other hand, from a teaching and learning perspective, Ball indicates that most prospective teachers that she studied know about teaching and learning and about the teacher's role in helping students learn mathematics. However, most prospective teachers tended to think of learning mathematics as entirely an individualistic process of acquiring information and technique and the need for repetition and practice. Therefore, for these prospective teachers, the teachers’ role is to tell students definitions and "how to do it” (Ball, 1988).

Ball (1988) found that prospective teachers have their own beliefs about mathematics teaching and learning from their previous experience. Dewey (1938) notes, "experience and education cannot be directly equated to each other. For some experiences are miseducative. Any experience is miseducative that has the effect of arresting or distorting the growth of further experience" (p. 26). Ball (1989) also indicates that experiences may inhibit open mindedness, freeze ways of looking, or engender undesirable attitudes. In the case of prospective mathematics teachers, their experiences have often “persuaded them that mathematics is a fixed body of rules, a dull and uninteresting subject best taught through memorization and drill, and that they themselves are not good at math (Ball, 1989, p. 4).

According to Kinach (2002), teacher education should both transform and deepen prospective teachers' understanding of subject matter and redirect their habitual ways of thinking about subject matter for teaching. He indicates that the knowledge transformation process, which is from subject matter knowledge to pedagogical content knowledge, is very important in teacher education program. Lampert and Ball (1999) criticized teacher education, saying that it did not change in the way that fits the new stance of mathematics education and effective teaching. Ball (1991) also points to the weak impact of professional education on teachers, stating that the university teacher preparation program, the prospective teachers' school experience of mathematics has instilled not only traditional images of teaching and learning but has also shaped their understandings of mathematics.

Capraro, Capraro, Parker, Kulm, and Raulerson (2005) also indicate that undergraduate mathematics teacher preparation programs may choose to focus mainly on presenting mathematics content, with little consideration of preparing teachers to actively inquire about mathematics teaching and learning, or focus on presenting pedagogical issues with little regard for depth of mathematical content. Also, they criticize teacher preparation programs that are often measured by teacher certification examinations, which may not align well with specific grade levels or required content-specific subtests for prospective teacher.

They also claim that the way pedagogical awareness is taught should relate to deeper and broader understandings of mathematical concepts for prospective teachers.

Some teacher preparation programs in the university have little connection to actual teaching practice and no preparation for further teaching practice.

Kinach (2002) focus on prospective teachers' knowledge transformation process. He used a modification of Pekins and Simmons' *level of understanding* framework. In this framework, it included four different types of subject matter understanding. They are 1) concept-level understanding, which refers to "knowledge about experience with the generalized ideas that define, bound, and guide inquiry in a discipline (Kinach, p.55); 2) problem solving level understanding, which "refers to general and domain-specific strategies and heuristic schemas for monitoring one's own thinking" (pp. 55-56); 3) epistemic-level understanding "refers to the warrants for evidence in a discipline" (p.56); 4) inquiry-level understanding "refers to the generation of new knowledge that advances thinking in the field" (p.56). Based on this framework, the researcher indicates the change process. He considers qualitative data of students' written journals, written homework assignments on instructional explanations, and transcribed video-recordings of classroom discussion. He finds that if prospective teachers bring undesirable forms of PCK, the undesirable forms of PCK do not lead to the deeper kinds of relational SMK of concept, problem, epistemic and inquiry levels of understanding.

Magnusson, Krajcik and Borko (1999) describe a model of PCK development. For the case of teacher A, they indicate that this teacher has more subject matter knowledge than the two other types of knowledge, which influence her/his PCK. In contrast, teacher B has more pedagogical knowledge, which has a greater influence on the transformation of his/her knowledge into PCK. They conclude that there are different routes or multiple pathways to developing PCK for a specific content topic.

As mentioned earlier, most researchers critique teacher education for its low quality and weak effects of teacher education from multiple perspectives (Ball, 2007). Also, many researchers focus on middle grade prospective teachers' knowledge (e.g., Borko et al., 1992) and indicate that prospective teachers were not well prepared in both conceptual and procedural knowledge. However, researchers indicate that no research directly assesses what teachers learn and acquire in their teacher preparation program and then evaluates the relationship of that knowledge to student learning or teacher behavior (e.g., Wilson, Floden, &Ferrini-Mundy, 2001; 2002). Moreover, teaching is a complicated activity, and through teaching, prospective teachers' knowledge continuously develops. The study about knowledge development of prospective teachers in their student teaching process is limited.

2.2 Teachers' Beliefs about Teaching Mathematics

If knowledge is the "cognitive outcome of teacher education" (Ernest, 1989), then "beliefs and attitudes represent the affective outcome" (Ernest, 1989).
Although beliefs are hard to change (Pajares, 1992), researchers have found that after participating in the undergraduate mathematics methods course, prospective teachers changed their beliefs in a way that was more consistent with current mathematics education reform (e.g., Wilkins & Brand, 2004). Since teachers' beliefs about teaching and learning mathematics, as well as the nature of mathematics, influence teachers'

instructional behavior (e.g., Thompson, 1984, 1992), an important part of the teacher education program, especially the mathematics method course, should focus on the development of prospective teachers' beliefs (e.g., Wilkins & Brand, 2004). To help prospective teachers develop beliefs that are consistent with mathematical education reform, it is important to study how the beliefs change during teacher education program.

Researchers view teachers' beliefs in different ways. For example, some researchers understand teachers' belief as a part of teachers' PCK (e.g., Ball, 1990; Fennema & Franke, 1992; Gess-Newsome & Lederman, 1999). Others make an explicit distinction between knowledge and beliefs (e.g., Thompson, 1984, 1992).

2.2.1 Terminology for teachers' beliefs

A teachers' belief system includes conceptions, values and ideology, which are referred to as teachers' "dispositions" (Kuhns & Ball, 1986). Philipp (2007) comments on Thompson's (1992) characterization of beliefs and suggested different terminology be used in educational research. Thompson states that "(f)or the most part, researchers have assumed that readers know what beliefs are" (p.129). According to Philipp, there are several definitions or descriptions of terms, including *affects, beliefs, conception, identity, knowledge* and *value.*

Affect is "a disposition or tendency or an emotion or feeling attached to an idea or object" (Philipp, 2007, p.259), including two components, emotions and attitudes. In *handbook of mathematics teaching and learning,* McLeod (1992) analyzed the research on *affect* in mathematics education. Based on previous research, McLeod indicated that beliefs, attitudes, and emotions are used to describe a wide range of affective responses to mathematics. However, these terms "vary in the stability of the affective response" (p. 578). Beliefs and attitudes are generally stable, but emotions may change rapidly. Further, McLeod indicates that many researchers use attitudes as a general term that includes beliefs about mathematics and about self. Attitudes toward mathematics mainly indicated the way students respond to the task. Emotions, on the other hand, lack a theoretical framework to interpret the role of the emotions in learning of mathematics. The study of mathematical affects mainly focused on students' affective characteristics in order to provide useful information to teachers and classroom teaching. Moreover, beliefs are more cognitive in nature than attitude. McLeod concludes that beliefs tend to develop gradually and that cultural factors play a key role in their development.

Philipp (2007) indicates that although Thompson (1992) uses both term *belief* and *conceptions* and her definition of *conceptions* included *beliefs.* She refers to teachers' conceptions "as a more general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like" (p.130). Belief, according to Phillip (2007), "held understandings, premises, or propositions about the world that are thought to be true" (p.259).

> "Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitude" (p.259).

As mentioned previously, Thompson (1992) provides a clear distinction between beliefs and knowledge through *conviction* and *consensuality.*

Value is another component, which is considered an influential factor in mathematics classroom instruction. Philipp (2008) emphasizes that value is “less context-specific than beliefs” (p.259). “Whereas beliefs are associated with a true/false dichotomy, values are associated with a desirable/undesirable dichotomy” (p.259). Bishop, Seah, and Chin (2003) indicates that the value mathematics teachers brought to affect what and how they teacher, and further, “those values influence the next generation on matters such as the nature of mathematics an how it is best taught and learned” (p.718). It means that “beliefs are true/false statements about constructs whereas the choice of the particular constructs one finds desirable or undesirable represents one’s more context-independent value” (p.265). They also indicate that researchers view these two terms similar and can be interchanged to each other.

Overall, some distinctions have been drawn among affects, beliefs, concepts and values. Thompson (1992) note there is a higher degree of consistency of the relationship between teachers’ conceptions of mathematics and their instructional practices than by teachers’ conceptions about teaching and learning and their practice. This study uses the term “beliefs” to mean *what* beliefs teachers hold and *how* teachers hold these beliefs.

2.2.2 Research on teachers’ beliefs

Researchers are interested in studying teachers’ beliefs about teaching, learning, and the nature of mathematics because these affective factors can affect teachers’ instructional behaviors (Thompson, 1984, 1992). Teachers' beliefs have been studied for several decades (e.g., Thompson, 1984). Research has focused on teachers' thinking and decision-making process (e.g., Clark & Peterson, 1986). Thompson (1992) notes the importance for researchers studying mathematics teachers' beliefs to make explicit to themselves and others that the perspectives they hold about teaching, learning, and the nature of mathematics.

Gess-Newsome (1999) indicates that the "differences in teachers' beliefs and conceptions about subject matter were directly linked to teachers' judgments about content and were noted as a primary factor influencing planning" (p.52). Gess-Newsome (1999) interprets Shulman's definition of SMK linked between "the knowledge teachers possess, the instructional actions they employ, and the learning, attitudes, and beliefs of the students they teach" (p.52). Peterson, Fennema and Carpenter (1991) indicated teachers' beliefs about students' knowledge, as well as their thinking about instruction, learning, and assessment were all influenced by their beliefs and their knowledge.

Ball (1991) considers the idea of teachers' beliefs as part of their knowledge. She distinguishes between the knowledge *of* mathematics (the subject matter) and knowledge *about* mathematics (its nature and discourse). According to Ball, knowledge about mathematics includes:

> Understanding about the nature of mathematical knowledge and activity: what is entailed in doing mathematics and how truth is established in the domain. What counts as a solution in mathematics? How are solutions justified and conjectures disproved? Which ideas are arbitrary or conventional and which are necessary or

> logical? Knowledge about mathematics entails understanding the role of mathematical tools and accepted knowledge in the pursuit of new ideas, generalizations, and procedures. (p.7)

However, Thompson (1992) discussed the distinction between teacher knowledge and beliefs. According to Thompson, “one feature of beliefs is that they can be held with varying degrees of conviction” (p.129), where knowledge is generally not thought of in this way. Philipp (2007) provides an example that one might say that he or she believed something strongly, while one might not know a fact strongly. “Another distinctive feature of beliefs is that they are not consensual” (Thompson, 1992, p.129). For example, one is generally aware that others may believe differently and that their stances cannot be disproved, whereas with respect to knowledge, one finds “general agreement about procedures for evaluating and judging its validity” (Thompson, 1992, p.130). Thompson (1992) further discusses two very different views of the nature of mathematics. First, “Mathematics is a discipline characterized by accurate results and infallible procedures… Knowing mathematics is equivalent to being skillful in performing procedures and being able to identify the basic concepts of the discipline” (p.127). This concept of mathematics leads classroom teaching, “where concepts and procedures are presented in a clear way and opportunities are afforded the students to practice identifying concepts and performing procedures” (p.127). Another view of mathematics is a “constructing mathematics” view. According to Thompson (2007), here mathematics is considered to be “a social construction involving conjectures, proofs, and refutations, whose results are subject to revolutionary change and whose validity, therefore, must be judged in relation to a social and cultural setting" (p.127). Therefore, it changes the view of people's conceptions of the nature of mathematics from the knowing mathematics view to the constructing mathematics view (Sowder, Philipp, Flores, & Schappelle, 1999).

Ernest (1988) notes that three key elements influenced mathematics classroom teaching. They are 1) The teachers' mental contents or schemas, particularly the system of beliefs concerning mathematics and its teaching and learning; 2) The social context of the teaching situation, particularly the constraints and opportunities it provides; and 3) The teacher's level of thought processes and reflection. (p.1). Ernst (1990) further listed the model for both knowledge and belief. The knowledge component includes knowledge of mathematics; knowledge of other subject matter; knowledge of teaching mathematics (mathematics pedagogy and mathematics curriculum); knowledge of classroom organization and management for mathematics teaching; knowledge of the context of teaching mathematics; and knowledge of education (educational psychology, education and mathematics education). The beliefs component includes conception of the nature of mathematics; beliefs models of teaching and learning mathematics; and principles of education. Ernst (1988, 1990) also indicates that teachers' approaches to mathematics teaching depend fundamentally on their systems of belief, in particular on their conceptions of the nature and meaning of mathematics, and their mental models of teaching and learning mathematics.

Stipek, Givvin, Salmon, and MacGyvers (2001) assess the relationship between teachers' beliefs and practices from six perspectives, which include teachers' beliefs about 1) the nature of mathematics (i.e., procedures to solve problems versus a tool for thought); 2) mathematics learning (i.e., focusing on getting correct solutions versus understanding mathematics concepts); 3) who should control students' mathematical activity; 4) the nature of mathematical activity (i.e., fixed versus malleable); 5) the value of extrinsic rewards for getting students to engage in mathematics activities, and 6) teachers' self-confidence. Using a four-page survey of "beliefs about mathematics and teaching," the researchers measured the relation between beliefs and practice. The results show that there is a degree of stability. Moreover, there is "coherence in the beliefs that mathematics is a tool for thought, that students' goal is to understand, that student should have some autonomy, that mathematics ability is amenable to change, and that, in the absence of rewards, students will want to engage in mathematics if the tasks are interesting and challenging" (p.222). Thus, "the associations between teachers' beliefs and their classroom practices were all in the predicted direction" (p.223).

Philipp (2007) states that research studies on teachers' beliefs mostly use a case-study methodology to provide detailed descriptions of the beliefs of a small number of teachers. Further, Raymond (1997) finds that a teacher's practices were more dependent on her beliefs about mathematics than her beliefs about mathematics teaching and learning. Also, Philipp concluded that researchers indicated that changing beliefs may change teachers' classroom practice. Reflection is a factor that supports teachers' changing beliefs. "Through reflection, teachers learn new ways to make sense of what they observe, enabling them to see differently those things that they had been seeing while developing the ability to see things previously unnoticed (Philipp, p.281). Thus, according to Philipp, we need to understand not only what beliefs teachers hold but also how they hold them.

Thompson (1992) also indicates the distinctions between beliefs and knowledge through two perspectives. First, beliefs can be held with vary degrees of conviction, whereas knowledge is generally not thought of in this way. Also, beliefs are not consensual, but knowledge is (Thompson, 1992). Thompson (1992) addresses the important relationship between knowledge and beliefs. "To look at research on mathematics teachers' beliefs and conceptions in isolation from research on mathematics teachers' knowledge will necessarily result in an incomplete picture" (p.131). However, the study focusing on relationship and connection between these two aspects can be explored.

Thompson (1992) defined a teacher's conception of the nature of mathematics, conception of mathematics teaching and learning, as "that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (p.132), including 1) a dynamic, problem-driven discipline, 2) a static, unified body of knowledge, or 3) a bag of tools.

2.2.3 Research on prospective teachers' beliefs

As mentioned earlier, teacher education is an important stage for prospective teachers' to construct their knowledge and change their beliefs. Although researchers indicated that beliefs are generally stable (e.g., McLeod, 1992), Thompson (1992) states that prospective teachers often assimilate new ideas to fit their existing schemata

(Philipp, 2007). Also, when teachers learned about children's thinking, their beliefs and practice changes (Phillipp, 2007). Therefore, beliefs systems should be considered as dynamic mental structure instead of static entities.

Raymond (1997) indicates that beginning teachers reveal much about their beliefs as they struggle to develop their teaching practice. Moreover, beginning teachers' beliefs about mathematics and mathematics pedagogy are likely to be challenged during the first few years of teaching. She collected the data through an introductory phone interview, audio-taped interviews, and classroom observations, an analysis of several examples of lesson planning, a concept mapping activity in which teachers presented their views of the relationships between mathematics beliefs and practices, and the questionnaire on mathematics beliefs and factors that influence teaching practice. Four categories were created, including teachers' beliefs about the nature of mathematics, teachers' beliefs about teaching mathematics, teachers' beliefs about mathematics, and teachers 'mathematics teachers' practices with five-level scale ranging from traditional to nontraditional. Raymond found a major inconsistency between beliefs about mathematics teaching and learning and traditional practices.

Raymond's model (see Figure 4) indicated that teachers' beliefs were directly influenced by their prior school experience (e.g., Borko et al., 1992), either including experiences as a mathematics student, or the influence of prior teachers and of teacher preparation programs.

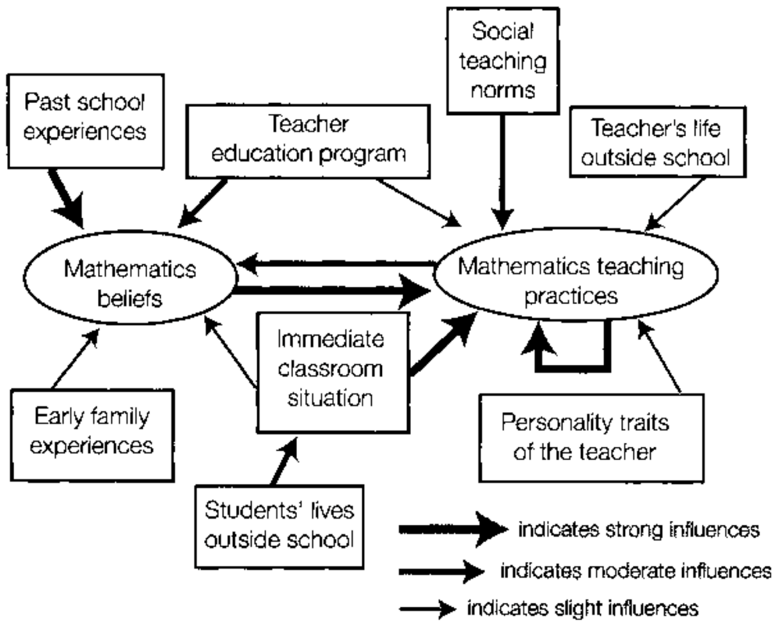


Figure 4. Raymond's model of beliefs and practice.

According to Raymond (1997), *social teaching norms* and *immediate classroom situation* can affect the relationship between beliefs and practice of the beginning elementary school teachers. Socialization within the teaching profession is another factor that influences classroom practice. In his study, Raymond concludes that there is strong influence from mathematics beliefs to mathematics teaching practice. Other factors may cause the inconsistencies between beliefs and instructional practices. Further, she suggested that during teacher education, prospective elementary school teachers become aware of the beliefs with which they enter and attend to how these beliefs begin to change during this important period of growth.

Some researchers have argued that teachers' beliefs about teaching and learning mathematics significantly affect the form and type of instruction (Clark & Peterson, 1986). According to Richardson (1990), if beliefs are based on the underlying philosophy and materials of a curriculum, teachers would construct their instruction to implement the curriculum. Changing teachers' beliefs is essential for their development, and it is important to understand not only what teachers believe but also how their beliefs are structured and held.

Researchers who have described changes in mathematics teachers' beliefs and practices (e.g., Cooney, Shealy, & Arvold, 1998; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Schifter & Simon, 1992; Schram, Wilcox, Lappan, & Lanier, 1989; Thompson, 1992) agree that there may be several developmental stages of teaching. Cooney et al. (1998) emphasize that reflection plays an important role in the growth of prospective secondary school teachers over their last year in an undergraduate teacher preparation program. They focus on not only what beliefs they held but also how the beliefs were held. They present four characterizations for how the prospective teachers held their beliefs, including isolationist, naïve idealist, naïve connectionist, and reflective connectionist. The Isolationist:

> Tends to have beliefs structured in such a way that beliefs remain separated or clustered away from others. Accommodation is not a theme that characterizes an isolationist. For whatever reason, the isolationist tends to reject the beliefs of others at least as they pertain to his/her own situation. (Cooney, 1999, p.172)

Naïve idealist "tends to be a received knower in that, unlike the isolationist, he/she absorbs what others believe to be the case but often without analysis of what he/she believes" (Cooney, 1999, p.172). "The naïve connectionist fails to resolve conflict or differences in beliefs whereas the reflective connectionist resolves conflict through reflective thinking (p.172). Cooney suggests that in order to change their beliefs; teachers must become more reflective when they were thinking about teaching, and rebuild them in a rational ways.

In summary, beliefs have been studied for decades. Although some researchers indicated that knowledge and beliefs are closely related to each other, few researchers emphasize the relationships and the interaction between them, especially in the last year of the teacher preparation program. Moreover, researchers indicate the inconsistency of teacher' beliefs and the change of their beliefs, studies focus on how these changes happened still need to be explored.

2.3 Teaching and Learning Division of Fractions (DoF)

Division of fractions (DoF) is one of the most mechanical and least understood topics in middle school (e.g., Fendel, 1987). The difficulty is due to the fact that DoF requires conceptual proficiency in both division and fraction concepts (e.g., Armstong & Bezuk, 1995). Moreover, fractions, as part of the rational number set, have several different interpretations (e.g., Kieren, 1993) making division on that set difficult. Thus, fraction division has many interpretations (Sinincrop, Mick & Kolb 2002) and the use of fractions in division makes this concept even more complicated for the learners (e.g., Borko et al., 1992; Ma, 1999; Sowder, 1995). In this session, I mainly focused on the study of teaching and learning DoF.

2.3.1 Research on teaching and learning DoF

As stated by Siebert (2002), "children often lack a ready understanding for operations involving rational numbers, because these operations are frequently equated with seemingly nonsensical algorithms, such as the algorithm for division of fraction" (p.247). Students learn it only by simply remembering the rule of algorithm and do not know why the "invert and multiply" algorithm for fraction made sense. Like whole-number division, DoF problems can be categorized as "*measurement* division (determining the number of groups); *partitive* division (determining the size of each group); or the inverse of a *Cartesian product* (determining a dimension of a rectangular array)" (Sinincrop, Mick, &Kolb, 2002, p.153). DoF also can be interpreted as the inverse of multiplication. Therefore, fraction division has many interpretations (Sinincrop et al., 2002) and the use of fractions in division makes this concept even more complicated for the learners (e.g., Borko et al., 1992, Ma, 1999, Sowder, 1995). As stated by Siebert (2002), "Children often lack a ready understanding for operations involving rational numbers, because these operations are frequently equated with seemingly nonsensical algorithms, such as the algorithm for division of fraction" (p.247).

Warrington (1997) uses measurement interpretation in her instruction. She starts with the whole number division and indicated 4÷2 means how many times does 2 fit into 4 or how many groups of two fit into four. Warrington further helps students to understand that it is the same meaning of DoF. Sinicrope et al. (2001) provided the pattern blocks to help students understanding. Further, they provides the common-denominator algorithm for DoF understanding

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{ad} = \frac{ad}{bc}$$

The first step in the common-denominator algorithm is to express both the divisor and the dividend as fractions with like denominators. Once the denominators are the same, the numerators are divided. According to Sinicrope et al. (2001), “it is possible to relate the procedural reasoning used in the solution of measurement divisions to the invert-and-multiply algorithm” (p.155).

Flores (2001) also mentions that DoF needs to make connection with “fraction and quotients, fractions and ratios, division as multiplitative comparison, reciprocal (inverse elements), and inverse operation” (p.238), because this connection will help students’ learning be coherent.

Overall, measurement and partative interpretation are viewed as the meaning of DoF in the classroom instruction (e.g., Flores) although it is one way of interpretation to understand. Next, I discuss how teachers understand this content topic.

2.3.2 Research on teachers’ understanding and teaching of DoF

Researchers have addressed the issue that although they can do algorithms well, prospective teachers were not well-prepared to explain DoF to students conceptually (e.g., Borko et al., 1992; Tirosh, 2000). Tirosh (2000) conducts research focused on prospective elementary school teachers’ SMK and PCK of DoF, before and after instruction. She identifies the students’ mistakes into three parts, 1) algorithmically-based mistakes, 2) intuitively-based mistakes, and 3) mistakes based on formal knowledge. The data indicates that "before the prospective teachers entered the course most mentioned only algorithmically based mistakes, and by the end of the course with developing the prospective teachers' knowledge of common ways of child thinking, most participants were familiar with various sources of incorrect responses.

Borko et al., (1992) discuss evidence that a student teacher was unsuccessful in providing a conceptually based justification for standard DoF algorithm. The authors interviewed the student teacher in order to understand why the lesson failed. A student in the class asked why the 'invert-and-multiply' algorithm worked when this student teacher reviewed the DoF algorithm using the problem $\frac{3}{4} \div \frac{1}{2}$. She tried to use a concrete example to explain the algorithm but failed. Finally, she asked students to memorize the rule for the algorithm mechanically. Through the interview, the researchers found that this student teacher did not understand DoF in a meaningful way.

In-service teachers also have unclear understanding of DoF. Ma (1999) found that American teachers' failed to come up with a representation of division of fractions $1\frac{3}{4} \div \frac{1}{2}$. According to Ma, there are three misconceptions. First, teachers are confused with meaning of measurement and the partitive algorithm. Here, it means how many ½s goes to $1\frac{3}{4}$, while many teachers try to make up story problems that represent the partitive idea. Thus, teachers confused divided by $\frac{1}{2}$ with divided by 2. Second, Ma argued that teachers may confuse of division by $\frac{1}{2}$ with multiplication by $\frac{1}{2}$.

Overall, teachers with profound understanding of DoF need to have the basic concepts and principle of fractions and divisions, such as “identity” element for multiplication, reciprocal (multiplicative inverses), and the inverse nature of the operations of division and multiplication (Flores, 2000).

2.4 Summary

In this chapter, I review the previous studies focusing teachers’ knowledge, beliefs and teaching and learning the content topics of DoF. Knowledge and beliefs are two major research genres that are directly related to classroom instruction and students achievement. They were also well studied by researchers for decades.

Researchers create and explore the model of SMK, PCK, and MKT to understand what knowledge teachers held. Ball (2005) and her colleagues develop Shulman’s (1986) SMK and PCK to MKT. MKT requires teachers not only explain why the algorithm works, but also use “an effective way to represent the *meaning* of the algorithm” (Ball, Hill, & Bass, 2005, p.20). That is, teachers need to think from a students’ perspective and consider what helps to understand a mathematical idea for someone first seeing it. Moreover, it requires teachers to recognize students’ errors and to analyze the source of the error. According to Ball, knowledge *about* mathematics includes teachers’ understanding of what is mathematics as a discipline. Thus, it includes teachers’ beliefs somehow. However, the studies of this part still need to explore. For the prospective teacher, the way that both MKT (including CCK and SCK) and PCK develop in their undergraduate teacher preparation and field practice teaching need to be explored.

Beliefs, as another factor, have been studied from different aspects using different terminology. Researchers indicate that knowledge and beliefs are closely related to other. Few studies on teaches' beliefs focus on how knowledge and beliefs interacted to each other. In other words, what is the process for knowledge development and beliefs change? Moreover, research on study beliefs and their practice always focused on mathematics in general. Moreover, they mainly focus on the relation between teaching practice and beliefs, instead of the beliefs change process.

I choose the topic of DoF for this study. It can present "a rich context for exploring possible depth and limitation in prospective teachers' knowledge in mathematics and pedagogy" (Li & Smith, 2007, p.185). It requires middle grade mathematics teachers have a profound understanding of fundamental mathematics (Ma, 1999) to represent this content topic. Thus, middle grade teachers should have a deep, broad, and thorough understanding of DoF and other topic are able to reveal and represent connections among concepts and procedures to their students.

CHAPTER Ⅲ

METHODOLOGY

In order to answer the research questions, this study use qualitative methodology. While primarily a qualitative study, I also report the quantitative results. This research method can help develop a deep understanding of prospective teachers' cognitive development and the changes of meta-cognitions. I indentify in what way knowledge and beliefs impact to each other. In other words, I focus on the correlation between the knowledge development and the changes of beliefs. The qualitative data include their textual analysis (pre-test and post-test, writing assignment, and concept mapping), interview data and classroom observation. The quantitative data include survey and test. For the interview and classroom observation, attention was also given to non-verbal clues because such information can reflect their value patterns (Lincoln & Guba, 1985).

I mainly study prospective teachers' SMK mathematics content knowledge (CCK and SCK) and the knowledge development based on the textual documents and the interview data and study prospective teachers' beliefs of teaching and learning DoF and its changes based on the survey and interview data in the method course. I intend to study one prospective teacher's SCK, PCK (knowledge of content and curriculum, KCS, and KCT), and his beliefs from classroom observation of the prospective teacher's field practice and interview data after the field practice.

Part of the data (pre- and post-test, survey) in this study is from a funded project: Math TEKS Connection Preservice Modules (MTC project). The module is developed in order to help prospective teachers develop the understanding of the precise meaning of the TEKS (Texas Essential Knowledge and Skill). The module focuses on the clarification of the meaning of the TEKS. The clarification includes: 1) the mathematical content; 2) the level of sophistication expected; 3) necessary prerequisite knowledge and skills; and 4) possible misconceptions or difficulties that students might have. Thus, the prospective teachers are expected to develop a strategy for clarifying TEKS objectives so that they can plan and teach mathematics lessons based on TEKS objectives.

The participants in this study are the prospective teachers preparing for middle grades. They are taught by the instructional module, which is developed by mathematics education faculty for grade 5^{th} -8^{th} prospective teachers. In this module, the instructors focused on several aspects and one is the content topic of DoF. The module intends to provide the foundation for the division of fractions, develop a deeper understanding of the division of fractions including justification of the "invert-and-multiply (IM)" algorithm, explaining DoF by using multiple representations, and examine students' errors in the division of fractions.

Based on the module, the instructor used five lessons for teaching division of fractions (DoF). The topic started in the fifth-week-class in the semester and the instructor provided a problem $1\frac{3}{4} \div \frac{1}{2}$ and raised the following question. "Why did you get an answer that is larger than either of the original numbers in the problem? Doesn't dividing make numbers smaller?" By using the module, these prospective teachers started to consider the concept underlying the algorithm of DoF. The instructor tried to help students understand the concept of DoF while mainly focusing on using two different models of division problems. One is measurement model of division and the other is partitive model of division. For each model, the instructor provided a real-world problem (See below) for each type of division in the class.

"Amy is planning a party to celebrate surviving half of the semester of senior methods. She ordered $2\frac{1}{4}$ pints of ice cream from Ben and Jerry's ice cream factory. If she serves each guest $\frac{3}{4}$ pint of ice cream, how many guests can be served?"

" Lauren has $\frac{1}{2}$ pounds of candy left, she divided to share with Janet and Casey, how much did each of them get?"

Although the instructor provided other scenarios to show other quantity relationships in DoF, she still emphasized to distinguish between "grouping" (measurement) and sharing (partitive) division problems to help prospective teachers in their understanding and effective teaching. For the first example, it means forming groups of $\frac{3}{4}$ pint of ice cream out of $2\frac{1}{4}$ of pint of ice cream. For the second example, it means 2 groups out of $\frac{1}{2}$ pound of candy. However, the real-world problem in the test goes beyond the partitive and measurement relationship. It requires the prospective teachers to truly understand the quantity relationship in the problem. Further, the instructor connected TEKS with prospective teachers' understanding.

In this study, I focus on two time periods. In the first time period, in which the prospective teachers were in their math method course, the data include pre-instructional and post-instructional test, surveys, students writing, concept mapping, and prospective teachers' post-instructional interview. In order to study the relationship between their beliefs and the knowledge, the surveys and the test assessing their beliefs and knowledge were taken respectively. Both the survey and test were taken both before DoF topic was introduced and at the end of the semester. Students writing, concept mapping, and prospective teachers' post-instructional interview were taken for complementing the understanding.

Overall, the context for the study is a middle grade mathematics method course that used a constructivist approach to model and develop students' knowledge about mathematics. The instructor and the researchers design modules to focus on important content topics in middle grade mathematics curriculum. Classroom activities also are observed in an attempt to identify the way that the course influences the prospective teachers' knowledge and beliefs. Although this course is not the only one that influences prospective teachers' knowledge and beliefs, it is the one course that is designed to focus on DoF topic.

3.1 Participants

Purposive sampling is used in selecting prospective teachers who were enrolled the subject-specific method course, where the instructor used a module focusing on DoF, and who taught DoF in their field practice period. 'Purposive sampling can be pursued in ways that will maximize the investigator's ability to devise ground theory that takes adequate account of local conditions, local mutual shaping's, and local values (for possible transferability) (Lincoln & Guba, 1985, p.40).

First, a total of 27 prospective teachers participate the study in their method course. Except one, other 26 are in their senior year of the college. There are five male students and 21 female students in the class. They take pre-survey and pos-survey to reveal their beliefs of DoF, beliefs of teaching and learning mathematics and DoF, and its changes. They also take pre- and post-test and pre- and post-concept mapping to assess their CCK and SCK of DoF and the development in general.

Next, in order to study the correlation between knowledge and beliefs, six participants are chosen from all 27 based on the performance of the tests (mainly CCK) and their achievement in the class. They are three males (Eric, Mark, David) and three females (Amy, Mary, Lily). Eric and Amy are considered as those who perform well in the method course; Mark and Mary are considered as average students; and David and Lily are considered as those whose knowledge is limited. Besides the test and surveys, they are asked to provide the paper assignment regarding their understanding development of DoF. Further, all six have a post-instructional interview. The post-instructional interview mainly focus on assessing their SCK based on the test. Further, the data of paper assignment and interview also provide the information about their beliefs of teaching and learning mathematics and DoF. I intend to reveal 1) whether the participants' knowledge differences impact their beliefs changes. In other word, the participants who performs well or has high CCK or SCK may change his/her beliefs at the end of the method course; 2) how the correlation between knowledge development and beliefs changes happen.

Further, I follow up one prospective teacher, Mark, from those six participants in his field practice. I choose Mark because he is the only one who actually teaches DoF his filed practice.

3.2 Instrumentation and Data Collection

Mentioned in previous section, I collect data from two time periods. The first time period is the time in the method course and the second time period is the time in field practice. Next, I explain the instrumentation and data collection based on these two time periods.

3.2.1 Instrumentation during the method course

Mentioned in the previous section, the context for the study is a middle grade mathematics method course that used a constructivist approach to model and develop students' knowledge about mathematics. The instructor and the researchers design the module to focus on important content topics in middle grade mathematics curriculum. Classroom activities are observed in an attempt to identify the way that the course influenced the prospective teachers' knowledge and beliefs. Although this course is not the only one that influenced prospective teachers' knowledge and beliefs, it is the one course that is designed to focus on DoF topic.

During the method course, the data include pre- and post-survey, pre- and post-test, pre- and post-concept mapping, writing assignment, and post-instructional interview. Below is each instrument and data collection in the method course.

Survey items using in this study (see Appendix 1) - Mathematics Curriculum and Instruction Survey is developed by mathematics education faculty at Texas A&M University for MTC project. The survey includes seven items. Each item includes multiple statements. For example, the sixth item includes 10 statements, which refer to prospective teachers' beliefs about teaching and learning mathematics. Participants need to indicate whether they agree or not agree with the statement. In the current study, I mainly focus on two parts. The first part is prospective teachers' self-evaluation about their understanding of the curriculum (Mathematics Texas Essential Knowledge and Skills – TEKS). It is the first and fifth item in the survey. The second part reveals prospective teachers' beliefs and attitude towards to teaching and learning mathematics. Thus, I choose six statements from the sixth item. Combining with prospective teachers' writing assignment and interview, the survey intends to report the changes of prospective teachers' beliefs in their method course. The total 27 prospective teachers' pre- and post-survey are collected for this study.

Text items of Pre-test and Post-test (see Appendix 2) - The test is developed based on both Ball's (1989) and Ma's (1999) study by mathematics education faculty at Texas A&M University for MTC project. The test contained two sections, which assess prospective teachers CCK and SCK. The pre- and post-test mainly examine the participants' CCK and SCK. The first section mainly assesses prospective teachers'

CCK. More specific, it assesses the participants' ability of 1) perform the DoF algorithm, and 2) real world application.

There are four items for assessing algorithm fluency. Three items are simple computation, represented as fraction divided by a whole number (i.e., $\frac{1}{5} \div 5 =$), fraction divided by fraction (i.e., $\frac{7}{9} \div \frac{2}{3} =$), mixed-number division (i.e., $5\frac{1}{4} \div 3\frac{1}{2} =$). One item requires to find a missing numerator of fraction division equation (i.e., if $\frac{14}{15} \div \frac{?}{9} = \frac{3}{10}$, find ?). The first three questions assess the participants' procedural knowledge of DoF. To solve the fourth question, the participants not only need to remember the algorithm rule as IM, they also need to solve it in an algebraic way.

Beyond the procedural frequency, the test further requires the participants to solve questions in order to assess their understanding of fraction division concept and the features of fraction division algorithm. There are six items of word problems. The questions can be categorized into three types, including two simple word problems or word problems without a real world context, one for problem solving, and three word problems in a real world context.

The first type is the word problem without a real world context. There are two problems in this category (i.e., How much $\frac{1}{2}$'s are in $\frac{1}{3}$; $\frac{5}{6}$ of a number equals to $\frac{5}{24}$, find the number). The first question in the question assesses prospective teachers' understanding of measurement model of division. In this problem, it requires prospective teachers the understanding of forming groups of a certain size. In other words, the problem asks how many groups of $\frac{1}{2}$ can be formed in size $\frac{1}{3}$. The second question requires the prospective teachers' understanding the meaning of multiplication in order to write the number sentence. In this case, the participants should be able understand that the question asks $\frac{5}{6}$ of a set of a number equals to $\frac{5}{24}$. It is multiplication operation. In order to find the number, the participants should do the inverse operation of multiplication, which is division. Thus, the participants are assessed both their quantitative relation of fraction division beyond the algorithm and DoF algorithm.

The second type of questions is three word problems with a real-world context. They are 1) A five-meter rope was divided into 15 equal pieces. What is the length of each?; 2) Andrew bought 7 apples, which is $\frac{1}{3}$ of the number of oranges he bought. How many oranges did Andrew buy?; 3) Johnny's Pizza Express sells several different flavored large-size pizzas. One day, it sold 24 pepperoni pizzas. The number of plain cheese pizzas sold on that day was 3/4 of the number of pepperoni pizzas sold, and 2/3 of the number of deluxe pizzas sold. How many deluxe pizzas did the pizza express sell on that day?

The first question in this category is partitive model of the division. The question asked how much in each of the 15 sets for the whole, which is the size of groups unknown. For the second question, it is multiplicative comparison problem. There are two different sets (apples and oranges), as there were with comparison situation with addition. In this case, the comparison is based on the set of apples being a particular multiple ($\frac{1}{3}$) of the set of oranges. In this case, the unknown is reference set (the number of oranges). The number of apples is $\frac{1}{3}$ of the number of orange, in other words, it means the number of oranges is three times greater than the product (the number of the apples). Thus, it can be consider the partition division. Prospective teachers can solve this question either using the algebraic way ($\frac{1}{3}x = 7$) or using the arithmetic way ($7 \div \frac{1}{3}$).

The last question is also the multiplicative comparison problem and it takes two steps. Prospective teachers should understand the quantitative relationship among three different types of pizzas. The question asks the number of deluxe pizzas. However, in order to get the number of deluxe pizzas, the participants need to know the number of plain cheese pizzas first. It is because the number of the plain cheese pizzas is $\frac{2}{3}$ of the number of deluxe pizzas sold. In this part, the reference set size (the number of deluxe pizzas sold) is unknown. Thus, it is division problem. In other words, the number of plain cheese pizzas is $\frac{2}{3}$ times greater than the number of deluxe pizzas sold. Then, in order to get the number of the plain cheese pizzas, the participants need to understand another quantitative relationship between the number of the plain cheese pizzas and the number of the pepperoni pizzas. Here, the reference set is the number of pepperoni pizzas, and the number of plain cheese pizzas is $\frac{3}{4}$ times greater than the number of pepperoni pizzas.

The last type of question is problem solving. It requires prospective teachers' understanding of the characteristics of fraction division algorithm. In particular, it requires identifying which quotient is greater without calculating (i.e., "$\frac{9}{11} \div \frac{2}{3}$" and "$\frac{9}{11} \div \frac{3}{4}$") and explain their answer. Prospective teachers should be able to identify that the answer of $\frac{9}{11} \div \frac{2}{3}$ is greater than that of $\frac{9}{11} \div \frac{3}{4}$. Further, they also need to understand the relation between the dividend and the divisor. That is, when the dividend keeps the same and if the divisor is greater, the quotient becomes smaller.

Next, the second section of the test mainly assesses SCK. In other word, the questions intend to evaluate whether the participants 1) could explain the conceptual meaning underlying the algorithm procedure and 2) have knowledge of students' errors and misconception.

The first subsection includes four questions and is used to assess prospective teachers' knowledge of giving mathematical explanations and using mathematical representations. Two questions ask the participants to explain *why* the computation pf DoF works and the other two questions ask the participants to explain *how* the computation works by using mathematical representation.

The first category of prospective teachers' knowledge of giving mathematical explanation of *why* the computation works includes two questions regarding to the special case and the general case. For example, one question asks prospective teachers to define whether the operation of fraction addition, subtraction, and division can be done like fraction multiplication (i.e., multiplying the numerators as numerator and multiplying the denominators as denominator). Thus, the participants should first identify that DoF can be done dividing straight across as a special case and then explain why this special case works, which is in the case of both the dividend in the denominator and the numerator is divisible by the divisor in the denominator and numerator.

The other question is a situation that the prospective teachers may face in their future teaching. The participants face to a scenario that if the students ask why 'invert' the second number and change from 'division' to 'multiplication' as a computation rule. This requires a general explanation, which is, why division changes to multiplication and the divisor becomes its reciprocal. The prospective teachers are encouraged to use multiple representations that they feel comfortable.

The second category requires prospective teachers to use pictures and create a word problem to explain *how* each specific computation works. Both questions provide a specific DoF equation. The first question in this category asks why $\frac{2}{3} \div \frac{1}{6} = 4$ and why $\frac{2}{3} \div 2 = \frac{1}{3}$. The participants are required to understand how the computation works. In other words, they should explain by using multiple representations to show why the quotient in a division problem goes greater than the dividend in the equation $\frac{2}{3} \div \frac{1}{6} = 4$.

It also assess whether the prospective teachers understand measurement model and partitive model of division.

Another question in this category requires prospective teachers to make up a story problem of $\frac{3}{4} \div \frac{1}{2}$ and draw a picture representation that represents “three-fourths divided by one-half”. It assesses whether prospective teachers understand the meaning of $\frac{3}{4} \div \frac{1}{2}$, which means, “how many groups of $\frac{1}{2}$’s goes into $\frac{3}{4}$”. In other words, it is a measurement model of fraction division. The participants should first understand the mathematical relationship from the given expression. Then, using different representations, prospective teachers should be able to explain how $\frac{3}{4} \div \frac{1}{2}$ works.

The second subsection includes two questions in order to assess prospective teachers’ knowledge of analyzing students’ errors and identify the error sources from different ways. The third question in the test provides a word problem in a real world situation (i.e., Six pounds of sugar were packed in boxes, each box containing $\frac{3}{4}$ pound. How many boxes were needed to pack all the sugar?). It requires the participants to write the number sentence to show the quantitative relation and anticipate an incorrect expression showing the wrong quantity relation. Another question in this subsection is to find the error patterns (i.e., $\frac{7}{10} \div \frac{1}{2} = \frac{7}{10} \div \frac{5}{10} = \frac{2}{10} = \frac{1}{5}$; $\frac{1}{3} \div \frac{5}{7} = \frac{7}{21} \div \frac{15}{21} = \frac{8}{21}$).

All 27 participants completed a pre- and post-test at the beginning and the end of the spring semester, 2007. The participants could spend as long as possible to finish the test. Further, the participants were told the test was not the part of a final exam and also it was not counted toward their grade.

Pre- and Post-concept mapping (see Appendix 3) – In order to get a deep understanding of prospective teachers' knowledge development, I further collect the data of the concept mapping of all 27 participants. This open-ended question developed based on Liping Ma's (1999) study. Concept mapping is to assess how the participants connect their prior knowledge to DoF and how this connection develops at the end of the method course.

In the pre-instruction concept mapping, the participants are given two examples of concept mapping. One is for "water" and the other is for "function". The examples are taken from the studies of Novak et al (1984) and Williams (1998). Thus, the participants are provided the idea of concept mapping in a general topic and continually help them to be familiar with constructing a concept mapping related to mathematics content topic.

Further, the participants are given seven possible concepts that they learned previously. These are "concept of unit", "concept of fraction", "concept of addition", "concept of inverse operations", "multiplication of whole number", "multiplication of fraction", and "division with whole number" that Ma (1999) mentioned in her study. These possible concepts may help them to develop their organization of their knowledge. The participants were not limited to only use the keynote provided. The participants are encouraged to use other keynotes to indicate the concept that may relate to DoF. Thus, the participants are asked to draw a concept map of DoF that contains some concepts and linking words connecting with the concepts.

The rationale for using the concept mapping in this study is to maximize the participant involvement. In drawing and labeling the linking lines and keynotes, the prospective teachers can explicitly state the conceptual relationships they consider regarding to DoF.

***Writing assignments* –** To get a deep understanding of prospective teachers' knowledge development and their beliefs changes. I further collect six participants writing assignments. The writing assignments include a reflection paper entitled "Division of fractions and me," in which they discuss what they learned in the class and the development of their understanding of DoF. Further, the reflection paper also reveals their beliefs about DoF as a subject matter content, their beliefs of teaching and learning DoF before and after their learning. Besides the reflection paper, six participants also provide the reflection papers regarding their field observation or classroom observation. In these papers, the participants mainly indicate their thought towards to their mentor teachers' instruction. Some also indicate the concern about classroom management.

***Post-instructional interviews (see Appendix 4)* –** Based on the previous data (i.e., survey, test, and writing assignments), the six participants take a post-instructional interview at the end of the semester. Mentioned previously, the prospective teachers are chosen based on their willingness to participate, gender, and the levels of response to their performance. Notes and audiotape recordings are taken during the interviews.

Overall, the interview is conducted to understand prospective teachers' both knowledge and their beliefs. To understand their knowledge and its development, the interview intends to clarify the unclear answer that the participants wrote in their tests. In other words, the first main purpose of the interview is to clarify and learn more about their CCK and SCK of DoF based on their test. Thus, the interview questions are similar with the second part of the test. In particular, I ask the participants to explain "why" the computation of DoF works by using different representation. I also provide two equations represent measurement model and partitive model of division. Based on the equations, I ask them to create a word problem respectively and explain each by using different representations. Further, I provide some students' error for them to identify.

Another purpose is to help me understand the participants' beliefs towards to DoF, beliefs towards to teaching and learning DoF and its changes through their own learning experiences. In particular, what were their beliefs of DoF and learning DoF at the beginning of the semester? What happened when their knowledge and beliefs encounter reform-oriented ideas? What did they struggle with their previous beliefs? How were their beliefs towards to DoF the end of semester? Thus, besides the explanation of the content topic, the interviews provide detailed information about what participants think they learned from the course. These answers also help to show how they understand what they had been taught, what kind of perspective they had of the substance of their courses, the way they organized that knowledge, and how articulate they were in talking about it. Finally, the interview reveals the participants' beliefs and attitude about good (or effective) teaching in DoF and mathematics, how they consider mathematics should be learned. Some participants also provide detailed information about what they remembered about learning math or division of fraction from high school.

3.2.2 Instrumentation during the field practice

In the second time period, I focus on Mark in his field practice. As mentioned in the previous section, Mark is chosen because he is the only one who teaches in the seventh grade in his field practice among six participants. I intend to study the way that knowledge and beliefs that Mark holds affects his instructional behavior in the field practice and further, the way that the practical experiences in the field practice influences Mark's knowledge development and beliefs changes. Thus, Mark's reflective practice (Lerman, 2001) may also offer them opportunities to reconstruct beliefs and knowledge.

Mark gives permission to observe his field practice when he teaches the content topic of DoF. I focus on Mark transition to classroom practice, referring to the teaching experience that the student teachers gain while trying out classroom teaching interventions (Simon & Tzur, 1999). Research has shown that these activities have contributed to the teacher's development by challenging existing beliefs and practices (Simon, 1995). Thus, I focus on the learning process and the transition for teaching Mark and study how his university study influences his classroom practice and how the field experience further influences his knowledge development and beliefs changes.

Mark started his field observation when he was in the method course and continues to do his field practice in the same school with the same mentor. He observes the mentor's classes for a whole semester during the time in his method course and indicate his beliefs about mathematics teaching and learning, in particular in DoF. Mark is considered as an average student in the method class.

In order to examine the kind of knowledge and beliefs that are acquired through the field practice, I observe Mark's classroom instruction and then interview Mark after his teaching.

Classroom observation - Classroom observation is another way to assess teacher knowledge (e.g., Borko et al., 1992; Leinhardt & Smith, 1985; TIMSS, 1999). Hill et al. (2007) indicated "analysis that appeared in print was primarily qualitative, with researchers using methods and coding systems tailored specifically to the mathematical topics and questions at hand" (p.125). Also, researchers mainly combine observational data with other data to gain insight into teacher mathematical knowledge for teaching (Hill et al., 2007). I adopt the instructional observation rubric created by Sowder and her colleagues (1998) (see Appendix 5).

The class that Mark was teaching in the field practice is called "block" class. Comparing with the regular class, it is 90 minutes instead of 45 minutes for a lesson. The size of the "block" class is smaller then the regular class. All students in the "block" class have difficulties of learning mathematics. There are 12 students in the block class, three are African American and nine are Latino students.

Mark teaches the "block" class for DoF content topic for three days**.** The first lesson focuses on the content of a whole number divided by a fraction; the second lesson focuses on the content of a fraction divided by a fraction; and the last lesson is for students' application. Discussing with his mentor, Mark chooses to use the activities and problems from textbook *Rethinking middle school mathematics: Numerical reasoning* (TEXTEAMS), developed by the mathematics educators in the University of Texas at

Austin. In the field practice, Mark is encouraged by his mentor to choose the activity for his instruction. Therefore, based on his learning experience, he decides to teach DoF by solving real world problem (see Appendix 6).

The instructional practice is similar in first and second lesson. In the first lesson, Mark intends to create a group discussion and encourage students to find the answer of the problem at the beginning. However, he changes his instruction in the middle of the lesson and the second lesson. He mainly explains to the class and guides his students thinking for the rest of the lessons. In his writing and interview, he mentioned his concern about classroom management in his writing.

Interviews (see Appendix 7) – During their teaching and after finishing the teaching period, I interview Mark based on the observation notes that I wrote. The interviews are open-ended questions and the reflection of the teaching. They are conducted from several episodes that happened in the classroom, including Mark's explanations, the responses to students' answers, and confirming students understanding for the concept. I also ask Mark the way he is planning for the instruction and interacting with the mentor teacher. The interview also includes his personal history of learning DoF and university experiences, including his mathematics methods course.

3.3 Data Analysis

In this study, I use constructivism perspective for understanding their knowledge development and beliefs change. According to Torish and Even (2008), the constructivist perspective emphasizes the development of different forms of knowledge such as conceptual knowledge, problem-solving strategies, and meta-cognitive abilities. The data are analyzed to answer the research questions.

3.3.1 Data analysis for the instrumentation in the method course

Table 1 showed the purpose and the data analysis during the method course.

Table 1

Purpose of each instrument and the data analysis during the method course

	Pre- and Post-survey	Pre- and Post-test	Pre- and post concept mapping	Writing assignment	Interview
Purpose	The beliefs and its changes	Knowledge development	Knowledge development; Perception	Correlation	Correlation
Data analysis	Beliefs (teaching and learning mathematics; what to learn and how to learn); Self evaluation (ready to teach DoF; knowledge of TEKS);	CCK (procedural proficiency; application) and its development; SCK (explaining why and how DoF works; Presenting DoF; Presenting student error pattern) and its development	Concept unit (i.e., concept of inverse operation); Procedural unit (i.e., IM; reciprocal); Representation and modeling (i.e., manipulatives); Structure and links	Meta-cognition of beliefs changes; Meta-cognition of development of their knowledge; Correlation	Beliefs changes or meta-cognition of its changes; Knowledge development or meta-cognition of its development

Data analysis for survey - The survey is mainly used to evaluate the participants' beliefs and its changes. Therefore, based on the previous study (Thompson, 1992; Philips, 1999), I categorize the survey items in two groups. The first group contains the beliefs of DoF, beliefs and attitude towards to learning and teaching mathematics. In particular, it intends to assess the prospective teachers' thinking about *what* to learn about mathematics and *how* to learn mathematics. The second group focuses on the participants' self-evaluation. It contains whether they consider they are ready to teach DoF and their self-evaluation of their understanding of TEKS.

Data analysis for content knowledge test - The pre- and post-test mainly examine the participants' CCK and SCK. Table 2 developed based on Ball and her colleagues' study (Ball et al., 2008). It reveals the dimensions for evaluating prospective teachers' content knowledge.

Table 2

The dimensions for evaluating prospective teachers' content knowledge (CCK and SCK)

Domains for evaluating the content knowledge for teaching
Doing the operation
Solving non-context based word problems
Solving context based word problems
Recognizing a non-standard (short-cut) approach of solving DoF and understanding the concept underlying
Explaining why the computational rule works
Recognizing wrong answers (errors)
Using notation correctly
Analyzing errors and evaluating alternative ideas
Giving mathematical explanations and using mathematical representations
Being explicit about mathematical language

The first section in the test mainly assesses prospective teachers' CCK. More specific, it assesses the participants' ability of 1) perform the DoF algorithm, and 2) real world application. Thus, it includes the procedural frequency of DoF by doing algorithm, the characteristics of the algorithm, and the conceptual understanding for mathematics relationship by solving word problems. First, they must be able to do simple calculation of DoF or, correctly solving problem of DoF. For the procedural frequency, it mainly assesses whether they can do the algorithm of DoF on the test. I focus on the percentage of accuracy each question. Further, by solving word problems, the prospective teachers are required to understand the mathematical relationship and conceptual understanding. It is also considered as a part of CCK. There are two types of the word problems on the test to assess application of DoF. One is word problem without a real world situation or word problems of mathematics relationship (simple word problems) and the other is world problem within a real world situation. They mainly assesses whether the prospective teachers understand the word problem in order to write down the number sentence and solve them. In this part, I not only focus on the accuracy of the computation but also identify the students' understanding of mathematical relationship.

Besides CCK, there are six problems intends to assess prospective teachers' SCK. Ball (2006) indicated that SCK is a knowledge base mainly needed by teachers in their work and beyond that expected of any well-educated adult. This knowledge base supports teachers "presenting mathematical ideas", "responding to students' 'why' question", "finding an example to make a specific mathematical point", and " linking representations to underlying ideas and to other representations" (Ball et al., 2008, p.400). These six problems can be categorized into two subgroups based on different purposes. Four are assessed prospective teachers' knowledge for "presenting mathematical ideas", "responding to students' 'why' question', "finding an example to make a specific mathematical point", " linking representations to underlying ideas and to other representations". In other words, these problems assessed prospective teachers' understanding of why the algorithm of DoF works and knowledge to explain and represent to the students, which was categorized in the first subgroup. The other three questions were mainly to assess prospective teachers' knowledge of recognizing and analyzing errors and evaluate alternative ideas (Ball, 2006), which were categorized in the second subgroup.

Therefore, these problems require prospective teachers to explain why the computational rule works by using multiple representations, to give a mathematical explanation and to use mathematical representations, to recognize with a non-standard approach (short-cut approach) and understand the concept underlying the approach, to recognize students' errors, to analyze errors and evaluate alternative ideas, and to use explicit mathematical language. In other words, it requires mathematical reasoning to do the explanation and to uncover students' error.

I mainly analyze the second part of test by using contextual analysis. For the question to identify whether $\frac{a}{b} \div \frac{c}{d} = \frac{(a \div c)}{(b \div d)}$ is correct and explain why, I first developed five categories for each answer. They are correct identifying with correct explanation; correct identifying without explanation; no identification but only indicating IM algorithm; wrong identification (with or without indicating IM); and no answer or indicating does not know the answer. Next, I analyze each explanation and analyze the participants' explanations in detailed. Especially, I focus on the six participants' answers.

For the questions of explaining why $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, four categories are developed as four categories. They are correct explanation with different representations, only indicating IM algorithm; inappropriate explanation, and no answer or indicating does not know the answer. Further, I focus on the different representations that the participants use (i.e., symbolic representation, pictorial representation, or verbal representation) and especially focus on the selected six participants and their answers. Same categories are developed for the questions of identifying students' error patterns and its sources.

For the question that requires the prospective teachers to explain how the computation works (i.e., $\frac{2}{3} \div 2 = \frac{1}{3}$ and $\frac{2}{3} \div \frac{1}{6} = 4$"). I also develop the four categories similar with the previous one. This question was used to assess prospective teachers' understanding of two different types of DoF provided by the instructor. The first equation can be considered as a partitive division problem and the second equation can be considered as the measurement division problem. Further, I analyze their explanation with the different representation and mainly focus on the selected six participants.

For the question that requires the prospective teachers to make up a story problem of $\frac{3}{4} \div \frac{1}{2}$ and draw a picture, I develop four categories including correct scenario with correct pictorial representation, correct scenario with no pictorial representation, scenario of wrong mathematical relationship, and no answer or indicating does not know about the answer. Further, I present the representative scenario that the participants create and the pictorial representation they use. Six selected participants answers also are analyzed in particular.

Data analysis of concept mapping - The rationale for using concept mapping in this study was to maximize participant involvement. In drawing and labeling the linking lines and key terms, the prospective teachers could explicitly state the conceptual relationships they saw regarding DoF. Concept mapping is widely used in science education to understand assess students' cognitive development (e.g., Novak & Gowin, 1984; Novak, 1990). It is essentially as a method to make science teaching more effective, with the intention to map something from the outside world into the students' mind (Novak, 1990). The underling idea is that something that is inside the mind can be mapped to the outside. A concept map is a diagram representing the conceptual structure of a subject discipline as a graph in which nodes represent concepts and connections represent cognitive links between them. It is also considered as a method to enhance learning in the science education. Therefore, concept mapping in the sense of Novak (Novak & Gowin, 1984) means to present to learners new knowledge in the form of structural networks, or to encourage them to construct such networks by themselves, with the expectation that the similarity of external and internal representation makes the acquisition of knowledge easier or more effective.

The use of concept mapping is often linked to the 'constructivist' view of learning (Kinchin et al., 2000). Individuals construct and reconstruct the meaning of object that they learn. The map usually has each idea in a separate box or oval with lines connecting with lines connecting related ideas and often labeled with "connective" terms (e.g., leads to, results from, is a part of, etc.). Researchers found that concept maps were a good way for students to find the key concepts and principles in lectures, readings, or other instructional materials (e.g., Novak, 1998). Further, Ma (1999) mentioned in her study that mathematics teachers in China should have a knowledge "package" to organize a new piece of knowledge. They should know that the knowledge is supported by which ideas or procedure. Thus, teachers should have a knowledge network or package including basic ideas of the subjects.

Unlike other measurement, the concept mapping requires students to state their understanding of the concept knowledge and thus, concept mapping should open-ended. The individual draws a picture of all the ideas related to some general theme and shows how these are related (e.g., Jackson & Trochim, 2002). Concept mapping usually has each idea (keynote) in a separate box with lines connecting related ideas.

Thus, the concept mapping reveals prospective teachers' knowledge of "connecting a topic being taught to topics from prior or future years" (Ball et al., 2008, 400). The frameworks of analysis provided by Kinchin and his colleagues and McGowen and Tall (1999) were used to evaluate the participants' concept maps. Kinchin et al., (2000) provides guidelines for the qualitative classification of concept maps. These authors identified three types of structures, which are 'spokes', 'chains' and 'nets'. 'A spoke', according to Kinchin et al., (2000), is defined as a "radial structure in which all the related aspects of the topic are linked directly to the core concept, but are not directly linked to each other" (p.47). A "chain' is "a linear sequence of understanding in which each concept is only linked to those immediately above and below. Although a logical sequence exists from beginning to end, the implied hierarchical nature of many of thelinks is not valid" (p.47). A 'net' is "a highly integrated and hierarchical network demonstrating a deep understanding of the topic" (p.47). Kinchin further indicates the distinctions between these three types. According to the authors, if a student holds a spoke structure, then the addition of new knowledge will not cause any disturbance to the existing framework. It can simply be added in with a link to the core concept, but without any links to associated concepts. The result would be that the knowledge can be assimilated quickly, but only be accessed by reference to the core concept and not by reference to one or other of the associated concepts. For the chain structure, the addition of the new knowledge will be easy if there is an obvious break in the sequence, but may be problematic if a workable sequence is already in place as the additional concept may appear superfluous. Therefore, the addition of a concept near the beginning of the sequence may be so disruptive to the knowledge structure low down that incorporation of the new knowledge is rejected. Also, it is difficult for a learner to understand a concept.

I first identify the categories of concept mapping into three parts based on Ma's (1999) study, which are procedural knowledge based or procedural units; conceptual knowledge based or conceptual units; and basic principles. Procedural units are including the units to support the procedural learning. Conceptual units are included mainly for a thorough understanding of the rationale underlying the algorithm. Basic principles included what is important for DoF learning or what knowledge may be basic of learning DoF and understanding for teaching DoF. The analysis also focuses on the features and the changes of conceptual mapping in an attempt to reveal how the knowledge is constructed and reconstructed in the understanding of DoF by the prospective teachers after their method course. Based on the data of post-instructional concept mapping, the representations are added as a fourth category. Thus, the categories were 1) procedural understanding; 2) previous knowledge understanding; 3) basic principles, and 4) representing DoF to students using models.

Further analyses are then conducted to specify how the prospective teachers developed their understanding of fraction division, especially, their knowledge development for teaching. Specific attention was given to the difference of the level or the degree of links (hierarchy of the links), relations between each keynotes (process or linkage), increasing or decreasing of links and notes between pre- and post- concept mapping, and the complexity of the shape in order to show their development of understanding for the teaching and learning of fraction division.

The first focused was on the keynotes based on the previous categories, looking mainly at the characteristics of each keynotes and their exploration of keynotes. I intend to identify what other pieces of knowledge directly or indirectly connected to the core concept. A keynote and a linkage are identified as a unit. A unit of analysis consists of a phrase containing only one relation between to concepts or phases and it based on the previous categories. It reveals the participants' idea for each concern or opinion, their understanding of the relation between the previous knowledge and the new knowledge, and their teaching strategies. The focus was not only on the frequency of the change of the linkage and keynotes, but also on the way each dimension changed. Thus, the qualitative description would provide more informative results. In other words, unitizing is done by breaking each linkage and the unitizing process is a set of single-concept statements putting into each categories in an attempt to identify the features of each unit and the reconstruction of the post- map.

After the analysis of the units, further focus was on the process or hierarchy of the each unit. The hierarchy of the unit can help to understand how each unit directly or indirectly connected to the core concept, DoF. The hierarchy of the links shows how many justifiable levels holding in each concept mapping. The process indicates the interactions at different conceptual levels, that is, whether and how the complex interactions happened at different level. The map is also analyzed to determine whether and how it integrated, which examines the complexity of the map. Based on the complexity aspect of the map, further study was done of conceptual development or procedural development for DoF. Based on the study Kinchin et al., knowledge revealed by the concept mapping was determined. Extending Novak's (1998) framework, I both examine "valid linkages" and "invalid linkages" in order to see the knowledge relationships constructed by the prospective teachers.

The example provided by Ma (1999) was used as a 'criterion' to compare the prospective teachers' concept mapping to see the differences between them. As Ma

(1999) indicated, teachers would have a knowledge package related to DoF. For example, in Ma's study, Chinese teachers considered that the two concepts as important previous knowledge for teaching and learning DoF. One is the meaning of division fraction, which is considered as the basis for understanding the meaning of DoF. The other is the concept of division, which is the inverse of multiplication. Based on these "criterion", we examine prospective teachers' perceptions of the concept topic of DoF.

Then, I compare the pre-instructional concept mapping and the post-instructional concept mapping across the whole class level. I first identify the structure of both pre- and post-concept mapping from several dimensions. I study the change of the structures across the prospective teachers. For example, if the prospective teachers hold a spoke type of structure, "the new knowledge will not cause any disturbance to the existing framework" (Kinchin et al., 2000, p.47). Therefore, "it can simply be added in with in a link to the core concept, but without any links to associated concepts" (p.47). Thus, the result would be that the knowledge may be "assimilated quickly, but only be accessed by reference to the core concept and not be reference to one or other of the associated concepts" (p.47). If the prospective teachers' concept mapping has chain structure, "the addition of new knowledge will be easy if there is an obvious break in (or premature end to) the sequence, but may be problematic if a workable sequence is already in place as the additional concept may appear superfluous" (p.47). If the prospective teachers' concept mapping has networks, this shows that prospective teachers "access to a particular concept may be achieved by a number of routes, making the knowledge more flexible" (p.48). This showed that the prospective teachers have the "understanding of the associated concepts beyond their link with the core concept and so implies a wider understanding" (p.48). Through different structure of the concept mapping, we examine how these prospective teachers constructed and organized the knowledge.

An analysis of the different aspects of the maps identified *what* knowledge of DoF the prospective teachers held and *how* the knowledge is arranged in their mind. Comparing with the pre- and post-concept mapping, an attempt was made to indicate *what* knowledge of DoF that prospective teachers developed and *how* the knowledge was developed.

By comparing their concept maps, new knowledge can be seen that is integrated with the knowledge already in their cognitive construction and reconstruction of their knowledge of DoF. A special focus is placed on participants' meta-cognition related to developing pedagogical content knowledge and learning fraction division. Also, concept mapping reveals the participants' perception of fraction division. Concept mapping also provides some perspective about perceptions on this content topic.

Finally, based on the results of their pre- and post-instructional tests and recommendation from the instructor, six participants with different CCK level are analyzed in detail.

Data analysis of writing assignments and interview – The content analysis is used in six participants writing assignments combining with the interview. Qualitative interview method was used in this study. The analysis of information from the interviews was based on Thompson's (1992) work. She first defined a teacher's beliefs of the nature of mathematics as "that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (Thompson, 1992, p.132). She explained three beliefs of mathematics as (a) a dynamic, problem-driven discipline, (b) a static, unified body of knowledge, and (c) a bag of tools. She looked at teachers' beliefs of mathematics teaching and cited from Kuhs and Ball's (1986) point of view. There are four views, (a) learner-focused, which is mathematics teaching focused on the learner's personal construction of mathematical knowledge, (b) content-focused with emphasis on conceptual understanding, which is mathematics teaching is driven by the content itself but emphasizes conceptual understanding, (c) content-focused with emphasis on performance, which is mathematics teaching emphasize student performance and mastery of mathematical rules and procedures, and (d) classroom-focused, which based on knowledge about effective classrooms (Kuhs & Ball, 1986).

The post-instructional interviews were organized into 1) clarifying SMK in test, both CCK and SCK, about DoF, 2) Beliefs about DoF, and 3) Beliefs about teaching and learning DoF.

Both writing assignments and the interview focus on prospective teachers' meta-cognition of beliefs and meta-cognition of development of their knowledge.

3.3.2 Data analysis for the instrumentation in the field practice

In this time period, I try to identify Mark's PCK and how his PCK developed through the field practice. Therefore, Table 3 showed the domains of SCK and PCK for teaching DoF.

Table 3

The domains of SCK and PCK for teaching DoF

SCK	PCK (KCC, KCS and KCT)
Responding to students' 'why' questions of DoF algorithm works	Anticipating what ***students*** might be interested in about learning DoF (KCS)
Analyzing errors and evaluating alternative ideas	Anticipating ***student*** errors and common misconceptions for DoF (KCS)
Giving mathematical explanations and using multiple mathematical representations	Interpreting ***student*** incomplete thinking about DoF (KCS)
Being explicit about mathematical language	Assessing and Evaluating ***students*** learning to achieve teaching objectives (KCS)
Linking presentations to underlying ideas and to other presentations	Knowing different instructionally viable model or manipulatives for DoF
Connecting a topic being taught to topics from prior or future years.	Knowledge about curriculum

Table 4 indicates the purpose and the data analysis in the field practice. I use Sowder, Philipp, Armstrong, and Schappelle's (1998) observation form (see Appendix 6) for Mark's field practice observation to assess his SCK and PCK. It can be categorized into three groups. First is mathematics content, DoF. Here I mainly focus on the goal of the lesson, which is what students are supposed to be learning to be able to do or to understand. It reveals Mark's beliefs towards to teaching and learning DoF. Further, I focus on whether the underlying of DoF concept is emphasized in the lesson or whether procedural steps and facts are emphasized. Thus, the next item is to identify whether the lesson is the emphasis on "doing mathematics" (e.g., framing problems, making conjectures, looking for patterns, examining constraints, determining whether an answer is valid or reasonable, knowledge when a problem is solved, justifying, explaining, challenging) or whether the lesson is the emphasis on getting right answers. Finally, I indicate whether the content of the lesson connected to other class has been dealing with.

Table 4

Purpose of the instrumentation and the data analysis in the field practice

	Classroom observation	Interview
Purpose	Understanding participants PCK/beliefs towards to DoF, teaching and learning DoF	Reflection of PCK and his teaching beliefs
Data analysis	Categorized: * Mathematical content (DoF): Purpose or goal/emphasis underlying meaning or procedural steps and facts/doing mathematics or getting answers/understanding assess * Instructional representations (What instructional representation did he use? How did he use?) * Classroom discourse (Did the teacher frequently and correctly verbalize reasons, understandings, and solution strategies himself? Did students do this frequently in response to encouragement from the teacher? How did he respond to students' answer? In what way was the discourse convergent?)	The objectives for the lessons. Select the materials and examples? Why? Self-evaluation of using representations and examples? Why? Indicate and predict the common students' misconceptions about the content and how to deal with? Why? Meta-cognition of knowledge and its development Meta-cognition of beliefs

For the instructional representation, I focus on what instructional representations (concrete, pictorial, real-world, or symbolic) Mark used in his lesson and the purpose for the representation used for connecting the content topic of DoF. I describe the strength and weakness from my perspective of each instructional representation for the lesson.

To analyze the classroom discourse, I mainly focus on Mark's discourse and the interaction between Mark and his students. I identify whether he frequently verbalize reasons, understandings, and solution strategies himself or the students do this frequently in response to encouragement from Mark. I also focus on the way Mark responded to students' errors.

Finally, I organize the reflective interview to identify 1) the objectives for the lesson; 2) the way selecting the materials and examples, and why he choose them, 3) self-evaluation of using representation and examples in the classroom of DoF and why, 4) responding to students' errors, 5) their understanding (or changing thinking) of teaching and learning DoF; 6) their understanding (or changing thinking) about teaching and learning. Since university and public middle school are two different contexts for prospective teachers (e.g., Borko et al., 1992), the emphasis of data analysis focused on how the knowledge (conceptual based or procedure based) of prospective teachers developed and how their beliefs changed in these different contexts.

3.4 Summary

Overall, I provide the instrumentations for this study. Mentioned previously, this study is divided into two time periods, the undergraduate mathematics method course and the prospective teacher's field practice. I further analyze and report the data in three steps. First, I focus on a total of 27 prospective teachers in the method course and reveal the way knowledge developed and beliefs changes in general by analyzing the data of test, the concept mapping, and the survey. Next, by choosing six participants from the total 27 based on CCK, I intend to reveal the correlation between knowledge development and beliefs changes. I focus on whether and how the differences of CCK influence the development of SCK. Moreover, I intend to reveal whether and how beliefs changes based on the development of CCK and SCK. Finally, I follow up on prospective teacher in his field practice in order to identify how the beliefs he hold impacted his teaching behavior and how SCK he developed influenced his PCK and its development. Further, I try to identify how PCK and beliefs developed from his own reflective thinking after his teaching experiences.

CHAPTER Ⅳ

RESULTS

In this chapter, I first report 27 prospective teachers' knowledge development and belief changes through their learning and teaching of fraction division (DoF) in general by reporting the results of the test, the concept mapping, and the survey. Next, I report the results of their writing assignments and interviews of six participants who are considered as high-achievement, average, and low-achievement. Thus, I study the correlation between knowledge development and beliefs changes. I focus on whether and how the differences of CCK influence the development of SCK. Moreover, I intend to reveal whether and how beliefs changes based on the development of CCK and SCK. Finally, I report one prospective teacher classroom instruction and his reflective thinking in order to identify how the beliefs he hold impacted his teaching behavior and how SCK he developed influenced his PCK and its development. Especially, I focus on KCS and KCT as part of PCK. Further, I try to identify how PCK and beliefs developed from his own reflective thinking after his teaching experiences.

4.1 Knowledge Development and Beliefs Changes in the Method Course

First, I will report the results of those prospective teachers' knowledge development and beliefs changes in the method course.

4.1.1 Prospective teachers' beliefs and its changes

The results will be reported for the prospective teachers' meta-cognition about their understanding of TEKS and their beliefs and attitudes to mathematics teaching and learning. These results included the following aspects: 1) self-evaluation of their understanding of curriculum (TEKS), 2) self-efficiency of teaching DoF, and 3) beliefs and attitudes about teaching and learning mathematics.

Understanding of curriculum (TEKS) - As shown in Table 5, about three-fourths of the prospective teachers considered that they were proficient in their understanding of TEKS by the end of the course, representing a notable improvement from the pretest. On the posttest, a total of 26 (96%) prospective teachers considered that their understandings of TEKS. These results show that a great majority of these prospective teachers became more confident for their understandings of TEKS after their undergraduate method course.

Table 5

Prospective teachers' perception of their understanding of mathematics TEKS

Level of Understanding	Pre-test	Post-test
High	4%	22%
Proficient	37%	73%
Limited	18%	4%
Low	4%	/

Beliefs and Attitudes about Teaching Mathematics - The results for four aspects of teaching and learning are reported: 1) use of multiple representations (picture, concrete materials, symbols, etc); 2) the role and use of representations; 3) knowledge about students' common misconception/difficulties; and 4) use of the models.

As illustrated in Table 6, all of the prospective teachers in both pre-survey and post-survey indicated that they were ready for representing and explaining computations with fractions using words, numbers, or models based on their training and experience in both mathematics and instruction. From the pre- to the post-survey, 7 participants (26%) in the pre-test considered that they were very ready, while the number increased to 16 (59%) in the post-survey. Further, 20 (44%) of them considered that they were ready in the pre-survey and 11 (41%) were very ready in post-survey. The results showed that more prospective teachers felt confident to teach and represent fraction computation.

Table 6

Prospective teachers' self-efficacy of using multiple representations

How ready do you feel you are to represent and explain computations with fractions using words, numbers, or models	Pre-test	Post-test
Very ready	26%	59%
Ready	74%	41%
Not ready	/	/

As shown in Table 7, a large majority of the prospective teachers agreed that using manupulatives can help students avoid abstract mathematics. The percentage of increased from 60% on the pre-survey to 75% on the post-survey. It showed that the majority in this group did not consider connecting the manipulatives and abstract mathematic ideas in their teaching.

Table 7

Prospective teachers' perception regarding the role of manipulatives

Use of manipulatives can help students avoid abstract mathematics	Pre-test	Post-test
Strongly agree	27%	41%
Agree	33%	33%
Disagree	33%	18%
Strongly disagree	7%	7%

The majority of the prospective teachers did not consider that teachers should prevent students from making errors in their learning of a mathematics topic. Specifically, as shown in Table 8, a total of 20 (71%) in the pre-survey and 18 (67%) in the post-survey disagreed or strongly disagreed to prevent students from making errors. The percentage of disagreement slightly decreased in the post-survey.

Table 8

Prospective teachers' perception regarding preventing students from making errors

Teacher should prevent students from making errors in their learning of mathematics	Pre-test	Post-test
Strongly agree	15%	11%
Agree	11%	22%
Disagree	41%	52%
Strongly disagree	30%	15%

In summary, there were few differences between the results of pre- and post-survey in terms of prospective teachers' perception about the use of representation, the students' common misconception and learning difficulties, and the use of real-world problems. The majority of prospective teachers agree that teachers should use multiple representations (picture, concrete material, symbols, etc) in teaching a mathematics topic, teachers need to know students' common misconception/difficulty in teaching a mathematics topic, and that modeling real-world problems is essential to teaching mathematics.

Beliefs and Attitudes about Learning Mathematics - From the learning perspectives, the prospective teachers provided their perceptions of 1) the role of memorization, and 2) the role of algorithm and rules. It reveals whether these prospective teachers consider that mathematics should be learned as sets of algorithms or rules that cover all possibilities. It showed the prospective teachers belief about how to learn mathematics and what to learn.

There were few differences between their pre- and post-survey on memorizing. As shown in Table 9, most participants either disagreed or strongly disagreed that learning mathematics mainly involved memorizing. All but one participant believed that there were multiple ways that students should master to solve most mathematics problems.

Table 9

Prospective teachers' perceptions regarding memorizing

Learning mathematics mainly involves memorizing	Pre-test	Post-test
Strongly agree	/	11%
Agree	11%	15%
Disagree	48%	48%
Strongly disagree	41%	26%

As shown in Table 10, both nine (33%) of the prospective teachers in the pre-survey and 11 (41%) in the post-survey indicated either agree or strongly agree that mathematics should be learned as sets of algorithms or rules. More than half of the participants disagreed.

Table 10

Prospective teachers' perceptions regarding algorithms and rules

Mathematics should be learned as sets of algorithms or rules that cover all possibilities	Pre-test	Post-test
Strongly agree	/	15%
Agree	33%	26%
Disagree	52%	44%
Strongly disagree	11%	15%
Not sure	4%	/

In summary, from the results, on one hand, prospective teachers agree that mathematics should not be learned only by memorization. On the other hand, they considered that the mathematics content that students should learn is the algorithm and rules.

4.1.2 Prospective teachers' SMK (CCK and SCK) and its development

***Procedural proficiency* -** Overall, the results show that there are no differences between pre- and post-tests for the prospective teachers' CCK. In particular, the majority of these prospective teachers did well on first three problems, which were DoF computation. As shown in Table 11, on the pre-test, all but four students (81%) answered the first question correctly, all but two students (93%) answered the second question correctly, and all but two students answered the third correctly (93%). On the post-test, 74% answered the first item correctly, 93% answered the second item correctly, and 74% answered the third correctly.

Table 11

The accuracy of DoF computation in the pre- and post- test

Test	$\frac{1}{5} \div 5 =$	$\frac{7}{9} \div \frac{2}{3} =$	$5\frac{1}{4} \div 3\frac{1}{2} =$	$\frac{14}{15} \div \frac{?}{9} = \frac{3}{10}$, find ?
Pre-	85%	93%	93%	48%
Post-	74%	93%	74%	33%

The results showed that there were several error patterns that prospective teachers made. Three out four in the pre- and one out of seven in the post-test who gave the incorrect answer for the first item ($\frac{1}{5} \div 5 =$) answered "$\frac{1}{10}$". It can be assumed that these students inverted the second fraction and added the denominators together instead of multiplying the denominators. On the post-test, five students answered “1”, which can be assumed that they considered “÷” as “×” and multiplied two numbers to get answer.

Only one student had an incorrect answer for the second item in both pre- and post-test. Except arithmetic errors, one student answered $\frac{7}{9} \div \frac{2}{3} = \frac{7}{9} \div \frac{6}{9} = \frac{7}{9} \times \frac{9}{6} = \frac{42}{9}$ on her pre-test. She found the common denominator and inverted the second fraction. Instead of simplifying 9, she kept the 9 as the denominator and cross-multiplied 7 and 6 as the numerator. This student used the same approach to solve the third item on her post-test.

For the third problem, except the arithmetic errors and the pattern mentioned previously, prospective teachers had several other error patterns. One of them inverted the dividend and then multiplied the second fraction. This prospective teacher memorized the DoF algorithm rule as “invert and multiply” but confused of which number she should invert. Two had problems when they tried to convert mixed numbers to improper fractions. One student did not give the answer in both pre- and post-test.

The prospective teachers had some difficulties with the fourth item (i.e., if $\frac{14}{15} \div \frac{?}{9} = \frac{3}{10}$, find ?). Only 13 students (48%) on the pre-test and nine students (33%) in the post-test got the answer correctly.

There were four types of students’ incorrect or incomplete answers. The first error was providing the equation ($\frac{14}{15} \cdot \frac{9}{x} = \frac{3}{10}$) without solving the problem. Four prospective teachers (15%) on the pre-test and two 7% on the post-test made this error. The second pattern was an arithmetic mistake. Six (22%) on the pre-test and five (19%) on the post-test either wrote the wrong number to do the computation (i.e., writing 13 instead of 3) or missed the number (i.e., $\frac{14}{15}\times\frac{9}{?}=\frac{3}{10}\Rightarrow\frac{14}{5}\times\frac{3}{?}=\frac{3}{10}\Rightarrow\frac{42}{5?}=\frac{3}{10}\Rightarrow 420=15?\Rightarrow ?=420$). Further, the answers of four (15%) in the pre-test and three (11%) in the post-test were not clear. Also, three (11%) in pre-test and five (19%) in the post-test did not answer this question or indicated that they did not know how to do it.

On the post-test, besides the previous types of errors, three (11%) prospective teachers had a conceptual misunderstanding and used the method of cross-multiply to get their answer. They inverted the second fraction and multiplied. However, when they multiplied, they also remembered the rule of cross-multiply. One provided the answer $\frac{15x}{126}=\frac{3}{10}$; and one prospective teacher was not sure which number would be the numerator after the cross-multiplication. Thus he wrote "$4x$" and "9(15)". The third student confused the expression $\frac{14}{15}\times\frac{9}{x}$ with the algebraic equation ($\frac{14}{15}=\frac{9}{x}$). She wrote the equation $14x=15(9)$ and ignored the number $\frac{3}{10}$. Although prospective teachers did not use the cross-multiplication method in the pre-test, I assumed that some held the misconception. In other words, four wrote inverting second fraction process without the computation ($\frac{14}{15} \times \frac{9}{x}$). It is hard to tell whether they would use cross multiplication for the next step.

Overall, the accuracy in the post-test slightly decreased slightly. This result may be due to the fact that the post-test was taken at the same time as the final exam and this may have disturbed students' attention and thus, it may have influenced the results of the post-test.

Knowledge of DoF applications - Beyond the procedural fluency, the test required prospective teachers to solve problems in order to assess their understanding of the concept of fraction division and the features of fraction division algorithm. The questions can be categorized into three four types.

The first type is the word problem without a real world context. There were two problems in this category (i.e., How much $\frac{1}{2}$'s are in $\frac{1}{3}$; $\frac{5}{6}$ of a number equals to $\frac{5}{24}$, find the number). The results for these items are shown in Table 12.

There was a notable difference between the pre- and post-test for the first word problem. Only 10 (37%) of the prospective teachers both wrote the expression and did the algorithm correctly in the pre-test, while 24 (89%) answered correctly on the post-test. Most prospective teachers could not find the mathematical relationship for this problem. On the pre-test, 17 (63%) prospective teachers either answered incorrectly or did not answer this question. There were six different types of inaccurate answers. Seven (26%) participants wrote "none" without any explanation or number sentence; three

(11%) wrote “1 or one” for the answer; one (4%) wrote “2” for the answer without the explanation; two (7%) understood the problem as “ how many $\frac{1}{3}$’s are in $\frac{1}{2}$ instead and

Table 12

Results of the simple word problems in the pre- and post-test

Word problems without a context-based situation		How many $\frac{1}{2}$’s are in $\frac{1}{3}$	$\frac{5}{6}$ of a number equals to $\frac{5}{24}$, find the number
Both expression and Result are correct	Pre-	37%	64%
	Post-	89%	64%
No expression Correct result	Pre-	/	11%
	Post-	/	/
No/wrong result Correct expression	Pre-	/	7%
	Post-	/	7%
Both procedures and the result are wrong	Pre-	56%	11%
	Post-	11%	25%
No Answer	Pre-	7%	7%
	Post-	/	4%

wrote the number sentence as "$\frac{1}{2} \div \frac{1}{3} = \frac{3}{2} = 1\frac{1}{3}$"; two (7%) prospective teachers used the multiplication instead of division and wrote "$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$" and the last two did not write any answer. A total of 17 prospective teachers on the pre-test did not correctly understand the mathematical relationship in the problem and solve it.

On the post-test, most prospective teachers successfully wrote the expression to show the quantity relationship. Only three prospective teachers could not understand the mathematical relationship and wrote the expression in the post-test correctly. Two of them had the same wrong answer (i.e., "none" or "one") on both pre- and post-test, indicating that they still had difficulties understanding the questions and solving them.

In comparison with the first problem, prospective teachers did not show improvement from the pre- to the post-test for the second word problem without a real world context. As shown in Table 12, 20 (74%) prospective teachers answered this problem correctly in pre-test. 17 (64%) wrote the expression and did the computation correctly; three (11%) of them gave the correct answer without showing the expression), while 17 (64%) prospective teachers wrote the expression and did the algorithm correctly in the post-test. Moreover, all who provided the number sentence and answered correctly in both pre- and post-test used the algebraic method to solve this problem. It shows that most students understood that the quantitative relationship of "$\frac{5}{6}$ *of* a number" is $\frac{5}{6}$ multiplying a number.

There were three (11%) prospective teachers in pre-test and seven (25%) in the post-test who did not provide the correct expression. There were two patterns. One pattern was that the prospective teachers noticed the multiplicative relationship by the word "*of* ". However, they simply multiplied $\frac{5}{6}$ with $\frac{24}{25}$ and found the answer. The other pattern was that the students divided $\frac{5}{6}$ by $\frac{24}{25}$.

Two prospective teachers (7%) on pre-test provided the expression correctly without doing the computation, while two (7%) in the post-test provided the expression correctly with the wrong computation. Instead of dividing $\frac{24}{25}$ by $\frac{5}{6}$, they multiplied these two numbers. Finally, two prospective teachers (7%) on pre-test and one (4%) on post-test did not provide any answer for this problem. The third item assessed participants on their understanding about features of fraction division. The problem required students to identify which quotient is greater without calculating (i.e., "$\frac{9}{11} \div \frac{2}{3}$" and "$\frac{9}{11} \div \frac{3}{4}$") and explain why. As shown in Table 13, there was not much difference between pre- and post-test. 11 (41%) prospective teachers in the pre-test and 12 (44%) on the post-test correctly identified and explained the features of DoF (i.e., "if the dividend keeps the same, then the quotient would be bigger if the divisor is smaller"). Six (22%) prospective teachers on the pre-test and 4 (15%) on the post-test correctly identified the answer (see Table 13). But they either did not provide an explanation or obtained the correct answer through calculation. 10 (37%) of all prospective teachers in the pre-test and 11 (41%) in the post-test failed to identify and explain the answer. Five students were wrong in both the pre- and post-test.

Table 13

Understanding of characteristics of Dof in the pre- and post-test

Problem solving		Identify whether $\frac{9}{11} \div \frac{2}{3}$ is greater than or less than $\frac{9}{11} \div \frac{3}{4}$ without solving, explain why.
Both identification and explanation are correct	Pre-	41%
	Post-	44%
Correct identification but no explanation	Pre-	22%
	Post-	15%
Both identification and explanation are wrong	Pre-	37%
	Post-	41%
No Answer	Pre-	/
	Post-	/

The third type of problems is the word problems within a real world application. Two are one-step problems and one is a two-step problem (see Table 14).

Table 14

Knowledge of solving real world application in the pre- and post-test

(One Step Problem)

Problem solving		A five-meter-long rope was divided into 15 equal pieces. What is the length of each piece?	Andrew bought 7 apples, which is 1/3 of the number of oranges he bought. How many oranges did Andrew buy?
Right expression and results	Pre-	70%	52%
	Post-	78%	59%
No expression / right result	Pre-	11%	40%
	Post-	7%	27%
No result / right expression	Pre-	4%	4%
	Post-	/	7%
Incorrect answer/ NA	Pre-	15%	4%
	Post-	15%	7%

As shown in Table 14, for the first two questions, there was no difference between the pre- and post-test. A total of 22 (81%) in the pre-test and 23 (85%) in the post-test answered correctly. There were 19 (70%) in the pre-test and 21 (78%) in the post-test who correctly wrote the expression and solved it. The other three (11%) prospective teachers in the pre-test and two (7%) in the post-test gave the correct answer without any expression or with the wrong expression. Among those who answered correctly, seven prospective teachers in the pre-test and nine in the post-test provided a pictorial representation. There was one student (4%) in the pre-test who correctly gave the expression for the problem but did not solve it.

There were four (15%) students both in the pre- and post-test who either could not solve this problem or answered incorrectly. Only two students did wrong on both tests. The main error they made was failing to understand the quantity relationship. A prospective teacher wrote the following on her test: " A meter stick is 3ft long, so 5 meters times three feet in each meter gives 15 equal feet".

Thus, they provided the expression as "3×5=15" for the answer.

The second question was to assess students' understand of the quantity relationship of the word "*of*" in a real world context. Most of the students who solved the problem correctly noticed that the number of the apples is $\frac{1}{3}$ *of* the number of oranges. This means that $\frac{1}{3}$ multiplying the number of oranges is the number of apples. A total of 25 prospective teachers in the pre-test and 23 in the post-test gave the correct answer. Among them, 14 (52%) in the pre-test and 16 (59%) in the post-test provided both the accurate expression and solved it. The number of prospective teachers using an algebraic approach (i.e., $\frac{1}{3}x = 7$ or $\frac{x}{3} = 7$") and the number of those using the arithmetic way ($7 \div \frac{1}{3} = 7 \times 3 = 21$) were almost the same in both the pre- and post-test. 11 (41%) in the pre-test and 7 (27%) in the post-test provided the correct answer without number sentence.

One participant on both the pre-test and post-test solved the problem correctly. However, she considered that the number of apples (7) is $\frac{1}{3}$ of the number of *all* fruits that Andrew bought and got the wrong answer (21-7=14). There was one in the post-test who only provided the correct expression without the answer. There was one student in the pre-test and two in the post-test who could not identify the quantity relation and wrote the wrong number sentence for this problem ($7 \times \frac{1}{3} = 2\frac{1}{3}$). Looking across the pre- and post-test, those who could not solve the problem in the pre-test successfully answered the problems in the post-test.

The last type of word problem in the real world context entails two steps, which is more complicated for the students. The majority of these prospective teachers could solve the first step correctly, while they had difficulties with the second step (see Table 15).

Table 15

Knowledge of solving real world application in the pre- and post-test

(Two-Step Problem)

Word problem within a real world context		Johnny's Pizza Express sells several different flavored large-size pizzas. One day, it sold 24 pepperoni pizzas. The number of plain cheese pizzas sold on that day was 3/4 of the number of pepperoni pizzas sold, and 2/3 of the number of deluxe pizzas sold. How many deluxe pizzas did the pizza express sell on that day?
Both first /second step were correct	Pre-	30%
	Post-	30%
First step was correct	Pre-	56%
	Post-	78%
Both steps were wrong	Pre-	44%
	Post-	19%
No answer	Pre-	/
	Post-	4%

Finding the number of deluxe pizzas sold requires knowing the number of the plain cheese pizzas sold on that day. Since the number of plain cheese pizzas was $\frac{3}{4}$ *of* the number of deluxe pizzas, it would be the multiplicative quantity relation. Fifteen participants (56%) found the correct answer for the first step in the pre-test, while 21 (78%) solved the first step in the post-test. It can be assumed that most students developed the understanding of quantity relation for "*of"* in word problem. However, only 8 (30%) solved both first and second step in the pre-test and 8 (30%) solved both steps in the post-test. There were 12 (44%) prospective who teachers failed to solve either step in the pre-test, while 5 (19%) failed to solve either step in the post-test. There was one prospective teacher (4%) who did not provide any answer for this problem.

The main error was a misunderstanding of the quantity relationship. There were several types of student errors. For instance, some of them wrote "$18\times\frac{2}{3}=12$". Here, those prospective teachers noticed, "$\frac{2}{3}$ *of* the number of deluxe pizzas sold". S/He may consider it would require a multiplicative relation in this problem. However, s/he did not recognize that "$\frac{2}{3}$ *of* the number of deluxe pizzas sold" has a different quantity relation with "$\frac{2}{3}$ *of* the number of plain cheese pizzas sold". Some prospective teachers directly wrote the number sentence that "$24\times\frac{2}{3}=16$". Again, these prospective teachers understand the multiplicative relation in this problem but find "$\frac{2}{3}$ *of* the number of pepperoni pizzas sold". Others wrote "$24\div\frac{2}{3}=32$". They notice that "$\frac{2}{3}$ *of* the number of pepperoni pizzas sold" would be $\frac{2}{3}x$, but misunderstood that it equals to the number of plain cheese pizzas instead of the number of pepperoni pizzas.

Connecting knowledge of DoF with previous knowledge - The results of concept mapping showed prospective teachers understanding of the connection between knowledge of DoF and previous knowledge. The concept mapping required that the prospective teachers construct an integrated and hierarchical network demonstrating an understanding of DoF. All but four students explored the keynotes and links in their post-concept mapping, indicating that many prospective teachers developed a hierarchical network in the post-concept mapping (see Table 16).

Table 16

Levels of links in the pre- and post-concept map

	Pre-concept mapping	Post-concept mapping
1-level	12%	3%
2-level	34%	12%
3-level	46%	46%
4-level	8%	19%
5-level	0	12%
Net	0	8%

As shown in Table 17, the concept units decreased in the post-concept mapping. More than two-thirds of the prospective teachers used at least one concept unit provided in their pre-concept mapping. In the pre-concept mapping, the three most- used concept units were “Concept of fraction”, “Concept of inverse operation”, and “Whole number division”. 15 prospective teachers mentioned “whole number division” (57.7%) instead of “multiplication of fraction”. 12 (46.2%) prospective teachers mentioned “the concept of inverse operation" in the pre-concept mapping. However, seven (26.9%) explained it as the "inverting and multiplying algorithm" or "reciprocal" and one (3.8%) addressed it without further explanation. It showed that many students considered it more procedural instead of the basic principle in the pre-concept mapping.

In post-concept mapping, all seven (26.9%) prospective teachers indicated "concept of inverse operation" correctly. Besides the concepts units provided to the prospective teachers, five (19.2%) indicated the ideas of "partitive" and "measurement" in the post-one.

Table 17

Concept units in the pre- and post-concept map

Concept units	Pre-	Post-
Concept of units	9 (34.6%)	7 (26.9%)
Concept of fraction	16 (61.5%)	5 (19.2%)
Concept of addition	5 (19.2%)	4 (15.4%)
Concept of inverse operation	12 (46.2%)	7 (26.9%)
Multiplication of fraction	11 (42.3%)	9 (34.6%)
Whole number multiplication	8 (30.8%)	9 (34.6%)
Whole number division	15(57.7%)	6 (23.1%)
Partitive (Forming a certain number of groups)	1 (3.8%)	5 (19.2%)
Measurement (Forming groups of a certain size)	1 (3.8%)	5 (19.2%)
Others	2 (7.7%)	4 (15.4%)

Overall, the results for the CCK showed that before they received the instruction, the participants' knowledge for teaching fraction division was relatively procedural. They remember, "invert-and-multiply" as the rule for DoF computation. However, some prospective teachers showed they had problems with doing the algorithm. After inverting the second number, some prospective teachers cross-multiplied two numbers. They confused DoF computation with solving an equation problem. Although most problems had no significant differences between pre- and post-test, after the instruction, the prospective teachers showed the development to the understanding of the quantity relationship for the word problems. Most prospective conceptual understanding of this topic was relatively weak in the pre-test. It reflected that they could correctly identify the quantity relation of word problems. However, after the instruction, some prospective teachers developed their understanding of quantity relation of DoF.

Unlike the concept units, the procedural units generally increased in the post-concept mapping except for one item, the reciprocal (see Table 18). After the methods course, the prospective teachers developed their understanding of common denominator algorithm as another way for doing computation. Nine (34.6%) students mentioned the common denominator algorithm in the post-mapping, while there is not one indicated in the pre-one.

Further, three students considered the common denominator algorithm as the modeling and representational units. They can explain using this method and show why the algorithm of fraction division works. Students still considered the inverse and multiply algorithm when they thought about fraction division.

Table 18

Procedural units in the pre- and post-concept map

Procedural units	Pre-concept mapping	Post-concept mapping
Algorithm	4 (15.4%)	5 (19.2%)
Inverse & Multiply (Flip &Multiply)	6 (23.1%)	10 (38.5%)
Common Denominator Algorithm	0	9 (34.6%)
Reciprocal	9 (34.6%)	5 (19.2%)
Others	2 (7.7%)	2 (7.7%)

All prospective teachers in this group developed the category of modeling units in the post-concept mapping was collected (see Table 19). The majority of the prospective teachers addressed the modeling units in the post-instructional concept mapping, while the majority did not address it in the pre-one. In the items of the modeling units, *"area model"* and "*manipulative"* are most used in the post-one. The prospective teachers considered more about how to represent the content topic besides content knowledge itself.

Table 19

Presentational units in the pre- and post-concept map

	Pre-instruction	Post-instruction
Area Model	0	18 (69.2%)
Array Model	0	1 (3.8%)
Manipulatives	2 (7.7%)	17 (65.4%)
Problem solving	0	3 (11.5%)
Others pictorial representation	6 (23.1%)	1 (3.8%)

Development of SCK - Six items on the second part of the test focused on Specialized Content Knowledge (SCK) and were categorized into two subgroups. Four item assessed "presenting mathematical ideas", "responding to students' 'why' question', "finding an example to make a specific mathematical point", " linking representations to underlying ideas and to other representations". Thus, these problems assessed prospective teachers' understanding of why the algorithm of DoF works and the ability to explain and represent to the students how the algorithm works by using different models. The other two items assessed prospective teachers' ability to identify and analyze student errors and evaluate alternative ideas.

Providing mathematical explanations and using mathematical representation

Overall, most prospective teachers had difficulties in explaining both "why"-questions in both pre- and post-test. As shown in Table 20, on the pre-test, only two prospective teachers clearly indicated that the division operation can be done by dividing across straight as a special approach. However, none of them provided an explanation why it works and the limitation for this operation. Sixteen (59%) did not identify whether dividing across worked and only indicated that fraction division should be done with the operation of “inverting-and-multiplying”. Six of them used “*must*” to indicate that ‘invert-and-multiply’ algorithm is the only way to compute. Four (15%) said that the operation of dividing across was wrong. Among these three, one indicated that one should invert and multiply after the identification; one provided the incorrect counterexample to indicate after they identify it was wrong; and one had no explanation after the identification. Five (19%) prospective teachers either did not answer this question or indicated that they did not know (Table 20).

Table 20

Knowledge for explaining why the computation works (special case)

$\frac{a}{b} \div \frac{c}{d} = \frac{(a \div c)}{(b \div d)}$	Pre-test	Post-test
Identifying with explanation	/	22%
Identifying without explanation	7%	11%
Without identification and indicating IM	59%	11%
Wrong identification	15%	19%
No answer/ Do not know/Not clear	19%	37%

Comparing with pre-test, the prospective teachers' post-test for this problem had some improvement in the part of explaining the operation of fraction division. The prospective teachers who only indicated the 'invert-and-multiply' algorithm rule reduced. Nine (33%) clearly indicated that the operation did work although it had some limitation and may not always work. Among them, six (22%) prospective teachers explained that it works because fraction division is the inverse operation of multiplication; three (11%) indicated if b=d, using common denominator, it can work. However, five (19%) still indicated that dividing across was the wrong operation for DoF and three prospective teachers (11%) did not identify whether dividing across worked but indicated that "inverting and multiplying" is the way to do fraction division. Seven (26%) did not provide any answer for this part and the other three (11%) provided an unclear explanation.

Although the prospective teachers improved somewhat on the post-test, the majority of them still had difficulties to identify whether dividing across worked for DoF and to response why it works. They can easily find counterexamples for fraction addition and subtraction, but they gave wrong counterexamples for fraction division (i.e., $\frac{1}{2} \div \frac{1}{4} = \frac{1}{8}$). Although some recognized that inverting and multiplying was not the only way for doing DoF, many of them still considered that it was the most important way to solve DoF.

The fifth question asked to explain why DoF algorithm works in general. Most of prospective teachers developed the understanding for this question in the post-test (see Table 21).

Table 21

Knowledge for explaining why the computation works (general case)

Explaining why $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$	Pre-test	Post-test
Correct explanation	11%	41%
Inert and multiply algorithm	25%	11%
Inappropriate explanation	4%	4%
No answer /Don't know/Not clear	60%	44%

The percentage for both using the "invert-and-multiply" algorithm and answering "do not know" slightly decreased in the post-test, while the percentage of using different representation to explain this "why" question increased in the post-test. Only three (11%) prospective teachers answered this question correctly in the pre-test, while 41% of the participants can explain the computation by using representations.

Although the number of the prospective teachers who could use different representations to explain DoF increased in the post-test, the majority of these prospective teachers still had difficulties to explain why the computational rule works. Moreover, most prospective teachers used the common denominator method in the post-test to explain why the computational rule works in general. They also indicated that the instructor provided the common denominator method as an example in their writing assignment.

Knowledge to Explain How the DoF Computation Works

One item asked prospective teachers to use pictures, and create a word problem to explain *how* each specific computation works. For the second item the prospective teachers were provided two equations (i.e., $\frac{2}{3} \div 2 = \frac{1}{3}$ and $\frac{2}{3} \div \frac{1}{6} = 4$”). This item asked participants to explain the answer of $\frac{2}{3} \div 2 = \frac{1}{3}$ and why $\frac{2}{3} \div \frac{1}{6} = 4$. The prospective teachers can use different representations to explain how each computation works. The first equation can be considered as a partitive division problem and the second equation can be considered as the measurement division problem. As shown in Table 22, although many prospective teachers only showed the “inverting and multiplying” algorithm instead of “explaining why”, many students tried to use different representations for explanation. The prospective teachers easily explained the first equation comparing with the second one.

Table 22

Knowledge to explain how the computation works

Explaining		Why $\frac{2}{3} \div 2 = \frac{1}{3}$?	Why $\frac{2}{3} \div \frac{1}{6} = 4$
Using multiple representations	Pre-	44%	15%
	Post-	52%	***56***%
Invert and multiply algorithm	Pre-	52%	56%
	Post-	26%	22%
Inappropriate explanation	Pre-	4%	/
	Post-	22%	/
No answer /Don't know/Not clear	Pre-	/	29%
	Post-	/	22%

On the pre-test, 12 prospective teachers (44%) gave at least one representation for explanation. Among them, four used only pictorial representation, four used only verbal explanation, and other four used both verbal and pictorial for their explanation. However, 14 (52%) of them only showed the "invert and multiply" algorithm to get the answer and used the verbal section to explain how to do the algorithm step by step. One student explained the problem inappropriately and considered that $\frac{2}{3} \div 2$ was a measurement problem (i.e. You are say that the value 2 is in the value $\frac{2}{3}$ $\frac{1}{3}$times).

The post-test showed that the percentage of using multiple representations or modeling slightly increased. However, the inappropriate explanation to the first equation increased at the same time. Six (22%) prospective teachers' explanation for the first equation was inappropriate (i.e., how many groups of 2 are into $\frac{2}{3}$ or 2 goes into $\frac{2}{3}$ $\frac{1}{3}$ time). It showed that those prospective teachers developed their understanding of measurement division problems. However, they did not consider that it was difficult for middle school students to understand the quantity relationship if they only explained it in the previous way (measurement division problem).

The results showed that the majority of those prospective teachers developed their knowledge of giving mathematical explanations and using mathematical representations. Some prospective teachers, like one of the interview participants, used the measurement division idea to explain the quantity relationship between dividend and quotient (i.e., $\frac{1}{3}$ can go into $\frac{2}{3}$ twice); some provided word problem or numerical representation (i.e., $\frac{2}{3} \div 2 = \frac{2}{3} \div \frac{2}{1} = \frac{2 \div 2}{3 \div 1} = \frac{1}{3}$) and represented the idea of dividing across straight; others provided a word problem that showed their understanding of the equation as a partitive division. (i.e., "I think(s) it is so much easier if its in words. We have $\frac{2}{3}$ cookie and 2 kids, so how much would each kid get?").

Compared with the first item, many prospective teachers had difficulties to explain using multiple representations in their pre-test. Only four students (15%) provided at least one representation for explanation. Two of them used verbal explanation and the other two used both pictorial and verbal explanations. 15 (56%) of the prospective teachers can only show the invert and multiply algorithm for this problem either without any explanation or explained step-by-step process of doing the algorithm. Seven (26%) did not provide any answer for the second equation. The left (3%) indicated s/he did not know how to do the problem. Among these six interviewees, only one provided a verbal explanation and other either mentioned they did not know how to show or they just provided invert and multiply algorithm.

> $\frac{2}{3}$ is equivalent to $\frac{4}{6}$, which is made of 4 partitions of 6. So when you want to see how many $\frac{1}{6}$ are in $\frac{2}{3}$, you find there are 4 by looking at the partitions of the whole.

There were some differences between the pre- and post-test for the questions, in particular, for the second equation. Comparing with pre-test, prospective teachers developed their explanation by using multiple representations for both equations. Especially for the second equation (i.e., Why $\frac{2}{3} \div \frac{1}{6} = 4$), 15 prospective teachers (56%) correctly provided at least one representation. However, the frequency of using the invert and multiply algorithm decreased (22%). Nine people used verbal explanation; three used pictorial representation; one used numerical; and two used both pictorial and verbal explanations.

Most prospective teachers developed their understanding of measurement division problems, thus, the majority of them used verbal explanation. For instance, "how many times does $\frac{1}{6}$ go into $\frac{2}{3}$". Below is the only numerical explanation.

> "$\frac{2}{3} \div \frac{1}{6} = 4$ divide numerator: 2÷1=2 divide denominator: 2÷1=$\frac{1}{2}$

$\dfrac{2}{\frac{1}{2}}$ get fraction out of bottom: $\dfrac{2 \cdot 2}{\frac{1}{2} \cdot \frac{2}{1}} = \dfrac{4}{1} = 4$ or $\dfrac{2}{3} \div \dfrac{1}{6} = \dfrac{4}{6} \div \dfrac{1}{6} = \dfrac{4 \div 1}{6 \div 6} = \dfrac{4}{1} = 4$.

Knowledge to provide an example scenario and using manipulatives or models

One question required prospective teachers to make up a story problem of $\frac{3}{4} \div \frac{1}{2}$ and draw a picture that will go with "three-fourths divided by one-half". As shown in Table 23, most prospective teachers had difficulties making a measurement division problem in their pre-test. It revealed that prospective teachers did not understand the quantity relation in this expression. Moreover, they had difficulties using pictorial representation to show "three-fourths divided by one-half". In the post-test, almost half of them still made a scenario that did not represent $\frac{3}{4} \div \frac{1}{2}$ mathematical relationship (see Table 23). There is no difference when comparing with pre-test. In the post-test, the percentage of not providing the information decreased.

On both pre- and post-test, no prospective teacher provided the correct pictorial representation. Only two created an appropriate scenario for $\frac{3}{4} \div \frac{1}{2}$. The percentage

Table 23

Knowledge of real world applications of DoF

Make up a story problem $\frac{3}{4} \div \frac{1}{2}$ and draw a picture	Pre-test	Post-test
Correct scenario with pictorial representation	/	/
Correct scenario with no pictorial representation	7%	44%
Scenario of different quantity relation	52%	48%
No answer /Don't know/Not clear	41%	8%

increased to 44% on the post-test. Twelve prospective teachers developed their understanding of this real life application for this expression. However, the percentage incorrect did not change in the post-test. Only one of the six interviewees provided a correct scenario in the pre-test and two on the post-test.

There were several major incorrect scenarios that prospective teachers provided in both pre- and post-test.

1) The scenario of $\frac{3}{4} \div 2$*, instead of* $\frac{3}{4} \div \frac{1}{2}$.

The most common error that prospective teachers made was confusing $\frac{3}{4} \div \frac{1}{2}$ with $\frac{3}{4} \div 2$. Nine out of 14 prospective teachers (33%) on the pre-test and 12 out of 13 (44%) on the post-test who created an inappropriate scenario provided a scenario of partitive division problem such as the following.

Jerry has $\frac{3}{4}$ of a pizza left. He wants to split it with Kristi. How much pizza should each of them get if it is split evenly?

Similar with the results of Ma's finding, the prospective teachers' used the phrases like "divide (share) evenly between two" or "divide into half". Most prospective teachers did not notice the difference between $\frac{3}{4} \div \frac{1}{2}$ and $\frac{3}{4} \div 2$.

2) The scenario of multiplication problem, instead of fraction division.

The scenarios of two different multiplication problems were made up in the pre-test.

I have three-fourths of a pizza left. I want to double how much I have. How big will the new pizza be?

There was $\frac{3}{4}$ of leftover pie and Suzzie wanted to eat $\frac{1}{2}$ of what was left. How much pie did Suzzie eat?

The first scenario was created to represent the quantity relationship of $\frac{3}{4} \times 2$. It may be that this prospective teacher did the computation and made up the scenario based on the computation. Although the answer of $\frac{3}{4} \div \frac{1}{2}$ and $\frac{3}{4} \times 2$ are the same, the concept of these two express are totally different.

The second scenario was created to represent $\frac{3}{4} \times \frac{1}{2}$ instead of $\frac{3}{4} \div \frac{1}{2}$. It is to find a certain portion ($\frac{1}{2}$) of a unit ($\frac{3}{4}$), which is multiplication by fractions.

3) The scenario of subtraction problem.

One prospective teacher created this scenario to represent $\frac{3}{4} - \frac{1}{2}$.

Candie has $\frac{3}{4}$ a bag of candy. She decided to give her friend Mandi half that bag. How much of the bag will Candie have left?

4.1.3 Summary

Overall, in general, both CCK and SCK of the prospective teachers in the method course developed at the end of the semester. SCK, in particular, the knowledge of explaining and representing why DoF works developed significant after learning. However, the majority of the participants still had difficulty to create a real-world problem based on measurement model of division and all the participants had difficulties to use a pictorial representation for the measurement model of division.

4.2 The Correlation Between Knowledge Development and Beliefs Changes

Next, I report six students' SMK (CCK and SCK) development and beliefs changes.

4.2.1 Six prospective teachers' SMK (CCK and SCK) and its development

Mentioned previously, six participants are selected based on their mathematics achievement. In other words, they are selected based on their CCK. There are no significant differences of CCK development between pre-test and post-test. However, it reveals the differences of SCK in certain points. Table 24 showed six participants' SCK in several item. I combine the interview and writing assignment data for the further explanation.

Knowledge for explaining why the computation works - For the first question of identifying and explaining $\frac{a}{b} \div \frac{c}{d} = \frac{(a \div c)}{(b \div d)}$. Only Eric, who has high mathematics achievement, identifies this question correctly in the pre-test. Mark answers wrong (i.e., it's not correct). David does not identify but indicates IM algorithm instead. Amy and Mary either indicates she doesn't know or without answer. Lily answers this problem unclearly. In the post-test, Eric, Amy and Mark identify this equation by using either verbal explanation or symbolic presentations. However, Mary and Divid still indicate IM algorithm instead. Lily, on the other hand, still provides unclear explanation.

Table 24

Six participants' SCK in the pre- and post-test

	$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$	Explain why $\frac{2}{3} \div \frac{1}{6} = 4$ and why $\frac{2}{3} \div 2 = \frac{1}{3}$.	$\frac{3}{4} \div \frac{1}{2}$
Eric (pre)	Identify without explanation	IM	Correct verbal/
(post)	Identify with verbal explanation	Verbal, pictorial (1) Numerical for (2)	Correct verbal/ measurement explanation
Amy(pre)	Don't know	IM	/
(post)	Identify with symbolic representation	Pictorial for (1)	Correct verbal/Consider the real world situation
Mark (pre)	Wrong	IM	/
(post)	Symbolic representation	IM	Wrong verbal
Mary (pre)	/	Verbal for both (2)	Wrong
(post)	IM (common denominator)	Pictorial & verbal (2)	Wrong
David (pre)	IM	Verbal (1)	/
(post)	IM	Inappropriate (1) Verbal (2)	Correct verbal
Lily (pre)	Not clear	Verbal (2)	Wrong
(post)	Not clear	Verbal (2)	Wrong

* (1) refers to the first equation and (2) refers to the second equation

Further, in the interview, Eric, Amy, Mark, and Mary mentioned that dividing across straight is a special case in DoF and they would teach this strategy carefully. Eric indicated the importance to let students to realize that dividing across straight is only a special case. The students should understand that it could not be used in general. On the other hand, Mark and Mary provided common denominator algorithm as example to show dividing across straight works. Eric explained:

> I would (teach this special case), but I would teach them be careful, because like $\frac{1}{2} \div \frac{3}{4}$, it would be just circle over and over. (Eric, interview, 04/24/2007)

It shows that Eric, Amy and Mark, who have high CCK, easily to develop their understanding of explain why DoF computation works in this special case. Mary, although she provides IM in her post-test, she dose include common denominator method for her explanation in the interview. Comparing to the other four participants, David and Lily still struggle with the explanation in their posttest.

Next, in order to explain why $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ works,

Eric explained verbally in the pre-test. None of except Eric explained in the pre-test:

> I would use an example, we have 2 whole pizzas. I want to know how many $\frac{1}{3}$ pieces I can have. So now, with 1 pizza I have $\frac{3}{3}$, and same with other. I have $\frac{6}{3}$. Another way to say this is I have 6 $\frac{1}{3}$ pieces. This is answering our question.
>
> (Eric, pre-test)

However, his explanation actually did not explain why you change from "division" to "multiplication" and flip the second fraction. However, using an example, Eric showed how “division” to “multiplication”, flip the second fraction and get the answer. There is only one explanation as to why DoF computation works in this study. Except Eric, other five participants either indicate IM algorithm or indicate they do not know why. In the post-test, five students answered this question.

Mark explained in a different way:

> We can use the logic that $\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$ and since $a \div c$ can be written $\frac{a}{c}$ and $b \div d$ can be written $\frac{a}{c}$, and that leads us to $\frac{\frac{a}{c}}{\frac{b}{d}} = \frac{a}{\frac{b}{d} \cdot c}$ and $\frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}$. (Mark, post-test)

He used the knowledge of division relation between fraction and division. Division can be written as the fraction. Instead of simplifying and getting the denominator as 1, the student put the denominator c to multiply $\frac{b}{d}$. He did not explain why d is changed as the numerator in his explanation. Although his explanation was incomplete, he did have one aspect of SCK, which was giving mathematical explanations and using mathematical representation.

Amy, Mark, Mary, David, and Lily also answered correctly. There were three different ways. First, nine (including Mark, Mary, David and Lily) of those who answered correctly used the numerical representation $\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{ad}{bc}}{1} = \frac{ad}{bc}$. They mainly used numerical expression to show how the denominator cancelled out. Including Amy, several prospective teachers explained it by verbally.

Amy used the concept of the equation to indicate this problem. She linked the concept of the equation to multiplying the same number $\frac{d}{c}$ on both sides of the equal sign and explained it linking previous knowledge.

> When you think of division as what times $\frac{c}{d}$ will give me $\frac{a}{b}$ or $\frac{a}{b} = \frac{c}{d}x$. We want x, so we multiply by the reciprocal $\frac{d}{c} \cdot \frac{a}{b} = \frac{c}{d}x \cdot \frac{d}{c}$. So on one side we have x, the other we have $\frac{a}{b} \cdot \frac{d}{c}$ (which is reciprocal of $\frac{d}{c}$). (Amy, post-test)

Eric tried to explain in a way by cancelling out the denominator. In order to cancel out the numerator $\frac{b}{d}$, both the numerator and denominator should multiply the reciprocal $\frac{d}{b}$, because $\frac{\frac{d}{b}}{\frac{d}{b}}$=1. However, Eric remembered that the fraction should multiply 1 to not change the quantity, but did not understand what it is.

> Show them the division across $\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d} = \frac{\frac{a}{c}}{\frac{b}{d}}$. To get rid of the denominators, we multiply by d/d and c/c $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ (Eric, post-test).

This question shows that most participants can use symbolic representation to show why the computational rule of DoF works. It can be considered that the participants can understand conceptually by themselves.

Knowledge for explaining how the computation works - Next, I focus on whether or not they can show it by using different representation.

For the question of explaining why $\frac{2}{3} \div \frac{1}{6} = 4$ and why $\frac{2}{3} \div 2 = \frac{1}{3}$, in particular, all six prospective teaches intended to explain how the second equation works. Eric, Amy, and Mark simply provided the 'invert-and-multiply' algorithm. Mary, David, and Lily explained verbally for the second equation, which is the partitive model of division.

> $\frac{2}{3}$ is made up of 2 pieces, $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. So dividing $\frac{2}{3}$ by 2 gives you 2 pieces $\frac{1}{3}$ and $\frac{1}{3}$. (Mary, pre-test)
>
> $\frac{2}{3} \div 2 = \frac{1}{3}$ is easy because $\frac{1}{2}$ of $\frac{2}{3}$ is $\frac{1}{3}$. (David, pre-test)
>
> Since $\frac{2}{3}$ is two pieces that are the size $\frac{1}{3}$ ($\frac{1}{3}$ and $\frac{1}{3}$), then dividing it by 2 will get two $\frac{1}{3}$. (Lily, pre-test)

David indicated that $\frac{2}{3} \div 2$ means $\frac{1}{2}$ of $\frac{2}{3}$. He actually made the connection between fraction multiplication and division and explained why the algorithm works in a conceptual way.

The post-test showed that the using multiple representations or modeling slightly increased. However, the inappropriate explanation to the first equation increased at the same time.

Among six prospective teachers, Eric, Amy, Mary, and Lily gave at least one way (pictorial, verbal or numerical) to represent their understanding. Eric and Amy even provided both pictorial and numerical representations. In the post-test, (see Figure 5)

2. How would you explain to your students why $\frac{2}{3} \div 2 = \frac{1}{3}$? Why $\frac{2}{3} \div \frac{1}{6} = 4$?

Pose it as a situation with manipulatives

, 2 out of three divided into 2 groups gives you $\frac{1}{3}$ groups

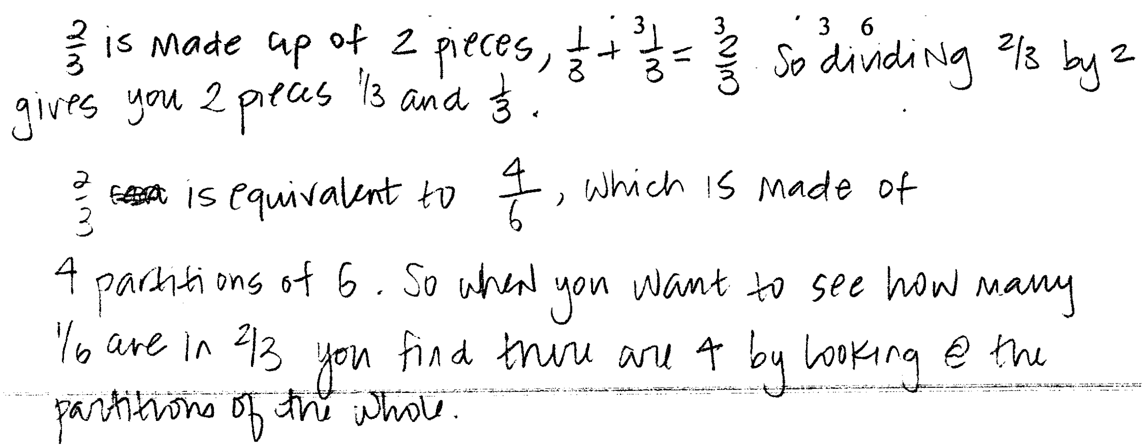
If you divide straight agross with $\frac{2}{3} \div \frac{1}{6}$ you get $\frac{2}{3 \div 6}$. We know $3 \div 6 = .5$ so 2 into half groups gives you 4 groups

Figure 5. Eric's explanation of how the computation works in the post-test.

Mark still indicated IM algorithm, while David provided an inappropriate explanation (i.e., 2 goes into $\frac{2}{3}$ $\frac{1}{3}$ times) in post-test.

For the first equation, only Amy (see Figure 6) provided a verbal explanation and other either mentioned they did not know how to show or they just provided IM algorithm.

> $\frac{2}{3}$ is equivalent to $\frac{4}{6}$, which is made of 4 partitions of 6. So when you want to see how many $\frac{1}{6}$ are in $\frac{2}{3}$, you find there are 4 by looking at the partitions of the whole. (Amy, post-test)



$\frac{2}{3}$ is made up of 2 pieces, $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$ So dividing 2/3 by 2 gives you 2 pieces 1/3 and $\frac{1}{3}$.

$\frac{2}{3}$ is equivalent to $\frac{4}{6}$, which is made of 4 partitions of 6. So when you want to see how many 1/6 are in 2/3 you find there are 4 by looking @ the partitions of the whole.

Figure 6. Amy's explanation of how the computation works in the post-test.

Most prospective teachers developed their understanding of measurement division problems, thus, the majority of them used verbal explanation. For instance, "how many times does $\frac{1}{6}$ go into $\frac{2}{3}$". Below is the only numerical explanation.

$\frac{2}{3} \div \frac{1}{6} = 4$ divide numerator: 2÷1=2 divide denominator: 2÷1=$\frac{1}{2}$

$\frac{2}{\frac{1}{2}}$ get fraction out of bottom: $\frac{2 \cdot 2}{\frac{1}{2} \cdot \frac{2}{1}} = \frac{4}{1} = 4$ or $\frac{2}{3} \div \frac{1}{6} = \frac{4}{6} \div \frac{1}{6} = \frac{4 \div 1}{6 \div 6} = \frac{4}{1} = 4$.

(Amy, post-test)

Except Mark, the other four prospective teachers provided the explanation either by pictorial, verbal, or numerical way. Mark provided IM algorithm without any explanation. Mary, David and Lily explained the equation with the idea of measurement, while Eric explored his understand and explained in a numerical way.

> If you divide straight across with $\frac{2}{3} \div \frac{1}{6}$, you get $\frac{2}{3 \div 6}$. We know $3 \div 6 = .5$, so 2 into half groups gives you 4 groups. (Eric, post-test)

Overall, for this question, we can see most participants had difficulty to explain the measurement model of division ($\frac{2}{3} \div \frac{1}{6} = 4$) at the pre-test, while they can explain ($\frac{2}{3} \div 2 = \frac{1}{3}$). However, the post-test showed that most participants show more confident to explain two different models of divisions. Moreover, the participants who had high CCK represent multiple representations for explanation. On the other hand, the participants who had low CCK (David and Lily) had little progress.

For the question that required prospective teachers to make up a story problem of $\frac{3}{4} \div \frac{1}{2}$ and draw a picture that will go with "three-fourths divided by one-half", none of them provide a correct pictorial representation both in pre- and post-test.

It requires that prospective teachers understand the meaning of $\frac{3}{4} \div \frac{1}{2}$, which means "how many $\frac{1}{2}$ goes into $\frac{3}{4}$". Most prospective teachers had difficulties making a measurement division problem in their pre-test. It revealed that prospective teachers did not understand the quantity relation in this expression. Only Eric provided a correct scenario in the pre-test, while Eric and David in the post-test. In the pre-test, Amy, Mark, and David indicated that they did not know. Mary and Lily created a scenario for $\frac{3}{4} \div 2$ instead. In the post-test, Amy, Mary, Mary, and Lily still created a scenario of $\frac{3}{4} \div 2$.

Eric provided a model and indicated as below in the interview:

> Ok, let's see, three fourths, I am going to try a model to see if… (ok) (Drawing a pie model). Three fourths, now I want to divide it into, divide into half-groups. Uhh, well, I can see here in my model, that I have one half group, and then I have half of another half group, so I can actually see here by redrawing, I have one group plus half of another group. So my answer will then, I will say it would be one and one half (times). (Eric, 04/24/2007)

By redrawing the model, Amy actually explained the expression in measurement model in DoF in the interview, which is, $\frac{3}{4} \div \frac{1}{2}$ means how many groups of $\frac{1}{2}$ can be taken away (or fit into) $\frac{3}{4}$:

> Susan has three-fourths ounces of chocolate. She wants to place one-half ounces of chocolate in individual cups. How many cups will she need? This is the one you have to, I guess, I guess you could say you have to, hold on,… ok, oh, no, it wouldn't really matter. How man cups will she need? Because I was saying sometimes you have problems when you have a person who will be left out. Or you had to add some extra car something like that. But she divided it into cups, evidently, this would be, this would be the same. You see what I am saying? (Amy, 04/24/2007)

Mary and Lily also provide the measurement model of division fraction to express $\frac{3}{4} \div \frac{1}{2}$:

> We can say, uhh, (writing) Carrie's cooking a meal, and she needs, let's see, she needs three fourths cup of oil, but only has half cup measuring container, how many times she have to fill in the half cup in order to get her three fourths of oil? (Mary, interview, 04/24/2007)

> Uhh, You have a container that holds one-half gallon of water. You have a bucket holds three fourths of a gallon… That wouldn't work. Yeah, works, how many times, how many times, would it take to fill the larger bucket using the

> smaller container?.... It make sense because you take half gallon and put once, right, and then you have filled back up and you've already have half gallon in the bucket of whole three fourths gallons, you have to take half of half gallon to put in, which is a quarter of a gallon. So it would be one and half times, which is three over two, so that way makes sense. (Lily, interview, 04/26/2007)

Amy, Mary, and Lily provided the similar scenario in real life situation. It showed that Amy not only represented the concept correctly by creating a scenario, she also considered whether the representation be a pedagogically problematic. That it, she realized if she made a scenario as person, the results should not be a fractional number, because in real life a number of persons will never be a fraction:

> This one is three fourths of chocolate. She wants to place one half of the chocolate in individual cup. So you have three fourths out of the chocolate. One, two, three, four. So one fourth, one fourth, one fourth, one fourth, and you want place that much of chocolate in individual cups. So you have cups. And you want to place half ounces into each cup. So you have to figure out that two fourths equal to one half. So how many of these, or how many cups should she need to fit. This is three fourths, So how many cups does she need, you will see she needs one fourth and one fourth that is one half ounce and you have one fourth left, you just can't leave it there. So have to put it into another cup. So she will need two whole cups and each one holds one-half ounces, so these one fourth only filled half cup. So it will be this cup and this half. So that's how I do it. (Amy, interview, 04/24/2007)

On the other hand, Mark and Lily provided a scenario that presented $\frac{3}{4} \div 2$ instead.

Somebody used three-fourths pound of cookie something, and wants to divide between Humin and Rain. (Writing), so that would be you how much would you get. (Mark, interview, 04/26/2007)

Lily used unclear expression although the idea is correct (i.e., "how many times, how many times, would it take to fill the larger bucket using the smaller container?").

Pictorial representations for measurement and partitive model of DoF- Many prospective teachers indicated that the use of pictures has always helped their understanding (i.e., Amy**)** of modeling in their post-concept map for teaching fraction division. Especially, many students focused on the area model in their post-concept mapping. In their paper assignment, all of them tried to explain the concept of measurement division idea. Based their concept map, I assumed that the concept map would be the most familiar pictorial representation for prospective teachers. Therefore, the results were studied from area modeling and other modeling from six prospective teachers.

All participants indicated that the area model was one of the hardest models for them to comprehend. However, three prospective teachers stressed their struggle for using this model. David indicated "it is difficult to understand and can get rather crowded with fractions that don't divide nicely". Mark and Amy also addressed their struggle to understand and use the area model. Amy further indicated the reason for her misunderstanding. According to Amy, prospective teachers face the most difficulty when trying to visualize the idea of area and the dimensions of a rectangle with the division problem. For instance, in the problem $\frac{4}{5} \div \frac{1}{4}$, she considered that both $\frac{4}{5}$ and $\frac{1}{4}$ were indicated as the dimensions of the rectangle and she needed to find the area of the rectangle as the answer.

Although they indicated that the area model is difficult to visualize, four (Eric, Amy, Mark, and Mary) addressed this method in their paper assignment. Amy demonstrated the problem of $\frac{4}{5} \div \frac{1}{4}$. She drew out the rectangle with five columns and four rows and created 20 squares in the rectangles. Amy drew two rectangles to show the fractions $\frac{4}{5}$ and $\frac{1}{4}$. She restated the problem with common denominators. That is " How many sets of $\frac{5}{20}$ are in $\frac{16}{20}$?" From her illustration, since the unit whole is the same, thus, the problem became how many sets of five were in 16. There were three sets of five and $\frac{1}{5}$ left. Therefore, the result is $\frac{16}{5}$. Therefore she shaded the four of five columns by using black pen as $\frac{4}{5}$ and one of the four rows by using red pen as $\frac{1}{4}$. The overlapped shaded square would be four. Thus, she made a rectangle of one dimension as $\frac{1}{4}$ and the area is $\frac{4}{5}$. The missing factor is the other unknown dimension (see Figure 7).

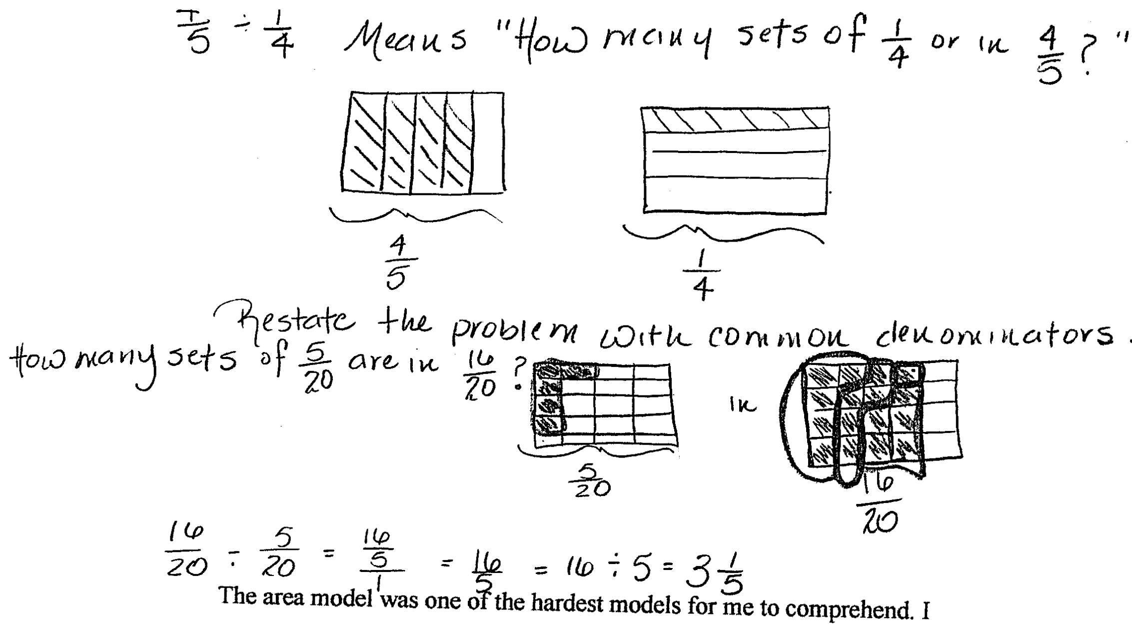


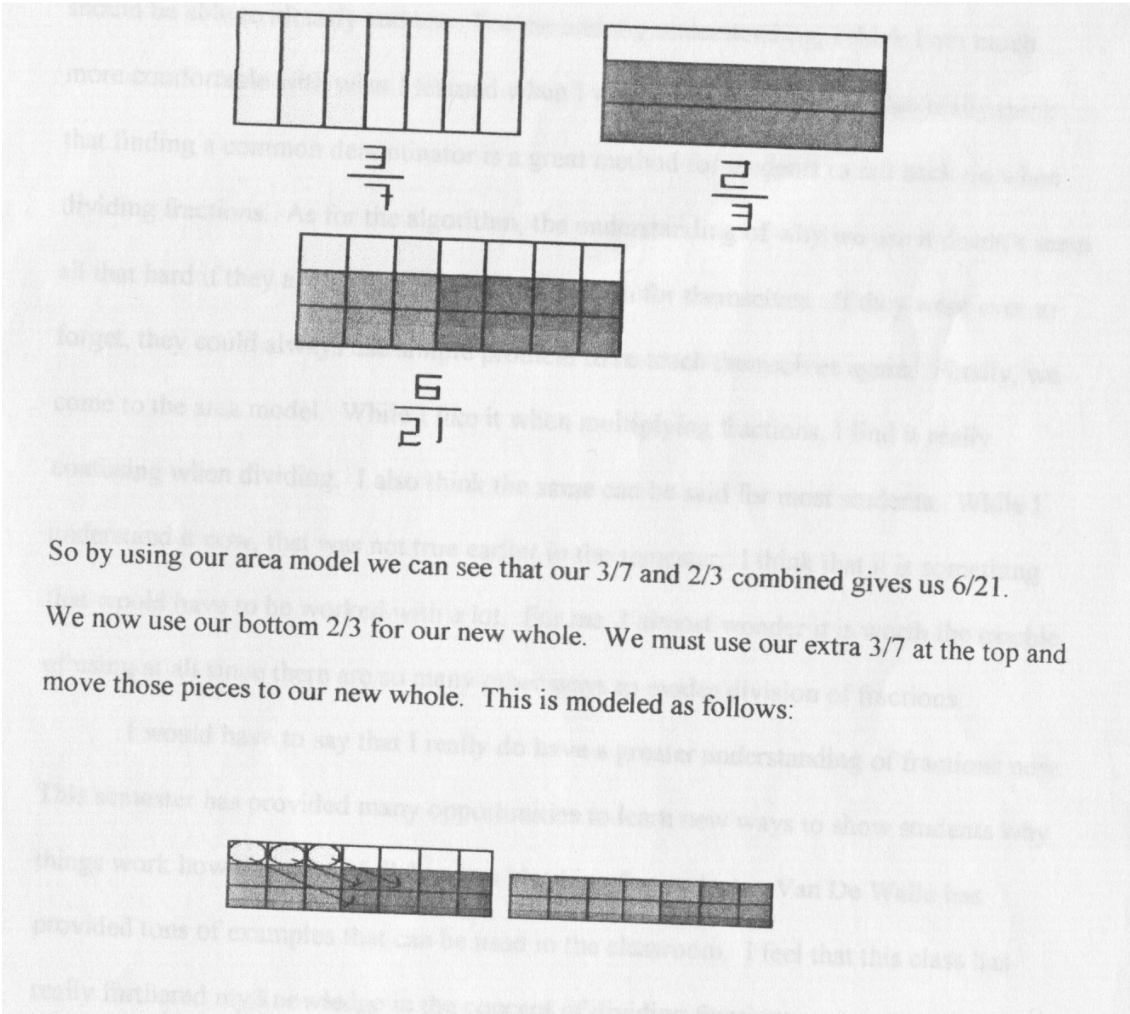
Figure 7. Representation of measurement model of DoF in Amy's work.

In order to find the unknown dimension, from the figure, it could be seen that the black shaded squares should be moved into $\frac{1}{4}$ dimension of the square. From this figure, the black shaded squares should move into leftover space. However, "right now I can only fit one square" (shaded by the red pen). Furthermore, she had to "add another rectangle of the same dimensions on to the original one or extend the rectangle". Five more shaded squares fit into it. She added the same size rectangle twice to fit all squares shaded only by the black pen.

Now all the rectangles are within the $\frac{1}{4}$ dimensional space of the original rectangle. Since the original rectangle was divided into 5 columns or $\frac{1}{5}$ pieces I need to find out the total number of shaded $\frac{1}{5}$ pieces I have added with each new rectangle that had one dimension of $\frac{1}{4}$. There are a total of 16 shaded $\frac{1}{5}$ pieces or $\frac{16}{5}$ pieces. The other dimension of the new rectangle is $\frac{16}{5}$ or 3 and $\frac{1}{5}$. (Amy, writing assignment)

Eric, Mark, and Mary also used area model demonstration.

In comparison with with Amy, Mark's explanation was unclear. He used $\frac{3}{7} \div \frac{2}{3}$. Unlike Amy, Mark drew a rectangle to represent $\frac{3}{7}$ and $\frac{2}{3}$ (see Figure 8).



So by using our area model we can see that our 3/7 and 2/3 combined gives us 6/21. We now use our bottom 2/3 for our new whole. We must use our extra 3/7 at the top and move those pieces to our new whole. This is modeled as follows:

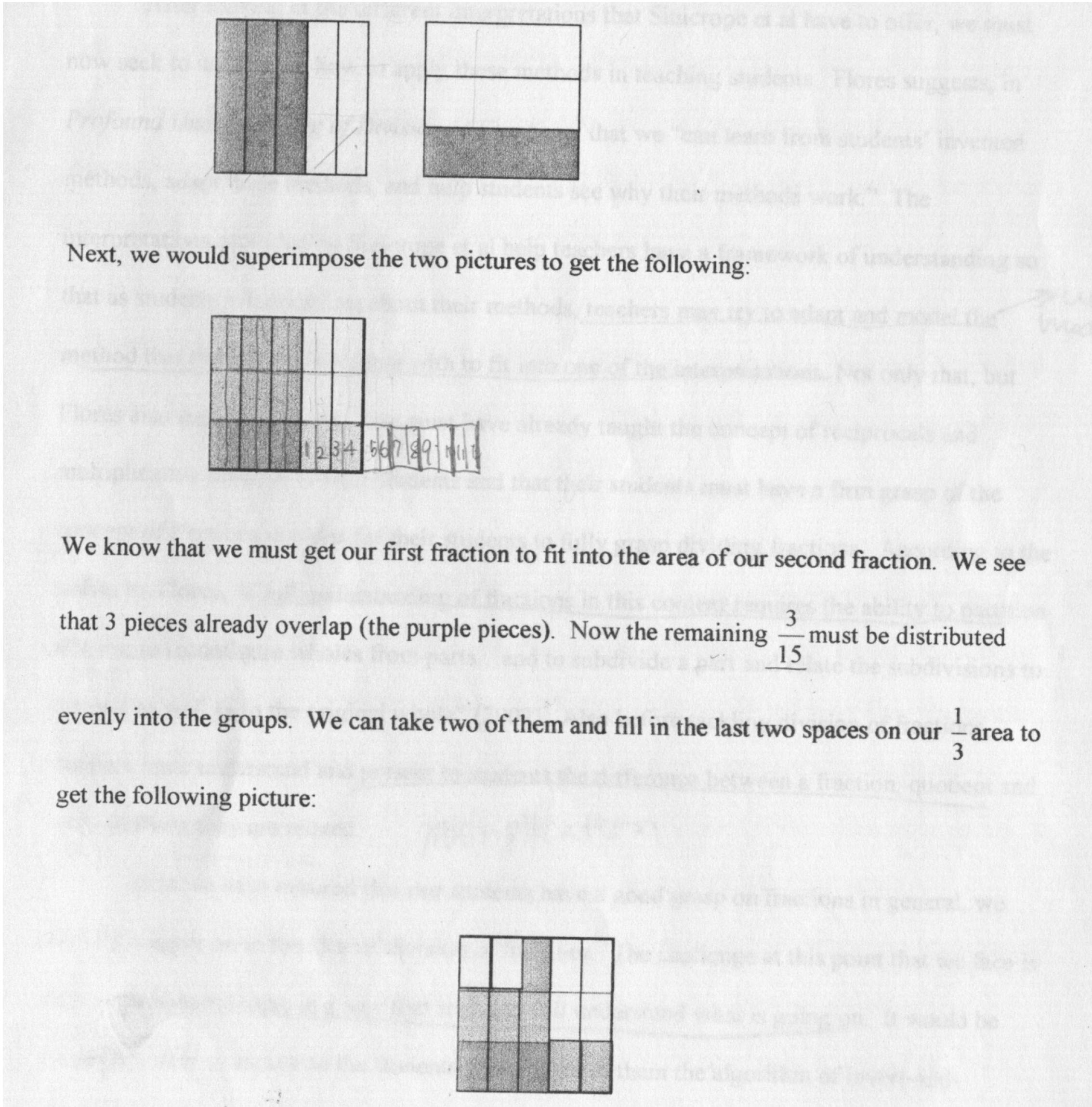
Figure 8. Representation of measurement model of DoF in Mark's work.

In order to represent it as an area model, he indicated "(s)o by using our area model we can see that our $\frac{3}{7}$ and $\frac{2}{3}$ combined give us $\frac{6}{21}$" (Mark). Here, what he wanted to indicate was the shaded overlap was 6 of 21 squares. Yet, there were 3 left pieces of squares to the new whole (base), which was $\frac{2}{3}$ of 21 and was 14. So $\frac{9}{14}$ is the answer.

Eric drew two rectangles and shaded them as $\frac{3}{5}$ and $\frac{1}{3}$. Next, he superimposed the two pictures (see Figure 9). He used the measurement division idea for his explanation.

> We now have 1 $\frac{5}{5}$ group filled up completely with 4 smaller pieces left over. (Eric, writing assignment).

He explained it step by step. However, it was not very clear, especially, he addressed that distribute the $\frac{3}{15}$ evenly into the groups, but did not give further explanation. In his verbal explanation, it showed that he moved the remaining $\frac{6}{15}$ instead of $\frac{3}{15}$ to fit into.



Next, we would superimpose the two pictures to get the following:

We know that we must get our first fraction to fit into the area of our second fraction. We see that 3 pieces already overlap (the purple pieces). Now the remaining $\frac{3}{15}$ must be distributed evenly into the groups. We can take two of them and fill in the last two spaces on our $\frac{1}{3}$ area to get the following picture:

Figure 9. Representation of measurement model in Eric's work.

Mary also tried to illustrate by using area model method (See Figures 10, 11)

$\frac{3}{5} \div \frac{1}{3}$.

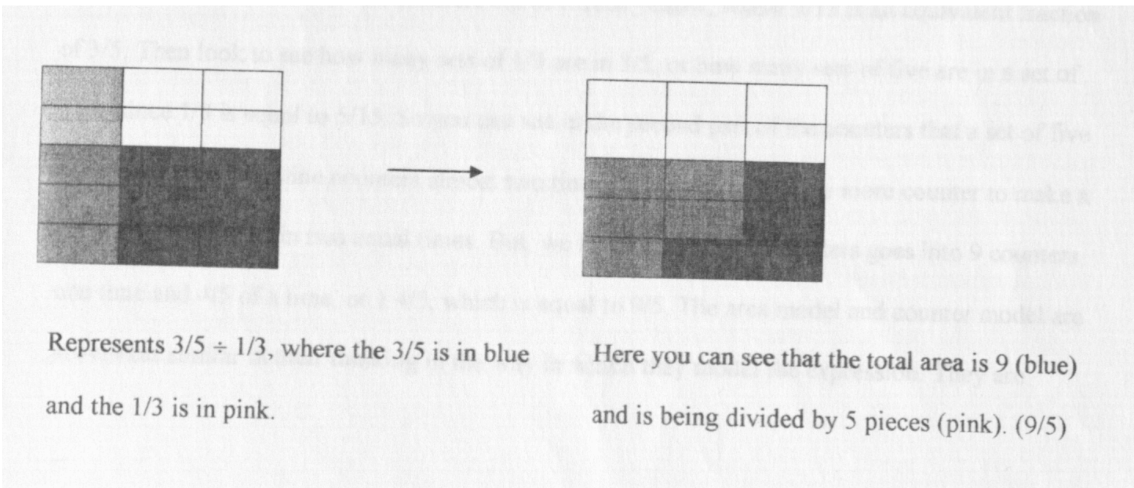


Figure 10. Representation of measurement model in Mary's work–part 1.

She created a rectangle with five rows and three columns. She shaded three rows in blue as $\frac{3}{5}$ and one column in pink as $\frac{1}{3}$. Thus, there were 11 small rectangles shaded and three of them were overlapped. Instead of putting blue rectangle into pink ones, Mary moved the pink rectangles into blue ones. She explained "(h)ere you can see that the total area is 9 (blue) and being divided by 5 pieces (pink). ($\frac{9}{5}$)."

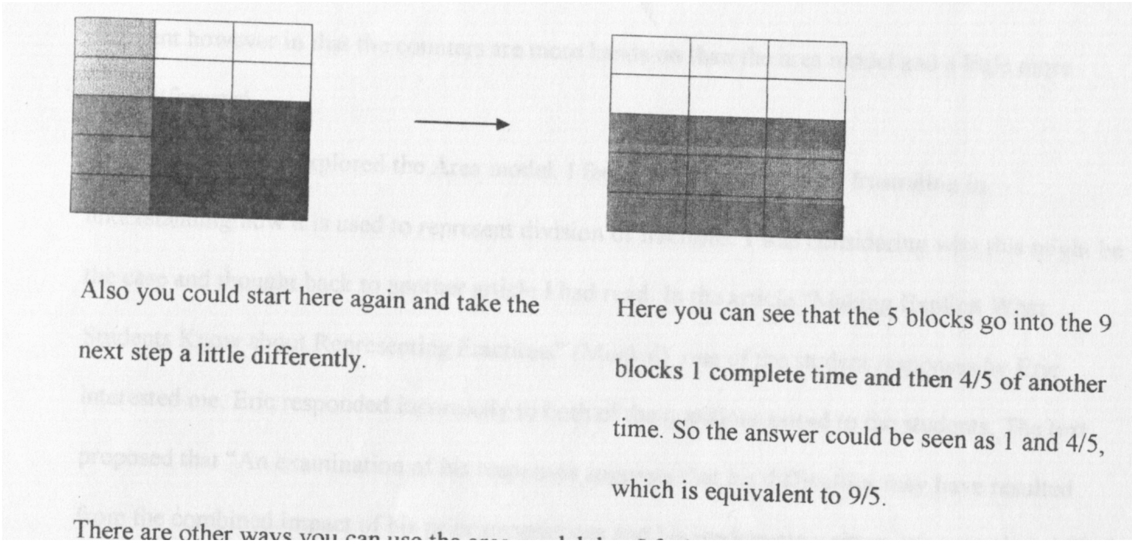


Figure 11. Representation of measurement model in Mary's work - part 2.

Her illustration showed $\frac{1}{3} \div \frac{3}{5}$ instead of the original problem. She further used a measurement division idea to explain the illustration.

> Here you can see that the 5 blocks go into the 9 blocks 1 complete time and the $\frac{4}{5}$ of another time. So that answer could be see as 1 and $\frac{4}{5}$, which is equivalent to $\frac{9}{5}$. (Mary, writing assignment)

Actually, her illustration is explained in a different way. It showed that $\frac{1}{3}$ goes into $\frac{3}{5}$ $\frac{5}{9}$ times.

Lily created a measurement word problem to explain how many parts would be divided out. She indicated that she created a word problem for students to solve:

You have been assigned the task of sharing pizza with a group of a kindergartener's who have come to visit our class. Your group has been given one pizza is to cut, but someone has already eaten part of it. There is $\frac{3}{4}$ of the pizza left and the kindergartener's will eat slices that are $\frac{1}{8}$ of the pizza. How many slices will your group contribute this lunch? (Lily, writing assignment)

In combination with the pie graph (see Figure 12), Lily indicated that six pieces can be divided into. She further explained it in another way, "In this case I am measuring out $\frac{1}{8}$ of a liter to every person and therefore six people can enjoy it".

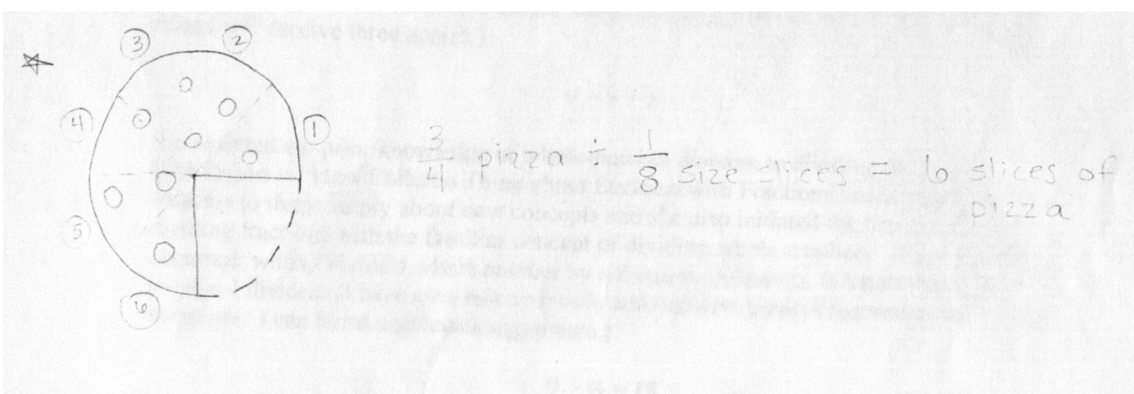


Figure 12. Representation of measurement model in Lily's work.

David also used the pie graph method to illustrate the problem $3\frac{1}{3} \div \frac{1}{6}$ (see Figure 13). The process is "to count how many $\frac{1}{6}$ will go into $3\frac{1}{3}$. He drew three whole pies and one pie cut into three pieces and shaded one piece of it on the left of the paper. He put a pie graph divided into six pieces and shaded one piece of it. Without any further explanation, he indicated that "$\frac{1}{6}$ will go into $3\frac{1}{3}$ 20 times".

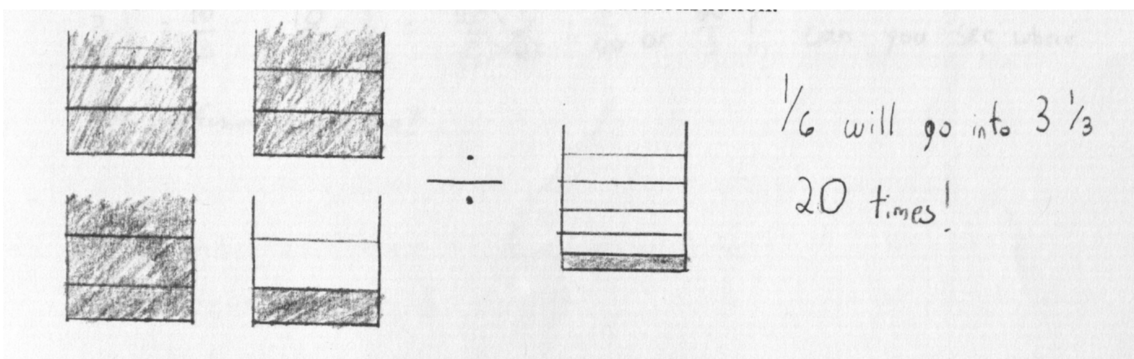
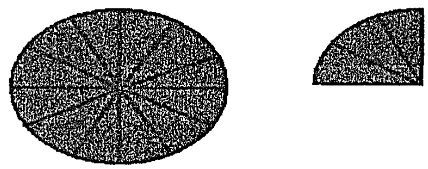


Figure 13. Representation of measurement model in David's work.

Mark is the only one person among six prospective teachers to explain partitive division idea. He first drew the pie graph with a whole pie and $\frac{1}{4}$ of a pie. He divided a whole pie into 12 equal pieces and subdivided $\frac{1}{4}$ of a pie into three equal parts and shaded all of them. For the next step he divided it into three equal parts (see Figure 14).



The first thing that I you would of noticed is that the pieces must be even further subdivided. Our fourths are subdivided into three equal parts. This gives us twelfths. I think that this is probably one of the harder steps for both me then and students now. Our extra fourth is denoted by the three pieces grouped together. Using a circle is easy to figure out our ¼ because of the way it naturally divides. The next step is to divide it into our three equal parts. For me, I remember giving one group one piece at a time until all the pieces are gone.

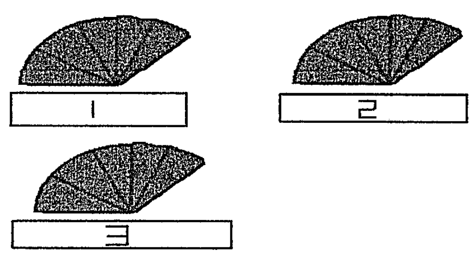


Figure14. Representation of partitive model of DoF in Mark's work.

He mentioned that he would give one group one piece at a time until all the pieces are gone. He pulled one piece at a time and placed them into three groups. When all of the pieces were used each group contained five total pieces. Therefore, it concluded that there were 3 groups of $\frac{5}{12}$.

Numerical/symbolic representation - All six prospective teachers clearly explained the logic behind the computation by using numerical/ symbolic justification. There were three major methods that six prospective teachers provided. Most used more than one method for their explanation.

1) Common denominator method

The common denominator method was mentioned as a way for teaching students conceptual understanding of DoF computation in many prospective teachers' post-concept maps. Five prospective teachers (Eric, Amy, Mark, Mary, and Daivd) provided an example to illustrate this method.

Among them, prospective teachers either combined numerical illustration with the verbal explanation or used one method for explanation.

> You can come to see that the first step in working the problem $\frac{3}{5} \div \frac{1}{3}$ would be to find the common denominator for both of the fractions so that 'like can divide like'". Therefore, you would calculate that the common denominator for 5 and 3 would be 15 and would end up with the equation $\frac{3}{5} \div \frac{1}{3} = \frac{9}{15} \div \frac{5}{15} = \frac{9}{5}$. Since you now have denominators that are alike, you can divide the numerators straight across and get the answer of $\frac{9}{5}$. (Eric, writing assignment)

Comparing with the mathematical knowledge test, Eric was one of whom correctly identified $\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$ as one form when b=d. He also indicated the limitation of the computation. Thus, he combined the numerical illustration in his verbal explanation. In his explanation, Eric indicated that since the denominators are alike, the numerators can be divided straight across, which showed $\frac{9}{15} \div \frac{5}{15} = \frac{9}{5}$. Eric directly got the answer of $\frac{9}{5}$ without explaining $\frac{9}{5}$ coming from $\frac{\frac{9}{5}}{1}$.

> Another way to divide fractions that I do not remember ever using, is the process of finding a common denominator. … So if we find our common denominator it would look like this: $\frac{3}{7} \div \frac{2}{3} = \frac{9}{21} \div \frac{14}{21}$. We can use our method of dividing straight across to get: $\frac{\frac{9}{14}}{\frac{21}{21}}$, we know 21 divided by 21 is 1, so $\frac{\frac{9}{14}}{1}$. The division of $\frac{9}{14}$ is the same as a fraction…. I really like how simple finding a common denominator seems to be. I think that this method would work really well for any student who may not be able to remember the algorithm. (Mark, writing assignment)

This prospective teacher also used the concept of fraction and dividing straight across algorithm. Comparing to Eric, Mark explicitly indicated that the denominator was divided straight across and became one.

Mary also explained how she understood by using the common denominator method.

> The example $\frac{3}{4} \div \frac{2}{5}$ was given to work with. I noticed that in finding a common denominator, which in this case would be 20, you are multiplying the 4 by 5 to get the common denominator and also multiply the numerator 3 by 5 to obtain the appropriate representation for the fraction of $\frac{3}{4}$. Then, for $\frac{2}{5}$ you are multiplying the denominator 5 by 4 and multiplying the numerator 2 by 4 to get the equivalent representation. This really clarified what was going on in the invert and multiply algorithm because I saw that you are multiplying 3×5 and 2×4which is the same as saying $\frac{3}{4} \times \frac{5}{2}$. (Mary, writing assignment)
>
> You can also check the invert and multiply be the common denominator method. If you found the equivalent fractions with a common denominator or 20, they are $\frac{15}{20} \div \frac{8}{20}$. Then since the whole has 20 parts each, you just divide the parts, which is $\frac{15}{8}$. This is also what you get when you multiply by the reciprocal, $\frac{3}{4} \cdot \frac{5}{2}$. (Mark, writing assignment)

David only presented the numerical illustration without verbal explanation. David indicated below, The common denominator algorithm is one that I don't particularly agree with simply because it is so similar to addition; I think that it could become easily confused. (David, writing assignment)

2) The method of making the denominator as one (common denominator method)

Some prospective teachers also considered the second method as another approach of the "common denominator method". Here, I used the category of "the method of making denominator as one method". The results showed that five provided an example for this method (Eric, Amy, Mark, Mary, Lily). Yet, there were different approaches among their illustration.

1) Using the concept of fraction

Most prospective teachers provided the symbolic explanation in this way. First, the division was changed DoF form in terms of fraction form like "$\frac{\frac{3}{5}}{\frac{1}{3}}$". Then, both denominator and numerator of this complex fraction were multiplied by $\frac{3}{1}$ to get the denominator to 1.

(W)e can simply try to find a way to make the denominator equal one. The only way we can possibly do that is to multiply by $\frac{3}{1}$ or 3. That would give us the following equation: $\dfrac{\frac{3}{5}}{\frac{1}{3}} = \dfrac{\frac{3}{5} \times \frac{3}{1}}{\frac{1}{3} \times \frac{3}{1}} = \dfrac{\frac{9}{5}}{1} = \frac{3}{5} \times \frac{3}{1} = \frac{9}{5}$. If you notice, this gives us the rationale behind the invert-and-multiply algorithm. (Eric, writing assignment)

Using the concept of fraction, Eric converted the division form to the complex fraction form. In order to "make the denominator equal one", Eric multiplied both the denominator and numerator by "$\frac{3}{1}$ or 3". Instead of using "taking the denominator away", Eric used the language of "make the denominator equal one" for explicit mathematics language using. Mark did the similar process.

> We said that $\frac{1}{2} \div \frac{2}{3} = \frac{1}{2}$over $\frac{2}{3}$. So then we must multiply the bottom $\frac{2}{3}$ by $\frac{3}{2}$ and the top $\frac{1}{2}$ by $\frac{3}{2}$. So $\dfrac{\frac{1}{2} * \frac{3}{2}}{\frac{2}{3} * \frac{3}{2}}$. The bottom portion cancels out and that leaves the top $\frac{1}{2} * \frac{3}{2}$, which is also the same as multiplying by the reciprocal. (Mark, writing assignment)

Lily also used the same for explanation. She set up the algebraic sentence, her explanation did not relate to:

> Dividing fractions is related to multiplying fractions. In other words, I can substitute the variable x for the missing answer to set up the division as multiplication. $\frac{3}{4} \div \frac{1}{8} = x$ or $\frac{1}{8}x = \frac{3}{4}$. This division problem can also be set up in order to cancel out the denominator if the two fractions are in

numerator/denominator form. The concept of one is used here in two forms. *First, I multiply by one, or $\frac{8}{1}$ divided by $\frac{8}{1}$. Second, I chose to multiply by $\frac{8}{1}$ because $\frac{1}{8}$ times $\frac{8}{1}$ is equal to one. (1×8=8 and 8×1=8, therefore 8÷8-1). When I do this I am left with $\frac{3}{4} x \frac{8}{1}$ divided by 1. Because $\frac{8}{1}$ is the inverse of $\frac{1}{8}$. I had found one reason why I can invert and multiply. (Lily, writing assignment)

$$“\frac{3}{4} \div \frac{1}{8} = \frac{\frac{3}{4}(numerator)}{\frac{1}{8}(deno\min ator)}; \left[\frac{\frac{8}{1}}{\frac{8}{1}} = 1\right] \left[\frac{1}{8} \rightarrow \frac{8}{1}\right]$$

inverse

$$\frac{\frac{3}{4}}{\frac{1}{8}} \times \frac{\frac{8}{1}}{\frac{8}{1}} = \frac{\frac{3}{4} \times \frac{8}{1}}{\frac{1 \times 8}{8 \times 1} = \frac{8}{8} = 1} = \frac{\frac{3}{4} \times \frac{8}{3}}{1} = \frac{3}{4} \times \frac{8}{1}.$$

Lily did not use the algebraic number sentence and simply multiply by the reciprocal like previous others. Yet, she considered multiplying by one $\left[\frac{\frac{8}{1}}{\frac{8}{1}} = 1\right]$. Instead, she changed the division form to the fraction form and multiplied across by $\frac{8}{1}$. She explained that it means that $\frac{\frac{3}{4}}{\frac{1}{8}}$ multiplied by one, because of $\left[\frac{\frac{8}{1}}{\frac{8}{1}} = 1\right]$. Since both numerator and denominator multiplied by $\frac{8}{1}$, the denominator of the complex fraction was simplified and became one.

2) Dividing straight across and making the denominator as one

Amy and Mark used another approach. They first divided straight across, thus, the quotient of the numerators became the numerator of the complex fraction and the quotient of the denominator became the denominator of the complex fraction. Next, they re-wrote the division as the fraction form. It required multiplying by the reciprocal of the denominator to get the denominator as one. Thus, the numerator also needs to be multiplied by the reciprocal.

> If you take $\frac{4}{5}$ and $\frac{1}{4}$ and divide them straight across, similar to what would happen in the common denominator algorithm after like denominators you would get a fraction of $\frac{\frac{4}{1}}{\frac{5}{4}}$. In order to get the fraction out of the bottom you have to multiply the numerator and the denominator by the reciprocal of the fraction in the denominator or $\frac{4}{5}$. Illustration: $\frac{4}{5} \div \frac{1}{4} = \frac{\frac{4}{1}}{\frac{5}{4}} \cdot \frac{\frac{4}{5}}{\frac{4}{5}} = \frac{4}{1} \cdot \frac{4}{5} = \frac{16}{5} = 3\frac{1}{5}$. The denominator becomes one and what is left is $\frac{4}{1} \cdot \frac{4}{5}$ which is equal to $\frac{16}{5}$ or 3 and $\frac{1}{5}$. (Amy, writing assignment)

Like mentioned previously, she divided two numbers straight across (i.e., $\frac{4 \div 1}{5 \div 4}$). Using the concept of fraction, he got a complex fraction that the quotient of the denominator was the denominator of the complex fraction and the quotient of the numerator was the numerator of the complex fraction (i.e., $\frac{\frac{4}{1}}{\frac{5}{4}}$). Instead of finding the common denominator (20) here, she multiplied $\frac{5}{4}$ both the numerator and denominator by 5/4. Although the result showed that the denominator of the complex fraction become one, she did not show this process in her numerical illustration. Mark also did the similar process.

> So if we took our problem $\frac{3}{7} \div \frac{2}{3}$, and divided across we would get $\frac{\frac{3}{7}}{\frac{2}{3}}$. We do not want the denominator to be a fraction, so we must then find a way to remove it. To do this we would multiply by its reciprocal. This is assuming we know that a fraction multiplied by its reciprocal is equal to one. … However, we must also multiply the top by the reciprocal as well because what we do to the bottom, we must do to the top. So we would have $\frac{\frac{3}{2}}{\frac{7}{3}} \times \frac{\frac{3}{7}}{\frac{3}{7}} = \frac{\frac{3}{2} \times \frac{3}{7}}{1}$. We know that any number divided by 1 is itself, so we can disregard the 1 in our denominator to arrive at: $\frac{3}{2} \times \frac{3}{7}$. After multiplying our fractions straight across we arrive at our answer $\frac{3}{2} \times \frac{3}{7} = \frac{9}{14}$. (Mark, writing assignment)

3) Other numerical justification

Besides the previous method, some previous used other unique ways to justify DoF computation.

Eric used the concept fraction to change the division to a fraction form ($\frac{3}{5\times\frac{1}{3}}$). Then he both multiplied by 3 to simply the denominator of the complex fraction. He showed that "$\frac{3}{5}\div\frac{1}{3}=\frac{3}{5\times\frac{1}{3}}=\frac{3}{\frac{5}{3}}=\frac{3\times 3}{(\frac{5}{3})\times 3}=\frac{9}{5}$". He explained below:

> We can notice that in order to get rid of the fraction in the denominator, we multiply both the numerator and the denominator by 3. This re-enforces the idea of the invert-and-multiply algorithm. (Eric, writing assignment)

He used the equivalent fraction algorithm (i.e., multiplying the same number with both numerator and denominator) or the concept of multiply by one (i.e., considering $\frac{3}{3}$ as one). Thus, the numerator of the complex fraction was simplified.

Amy used an algebraic way to explain in another way. She wrote the equation in an equivalent form as a product with a missing fact.

> For my problem the equation would be $\frac{1}{4}x(\ )=\frac{4}{5}$. I would have to multiply both sides by 4 which is the reciprocal of $\frac{1}{4}$ in order to get 1x. The reciprocation part was a very important discovery to me. It is the essence of the invert and

multiply algorithm and should be emphasized in the classroom along with other models. (Amy, writing assignment)

Developing the area model in her pictorial representation, she indicated the dividend is the area and the divisor can be considered as one dimension. The question is to ask to find the missing dimension (factor). She created the algebraic number sentence $\frac{1}{4}x(\ ) = \frac{4}{5}$, which is the equation in an equivalent form as a product with a missing factor. Then, she multiplied both sides by 4 based on the concept of the equal sign. She wrote $\frac{1}{4}x(\ ) = \frac{4}{5}$ instead of $\frac{1}{4}x = \frac{4}{5}$ for her number sentence. From the whole context, it can be assumed as $x = (\ )$, or $\frac{1}{4}x(\ ) = \frac{4}{5}(\ )$.

Lily also provided an algebraic sentence, yet, explained in a different way like mentioned previously.

Overall, all six teachers provided both pictorial illustration and numerical justification to demonstrate DoF computation. It showed that most prospective teachers were more familiar and comfortable using numerical justification, in particular, using the common denominator method. The results were consistent with the results of prospective teachers' post-test of SMK.

4.2.2 Six prospective teachers' beliefs and its change

The writing assignment and interviews revealed changes in the participants' perceptions about DoF. The interviews showed that all five of the six participants believed that students should not only memorize the algorithm or rules, they also need to understand the underlying concept. David participant said: "the best way to teach is the 'invert-and-multiply' algorithm". Other participants indicated that their attitudes about learning mathematics had changed (see Table 25).

Table 25

Six participants' beliefs in the pre- and post-test

	DoF	What to learn	How to learn	Meta-cognition
Eric (pre)	Algorithm	IM	Memorization	Procedural based
(post)	Concept/Real word application	Concept /IM	Model /Concept /Application	Conceptual understanding Multiplication & Division Model use (Struggle)
Amy(pre)	Algorithm	IM	Memorization	Procedural based
	Concept/Model	Concept/ IM	Model	Model use for understanding Model use
Mark (pre)	Algorithm	IM	Model	Procedural based
(post)	Algorithm/Model	IM/Model	Model	Conceptual understanding Model use (struggle)
Mary (pre)	Algorithm	IM	Model	Procedural based
(post)	Algorithm & concept/Model	Concept/IM	Model/IM	Conceptual understanding Model use (Struggle)
David (pre)	Algorithm	IM	Memorization	Procedural based
(post)	Algorithm/Model	IM	Memorization	Procedural based
Lily (pre)	Algorithm	IM	Memorization	Procedural based
(post)	Concept/algorithm	Concept/IM	Model	Conceptual understanding Model use (struggle)

It showed that most participants consider DoF as either algorithm based or computational rule. Thus, before their learning, they consider IM algorithm is the only thing they need to teach to their students. However, after the learning of Module, they start to doubt what they believed before and recognize that DoF had very important concept to support its algorithm. Thus, except David, all other five participants consider to teach concept back underlying the algorithm.

Mary indicated that, based on her own learning experience in the course, students also need to learn the concept underlying the algorithm. She stated:

> I definitely agreed the idea that I need to understand ***why*** behind things, and I believe students need to know **why** behind things. So, I expected it's helpful on the other different ways how we learned, how to learn. That was helpful. (Mary, 04/24/2007)

The results from the previous session showed the majority of these prospective teachers still considered that algorithms and rules were the mathematics content for student learning. However, they also considered the importance of students' conceptual understanding. Thus, they agreed that memorizing was not a good way for students to learn. They also noticed the importance of manipulatives and multiple representations and models for students' conceptual understanding. Yet, they could not make the connection between the manipulatives and abstract mathematics. The results also showed in the prospective teachers' perception towards to the content topic DoF from the interviews and writing.

All six prospective teachers considered the content topic DoF only as algorithm with doing 'invert-and-multiply (IM)' at the beginning of the semester. Eric doubted whether DoF would be used in a real-life application at the beginning. Most prospective teachers did not know or consider the concept behind the algorithm. The perception changed through the methods course and other courses. Three students mentioned and explained their understanding to the idea of measurement and partition in DoF either in their writing or in interviews.

Since the perceptions about DoF were procedural based, their attitudes towards to learning this content topic were also procedural at the beginning. Further, they did not realize that their understanding and knowledge about DoF was limited until the pre-test. During the interview, all six teachers indicated that they learned DoF as a procedurally based concept and considered DoF only as an algorithm. They simply memorized the IM algorithm to solve. They mentioned that they never considered DoF in a conceptual way and only learned to do the algorithm. Therefore, they also considered that IM was the only thing they need to teach. However, they considered that they developed their conceptual understanding through the semester by manipulatives and real-world examples. They realized that they did not know how to explain DoF by any representations and models at the beginning but much more confident at the end of the semester.

> I know when I was taught DoFs in school, uhh, that was just IM. Never went too much into detail, with what just we did (IM)… Because before I was like DoF is just DoF and didn't know there were different types, like partitioning and sharing

> (partitive), grouping (measurement). I don't know these different types. I just thought of the numbers dividing, really the most I know about DoF is IM. I really don't know too much about how to use the model. ..I think I learned more about it. (Eric, interview, 04/24/2007)

> It's like the only thing that I knew before that I remember really was the algorithm. ..So when I actually go back and relearn about the models and some of the models, that kind of gave me a greater understanding of DoF cause before then it was just divide, flip, and then invert and multiply. But now I have some kind of understanding. (Eric, interview, 04/24/2007)

> In fact, on that first pre-test, I couldn't even come up with the word problem to use, because I didn't, couldn't visualize a picture of two different fractions and dividing of two of them would be. (Mary, interview, 04/24/2007)

Except one participant (David), all other five prospective teachers considered that teachers must find a way to present the topic of DoF in such a way that their students must face what they do not know. By giving them a problem in which they must learn new mathematics to solve the problem, it will ensure that they learn it. Moreover, Mark considered the overuse of only one model may mislead students' learning so that students may consider that DoF can only be represented as one way (i.e., pizza or bar graph). He said:

> I remember when learning to divide fractions, I cannot remember too much dealing with modeling. I know that I was familiar with fraction blocks or bars. However, I feel that these were most used when modeling a whole number

divided by a fraction. Some examples that seem to be overused are that of pizzas and cookies. I think that these can become misleading to students at times. (Mark, interview, 04/26/2007)

All six prospective teachers indicated that their understanding of DoF now is obviously much greater and more in depth than what they learned previously. Thus, they all felt more comfortable with what they learned and felt more comfortable to teach this content topic. All six teachers used the method of finding a common denominator in both paper assignments and their interview. They considered that it was a great method for students to fall back on when dividing fractions. They believed that understanding of why to use the method doesn't seem difficult if they are allowed to find that pattern for themselves.

In both writing assignments and interviews, all six prospective teachers indicated that they developed their understanding of why the 'invert-and-multiply' algorithm works by using different representations in their own learning. All six of the prospective teachers indicated that the common denominator method that helped them to understand how the algorithm works.

The participants' reflections about the limitation of their understanding caused concerns about teaching. One participant indicated he was in a panic because he was unsure of how to properly teach this concept because he did not understand this topic conceptually. Initially, the participants thought they would teach only the algorithm and rule for the students like they were taught before. However, when they were asked to why it worked, they realized that they were not able to explain why. All six participants tried to provide numerical and pictorial representations for explanations in their paper assignment. Thus, all six teachers mentioned that their attitude of teaching DoF would change after their own learning experiences. They considered that teachers must understand the concept behind the algorithm and present clearly to the students, but also realized the importance of connecting the previous topics, such as the concept of fractions, and multiplication. Overall, the beliefs or attitudes of teaching DoF by using different modeling in order to understand the algorithm became stronger after their learning. For example, Mark stated:

> Something I think that all students should do once they have practiced using modeling is finding the algorithm. …I know that most people do not really understand why we use the multiplicative inverse of our second fraction when dividing. I can recall being able to use an example to always find the algorithm. This is a good strategy for students to be able use for any problem solving situation. (Mark, interview, 04/26/2007)

Although all six prospective teachers agreed that manipulatives and multiple modeling should be used in teaching the DoF algorithm, they had different perceptions toward the use of modeling. In the interviews, there were two different perceptions ofusing common denominator models. Four of the prospective teachers (Eric, Amy, Mark and Lily) indicated that the common denominator method was easy for them to understand and explain to students. Mark indicated it would be helpful since students have already learned the idea of common denominator in fraction addition and subtraction. Although he agreed that the common denominator method might cause students' confusion, he still indicated that it was the easy way to start for student understanding.

> Something I kind of like, is finding the common denominator. Because, uhh, they've already found CD when they add and subtract, so why not just keeping doing it, you know. (Mark, interview, 04/26/2007)

In contrast, two of the participants clearly mentioned that the common denominator algorithm would be confused with addition and subtraction to the students. David and Mary indicated that the common denominator method was clear for her as a university student, but it may cause students' confusion:

> I looked back, the most of representation came down to finding common denominator, which I still clear from, at the first very beginning, because I didn't; like that if I brought that up to my students, it wouldn't, uhh, be very clear for them, cause it sounds like we're going back to addition and subtraction. But it really does seem to work that as long as you can get the common denominator, I mean, of course it works, but it just seems to be how it's most easily represented before actually going to the algorithm, mathematical part of it when you're doing representation. (Mary, interview, 04/24/2007)

Daive stated:

> Because to me it's too much like addition. When you get the common denominator and then you add, to me, it seems.. (What do you mean). Like, when you, when you use two thirds and three fourths. Two thirds plus three fourths, the common denominator becomes twelve. . So you have to multiply two by four,

> and you get eight twelfths and plus, it would be, uhh, nine twelfths. , and you add, you get seventeen twelfths. And division, you did same thing to get the common denominator, so you have eight twelfths divided by nine twelfths, and that equals eight ninths. (David, interview, 04/26/2007)

From the survey, students' writing assignment, and the interview, it was shown that the majority of prospective teachers felt confident to explain and represent DoF for their students by using at least one model. During the semester, the prospective teachers had opportunities to use different models and representations. Although most prospective teachers in their writing assignment and in the interview mentioned that they were struggling with area model, they did build confidence to explain or represent to students by using at least one model.

Overall, the majority of prospective teachers considered that their understandings of TEKS were high or proficient. Moreover, after one semester studying, their self-efficacy of curriculum and the topic of fraction division also increased. They felt more confident of teaching DoF in the post-survey. Further, the majority of prospective teachers considered that mathematics topics should be taught by using multiple representations and should be modeled by real-world problems in their teaching. They also agreed that teachers should have knowledge about students' common misconception/difficulty in teaching a mathematics topic. However, most prospective teachers disagreed that teachers should prevent students from making errors in their learning of mathematics. Some of them considered that students should learn from their errors.

From a mathematics learning perspective, the results revealed that the majority of the prospective teachers in both pre- and post-survey disagree that mathematics learning only involves memorizing. However, conflicting with this item, the results showed that most prospective teachers considered that mathematics should be learned as sets of algorithms or rules that cover all possibilities.

4.2.3 Summary

Overall, the results of prospective teachers' performance on the mathematics test revealed that their mathematical knowledge was procedural and they lacked conceptual understanding in the pretest. There was some inconsistency of the results between the mathematics knowledge test and the survey of prospective teachers' perceptions. Most prospective teachers disagreed that mathematics should be learned as the set of algorithm and learned by the memorization, even though they were more familiar with procedural knowledge and their knowledge was constructed by memorization. Moreover, most prospective teachers showed confidence in explaining and modeling fraction computation topic, yet, their knowledge, especially SCK was not yet fully developed in preparation for teaching.

Moreover, the results showed that the participants who had high CCK increased their SCK significantly by either providing multiple representation and explanation. Moreover, the development of SCK also causes students rethink about their beliefs towards to DoF and learning and teaching DoF. Thus, the participants who develops his/her SCK significantly more consider understand and learn DoF in a conceptual way.

4.3 A Case Study:

Mark's PCK Development and Beliefs changes in the Field Practice

In this section, the focus is on one of the prospective teacher's transition to classroom practice, referring to the teaching experience that student teachers gain while trying out classroom teaching interventions (Simon & Tzur, 1999). Research has shown that these activities have contributed to the teacher's development by challenging existing beliefs and practices (Simon, 1995). This case study focused on the learning process and how this prospective teacher's university study influenced his classroom practice in teaching in seventh grade students.

Like several prospective teachers in his class, Mark (pseudonym) continued to do his student teaching in the same school with the same mentor, he observed for the semester during his method course. As indicated by his beliefs about mathematics teaching and learning, in particular in DoF, Mark was an average student in the method class, compared to other prospective teachers. He was only one of x males in the methods course.

During the student teaching period this case study focused on, Mark development from SCK to knowledge of content and students (KCS) and knowledge of content of teaching (KCT) (Ball, 2006). Moreover, with knowledge of curriculum, I study how PCK developed to bridge his content knowledge and the practice of teaching (Shulman, 1986).

4.3.1 Mark's beliefs of teaching and learning mathematics in the method course

Like the other prospective teachers in his class, Mark did not have high self-evaluations of his understanding of curriculum (TEKS) in the pre- and post-surveys. He did not consider that using manipulatives could help students avoid abstract mathematics and did not agree teacher should prevent students from making errors. Although he disagreed that learning mathematics mainly involves memorizing, he considered that students should learn algorithms and rules. In particular, he believed that students should learn the algorithmic rule of DoF and the use of modeling, because the manipulatives and modeling can help students to recall the algorithm.

> Something I think that all students should do once they have practiced using modeling is finding the algorithm. This is something that I did in my first lesson plan after we worked with modeling division of fractions. I know that most people do not really understand why we use the multiplicative inverse of our second fraction when we dividing. I can recall being able to use an example to always find the algorithm. (Mark, writing assignment, 03/22/2007).

He provided the example $\dfrac{\frac{3}{2}}{\frac{7}{3}} \times \dfrac{\frac{3}{7}}{\frac{3}{7}} = \dfrac{\frac{3}{2} \times \frac{3}{7}}{1}$ and indicated the students should learn to do multiple ways of modeling. Thus, they could recall the algorithm by doing the modeling. Through his own learning experience, Mark recognized that modeling and manipulatives is a way to connect students' understanding of why the algorithm works. Therefore, without memorization, the students can still recall and solve the problem based on their understanding.

Through observing the classroom instruction and working with the mentor teacher, Mark indicated that his concern was classroom management in practice in both beginning and at the end of the semester. He was confident with the content knowledge that he would teach, however, he noticed it's not enough as a teacher.

> Everyday it is a struggle to get them to participate. They know they are supposed to do it but choose to gripe and complain about it instead. Some other procedures are raising hands, not talking with others are, and other general things like that. It seems to be a day-to-day thing on how the students respond and act (Mark, writing assignment, 01/26/2007).

> I am still mostly concerned about classroom and time management. I feel that time management will come after having some more time in the classroom. I think that after getting to know the flow of my own class, the time management will come. Classroom management is something that I cannot really master until I am teaching. (Mark, writing assignment, 04/26/2007).

Besides the classroom management, Mark also noticed the consideration of using activities in the beginning of the semester.

> Also, I noticed that the same thing doesn't work for all classes. Some activities work really well for some teachers. Others may choose not to use these because they don't seem to mesh with their classroom. Even among different classes for the same teacher, not all the activities work (Mark, writing assignment, 01/26).

He indicated that the different activities used by the teacher in different situation and different student groups. However, he wanted to find "some good activities that all the classes would do".

4.3.2 Mark's knowledge development in method course

Mark felt confident about his understanding and teaching DoF after the methods course. In the pre- and post-test, he performed well on CCK, successfully applying the DoF algorithm and solving simple word problems. However, he had difficulty solving the two-step word problem on both pre- and post-test. This result showed that he had difficulty identifying the quantity relationship and writing the number sentences correctly.

Mark developed his ability to explain why the algorithm works using different modeling approaches. For instance, to verify the equation $\frac{a}{b} \div \frac{c}{d} = \frac{(a \div c)}{(b \div d)}$ works and explain why, he identified that dividing straight across was incorrect and indicated that it should not just divide top-by-top and bottom-by-bottom when dividing. He emphasized the IM algorithm in his pre-test. In his post-test, he wrote:

> I would tell him that this (dividing straight across) does work for division. As long as he makes sure to follow the correct the procedure when dividing. Ex.
>
> $$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d} = \frac{\frac{a}{c}}{\frac{b}{d}} = \frac{\frac{a}{c}}{\frac{b}{d}} \cdot \frac{\frac{c}{d}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$"

In the interview, he made a further explanation in his interview and writing assignment:

Like, $\frac{4}{8} \div \frac{2}{8} = \frac{2}{1}$. So four divide by two is two, and eight divide by two is four. This one would be, correct, too. ... But it won't always work. Like, $\frac{8}{10} \div \frac{2}{4}$, because 10 is not divisible by 4. … I don't think it works for every problem, I mean, it works for this problem, but you have to go and happen to work for this one. But it doesn't work for this one.

So if we took our problem $\frac{3}{7} \div \frac{2}{3}$, and divided across we would get: $\frac{\frac{3}{2}}{\frac{7}{3}}$. We do not want the denominator to be a fraction, so we must then find a way to remove it. To do this we would multiply by its reciprocal. This is assuming we know that a fraction multiplied by its reciprocal is equal to one. They (the students) should know this at this time. However, we must also multiply the top by the reciprocal as well because what we do to the bottom, we must do the top. So we would have $\frac{\frac{3}{2}}{\frac{7}{3}} \times \frac{\frac{3}{7}}{\frac{3}{7}} = \frac{\frac{3}{2} \times \frac{3}{7}}{1}$. We know that any number divided by 1 is itself, so we can disregard the 1 in our denominator to arrive at: $\frac{3}{2} \times \frac{3}{7}$. After multiplying our fractions straight across we arrive at our answer $\frac{3}{2} \times \frac{3}{7} = \frac{9}{14}$. (Mark, interview, 04/26/2007)

Combining with the post-test, interview, and his writing assignment, the results showed that Mark developed the understanding that dividing straight across works in some special cases in which the dividend is divisible by the divisor in both numerator and denominator respectively. If the dividend is not divisible by the divisor, the IM algorithm then works. Mark also mentioned the common denominator method:

> Something I kind like, is finding the common denominator. Because, uhh, they've already found CD when they add and subtract, so why not just keeping doing it. (Mark, interview, 04/26/2007)

In neither the pre- or post-test, did Mark provide an explanation for *how* each specific computation works (i.e., $\frac{2}{3} \div 2 = \frac{1}{3}$ and $\frac{2}{3} \div \frac{1}{6} = 4$"). He only showed the IM algorithm.

These results showed that Mark was more confident to use the numerical representation to explain why the computation works comparing with the pictorial representation. He also indicated that he was struggling with using area model. On the other hand, in both pre- and post- test, Mark provided a scenario that presented $\frac{3}{4} \div 2$ instead of $\frac{3}{4} \div \frac{1}{2}$:

> John has $\frac{3}{4}$ of an apple. He and his brother split in half. How much of an apple does each get. (Mark, post-test)

In the interview, he also failed to think of a scenario of $\frac{3}{4} \div \frac{1}{2}$:

> I was like to be like, you get three fourths pound of something, (Uh, whatever you have), that was I used, somebody used three fourths pound of cookie

something, and wants to divide between Humin and Rain. (Writing), so that would be you how much would you get. …Yes, so that was three fourths divided be one half. Uhh, three fourths divide by two, like, I am terrible of making story problem. I am not very good at it. Uhh, … (Ok, you can just draw the picture for this one?) Uhh, what picture what I do was, three fourths something, and there is two people, there is the first person, there is the second person. And each one gets , you divide by half, so you'd say, ok, you take the first piece, give to this person, you take the next piece and gave to another person. And you get this piece half. So, uhh, each person gets half of that. So, what of this, this is fourth, so what is half of fourth. So half of fourth, is would be, uhh, eighths, two eighths? And, yeah, no, what half of fourth, ….Am I right? So sending this person,… hold on. This is one and half. Here we go, makes we change this into mixed-number, so each person gets one and each person gets half. So that's one and half. Does it make sense? Let me draw it again. (Mark, interview, 04/26/2007)

In the interviews, Mark revealed difficulty in finding a scenario for the equation. He tried to explain the algorithm from the pictorial representation. He drew a rectangle as a unit whole, divided into four equal parts, and shaded three parts of them. It showed three fourths. He evenly divided two fourths by two (each person gets one fourth part) and tried to divide the left one fourth into half. He realized that means half of one fourth, which is one eighth and knew the answer of the problem $\frac{3}{4} \div \frac{1}{2} = 1\frac{1}{2}$. Thus, he tried to fit the answer into the picture. He noticed that three parts evenly divided by two and got one and half, without considering that three parts means three parts of the whole. This also showed that Mark had difficulties with connecting the concept of fraction with a pictorial representation.

4.3.3 Mark' SCK and PCK in field practice

During student teaching, Mark was assigned to teach the DoF unit. The class that he taught was organized as "block-scheduled" class that met for 90 minutes. The class had 12 students, three were African American and nine were Latino. All of the students in the class had difficulties learning mathematics.

Mark taught the DoF content topic for three days**.** The first lesson focused on the content of a whole number divided by a fraction; the second lesson focused on the content of a fraction divided by a fraction; and the last lesson was for the application. Based on discussion with his mentor teacher, Mark decided to use the activities and problems from the textbook *Rethinking Middle school Mathematics: Numerical Reasoning.* He tried to teach DoF by solving real world problems He provided two problems and each included four situations respectively with different numbers. The problems required students to think about the situation in their own words, to draw the picture to represent the problem, write the number sentences, do the algorithm and provide the answer. The problems that Mark chose can be categorized as DoF of measurement (see Appendix 6)**:**

Activity 1: Naylah plans to make small cheese pizzas to sell at a school fund-raiser. She has 9 bars of cheddar cheese. How many pizzas can she make if each takes.

Activity 2: Rashhed takes a customer order to cut ribbons for conference badges. It takes $\frac{1}{6}$ of a yard to make a ribbon for a badge. How many badge ribbons can he make from the given remnants of ribbon?

Overall, the instruction was similar in first and second lessons. In the first lesson, Mark intended to create a group discussion and encourage students to find the answer to the problem at the beginning. However, he changed his instruction in the middle of the lesson and the second lesson. He mainly explained the problems to the class and guided students thinking.

In the interview, Mark indicated that he expected the students to understand and remember the algorithm rule and model it by themselves. He also expected the students would do the application for DoF from previous problems. Thus, the activity and problems that he chose focused on understanding the quantity relation in the problem and how the algorithm rule works, instead of the algorithm rule itself.   Mark tried to achieve this instructional goal by teaching DoF through the steps of asking the students to: 1) think about the problem in own words; 2) Model or Picture the problem; 3) Write the problem in symbols; 4) Process or dothe algorithm, and 5) Solve the problem.

The development and limitations of Mark' explanation - Mark expected the students understand how the computation of DoF works and he expected students to understand the quantity relationship by solving word problems. . The problems and activities that C chose were similar for both first and second lessons. Mark intended to guide students thinking in the five processes. However, both two lessons had few teacher-students interaction partly due to difficulties with classroom management.

Mark' unclear explanation can be found in his lesson instruction. For example, he used a scenario (a whole number divided by a fraction) to introduce a whole number divided by a fraction in the first lesson. He showed four circles and provided a story problem:

> Fred's mom ordered 4 circular chocolate cakes for celebration. Each person will receive a slice is $\frac{1}{8}$ of a cake. Mom ordered enough to feed the team and family members attend. How many people did mom serve? (First class)

He asked students to discuss with their peers in each group and provide their answer. Mark first asked how many $\frac{1}{8}$'s are in each circle. The students answered that it should be divided into eight pieces. Mark divided one circle into eight pieces and asked how many pieces of $\frac{1}{8}$ are in all the circles. The students counted and answered 32. Mark further asked what they did and confirmed the answer 32 comes from 8×4. He did not explain what this problem meant (i.e., how to present write a number sentence) and moved to the next problem.

Students can easily count and get the answer from the pictorial representation. However, Mark failed to explain what the model/picture meant. Students got the answer easily by counting from the picture. They also found the expression of 8×4 based on the answer and the given umber. Mark focused on the process of how to get the answer and failed to explain why $\frac{1}{8}$ in the problem became 8 in the expression. The expression did not represent the problem but how the answer was found. Few students can connect the expression that Mark provided with division problem that they would learn in this class. Without further explanation, Mark provided the next problem.

Naylah plans to make small cheese pizzas to sell at school fund-raiser. She has 9 bars of cheddar cheese. How many pizzas can she make if each takes $\frac{1}{3}$ bar of cheese?

The discussion of the problem proceeded as follows:

M: Alright, so we will draw the picture. You all said 9 bars, right? (Drawing) This is 1 bar, two bar, three, four, five, six, seven, eight, nine. And now we're dividing it into three pieces. You all say there are three pieces in each one, right? (Drawing) Alright, so if we want to… if we look at our picture. What can we can about how many pieces we have now.

S: We have 27.

M: 27? Ok, you're saying we have 27 total pieces now? Let's count…..

S: 9 times 3

M: Right, M said that we can say 9 times 3, because we have one group 3, and total we have 9 of them.

As with the first problem, Mark intended to help students get the answer from the picture. He suggested dividing each bar into three pieces. However, he did not provide an explanation of why it should be divided into three pieces. Thus, students had difficulty making connection that the question is asking how many of the size $\frac{1}{3}$ can be formed in the amount of 9. Getting the result of 27 from counting in the pictures and getting the equation of $9 \times 3 = 27$ from the answer 27 and the given number 9. The process

Mark did not indicate the division idea for the problem. It may cause students' confusion.

In the second lesson, Mark improved his explanation.

M: Ok, so all we have half a yard right?... So for.., we have half a yard. (writing on the OHP) and we got $\frac{1}{6}$ of a yard makes one badge, right? So there are two fractions. So anyone can find the relation in you own words? Where will we put now? We try to figure out how many badges we can make. If half yard and each badge is $\frac{1}{6}$ yard?

S: Is it three?

M: You are right, it is 3. But how do you get that?

S: multiply. Two times three. One half times one third.

M: But how do you do that though? Why do you do one half times one third? Our question is how many times of this?

S: I don't know

M: No? You got the right answer. Alright, In our own words, we all write here, we know we have half yard of ribbon. And it takes $\frac{1}{6}$ to make a badge, so we need to know **how many $\frac{1}{6}$'s go into what?**

S: $\frac{1}{2}$

M: $\frac{1}{2}$. exactly. So in our words, we say (writing) how many $\frac{1}{6}$'s are in $\frac{1}{2}$. …in you own words, how many $\frac{1}{6}$ are in $\frac{1}{2}$.

….

M: Three sixths, right? (Writing $\frac{1}{2} = \frac{3}{6}$). Three times one is three, and two times one is six. So we also could say in our own words, how many $\frac{1}{6}$'s are in $\frac{3}{6}$, you can say that too (Writing). How many $\frac{1}{6}$ are in $\frac{1}{2}$, or $\frac{3}{6}$. The same thing, right?

S: Three.

In the second lesson, Mark asked the students to repeat the relationship between two fractions in an idea of measurement. He identify that how many $\frac{1}{6}$'s are in $\frac{1}{2}$. Further, he provided the idea of common denominator in order to visualize the quantity relationship between many $\frac{1}{6}$ and $\frac{1}{2}$.

However, and unclear pictorial representation caused confusion for the students.

Mark: So we will draw a picture, what will we divide up do. How many pieces do we need? Six, right? So our picture, (drawing a rectangle and divided into six) one six, and one half, and you have six out of it (see Figure 15)

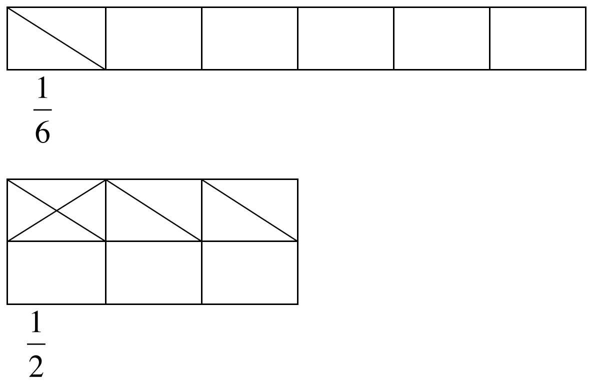


Figure 15. Representation Mark used for his teaching (1).

M: Ok, we say how many $\frac{1}{6}$'s are $\frac{3}{6}$. So what're we looking first? How we represent $\frac{3}{6}$ in our chart? Three, right? So we colored three of them (Colored). That's $\frac{3}{6}$, right? So we want to know how many $\frac{1}{6}$ will go to this (three sixths). So we say, $\frac{1}{6}$ is one group, right? So $\frac{1}{6}$ is our first group, right? So one group is $\frac{1}{6}$, so this is one group, how many $\frac{1}{6}$ go into this ($\frac{3}{6}$)? (Two? One?) Ok, if you looked at the box, what is the value of one box?

S: One? …

The pictorial representation that Mark provided is difficult for students to understand the concept of the unit whole. Thus, it failed to provide the help for students' concept understanding.

Further, Mark explained the second express $\frac{3}{4} \div \frac{1}{6}$. He used the common denominator idea for the pictorial representation (see Figure 15).

M: So we have to draw a picture with 12 pieces right? (Draw the picture) and we wantta know how many $\frac{2}{12}$ are in $\frac{9}{12}$. How many shade this one? (Two, three, twelve, eleven) We have twelve. We're shading nine, right?

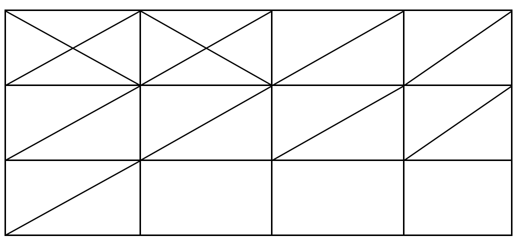


Figure 16. Representation Mark used for his teaching (2).

Mark: Two, right? This is $\frac{1}{12}$, this is $\frac{2}{12}$, so $\frac{2}{12}$ is that much, right. It takes, two to make, how many groups to make, two to make one group, right? So we can say this is gonna equal how many groups we can make where two in one group, right? So we can make how many groups and each groups are two. So, we have, one, two, ….(Students confused) we have nine pieces total. We have one, two, three, four, we left (one), one, and how many can we make a group? (Two). Two this is another group. How much are in this group? (Five). (Ten). Ok, just look at the last group. We have one of them, we need two. How much do we have? (One half) $\frac{1}{12}$. We have half of a group. If we change to change to mixed number. To write the symbol, we write two make one group, we have four groups and one extra in a group. So 9 pieces we can use and two group include 2. $\frac{9}{2}$. How do we write our division sentence. It's one your process…. What would be…

S: One?

Both representations may easily cause students' confusion and misunderstanding. With the first picture, Mark' did not explain that both fraction had the same unit whole.

Although Mark intended to develop students' conceptual understanding, he failed to help students come up with the number sentences due to the inappropriate presentation.

M: So $\frac{1}{6}$ goes into this…

S: Three times.

M: Three times. That makes sense? So we know that $\frac{1}{6}$ is one group and how many groups do we have now?

S: three of them.

M: So what's 3 divided by 1?

S: Three.

M: Exactly, 3. So we write it in symbols. We say that we know, you have three divided by one. One is our group. Ok, 3 over the group, so write it in symbols, you should have 3 one sixths in our group, $\frac{3}{1}$=3 in our symbols. For *the process it* that we actually do, what did we do for our process to solve this? What would we write the equation? What would we write for dividing the fractions? What're we dividing here? We said how many $\frac{1}{6}$'s are go into $\frac{1}{2}$, so our fraction sentences be,

S: $\frac{1}{6} \div \frac{1}{2}$

M: We divided $\frac{1}{2}$ by $\frac{1}{6}$ ($\frac{1}{2} \div \frac{1}{6}$) or divided $\frac{1}{6}$ by $\frac{1}{2}$($\frac{1}{6} \div \frac{1}{2}$). $\frac{1}{6}$ goes into $\frac{1}{2}$, so how do we write it? … We would say $\frac{1}{2} \div \frac{1}{6}$. How we do this?

S: Multiply.

M: We want to know how many $\frac{2}{12}$'s are in there?... What would be $\frac{2}{12}$? How many $\frac{2}{12}$'s are of whole thing. Two, right? This is $\frac{1}{12}$, this is $\frac{2}{12}$, so $\frac{2}{12}$ is that much, right. It takes, two to make, how many groups to make, two to make one group, right? So we can say this is gonna equal how many groups we can make where two in one group, right? So we can make how many groups and each groups are two. So, we have, one, two, …. (Students confused) we have nine pieces total. We have one, two, three, four, we left (one), one, and how many can we make a group?(Two). Two this is another group. How much are in this group? (Five). (Ten). Ok, just look at the last group. We have one of them, we need two. How much do we have? (One half) $\frac{1}{2}$. We have half of a group. If we change to change to mixed number. To write the symbol, we write two make one group, we have four groups and one extra in a group. So 9 pieces we can use and two groups include 2. $\frac{9}{2}$. How do we write our division sentence. It's one your process…. What would be…

S: $\frac{1}{6} \div \frac{3}{4}$

M: So it's $\frac{3}{4} \div \frac{1}{6}$. K, how can we rewrite this to solve it?

S: Flip…

In order to help students think the situation in their own words, Mark provided multiple hints.

M: Ok, you can say we want, we want how many.. .when we have 9 bars, how many we aretaking for each?

S: $\frac{1}{3}$.

M: Alright, that you all want to say that, how do you say that? ….We **want $\frac{1}{3}$ of 9 bars.** So we say that.. Do we want $\frac{1}{3}$ of 9 bars, is that we want?... We want 9 bars divided by how many bars?

S: two, three, four…

M: Alright, we want 9 bars divided into…

S: three pieces.

M: Three pieces? Say that again. We want 9 bars divided into $\frac{1}{3}$ (Writing).

There are two places that he made unclear explanations. First, he used that "$\frac{1}{3}$ *of 9 bars"* to express the problem. It may cause misconception between division and multiplication. Second, he indicated that the problems represented that how many total would be divided into $\frac{1}{3}$. However, he failed to give a further explanation for what it is and why it is a division relation in this case. Students could not make the connection between these problems with the division fractions. Thus, for Mark's question that 9 bars divided by how many bars, the students responded to him without thinking. He explained further by pictorial representation.

In summary, although Mark developed SCK in the methods course it seemed insufficient for teaching in at the beginning of the class. In his teaching, the model and manipulatives were used to find the answer and Mark failed to explain what each representation showed. Thus, students may have difficulties make connection between the manipulatives and DoF. On the other hand, Mark developed verbal explanations by using accurate mathematical language through the teaching. Mark mentioned in the interview that his understanding of modeling using developed through his teaching.

KCS and KCT in classroom instruction - From the interview and classroom instruction, it was evident that Mark' KCS and KCT was limited at the beginning of teaching. It showed from the several aspects. In KCS domain, when he chose the materials for students learning, he could not anticipate what students are likely to think about using this materials and whether it would cause students' confusion of learning (Ball, Thames, & Phelps, 2008).

> I chose this (the materials) because I understand it. Uhh, I didn't think, I didn't realize that might be difficult for them (the students). …It just, I don't know, I didn't present well at the beginning. I think I would do better. (Mark, interview, 11/19/2007)

At the beginning of the lesson, Mark set up the purpose of the learning that helped students to both remember the algorithm and understand why the algorithm works. This was consistent with his beliefs towards teaching and learning DoF in his method course. Thus, he chose the examples could help students come up with the computational rule by themselves. However, the limitation of the introduction and the explanation failed to make the connection. In the interview, Mark reflected his understanding of teaching DoF:

> You know, if we had a problem $1 \div \frac{2}{3}$, we really want to stress them this is how many groups of, we only use the word of the GROUPS, how many groups of $\frac{1}{3}$ can we put into 1. So that way, whenever they had that, first part, when they do their modeling, they can relate their modeling to not only words using "groups", you know, how many groups of what into.. , but only the number sentence they came up with later, you know, the process that when you divided 1 by $\frac{1}{3}$ is also 1

times 3. So, the model apply, not only to the dividing, but the multiply (How to do the algorithm). You know, cause whenever you dividing the fraction, you end up multiply by their reciprocal. I want to show them the picture of the division sentence 1 divided by one third, and the others, the multiplication would be 1 times 3 over 1… we want to stress the keyword and we want to stress how the model to the number sentence. (Mark, interview, 11/22/2007)

Furthermore, when he reflected on his teaching, he realized that the unclear explanation caused students' confusion about DoF:

I think for this, the particular lesson, for dividing and conquer, uhh, the hardest part for them, maybe is not necessarily drawing the picture, but when the picture drawn, they may be confused by the part of applying the picture to dividing the fraction. Why to divide the fraction? How the number sentence related to your picture, which is what we want them to see, which is one way to teach…. I think applying the pictures to the number sentences. Some of them may be confused with the picture. But I think, the main problem is to (understand) the applying the picture to the number sentence. That's why I think, going through it and seeing where their trouble is. The second time when I teach, …I really focus on showing them the concept of how the picture related to the number sentences. It just, I don't know, I didn't present well at the beginning. I think I would do better…I think the picture makes more complicated. … So that was my fault, I shouldn't draw it like that… (Mark, interview, 11/22/2007).

Mark reflection showed that he lacked understanding about the students who he would teach. On one hand, he considered that student should learn mathematics through different activities and modeling so he purposely planned to use the problems of DoF in a real world application. On the other hand, he did not consider whether the problems were appropriate for them. Further, when he first chose these problems, he did not anticipate that what the students would find interesting and motivating (Ball, Thames, & Phelps, 2008).

In KCT domain, teachers are required to have knowledge to design mathematics tasks in classroom instruction (Ball, Thomes, & Phelps, 2008). In this case, Mark needed to consider which examples to start with and which examples take students deeper into the content, in other words, how to sequence DoF instruction. Moreover, KCT also requires him to consider how to use appropriate model to present to the students. Mark chose two types of DoF by problem solving. One type is a whole number is divided by a fraction and the other type is a fraction is divided by a fraction. Both types can be considered a measurement model of division. In this case, the instruction practice may cause students' intuitive misconception that the dividend must be the big number. However, C could not identify the limitation of the examples. It can be assumed to his understanding of difference between two models of the division.

> I mean, right, there's grouping and sharing, right? And I understand that one is more easier than the other to understand, I was never taught grouping and sharing, the differences between them. Using the keywords to write, which one make small pieces. ... I didn't really do that. Method class, maybe, yes, it's the

first time that the idea of grouping and sharing came up. It's kind of confusing. (Mark, interview, 11/22/2007)

Mark indicated his concern about students' confusing about the quantity relationship:

The problems, they need to read it and tried to figured out, what they put the fraction in and they will showing they have the process. It showed that they have written the wrong fraction first… So I had, like, and work to the problems. I noticed that what they do is the first fraction they see they put as the dividend, and second fraction as second. That's just what they do, they don't really be carful. … And that another thing, because, let's see, maybe they see the whole number, and it's bigger than a fraction. So they put first… That's definitely a misconception, what we did, we went through and they took a quiz, and they're doing their right in algorithm, but they put the fraction in an wrong order, or fraction in a wrong order, a mixed number in a wrong order. So, I noticed that (misconception exists), and I went back, and I wrote two problems. And first problem, ….well, I don't remember exactly I made the first question where I put the whole amount as the first, and then the small pieces, and then, the second question, I just inverse, I put the small pieces at first. That way, I will just show that it wasn't matter which appears first, it matters what the keyword around, and what it started with. We tried, you know, which one.. I asked them why you think this one goes first, they will like because it's bigger. That's another misconception, not necessary bigger number that go first. Just because this

> fraction is bigger, doesn't mean it go first. You know, so. We went back and did that, and we wrote whole. Some of them has improvement, others haven't. (Mark, interview, 11/22/2007)

4.3.4 Mark's beliefs and its changes in the field practice

The results of the teaching experience showed that Mark considered that modeling and manipulatives not only developed his understanding but also students' understanding of why the algorithm works:

> I always say modeling is more confusing, but having the chance to teach and use them, I mean I understand now. I've never understood since method course. I've never got it, …but when I teach the students, I think I understood, and I think they understood dividing fraction. They knew how to draw it and they can do it. …After they learned the rule for the algorithm, they still draw the pictures. So I mean it's students by students bases. And I tried to tell them "if you have 25 divided by.., how can you keep modeling that". I mean it's fine that they can still do this, some points, I want them to be able to use both algorithm rule and modeling. Some cases pictures make more sense…. (Mark, 11/22/2007).

CHAPTER V

DISCUSSION AND CONCLUSION

The purpose of this study was to understand prospective teachers' knowledge development and change in beliefs both in their method courses and student teaching practice. In particular, results are focused on the content topic division of fractions (DoF), summarizing how teacher's Common Content Knowledge (CCK) and Mathematics Knowledge of Students (KCM) developed in their methods course. In addition, the discussion addresses prospective teachers' SMK development correlated with their beliefs towards teaching and learning mathematics and DoF. Finally, a case study is presented of one prospective teacher's SMK and development of Pedagogical Content Knowledge (PCK) through classroom instruction. The chapter concludes with an argument that the limitation of prospective teacher's SMK prevented the development of PCK during classroom instruction. The discussion and conclusions are presents for each of the research questions for this study.

5.1 Prospective teachers' CCK Influences SCK and beliefs Changes

The results showed that the majority of the prospective teachers remembered the computational rule and were able to do the DoF algorithm in both pre- and pos-test. However, there many students had difficulties with the application of DoF in pre-test. The results showed that there was no significant difference regarding to prospective teachers' procedural knowledge between pre- and post-test. The prospective teachers in this study also considered DoF as a mechanical topic. Therefore, the majority of the prospective teachers memorized the computational rule for doing the algorithm. It was the only thing they were taught in their middle school, and before the method course, they also considered that it would be the only thing they need to teach. The results also revealed misconceptions that the prospective teachers held.

Although they memorized the computational rule as "invert-and-multiply", they failed to do MoF algorithm. They cross-multiplied instead of multiplied across straight. This "bug" more often appeared in the equation problem. They confused multiplication expression (i.e. $\frac{14}{15} \times \frac{9}{x}$) with the algebraic equation or proportion ($\frac{14}{15} = \frac{9}{x}$). These results support previous studies that indicated the misconceptions that children might hold (Barash & Klein, 1996; Tirosh, 2000). However, the prospective teachers may also hold the misconception due to their own learning experience. The results also revealed that it doesn't necessary mean that prospective teachers can do the algorithm even if they memorized the computational rule. The incorrect performance in this case is due to inadequate knowledge related to the proportion.

Further, the knowledge limitation influenced the prospective teachers' performance on the application of DoF. Division often can be learned conceptually as two different models, measurement model of division and partitive model of division. The measurement model of division can be considered as how many groups of a certain size can be formed and the partitive model can be categorized as determining the size of each group. The results showed that many prospective teachers had the difficulties with the measurement model of division (i.e., How many $\frac{1}{2}$'s are in $\frac{1}{3}$) in the pre-test. These results support prior research studies that prospective teachers possess shallow understanding of fraction division (e.g., Ball, 1990; Simon, 1993). The interview results revealed that they did not consider DoF as a measurement model of division or a partitive model of division in their learning experiences. It was a new idea for most of them. Therefore, they had difficulty identifying the quantity relationship between two fractions in the word problem. Moreover, some prospective teachers wrote the number sentence as "$\frac{1}{2} \div \frac{1}{3}$" instead of "$\frac{1}{3} \div \frac{1}{2}$" and some answers provided as "one" in the pre-test. It can be assumed that prospective teachers may hold a misconception from intuition that "one can not divide a small number by a larger number because it is impossible to share less among more (e.g., Tirosh, Fischbein, Graeber, & Wilson, 1993, p.18). In this case, it can be interpreted as how many times of a given quantity ($\frac{1}{3}$) is contained in a larger quantity ($\frac{1}{2}$). This intuitively based mistake or conception (Tirosh, 2000) also showed the influences when they chose the examples for their teaching.

This lack of conceptual understanding was also revealed when the prospective teachers provided a scenario for the expression $\frac{3}{4} \div \frac{1}{2}$. All but two of them failed to identifys how many times of a given quantity ($\frac{1}{2}$) is contained in a larger quantity ($\frac{3}{4}$). Over half of the prospective teachers failed to make a scenario and some of them still confused and made a partitive model of division $\frac{3}{4} \div 2$ on the pro-test. It may be considered that the prospective teachers more familiar with partitive interpretation than measurement model (e.g., Ball, 1990; Tirosh & Glover, 1989).

Compared with the previous problems, the results showed that most prospective teachers had a clear understanding of the quantity relationship of “a number *of* another number”. Most prospective teachers solved one-step problems of this type on both the pre- and post-test. However, the prospective teachers had difficulties identifying the quantity relationship in a two-steps word problem.

The results showed that over half of the prospective teachers could identify the multiplicative relationship. However, most prospective teachers did not notice that these results showed their conceptual knowledge regarding DoF application was limited.

Overall, there was no difference between pre- and post-test of the prospective teachers understanding of the definition of DoF (i.e., measurement model of division). However, the knowledge was fragile making it difficult for them to apply in other contexts such as creating a word problem and representing with in pictorial diagrams.

The results showed although the prospective teachers improved the SCK to explain *why* and *how* DoF works, although they still faced difficulties ing providing a scenario and modeling the scenario.

On the pre-test, the majority of the prospective teachers used only the “invert-and-multiply” algorithm rule to explain both *why* they think $\frac{a}{b} \div \frac{c}{d} = \frac{(a \div c)}{(b \div d)}$ was wrong and *how* $\frac{2}{3} \div 2 = \frac{1}{3}$ and $\frac{2}{3} \div \frac{1}{6} = 4$ works. Only three prospective teachers explained why $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ in the pre-test.

None of them could explain that the previous equation is a special case of DoF and identify DoF is the inverse operation of MoF in their pre-test. Most of them relied on the "invert-and-multiply" algorithm. A few of them clearly indicated that the "invert-and-multiply" algorithm is only way to do DoF. Although the results improved on the post-test, over half of them either did not provide a correct answer or did not answer this question.

Compared with the first question, the results for explaining why $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ works showed that prospective teachers improved on the post-test. Combined with the results of their writing assignment and the interview, all prospective teachers had impact on this problem. As foundd in other studies (i.e., Ball, 1990; Borko et al., 1992), these prospective teachers had little chance to learn the underlying concept before entering the methods course. In fact, this problem was most mentioned in their writing assignment. Although they used different representations to explain this problem, most of them used a numerical representation for the explanation. Only a few prospective teachers used manipulatives to illustrate the area model. All who explained the problem correctly on the post-test used a numerical explanation. Most of the prospective teachers developed their own understanding of why DoF works. Even though the explanation showed their understanding of why $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, this did not connect with how they would explain it to their students in classroom instruction.

On the equation $\frac{a}{b} \div \frac{c}{d} = \frac{(a \div c)}{(b \div d)}$, over half of the prospective teachers gave incorrect answers or did not answer. This problem required prospective teachers to connect to their knowledge and understanding of division and fraction. It also required them to identify the limitations of the algorithm.

Most of the prospective teachers failed to make a connection between $\frac{\frac{a}{c}}{\frac{b}{d}}$ and $\frac{(a \div c)}{(b \div d)}$, revealing that the knowledge that prospective teachers developed in their methods course was limited.

For the problem that explain *how* division of a fraction by a whole number or another fraction works, the results showed differences between pre- and post-test. On the post-test, the number of the prospective teachers using representations for both equations to show how the answer comes from increased and the number using "invert-and-multiply" algorithm to get answer decreased. Most of them tried to use multiple ways to explain how the answer is obtained. It was easier for the prospective teacher to present division by a whole number pictorially and verbally in the pre-test comparing with division by a fraction. This results supported Ball and other researchers' (Ball et al, 2008) findings and the findings of CCK for this study that prospective teachers tended to think in partitive interpretation. The prospective teachers developed their understanding for the measurement model of division and correctly explained how many times one fraction goes into another. The difference between two equations also appears in the representation they chose. The prospective teachers used both pictorial and verbal to represent division by a whole number, while they mainly used verbal and numerical representations for division by a fraction.. The interview results also showed that they struggled with using a pictorial area model representation for the measurement model of division. These results were also consisted with the question that required prospective teachers to make up a story problem and draw a picture for one fraction divided by another. None could provide an appropriate pictorial representation on either the pre- or the post-test. Half of the prospective teachers failed to provide an appropriate scenario for one fraction divided by another.

The interview results revealed that prospective teachers constructed the pictorial representation from the answer. In other words, they did the computation first and intended to use the answer to fit the picture.

Overall, the previous results revealed that prospective teachers had relatively weak conceptual understanding of DoF. The weak conceptual understanding affected both their CCK and SCK on the pre-test. After the methods course, their own understanding of DoF developed and they were able to use abstract numerical representation for explanation. However, the conceptual understanding they developed was limited and it affected in their SCK in the post-test.

The results of survey, concept mapping, writing assignment, and interview revealed the prospective teachers' beliefs and attitude towards to teaching and learning mathematics, in particular, DoF. Previous research (Brown & Borko, 1992; Brown, Cooney, & Jones, 1990) suggest that teachers' beliefs about mathematics and how to teach mathematics are influenced in significant ways by their experiences with mathematics and schooling long before they enter the formal world of mathematics education. Moreover, some researchers have argued that beliefs seldom change dramatically without significant intervention (cf. Lappan et al., 1988).

Among the items on teaching perspectives, the majority of prospective teachers considered manipulatives and abstract mathematics separately. Instead of thinking that manipulatives could help students to understand abstract mathematics, they agreed that manipulatives could help students avoid abstract mathematics. However, they indicated the importance of using different manipulatives and modeling. They emphasized it is important to teach students the concept underlying the algorithm based on different modeling approaches.

From the learning perspective, most prospective teachersI on both pre- and post-survey indicated that they considered that mathematics should not be learned by memorizing. Although some prospective teachers considered that mathematics should be learned as sets of algorithm and rules, most of them indicated that students should not only learn how to do algorithm, but also learn why underlying the algorithm.

In particular, the prospective teachers indicated their conception towards DoF. Reflecting on their own learning experiences, most prospective teachers indicated that they learned DoF as algorithm-based and never thought of the way that DoF applied in the real-life contexts. Most of them did not realize the limitation of their knowledge until the method course. The question of *why* of the DoF algorithm works made them to rethink their own knowledge and understanding. Thus, they relearned this content topic in methods course.

From their own learning experience, which can be considered the intervention with their long schooling experiences (e.g., Lappan et al., 1988), they indicated that their beliefs and attitudes towards teaching this content topic changed and learning this content topic changed. They considered teaching students by using different models and munipulatives. They believed that they learned from the models that represented how DoF works and felt confident to teach this content topic. Furthermore, they also considered that if students were taught and used model by themselves, they could solve the computation even they forgot the computational rule.

The results from the concept mapping also showed that the prospective teachers changed their perception to this content. They described DoF from procedural based units to more conceptual based units. Moreover, the units related to classroom practice increased at the end of the semester.

The results indicated the prospective teachers improved their SCK and the development of SCK changed their beliefs towards to learning and teaching DoF. The development of SCK also influenced their self-confidence in teaching.

The results from six participants showed that students who have high achievement of CCK would develop his/her SCK significant comparing with those who has low achievement.

For example, Eric, Amy and Mark, who have high CCK, easily to develop their understanding to explain why DoF computation works in either special case or general case. Especially, the participants who developed SCK significant provide more multiple representations for their explanation. Moreover, Like Amy indicated in her interview, she understand what she is struggling in her learning and would consider more in her future teaching. On the other hand, the participants who had low CCK have less progress for their SCK development. Moreover, like Cooney (2001) indicated in his study, knowledge development influences prospective teachers' beliefs towards to the subject content and teaching and learning. In particular, this study reveals that SCK has more impact on prospective teachers' beliefs changes

5.2 Prospective Teachers' SCK development Impacts PCK

The results showed that the case study participant Mark developed his SCK through student teaching experiences. His KCS and KCT were limited at the beginning of the teaching and this affected the development of his teaching and reflective thinking. Mark was considered an average student and was proficient with DoF algorithm and could solve word problems. He was confident for his mathematics content knowledge and considered he was ready for teaching. However, as with prospective teacher mentioned in previous research (Brown, Cooney, & Jones, 1990; Coonney, 2001), Mark lacked mathematical sophistication. His limited SCK also influenced their KCS and KCT through the teaching practice. The lack of SCK (Ball, Thames, & Phelps, 2008) was revealed in several aspects.

First, a limited SCK contributed to the failure to link representations to underlying ideas and to other ideas. During the instruction practice, Mark used verbal, pictorial, and numerical representation to represent DoF. However, the unclear verbal explanation could not successfully present DoF instead of finding the answer. The unclear pictorial representation also led to students' misunderstanding. Second, limited SCK was evident from Mark' unclear and inappropriate explanation. Like other researchers have indicated (e.g., Ball, Thames, & Phelps, 2008), teaching involves the use of decompressed mathematical knowledge to make students develop their understanding. Finally, the limited SCK was shown by the failure to identify students' errors during instruction. This outcome might have been due to Mark' concern about classroom management, not attending to student errors or misstatements.

On the other hand, Mark developed his own understanding through teaching. The results of his writing and post-test showed that Mark was more confident in using numerical representations to show why DoF computation works. He also mentioned that area model was the one that he was struggling with during his learning. He still struggled at the end of the method course, however, his teaching practice developed his understanding of using models.

The results showed that his beliefs about DoF and teaching DoF influence what he would teach (the goal) of his lesson and how would he teach (e.g., Ernst, 1990; Gess-Newsome, 1999). The insufficient CCK and SCK also affect the prospective teacher's KCC, KCS and KCT (Ball, et al, 2008). For the implication, undergraduate teacher preparation should consider combine these two factors in the future.

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APPENDIX 1

SURVEY ITEMS USED IN THIS STUDY

1. How would you rate yourself the degree of your understanding of the Mathematics Texas Essential Knowledge and Skills (TEKS)?

(1) High (2) Proficient (3) Limited (4) Low

5. Considering your training and experience in both mathematics and instruction, how ready do your feel you are to teach these topics?

(a) Very ready
(b) Ready
(c) Not ready

Number – Representing decimals and fractions using words, numbers, or models

Number – Representing and explaining computations with fractions using words, numbers, or models.

6. To what extent do you agree or disagree with each of the following statements?

(a) Strongly agree
(b) Agree
(c) Disagree
(d) Strongly disagree

- More than one representation (picture, concrete materials, symbols, etc.) should be used in teaching a mathematics topic.
- Use of manipulatives can help students avoid abstract mathematics
- Mathematics should be learned as sets of algorithms or rules that cover all possibilities.
- Teacher should prevent students from making errors in their learning of mathematics
- Learning mathematics mainly involves memorizing
- Teachers need to know students' common misconception/difficulty in teaching a mathematics topic.

APPENDIX 2

TEST ITEM

Part 1

1. Find the following value (no calculator).

   (1) $\frac{1}{5} \div 5 =$; (2) $\frac{7}{9} \div \frac{2}{3} =$; (3) $5\frac{1}{4} \div 3\frac{1}{2} =$

   (4) How much $\frac{1}{2}$'s are in $\frac{1}{3}$?

   (5) If $\frac{14}{15} \div \frac{?}{9} = \frac{3}{10}$, find ?.

2. Solve the following problems (no calculator). Be sure to show your solution process.

   (1) A five-meter-long rope was divided into 15 equal pieces. What is the length of each piece (in meter)?

   (2) Tell whether $\frac{9}{11} \div \frac{2}{3}$ is greater than or less than $\frac{9}{11} \div \frac{3}{4}$ without solving. Explain your reasoning.

   (3) $\frac{5}{6}$ of a number equals to $\frac{5}{24}$, find the number.

   (4) Andrew bought 7 apples, which is $\frac{1}{3}$ of the number of orange he bought. How many oranges did Andrew buy?

   (5) Jonny's Pizza Express sells several different flavor large-size pizzas. One day, it sold 24 pepperoni pizzas. The number of plain cheese pizzas sold on that day was $\frac{3}{4}$ of the number of pepperoni pizzas sold, and $\frac{2}{3}$ of the number of deluxe pizzas sold. How many deluxe pizzas did the pizza express sell on that day?

Part 2

1 You are discussing operations with fractions in you class. During this discussion, John says *It is easy to multiply fractions; you just multiply the numerators and the denominators. I think that we should define the other operations on fractions in the similar way:*

Addition: $\frac{a}{b}+\frac{c}{d}=\frac{(a+c)}{(b+d)}$

Subtraction: $\frac{a}{b}-\frac{c}{d}=\frac{(a-c)}{(b-d)}$

Division: $\frac{a}{b}\div\frac{c}{d}=\frac{(a\div c)}{(b\div d)}$

How will you respond to John's suggestions? (Deal with each separately).

2 How would you explain to you students why $\frac{2}{3}\div\frac{1}{6}=4$ and why $\frac{2}{3}\div 2=\frac{1}{3}$.

3 For the following word problem:

Problem: *Six pounds of sugar were packed in boxes, each box containing* $\frac{3}{4}$ *pound. How many boxes were needed to pack all the sugar?*

1) Write an expression that solve the problem
2) Write common incorrect responses, and
3) Describe possible sources of these incorrect response

4 Examine a student's work in Exercise A-B below, and answer the following three questions.

A. $\frac{7}{10}\div\frac{1}{2}=\frac{7}{10}\div\frac{5}{10}=\frac{2}{10}=\frac{1}{5}$ B. $\frac{1}{3}\div\frac{5}{7}=\frac{7}{21}\div\frac{15}{21}=\frac{8}{21}$

1) Describe the Error Pattern for this student
2) Describe possible sources of these incorrect responses
3) What could you as the teacher do to help this student correct his/her procedure?

5) During the lesson when you teach the algorithm for "division of fractions" (i.e., $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$, students asked why you change from "division" to "multiplication" and flip the second fraction. How would you explain to students?

6) Calculate the division expression: $\frac{3}{4}\div\frac{1}{2}=$, then make up a story problem and draw a picture that would go with "three-fourths divided by one-half".

APPENDIX 3

CONCEPT MAPPING

Draw a concept map of "Division of fractions" that contains some concepts and linking words on the line connection the concepts. Possible concepts related to "division of fractions: include "Division of fractions", "Concept of unit", "Concept of fraction", "Concept of addition", "Concept of inverse operations", "Multiplications of whole numbers", "Multiplication of fractions", "Division with whole numbers".

You can feel free to decide whether you want to select all or some concepts from the above list, or add some other concepts as needed. And you could draw your map however you wish.

APPENDIX 4

POST-INSTRUCTIONAL INTERVIEW

Questionnaires

1) Is there any different of learning DoF that you studied in this course from your previous experiences of learning DoF? If there is the different, what is it? How do you think about it?

2) Through the whole semester, Ms. XX put most part of time on the division of fraction, both from the representation, and problem solving. I want to know your understanding why Ms. XX focused on DoF instead of other topics?

3) Did your understanding on the division of fraction change? If it changed, how did it change?

4) How do you think you get more understanding of the concept of the division, the division of the whole number, the multiplication of the fraction? Give the specific example to show your understanding?

5) Can you describe a little bit about how you would approach this if you were teaching, say DoF?

6) How do you think what you learned in the method course would influence your teaching in teaching?

7) Not only knowing the algorithm, but also knowing why it works, does it means, you make sense of the algorithm? If it makes sense to you, please explain using any representations (as many as you can)?

8) What kind of things do you still struggle, or do not feel very confident with? Why does it not make sense to you?

$\frac{3}{4} \div \frac{1}{2}$ $\qquad\qquad$ $\frac{3}{4} \div 2$

Please do the algorithm.

Make up a story problem for each expression and drawing a picture for each expression to explain. What is the difference between these two expressions?

9) How did you prepare your lesson? What kind of materials do you use?

10) You used the lesson plan that you created in Ms. XX's class, how did you use? Why do you use?

11) How did you construct each lesson? What were the students' difficulties? How did you represent to the students?

12) What were difficulties for you to present to the students the new idea?

13) What kind of things you think you can improve?

14) Three teachers express $\frac{3}{4} \div \frac{1}{2}$ by story problem,

1) There is three fourths of a pie and there have two people, gets the pie divided evenly, so that each person gets an equal share.
2) There is three fourths of a pie and you have to take one half of it.

Pattern 1

$\frac{4}{8} \div \frac{2}{2} = \frac{2}{4}$ $\quad \frac{4}{8} \div \frac{2}{8} = \frac{2}{1}$ $\quad \frac{8}{10} \div \frac{2}{4} = \frac{4}{2}$ $\quad \frac{5}{7} \div \frac{3}{2} = \frac{2}{2}$

How could estimation have warned that this student computation pattern was not always giving her the correct answer?

What could you as the teacher do to help this student clarify this procedure?

Pattern 2

$\frac{2}{5} \div \frac{3}{8} = \frac{5}{2} * \frac{3}{8} = \frac{15}{16}$

Describe the error patter.
With what other procedure might this student have confused this?
What could you as the teacher do to help this student clarify this procedure?

APPENDIX 5

INSTRUCTIONAL OBSERVATION FORM

1 MATHEAMTICS CONTENTS

1) What was being taught?
2) What seemed to be the goal? (What are students supposed to be learning to be able to do, to understand, etc?)
3) Was the emphasis on "doing mathematics" (e.g., framing problems, making conjectures, looking for patterns, examining constraints, …) or was the emphasis on getting right answers? (Given specific examples.)
4) Was the content of this lesson connected to other things that the class has been dealing with? Give specific examples.
5) How was understanding assessed?

2 INSTRUCTIONAL REPRESENTATIONS AND MATHEAMTICAL TOOLS

1) What instructional representations (concrete, pictorial, real-world, or symbolic) did the teacher or the students use in this lesson and what mathematics ideas were they targeting?
2) Itemize the mathematical tools that teacher or students used in this lesson. (concrete pedagogical materials; pictorial tools; 'real-world' situation or stories; measurement tools and other mathematical objects; calculators;
3) Mathematical discourse use.
   (special terms used to refer to the substance of mathematics; mathematical symbols and notation; language and skills of mathematics discourse- formulating hypotheses; challenging solutions; providing counterexamples)

3 CLASSROOM DISCOURSE

1) Did the teacher frequently verbalize reasons, understandings, and solution strategies himself? Did the students do this frequently in response to prompting/encouragement from teacher? How did the teacher respond to students when they did this?
2) Did the teacher frequently make conjectures, challenge ideas, validate and justify solutions himself? Did the students do this frequently in response to prompting/encouragement from teacher? How did the teacher respond to students when they did this?
3) What were the students doing and what was the teacher's role during discussions?

APPENDIX 6

CLASSROOM ACTIVITIES

1 Divide and Conquer! Activity

Use the chart to organize your thought about problem.

1) Naylah plans to make small cheese pizzas to sell at a school fund-raiser. She had 9 bars of cheddar cheese. How many pizzas can she make if each takes:

Situation	Think about it? (in your own words)	Picture it	Write it in symbols	Process it (What do you actually do?)
a) $\frac{1}{3}$ bar of cheese?				
b) $\frac{1}{6}$ bar of cheese?				
c) $\frac{1}{4}$ bar of cheese?				
d) $\frac{3}{4}$ bar of cheese?				

2) Rasheed and Jode have a summer job at a kiosk called Ribbon Remnants. This is a place where you can get small amount of ribbon very inexpensively from end-of-bolt pieces of ribbon. In each situation that follows, use the chart to organize your reasoning and find your solution.

Rasheed takes acustomer order to cut ribbons for conference badges. It takes $\frac{1}{6}$ of a yard to make a ribbon for a badge. How many badge ribbons can he make from the given remnants of ribbon? For each answer that has a remainder- some ribbon left over- tell what fractional part of another badge ribbon you could make with the amount left over.

Situation	Think about it? (in your own words)	Picture it	Write it in symbols	Process it (What do you actually do?)
a) $\frac{1}{2}$ yard?				
b) $\frac{3}{4}$ yard?				
c) $\frac{5}{8}$ yard?				
d) $2\frac{2}{3}$ yard?				

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