DIMENSIONS OF BIVARIATE SPLINE SPACES AND ALGEBRAIC GEOMETRY

A Dissertation

by

YOUNGDEUG KO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2009

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Peter Stiller
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	John Keyser
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ABSTRACT

Dimensions of Bivariate Spline Spaces and Algebraic Geometry. (December 2009)

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Splines are piecewise polynomial functions of a given order of smoothness r. Given complex Δ the set of splines of degree less than or equal to d forms a vector space and is denoted by $S_d^r(\Delta)$. For a simplicial complex Δ, Strang conjectured a lower bound on the dimension of spline space $S_d^r(\Delta)$ and it is known that the equality holds for sufficiently large d. It is called the dimension formula.

In this dissertation, we approach the study of splines from the viewpoint of algebraic geometry. This dissertation follows the works of Lau and Stiller. They introduced the conformality conditions which lead to the machinery of sheaves and cohomology which provided a powerful type of generalization of linear algebra.

First, we try to analyze effects in the dimensions of spline spaces when we remove or add certain faces in the given complex. We define the cofactor spaces and cofactor maps from the given complexes and use them to interpret the changes in the dimensions of spline spaces.

Second, given polyhedral complex Δ, we break it into two smaller complexes Δ^1 and Δ^2 which are usually easier to handle. We will find conditions for Δ^1 and Δ^2 which guarantee that the dimension formula holds for the original complex Δ.

Next, we use the previous splitting method on certain types of triangulations. We explain how to break the given triangulation and show what kind of simple complexes we end up with.

Finally, we study the "$2r+1$" conjecture on a certain triangulation. The "$2r+1$"

conjecture is that the dimension formula holds on any triangulation for $d \geq 2r + 1$. We know that the conjecture is sharp because the dimension formula fails on a certain triangulation for $d = 2r$, but we do not know if it holds on the same triangulation when $d = 2r + 1$. It is related to a Toeplitz matrix.

To my parents and my brother

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CHAPTER I

INTRODUCTION

For a complex Δ whose support is a simply connected polygonal domain Ω in $\mathbb{R}^n$, splines are piecewise polynomial functions of a given order of smoothness r. The set of splines of degree less than or equal to d forms a vector space and is denoted by $S_d^r(\Delta)$:

$$S_d^r(\Delta) := \{s \in C^r(\Omega) : s|_\sigma \in P_d,\ \sigma \in \Delta_n\},$$

where P_d is the space of n-variable polynomials of degree at most d and σ is an n-dimensional face.

Splines have many applications. For example, they have been used for solving differential equations, in computer graphics, and to fit scattered data.

The main purpose of this dissertation is to explore the dimensions of certain bivariate spline spaces.

We denote by Δ_i, Δ_i^0 and Δ_i^∂ respectively, the set of i-dimensional faces, i-dimensional interior faces and i-dimensional boundary faces in Δ. Also, let $f_i(\Delta)$, $f_i^0(\Delta)$ and $f_i^\partial(\Delta)$ denote the cardinalities of the preceeding sets. We will omit Δ in these notations if there is no ambiguity.

For a simplicial complex Δ, Strang made in [11] and [12] a conjecture for a lower bound on $dim\ S_d^r(\Delta)$:

$$dim\ S_d^r(\Delta) \geq \binom{d+2}{2} + f_1^0\binom{d-r+1}{2} - f_0^0\left[\binom{d+2}{2} - \binom{r+2}{2}\right] + \delta, \qquad (1.1)$$

where $\delta = \sum_{i=1}^{f_0^0}\sum_{j=1}^{d-r}(r+j+1-je_i)_+$, $(m)_+ = max(m,0)$, and e_i is the number of

This dissertation follows the style of Advances in Computational Mathematics.

distinct slopes at an interior vertex v_i. We will denote the right-hand side in (1.1) by $lb^r_d(\Delta)$.

Since this conjecture was made, the dimension of $S^r_d(\Delta)$ has been studied for a long time and Schumaker proved Strang's conjecture in 1979 [10]. Later, Alfeld and Schumaker proved the equality holds for $d \geq 4r + 1$ [1].

This was extended to $d \geq 3r + 2$ [5]. To date, it is known that the formula still holds for $d \geq 3r + 1$ for a generic triangulation [2].

In [9] it was conjectured that the formula actually holds for $d \geq 2r+1$ and in [13] Tohaneanu showed that the "$2r + 1$" conjecture is tight, *i.e.,* there is a triangulation Δ such that $S^r_{2r}(\Delta) > lb^r_{2r}(\Delta)$. We call this the "$2r + 1$" conjecture. In [15] Whiteley showed that $dim\ S^1_3(\Delta) = lb^1_3(\Delta)$ for a generic triangulation Δ.

A. Definitions

For a more general case, we introduce the polyhedral complexes. [3]

Definition I.1. A finite family of polytopes, Δ is called a polyhedral complex if every face of a polytope in Δ is a polytope in Δ and the intersection of any 2 elements of Δ is a face of each of them.

A triangulation (simplicial complex) is a special case of a polyhedral complex where all the polytopes in Δ are simplexes.

Definition I.2. For any face τ in Δ, Star(τ) is the complex of the collection of all the members of Δ containing τ.

Note that for $\Delta \subset \mathbb{R}^2$ if v is an interior vertex then $Star(v)$ has only one interior vertex v. In particular, if Δ has at most one interior vertex, then it is known that in [7]

$$dim\ S^r_d(\Delta) = lb^r_d(\Delta) \quad \text{for } d \geq r.$$

Because we are concerned with bivariate spline spaces in this dissertation, we will assume our polyhedral complex is planar.

From now on, we place a few restrictions on the polyhedral complex Δ.

First, we assume it is pure. That is, all the maximal faces must be of dimension 2. Second, we require that complex is strongly connected, which means for any two 2-faces σ and σ', there is a sequence of 2-faces, $\sigma_1, \ldots, \sigma_k$ such that $\sigma_1 = \sigma$, $\sigma_k = \sigma'$ and $\sigma_i \cap \sigma_{i+1}$ is an edge for $1 \geq i < k$. In this case, we will say that σ_i and σ_{i+1} are adjacent. Finally we assume the complex is simply connected. So, it does not have any holes in it.

Definition I.3. A polyhedral complex is called regular if it satisfies the above conditions.

From now on we will assume that all polyhedral complexes in this dissertation are regular.

B. Conformality conditions

Let Δ be a polyhedral complex in $\mathbb{R}^2$. Suppose $f \in S_d^r(\Delta)$ and $\sigma_i, \sigma_j \in \Delta$ are 2-faces which are adjacent and sharing a common interior edge e_{ij}. Then, when l_{ij} is a defining equation vanishing on e_{ij}, we have

$$l_{ij}^{r+1} | (f_i - f_j)$$

where $f_i = f|_{\sigma_i}$ and $f_j = f|_{\sigma_j}$. [7]

Using this fact gives the following:

Lemma I.4. *(Lau and Stiller [7]) For a polyhedral complex Δ, $S_d^r(\Delta)$ is isomorphic to the subspace $W \subset P_d \oplus \cdots \oplus P_d$ ($f_2(\Delta)$ copies) defined by the conditions that*

$l_{ij}^{r+1}|(f_i - f_j)$ *for every interior edge* $e_{ij} \in \Delta_1^0$ *indexed by* i *and* j *which correspond to the two 2-faces* σ_i *and* σ_j*, and where* l_{ij} *is a defining equation vanishing on* e_{ij}*.*

Proof. See [7] □

Note that $l_{ij}^{r+1}|(f_i - f_j)$ implies that there exists $g_{ij} \in P_{d-r-1}$ satisfying

$$f_i - f_j = l_{ij}^{r+1} g_{ij}. \tag{1.2}$$

Lau and Stiller gave in [7] the necessary and sufficient conditions on the g_{ij}'s to construct an element of $S_d^r(\Delta)$. In order to do that, we need the following Lemma which states the local conformality condition and we will extend it to the triangulation Δ in Theorem I.7.

Lemma I.5. *(Lau and Stiller [7]) Let* v *be an interior vertex of* Δ*. The equation*

$$\sum_{e_{ij}} \pm l_{ij}^{r+1} g_{ij} \equiv 0, \tag{1.3}$$

where the sum runs over all interior edges e_{ij} *having* v *as a face,*
is the necessary and sufficient condition to construct a unique spline f *on Star(*v*).*

Proof. See [7] □

The above condition is called the local conformality condition at v and we will call g_{ij} a smoothing cofactor associated with the edge e_{ij}.

The choice of a sign above is determined by the orientation after we have put an orientation on the given complex in some sense.

To get an idea of how this is done, let's look at the triangulation above which has only one interior vertex (see Figure 1):

Example I.6. Let $f = (f_1, f_2, f_3) \in S_d^r(\Delta)$ and e_{ij} be the interior edge contained in both σ_i and σ_j. Suppose that l_{ij} are defining equations vanishing on e_{ij}.

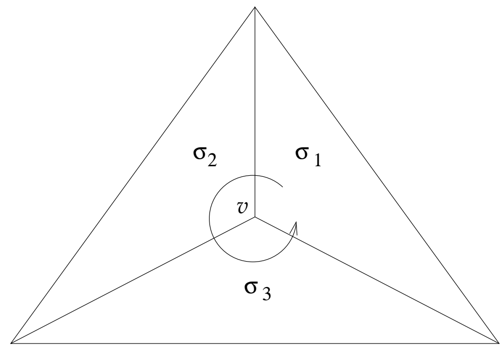

Fig. 1. One interior vertex case

Then, the following relations should hold by Lemma I.5

$$f_1 - f_2 = l_{12}^{r+1} g_{12} \tag{1.4a}$$

$$f_2 - f_3 = l_{23}^{r+1} g_{23} \tag{1.4b}$$

$$f_1 - f_3 = l_{13}^{r+1} g_{13} \tag{1.4c}$$

Then, we have

$$\text{LHS of } ((1.4a) + (1.4b) - (1.4c)) \equiv 0 \tag{1.5}$$

$$\Longrightarrow l_{12}^{r+1} g_{12} + l_{23}^{r+1} g_{23} - l_{13}^{r+1} g_{13} \equiv 0 \tag{1.6}$$

Note that we got the minus sign above because the counterclockwise rotation goes from higher indexed face to lower indexed face.

Conversely, suppose g_{12}, g_{23}, and g_{13} satisfy (1.6) and a polynomial $h \in P_d$ is given. If we define a piecewise polynomial function f by

$$f|_{\sigma_1} = h$$

$$f|_{\sigma_2} = h + l_{12}^{r+1} g_{12}$$

$$f|_{\sigma_3} = f|_{\sigma_2} + l_{23}^{r+1} g_{23} = h + l_{12}^{r+1} g_{12} + l_{23}^{r+1} g_{23},$$

then it is easy to show that $f \in S_d^r(\Delta)$. Now, let V be a subset of $\oplus_3 P_{d-r-1}$ satisfying the local conformality condition at v as in (1.6) then V is a vector space. Moreover, by Lemma I.5 we have

$$S_d^r(\Delta) \cong P_d \oplus V.$$

If we keep this orientation at every interior vertex, we can extend it to the global case.

Theorem I.7. *(Lau and Stiller [7]) Suppose that Δ is a polyhedral complex. Then,*

$$S_d^r(\Delta) \cong P_d \oplus V(\Delta) \subset P_d \oplus (\oplus_{e_{ij} \in \Delta_1^0} P_{d-r-1})$$

where $V(\Delta)$ $(\subset \oplus_{e_{ij} \in \Delta_1^0} P_{d-r-1})$ is the subspace consisting of tuples $(\ldots, g_{ij}, \ldots)$ which satisfy (1.3) in Lemma I.5 at every interior vertex.

Proof. See [7] □

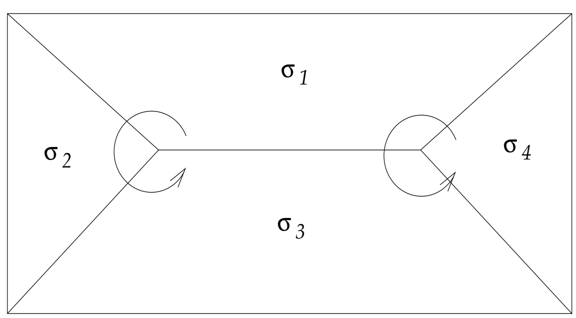

Fig. 2. Two interior vertices case

We call $V(\Delta)$ the cofactor space associated with $S_d^r(\Delta)$. For example, let's look at the polyhedral complex in Figure 2. By Theorem I.7 we have

$$S_d^r(\Delta) \cong P_d \oplus V(\Delta),$$

where the cofactor space $V(\Delta)$ is subject to all local conformality conditions at interior vertices, *i.e.*,

$$V(\Delta) =$$

$$\{(g_{ij}) \in \oplus_5 P_{d-r-1} |\ l_{12}^{r+1} g_{12} + l_{23}^{r+1} g_{23} - l_{13}^{r+1} g_{13} \equiv 0,\ l_{13}^{r+1} g_{13} + l_{34}^{r+1} g_{34} - l_{14}^{r+1} g_{14} \equiv 0\} \tag{1.7}$$

This leads to an alternative approach to the construction of an element of $S_d^r(\Delta)$. Instead of finding polynomial functions on 2-faces, we look instead for smoothing cofactors on edges constrained to satisfy the global conformality condition.

Starting with $h_i \in P_d$ on σ_i, we move to an adjacent 2-face, say σ_j by adding (or subtracting depending on orientation) $l_{ij}^{r+1} g_{ij}$ to (from) h_i to get h_j on σ_j, where $\{g_{ij}\}$ is an element in $V(\Delta)$, *i.e.*, the g_{ij}'s satisfy the equation (1.3) at every interior vertex.

C. Cofactor maps

In this section, we will go deeper into the relationship between cofactor spaces which come from different complexes.

For the given polyhedral complex Δ, let $V(\Delta)$ be a cofactor space associated with $S_d^r(\Delta)$, a subspace of $\oplus_{f_1^0} P_{d-r-1}$, as in the Theorem I.7. Then, for any interior edges $\{e_1, e_2, \ldots, e_f\}$ we can define a linear map :

$$\phi_{\{e_1,e_2,\ldots,e_f\}}(\Delta) : V(\Delta) \to \oplus_f P_{d-r-1}$$

which maps a smoothing cofactor on e_i in $V(\Delta)$ to itself in the ith component. We call this map a cofactor map of $S_d^r(\Delta)$ over $\{e_1, e_2, \ldots, e_f\}$.

Proposition I.8. *Let W be the kernel of a cofactor map of $S_d^r(\Delta)$ over $\{e_1, e_2, \ldots, e_f\}$*

and let Δ' be a polyhedral complex obtained by removing e_i's from Δ. Suppose that Δ' is regular and it has the same interior vertices as Δ has. Then, we have

$$S_d^r(\Delta') \cong P_d \oplus W.$$

Proof. It suffices to show that $V(\Delta') \cong W$ due to Theorem I.7.

We can think of $V(\Delta')$ as a subspace of $\oplus_{f_1^0(\Delta)} P_{d-r-1}$ by putting zero polynomials in the places corresponding to e_i's. Then, $V(\Delta') \subset V(\Delta)$ and $\phi_{\{e_1, e_2, \ldots, e_f\}}(\Delta)|_{V(\Delta')} = 0$. Thus, $V(\Delta') \subset W$.

On the other hand, we can see W as a subspace of $\oplus_{f_1^0(\Delta')} P_{d-r-1}$ by removing zero polynomials corresponding to e_i's. Since Δ and Δ' have the same interior vertices and any element w in W satisfies the global conformality condition from Δ, regarding w as an element in $\oplus_{f_1^0(\Delta')} P_{d-r-1}$, w satisfies the global conformality condition for Δ'. Thus, $W \subset V(\Delta')$. ∎

Example I.9. Look at the two triangulations in Figure 3.

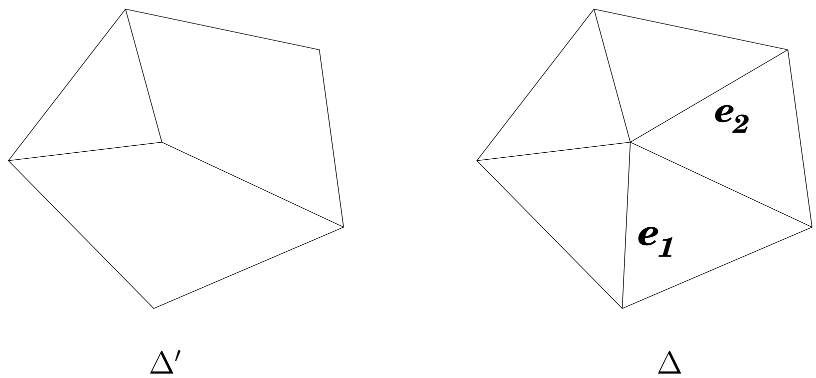

Fig. 3. Example of the kernel of the cofactor map $\phi_{e_1,e_2}(\Delta)$

D. An interior vertex with two edges

In this section, we will show that if the interior vertex v has only two edges with distinct slopes then we can ignore the vertex v in the spline spaces with $d \leq 2r+1$.

As we mentioned before, for a polyhedral complex Δ with only one interior vertex we know that $dim\ S_d^r(\Delta) = lb_d^r(\Delta)$:

Lemma I.10. *Let Δ be a polyhedral complex with one interior vertex v and f_1^0 interior edges. Then for $d \geq r$*

$$dim\ S_d^r(\Delta) = \binom{d+2}{2} + f_1^0\binom{d-r+1}{2} - \left[\binom{d+2}{2} - \binom{r+2}{2}\right] + \delta_e, \qquad (1.8)$$

where $\delta_e = \sum_{j=1}^{d-r}(r+j+1-je)_+$ and e is the number interior edges with distinct slopes emerging from v.

Proof. See [7] □

Lemma I.11. *Let Δ be a polyhedral complex with one interior vertex v and 2 interior edges e_1 and e_2 with distinct slopes. Then, for $r \le d \le 2r+1$ we have*

$$dim\ S_d^r(\Delta) = \binom{d+2}{2}$$

Proof. If $d = r$, then it is clear from (1.8).

If $r < d < 2r$, then from (1.8) in Lemma I.10 we get

$$\begin{aligned} dim\ S_d^r(\Delta) &= \binom{d+2}{2} + 2\binom{d-r+1}{2} - \left[\binom{d+2}{2} - \binom{r+2}{2}\right] \\ &\quad + (r + (r-1) + \cdots + (2r+1-d)) \\ &= 2\binom{d-r+1}{2} + \binom{r+2}{2} + \frac{(3r+1-d)(d-r)}{2} \\ &= \binom{d+2}{2}. \end{aligned}$$

If $2r \le d \le 2r+1$, then we have

$$\begin{aligned} dim\ S_d^r(\Delta) &= \binom{d+2}{2} + 2\binom{d-r+1}{2} - \left[\binom{d+2}{2} - \binom{r+2}{2}\right] + \binom{r+1}{2} \\ &= 2\binom{d-r+1}{2} + \binom{r+2}{2} + \binom{r+1}{2} \\ &= d^2 - 2dr + 2r^2 + d + r + 1 \\ &= \begin{cases} \binom{2r+2}{2} & \text{if } d = 2r \\ \binom{2r+3}{2} & \text{if } d = 2r+1 \end{cases} \\ &= \binom{d+2}{2}. \end{aligned}$$

□

This can be shown by means of conformality condition: let l_1, l_2 be defining equations for two edges and then we need to solve the following conformality condition;

$$l_1^{r+1} g_1 + l_2^{r+1} g_2 \equiv 0 \tag{1.9}$$

where $g_i \in P_{d-r-1}$; however, if $d \leq 2r+1$ then equation (1.9) has no solutions because degree of g_i is not more than r and l_i's are distinct, *i.e.,* splines should be trivial.

Remark I.12. This says that if there are only 2 interior edges emerging from an interior vertex v with distinct slope then the conformality condition at v forces the smoothing cofactors corresponding to two edges to become zero. So if we have such vertex v and edges e_1, e_2, we can ignore $v, e_1,$ and e_2 for the spline space.

Example I.13. Let Δ^1 and Δ^2 be polyhedral complexes as in Figure 4. Then, for $r \leq d \leq 2r+1$

$$S_d^r(\Delta^1) = S_d^r(\Delta^2).$$

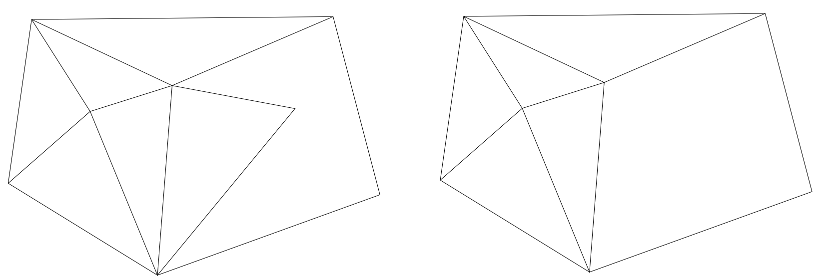

Fig. 4. Δ^1 and Δ^2 in Example I.13

E. Splines and cohomology groups

We now move onto algebraic geometry, where the machinery of sheaves and cohomology provide a generalization of linear algebra that will allow us to approach questions concerning the dimension of $S_d^r(\Delta)$ and its behavior as r, d or Δ change.

For any linear form l_{ij} which vanishes on an interior edge of Δ we can homogenize it and regard it as a homogeneous polynomial of degree 1 over $\mathbb{C}$.

We can also identify the space of complex polynomials of degree d, P_d, with the global sections of the line bundle $\mathcal{O}_{\mathcal{P}^2_{\mathbb{C}}}(d)$ on $\mathcal{P}^2_{\mathbb{C}}$, complex projective 2-space. For notational simplicity, I will omit the subscript $\mathcal{P}^2_{\mathbb{C}}$ in $\mathcal{O}_{\mathcal{P}^2_{\mathbb{C}}}(d)$.

Let Δ be a polyhedral complex. We construct a map of vector bundles on $\mathcal{P}^2_{\mathbb{C}}$

$$\mathcal{O}^{f_1^0}(d-r-1) \to \mathcal{O}^{f_0^0}(d), \tag{1.10}$$

which on the global section maps

$$(\ldots, g_{ij}, \ldots) \longmapsto (\ldots, \sum \pm l_{ij}^{r+1} g_{ij}, \ldots), \tag{1.11}$$

where the signs have been fixed by the orientation of Δ and the choice of an oriented loop around each vertex as before. Here note that the kernel of the above map (1.11) is isomorphic to the cofactor space $V(\Delta)$.

If we twist this map by tensoring with $\mathcal{O}(-d)$ we get

$$\mathcal{O}^{f_1^0}(-r-1) \to \mathcal{O}^{f_0^0}$$

and we denote the kernel of this map by $\mathcal{K}$, the cokernel by $\mathcal{C}$, and the image by $\mathcal{R}$. Then, we have the exact sequence,

$$0 \to \mathcal{K}(d) \to \mathcal{O}^{f_1^0}(d-r-1) \to \mathcal{O}^{f_0^0}(d) \to \mathcal{C}(d) \to 0. \tag{1.12}$$

It breaks down into two short exact sequences:

$$0 \to \mathcal{K}(d) \to \mathcal{O}^{f_1^0}(d-r-1) \to \mathcal{R}(d) \to 0 \tag{1.13}$$

and

$$0 \to \mathcal{R}(d) \to \mathcal{O}^{f_0^0}(d) \to \mathcal{C}(d) \to 0. \tag{1.14}$$

Let's look at the sequence of cohomology groups from (1.13):

$$0 \to H^0(\mathcal{K}(d)) \to H^0(\mathcal{O}^{f_1^0}(d-r-1)) \to H^0(\mathcal{R}(d)).$$

By the definition of the map in (1.10) and (1.11) we have

$$H^0(\mathcal{K}(d)) \cong V(\Delta). \tag{1.15}$$

Then using Theorem I.7 we have

$$S_d^r(\Delta) \cong H^0(\mathcal{K}(d)) \oplus P_d. \tag{1.16}$$

We will use the notation $h^i(\mathcal{F})$ to stand for the dimension of the cohomology group $H^i(\mathcal{F})$. Then from (1.16) we get

$$dim\ S_d^r(\Delta) = h^0(\mathcal{K}(d)) + \binom{d+2}{2} \tag{1.17}$$

So, for the dimension of $S_d^r(\Delta)$ we just need to compute $h^0(\mathcal{K}(d))$.

Let's look at the vector bundle maps from the local conformality conditions at interior vertices and take a direct sum.

For each interior vertex $v_i, i = 1, 2, \ldots, f_0^0$, we can get the following exact sequence from $Star(v_i)$:

$$0 \to \mathcal{K}_i(d) \to \mathcal{O}^{s_i}(d-r-1) \to \mathcal{O}(d) \to \mathcal{C}_i(d) \to 0, \tag{1.18}$$

where s_i is the number of interior edges of $Star(v_i)$. In [7] it was shown that if Δ is a simplicial complex then

$$\mathcal{C}(d) \cong \oplus \mathcal{C}_i(d)$$

which induces

$$\mathcal{R}(d) \cong \oplus \mathcal{R}_i(d).$$

Furthermore, it follows that $H^1(\mathcal{R}(d)) = H^2(\mathcal{K}(d)) = 0$ for $d \geq 2r$ by using the above facts.

Let us look at the long exact sequence of cohomology groups of the exact sequence (1.13):

$$\begin{aligned} 0 &\to H^0(\mathcal{K}(d)) \to H^0(\mathcal{O}^{f_1^0}(d-r-1)) \to H^0(\mathcal{R}(d)) \\ &\to H^1(\mathcal{K}(d)) \to H^1(\mathcal{O}^{f_1^0}(d-r-1)) = 0 \to H^1(\mathcal{R}(d)) \\ &\to H^2(\mathcal{K}(d)) \to H^2(\mathcal{O}^{f_1^0}(d-r-1)). \end{aligned}$$

Similarly, from the exact sequence (1.14) we have the long exact sequence:

$$0 \to H^0(\mathcal{R}(d)) \to H^0(\mathcal{O}^{f_0^0}(d)) \to H^0(\mathcal{C}(d)) \to H^1(\mathcal{R}(d)) = 0.$$

Thus, using these in (1.17) yields that for $d \geq 2r$

$$\begin{aligned} dim\ S_d^r(\Delta) &= \binom{d+2}{2} + h^0(\mathcal{K}(d)) \\ &= \binom{d+2}{2} + h^0(\mathcal{O}^{f_1^0}(d-r-1)) - h^0(\mathcal{R}(d)) + h^1(\mathcal{K}(d)) \\ &= \binom{d+2}{2} + h^0(\mathcal{O}^{f_1^0}(d-r-1)) - [h^0(\mathcal{O}^{f_0^0}(d)) - h^0(\mathcal{C}(d))] + h^1(\mathcal{K}(d)) \end{aligned}$$

$\Longrightarrow$

$$dim\ S_d^r(\Delta) - h^1(\mathcal{K}(d)) = \binom{d+2}{2} + h^0(\mathcal{O}^{f_1^0}(d-r-1)) - h^0(\mathcal{O}^{f_0^0}(d)) + h^0(\mathcal{C}(d)) \quad (1.19)$$

Lau and Stiller [7] showed that the right-hand side of the equation (1.19) is the same as the lower bound in (1.1), $lb_d^r(\Delta)$.

Thus, the "$2r+1$" conjecture is equivalent to showing that

$$H^1(\mathcal{K}(d)) = 0 \text{ for } d \geq 2r+1.$$

Remark I.14. For $d \geq 2r$ if $h^1(\mathcal{K}(d)) = 0$ then $dim\ S_d^r(\Delta) = lb_d^r(\Delta)$.

Moreover, by standard cohomology vanishing theorems [4] if d is sufficiently large then $H^1(\mathcal{K}(d))$ will be zero and so we have $S^r_d(\Delta) = lb^r_d(\Delta)$ for d sufficiently large.

CHAPTER II

PRELIMINARY

A. Introduction

The main idea we are going to use later is splitting the given partition into two pieces and so reduce the given complex to a simpler one. In order to do that, we need to define a new type of polyhedral complex which can be obtained from a triangulation in a certain manner. First, we will define the following:

Definition II.1. For the given complex Δ, an interior edge σ is called totally interior edge if its two end points are interior vertices. Otherwise, it is called a non-totally interior edge.

B. Flawed triangulations

Definition II.2. A polyhedral complex Δ is called flawed triangulation if Δ has only one non-simplicial 2-face σ and σ has a boundary vertex.

Note that for any flawed triangulation Δ with non-simplicial face σ we can construct a triangulation $\bar{\Delta}$ from Δ by adding edges the e_i's inside σ so that in $\bar{\Delta}$ e_i's are non-totally interior edges and $\bar{\Delta}$ and Δ have the same interior vertices. Conversely, we can obtain Δ from $\bar{\Delta}$ by removing non-totally interior edges e_i's. See Figure 5.

In the next theorem we will show that for any flawed triangulation Δ the dimension of the spline space on it has a lower bound which has the same form as the triangulation case. *i.e.*, it is bounded below by $lb_d^r(\Delta)$.

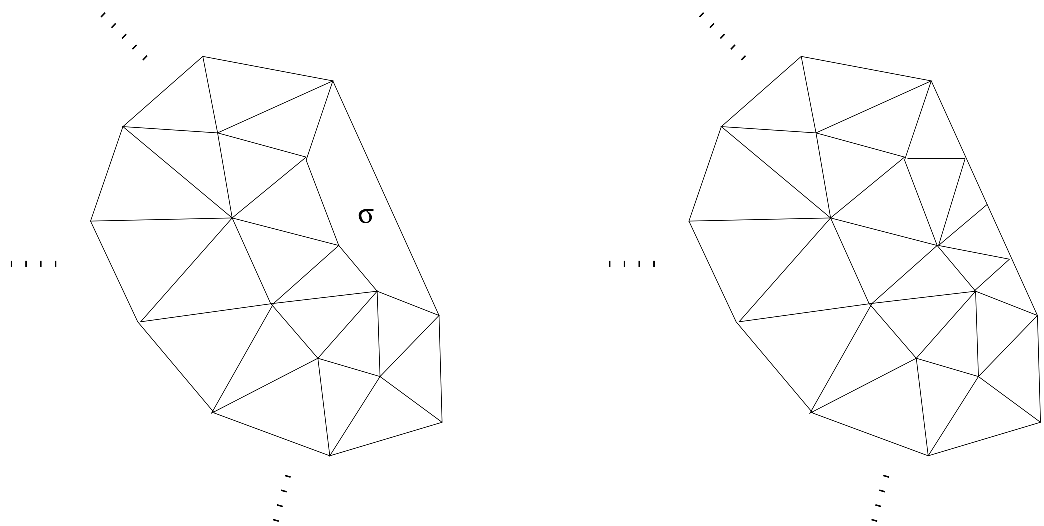

Fig. 5. Δ and $\bar{\Delta}$

Theorem II.3. *Let Δ be a flawed triangulation. Then,*

$$dim\ S_d^r(\Delta) \geq lb_d^r(\Delta).$$

Proof. Let $\bar{\Delta}$ be a triangulation from which we are able to get Δ by removing some non-totally interior edges $e_1, \ldots, e_m$. Set $E = \{e_i\}_{i=1}^m$.

By Proposition I.8 we have

$$ker\ \phi_E(\bar{\Delta}) \cong V(\Delta)$$

which implies

$$dim\ V(\Delta) = dim\ V(\bar{\Delta}) - dim\ img\ \phi_E(\bar{\Delta}) \tag{2.1}$$

because we have

$$dim\ ker\ \phi_E(\bar{\Delta}) = dim\ V(\bar{\Delta}) - dim\ img\ \phi_E(\bar{\Delta}).$$

Let $v_1, \ldots, v_n$ be interior vertices of $\bar{\Delta}$ such that v_j is one of the end points of some e_i and let E_j be the collection of e_i's which are emerging from v_j. Then, we have

$\{E_j\}_{j=1}^n$ is disjoint and $\cup E_j = E$ because e_i's are not totally interior edges.

Let $\Delta_{v_j} = Star(v_j)$ in Δ and $\bar{\Delta}_{v_j} = Star(v_j)$ in $\bar{\Delta}$. Now for each v_j consider the map $\phi_{E_j}(\bar{\Delta}_{v_j})$ and its image. Since the global conformality condition is subject to the local conformality condition at every interior vertex, we have

$$img\ \phi_E(\bar{\Delta}) \subset \oplus_{j=1}^n img\ \phi_{E_j}(\bar{\Delta}_{v_j}) \tag{2.2}$$

which gives

$$dim\ img\ \phi_E(\bar{\Delta}) \leq \sum_{j=1}^n dim\ img\ \phi_{E_j}(\bar{\Delta}_{v_j}). \tag{2.3}$$

Using this in equation (2.1) gives

$$dim\ V(\Delta) = dim\ V(\bar{\Delta}) - dim\ img\ \phi_E(\Delta) \tag{2.4}$$

$$\geq dim\ V(\bar{\Delta}) - \sum_{j=1}^n dim\ img\ \phi_{E_j}(\bar{\Delta}_{v_j}) \tag{2.5}$$

$$\geq lb_d^r(\bar{\Delta}) - \binom{d+2}{2} - \sum_{j=1}^n dim\ img\ \phi_{E_j}(\bar{\Delta}_{v_j}). \tag{2.6}$$

In the last step above we used that $dim\ V(\bar{\Delta}) + \binom{d+2}{2} = dim S_d^r(\bar{\Delta}) \geq lb_d^r(\bar{\Delta})$. Since $\bar{\Delta}_{v_j}$ and Δ_{v_j} have only one interior vertex v_j, we can compute $dim\ img\ \phi_{E_j}(\bar{\Delta}_{v_j})$ by using the fact that $dim\ img\ \phi_{E_j}(\bar{\Delta}_{v_j}) = dim\ V(\bar{\Delta}_{v_j}) - dim\ V(\Delta_{v_j})$. Then,

$$dim\ img\ \phi_{E_j}(\bar{\Delta}_{v_j}) = |E_j|\binom{d-r+1}{2} + \delta(\bar{t}_j) - \delta(t_j), \tag{2.7}$$

where $\bar{t}_j$ is the number of distinct slopes at v_j in $\bar{\Delta}$, t_j is the number of distinct slopes at v_j in Δ, and $\delta(t) = \sum_{k=1}^r (r+k+1-kt)_+$.

Then, from (2.6) we get

$$\begin{aligned} dim\ V(\Delta) &\geq lb_d^r(\bar{\Delta}) - \binom{d+2}{2} - \sum_{j=1}^{n}(|E_j|\binom{d-r+1}{2} + \delta_{\bar{t}_j} - \delta_{t_j}) \\ &= lb_d^r(\bar{\Delta}) - \binom{d+2}{2} - m\binom{d-r+1}{2} - \sum_{j=1}^{n}(\delta_{\bar{t}_j} - \delta_{t_j}) \\ &= lb_d^r(\Delta) - \binom{d+2}{2}. \end{aligned}$$

because, even though $\bar{\Delta}$ has m more interior edges than Δ has, Δ and $\bar{\Delta}$ have the same interior vertices and same number of distinct slopes at all interior vertices except v_j's. Therefore,

$$dim\ S_d^r(\Delta) = dim\ V(\Delta) + \binom{d+2}{2} \geq lb_d^r(\Delta). \tag{2.8}$$

□

For a simplicial complex Δ, we have $\mathcal{C}(d) \cong \oplus \mathcal{C}_i(d)$ and $\mathcal{R}(d) \cong \oplus \mathcal{R}_i(d)$. However, these decompositions are also true for a flawed triangulation. Next, we will see why.

Definition II.4. Let Δ be a polyhedral complex and $p \in \mathbb{R}^2$. $G_p(\Delta)$ is a graph where vertices correspond to those $\sigma \in \Delta_2$ such that there exists an edge of σ whose linear span contains p. Two vertices of $G_p(\Delta)$ are joined if and only if the corresponding 2-faces share a common edge whose linear span contains p.

In [8], it was shown that the graph $G_p(\Delta)$ is homotopic to a disjoint union of circles and segments. In [7] Lau and Stiller showed that the skyscraper sheaf $\mathcal{C}$ is supported at a point p whose corresponding graph $G_p(\Delta)$ has a component homotopic to a circle.

Now look at a flawed triangulation Δ. Suppose that $G_p(\Delta)$ has a component ψ which is homotopic to a circle. Note that by the Definition II.2 ψ has a node , say v_σ, which should be corresponding to a triangular face σ in Δ. Then, two edges connected to v_σ are corresponding to edges belonging to σ in Δ. Since σ is a triangle, p should be an interior vertex contained in σ.

This argument gives $\mathcal{C}$ is supported only on interior vertices of Δ and we have the following:

Lemma II.5. *Let Δ be a flawed triangulation with interior vertices $v_1, v_2, \dots, v_{f_0^0}$. Let $\mathcal{R}, \mathcal{C}$ be the sheaves discussed above from Δ and $\mathcal{R}_i, \mathcal{C}_i$ be from $Star(v_i)$. Then, we have*

$$\mathcal{C} \cong \oplus \mathcal{C}_i \text{ and } R \cong \oplus \mathcal{R}_i.$$

Moreover, for $d \geq 2r$

$$H^1(\mathcal{R}(d)) = H^2(\mathcal{K}(d)) = 0.$$

Thus, the statements in Remark I.14 still hold for a regular flawed triangulation. That is, for $d \geq 2r$ if $H^1(\mathcal{K}(d)) = 0$ then $dim\ S_d^r(\Delta) = lb_d^r(\Delta)$.

C. Splitting method

Let Δ be a flawed triangulation with interior vertices $\Delta_0^0 = \{v_1, v_2, \dots, v_{f_0^0}\}$. Let A and B be disjoint subsets of Δ_0^0 such that $A \cup B = \Delta_0^0$ and suppose $\Delta^1 = \cup_{v_i \in A} \text{Star}(v_i)$ and $\Delta^2 = \cup_{v_i \in B} \text{Star}(v_i)$ are regular flawed triangulations.

Then, let $e_1, e_2, \dots, e_m$ be interior edges of Δ contained in both Δ^1 and Δ^2 and let W_1 and W_2 be the images of the cofactor maps $\phi_{\{e_1,\dots,e_m\}}(\Delta^1)$ and $\phi_{\{e_1,\dots,e_m\}}(\Delta^2)$, respectively. Note that W_1 and W_2 are subspaces of vector space $\oplus_m \mathcal{P}_{d-r-1}$.

Also, let the kernel bundles that arise from Δ, Δ^1, and Δ^2 be denoted by $\mathcal{K}, \mathcal{K}_1$, and $\mathcal{K}_2$, respectively. Then, we have the following Lemma:

Lemma II.6. *For given r and d, suppose that $H^1(\mathcal{K}_1(d)) = 0$ and $H^1(\mathcal{K}_2(d)) = 0$.*

Then,

$$W_1 + W_2 = \oplus_m \mathcal{P}_{d-r-1} \Longleftrightarrow H^1(\mathcal{K}(d)) = 0.$$

In particular, if $d \geq 2r$ then we have

$$W_1 + W_2 = \oplus_m \mathcal{P}_{d-r-1} \Longleftrightarrow dim\ S_d^r(\Delta) = lb_d^r(\Delta).$$

Proof. Let f_1^0, f_{1s}^0, and f_{1t}^0 be the numbers of interior edges of Δ, Δ^1, and Δ^2, respectively.

As before, from Δ, Δ^1, and Δ^2 we can construct the exact sequences:

$$0 \to \mathcal{K}(d) \to \mathcal{O}^{f_1^0}(d-r-1) \to \mathcal{R}(d) \to 0 \tag{2.9a}$$

$$0 \to \mathcal{K}_1(d) \to \mathcal{O}^{f_{1s}^0}(d-r-1) \to \mathcal{R}_1(d) \to 0 \tag{2.9b}$$

$$0 \to \mathcal{K}_2(d) \to \mathcal{O}^{f_{1t}^0}(d-r-1) \to \mathcal{R}_2(d) \to 0. \tag{2.9c}$$

Take the direct sum of (2.9b) and (2.9c), we have

$$0 \to \mathcal{K}_1(d) \oplus \mathcal{K}_2(d) \to \mathcal{O}^{f_{1s}^0 + f_{1t}^0}(d-r-1) \to \mathcal{R}_1(d) \oplus \mathcal{R}_2(d) \to 0. \tag{2.10}$$

Now consider a map of vector bundles

$$\mathcal{O}^{f_1^0}(d-r-1) \to \mathcal{O}^{f_{1s}^0 + f_{1t}^0}(d-r-1)$$

which on the global section maps

$$(g_1, \ldots, g_{f_1^0}) \longmapsto (g_1, \ldots, g_m, g_1, \ldots, g_m, g_{m+1}, \ldots, g_{f_1^0}),$$

where the map sends the first m coordinates twice and $g_1, \ldots, g_m$ are cofactor polynomials on common interior edges of Δ^1 and Δ^2, $e_1, \ldots, e_m$; the first m polynomials in the target are for the interior edges, e_i in Δ^1 and the second m polynomials in the target are for the interior edges, e_i in Δ^2. This map is obviously injective with cokernel $\mathcal{O}^m(d-r-1)$ and the map

$$\mathcal{O}^{f^0_{1s}+f^0_{1t}}(d-r-1) \to \mathcal{O}^m(d-r-1) \tag{2.11}$$

is defined on the global section by

$$(g_1, \ldots, g_{f^0_{1s}+f^0_{1t}}) \longmapsto (g_1 - g_{m+1}, \ldots, g_m - g_{2m}).$$

Also, it induces an injective map $\beta : \mathcal{K}(d) \to \mathcal{K}_1(d) \oplus \mathcal{K}_2(d)$. Denote the cokernel of this map by $\mathcal{J}$.

Then, together with Lemma II.5, we have the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{K}(d) & \longrightarrow & \mathcal{O}^{f^0_1}(d-r-1) & \longrightarrow & \mathcal{R}(d) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\theta} & & \| & & \\
0 & \longrightarrow & \mathcal{K}_1(d) \oplus \mathcal{K}_2(d) & \longrightarrow & \mathcal{O}^{f^0_{1s}+f^0_{1t}}(d-r-1) & \longrightarrow & \mathcal{R}_1(d) \oplus \mathcal{R}_2(d) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{J} & & \mathcal{O}^m(d-r-1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

In the above diagram we have $\mathcal{R}(d) \cong \mathcal{R}_1(d) \oplus \mathcal{R}_2(d)$ by Lemma II.5.

By the snake Lemma, we get $\mathcal{J} \cong \mathcal{O}^m(d-r-1)$. Let us look at the long exact sequence of cohomology groups of the exact sequence involving the map β:

$$\begin{aligned}0 \to H^0(\mathcal{K}(d)) \to H^0(\mathcal{K}_1(d)) \oplus H^0(\mathcal{K}_2(d)) \xrightarrow{\alpha} H^0(\mathcal{J}) \cong H^0(\mathcal{O}^m(d-r-1)) \\ \to H^1(\mathcal{K}(d)) \to H^1(\mathcal{K}_1(d)) \oplus H^1(\mathcal{K}_2(d)).\end{aligned} \tag{2.12}$$

Here, we have $H^1(\mathcal{K}_1(d)) \oplus H^1(\mathcal{K}_2(d))) = 0$ by our assumption.

Thus, we have that the map α in (2.12) is surjective *iff* $H^1(\mathcal{K}(d)) = 0$.

Notice that $H^0(\mathcal{K}_1(d)) \cong V(\Delta^1)$ and $H^0(\mathcal{K}_2(d)) \cong V(\Delta^2)$. Also, α is equivalent to the composition map of the following two maps

$$H^0(\mathcal{K}_1(d)) \oplus H^0(\mathcal{K}_2(d)) \hookrightarrow H^0(O^{f^0_{1s}+f^0_{1t}}(d-r-1)) \twoheadrightarrow H^0(\mathcal{O}^m(d-r-1)),$$

where the second map is from θ in the above diagram.
So, the image of α is isomorphic to $W_1 + W_2$.

Therefore, we have

$$W_1 + W_2 = \oplus_m \mathcal{P}_{d-r-1} (\cong H^0(\mathcal{O}^m(d-r-1))) \Longleftrightarrow H^1(\mathcal{K}(d)) = 0.$$

In particular, if $d \geq 2r$ then by Remark I.14 we have

$$W_1 + W_2 = \oplus_m \mathcal{P}_{d-r-1} \Longleftrightarrow dim\ S^r_d(\Delta) = lb^r_d(\Delta).$$

□

In other words, for $d \geq 2r$ if $dim\ S^r_d(\Delta^1) = lb^r_d(\Delta^1),$ and $dim\ S^r_d(\Delta^2) = lb^r_d(\Delta^2)$ then $img\ \phi_E(\Delta^1) + img\ \phi_E(\Delta^2) = \oplus_{|E|} P_{d-r-1}$ implies $dim\ S^r_d(\Delta) = lb^r_d(\Delta)$.

Corollary II.7. *Let Δ, Δ_1, and Δ_2 be the same as the above Theorem. If Δ_1 and Δ_2 have no common edges, then*

$$H^1(\mathcal{K}_1(d)) = H^1(\mathcal{K}_2(d)) = 0 \Longrightarrow H^1(K(d)) = 0.$$

Note that if the given complex Δ is a triangulation, then Δ_1 and Δ_2 will be triangulations and we will get the same conclusion as well.

This gives some idea of relationship of the dimension of the spline spaces on Δ to those of its subtriangulations.

Lemma II.8. *Let Δ be a flawed triangulation with a non-simplicial 2-face σ and let v be an interior vertex of Δ belonging to σ.*

Let $\bar{\Delta}$ be a complex obtained by adding some edges emerging from v inside of σ without generating any interior vertices. (See Figure 6)

Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be kernel bundles from Δ and $\bar{\Delta}$, respectively. Then, for fixed r and $d \geq 2r$ we have

$$H^1(\mathcal{K}(d)) = 0 \Longrightarrow H^1(\bar{\mathcal{K}}(d)) = 0$$

and so

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta) \Longrightarrow dim\ S_d^r(\bar{\Delta}) = lb_d^r(\bar{\Delta}).$$

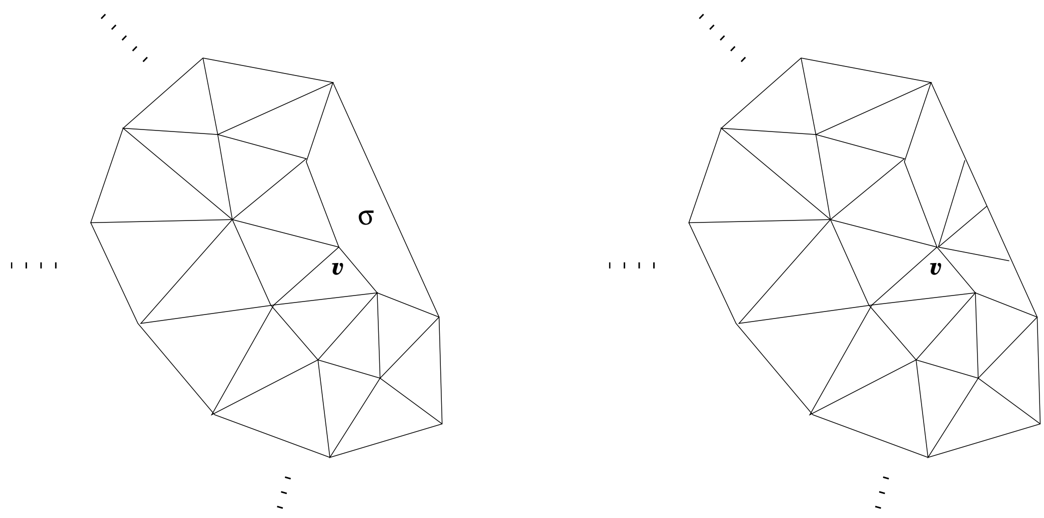

Fig. 6. Δ and $\bar{\Delta}$

Proof. Let $e_1, e_2, \ldots, e_m$ be additional edges inside of σ and let f_1^0 be the number of interior edges of Δ.

Consider the following map:

$$\mathcal{O}^{f_1^0}(d-r-1) \rightarrow \mathcal{O}^{f_1^0+m}(d-r-1) \tag{2.13}$$

which on the global section maps

$$(g_1, \ldots, g_{f_1^0}) \longmapsto (g_1, \ldots, g_{f_1^0}, 0, \ldots, 0). \tag{2.14}$$

This map is obviously injective with cokernel $\mathcal{O}^m(d-r-1)$. Also, it induces an injective map $\mathcal{K}(d) \to \bar{\mathcal{K}}(d)$. Let $\mathcal{J}$ be the cokernel of this map.

Then, we have the following diagram:

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}(d) & \longrightarrow & \mathcal{O}^{f_1^0}(d-r-1) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bar{\mathcal{K}}(d) & \longrightarrow & \mathcal{O}^{f_1^0+m}(d-r-1) \\ & & \downarrow & & \downarrow \\ & & \mathcal{J} & & \mathcal{O}^m(d-r-1) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

Since $H^1(\mathcal{K}(d)) = 0$, we have the exact sequence:

$$0 \to H^0(\mathcal{K}(d)) \to H^0(\bar{\mathcal{K}}(d)) \to H^0(\mathcal{J}) \to 0. \tag{2.15}$$

Note that, by equation (2.14), $H^0(\mathcal{J})$ is isomorphic to the image of cofactor map, $\phi_{\{e_1,\ldots,e_m\}}(\bar{\Delta})$.

Let Δ_v be the $Star(v)$ in Δ and $\bar{\Delta}_v$ be the $Star(v)$ in $\bar{\Delta}$. Also, let $\mathcal{K}_v$ and $\bar{\mathcal{K}}_v$ be the kernel bundles from Δ_v and $\bar{\Delta}_v$, respectively. Then in the similar way as above we have the following diagram:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{K}_v(d) & \longrightarrow & \mathcal{O}^{g_1^0}(d-r-1) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \bar{\mathcal{K}}_v(d) & \longrightarrow & \mathcal{O}^{g_1^0+m}(d-r-1) \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{J}_v & & \mathcal{O}^m(d-r-1) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

where g_1^0 be the number of interior edges of Δ_v.

Since Δ_v and $\bar{\Delta}_v$ are complexes with only one interior vertex, we have

$$H^1(\mathcal{K}_v(d)) = H^1(\bar{\mathcal{K}}_v(d)) = 0.$$

As a result, we have

$$0 \to H^0(\mathcal{K}_v(d)) \to H^0(\bar{\mathcal{K}}_v(d)) \to H^0(\mathcal{J}_v) \to 0. \tag{2.16}$$

It is easy to see that $H^0(\mathcal{J}_v)$ is isomorphic to the image of cofactor map, $\phi_{\{e_1,\ldots,e_m\}}(\bar{\Delta}_v)$.

Since $\bar{\Delta}_v$ is a subtriangulation of $\bar{\Delta}$ and the global conformality condition from $\bar{\Delta}_v$ is subject to only one local conformality condition at v, we have

$$H^0(\mathcal{J}) \subset H^0(\mathcal{J}_v)$$

which implies

$$h^0(\mathcal{J}) \leq h^0(\mathcal{J}_v). \tag{2.17}$$

Consequently, starting from (2.15) and using (2.16) and (2.17), we have

$$\begin{aligned} h^0(\bar{\mathcal{K}}(d)) &= h^0(\mathcal{K}(d)) + h^0(\mathcal{J}) \\ &\leq h^0(\mathcal{K}(d)) + h^0(\mathcal{J}_v) \\ &= h^0(\mathcal{K}(d)) + h^0(\bar{\mathcal{K}}(d)) - h^0(\mathcal{K}_v(d)). \end{aligned} \tag{2.18}$$

Since we have

$$H^1(\mathcal{K}(d)) = H^1(\bar{\mathcal{K}}_v(d)) = H^1(\mathcal{K}_v(d)) = 0,$$

we can compute the last values in the equation (2.18);

$$\begin{aligned} h^0(\mathcal{K}(d)) = dimV(\Delta) = lb_d^r(\Delta) - \binom{d+2}{2} \\ h^0(\bar{\mathcal{K}}(d)) = dimV(\bar{\Delta}) = lb_d^r(\bar{\Delta}) - \binom{d+2}{2} \\ h^0(\mathcal{K}_v(d)) = dimV(\Delta_v) = lb_d^r(\Delta_v) - \binom{d+2}{2}. \end{aligned}$$

It turns out that $h^0(\mathcal{K}(d)) + h^0(\bar{\mathcal{K}}_v(d)) - h^0(\mathcal{K}_v(d))$ is equal to the lower bound of the dimension of the spline space on $\bar{\Delta}$ minus $\binom{d+2}{2}$. Therefore,

$$h^0(\bar{\mathcal{K}}(d)) = lb_d^r(\bar{\Delta}) - \binom{d+2}{2}$$

which implies

$$dim\ S_d^r(\bar{\Delta}) = lb_d^r(\bar{\Delta}) \text{ and } H^1(\bar{\mathcal{K}}(d)) = 0.$$

□

Lemma II.9. *Let Δ be a flawed triangulation with a non-simplicial 2-face σ. Let $\bar{\Delta}$ be a triangulation obtained by adding some edges inside of σ without generating any interior vertices and where the added edges are not totally interior in $\bar{\Delta}$.*

Then, for fixed r and $d \geq 2r$ we have

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta) \Longrightarrow dim\ S_d^r(\bar{\Delta}) = lb_d^r(\bar{\Delta}).$$

Proof. Let $\mathcal{K}$ and $\bar{\mathcal{K}}$ be kernel bundles from Δ and $\bar{\Delta}$, respectively. Let $v_1, v_2, \ldots, v_k$ be interior vertices of Δ belonging to σ.

Let $\bar{\Delta}^1$ be a flawed triangulation obtained by adding edges connected to v_1 and $\bar{\mathcal{K}}_1$ be kernel bundle from $\bar{\Delta}^1$.

Then by Lemma II.8,

$$H^1(\bar{\mathcal{K}}_1(d)) = 0.$$

Keep doing this until we have $H^1(\bar{\mathcal{K}}(d)) = 0$. □

CHAPTER III

INDUCTION

A. Introduction

We are interested in the dimension of the space $S_d^r(\Delta)$ and the extent to which its dimension depends on Δ. This is a rather subtle question because a small perturbation of the interior vertices can result in a jump in the dimension.

A polyhedral complex is said to be generic if all sufficiently small perturbations of the vertices do not change the dimension of the spline space.

Let Δ be a polyhedral complex. Let $\mathcal{M}_\Delta$ be the moduli space of continuous deformation of Δ. In [7], it was shown that $\mathcal{M}_\Delta$ is an open set (in the Zariski topology) in $\mathbb{R}^{2f_0}$. Here f_0 is the number of vertices in Δ.

Theorem III.1. *$dim S_d^r(\Delta)$ will be an upper semi-continuous function for the natural Zariski topology on $\mathcal{M}_\Delta$ and there will be a generic dimension which will be minimum.*

Proof. See [7]. □

We will be considering if it is possible to use induction in some way to prove the "$2r+1$" conjecture. In this chapter we will try to use induction on the number of interior vertices to prove "$2r+1$" conjecture for generic triangulations.

B. The idea of induction

To grasp the idea, let's look at the following example. Let Δ and $\bar{\Delta}$ be triangulations as in Figure 7. Here, we obtain $\bar{\Delta}$ from Δ by adding three edges $\bar{e}_1, \bar{e}_2, \bar{e}_3$ inside σ. Let Δ' be a triangulation obtained by removing the simplex σ from Δ. Note that Δ and $\bar{\Delta}$ have the same interior vertices; furthermore, they have the same slopes at

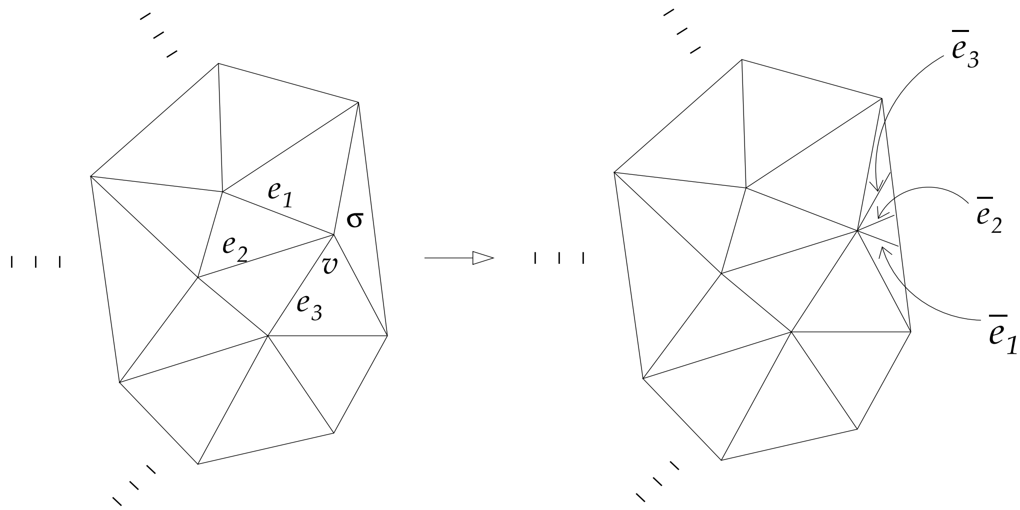

Fig. 7. Adding edges emerging from v

every interior vertex.

As before, we are focusing on the kernel sheaves. Let $\bar{\mathcal{K}}, \mathcal{K}'$, and $\mathcal{K}_0$ be the kernel sheaves from $\bar{\Delta}, \Delta'$, and $Star(v)$ (in Δ), respectively. (See Figure 8)

From Δ' we have the exact sequence

$$0 \to \mathcal{K}'(d) \to \mathcal{O}^{f_1^0-2}(d-r-1) \to \mathcal{O}^{f_0^0-1}(d) \to \mathcal{C}'(d) \to 0 \tag{3.1}$$

which gives two short exact sequences:

$$0 \to \mathcal{K}'(d) \to \mathcal{O}^{f_1^0-2}(d-r-1) \to \mathcal{R}'(d) \to 0 \tag{3.2}$$

and

$$0 \to \mathcal{R}'(d) \to \mathcal{O}^{f_0^0-1}(d) \to \mathcal{C}'(d) \to 0, \tag{3.3}$$

where $\mathcal{R}'$ is the image sheaf.

Also, from $Star(v)$ we have

$$0 \to \mathcal{K}_0(d) \to \mathcal{O}^5(d-r-1) \to \mathcal{O}(d) \to \mathcal{C}_0(d) \to 0 \tag{3.4}$$

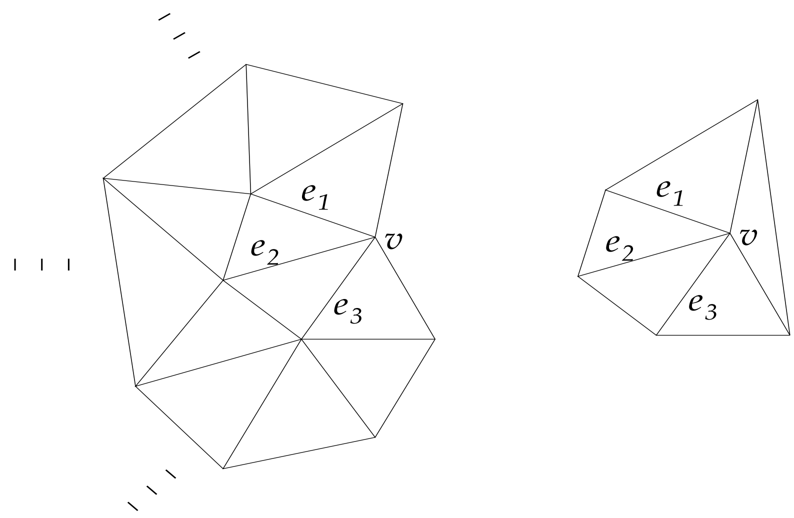

Fig. 8. Δ' and $Star(v)$

which gives two short exact sequences:

$$0 \to \mathcal{K}_0(d) \to \mathcal{O}^5(d-r-1) \to \mathcal{R}_0(d) \to 0 \tag{3.5}$$

and

$$0 \to \mathcal{R}_0(d) \to \mathcal{O}(d) \to \mathcal{C}_0(d) \to 0, \tag{3.6}$$

where $\mathcal{R}_0$ is the image sheaf.

Taking a direct sum of (3.2) and (3.5), we have

$$0 \to \mathcal{K}'(d) \oplus \mathcal{K}_0(d) \to \mathcal{O}^{f_1^0-2}(d-r-1) \oplus \mathcal{O}^5(d-r-1) \to \mathcal{R}'(d) \oplus \mathcal{R}_0(d) \to 0 \tag{3.7}$$

Next, from $\bar{\Delta}$ we have the following two exact sequences:

$$0 \to \bar{\mathcal{K}}(d) \to \mathcal{O}^{f_1^0+3}(d-r-1) \to \bar{\mathcal{R}}(d) \to 0 \tag{3.8}$$

and

$$0 \to \bar{\mathcal{R}}(d) \to \mathcal{O}^{f_0^0}(d) \to \bar{\mathcal{C}}(d) \to 0. \tag{3.9}$$

Define a map of vector bundles

$$\phi : \mathcal{O}^{f_1^0-2}(d-r-1) \oplus \mathcal{O}^5(d-r-1) \to \mathcal{O}^{f_1^0+3}(d-r-1) \tag{3.10}$$

by

$$(g_1,\ldots,g_{f_1^0-2},h_1,\ldots,h_5) \longmapsto (g_1,\ldots,g_{f_1^0-2},h_1,h_2,h_3-g_{f_1^0-4},h_4-g_{f_1^0-3},h_5-g_{f_1^0-2}),$$

where $g_{f_1^0-4}, g_{f_1^0-3}, g_{f_1^0-2}$ correspond to the edges e_1, e_2, e_3 and h_3, h_4, h_5 correspond to the edges $\bar{e_1}, \bar{e_2}, \bar{e_3}$.

Then ϕ is an isomorphism and we have the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & \mathcal{K}'(d) \oplus \mathcal{K}_0(d) & \to & \mathcal{O}^{f_1^0-2}(d-r-1) \oplus \mathcal{O}^5(d-r-1) & \to & \mathcal{R}'(d) \oplus \mathcal{R}_0(d) & \to & 0 \\
 & & \downarrow & & \downarrow \phi & & \downarrow & & \\
0 & \to & \bar{\mathcal{K}}(d) & \to & \mathcal{O}^{f_1^0+3}(d-r-1) & \to & \bar{\mathcal{R}}(d) & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

We are now going to show $\mathcal{R}'(d) \oplus R_0(d) \cong \bar{\mathcal{R}}(d)$.

To do that let's look at the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{R}'(d) \oplus \mathcal{R}_0(d) & \to & \mathcal{O}^{f_0^0-1}(d) \oplus \mathcal{O}(d) & \to & \mathcal{C}'(d) \oplus \mathcal{C}_0(d) & \to & 0 \\
 & & \downarrow & & \downarrow i & & \downarrow & & \\
0 & \to & \bar{\mathcal{R}}(d) & \to & \mathcal{O}^{f_0^0}(d) & \to & \bar{\mathcal{C}}(d) & \to & 0
\end{array}$$

Here, i is an identity map. We have $\mathcal{C}'(d) \oplus C_0(d) \cong \bar{\mathcal{C}}(d)$ because they are skyscraper sheaves and supported on the interior vertices. Therefore, we have $\mathcal{R}'(d) \oplus R_0(d) \cong \bar{\mathcal{R}}(d)$ because they have the same slopes at every interior vertex.

Consequently, for $d \geq r$ we have $H^1(\mathcal{K}'(d)) = 0$ iff $H^1(\bar{\mathcal{K}}(d)) = 0$ because it is already known that $H^1(\mathcal{K}_0(d)) = 0$ for $d \geq r$.

In other words, $dimS_d^r(\Delta') = lb_d^r(\Delta')$ iff $dimS_d^r(\bar{\Delta}) = lb_d^r(\bar{\Delta})$.

Next, consider the following diagram:

$$\begin{array}{cccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\Delta: & 0 & \to & \mathcal{K}(d) & \to & \mathcal{O}^{f_1^0}(d-r-1) & \to & \mathcal{R}(d) & \to & 0 \\
 & & & \downarrow & & \downarrow j & & \downarrow \cong & & \\
\bar{\Delta}: & 0 & \to & \bar{\mathcal{K}}(d) & \to & \mathcal{O}^{f_1^0+3}(d-r-1) & \to & \bar{\mathcal{R}}(d) & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \to & \mathcal{I} & \xrightarrow{\cong} & \mathcal{O}^3(d-r-1) & \to & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}$$

Here, j is the natural injective map. Again, here we have $\mathcal{R}(d) \cong \bar{\mathcal{R}}(d)$ because any corresponding pair of vertices in Δ and $\bar{\Delta}$ has the same slopes.

Using the long exact sequences of cohomologies from the left column sequence, we have

$$0 \to H^0(\mathcal{K}(d)) \to H^0(\bar{\mathcal{K}}(d)) \xrightarrow{\psi} H^0(\mathcal{O}^3(d-r-1)) \to H^1(\mathcal{K}(d)) \to H^1(\bar{\mathcal{K}}(d)) \to 0.$$

The map ψ can be interpreted as the projection map from $V(\bar{\Delta})$, smooth cofactor space, to degree $d-r-1$ homogeneous polynomial spaces over the additional interior edges. (See Figure 9.) Recall that it is called the cofactor map over $\bar{e}_1, \bar{e}_2, \bar{e}_3$ and denoted by $\phi_{\{\bar{e}_1,\bar{e}_2,\bar{e}_3\}}(\bar{\Delta})$.

So, assuming $dimS_d^r(\Delta') = lb_d^r(\Delta')$, we get that ψ is onto if and only if $H^1(\mathcal{K}(d)) = 0$ (*i.e.*, ψ is onto if and only if $dimS_d^r(\Delta) = lb_d^r(\Delta)$.) This gives an idea how to use

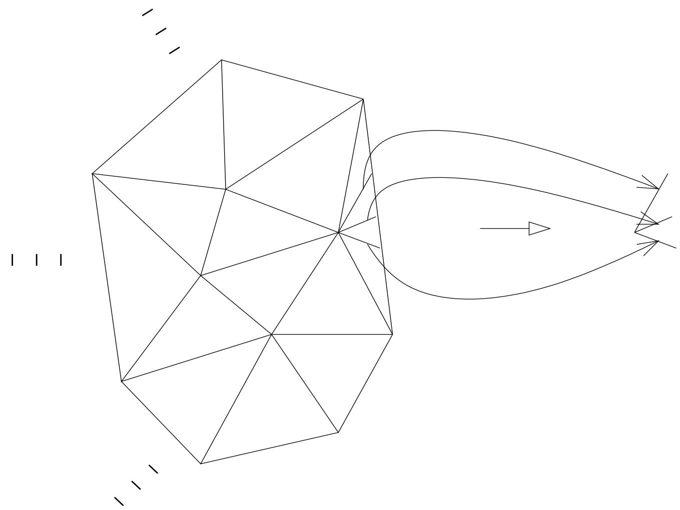

Fig. 9. The map ψ

induction. However, the given triangulation Δ may not allow us to construct $\bar{\Delta}$. However, if Δ is generic we can use Theorem III.1 to make $\tilde{\Delta}$ by a continuous deformation of Δ so that $S_d^r(\tilde{\Delta})$ and $S_d^r(\Delta)$ have the same dimension and we can construct $\overline{\tilde{\Delta}}$.

C. Necessary and sufficient condition for $dim\ S_d^r(\Delta) = lb_d^r(\Delta)$ for a generic triangulation

In this section, we give equivalent conditions for $dim\ S_d^r(\Delta) = lb_d^r(\Delta)$ when Δ is a generic triangulation.

Definition III.2. If a simplex in Δ has a boundary edge then it is called a boundary simplex.

For a given generic triangulation Δ and a boundary simplex σ in Δ, we denote by $\bar{\Delta}_\sigma$ the triangulation obtained by adding interior edges inside σ which are extended from existing total interior edges. (If necessary, we deform Δ to get $\tilde{\Delta}$, before

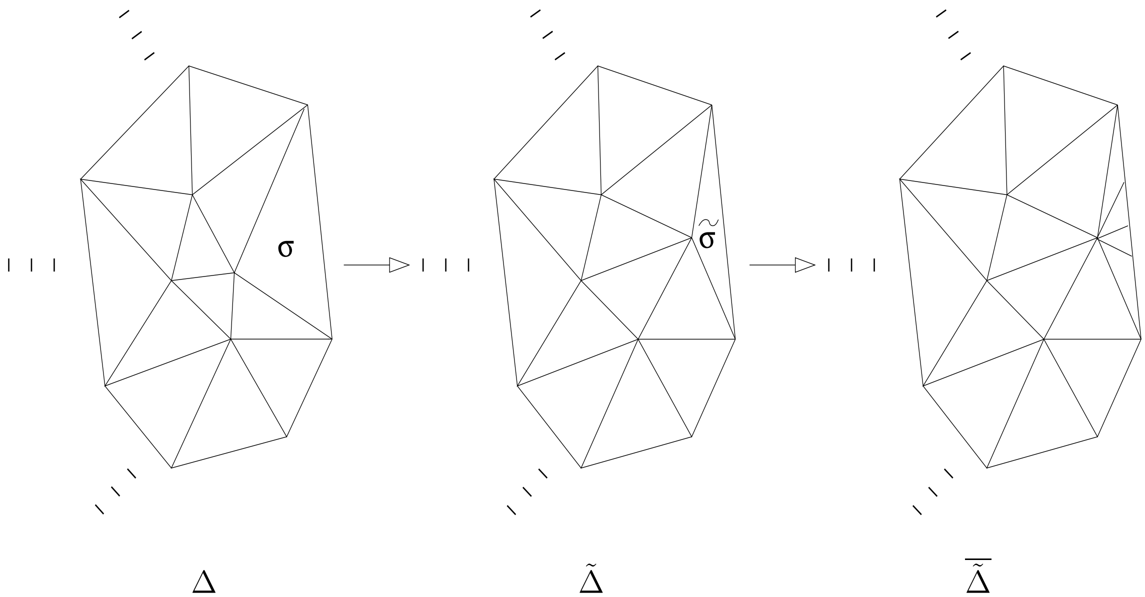

Fig. 10. Deformation $(\Delta, \tilde{\Delta}, and\ \bar{\Delta})$

performing this operation to get $\overline{\tilde{\Delta}}$. See Figure 10.)

Theorem III.3. *Let d and r be fixed non-negative integers. Then the following are equivalent:*

1. *$dimS^r_d(\Delta) = lb^r_d(\Delta)$ for any generic triangulation Δ.*
2. *For any generic triangulation Δ, there is a boundary simplex σ such that*

$$\phi_{\{e_1,\ldots,e_k\}}(\bar{\Delta}_\sigma) \text{ is onto,}$$

   where e_i's are interior edges added inside σ.

Proof. $1 \Rightarrow 2$. Assume that for any generic triangulation Δ we have $dimS^r_d(\Delta) = lb^r_d(\Delta)$. Let Δ be a generic triangulation and σ be any boundary simplex. Denote by Δ' the triangulation obtained by removing σ from Δ. WLOG, we can assume Δ' is also generic. Then, our hypothesis gives $dimS^r_d(\Delta') = lb^r_d(\Delta')$ which implies $dimS^r_d(\bar{\Delta}_\sigma) = lb^r_d(\bar{\Delta}_\sigma)$. That allows us to compute the dimensions of $V(\Delta')$ and

$V(\bar{\Delta}_\sigma)$. Consider the following cofactor map:

$$\phi_{\{e_1,\ldots,e_k\}}(\bar{\Delta}_\sigma) : V(\bar{\Delta}_\sigma) \to \oplus_k P_{d-r-1}.$$

Since the kernel of $\phi_{\{e_1,\ldots,e_k\}}(\bar{\Delta}_\sigma)$ is $V(\Delta')$, the dimension of the image of $\phi_{\{e_1,\ldots,e_k\}}(\bar{\Delta}_\sigma)$ is equal to $dim\ V(\bar{\Delta}_\sigma) - dim\ V(\Delta')$. However,

$$\begin{aligned} dim\ V(\bar{\Delta}_\sigma) - dim\ V(\Delta') &= dim S^r_d(\bar{\Delta}_\sigma) - dim P_d - \{dim S^r_d(\Delta') - dim P_d\} \\ &= lb^r_d(\bar{\Delta}_\sigma) - dim P_d - \{lb^r_d(\Delta') - dim P_d\} \\ &= k(dim\ P_{d-r-1}) \end{aligned} \tag{3.11}$$

This implies $\phi_{\{e_1,\ldots,e_k\}}(\bar{\Delta}_\sigma)$ is onto.

$2 \Rightarrow 1$. We are going to use mathematical induction on the number of interior vertices. For induction, Assume that if a generic triangulation Δ has $n-1$ interior vertices then $dim\ S^r_d(\Delta) = lb^r_d(\Delta)$. Let Δ be a generic triangulation that has n interior vertices and σ be a boundary simplex such that $\phi_{\{e_1,\ldots,e_k\}}(\bar{\Delta}_\sigma)$ is onto. Let Δ' be a triangulation obtained from Δ by taking off σ. Then, Δ' has $n-1$ interior vertices. By induction hypothesis, $dim\ S^r_d(\Delta') = lb^r_d(\Delta')$. Again, this implies $dim\ S^r_d(\bar{\Delta}_\sigma) = lb^r_d(\bar{\Delta}_\sigma)$. Since $\phi_{\{e_1,\ldots,e_k\}}(\bar{\Delta}_\sigma)$ is onto, we end up with

$$\begin{aligned} dim\ ker\ \phi_{\{e_1,\ldots,e_k\}}(\bar{\Delta}_\sigma) &= dim\ V(\bar{\Delta}_\sigma) - k(dim\ P_{d-r-1}) \\ &= lb^r_d(\bar{\Delta}_\sigma) - dim\ P_d - k(dim\ P_{d-r-1}) \\ &= lb^r_d(\Delta) - dim\ P_d \end{aligned} \tag{3.12}$$

Since the kernel of $\phi_{\{e_1,\ldots,e_k\}}(\bar{\Delta}_\sigma)$ is isomorphic to $V(\Delta)$ and $dim\ S^r_d(\Delta) = dim\ V(\Delta) + dim\ P_d$, we have

$$dim\ S^r_d(\Delta) = lb^r_d(\Delta).$$

□

CHAPTER IV

DEFORMED TYPE-1 TRIANGULATION

A. Introduction

In this chapter, we are going to define deformed type-1 triangulation and we will be concerned with the dimension of the spline space on it. We will provide sufficient conditions for $dim\ S_d^r(\Delta) = lb_d^r(\Delta)$ for $d \geq 2r+1$.

B. Uniform type-1 triangulation

Suppose

$$x_0 < x_1 < x_2 < \ldots < x_k, \qquad k \geq 1$$

$$y_0 < y_1 < y_2 < \ldots < y_l, \qquad l \geq 1.$$

Then, using a family of lines

$$x - x_i = 0,\ i = 0, 1, \ldots, k;$$

$$y - y_j = 0,\ j = 0, 1, \ldots, l$$

yields a rectangular partition Δ_{kl}.

If both the $\{x_i\}$ and the $\{y_j\}$ are uniformly spaced, then we call Δ_{kl} the uniform rectangular partition [14].

The type-1 triangulation is a triangulation yielded by connecting diagonally with positive slope at every rectangular cell based on the rectangular partition Δ_{kl}. It is easy to understand that if the original rectangular partition is uniform, then the type-1 triangulation yielded from uniform rectangular partition is a special cross-cut partition [14]. It is known in [6] that for a uniform type-1 triangulation Δ,

$$dim\ S^r_d(\Delta) = lb^r_d(\Delta) \quad for\ d \geq r \geq 0.$$

C. Deformed type-1 triangulation

In this section, we introduce a deformed type-1 triangulation which is obtained by perturbing vertices of the uniform type-1 triangulation. We denote it by $\Delta^{(1)}_{kl}$. (See Figure 11.)

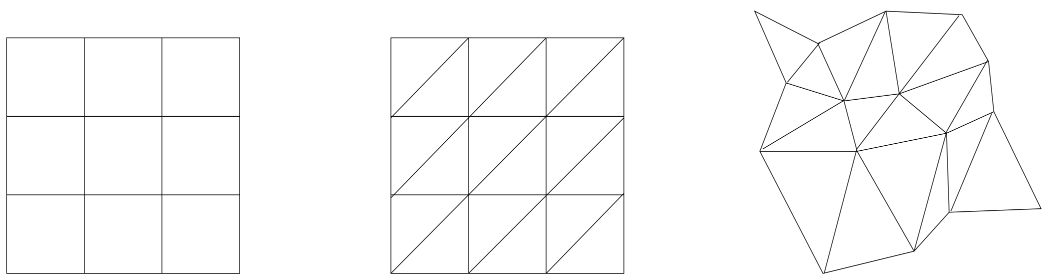

Fig. 11. Δ_{33} and $\Delta^{(1)}_{33}$

Let v_{ij} be the vertex of $\Delta^{(1)}_{kl}$ corresponding to the vertex (x_i, y_j) of the rectangular partition Δ_{kl}.

Let Δ be the triangulation obtained by removing two triangles at the top left and the bottom right (i.e., removing $(v_{0,l}, v_{1,l}, v_{0,l-1})$ and $(v_{k-1,0}, v_{k,0}, v_{k,1})$). (See Figure 12.)

It is easy to see that

$$dim\ S^r_d(\Delta^{(1)}_{kl}) = dim\ S^r_d(\Delta) + 2\binom{d-r+1}{2} \quad for\ d \geq r \geq 0.$$

In the above equation, $2\binom{d-r+1}{2}$ corresponds to the smoothing cofactors on the interior edges $(v_{0,l-1}, v_{1,l})$ and $(v_{k-1,0}, v_{k,1})$. Because neither of the end points of them

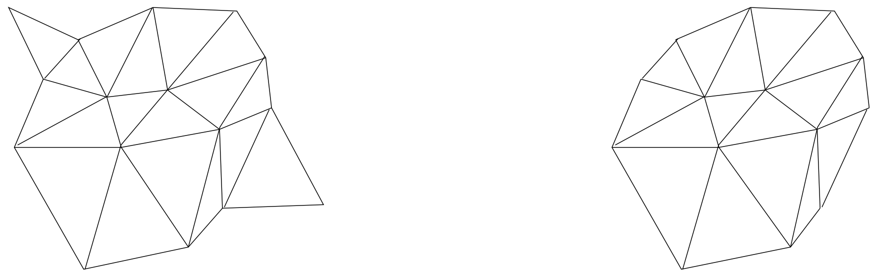

Fig. 12. $\Delta_{33}^{(1)}$ without two triangles

is an interior vertex, the set of smoothing cofactors on each edge is isomorphic to P_{d-r-1}.

It is easy to check that $dim\ S_d^r(\Delta_{kl}^{(1)}) = lb_d^r(\Delta_{kl}^{(1)}) \Longleftrightarrow dim\ S_d^r(\Delta) = lb_d^r(\Delta)$.

As long as we are concerned with the dimension of spline spaces, we can assume $\Delta_{kl}^{(1)}$ does not have these two triangles. From now on, we will use $\Delta_{kl}^{(1)}$ for a deformed type-1 triangulation of size k by l without these two triangles.

For the given $\Delta_{kl}^{(1)}$, let $E(\Delta_{kl}^{(1)})$ be the set of all interior edges having v_{ij} as the end point, *i.e.,* $E(\Delta_{kl}^{(1)})$ is the collection of interior edges corresponding to the very top row interior edges in the corresponding rectangular partition. Note that $|E(\Delta_{kl}^{(1)})| = 2k-2$.

For the given $\Delta_{kl}^{(1)}$, define $F(\Delta_{kl}^{(1)})$ as the polyhedral complex obtained by removing all edges in $E(\Delta_{kl}^{(1)})$ from $\Delta_{kl}^{(1)}$. Then, $F(\Delta_{kl}^{(1)})$ becomes a regular flawed triangulation. To show that $dim\ S_d^r(\Delta_{kl}^{(1)})$ coincides with the lower bound, it suffices to show that $dim\ S_d^r(F(\Delta_{kl}^{(1)}))$ coincides with the lower bound because of Lemma II.9.

Let $\mathfrak{M}_{kl}^{(1)}$ be the collection of all $F(\Delta_{kl}^{(1)})$, *i.e.,*

$$\mathfrak{M}_{kl}^{(1)} = \{F(\Delta_{kl}^{(1)}) : \Delta_{kl}^{(1)} \text{ is a deformed type-1 triangulation}\}.$$

By using induction on the second index l in $\mathfrak{M}_{kl}^{(1)}$, we are going to show that the two conjectures below are sufficient conditions for

$$dim\ S_d^r(F(\Delta_{kl}^{(1)})) = lb_d^r(F(\Delta_{kl}^{(1)})) \quad for\ d \geq 2r+1.$$

D. Conjectures

Definition IV.1. If v is a vertex of a complex Δ, then we say that v is of degree n provided that there are n edges with distinct slopes emerging from v.

Conjecture IV.2. *Let $\mathfrak{A}$ be the set of flawed triangulations having two interior vertices v_1 and v_2 such that degree$(v_1) = 4$ and degree$(v_2) = 3$. Then, for $d \geq 2r \geq 0$*

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta) \ \ \forall \Delta \in \mathfrak{A}.$$

Example IV.3. Let Δ be the polyhedral complex in $\mathfrak{A}$ depicted below in Figure 13.

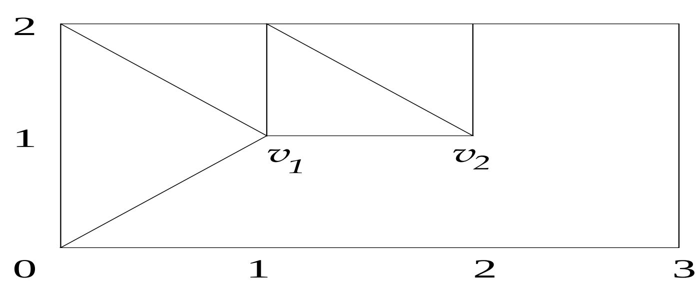

Fig. 13. A polyhedral complex in $\mathfrak{A}$

Table I gives the dimension of $S_{2r}^r(\Delta)$ and $lb_{2r}^r(\Delta)$ for $r \leq 10$.

For $\Delta \in \mathfrak{M}_{kl}^{(1)}$, let $R(\Delta)$ be a flawed triangulation obtained by removing all interior edges connected to v_{kj}, $j = 1, 2, \ldots, l-1$. (See Figure 14.)

Conjecture IV.4. *Define $\mathfrak{B} = \{R(\Delta) | \Delta \in \mathfrak{M}_{33}^{(1)}\}$.*

Table I. $dim\ S^r_{2r}(\Delta) = lb^r_{2r}(\Delta)$ for Δ, the flawed triangulation in Example IV.3

r	$dim\ S^r_{2r}(\Delta)$	$lb^r_{2r}(\Delta)$
1	6	6
2	16	16
3	31	31
4	51	51
5	75	75
6	105	105
7	139	139
8	178	178
9	222	222
10	271	271

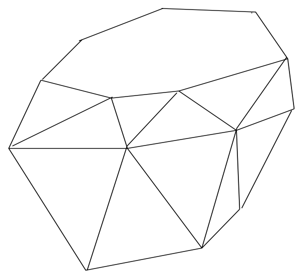

Fig. 14. $F(\Delta_{33}^{(1)}) \in \mathfrak{M}_{33}^{(1)}$ and $R(F(\Delta_{33}^{(1)}))$

Then, for any $d \geq 2r+1$ *and* $r \geq 0$

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta) \quad \forall \Delta \in \mathfrak{B}.$$

Example IV.5. In this example, we choose a polyhedral complex Δ in $\mathfrak{B}$ as in Figure 15 and compute the dimension of $S_{2r+1}^r(\Delta)$ and $lb_{2r+1}^r(\Delta)$ for $r \leq 10$.

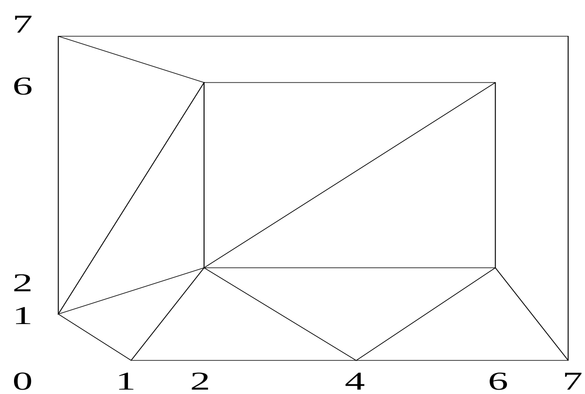

Fig. 15. A polyhedral complex in $\mathfrak{B}$

Table II gives the dimension of $S_{2r+1}^r(\Delta)$ and $lb_{2r+1}^r(\Delta)$ for $r \leq 10$.

Note IV.6. If $\Delta \in \mathfrak{M}_{33}^{(1)}$, then we have $dim\ S_d^r(R(\Delta)) = lb_d^r(R(\Delta))$ because $R(\Delta) \in \mathfrak{B}$ in Conjecture IV.4.

Table II. $dim\ S^r_{2r+1}(\Delta) = lb^r_{2r+1}(\Delta)$ for Δ, the flawed triangulation in Example IV.5

r	$dim\ S^r_{2r+1}(\Delta)$	$lb^r_{2r+1}(\Delta)$
1	18	18
2	34	34
3	56	56
4	83	83
5	115	115
6	154	154
7	197	197
8	245	245
9	299	299
10	359	359

E. Induction

Provided that the Conjecture IV.2 and Conjecture IV.4 are true, we are going to show that for $d \geq 2r+1$ the spline space on a deformed type-1 triangulation has the lower bound dimension by using induction on the second index l in $\mathfrak{M}_{kl}^{(1)}$.

First, consider a flawed triangulation in $\mathfrak{M}_{k1}^{(1)}$. Any flawed triangulation in $\mathfrak{M}_{k1}^{(1)}$ has one 2-face and so splines on it are trivial.

Secondly, consider a flawed triangulation in $\mathfrak{M}_{k2}^{(1)}$. For $k \leq 2$, it is easy to show that for $d \geq 2r$

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta) \quad \forall \Delta \in \mathfrak{M}_{k2}^{(1)}$$

because Δ has at most one interior vertex.

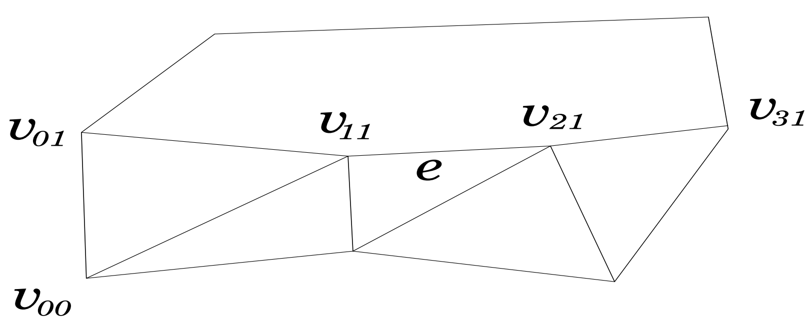

Fig. 16. $\Delta \in \mathfrak{M}_{32}^{(1)}$

Now consider $\Delta \in \mathfrak{M}_{32}^{(1)}$ as in Figure 16. It is clear that degree(v_{11}) ≥ 3 and degree(v_{21}) ≥ 3 because it is obtained by perturbing vertices of a type-1 triangulation. If the degree of v_{21} is 4 then by Conjecture IV.2 we have that for $d \geq 2r$

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta).$$

Suppose that the degree of v_{21} is 3. Then, two edges (v_{11}, v_{21}) and (v_{21}, v_{31}) should have the same slope. Now let's apply Lemma II.6 to $\Delta^1 = Star(v_{11})$ and $\Delta^2 =$

$Star(v_{21})$. It is clear that for $d \geq 2r$

$$dim\ S_d^r(\Delta^1) = lb_d^r(\Delta^1) \text{ and } dim\ S_d^r(\Delta^2) = lb_d^r(\Delta^2)$$

because they have one interior vertex. By Lemma II.6 all we need to show is

$$img\ \phi_{\{e\}}(\Delta^1) + img\ \phi_{\{e\}}(\Delta^2) = P_{d-r-1}.$$

However, it turns out that $img\ \phi_{\{e\}}(\Delta^2) = P_{d-r-1}$: let Δ' be a flawed triangulation yielded from Δ^2 by removing the edge e. Then,

$$\begin{aligned} dim\ img\ \phi_{\{e\}}(\Delta^2) &= dim\ V(\Delta^2) - dim\ V(\Delta') \\ &= dim\ S_d^r(\Delta^2) - \binom{d+2}{2} - \left[dim\ S_d^r(\Delta') - \binom{d+2}{2}\right] \\ &= \left[4\binom{d-r+1}{2} - \left[\binom{d+2}{2} - \binom{r+2}{2}\right] + \delta_3\right] \\ &\quad - \left[3\binom{d-r+1}{2} - \left[\binom{d+2}{2} - \binom{r+2}{2}\right] + \delta_3\right] \\ &= \binom{d-r+1}{2} \\ &= dim\ P_{d-r-1}, \end{aligned} \tag{4.1}$$

where $\delta_3 = \sum_{j=1}^{d-r} (r+j+1-3j)_+$.

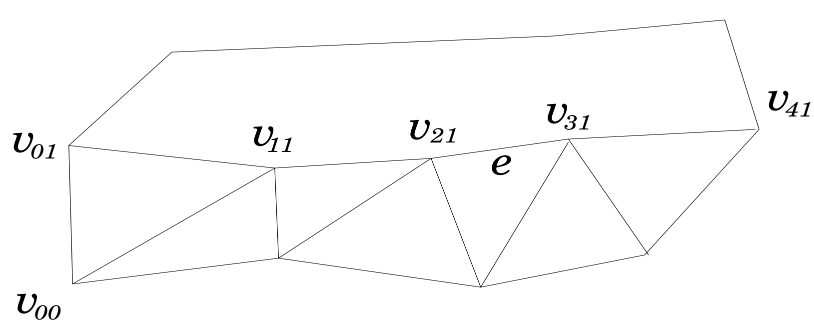

Fig. 17. $\Delta \in \mathfrak{M}_{42}^{(1)}$

Next, consider a flawed triangulation in $\mathfrak{M}_{42}^{(1)}$. (See Figure 17.)

Let's apply Lemma II.6 to $\Delta^1 = \cup_{i=1}^{2} Star(v_{i1})$ and $\Delta^2 = Star(v_{31})$. Since $\Delta^1 \in \mathfrak{M}_{32}^{(1)}$, we have for $d \geq 2r$

$$dim\ S_d^r(\Delta^1) = lb_d^r(\Delta^1)$$

from the above argument.

Also, since Δ^2 has only one interior vertex, $dim\ S_d^r(\Delta^2) = lb_d^r(\Delta^2)$ for $d \geq r$.

By Lemma II.6 we need to show that

$$img\ \phi_{\{e\}}(\Delta^1) + img\ \phi_{\{e\}}(\Delta^2) = P_{d-r-1}.$$

There are 4 edges emerging from v_{31}. If the degree of v_{31} is 3, then $img\ \phi_{\{e\}}(\Delta^2) = P_{d-r-1}$ by the same reason before as in (4.1) and we are done.

Suppose that the degree of v_{31} is 4. Define Δ' as a complex obtained from Δ^1 by removing all interior edges connected to v_{11}.

We still have $S_d^r(\Delta') \subset S_d^r(\Delta^1)$ which implies $img\ \phi_{\{e\}}(\Delta') \subset img\ \phi_{\{e\}}(\Delta^1)$. Notice that $\deg(v_{21}) \leq 3$ in Δ'. If $\deg(v_{21}) < 3$ there are two edges having the same slope as e and by the argument used in (4.1) we get $img\ \phi_{\{e\}}(\Delta') = P_{d-r-1}$. So we have

$$img\ \phi_{\{e\}}(\Delta^1) + img\ \phi_{\{e\}}(\Delta^2) \supset img\ \phi_{\{e\}}(\Delta') + img\ \phi_{\{e\}}(\Delta^2) = P_{d-r-1}.$$

If $\deg(v_{21}) = 3$ then $\Delta \cup \Delta' \in \mathfrak{A}$ because $\deg(v_{21}) = 3$ and $\deg(v_{31}) = 4$. By Conjecture IV.2 we have that for $d \geq 2r$

$$dim\ S_d^r(\Delta \cup \Delta') = lb_d^r(\Delta \cup \Delta').$$

Now, by Lemma II.6 we have $img\ \phi_{\{e\}}(\Delta')+img\ \phi_{\{e\}}(\Delta^2) = P_{d-r-1}$. Thus, for $d \geq 2r$

$$img\ \phi_{\{e\}}(\Delta^1) + img\ \phi_{\{e\}}(\Delta^2) \supset img\ \phi_{\{e\}}(\Delta') + img\ \phi_{\{e\}}(\Delta^2) = P_{d-r-1}.$$

We can do this for any $\Delta \in \mathfrak{M}_{k2}^{(1)}\ \ \forall k \geq 3$.

Therefore, we can conclude that $\forall \Delta \in \mathfrak{M}_{k2}^{(1)}\ for\ d \geq 2r \geq 0$

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta)$$

provided that Conjecture IV.2 holds.

Lemma IV.7. *Let $d \geq 2r \geq 0$ and $k \geq 1$. Then, for any $\Delta_{k2}^{(1)}$, a deformed type-1 triangulation of size k by 2*

$$dim\ S_d^r(\Delta_{k2}^{(1)}) = lb_d^r(\Delta_{k2}^{(1)}).$$

Proof. Use the above arguments and Lemma II.9. □

Thirdly, we are going to look at a flawed triangulation in $\mathfrak{M}_{k3}^{(1)}$. If $k = 1$, then for any $\Delta \in \mathfrak{M}_{13}^{(1)}$ the dimension of $S_d^r(\Delta)$ is the lower bound for all $d \geq r \geq 0$ since Δ has no interior vertices. For $k = 2$, any $\Delta \in \mathfrak{M}_{23}^{(1)}$ has two interior vertices and we can use the argument that was used for a flawed triangulation in $\mathfrak{M}_{32}^{(1)}$.

For $k = 3$ we can use Lemma II.9 based on some flawed triangulation in $\mathfrak{B}$: if Δ is in $\mathfrak{M}_{33}^{(1)}$, then $R(\Delta)$ is in $\mathfrak{B}$, where R is as in Conjecture IV.4. Thus, for $d \geq 2r+1$ we have

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta)\ \ \forall \Delta \in \mathfrak{M}_{33}^{(1)}$$

by Lemma II.9 and Conjecture IV.4.

Now, let $\Delta \in \mathfrak{M}_{43}^{(1)}$ like a flawed triangulation in Figure 18. We will deal with

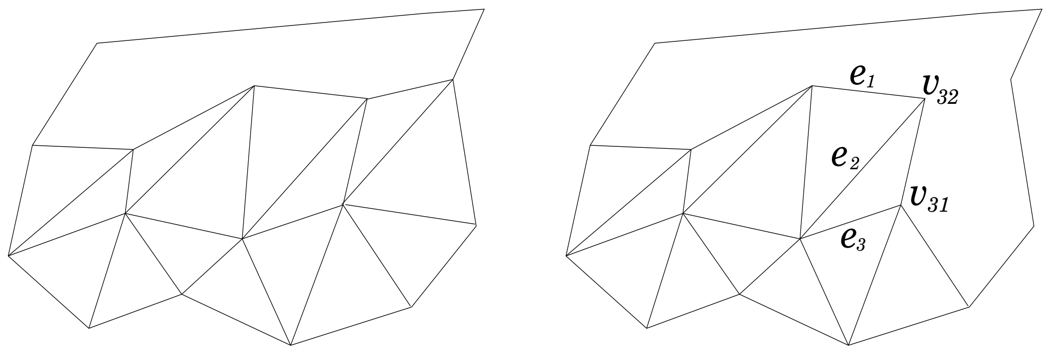

Fig. 18. $\Delta \in \mathfrak{M}_{43}^{(1)}$ and $R(\Delta)$

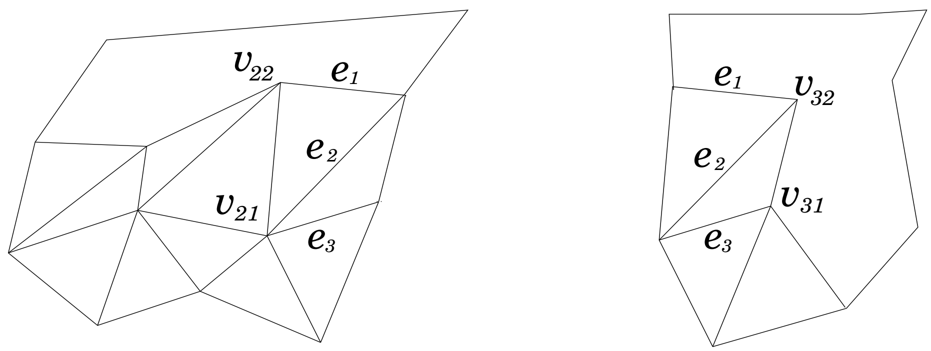

Fig. 19. Δ^1 and Δ^2 for $\mathfrak{M}_{43}^{(1)}$

$R(\Delta)$ (See Figure 18) and conclude that by Lemma II.9 for $d \geq 2r+1$

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta) \ \ \forall \Delta \in \mathfrak{M}_{43}^{(1)}$$

after showing $dim\ S_d^r(R(\Delta)) = lb_d^r(R(\Delta))$. Define Δ^1 and Δ^2 as in Figure 19:Δ^1 has four interior vertices, $v_{11}, v_{12}, v_{21},$ and v_{22}; Δ^2 has two interior vertices, v_{31} and v_{32}.

Notice that since there are 3 edges connected to v_{32} and 4 edges connected to v_{31} in Δ^2 we know that by using the same argument as for $\mathfrak{M}_{32}^{(1)}$ (if degree(v_{31}) $= 4$ use the Conjecture IV.2 and if degree(v_{31}) $= 3$ use Lemma II.6)

$$dim\ S_d^r(\Delta^2) = lb_d^r(\Delta^2) \ \ \text{for } d \geq 2r.$$

Also, since $\Delta^1 \in \mathfrak{M}_{33}^{(1)}$, for $d \geq 2r+1$ we have

$$dim\ S_d^r(\Delta^1) = lb_d^r(\Delta^1).$$

Let $E = \{e_1, e_2, e_3\}$, a set of common edges of Δ^1 and Δ^2.

By Lemma II.6 all we need to show is

$$img\ \phi_E(\Delta^1) + img\ \phi_E(\Delta^2) = \oplus_3 P_{d-r-1}.$$

Let $\Delta' = Star(v_{21}) \cup Star(v_{22})$. Then, $\Delta' in \mathfrak{M}_{23}^{(1)}$.

We are going to show that

$$img\ \phi_E(\Delta^1) = img\ \phi_E(\Delta')$$

and replace $img\ \phi_E(\Delta^1)$ with $img\ \phi_E(\Delta')$. It is clear that

$$img\ \phi_E(\Delta^1) \subset img\ \phi_E(\Delta') \tag{4.2}$$

because any spline on Δ^1 satisfies the conformality conditions at each interior vertex of Δ'. On the other hand, we have

$$\begin{aligned} dim\ img\ \phi_E(\Delta^1) &= dim\ V(\Delta^1) - dim\ V(R(\Delta^1)) \\ dim\ img\ \phi_E(\Delta') &= dim\ V(\Delta') - dim\ V(R(\Delta')) \end{aligned} \tag{4.3}$$

By Conjecture IV.2 we have for $d \geq 2r$

$$dim\ S_d^r(R(\Delta')) = lb_d^r(R(\Delta'))$$

and by Conjecture IV.4 we have for $d \geq 2r+1$

$$dim\ S_d^r(R(\Delta^1)) = lb_d^r(R(\Delta^1)).$$

Then, we know all values in the right-hand sides in the equation (4.3) so we can compute the values of the left-hand sides and it turns out that

$$dim\ img\ \phi_E(\Delta^1) = dim\ img\ \phi_E(\Delta') \tag{4.4}$$

Using (4.2) and (4.4), we have

$$img\ \phi_E(\Delta^1) = img\ \phi_E(\Delta')$$

Since $\Delta' \cup \Delta^2 \in \mathfrak{B}$, by Conjecture IV.4 along with Lemma II.6 we have that

$$img\ \phi_E(\Delta^1) + img\ \phi_E(\Delta^2) = img\ \phi_E(\Delta') + img\ \phi_E(\Delta^2) = \oplus_3 P_{d-r-1}.$$

Thus, spline space on $R(\Delta)$ has the lower bound dimension. We can do this inductively for $\{\mathfrak{M}_{k3}^{(1)}\}_{k\geq 3}$ in the same manner as above.

Finally, consider the general case, $\mathfrak{M}_{kl}^{(1)}$ for $l \geq 4$. First, to perform the inductive step, assume the inductive hypothesis that for $d \geq 2r+1$

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta)\ \ \forall \Delta \in \mathfrak{M}_{k,l-1}^{(1)}.$$

Let $\Delta \in \mathfrak{M}_{kl}^{(1)}$, $A = \{v_{i,l-1}\}_{i=1}^{k-1}$, and B be the complement of A in the set of interior vertices of Δ. Define $\Delta^1 = \cup_{v\in B} Star(v)$ and $\Delta^2 = \cup_{v\in A} Star(v)$. We are going to use Lemma II.6. Notice that $\Delta^2 \in \mathfrak{M}_{k2}^{(1)}$ and so the dimension of spline space on Δ^2 is the lower bound. On the other hand, since $F(\Delta^1) \in \mathfrak{M}_{k,l-1}^{(1)}$, by Lemma II.9 along with inductive hypothesis we have that the dimension of spline space on Δ^1 is the lower bound. According to Lemma II.6, all we need to show is

$$img\ \phi_E(\Delta^1) + img\ \phi_E(\Delta^2) = \oplus_{|E|} P_{d-r-1},$$

where $E = E(\Delta)$, the set of the common interior edges of Δ^1 and Δ^2. Let $\Delta' =$

$\cup_{i=1}^{k-1} Star(v_{i,l-2})$. We are going to show that

$$img\ \phi_E(\Delta^1) = img\ \phi_E(\Delta')$$

and replace $img\ \phi_E(\Delta^1)$ with $img\ \phi_E(\Delta')$. Since $img\ \phi_E(\Delta^1) \subset img\ \phi_E(\Delta')$, it suffices to show that they have the same dimension. However, we have

$$\begin{aligned} dim\ img\ \phi_E(\Delta^1) &= dim\ V(\Delta^1) - dim\ V(F(\Delta^1)) \\ dim\ img\ \phi_E(\Delta') &= dim\ V(\Delta') - dim\ V(F(\Delta')) \end{aligned} \tag{4.5}$$

Since $F(\Delta') \in \mathfrak{M}_{k2}^{(1)}$, we know $dim\ S_d^r(F(\Delta'))$ and $dim\ S_d^r(\Delta')$, which give $dim\ V(F(\Delta'))$ and $dim\ V(\Delta')$. The explicit computations of dimensions in the right-hand sides in (4.5) give

$$dim\ img\ \phi_E(\Delta^1) = dim\ img\ \phi_E(\Delta') \tag{4.6}$$

Since $\Delta' \cup \Delta^2 \in \mathfrak{M}_{k3}^{(1)}$, by Lemma II.6 we have

$$img\ \phi_E(\Delta') + img\ \phi_E(\Delta^2) = \oplus_{|E|} P_{d-r-1},$$

Thus, we have

$$img\ \phi_E(\Delta^1) + img\ \phi_E(\Delta^2) = img\ \phi_E(\Delta') + img\ \phi_E(\Delta^2) = \oplus_{|E|} P_{d-r-1}.$$

Now, we have the following Theorem;

Theorem IV.8. *If Conjecture IV.2 and Conjecture IV.4 are true, then for any* $d \geq 2r+1$ *the dimension of spline space on a deformed type-1 triangulation coincides with the lower bound.*

CHAPTER V

SEMI-DEFORMED TYPE-2 TRIANGULATION

A. Introduction

In the previous chapter, we provided sufficient conditions for $dim\ S_d^r(\Delta) = lb_d^r(\Delta)\ \forall d \geq 2r+1$ when Δ is a deformed type-1 triangulation. It was accomplished by reducing the given complex to a simpler one. In this chapter, we will define a semi-deformed type-2 triangulation which is obtained from a type-2 triangulation in some manner. Then, we will give sufficient conditions for $dim\ S_{2r}^r(\Delta) = lb_{2r}^r(\Delta)$.

The type-2 triangulation is a triangulation obtained by connecting two diagonals in every rectangular cell. If the original rectangular partition is uniform, then it gives a uniform type-2 triangulation and it is known that for a uniform type-2 triangulation Δ,

$$dim\ S_d^r(\Delta) = lb_d^r(\Delta) \quad for\ d \geq r \geq 0 \quad [6].$$

B. Semi-deformed type-2 triangulation

Let $\tilde{\Delta}_{kl}$ be a quadrilateral partition obtained by moving vertices of a rectangular partion Δ_{kl} such that each of the quadrilateral cells is strictly convex. The semi-deformed type-2 triangulation of size k by l is a triangulation obtained by adding two diagonals at every quadrilateral cell based on $\tilde{\Delta}_{kl}$ and we will denote it by $\Delta_{kl}^{(2)}$. For example, the complex in Figure 20 is one of our semi-deformed type-2 triangulation of size 3×3.

In this section, we are concerned with the dimension of $S_{2r}^r(\Delta_{kl}^{(2)})$.

For the given semi-deformed type-2 triangulation $\Delta_{kl}^{(2)}$, let v_{ij} be the vertex of

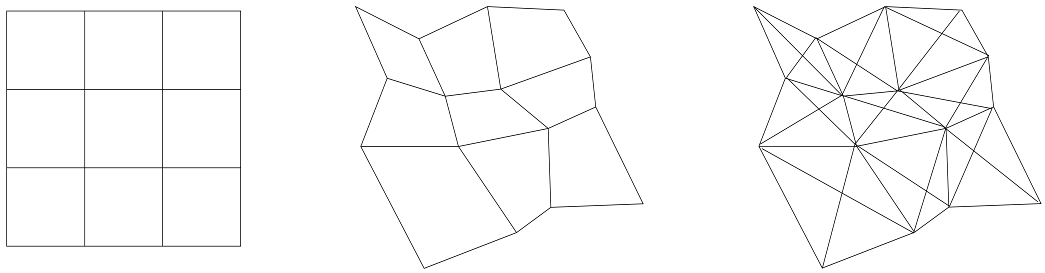

Fig. 20. Δ_{33}, $\tilde{\Delta}_{33}$, and $\Delta^{(2)}_{33}$

$\Delta^{(2)}_{kl}$ corresponding to the vertex (x_i, y_j) in Δ_{kl} and let w_{ij} be the vertex of $\Delta^{(2)}_{kl}$ that is the intersection of two diagonals $(v_{i-1,j-1}, v_{i,j})$ and $(v_{i-1,j}, v_{i,j-1})$.

For the given semi-deformed type-2 triangulation $\Delta^{(2)}_{kl}$, define $B(\Delta^{(2)}_{kl})$ as a triangulation obtained by removing from $\Delta^{(2)}_{kl}$ all triangles in $\Delta^{(2)}_{kl}$ which have a boundary edge. (See the second one in Figure 21.)

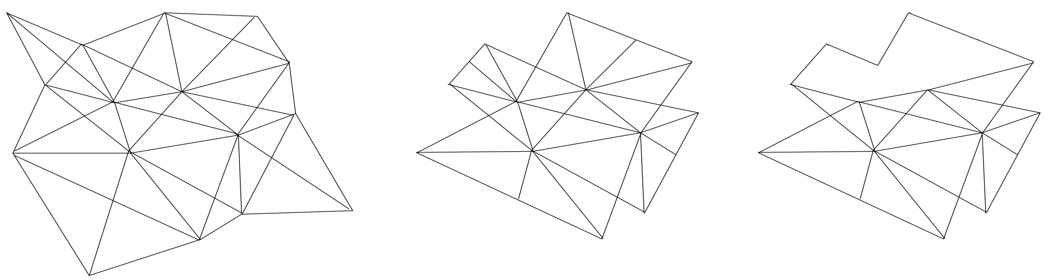

Fig. 21. $\Delta^{(2)}_{33}$, $B(\Delta^{(2)}_{33})$, and $F(\Delta^{(2)}_{33})$

Lemma V.1. *Let $\Delta^{(2)}_{kl}$ be a semi-deformed type-2 triangulation. If the dimension of $S^r_{2r}(B(\Delta^{(2)}_{kl}))$ coincides with the lower bound, $lb^r_{2r}(B(\Delta^{(2)}_{kl}))$, then we have that the dimension of $S^r_{2r}(\Delta^{(2)}_{kl})$ coincides with the lower bound, $lb^r_{2r}(\Delta^{(2)}_{kl})$.*

Proof. Let Δ^1 be a triangulation obtained by adding two triangles to $B(\Delta^{(2)}_{kl})$ so that Δ^1 has exactly one more interior vertex w_{11} than $B(\Delta^{(2)}_{kl})$ has. (See the second complex in Figure 22.) Now, applying Lemma II.6 to $Star(w_{11})$ and $B(\Delta^{(2)}_{kl})$ gives us that $dim\ S_{2r}^r(\Delta^1)$ is the lower bound. Next, let Δ^2 be a triangulation obtained by adding a triangle to Δ^1 so that Δ^2 has one more interior vertex w_{21}. (See the third complex in Figure 22.) Again, by Lemma II.6 applied to $Star(w_{21})$ and Δ^1 we have that $dim\ S_{2r}^r(\Delta^2)$ is the lower bound. We can repeat doing this inductively until we get the desired result that $dim\ S_{2r}^r(\Delta_{kl}^{(2)})$ is the lower bound. □

Note V.2. $dim\ S_{2r}^r(B(\Delta_{kl}^{(2)})) = lb_{2r}^r(B(\Delta_{kl}^{(2)}))$ implies that

$$dim\ S_{2r}^r(\Delta^i) = lb_{2r}^r(\Delta^i) \quad \forall i.$$

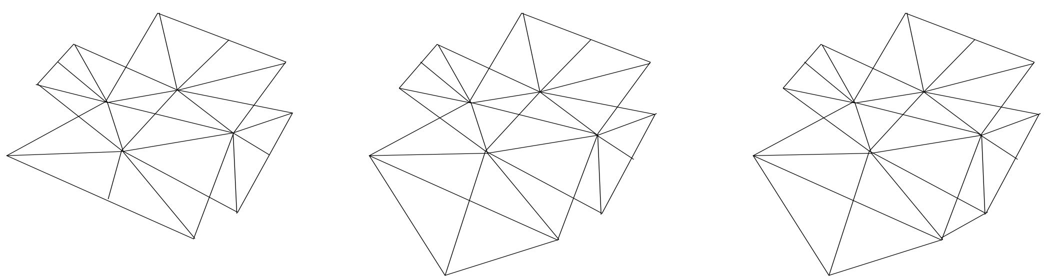

Fig. 22. $B(\Delta_{33}^{(2)})$, Δ^1, and Δ^2

For a semi-deformed type-2 triangulation $\Delta_{kl}^{(2)}$, let $E(\Delta_{kl}^{(2)})$ be the set of interior edges of $B(\Delta_{kl}^{(2)})$ connected to v_{il} or w_{jl} for $i = 1, 2, \ldots, k-1$ and $j = 1, 2, 3, \ldots, k$ and define $F(\Delta_{kl}^{(2)})$ as a polyhedral complex obtained from $B(\Delta_{kl}^{(2)})$ by removing all edges in $E(\Delta_{kl}^{(2)})$. (See the third complex in Figure 20.) Then, $F(\Delta_{kl}^{(2)})$ becomes a regular flawed triangulation. To show that the dimension of $S_{2r}^r(\Delta_{kl}^{(2)})$ is the lower bound, it suffices to show that the dimension of $S_{2r}^r(F(\Delta_{kl}^{(2)}))$ is the lower bound because of Lemma V.1 and Lemma II.9.

Let $\mathfrak{M}_{kl}^{(2)}$ be the collection of all $F(\Delta_{kl}^{(2)})$.

C. Conjecture

Like what we did for the deformed type-1 triangulation case, we are going to show that if Conjecture IV.2 and Conjecture V.3 (which will be stated below) hold then for $d = 2r$ the spline space on a semi-deformed type-2 triangulation has the lower bound dimension. We will do it by using induction on the second index l in $\mathfrak{M}_{kl}^{(2)}$.

For $\Delta \in \mathfrak{M}_{kl}^{(2)}$, let $R(\Delta)$ be a flawed triangulation obtained by removing all interior edges connected to v_{ki} or w_{kj} for $i = 1, 2, \ldots, l-1$ and $j = 2, 3, \ldots, l-1$. (See Figure 23)

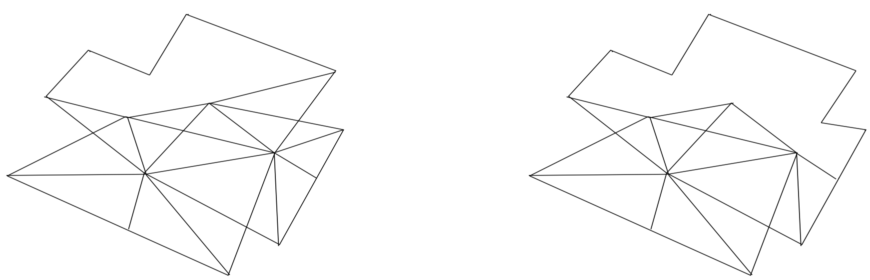

Fig. 23. $F(\Delta_{33}^{(1)}) \in \mathfrak{M}_{33}^{(2)}$ and $R(F(\Delta_{33}^{(1)}))$

Conjecture V.3. *Define* $\mathfrak{C} = \{R(\Delta) | \Delta \in \mathfrak{M}_{33}^{(2)}\}$.
Then, for any $r \geq 0$

$$dim\ S_{2r}^{r}(\Delta) = lb_{2r}^{r}(\Delta) \quad \forall \Delta \in \mathfrak{C}.$$

Example V.4. Let Δ be the polyhedral complex depicted below in Figure 24.

In the table III, we compute the dimension of $S_{2r}^{r}(\Delta)$ and $lb_{2r}^{r}(\Delta)$ for $r \leq 10$.

Note V.5. If the Conjecture V.3 holds, then for $\Delta \in \mathfrak{M}_{33}^{(2)}$ we have $dim\ S_{2r}^{r}(\Delta) = lb_{2r}^{r}(\Delta)$ by Lemma II.9.

Table III. $dim\ S^r_{2r}(\Delta) = lb^r_{2r}(\Delta)$ for Δ, the flawed triangulation in Example V.4

r	$dim\ S^r_{2r}(\Delta)$	$lb^r_{2r}(\Delta)$
1	10	10
2	28	28
3	54	54
4	90	90
5	135	135
6	189	189
7	252	252
8	325	325
9	406	406
10	497	497

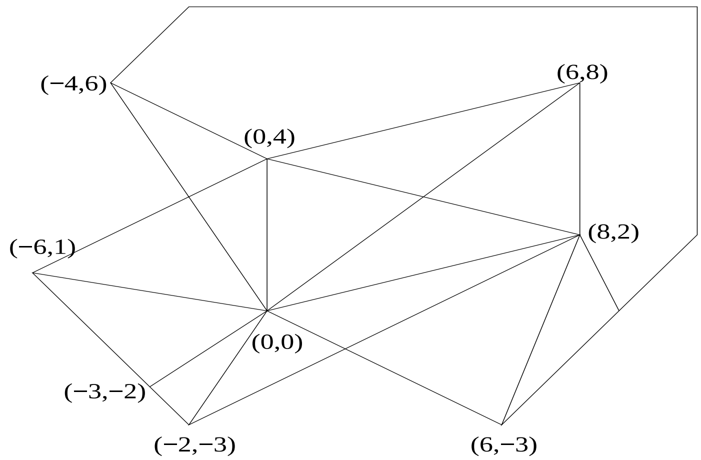

Fig. 24. A polyhedral complex in $\mathfrak{C}$

D. Induction

First, consider a semi-deformed type-2 triangulation of size $k\times 1$. In this case, we can directly deal with $\Delta^{(2)}_{k1}$ and show $dim\ S^r_{2r}(\Delta^{(2)}_{k1}) = lb^r_{2r}(\Delta^{(2)}_{k1})$ by using Lemma II.6. It can be done by induction on the first index: assume that for any semi-deformed type-2 triangulation of size $k\times 1$ the dimension of spline space on it is the lower bound. Let $\Delta^{(2)}_{k+1,1}$ be given. Let $\Delta^1 = \cup^k_{i=1} Star(w_{i,1})$ and $\Delta^2 = Star(w_{k+1,1})$. Applying Lemma II.6 to Δ^1 and Δ^2, we have the dimension of $S^r_{2r}(\Delta^{(2)}_{k+1,1})$ is the lower bound because Δ_1 and Δ_2 do not have any common edges.

Secondly, consider flawed triangulations of size $k\times 2$. However, by Lemma II.9, instead of dealing with flawed triangulations of size $k\times 2$, we can play with flawed triangulations in $\mathfrak{M}^{(2)}_{k2}$. It can be done by using induction on k in the same way as $\mathfrak{M}^{(1)}_{k2}$ in Chapter III by assuming that Conjecture IV.2 holds.

Thirdly, we are going to look at flawed triangulations in $\mathfrak{M}^{(2)}_{k3}$. If $k=2$, it can be done easily by using Conjecture IV.2.

Next, consider flawed triangulations in $\mathfrak{M}^{(2)}_{33}$. For any Δ in $\mathfrak{M}^{(2)}_{33}$, we can always get $\Delta' \in \mathcal{C}$ by removing non-totally interior edges in Δ. Now use Lemma II.9 and

Conjecture V.3 to get

$$dim\ S^r_{2r}(\Delta) = lb^r_{2r}(\Delta).$$

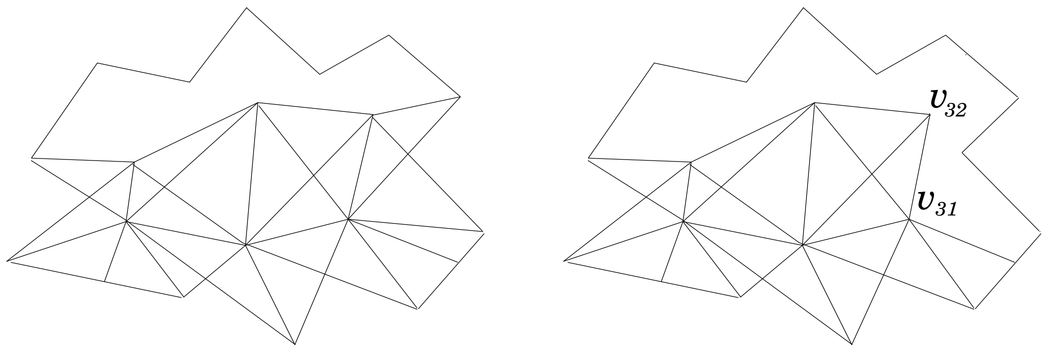

Fig. 25. $\Delta \in \mathfrak{M}^{(2)}_{43}$ and $R(\Delta)$

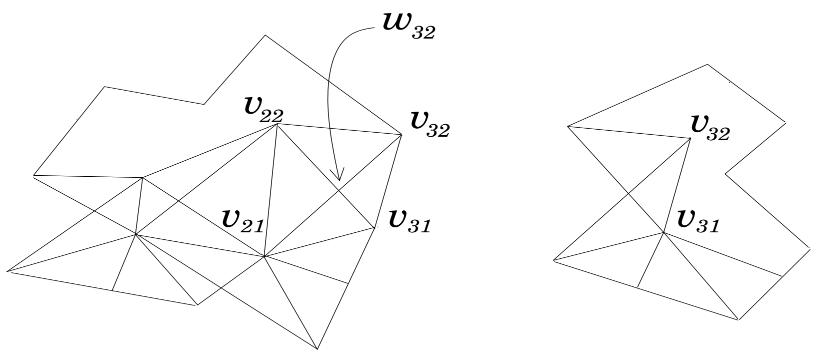

Fig. 26. Δ^1 and Δ^2 for $\mathfrak{M}^{(2)}_{43}$

Let $\Delta \in \mathfrak{M}^{(2)}_{43}$. We are going to show that spline space on $R(\Delta)$ has the lower bound dimension. (See Figure 25.) Define Δ^1 and Δ^2 as in Figure 26: Δ^1 has 6 interior vertices, $v_{11}, v_{12}, v_{21}, v_{22}, w_{22},$ and w_{32}; Δ^2 has two interior vertices, v_{31} and v_{32}. It is easy to show that by Conjecture IV.2

$$dim\ S^r_{2r}(\Delta^2) = lb^r_{2r}(\Delta^2).$$

Note that we can show that with Conjecture V.3

$$dim\ S^r_{2r}(\Delta^1) = lb^r_d(\Delta^1)$$

from the following fact: if we remove the triangle (w_{32}, v_{32}, v_{31}) from Δ^1, then we get a complex in $\mathfrak{M}^{(2)}_{33}$ and so we just use Note V.5 and the proof of Lemma V.1 for the desired result.

By Lemma II.6 all we need to show is

$$img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^1) + img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^2) = \oplus_4 P_{r-1}$$

where e_i's are the common interior edges of Δ^1 and Δ^2.

Let $\Delta' = Star(v_{21}) \cup Star(v_{22}) \cup Star(w_{32})$. (See Figure 27.) We are going to show that $img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^1) = img\ \phi_{\{e_i\}_{i=1}^4}(\Delta')$ and replace $img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^1)$ with $img\ \phi_{\{e_i\}_{i=1}^4}(\Delta')$.

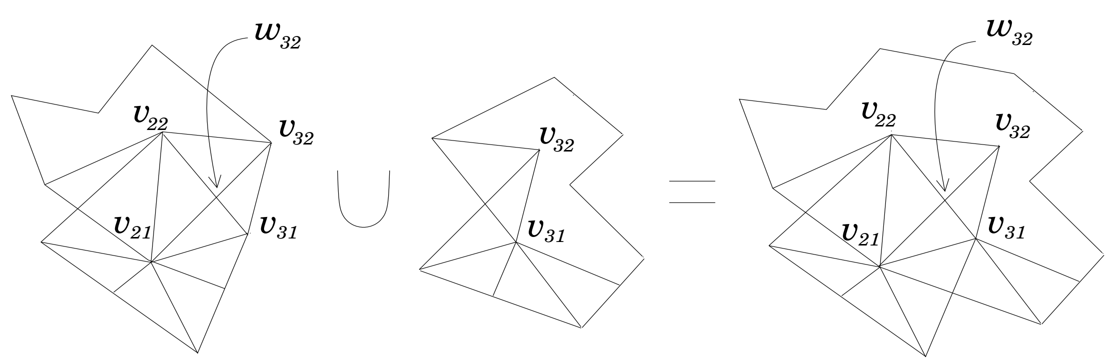

Fig. 27. Δ' and $\Delta' \cup \Delta^2$

It is clear that

$$img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^1) \subset img\ \phi_{\{e_i\}_{i=1}^4}(\Delta') \tag{5.1}$$

because any spline on Δ^1 satisfies the conformality condition at each interior vertex of Δ'. On the other hand, by Remark I.12 we have

$$\begin{aligned} dim\ img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^1) &= dim\ V(\Delta^1) - dim\ V(R(\Delta^1)) \\ dim\ img\ \phi_{\{e_i\}_{i=1}^4}(\Delta') &= dim\ V(\Delta') - dim\ V(R(\Delta')) \end{aligned} \tag{5.2}$$

From the fact that all flawed triangulations Δ in $\mathfrak{M}_{23}^{(2)}$ satisfy

$$dim\ S_{2r}^r(\Delta) = lb_{2r}^r(\Delta),$$

we get $dim\ S_{2r}^r(\Delta') = lb_{2r}^r(\Delta')$ by using Lemma II.9 and Lemma V.1.

On the other hand, by Conjecture V.3 we have that

$$dim\ S_{2r}^r(\Delta^1) = lb_d^r(\Delta^1).$$

by using Lemma II.9 and Lemma V.1.

Then, explicit computation gives us that

$$dim\ img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^1) = dim\ img\ \phi_{\{e_i\}_{i=1}^4}(\Delta'). \tag{5.3}$$

Using (5.1) and (5.3), we have $img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^1) = img\ \phi_{\{e_i\}_{i=1}^4}(\Delta')$.

Since $\Delta' \cup \Delta^2 \in \mathfrak{C}$ (See Figure 27), by Conjecture V.3 along with Lemma II.6 we have that

$$img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^1) + img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^2) = img\ \phi_{\{e_i\}_{i=1}^4}(\Delta') + img\ \phi_{\{e_i\}_{i=1}^4}(\Delta^2) = \oplus_4 P_{r-1}.$$

Thus, we can conclude that $dim\ S_{2r}^r(R(\Delta)) = lb_d^r(R(\Delta))$ and so $dim\ S_{2r}^r(\Delta) = lb_{2r}^r(\Delta)$ by Lemma II.9. We can repeat doing these inductively for $\{\mathfrak{M}_{k3}^{(2)}\}_{k\geq 3}$ in the same way as above.

Finally, consider the general case $\mathfrak{M}_{kl}^{(2)}$ for $l \geq 4$. To perform the inductive step, assume the induction hypothesis:

$$dim\ S^r_{2r}(\Delta) = lb^r_{2r}(\Delta) \ \ \forall \Delta \in \mathfrak{M}^{(2)}_{k,l-1}.$$

Let $\Delta \in \mathfrak{M}^{(2)}_{kl}, A = \{v_{i,l-1}\}_{i=1}^{k-1}$, and B be the complement of A in the set of interior vertices of Δ. Define $\Delta^1 = \cup_{v\in B} Star(v)$ and $\Delta^2 = \cup_{v\in A} Star(v)$. We are going to use Lemma II.6. Notice that $\Delta^2 \in \mathfrak{M}^{(2)}_{k2}$ and so

$$dim\ S^r_{2r}(\Delta^2) = lb^r_{2r}(\Delta^2).$$

On the other hand, by our hypothesis along with Lemma II.9 and Note V.2 we can show that

$$dim\ S^r_{2r}(\Delta^1) = lb^r_{2r}(\Delta^1).$$

According to Lemma II.6, all we need to show is

$$img\ \phi_E(\Delta^1) + img\ \phi_E(\Delta^2) = \oplus_{|E|} P_{r-1},$$

where E is the set of the common interior edges of Δ^1 and Δ^2 and $|E|$ is the cardinality of E.

Let $\Delta' = (\cup_{i=1}^{k-1} Star(v_{k,l-2}) \cup (\cup_{i=2}^{k-1} Star(w_{i,l-1}))$. We are going to show that $img\ \phi_E(\Delta^1) = img\ \phi_E(\Delta')$ and replace $img\ \phi_E(\Delta^1)$ with $img\ \phi_E(\Delta')$.

We have $img\ \phi_E(\Delta^1) \subset img\ \phi_E(\Delta')$ and by Remark I.12

$$dim\ img\ \phi_E(\Delta^1) = dim\ V(\Delta^1) - dim\ V(F(\Delta^1))$$

$$dim\ img\ \phi_E(\Delta') = dim\ V(\Delta') - dim\ V(F(\Delta'))$$

Since $F(\Delta^1) \in \mathfrak{M}^{(2)}_{k,l-1}$, by our hypothesis $dim\ S^r_{2r}(F(\Delta^1))$ is the lower bound. Also, $F(\Delta') \in \mathfrak{M}^{(2)}_{k2}$. Thus, we can compute the dimensions of $img\ \phi_E(\Delta^1)$ and $img\ \phi_E(\Delta')$ and it turns out that they are the same. So, we have $img\ \phi_E(\Delta^1) = img\ \phi_E(\Delta')$.

Since $\Delta' \cup \Delta^2 \in \mathfrak{M}_{k3}^{(2)}$, by Lemma II.6 we have

$$img\ \phi_E(\Delta^1) + img\ \phi_E(\Delta^2) = img\ \phi_E(\Delta') + img\ \phi_E(\Delta^2) = \oplus_{|E|} P_{r-1}.$$

Theorem V.6. *If Conjecture IV.2 and Conjecture V.3 are true, then for* $d = 2r$ *the dimension of spline space on a semi-deformed type-2 triangulation coincides with the lower bound.*

CHAPTER VI

A BASE CASE

A. Introduction

In [13], it was shown that the equality in (1.1) fails for $d = 2r$ for some triangulation. (See Figure 28.) That means the "$2r+1$" conjecture is sharp.

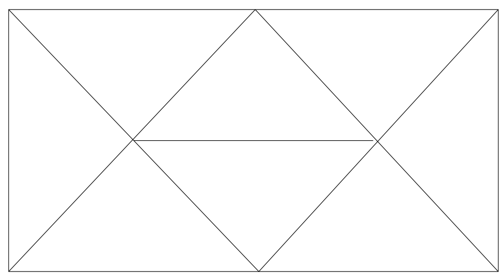

Fig. 28. Base triangulation

It is still unknown if the formula holds on the base triangulation for $d = 2r+1$ and we will explore the dimension of the spline space on the base triangulation for $d = 2r+1$.

We can embed Δ in $\mathbb{R}^3$ to form the cone $\hat{\Delta}$ of Δ with the origin. It is known that the set of splines on $\hat{\Delta}$ is a graded module $S^r(\hat{\Delta})$ over the polynomial ring $\mathbb{R}[x, y, z]$ and $S^r_d(\Delta)$ is isomorphic to $S^r(\hat{\Delta})_d$. Hence, it is sufficient to study $S^r(\hat{\Delta})$ when concerned with $S^r_d(\Delta)$.

Set the following two ideals occurring at the left and right interior vertices of $\hat{\Delta}$:

$$I_1 =< (x-y)^{r+1}, (x+y-2z)^{r+1} >: (y-z)^{r+1}$$

and

$$I_2 =< (x + y - 4z)^{r+1}, (x - y - 2z)^{r+1} >: (y - z)^{r+1}.$$

Notice that $y - z$ is the linear form vanishing on the totally interior edge.

From Lemma II.6, the following

$$(I_1)_{d-r-1} + (I_2)_{d-r-1} = \mathbb{R}[x, y, z]_{d-r-1}$$

is sufficient to show

$$dim\ S_d^r(\Delta) = lb_d^r.$$

Set $I = I_1 + I_2$.

Since I_1 and I_2 are homogeneous ideals, we have

$$(I_1)_{d-r-1} + (I_2)_{d-r-1} = \mathbb{R}[x, y, z]_{d-r-1}$$

$$\iff (\mathbb{R}/I)_{d-r-1} = 0$$

With the change of variables given by the matrix

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 2 \\ 1 & 1 & -4 \end{bmatrix}$$

we can suppose that

$$I_1 =< x^{r+1}, (x + y)^{r+1} >: y^{r+1} \text{ and } I_2 =< z^{r+1}, (z + y)^{r+1} >: y^{r+1}.$$

B. $(\mathbb{R}[x, y]/I_1)_s$

For any given $n \geq 1$, let T^n be an $n \times n$ upper triangular Toeplitz matrix such that $a_{ij} = \binom{n}{j-i}$.

$$T^r = \begin{pmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-2} & \binom{n}{n-1} \\ 0 & 1 & \binom{n}{1} & \cdots & \binom{n}{n-3} & \binom{n}{n-2} \\ 0 & 0 & 1 & \cdots & \binom{n}{r-4} & \binom{n}{r-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Let T_p^n be the $p \times p$ upper right corner submatrix of T^n.

We will show that if T_p^{r+1} is nonsingular for all $p \geq \frac{r+1}{2}$ then $N_{2r+1}(= (R/I)_r) = 0$. (i.e., the dimension of $S_d^r(\Delta)$ is given by the lower bound when $d = 2r+1$.)

We can look at the ideal I_1 as an ideal in $A = \mathbb{R}[x, y]$. Then we have the following Lemma.

Lemma VI.1. *Suppose that T_p^{r+1} is nonsingular for all $p \geq \frac{r+1}{2}$. Then, We have that*

$$dim\ (A/I_1)_s = r - s \quad \forall \quad \frac{r-1}{2} \leq s \leq r.$$

Moreover, if $\dfrac{r-1}{2} \leq s \leq r-1$ then

$$(A/I_1)_s = span\{y^s, y^{s-1}x, \ldots, y^{2s-r+1}x^{r-s-1}\}.$$

Proof. We need to break into two cases, depending on if r is odd or even.

First, let $r = 2n-1$ for some $n \in \mathbb{N}$. From [13], we get the minimal free resolution for A/I_1:

$$0 \to A(-2n) \to A(-n)^2 \to A \to A/I_1.$$

Calculating the Hilbert functions gives us

$$\begin{aligned} dim\ (A/I_1)_s &= \binom{s+1}{1} - 2\binom{s-n+1}{1} + \binom{s-2n+1}{1} \\ &= s+1-2(s-n+1) \\ &= r-s \end{aligned}$$

because $\frac{r-1}{2} \leq s \leq r$ and $r = 2n-1$.

Next, let $r = 2n$ for some $n \in \mathbb{N} \cup \{0\}$. From [13], we get the minimal free resolution for A/I_1:

$$0 \rightarrow A(-2n-1) \rightarrow A(-n) \oplus A(-n-1) \rightarrow A \rightarrow A/I_1.$$

In the similar way as above, we get

$$dim\ (A/I_1)_s = r - s.$$

Notice that $dim\ (A/I_1)_r = 0$.

For the second part, since we know that

$$dim\ (A/I_1)_s = r - s,$$

it suffices to show that

$$\sum_{k=0}^{r-s-1} c_k x^k y^{s-k} = 0 \text{ in } (A/I_1)_s \Longrightarrow c_k = 0\ \ \forall\ k = 0, \ldots, r-s-1.$$

Set up $g = \sum_{k=0}^{r-s-1} c_k x^k y^{s-k}$ and suppose $g = 0$ in $(A/I_1)_s$. That means $g \in I_1$.

Since $I_1 =< x^{r+1}, (x+y)^{r+1} >: y^{r+1}$, there are two homogeneous polynomials, h_1 and h_2 of degree s satisfying

$$g\ y^{r+1} = h_1 x^{r+1} + h_2 (x+y)^{r+1}. \tag{6.1}$$

Let $h_1 = \sum_{i=0}^{s} a_i x^i y^{s-i}$ and $h_2 = \sum_{j=0}^{s} b_j x^j y^{s-j}$. Then, substituting h_1, h_2 and g into (6.1) gives us

$$\sum_{k=0}^{r-s-1} c_k x^k y^{s-k+r+1} = \sum_{i=0}^{s} a_i x^{i+r+1} y^{s-i} + \sum_{j=0}^{s} \sum_{l=0}^{r+1} b_j \binom{r+1}{l} x^{j+l} y^{s-j+r+1-l}.$$

Comparing coefficients of $x^k y^{s-k+r+1}$ for $k = r-s, \ldots, s+r+1$ gives the following matrix equation:

$$\begin{matrix} x^{s+r+1} \\ x^{s+r}y \\ \vdots \\ x^{r+1}y^s \\ x^r y^{s+1} \\ \vdots \\ x^s y^{r+1} \\ \vdots \\ x^{r-s}y^{2s+1} \end{matrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \binom{r+1}{r} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \binom{r+1}{r+1-s} & \binom{r+1}{r+2-s} & \cdots & 1 \\ 0 & 0 & \cdots & 0 & \binom{r+1}{r-s} & \binom{r+1}{r+1-s} & \cdots & \binom{r+1}{r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \binom{r+1}{1} & \cdots & \binom{r+1}{s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \binom{r+1}{r-s} \end{pmatrix} \begin{pmatrix} a_s \\ a_{s-1} \\ \vdots \\ a_0 \\ b_s \\ \vdots \\ b_{2s+1-r} \\ \vdots \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} I_{s+1} & UL_{s+1} \\ 0 & T_{s+1}^{r+1} \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (6.2)$$

where $\vec{a} = (a_s, \ldots, a_0)^t, \vec{b} = (b_s, \ldots, b_0)^t$, $(\cdot)^t$ is the transpose matrix of $(\cdot)$ and UL is a unit lower triangular matrix.

Since $s \geq \frac{r-1}{2}$, we have $s+1 \geq \frac{r+1}{2}$ and (T_{s+1}^{r+1}) is nonsingular. Therefore, from the equation (6.2) we conclude

$$\vec{a} = \vec{b} = 0,$$

which implies

$$g \equiv 0 \ (i.e.,\ c_k = 0 \ \forall k = 0, \ldots, r-s-1).$$

□

Similarly, we can look at I_2 as an ideal in $B = \mathbb{R}[y,z]$. Then, we have

$$dim\,(B/I_2)_s = r - s \ \ \forall \ \frac{r-1}{2} \leq s \leq r$$

and if $\dfrac{r-1}{2} \leq s \leq r-1$ then

$$(B/I_2)_s = span\{y^s, y^{s-1}z, \ldots, y^{2s-r+1}z^{r-s-1}\}$$

provided that T_p^{r+1} is nonsingular for all $p \geq \frac{r+1}{2}$.

C. $(\mathbb{R}[x,y,z]/I)_r$

Note that if $\dfrac{r-1}{2} \leq s \leq r-1$ and

$$x^{\alpha}y^{s-\alpha} \equiv \sum_{i=0}^{r-s-1} a_i x^i y^{s-i} \ (mod\ I_1)$$

then

$$z^{\alpha}y^{s-\alpha} \equiv \sum_{i=0}^{r-s-1} a_i z^i y^{s-i} \ (mod\ I_2)$$

because I_2 can be obtained by replacing x with z in I_1.

Lemma VI.2. *Suppose that T_p^{r+1} is nonsingular for all $p \geq \frac{r+1}{2}$ and let*

$$P_{m,l} = y^{r-1-m}x^l z^{m-l} - y^{r-1-m}x^{m-l}z^l.$$

Then, $P_{m,l} \in I$ for all $0 \leq m \leq r-1$ and $0 \leq l \leq m$.

Proof. If $m = 0$ and $l = 0$, then it is clear because $P_{l,m} = 0$.

For the induction on m, suppose that $P_{k,l} \in I$ for all $k < m$.

Without loss of generality, for the given $P_{m,l}$, we can assume $2l < m$. (If not, consider $-P_{m,l} \in I$.)

Then, we have

$$\begin{aligned} r-1-l &> r-1-\frac{m}{2} \\ &\geq \frac{r-1}{2} \end{aligned}$$

because $m \leq r-1$.

Thus, by Lemma VI.1,

$$y^{r-1-m}z^{m-l} \equiv \sum_{i=0}^{l} a_i y^{r-1-l-i} z^i \ (mod\ I_2)$$

and

$$y^{r-1-m}x^{m-l} \equiv \sum_{i=0}^{l} a_i y^{r-1-l-i} x^i \ (mod\ I_1)$$

for some $a_i \in \mathbb{R}$.

Plugging the above into $P_{m,l}$ gives

$$\begin{aligned} P_{m,l} &\equiv x^l(\sum_{i=0}^{l} a_i y^{r-1-l-i} z^i) - z^l(\sum_{i=0}^{l} a_i y^{r-1-l-i} x^i) \ (mod\ I) \\ &\equiv \sum_{i=0}^{l} a_i (y^{r-1-l-i} x^l z^i - y^{r-1-l-i} x^i z^l) \ (mod\ I) \\ &\equiv \sum_{i=0}^{l} a_i P_{l+i,l} \ (mod\ I) \\ &\equiv 0 \ (mod\ I) \end{aligned}$$

since $2l < m$. □

Lemma VI.3. *Suppose that T_p^{r+1} is nonsingular for all $p \geq \frac{r+1}{2}$. Then,*

$$dim\ N_{2r+1} = 0.$$

That means dim $S^r_d(\Delta)$ *is given by lower bound in (1.1).*

Proof. Since $N_{2r+1} \cong (\mathbb{R}[x,y,z]/I)_r$, it suffices to show that

$$(\mathbb{R}[x,y,z]/I)_r = 0.$$

Let $f_{l,m,n} = y^l x^m z^n \in \mathbb{R}[x,y,z]$ with $l+m+n=r$.

We will use mathematical induction on n.

For the base case,

$$f_{l,m,0} = y^l x^m \in I$$

since $l+m=r$ and $(\mathbb{R}[x,y]/I_1)_r = 0$ from Lemma VI.1.

For the induction hypothesis, assume that

$$f_{l,m,n} \in I \ for \ all \ n \leq k.$$

Then, by symmetry of I_1 and I_2 we have

$$f_{l,m,n} \in I \ for \ all \ m \leq k. \tag{6.3}$$

Let l and m be any nonnegative integers satisfying $l+m+k=r$.

If $m=0$ then $f_{l,0,k} \in I$ and if $m>0$ then

by Lemma VI.2, we have

$$P_{r-1-l,m-1} \in I \tag{6.4}$$

because $r-1-l \leq r-1$ and $(r-1-l)-(m-1) = k \geq 0$.

Multiplying (6.4) by z gives

$$\begin{aligned} zP_{r-1-l,m-1} \in I &\Rightarrow z(y^l x^{m-1} z^k - y^l x^k z^{m-1}) \in I \\ &\Rightarrow y^l x^{m-1} z^{k+1} - y^l x^k z^m \in I \\ &\Rightarrow y^l x^{m-1} z^{k+1} \in I \\ &\qquad \text{since } y^l x^k z^m = P_{l,k,m} \in I \text{ from (6.3)} \\ &\Rightarrow f_{l,m-1,k+1} \in I. \end{aligned}$$

Since l and m were arbitrary, we are done. □

CHAPTER VII

SUMMARY

In this dissertation, we studied bivariate spline spaces. we approached the study of splines from the viewpoint of algebraic geometry by using the conformality condition. The conformality conditions leads to the machinery of sheaves and cohomology which provided a powerful type of generalization of linear algebra.

First, we introduced cofactor spaces and cofactor maps. They were used to interpret the change of dimensions when we removed non-totally interior edges. We used this fact to analyse the relationship of dimensions of spline spaces between complex and its subcomplexes.

Second, we built the sufficient and necessary condition of that the dimension of spline spaces are equal to the lower bound, which were conjectured true for $d \geq 2r+1$ in Theorem III.3. For the proof we used mathematical induction on the number of interior vertices.

Next, we defined a flawed triangulation and we splitted the given flawed triangulation into smaller flawed triangulations which are relatively easier to compute the dimension of spline spaces. Then, we applied this idea to deformed type-1 triangulations and semi-deformed type-2 triangulations. We reduced any sizes of them to specific type of smaller flawed triangulations. Also, we showed that if they are generic triangulations then the dimensions of spline spaces are same as the lower bound for $d = 2r + 1$ and $r \leq 10$.

Finally, we studied the "$2r + 1$" conjecture on a certain triangulation. We know that the conjecture is sharp because the dimension formula failed on a certain triangulation for $d = 2r$, but we do not know if it holds on the same triangulation when $d = 2r + 1$. It is related to a Toeplitz matrix.

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VITA

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