NORMALIZERS OF FINITE VON NEUMANN ALGEBRAS

A Dissertation

by

JAN MICHAEL CAMERON

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2009

Major Subject: Mathematics

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ABSTRACT

Normalizers of Finite von Neumann Algebras. (August 2009)

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For an inclusion $N \subseteq M$ of finite von Neumann algebras, we study the group of normalizers

$$\mathcal{N}_M(B) = \{u : uBu^* = B\}$$

and the von Neumann algebra it generates. In the first part of the dissertation, we focus on the special case in which $N \subseteq M$ is an inclusion of separable II_1 factors. We show that $\mathcal{N}_M(B)$ imposes a certain "discrete" structure on the generated von Neumann algebra. An analysis of the bimodule structure of certain subalgebras of $\mathcal{N}_M(B)''$ then yields a "Galois-type" theorem for normalizers, in which we find a description of the subalgebras of $\mathcal{N}_M(B)''$ in terms of a unique countable subgroup of $\mathcal{N}_M(B)$. We then apply these general techniques to obtain results for inclusions $B \subseteq M$ arising from the crossed product, group von Neumann algebra, and tensor product constructions. Our work also leads to a construction of new examples of norming subalgebras in finite von Neumann algebras: If $N \subseteq M$ is a regular inclusion of II_1 factors, then N norms M. These new results and techniques develop further the study of normalizers of subfactors of II_1 factors.

The second part of the dissertation is devoted to studying normalizers of maximal abelian self-adjoint subalgebras (masas) in nonseparable II_1 factors. We obtain a characterization of masas in separable II_1 subfactors of nonseparable II_1 factors, with a view toward computing cohomology groups. We prove that for a type II_1 factor N with a Cartan masa, the Hochschild cohomology groups $H^n(N, N)$=0, for all $n \geq 1$.

This generalizes the result of Sinclair and Smith, who proved this for all N having separable predual. The techniques and results in this part of the thesis represent new progress on the Hochschild cohomology problem for von Neumann algebras.

To Melly

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CHAPTER I

INTRODUCTION

The objective of this dissertation is to develop a structure theory for normalizers of subalgebras of finite von Neumann algebras. Our investigation will address a number of topics of ongoing interest in finite von Neumann algebra theory. The *normalizer* of a subalgebra B of a von Neumann algebra M is the group

$$\mathcal{N}_M(B) = \{u : uBu^* = B\}$$

of unitaries in M that stabilize B under conjugation. Dixmier [8] studied maximal abelian subalgebras (masas) of II_1 factors, and classified them according to the normalizer. He called a masa $A \subseteq M$ singular, regular (or Cartan), or semiregular, according to whether the von Neumann algebra generated by $\mathcal{N}_M(A)$ was equal to A itself, M, or a proper subfactor of M respectively. These definitions clearly also make sense when A is not necessarily abelian.

The study of normalizers of finite von Neumann algebras continues today. One of the main motivating questions of the subject is: what structure is imposed by normalizing unitaries on the von Neumann algebra they generate? It is also of interest to identify the structure and properties of the normalizing group $\mathcal{N}_M(N)$ itself in various examples of an inclusion $N \subseteq M$. Such questions originated in the seminal papers of Feldman and Moore [11, 12] and have been addressed more recently in the work of Smith, White, and Wiggins [43], and these three authors with Fang [10]. One problem is to identify the normalizers of the von Neumann algebra LH inside LG, when $H \subseteq G$ is an inclusion of discrete groups. In particular, what

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is the relation between the "analytic" normalizer $\mathcal{N}_{LG}(LH)$ of the subalgebra and the "algebraic" normalizer $\mathcal{N}_G(H)$ of the generating group? In the first part of this dissertation, which addresses the structure of the normalizer of a type II_1 factor, we answer this question in case LH is a II_1 factor. Our primary tool for obtaining that result will be our proof of a crossed product decomposition for normalizers; in particular, when $B \subseteq M$ is a subfactor, $\mathcal{N}_M(B)$ imposes a discrete crossed product structure on the von Neumann algebra it generates. By analyzing the structure of weakly closed bimodules in $\mathcal{N}_M(B)''$, this also leads to a "Galois-type" theorem for normalizers which generalizes work of Nakamura and Takeda [24] and Choda [2] on crossed products, in which we find a description of the subalgebras of $\mathcal{N}_M(B)''$ in terms of a unique countable subgroup of $\mathcal{N}_M(B)$.

The most important difference between our setting and that of previous work is the presence of a non-trivial relative commutant $N' \cap M$ which is not contained in N A counterexample in chapter III shows that a strict Galois-type correspondence is not possible in our setting. Thus, a new definition is required: that of a *seminormalizing partial isometry.* The main result of Chapter III states that intermediate subalgebras of the normalizer of any finite von Neumann algebra are parameterized by subsets of the groupoid $\mathcal{SN}_M(N)$ of seminormalizers.

This makes the problem of characterizing subalgebras of a normalizer interesting; what structure is displayed by $\mathcal{SN}_M(N)$ in various examples of inclusions $N \subseteq M$? The following three chapters of the dissertation are devoted to that problem and related questions. Chapter IV contains some preliminary facts about the von Neumann algebra generated by a II_1 factor $N \subseteq M$ and its relative commutant.

In Chapter V, we use the ideas of Chapter IV as part of a technique for "factoring out" the relative commutant, which allows us to decompose the normalizer of a subfactor as a twisted crossed product of an underlying von Neumann algebra by a certain countable, discrete group G of normalizing unitaries. These techniques allow us to use tools from crossed products to produce results for more general regular inclusions. In particular, Theorem V.9 gives a characterization of intermediate subalgebras between a II_1 factor N and its normalizer $\mathcal{N}_M(N)''$, in terms of certain sequences of partial isometries.

We will see in Chapter VI that the discrete decompositions of the normalizer in Chapter V are unique with respect to the group G. That observation leads to a new proof of a recent result of A. Wiggins, and a generalization of certain results of Smith, White, and Wiggins. One interesting aspect of our approach is that it is sufficiently general to yield information about inclusions arising from crossed product, group von Neumann algebra, and tensor product constructions. In Chapter VII, we apply ideas from previous chapters further to construct new examples of norming algebras. We show, in particular, that if M is a finite von Neumann algebra with a fixed, faithful trace and $N \subseteq M$ is a regular subfactor, then N norms M.

The second part of our investigation will be devoted to the normalizers of masas in nonseparable von Neumann algebras. A von Neumann algebra is *separable* if its predual is a separable Banach space. Our work is motivated by the Hochschild cohomology problem for von Neumann algebras, in which a characterization is sought for bounded, multilinear maps on von Neumann algebras. The study of Hochschild cohomology for von Neumann algebras can be traced back to a well-known theorem, due separately to Kadison [17] and Sakai [34], which states that every derivation $\delta : M \to M$ on a von Neumann algebra M is inner, that is, there exists an element $a \in M$ such that $\delta(x) = xa - ax$, for all $x \in M$. This corresponds to the vanishing of the first continuous Hochschild cohomology group, $H^1(N, N)$. It is natural to conjecture that the higher cohomology groups $H^n(N, N)$ are also trivial. This program was taken up in the seventies, in a series of papers by Johnson, Kadison, and Ringrose

([14, 19, 20]), who affirmed the conjecture for type I algebras and hyperfinite algebras. In the mid 1980's, a parallel theory of completely bounded cohomology was initiated. The groups $H^n_{cb}(N,N)$ are computed under the additional assumption that all cocycles and coboundaries, usually assumed to be norm continuous, are completely bounded. Christensen and Sinclair showed that $H^n_{cb}(N,N) = 0$ for all von Neumann algebras N(see [40] for details of the proof). It was proved in [6] that continuous and completely bounded cohomology coincide for all von Neumann algebras stable under tensoring with the hyperfinite II_1 factor R. Thus, it is known that the continuous Hochschild cohomology vanishes for all von Neumann algebras of type I, II_∞, and III, and for all type II_1 algebras stable under tensoring with R. This leaves open the case of the general type II_1 von Neumann algebra. By direct integral techniques, it is enough in the separable case to compute the groups $H^n(N,N)$ when N is a factor.

As in the Kadison-Sakai theorem, the vanishing of Hochschild cohomology groups gives structural information about certain bounded n-linear maps on the von Neumann algebra, in particular, that they can be formed from the $(n-1)$-linear maps. The vanishing of certain other higher cohomology groups also yields some nice perturbation results for von Neumann algebras (see [40, Chapter 7]). The Hochschild cohomology groups have been shown to be trivial for a few large classes of type II_1 factors, including those with property Γ, [7], and those which contain a Cartan maximal abelian subalgebra (masa) and have separable predual, [41]. In our work, we will extend this last result to include arbitrary II_1 factors with a Cartan masa. The techniques in [41] depended on separability, and so could not be modified to encompass nonseparable von Neumann algebras. The techniques we will use originated in [7] and [42], and can be viewed as part of a general strategy to relate properties of nonseparable type II_1 factors to their separable subalgebras.

CHAPTER II

PRELIMINARIES

A. Finite von Neumann algebras

The setting for this dissertation is a finite von Neumann algebra; a comprehensive account of finite von Neumann algebra theory is given in [37]. We will assume familiarity with the basic general theory of C^*-algebras and von Neumann algebras, as can be found in chapters 1-5 of [18]; this chapter contains a review of basic concepts and results of which we will make repeated use. The main references for this chapter are [18, 37], unless otherwise noted.

A von Neumann algebra is a self-adjoint subalgebra of the bounded operators on a Hilbert space $\mathcal{H}$ (which we will denote by $B(\mathcal{H})$) which is closed in the weak operator topology. We shall use the standard notation N' for the commutant

$$\{y \in B(\mathcal{H}) : xy = yx, \text{ for all } x \in N\}$$

of a von Neumann algebra $N \subseteq B(\mathcal{H})$, and $Z(N)$ for the center $N \cap N'$. A *factor* is a von Neumann algebra N for which $Z(N) = \mathbb{C}1$.

Recall that a net $\{x_\alpha\}$ in a von Neumann algebra N represented on a Hilbert space $\mathcal{H}$ converges to $x \in N$ in the weak operator topology if and only if the nets $|\langle x_\alpha \xi, \eta\rangle - \langle x\xi, \eta\rangle|$ converge to 0 in $\mathbb{C}$ for all vectors $\xi, \eta \in \mathcal{H}$. We say that $\{x_\alpha\}$ converges to $x \in N$ in the strong operator topology (or strongly), if $\lim_\alpha \|x_\alpha \xi - x\xi\| = 0$ for all $\xi \in \mathcal{H}$. If additionally, we have that $\|x_\alpha^* \xi - x^*\xi\|$ converges to zero for all $\xi \in \mathcal{H}$, we say that $\{x_\alpha\}$ converges to x $*$-strongly. Basic functional analysis shows that the strong and weak closures coincide for convex sets, hence also for subspaces. The following foundational result in von Neumann algebras characterizes this closure of a $*$-algebra in terms of the commutants of the algebra.

Theorem II.1 (von Neumann Double Commutant Theorem)**.** *Let A be a $*$-subalgebra of the bounded operators on a Hilbert space $\mathcal{H}$ containing the unit $1 \in B(\mathcal{H})$. Then the weak closure of A is equal to the double commutant A''. In particular, A is a von Neumann algebra if and only if $A = A''$.*

It is very difficult in general to write down the general form of an operator in the weak closure of a $*$-subalgebra of $B(\mathcal{H})$. Many von Neumann algebras are thus described in terms of generating sets or dense $*$-subalgebras. The following very useful theorem will allow us to pass from approximation arguments in dense subalgebras of von Neumann algebras to the whole von Neumann algebra, the critical feature being the norm estimate in the first part of the theorem.

Theorem II.2 (The Kaplansky Density Theorem)**.** *Let $N \subseteq B(\mathcal{H})$ be a von Neumann algebra and $A \subseteq N$ a strongly dense $*$-subalgebra, not assumed to be unital. Then*

(i) For any $x \in N$ there exists a net $\{x_\alpha\}$ in A converging $*$-strongly to x, such that $\|x_\alpha\| \leq \|x\|$ for all α.

(ii) If x is self-adjoint, then we may assume x_α to be self-adjoint for all α.

(iii) If $u \in \mathcal{U}(N)$ is a unitary operator, then there exists a net u_α of unitaries in the C^*-algebra generated by A converging to u.

The following result, which will be used in the averaging techniques developed in the following two chapters, says that averages of unitary conjugates of any element of the von Neumann algebra get arbitrarily close to the center of the von Neumann algebra. Recall the notation *conv* A for the convex hull of a set $A \subseteq B(\mathcal{H})$.

Theorem II.3 (The Dixmier Approximation Theorem)**.** *Let N be a von Neumann algebra with unitary group $\mathcal{U}(N)$. Then for any $x \in N$ the intersection of the operator norm-closure of the set*

$$\mathit{conv}\,\{uxu^* : u \in \mathcal{U}(N)\}$$

with the center $Z(N)$ is nonempty.

Projections are abundant in von Neumann algebras; it is well-known that every von Neumann algebra is the norm closure of the linear span of its projections. We say that projections e and f in a von Neumann algebra N are *Murray-von Neumann equivalent* (or simply *equivalent*) if there exists a partial isometry $w \in N$ such that $w^*w = e$ and $ww^* = f$.

Every von Neumann algebra decomposes as a direct integral of factors, which Murray and von Neumann classified according to equivalence of projections in the factor. A factor N is either of *infinite* or *finite* type, respectively, depending on whether or not the unit $1 \in N$ is equivalent to a proper subprojection of itself. Thus, a factor N is finite if and only if every partial isometry $w \in N$ satisfying either $ww^* = 1$ or $w^*w = 1$ is automatically a unitary; otherwise, the factor is infinite. A finite factor N is also characterized by the existence of a finite trace $\tau : N \to \mathbb{C}$. More specifically, $\tau : N \to \mathbb{C}$ is a linear functional with the following properties:

- *Normal:* The map τ is continuous on N, when N is given the weak-$*$ topology induced by N_*.
- *Positive:* $\tau(x^*x)$ is a non-negative real number for all $x \in N$.
- *Tracial:* $\tau(xy) = \tau(yx)$, for all $x, y \in N$.
- *Faithful:* For all $x \in N$, $\tau(x^*x) = 0$ if and only if $x = 0$.

In this work, all traces will be assumed to be *normalized*, that is, satisfying $\tau(1) = 1$.

The type $\mathrm{I_n}$ factors are all isomorphic to $\mathbb{M}_n(\mathbb{C})$, for $1 \leq n < \infty$, so admit such a trace. Type II_1 factors are infinite-dimensional and finite. We note that a finite von Neumann algebra is a factor if and only if it admits a *unique* finite trace τ. Factors of infinite type include the type III factors, in which any two nonzero projections are equivalent, the type I_∞ factors, which are $B(\mathcal{H})$ for an infinite-dimensional Hilbert space $\mathcal{H}$, and the type II_∞ factors, which are formed by taking the tensor product of a II_1 factor with a type I_∞ factor.

In this dissertation, we will be concerned exclusively with finite von Neumann algebras, and particularly with type II_1 factors. We construct examples of these objects in the next section.

B. Examples and basic properties of II_1 factors

The prototypical example of a type II_1 von Neumann algebra is the algebra $L^\infty[0,1]$ of bounded, Lebesgue measurable functions on the interval. As is well-known, this is a weakly closed $*$-subalgebra of the bounded operators on $L^2[0,1]$. The multiplication on $L^\infty[0,1]$, which is pointwise multiplication of functions almost everywhere, is clearly abelian; it is easy to see that the integral with respect to Lebesgue measure acts as a trace on $L^\infty[0,1]$. In fact, every finite, separable, infinite dimensional abelian von Neumann algebra is isomorphic to $L^\infty[0,1]$. In what follows, we construct the most common examples of non-abelian type II_1 von Neumann algebras, arising from group and matrix constructions.

1. Group von Neumann algebras

Let G be a countable, discrete group. For each $g \in G$, we define a function $\delta_g : G \to \mathbb{C}$ by $\delta_g(h) = 1$ if $g = h$ and $\delta_g(h) = 0$ otherwise. The collection $\{\delta_g\}_{g \in G}$ forms an orthonormal basis for the Hilbert space $\ell^2(G)$ of functions on G, defined

$$\ell^2(G) = \left\{ \phi : G \to \mathbb{C} : \sum_{g \in G} |\phi(g)|^2 < \infty \right\},$$

with inner product

$$\langle \phi, \psi \rangle = \sum_{g \in G} \phi(g)\overline{\psi(g)}.$$

For each $h \in G$, we define an operator λ_h on $\ell^2(G)$ by

$$\lambda_h \delta_g = \delta_{hg}, \text{ for all } g \in G.$$

This extends to a unitary operator on $\ell^2(G)$. The map $\lambda : G \to B(\ell^2(G))$ is thus a unitary representation of G, called the *left regular representation* of G. This group of unitary operators generates a self-adjoint subalgebra of $B(\ell^2(G))$, the weak closure of which is defined to be LG, the *left group von Neumann algebra* of G. By the von Neumann double commutant theorem, one has

$$LG = \{\lambda_g : g \in G\}''.$$

Moreover, LG is finite; a trace on the von Neumann algebra is given by

$$\tau(x) = \langle x\delta_e, \delta_e \rangle.$$

It is also possible to define a von Neumann algebra RG with respect to the *right regular representation* of G on $\ell^2(G)$, defined by the map $\rho : G \to \mathcal{U}(B(\ell^2(G)))$ defined

$$\rho_g \delta_h = \delta_{hg^{-1}}.$$

In a similar fashion, we obtain RG as $\{\rho_g : g \in G\}''$. It is easy to see that $LG' = RG$, and so by the double commutant theorem, we have $RG' = LG$ also.

Many properties of group von Neumann algebras are equivalent to corresponding properties of the generating discrete group. One important, and well-known observation along these lines characterizes which countable discrete groups give rise to factors under this construction.

Proposition II.4. *Let G be a countable discrete group. Then LG is a factor if and only if G is i.c.c., that is, if for all $g \neq e$, the set*

$$\left\{hgh^{-1} : h \in G\right\}$$

has infinite cardinality.

Two common examples of i.c.c. groups are the free group on two generators $\mathbb{F}_2$, and the group Π of all finite permutations of a countable set.

2. The hyperfinite II_1 factor R

The hyperfinite II_1 factor, denoted by R, is formed by an increasing union of matrix algebras. In particular, we have embeddings of the complex matrix algebras

$$M_2 \hookrightarrow M_4 \hookrightarrow \cdots M_{2^n} \hookrightarrow M_{2^{n+1}} \hookrightarrow \cdots$$

given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Then $\bigcup_{n\geq 1} M_{2^n}$ is a $*$-algebra of bounded operators on the Hilbert space $\bigotimes_{n=1}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n = \mathbb{C}^2$ for all n. A trace τ on the C*-algebra A generated by $\bigcup_{n\geq 1} M_{2^n}$ is defined by extending the normalized trace on M_{2^n}, for all n. One then represents A

by the GNS construction with respect to τ and defines R to be the weak closure of A in the GNS representation. Murray and von Neumann [23] proved that R is unique with respect to this property of approximate finite-dimensionality; that is, there is a unique II_1 factor R generated by $\bigcup_{n\geq 1} N_n$, where N_n is finite-dimensional for all n. Another important result, due to Schwartz [35], is that LG is hyperfinite if and only if the group G is amenable.

3. Automorphisms and crossed products

In what follows, we denote by $Aut(N)$ the group of automorphisms of a von Neumann algebra N.

Definition II.5. Let N be a von Neumann algebra, and let $\alpha : G \to Aut(N)$ be an action of a countable, discrete group G on N. We say that α is

(i.) *ergodic* if every $x \in N$ satisfying $\alpha_g(x) = x$ for all $g \in G$ lies in $\mathbb{C}1$.

(ii.) *outer* if $u \in \mathcal{U}(N)$ is a unitary and satisfies $\alpha_g(x) = uxu^*$ for all $x \in N$ implies that $g = 1$, the identity element of G, and $u = 1$, the unit of N.

(iii.) *free* if $y \in N$ is nonzero and satisfies

$$yx = \alpha_g(x)y$$

for all $x \in N$ implies that $y = 1$ and $g = 1$.

We will often denote by $Adu(x)$ the automorphism $x \mapsto uxu^*$. One way to identify the outer automorphism group of a von Neumann algebra N is to define an *inner* automorphism ϕ of N to be one for which there exists a unitary $u \in N$ such that $\phi = Adu$. One then defines $Inn(N)$ to be the group of inner automorphisms of N, and the group of outer automorphisms is the quotient group $Out(N) = Aut(N)/Inn(N)$.

Freeness of an automorphism is generally a stronger condition than outerness; an important result, the proof of which comes from [37], is that these concepts coincide for a factor N.

Lemma II.6. *An action α of a countable, discrete group G on a factor N is free if and only if it is outer.*

Proof. Suppose that a nonzero $y \in N$ satisfies

$$yx = \alpha_g(x)y,$$

for all $x \in N$. Then replacing x by $\alpha_{g^{-1}}(x)$ in the above equation gives a similar relation for y^* and $\alpha_{g^{-1}}$. It follows that yy^* and y^*y are in $\mathbb{C}1$, since N is a factor. Then if $\lambda = (y^*y)^{1/2}$, the operator $u = \lambda^{-1}y$ is a unitary in N satisfying $\alpha_g(x) = uxu^*$, for all $x \in N$. Since α is outer, this implies that $g = 1$. $\square$

If N is a von Neumann algebra represented on a Hilbert space $\mathcal{H}$ and G is a group acting by outer automorphisms on N, we may represent a new von Neumann algebra $N \rtimes_\alpha G$, called the *crossed product*, on the Hilbert space

$$\mathcal{H} \otimes \ell^2(G) = \left\{ \xi : G \to L^2(N) : \sum \|\xi(g)\|^2 < \infty \right\}.$$

We define a representation $\pi : N \to \mathcal{H} \otimes \ell^2(G)$ by

$$\pi(x)\xi(g) = \alpha_{g^{-1}}(x)\xi(g),$$

for $\xi \in \mathcal{H} \otimes \ell^2(G)$. We also define a representation of the group $u : G \to \mathcal{H} \otimes \ell^2(G)$ by

$$u_h\xi(g) = \xi(h^{-1}g).$$

One checks that u is a unitary representation, and that the relation

$$\pi(\alpha_h(x)) = u_h \pi(x) u_h^*$$

is satisfied for all $x \in N$ and all $h \in G$. The crossed product $N \rtimes_\alpha G$ is the von Neumann algebra generated by $\{\pi(x) : x \in N\}$ and $\{u_g : g \in G\}$. When there is no ambiguity we will drop the action in our notation and denote the crossed product $N \rtimes G$.

The crossed product construction may also be "twisted" by a 2-cocycle on the group G; a standard reference for the twisted crossed product construction below is [4]. Given groups G and K, a *2-cocycle* is a map $\omega : G \times G \to K$ satisfying

$$\omega(gh, k)\alpha_k^{-1}(\omega(g, h)) = \omega(g, hk)\omega(h, k),$$

for all $g, h, k \in G$. We call this equality the *cocycle relation*. In what follows, we will obtain a 2-cocycle from a normal inclusion of groups $L \subseteq K$, associated to an exact sequence

$$1 \to L \to K \to G \to 1.$$

There then exists a section $u : G \to K$ that picks one element from each coset in G. For $g, h \in G$ we have $(u_g L)(u_h L) = u_{gh} L$, so there exists a group element $\omega(g, h) \in L$ such that

$$u_g u_h = u_{gh}\omega(g, h).$$

One then deduces the cocycle relation for ω from associativity of the group. If G is a group of automorphisms of a von Neumann algebra N (represented on a Hilbert space $\mathcal{H}$), we may define a *semirepresentation* $\alpha : G \to Aut(N)$ with respect to a cocycle ω on G, which satisfies

$$\alpha_g \alpha_h = \alpha_{gh} Ad\omega(g, h),$$

for all $g, h \in G$. We define a representation $\pi : N \to \mathcal{H} \otimes \ell^2(G)$ by

$$\pi(x)\xi(g) = \alpha_{g^{-1}}(x)\xi(g),$$

for $\xi \in \mathcal{H} \otimes \ell^2(G)$, and a semirepresentation $u : G \to \mathcal{H} \otimes \ell^2(G)$ by

$$u_h\xi(g) = \omega(h, h^{-1}g)\xi(h^{-1}g).$$

The *twisted crossed product* $N \rtimes_\alpha^\omega G$ with respect to the cocycle ω is defined to be the von Neumann algebra on $\mathcal{H} \otimes \ell^2(G)$ generated by $\pi(N)$ and $u(G)$. Note that this construction produces the ordinary crossed product when $\omega = 1$ is the trivial cocycle.

4. The standard representation

The trace τ on a finite von Neumann algebra N induces an inner product on N by

$$\langle x, y\rangle = \tau(xy^*).$$

This defines a Hilbert norm on N, which we will denote by $\|\cdot\|_2$. The completion of N with respect to this norm is denoted $L^2(N)$. Notice that $N \subseteq L^2(N)$ is a dense linear subspace in the $\|\cdot\|_2$ norm by this construction. This can also be seen by letting $\zeta \in L^2(N)$ be the vector corresponding to the unit $1 \in N$ and defining an isometric embedding $N \mapsto N\zeta$. The von Neumann algebra N then also acts by left multiplication on the dense subspace $N\zeta$, hence also on $L^2(N)$. We call this the *standard representation* of N. One may also define a conjugate map $J : L^2(N) \to L^2(N)$ as an extension of the conjugate-linear isometry defined $Jx = x^*$, for $x \in N$. Notice that J^2 is the identity map; it can also be shown that $JNJ = N'$ (see [37] for details). We will often make use of the following examples of finite von Neumann algebras in standard representation.

Example II.7. If G is a countable, discrete group then $L^2(LG) = \ell^2(G)$.

Example II.8 ([4]). If G is a countable, discrete group and $\alpha : G \to Out(N)$ is a semirepresentation with respect to a cocycle $\omega : G \times G \to \mathcal{U}(N)$, then

$$L^2(N \rtimes_\alpha^\omega G) = L^2(N) \otimes \ell^2(G).$$

Elements of a crossed product von Neumann algebra are formal sums $\sum_{g\in G} x_g u_g$, where the u_g are the unitary representatives of elements of G and each $x_g \in N$. It follows from the standard representation above that these sums converge in the $\|\cdot\|_2$-norm of that Hilbert space, so are a kind of generalized Fourier series, but do not generally converge in the strong or weak topologies [22].

If $N \subseteq M$ is an inclusion of finite von Neumann algebras, the resulting inclusion of Hilbert spaces $L^2(N) \subseteq L^2(M)$ induces an orthogonal projection $e_N : L^2(M) \to L^2(N)$, called the *Jones projection*. The restriction of this map to M is a bounded, normal, completely positive conditional expectation $\mathbb{E}_N : M \to N$, which is characterized by the following two properties:

(i) For all $x \in M$ and $n_1, n_2 \in N$, $\mathbb{E}_N(n_1 x n_2) = n_1 \mathbb{E}_N(x) n_2$ (N-bimodularity)

(ii) For all $x \in M$, $\tau(\mathbb{E}_N(x)) = \tau(x)$ ($\mathbb{E}_N$ is trace-preserving).

We emphasize that $\mathbb{E}_N$ is unique among all maps from M into N with these two properties, and not just those assumed to be continuous and linear.

5. Separability of II$_1$ factors

A finite von Neumann algebra N is called *separable* if one of the following equivalent conditions holds [44]:

(i) There exists a countable set of projections in N generating a weakly dense subalgebra of N.

(ii) The Hilbert space $L^2(N, \tau)$ (on which N is faithfully represented) is separable in its norm.

(iii) The predual of N is a separable Banach space.

The weakly closed unit ball of N is then separable in this topology, being a compact metric space.

Lemma II.9. *Let N be a finite von Neumann algebra with a normal, faithful trace τ. Let M be a separable subalgebra of N. Suppose there exists a subset $S \subseteq N$ of unitaries such that $S'' = N$. Then there exists a countable subset F of S such that $F'' \supseteq M$.*

Proof. We first claim that $C^*(S)$ is $\|\cdot\|_2$-dense in $L^2(N)$. Let $x \in N$. Since $C^*(S)$ is strongly dense in N, by the Kaplansky density theorem, there exists a net $\{x_\alpha\}$ in $C^*(S)$ such that $\|x_\alpha\| \leq \|x\|$ and x_α converges to x $*$-strongly. Then also $(x_\alpha - x)^*(x_\alpha - x)$ converges to 0 weakly. Moreover, $\|(x_\alpha - x)^*(x_\alpha - x)\|$ is uniformly bounded and τ is a normal state (hence weakly continuous on bounded subsets of N by [18, Theorem 7.1.12]); then $\|x_\alpha - x\|_2^2 = \tau((x_\alpha - x)^*(x_\alpha - x))$ converges to zero. Thus, $x \in \overline{C^*(S)}^{\|\cdot\|_2}$, and the claim follows.

Note that $L^2(M)$ is a separable Hilbert subspace of $L^2(N)$. Let $\{\xi_n\}_{n=1}^\infty$ be a dense subset. For each n, there is a sequence $\{s_{nk}\}_{k=1}^\infty$ in $C^*(S)$ such that s_{nk} converges in $\|\cdot\|_2$-norm to ξ_n. The operators s_{nk} lie in the norm closure of $\mathrm{Alg}(F)$, for some countable subset F of S. It follows that $L^2(M) \subseteq L^2(\mathrm{Alg}(F)) \subseteq L^2(F'')$. We claim that this implies $M \subseteq F''$. Let $\mathbb{E}_M : N \to M$ and $\mathbb{E}_{F''} : N \to F''$ denote the respective trace-preserving conditional expectations, obtained by restricting the Hilbert space projections $e_M : L^2(N) \to L^2(M)$ and $e_{F''} : L^2(N) \to L^2(F'')$ to N. For any $x \in M$, we have

$$x = e_{F''}(x) + (1 - e_{F''})(x) = e_{F''}(x),$$

since $L^2(M) \subseteq L^2(F'')$. But then $x = \mathbb{E}_{F''}(x)$, so $M \subseteq F''$. $\square$

C. Subfactors and Galois theory

In the 1960's, Nakamura and Takeda initiated the study of inclusions of subfactors with their analysis of operator algebras generated by a finite group G acting outerly on a II_1 factor M. In particular, [24] showed that the lattice of intermediate subalgebras for inclusions $M^G \subset M$, where M^G is the fixed point algebra under the action of G, mirrors the lattice of subgroups of G. A similar result was proved in [25] for inclusions arising from crossed products $M \subseteq M \rtimes G$. The results began a subfield of operator algebras which might be called "noncommutative" or "quantized Galois theory."

Many authors have subsequently taken up the study of intermediate subalgebras. H. Choda [2] generalized the crossed product result of Nakamura and Takeda to the case of a group G of automorphisms acting by outer automorphisms on a II_1 factor N.

Theorem II.10 (Choda, [2])**.** *Let N be a II_1 factor and let G be a countable group acting on N by outer automorphisms. Then for any subgroup $H \subseteq G$, if P is a von Neumann algebra such that*

$$N \rtimes H \subseteq P \subseteq N \rtimes G,$$

then $P = N \rtimes K$ for some subgroup $H \subseteq K \subseteq G$.

Such results have been extremely useful in structure theory of II_1 factors. Wiggins [47] used the Choda result to construct examples of singular subfactors arising from crossed product inclusions $N \rtimes H \subseteq N \rtimes G$. The Choda result can also be used to construct examples of maximally injective subalgebras of crossed products [37].

D. Normalizers and the normalizing groupoid

If $B \subseteq M \subseteq Q$ is an inclusion of von Neumann algebras, we will denote the *normalizer* of B in Q by

$$\mathcal{N}_Q(B) = \{u \in \mathcal{U}(Q) : uBu^* = B\},$$

where $\mathcal{U}(Q)$ denotes the unitary group of Q. We will sometimes drop the Q subscript from this notation when the ambient von Neumann algebra is clear. Dixmier [8] initiated the study of normalizers in von Neumann algebras, when he classified the masas A of a von Neumann algebra M as regular (or Cartan), singular, or semi-regular according to whether $\mathcal{N}_M(A)'' = M$, $\mathcal{N}_M(A)'' = A$, or $\mathcal{N}_M(A)''$ is a proper subfactor of M. We will retain this terminology for a general von Neumann subalgebra A. In the case of an inclusion $N \subseteq M$, where N is an irreducible subfactor of M, Smith, White, and Wiggins showed [43] that there are unitaries $u \in M$ for which uNu^* is a proper subset of N. This necessitated the definition of the *one-sided normalizer* of N, which is denoted by

$$\mathcal{ON}_M(N) = \{u \in \mathcal{U}(M) : uNu^* \subseteq N\},$$

and is generally larger than $\mathcal{N}_M(N)$. In [43] an example was also given of a singular inclusion $N \subseteq M$ for which $\mathcal{ON}(N)'' = M$.

Another object of interest, given an inclusion $B \subseteq Q$, is the normalizing groupoid $\mathcal{GN}_Q(B)$, which consists of those partial isometries $w \in Q$ such that both w^*w and ww^* are in B and

$$wBw^* = Bww^* = ww^*B.$$

The following result of Dye [9] , the proof of which can also be found in [37], says that every element of the normalizing groupoid of a masa can be extended to a normalizing unitary. It will be central to the subject matter of this paper.

Theorem II.11. *Let N be a von Neumann algebra with a faithful, normal trace τ and let A be an abelian von Neumann subalgebra of N. Then a partial isometry $v \in N$ is in $\mathcal{GN}(A)$ if, and only if, there exists a projection $p \in A$ and a unitary $u \in \mathcal{N}(A)$, such that $v = pu$.*

Theorem II.11 was used by Jones and Popa [16] to obtain an operator algebraic proof of the following result of Dye, the so-called *Dye correspondence* for Cartan masas in a von Neumann algebra.

Theorem II.12. *Let A be a Cartan maximal abelian subalgebra of a finite von Neumann algebra M. If $N \subseteq M$ is a von Neumann subalgebra containing A, then A is a Cartan maximal abelian subalgebra of N.*

The techniques used in proving the latter two results depend heavily on commutativity of A, and in fact one can show that the first result is false for groupoid normalizers of general finite von Neumann algebras. However, we will show in what follows that a similar result to Theorem II.11 does hold for what we will call "seminormalizing" partial isometries of a subfactor. We present a generalization of Theorem II.12 in the next chapter.

CHAPTER III

SEMINORMALIZERS OF VON NEUMANN SUBALGEBRAS

We may be tempted to hope that a result analogous to the Dye correspondence holds for a regular inclusion of von Neumann algebras $N \subseteq M$ in which N is not a masa. A reasonable conjecture would be that if $N \subseteq M$ is a regular inclusion of von Neumann algebras, and P is a von Neumann subalgebra satisfying $N \subseteq P \subseteq M$, then there exists a subgroup $H \subseteq \mathcal{N}_M(N)$ such that $P = (N \cup H)''$. A simple example in the first section of this chapter shows that this conjecture is false, even when N and M are factors. We will see that the presence of a non-trivial relative commutant $N' \cap M$ makes the inclusion $N \subseteq M$ somewhat less rigid than an inclusion arising from a crossed product or a Cartan subalgebra. A new definition, that of a *seminormalizer* of a von Neumann subalgebra is explored in the first section. In the second section, it is shown that this is the "right" definition by which to obtain a generalized Dye correspondence; intermediate subalgebras of a normalizer are generated by sets of seminormalizing partial isometries.

A. The definition

Example III.1. Let N be the hyperfinite II$_1$ factor, which admits an outer action by any finite group. Let $\alpha : \mathbb{Z}_2 \to Out(N)$ be such an action and let $\alpha : \mathbb{Z}_2 \to M_3$ be the trivial action (sending both group elements to the identity) on the $3{\times}3$ matrices over the complex numbers. We let M be the von Neumann algebra

$$(N \otimes M_3) \rtimes_\alpha \mathbb{Z}_2,$$

and consider the inclusion $N \subseteq M$. Note that M is a factor. Let B be the subalgebra of M_3 of matrices of the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Note that B is the algebra $\mathbb{C} \oplus M_2$. Let Q be the von Neumann subalgebra of M generated by $N \overline{\otimes} B$ and vg, where g is the nontrivial group element of $\mathbb{Z}_2$, and v is the partial isometry

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We show that Q is a subfactor of M and N is not regular in Q.

First, NB embeds in Q as 2×2 matrices over M_3, and since $\mathbb{Z}_2$ acts trivially on M_3, a general element of NB looks like

$$t = \begin{pmatrix} \begin{matrix} m & 0 & 0 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{matrix} & 0 \\ 0 & \begin{matrix} \alpha(m) & 0 & 0 \\ 0 & \alpha(n_1) & \alpha(n_2) \\ 0 & \alpha(n_3) & \alpha(n_4) \end{matrix} \end{pmatrix}. \tag{III.1}$$

It is also easy to see that

$$vg = gv = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix},$$

where v is the 3×3 matrix above. By multiplying and adding elements of the form t, vgt, and tgv, where t and vg are as above, it is not hard to see that a general element of Q has the form

$$x = \begin{pmatrix} n & r \\ \alpha(r) & \alpha(n) \end{pmatrix}, \tag{III.2}$$

where n comes from NB, and r is of the form

$$\begin{pmatrix} 0 & r_1 & r_2 \\ r_3 & 0 & 0 \\ r_4 & 0 & 0 \end{pmatrix}, \tag{III.3}$$

where $r_i \in N$.

Suppose now that some $x \in Q$ of the form (III.2) is in the center of Q. We show that $r = \alpha(r) = 0$. Note that x commutes with all elements

$$\begin{pmatrix} 0 & b \\ \alpha(b) & 0 \end{pmatrix},$$

where b is of the form given in (3.3). The automorphism α fixes M_3, so $\alpha(b)$ has the same pattern of non-zero entries. By taking b to be the matrix

$$b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and equating top left matrix entries in the equation

$$\begin{pmatrix} n & r \\ \alpha(r) & \alpha(n) \end{pmatrix} \begin{pmatrix} 0 & b \\ \alpha(b) & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ \alpha(b) & 0 \end{pmatrix} \begin{pmatrix} n & r \\ \alpha(r) & \alpha(n) \end{pmatrix}, \tag{III.4}$$

we get that $r\alpha(b) = b\alpha(r)$, from which it follows (by equating matrix entries in this equation) that $r_1 = r_2 = \alpha(r_2) = \alpha(r_1) = 0$. Repeating this same argument, but now with b as the matrix

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

implies that $r_3 = r_4 = \alpha(r_3) = \alpha(r_4) = 0$. We have thus shown that if $x \in Z(Q)$, then it must be of the form

$$\begin{pmatrix} n & 0 \\ 0 & \alpha(n) \end{pmatrix},$$

for some $n \in NB$. By following a similar argument to the one above, we can show that $n \in \mathbb{C}1$, and hence $\alpha(n) = n$, so $x \in \mathbb{C}1$. Thus, Q is a factor.

We now show that Q is not generated by normalizing unitaries of N. An algebraic calculation, while noting that outerness of the action on N is equivalent to freeness, shows that every normalizing unitary of N in M is of the form ug, where $g \in \mathbb{Z}_2$ and $u \in N \otimes M_3$ is a unitary. Then every such normalizer of Q that lies outside of $N \otimes M_3$ must be of the form

$$v = \begin{pmatrix} 0 & u \\ \alpha(u) & 0 \end{pmatrix}, \tag{III.5}$$

where u is a unitary from $N \otimes M_3$. But our previous arguments showed that such a unitary must be of the form (3.3), and a few computations show that such a matrix over N cannot be unitary. Thus Q contains no normalizing unitaries of N that lie outside of $N \otimes M_3$, so N is not a regular subalgebra of Q.

The preceding example shows that a Galois correspondence, in the strictest sense, is not possible for a regular inclusion of subfactors. We will see in Chapter V, however, a complete description of intermediate subalgebras of the normalizer of a II_1 factor that can be viewed as a generalized Galois correspondence. Notice that the subalgebra Q from the example is not generalized by normalizing unitaries, but is generated by partial isometries v satisfying $vxv^* = \phi(x)vv^*$, for all $x \in N$ and some automorphism ϕ of N. We will show in the following chapters that this example is prototypical for the normalizer of a subfactor. For a regular inclusion $N \subseteq M$, it turns out intermediate subalgebra is generated by N and a special set of partial isometries lying in the normalizing algebra. We will call this set the seminormalizer of N, defined as follows.

Definition III.2. We define $\mathcal{SN}_M(N)$ to be the set of partial isometries $w \in M$ satisfying the following conditions:

1. $wxw^* = \phi(x)ww^* = ww^*\phi(x)$, where $\phi = Adu$ for some unitary $u \in \mathcal{N}_M(N)''$.
2. w^*w and ww^* are in $N' \cap M$.

We will call these partial isometries 'seminormalizing' partial isometries.

That is, $\mathcal{SN}_M(N)$ is the set of partial isometries that normalize elements of N "up to" a projection in the relative commutant of N. It is easy to see that $vNv^* = Nvv^* = vv^*N$, for all $v \in \mathcal{SN}_M(N)$. Moreover, a partial isometry v is in $\mathcal{SN}_M(N)$ if and only if $v^* \in \mathcal{SN}_M(N)$. When N is an irreducible subfactor of M, it is immediate from the definition that $\mathcal{SN}(N) = \mathcal{N}(N)$. If N is a masa in M, then Theorem $II.11$ implies that $\mathcal{SN}(N) = \mathcal{GN}(N)$.

B. Seminormalizers and intermediate subalgebras

In this section, the setting is a II_1 factor M, with a fixed von Neumann subalgebra N. Note that no additional assumptions are placed on N. Our aim is to find a form for the intermediate subalgebras of the normalizing algebra of N, i.e. the von Neumann subalgebras P of M satisfying

$$N \subseteq P \subseteq \mathcal{N}_M(N)''.$$

Our main result of this section is as follows. Observe that our setting generalizes the situation of an inclusion $N \subseteq N \times_\alpha G$ from classical Galois theory of subfactors, in addition to that of an inclusion of a Cartan subalgebra in a II_1 factor from the Dye correspondence.

Theorem III.3. *Let N be a von Neumann subalgebra of a II_1 factor M. Suppose that P is a von Neumann algebra such that*

$$N \subseteq P \subseteq \mathcal{N}_M(N)''.$$

Then there exists a set of partial isometries $S \subseteq \mathcal{SN}_M(N)$ such that P is the von Neumann algebra generated by N and S.

In what follows, we will denote by $\mathbb{E} : M \to P$ the trace-preserving conditional expectation. Our strategy will be to first apply $\mathbb{E}$ to general normalizing unitaries of N. We will show that for such a unitary u, $\mathbb{E}(u)$ implements a "cutdown" of the automorphism Adu. Using the polar decomposition in P, for every such u, we will isolate a partial isometry that does the same. Finally, a sort of maximality argument will show that P is generated by such partial isometries.

Lemma III.4. *Let $u \in \mathcal{N}_M(N)$. Then $\mathbb{E}(u)$ has the following properties:*

(i) $\mathbb{E}(u)\mathbb{E}(u)^*, \mathbb{E}(u)^*\mathbb{E}(u) \in N' \cap M$.

(ii) $E(u)xE(u)^* = \phi(x)\mathbb{E}(u)\mathbb{E}(u)^*$, *for all $x \in N$, where $\phi = Adu$.*

Proof. With notation as above, for all $x \in N$, one has

$$uxu^* = \phi(x). \tag{III.6}$$

Multiplying this equation by u on the right and applying the conditional expectation gives

$$\mathbb{E}(u)x = \phi(x)\mathbb{E}(u),$$

for all $x \in N$, and multiplying by $\mathbb{E}(u)^*$ on the right proves (ii). Multiplying equation (0.1) on the left by u^* and following a symmetric argument gives

$$\mathbb{E}(u)\mathbb{E}(u)^*\phi(x) = \mathbb{E}(u)x\mathbb{E}(u)^*,$$

from which it follows that $\mathbb{E}(u)\mathbb{E}(u)^* \in N' \cap M$. Considering the equation

$$u^*xu = \phi^{-1}(x)$$

and following a similar argument shows that $\mathbb{E}(u)^*\mathbb{E}(u) \in N' \cap M$. □

We are now in a position to prove theorem $III.3$. Let $z = \mathbb{E}(u)$, and let $z = tw$ be its polar decomposition in P, where $t = (\mathbb{E}(u)\mathbb{E}(u)^*)^{1/2}$. The previous lemma implies that $t \in N' \cap M$. Considering the relation

$$uxu^* = \phi(x),$$

where $\phi = Adu$ and observing that $\mathbb{E}(u^*) = \mathbb{E}(u)^*$ gives the equation

$$xw^*t = w^*t\phi(x).$$

The final projection ww^* of w is also the range projection $[t]$, so multiplying this equation on the left by w gives

$$wxw^*t = t\phi(x), \tag{III.7}$$

for all $x \in N$. It then follows that

$$twxw^*t = t^2\phi(x),$$

for all such x. If $x \in N$ is positive, then we may apply the adjoint of equation (0.2) to the left hand side of this last equation, which is then equal to $\phi(x)t^2$. That is to say,

t^2 commutes with $\phi(x)$ for all positive $x \in N$, hence for all $x \in N$. So also t commutes with $\phi(x)$ for all $x \in N$, by functional calculus. Then for all such x one has

$$wxw^*t = \phi(x)t.$$

In other words,

$$wxw^*\xi = \phi(x)\xi,$$

for all vectors ξ in the range of t. That is,

$$wxw^* = \phi(x)[t] = \phi(x)ww^*.$$

We have shown above that $t \in N' \cap M$, and this last von Neumann algebra contains all of its corresponding range projections, so in particular $[t] = ww^* \in N' \cap M$. A similar argument shows that $w^*w \in N' \cap M$. Thus, we have shown that to each normalizing unitary u for N, there corresponds a partial isometry $w \in P$ that comes from $\mathcal{SN}_M(N)$. Clearly, there will be "many" such partial isometries. We show that there are enough to generate P as a von Neumann algebra. We first note that, given $u \in \mathcal{N}_M(N)$, the operator $t = (\mathbb{E}(u)\mathbb{E}(u)^*)^{1/2}$ from the above argument is in $N' \cap M$. Since this von Neumann algebra is generated by normalizing unitaries, it will be enough to show that P is generated by the set

$$F = \{\mathbb{E}(u) : u \in \mathcal{N}_M(N)\}.$$

Denote by $\mathcal{A}$ the $*-$algebra generated by $\mathcal{N}_M(N)$. Since $\mathcal{N}_M(N)$ is closed under multiplication and involution, we have that $\mathbb{E}(\mathcal{A})$ is contained in the algebra generated by F. Now let $y \in P$ be arbitrary. By the Kaplansky density theorem, there exists a net $x_\alpha \in A$ (not necessarily in P) such that $\|x_\alpha\| \leq \|y\|$ and x_α converges to y $*$-strongly, hence also in $\|\cdot\|_2$. Then also $\mathbb{E}(x_\alpha)$ converges to $y = \mathbb{E}(y)$ in $\|\cdot\|_2$. Let

B denote the von Neumann algebra generated by $\mathbb{E}(\mathcal{A})$. We have just shown that $dist_2(y, B) = 0$, so $y \in L^2(B) \subseteq L^2(M)$. If we let $\mathbb{E}_B : M \to B$ denote the conditional expectation onto B induced by the Jones projection e_B, then we have

$$y = e_B(y) = \mathbb{E}_B(y),$$

since $y \in M$. Thus $y \in B$. This shows that P is generated by $\mathbb{E}(\mathcal{A})$, hence also by $\{\mathbb{E}(u) : u \in \mathcal{N}_M(N)\}$.

CHAPTER IV

ALGEBRA AND RELATIVE COMMUTANT

A. A useful lemma

We first consider the structure of the von Neumann algebra generated by a subfactor N and its relative commutant $N' \cap M$, which we will denote by C. The main observation of this chapter is that this von Neumann algebra, which we will by Q, is spatially isomorphic to the von Neumann algebra tensor product of the two algebras. The result is probably known to some experts, but we were not able to find a reference. A characterization of the intermediate subalgebras between N and Q will then follow immediately from the result of Ge and Kadison [13] characterizing intermediate subalgebras of tensor products. An essential ingredient of the proof is the von Neumann algebraic notion of orthogonality, originally defined in [33]. The following result, the proof of which comes from [33], gives the various equivalent conditions for orthogonality of subalgebras of which we will make use.

Proposition IV.1 (Lemma 2.1 of [33]). *Let A, B be von Neumann subalgebras of a finite von Neumann algebra M. The following are equivalent.*

(i) $\tau(ab) = 0$, *for* $a \in A$, $b \in B$ *satisfying* $\tau(a) = \tau(b) = 0$.

(ii) $\tau(ab) = \tau(a)\tau(b)$, *for all* $a \in A$, $b \in B$.

(iii) $\|ab\|_2 = \|a\|_2 \|b\|_2$, *for all* $a \in A$, $b \in B$.

(iv) $\mathbb{E}_A \mathbb{E}_B(x) = \tau(x)1_M$, *for all* $x \in M$.

(v) $\mathbb{E}_A(B) \subseteq \mathbb{C}1_M$.

Definition IV.2. Von Neumann subalgebras A, B of a finite von Neumann algebra M are *mutually orthogonal* if any of the conditions $(i) - (v)$ holds.

The first lemma of this section states that $N \subseteq M$ and its relative commutant are orthogonal subalgebras of M in this sense. We denote $L^2(N)$ and $L^2(C)$, respectively, the Hilbert spaces formed by restricting the trace τ of M to N and C. We also note our use below of the Dixmier Approximation Theorem. Recall that the theorem says that for a II_1 factor N, and $n \in N$. the norm closure of $conv\,\{unu^* : u \in \mathcal{U}(N)\}$ has non-empty intersection with $Z(N) = \mathbb{C}1$. Writing K for this (convex) intersection, note that as a convex subset of $\mathbb{C}$, K contains a unique element $T(n)$ of minimal absolute value. This defines a function $T : N \to \mathbb{C}$, which is a weakly continuous tracial state on N. By uniqueness of the trace τ on N, it follows that $T = \tau$.

Lemma IV.3. *For all $n \in N$ and $c \in C$, we have*

$$\tau(nc) = \tau(n)\tau(c)$$

Proof. By Dixmier's Approximation Theorem [18] if $n \in N$ then $\tau(n)$ is the norm-limit of expressions of the form

$$\alpha(n) = \sum_{i=1}^{k} \lambda_i u_i n u_i^*,$$

where $\sum_i \lambda_i = 1$, and each $u_i \in N$. Moreover, if $c \in C$ then

$$\begin{aligned}
\tau(\alpha(n)c) &= \tau([\sum_{i=1}^{k} \lambda_i u_i n u_i^*]c) \\
&= \sum \lambda_i \tau(u_i n u_i^* c) \\
&= \sum \lambda_i \tau(u_i n c u_i^*) \\
&= \tau(nc).
\end{aligned}$$

The result now follows by operator-norm continuity of the trace. $\square$

We conclude that elements n_1c_1 and n_2c_2 of Q are orthogonal in $L^2(Q)$ if and only if either n_1 and n_2 are orthogonal or c_1 and c_2 are orthogonal. In particular, construct an orthonormal basis $\{e_i\}$ for $L^2(C)$ by the Gram-Schmidt process. Note that the basis is countable (because C is separable), and we may choose $e_1 = 1$ and $e_i \in C$ for all $i \geq 1$. It follows from the previous lemma that for $i \neq j$ the spaces Ne_i and Ne_j, hence also the spaces $L^2(N)e_i$ and $L^2(N)e_j$, are orthogonal in $L^2(Q)$.

It will be convenient to have a representation of elements of Q as convergent sum of elements of the form nc, where $n \in N$ and $c \in C$. The following result states that we can decompose $L^2(Q)$ as an orthogonal direct sum of the spaces $L^2(N)e_i$ so that the set $\{e_i\}$ actually forms an orthonormal basis for $L^2(C)$ as a left module over N.

We denote by $\mathcal{H}$ the direct sum

$$\bigoplus_{i \geq 1} \mathcal{H}_i,$$

where $\mathcal{H}_i$ is the $\|\cdot\|_2$ closure of the space $L^2(N)e_i$.

Proposition IV.4. *(i)* $\mathcal{H} = L^2(Q)$

(ii) Each $x \in Q$ *has a unique Fourier expansion*

$$x = \sum_{i \geq 1} \eta_i e_i,$$

where $\eta_i = \mathbb{E}_N(xe_i^*) \in N$.

Proof. Our computations above showed that the spaces $L^2(N)e_i$ and $L^2(N)e_j$ are orthogonal for $i \neq j$. By standard Hilbert space theory, we also have

$$L^2(N)e_i \subseteq (L^2(N)e_j)^{\perp} = \mathcal{H}_j^{\perp},$$

hence also $\mathcal{H}_i \subseteq \mathcal{H}_j^{\perp}$. Symmetrically, $\mathcal{H}_j \subseteq \mathcal{H}_i^{\perp}$, so these spaces are orthogonal for

$i \neq j$. We now let $\mathcal{H}$ denote the internal direct sum of these subspaces of $L^2(Q)$. We claim that (i) holds. Suppose $\xi \in L^2(Q)$ is orthogonal to $\mathcal{H}$. Then, in particular, ξ is orthogonal to $L^2(N)$ and also $< \xi, e_i >= 0$, for all $i \geq 1$. Then also ξ is orthogonal to $L^2(Q)$, as we now show. Write $\mathcal{A}$ for the algebra generated by N and $\{e_i\}$. Then

$$\mathcal{A}'' = (N \cup \{e_i\})'' = Q.$$

Moreover, since every element of $\mathcal{A}$ is of the form

$$a = \sum_{i=1}^{m} n_i e_i,$$

with $n_i \in N$, we have $\langle \xi, a \rangle = 0$, for all $a \in \mathcal{A}$.

Now let $x \in Q$ be arbitrary. By the Kaplansky Density Theorem [37], there exists a net $\{x_\alpha\} \in \mathcal{A}$ such that $\|x_\alpha\| \leq \|x\|$ for all α and x_α converges to x $*$-strongly. A short computation with the trace shows that x_α also converges to x in $\|\cdot\|_2$. Then

$$\langle \xi, x \rangle = \lim_\alpha \langle \xi, \mathrm{x}_\alpha \rangle = 0,$$

so ξ is orthogonal to Q, from which it follows that ξ is orthogonal to $L^2(Q)$. This proves (i).

We now find a Fourier series over N for each $x \in Q$. By (i) we can write such x as a sequence $\{\eta_i\}$, with $\eta_i \in \mathcal{H}_i$ and $\sum_i \|\eta_i\|_2^2 < \infty$. Let e_N be the Jones projection on $L^2(Q)$. If η_j is of the form ne_j, for some $n \in N$, then for $i \neq j$ we have

$$e_N(\eta_j e_i^*) e_i = \mathbb{E}_N(n e_j e_i^*) e_i = n\tau(e_j e_i^*) e_i = 0.$$

By continuity of the Jones projection, this is true for all $\eta_j \in \mathcal{H}_j$. If $\eta_i \in \mathcal{H}_i$ is of the form ne_i, then

$$e_N(\eta e_i^*) e_i = \mathbb{E}_N(n e_i e_i^*) e_i = n\mathbb{E}_N(e_i e_i^*) e_i = ne_i = \eta_i,$$

so again by continuity we have $e_N(\eta_i e_i^*)e_i = \eta_i$ for a general element η_i of $\mathcal{H}_i$. Then if $x \in Q$ is a general element and $i \geq 1$ we compute

$$\mathbb{E}_N(xe_i^*)e_i = e_N(xe_i^*)e_i = \sum_{j\geq 1}(\eta_j e_i^*)e_i = \eta_i.$$

Our expression for the Fourier series of x follows immediately, and we note that

$$\sum \|\mathbb{E}_N(xe_i^*)\|_2^2 = \sum \|\mathbb{E}_N(\eta_i)\|_2^2 < \infty.$$

This completes the proof. □

As a consequence of the Fourier representation from Proposition $IV.4$, we may generalize Lemma $IV.3$. That is, if $x = \sum_i n_i c_i$ as above, then by $\|\cdot\|_2$-norm continuity of the trace, one has

$$\tau(x) = \sum_i \tau(n_i)\tau(c_i).$$

We now show that $L^2(Q) = \bigoplus_{i\geq 1}\mathcal{H}_i$ is isomorphic to $L^2(N) \otimes_2 L^2(C)$, and that the isomorphism induces a spatial isomorphism of M with the von Neumann algebra tensor product of N and C.

Lemma IV.5. *Let $N \subseteq M$ be an inclusion of* II_1 *factors. Then the von Neumann algebra Q generated by N and C is spatially isomorphic to $N\overline{\otimes}C$.*

Proof. Define a map U from the algebraic tensor product of $L^2(N)$ and $L^2(C)$ by the assignment

$$\sum_{i=1}^m n_i \otimes c_i \mapsto \sum_{i=1}^m n_i c_i.$$

By the above remark,

$$\left\|\sum_i n_i c_i\right\|_2^2 = \tau(\sum_{i,j} n_i n_j^* c_i c_j^*) = \sum_{i,j}\tau(n_i n_i^*)\tau(c_i c_j^*),$$

and it is easily seen that this last quantity is $\|\sum_i n_i \otimes c_i\|_2^2$, so U is an isometry. Thus, the extension of U to $L^2(N) \otimes L^2(C)$ is a Hilbert space isomorphism with $L^2(Q)$. To see that this induces an isomorphism of the represented von Neumann algebras, it is enough to examine the action of UxU^* on $L^2(Q)$ when x is an elementary tensor. We may restrict our attention to the set $\{nc : n \in N, c \in C\}$, whose linear span is dense in $L^2(Q)$. It is easy to check under these assumptions that $U(n \otimes c)U^*\xi = nc\xi$ for all $\xi \in L^2(Q)$, and so the map

$$n \otimes c \mapsto U(n \otimes c)U^* = nc$$

is a von Neumann algebra isomorphism of Q with $N \overline{\otimes} C$. $\square$

Ge and Kadison showed that if N and M are factors, and P is a von Neumann algebra satisfying $N \subseteq P \subseteq N \overline{\otimes} M$, then there exists a von Neumann subalgebra A of M such that $P = N \overline{\otimes} A$. We thus obtain the following corollary.

Corollary IV.6. *Let $N \subseteq M$ be an inclusion of* II_1 *factors. Suppose that P is a von Neumann algebra such that $N \subseteq P \subseteq Q$, where Q is the von Neumann algebra generated by N and $N' \cap M$. Then there exists a von Neumann subalgebra A of $N' \cap M$ such that $P = \{N \cup A\}''$.*

It is important to note here that Lemma $IV.5$ is particular to the case of an inclusion of type II_1 factors. For example, if $N \subseteq B(H)$ is a II_1 factor in standard representation, then its relative commutant C is just N'. Moreover, the von Neumann algebra generated by N and N' is $B(H)$ itself. On the other hand, it is known [13] that $B(H)$ is isomorphic to the von Neumann algebra tensor product of N and N' precisely when N has minimal projections, i.e., when N is type I. Thus, Lemma $IV.5$ is false for the case $M = B(H)$.

Lemma $IV.5$ is also false if N is not chosen to be a factor. For example, choose any finite von Neumann algebra N with nontrivial center (say, $Z(N) = \mathbb{C}^2$,) and consider the inclusion

$$N \subseteq M = N * L\mathbb{F}_2.$$

The relative commutant of N in M, then, is the center of N, and Lemma $IV.5$ would then imply that N is isomorphic to $N\overline{\otimes}Z(N) = N\overline{\otimes}\mathbb{C}^2$, which is absurd.

B. Bimodules and averaging techniques

In this short section, we describe an averaging technique we will use in what follows. The following result, which is essentially in [5] states that the projection of an element $x \in Q$ onto the relative commutant $N' \cap M$ can be computed by an "averaging process" over the unitary group of N.

Before embarking on the proof, it is important to consider whether the weak closure of the set

$$K_N(x) = conv\,\{uxu^* : u \in \mathcal{U}(N)\}$$

is taken in M or $L^2(M)$. The embedding of M into $L^2(M)$ is continuous, when both spaces are given their respective weak topologies. Thus, the (compact) weak closure of any bounded, convex set in M is weakly closed (hence also $\|\cdot\|_2$-closed, by convexity) in $L^2(M)$. Conversely, the preimage of a weakly closed (equivalently, $\|\cdot\|_2$-closed), bounded convex set in $L^2(M)$ is weakly closed in M. Thus, the weak closure in M, the weak closure in $L^2(M)$, and the $\|\cdot\|_2$ closure in $L^2(M)$ of $K_N(x)$ all coincide. We will use the common notation $K_N^w(x)$ for all three closures.

Proposition IV.7. *Let M be a finite von Neumann algebra, and let N be a von Neumann subalgebra of M.For any $x \in M$, $\mathbb{E}_{N'\cap M}(x)$ is the unique element of minimal $\|\cdot\|_2$-norm in the weak closure $K_N^w(x)$ of*

$$K_N(x) = \mathrm{conv}\{uxu^* : u \in \mathcal{U}(N)\}.$$

Proof. We exploit the uniqueness of the conditional expectation $\mathbb{E}_{N'\cap M}$; it will be enough to define a unital function $h : M \to N' \cap M$ which is trace-preserving and $N' \cap M$ bimodular. By standard Hilbert space theory, for each $x \in M$, there exists a unique element $h(x)$ of minimal $\|\cdot\|_2$ in $K_N^w(x)$. By the remark above, we also have $h(x) \in M$. We show that $h(x) \in N' \cap M$.

Indeed, if $v \in \mathcal{U}(N)$ is a unitary, the it is easy to see that $K_N(vxv^*) = K_N(x)$, from which it follows that $v^*K_N(x)v = K_N(x)$, for any such unitary. Moreover, since $\|v^*h(x)v\|_2 \leq \|z\|_2$ for all $z \in K_N(x)$, hence also for all $z \in K_N^w(x)$, by uniqueness of $h(x)$ we have that $v^*h(x)v = h(x)$. It follows by linearity that $h(x) \in N' \cap M$. Since $\tau(x) = \tau(z)$ for all $z \in K_N(x)$, we have that $h : M \to N' \cap M$ is a trace-preserving function. Clearly, also $h(1) = 1$.

To prove bimodularity, first consider unitary elements $u_1, u_2 \in N' \cap M$. Since these operators commute with N, we have that $K_N(u_1xu_2) = u_1K_N(x)u_2$. Moreover,

$$\|u_1h(x)u_2\|_2 = \|h(x)\|_2 \leq \|z\|_2 = \|u_1zu_2\|_2 ,$$

for all $z \in K_N(x)$. It follows that $\|u_1h(x)u_2\| \leq \|y\|_2$ for all $y \in K_N^w(u_1xu_2)$, and by uniqueness, we have $u_1h(x)u_2 = h(u_1xu_2)$. Thus, h is a bimodule map, so is equal to $\mathbb{E}_{N'\cap M}$. □

Proposition $IV.7$ will be used when we work with certain weakly closed bimodules over N. Note that the result implies that if $y \in \mathcal{M}$ for such a module $\mathcal{M}$, then also $\mathbb{E}_{N'\cap M}(x) \in \mathcal{M}$. This observation will be important in the next chapter.

CHAPTER V

THE NORMALIZER OF A SUBFACTOR

In this chapter we produce two of the main structural results of the dissertation. It was proved by M. Choda in [4] that every regular inclusion $N \subseteq M$ of II_1 factors satisfying $N' \cap M = \mathbb{C}1$ is a crossed product $M = N \rtimes_\alpha^\omega G$, for some countable, discrete group G. We use results from the previous chapter, and a generalization of Choda's construction, to obtain a similar result for a general regular inclusion of II_1 factors $N \subseteq M$. It should be noted that all of the following results hold, in fact, when M is simply a finite von Neumann algebra with a fixed tracial state. We will show that, given such an inclusion, we can decompose M as the crossed product $Q \rtimes_\alpha^\omega G$, where Q is the von Neumann algebra generated by N and its relative commutant. A key observation is that the action of a normalizing unitary on N determines, to an important extent, its action on Q.

This result will imply that the normalizer of a subfactor displays a "discreteness" phenomenon, as do all of its intermediate subalgebras containing N. We will prove, in particular, that any such intermediate subalgebra decomposes as a countable, discrete direct sum of weakly closed modules over a certain von Neumann subalgebra of Q. Simple algebraic relations among these modules will then determine the structure of the algebras themselves, as well as the lattice structure of the collection of intermediate subalgebras.

Lemma V.1. *Suppose $N \subseteq M$ is an inclusion of von Neumann algebras. Then the unitary group of the von Neumann algebra Q is invariant under conjugation by elements of $\mathcal{N}_M(N)$.*

Proof. Obviously $\mathcal{U}(N)$ is invariant under conjugation by elements of $\mathcal{N}_M(N)$. Let

$x \in N' \cap M$, and denote by ϕ the automorphism of N defined by conjugation by $u \in \mathcal{N}_M(N)$. Since $xn = nx$, for all $n \in N$, we also have $\phi(x)\phi(n) = \phi(n)\phi(x)$, for all $n \in N$. Then also $\phi(x) \in N' \cap M$. It follows that $\phi(x) \in Q$ for any element x of the algebra generated by N and $N' \cap M$. This algebra is weakly dense in Q, so by weak continuity of ϕ we have that $\phi(Q)$ is contained in Q. Since ϕ is conjugation by a unitary, it clearly fixes the unitary group of Q. $\square$

For the purpose of what follows, we would like to work with a countable subgroup of $\mathcal{N}_M(N)$. Since M is separable, there exists a countable subgroup $\tilde{K}$ of the normalizer which generates M as a von Neumann algebra. Let L be the group generated by the unitary groups of N and $N' \cap M$. Let K be the group generated by $\tilde{K}$ and L. It follows from the above lemma that L is a normal subgroup of K. Let G be the quotient group K/L. We do not know at this point what topology G inherits from M (it will turn out to be countable and discrete), but it is well-known that any exact sequence of groups

$$1 \to L \to K \to G \to 1$$

induces a "normalized section," i.e., a map $u : G \to K$ which chooses one (unitary) element $u_g \in K$ from the coset representing g, and u_e is the identity in K. We may also construct the map u so that $u_g^* = u_{g^{-1}}$, for all $g \in G$. The unitaries u_g induce automorphisms of L, hence also the von Neumann algebra Q generated by N and $N' \cap M$. We have a map $\alpha : G \to Aut(L)$ defined by $\alpha_g = Adu_g$.

We note that no α_g, $g \neq 1$ is inner. Indeed, suppose that $\alpha_g = Adu_g$ is inner on L. Then there exists a unitary v in the group L generated by $\mathcal{U}(N)$ and $\mathcal{U}(N' \cap M)$ such that $u_g x u_g^* = vxv^*$, for all $x \in N$. Then $v^* u_g \in N' \cap M$ so, in particular, $v^* u_g \in L$, so also $u_g \in L$. But u_g comes from a normalized section of G, so $u_g = 1$, hence also $\alpha_g = 1$. Now α is not a group action, because the section u is not a homomorphism.

In fact, there is a 2-cocycle $\omega : G \times G \to L$ satisfying $u_g u_h = \omega(g,h)u_{gh}$, and $\alpha_g\alpha_h = Ad\omega(g,h)\alpha_{gh}$ (See [3] for details). Since the u_g satisfy $u_g^* = u_{g^{-1}}$ for all g, we note that $\omega(g,g^{-1}) = 1$ for all $g \in G$.

In what follows, we will show that the normalizer of any II_1 factor can be written as a discrete twisted crossed product by this group of automorphisms. The first of the following lemmas is an application of our Lemma IV.5. The proof is to apply Lemma IV.5 and Lemma V.1 to see that we are in the situation of a group acting on each factor of a tensor product. A result of Kallman [21] will then complete the proof. We then prove discreteness of G, and employ a result of M. Choda [4] giving conditions under which an inclusion of finite von Neumann algebras is isomorphic to a crossed product. The proof of the following lemma is essentially identical to [21];we recall details for the reader's convenience.

Lemma V.2. *Let M be a II_1 factor, let $N \subseteq M$ be a subfactor, and let the von Neumann algebra Q be as above. If $\phi = Adu$ is a non-trivial outer automorphism of N given by $u \in \mathcal{N}_M(N)$, then ϕ acts freely on Q.*

Proof. By what we did in Chapter IV, an arbitrary $y \in Q = N\overline{\otimes}C$ can be written as a matrix (y_{ij}) over N indexed countably by elements of the relative commutant. Moreover, Lemma V.1 shows that ϕ also defines an automorphism of C. Thus, ϕ is an automorphism of Q. We show that ϕ acts freely. If $y \in Q$ satisfies

$$y\phi(x) = xy,$$

for all $x \in Q$, then by taking x to be a diagonal element of $N\overline{\otimes}1$, this equation becomes

$$(y_{ij})\begin{pmatrix} \phi(x) & & \\ & \phi(x) & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} x & & \\ & x & \\ & & \ddots \end{pmatrix}(y_{ij}).$$

By equating matrix entries, this gives the system of equations

$$y_{ij}\phi(x) = xy_{ij}.$$

Freeness and outerness are equivalent for an automorphism of a factor, so ϕ acts freely on N. It follows that $y_{ij} = 0$ for all $i, j \geq 1$. Thus, $y = 0$, and the proof is complete. $\square$

Lemma V.3. *Let G be the group K/L from above. If $g, h \in G$ are not equal then conjugation by $u = u_g u_h^*$ defines an outer automorphism of N.*

Proof. Suppose that $uxu^* = vxv^*$ for some $v \in \mathcal{U}(N)$. It follows that u is in the group L. But $u = u_g u_h^* = u_{gh^{-1}}\omega(g, h^{-1})$, and ω is L-valued, so this says $u_{gh^{-1}} \in L$, from which it follows that $g = h$, a contradiction. $\square$

Then by Lemma V.2, every such unitary acts freely on Q. This is the key observation in proving the following lemma.

Lemma V.4. *The group G is discrete in the $\|\cdot\|_2$-norm of M.*

Proof. For any $g, h \in G$ such that $g \neq h$ and for all $x \in Q$, we have

$$\mathbb{E}_Q(u)x = Adu(x)\mathbb{E}_Q(u),$$

where $u = u_g u_h^*$. By the previous remarks, then, $\mathbb{E}_Q(u_g u_h^*) = 0$. But then

$$\tau(u_g u_h^*) = \tau(E_Q(u_g u_h^*)) = 0,$$

for all such g and h, that is, u_g and u_h are orthogonal in $L^2(M)$. Then

$$\|u_g - u_h\|_2 = \sqrt{2},$$

for all such g, h. That is, G is discrete in $L^2(M)$. $\square$

Lemma V.5. *(Theorem 7 of [4]) Let M be a finite von Neumann algebra and let A be a von Neumann subalgebra. Suppose that there exists a group G and a normalized section $u : G \mapsto \mathcal{U}(M)$ such that*

(i) *M is generated by A and $\{u_g\}_{g\in G}$*

(ii) *Each u_g is a normalizer of A, and*

(iii) *There exists a conditional expectation $E : M \to A$ satisfying $E(u_g) = 0$, for all $1 \neq g \in G$.*

Then there exists a 2-cocycle $\omega : G \times G \mapsto \mathcal{U}(A)$ such that M is isomorphic to $A \rtimes_\alpha^\omega G$.

We refer the reader to [4] for the proof.

Theorem V.6. *Let $N \subseteq M$ be an inclusion of II_1 factors. Let Q be the von Neumann algebra generated by N and $N' \cap M$. Then there exists a countable, discrete group G, a map $\alpha : G \to Out(Q)$. and a 2-cocycle $\omega : G \times G \to \mathcal{U}(Q)$ such that*

$$\mathcal{N}_M(N)'' = Q \rtimes_\alpha^\omega G.$$

Proof. Writing $\tilde{M}$ for the normalizing algebra $\mathcal{N}_M(N)''$, we let $u : G \to \tilde{M}$ and $\alpha : G \to Aut(Q)$ be the maps from the discussion above. Let $\mathbb{E} : \tilde{M} \to Q$ be the restriction of the natural conditional expectation of M onto Q. By the previous lemma, it suffices to prove that $\mathbb{E}(u_g) = 0$, for all $g \neq 1$. Indeed, if $g \neq 1$, and

$$u_g x u_g^* = \alpha_g(x),$$

for all $x \in Q$, then multiplying on the right by u_g and applying the conditional expectation, we get

$$\mathbb{E}(u_g)x = \alpha_g(x)\mathbb{E}(u_g),$$

for all $x \in Q$. Since $g \neq 1$. By Lemma V.2, we have $\mathbb{E}(u_g) = 0$. $\square$

Recall from above that in any twisted crossed product by outer automorphisms, every element of $M = Q \rtimes_{\alpha}^{\omega} G$ can be written uniquely as a Fourier series

$$x = \sum_{g \in G} \eta_g u_g,$$

where the "coefficients" η_g are in Q and the unitaries u_g are as above. The convergence of the sum is in the norm on the Hilbert space $L^2(Q) \otimes \ell^2(G)$.

Now consider a II_1 factor $N \subseteq B(H)$ on a separable Hilbert space. Let $S = \{\phi_i\}$ be a countable set of automorphisms of N. We then have diagonal representations ρ of N and σ of N' on the Hilbert space $H \otimes \ell^2(S)$, defined as follows:

$$\rho(x) = \begin{pmatrix} \phi_1(x) & & \\ & \phi_2(x) & \\ & & \ddots \end{pmatrix}, x \in N,$$

and

$$\sigma(y) = \begin{pmatrix} y & & \\ & y & \\ & & \ddots \end{pmatrix}, y \in N'.$$

The following lemma is a minor modification of a result of White and Wiggins [46]. The version we will need is the following.

Lemma V.7. *Let $S = \{\phi_i\}$ be a countable set of automorphisms of a II_1 factor*

$N \subseteq B(H)$ such that $\phi_i \circ \phi_j^{-1}$ is outer for $i \neq j$. Then the von Neumann algebra T generated by $\rho(N)$ and $\sigma(N')$ is

$$T = \{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & \ddots \end{pmatrix} : t_i \in B(H),\ sup \|t_i\| < \infty \}$$

Proof. Clearly the von Neumann algebra M generated by $\rho(N)$ and $\sigma(N')$ is contained in T. If we show that the commutant of this von Neumann algebra is contained in T', then the result will follow from the double commutant theorem.

Now consider $t \in B(H \otimes \ell^2(S))$ be in the commutant of M. By our previous characterization of the tensor product, we may view t as an infinite matrix (t_{ij}) over S, with $t_{ij} \in B(H)$. But also each t_{ij} commutes with $\sigma(N')$, so lies in N. Moreover, each t_{ii} commutes with $\{\phi_i(x) : x \in N\} = N$, so $t_{ii} \in \mathbb{C}1$ for all i since N is a factor. For $i \neq j$, we have

$$\phi_i(x)t_{ij} = t_{ij}\phi_j(x),$$

for all $x \in M$, from which it follows that

$$\phi_i \circ \phi_j^{-1}(x)t_{ij} = t_{ij}x,$$

for all such x, by an easy substitution. By hypothesis $\phi_i \circ \phi_j^{-1}$ is an outer action, hence free since N is a factor, from which it follows that $t_{ij} = 0$ for all $i \neq j$. But now $t \in M'$ is a diagonal matrix with scalar entries, so is in T', as desired. □

Remark V.8. (i) In our situation, the automorphisms $\{\phi_i\}$ of the above lemma are $\phi_g = Adu_g$, where $g \in K$. Suppose that $g, h \in K$ and $g \neq h$. From the remarks preceding Theorem V.6, we have

$$\phi_g\phi_{h^{-1}} = Ad\omega(g, h^{-1})\phi_{gh^{-1}}$$

Write $\omega = \omega(g, h^{-1})$. If $\phi_g \phi_{h^{-1}}$ is an inner automorphism of N then there exists a unitary $v \in \mathcal{U}(N)$ such that

$$\omega^* u^*_{gh^{-1}} x u_{gh^{-1}} \omega = v x v^*,$$

for all $x \in N$. This implies that $u_{gh^{-1}} v^* \omega^* \in N' \cap M \subseteq Q$. Since v and ω are in Q, this says $u_{gh^{-1}} \in Q$, whence $g = h$, a contradiction. Thus, the hypotheses of Lemma $V.7$ are satisfied in our situation.

(ii) We will apply Lemma $V.7$ as follows. Given sequences of vectors ξ_g and η_g in $L^2(Q)$ which are indexed (countably) by elements of the group G and square-summable in norm, we will have an inner product equation

$$\sum_g \langle y\alpha_g(x)\xi_g, \eta_g \rangle = 0,$$

for all $x \in N$, $y \in N'$. This translates (up to a reordering of the group elements) into a matrix equation

$$\left\langle \begin{pmatrix} y\alpha_{g1}(x) & & \\ & y\alpha_{g2}(x) & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \xi_{g1} \\ \xi_{g2} \\ \vdots \end{pmatrix}, \begin{pmatrix} \eta_{g1} \\ \eta_{g2} \\ \vdots \end{pmatrix} \right\rangle = 0,$$

and by Lemma $V.7$, we will have

$$\left\langle \begin{pmatrix} t_{g1} & & \\ & t_{g2} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \xi_{g1} \\ \xi_{g2} \\ \vdots \end{pmatrix}, \begin{pmatrix} \eta_{g1} \\ \eta_{g2} \\ \vdots \end{pmatrix} \right\rangle = 0,$$

for any bounded sequence $\{t_i\} \in B(L^2(N))$. Then for any such sequence,

$$\sum_g \langle t_g \xi_g, \eta_g \rangle = 0.$$

The following is our main result of the chapter; we make a small notational convention. If $\{\mathcal{M}_i\}_{i\in I}$ is a collection of weakly closed submodules (or von Neumann subalgebras) of a II_1 factor M, we denote by

$$\sum_{i\in I}\mathcal{M}_i$$

their Hilbert space direct sum in M. That is, $\sum_i \mathcal{M}_i$ is a submodule (or subalgebra) of M in the obvious sense with the additional property that the $\mathcal{M}_i$ are mutually orthogonal in $L^2(M)$.

Theorem V.9. *Let $N \subseteq M$ be an inclusion of II_1 factors. If P is an intermediate subalgebra of the normalizer of N, i.e.*

$$N \subseteq P \subseteq \mathcal{N}(N).''$$

then

(i) There exists a subset $S \subseteq \mathcal{SN}(N)$ such that P is the von Neumann algebra generated by N and S.

(ii) There exists a subalgebra $A \subseteq N' \cap M$ and a set of partial isometries $\{v_g\}_{g\in G}$ such that

$$P = \sum_{g\in G}(N\overline{\otimes}A)v_g N,$$

where v_g is a weak limit of linear combinations $\sum n_i w_i$, where $n_i \in N$ and $w_i \in \mathcal{SN}(N)$.

Proof. The first step in the proof is to decompose P into a direct sum of weakly closed modules. To do this, we will make use of Lemma *V*.7 and the idea of the proof of Theorem 14.3.3 in [37]. As in the remarks from Chapter II each $x \in P$ admits a unique Fourier series representation

$$x = \sum_{g \in G} x_g u_g,$$

where each x_g is in Q. Denote by H the set of all $g \in G$ such that there exists an $x \in P$ with nonzero x_g coefficient. We will show that if $x \in P$ is as above, then $x_g u_g \in P$ for all $g \in H$.

Fix an element $y = \sum_g y_g u_g \in L^2(P)^\perp$. Note that $y_g \in L^2(Q)$, and $\sum \|y_g\|_2^2 < \infty$. If $x \in P$, then since P is invariant under multiplication by N, we can multiply by $m, n \in N$ on the left and right to get

$$mxn = \sum_g m x_g u_g n = \sum_g m x_g \alpha_g(n^*) u_g \in P,$$

where the multiplication makes sense by identifying N with $N \overline{\otimes} 1$ on $L^2(Q)$. We then have

$$\sum_g \langle m J \alpha_g(n^*) J x_g \xi, y_g \rangle = 0,$$

and applying J gives

$$\sum_g \langle J m J \alpha_g(n^*) J x_g \xi, J y_g \rangle = 0.$$

Recall from Chapter II that $JNJ = N'$, so by Lemma $V.7$ we obtain

$$\sum_g \langle t_g J x_g \xi, J y_g \rangle = 0,$$

for any sequence $t_g \in B(L^2(N))$ and $\sup_{\mathrm{g}} \|\mathrm{t_g}\|$ is finite.

Fix $k \in H$. We can choose $x \in P$ such that $x_k \neq 0$. Then in the above equation, choosing a sequence $\{t_g\}$ with $t_g = 0$ for $g \neq k$ and $t_k = 1$ gives

$$\langle x_k, y_k \rangle = \langle J x_k, J y_k \rangle = 0,$$

so $x_k u_k$ is orthogonal to $y = \sum y_g u_g$. Since $y \in L^2(P)^\perp$ was arbitrary, $x_k u_k \in L^2(P)$.

But then since $x_k u_k \in M$, we have

$$\mathbb{E}_P(x_k u_k) = e_P(x_k u_k) = x_k u_k,$$

so also $x_k u_k \in P$.

For $g \in H$ write

$$X_g = \{x \in Q : xu_g \in P\}.$$

It is easy to see that X_e is a von Neumann subalgebra of Q containing N, so by Lemma $IV.5$, X_e is $N\overline{\otimes}A$, for some von Neumann subalgebra A of C. Moreover, the spaces $X_g u_g$ and $X_h u_h$ are orthogonal in $L^2(P)$ for $g \neq h$, so we have a decomposition of P as a direct sum of subspaces

$$NB + \sum_{g \in G} X_g u_g.$$

Each X_g has the following properties:

(i) X_g is a left $N\overline{\otimes}A$-module, and is weakly closed.

(ii) X_g is a right N -module.

(iii) $E_C(X_g) \subseteq X_g$ (since E_C is obtained by averaging over $\mathcal{U}(N)$)

(iv) If $x, y \in X_g$ then $xy^* \in N\overline{\otimes}A$

We will show that X_g is $(N\overline{\otimes}A)v_g N$, for a certain partial isometry $v_g \in X_g$. We proceed in four steps. First, note that we cannot have $X_g \subseteq N$. If this were true, then since X_g is a left A-module, there would exist nonzero operators $x, y \in X_g$ and a projection $p \in A$ such that $px = y$. But since X_g is a right N-module, we may assume $x, y \geq 0$. Applying $\mathbb{E}_{N' \cap M}$ to the previous equation we obtain the nonzero equation

$$p\tau(x) = \tau(y),$$

a contradiction when we choose $p \in A$ not equal to 1.

We claim X_g then contains nonzero elements of $N' \cap M$. If $y \in X_g$ is not in N, write $y = \sum y_i e_i$, for some orthonormal basis $\{e_i\} \subseteq N' \cap M$ for $L^2(N' \cap M)$. Then since X_g is a left N-module, the element $y_j^* y \in X_g$ for any $j \geq 1$. We now average over the unitary group of N, and recall that the operator

$$\mathbb{E}_{N' \cap M}(y_j^* y) = \sum_{i \geq 1} \tau(y_j^* y_i) e_i$$

is the element of minimal $\|\cdot\|_2$ norm of $K_N^w(y_j^* y)$, and this last set is contained in X_g since X_g is a left N-module. Then also $\mathbb{E}_{N' \cap M}(y_j^* y)$ is an element of X_g since X_g is weakly closed, and is nonzero by orthogonality of the set $\{e_i\}$.

We now show that X_g is generated by partial isometries in the relative commutant. We adapt the proof of the polar decomposition in [26] to find partial isometries, and then use a maximality argument which is essentially a variation of those in [32]. For a nonzero element $x \in X_g \cap (N' \cap M)$, consider the sequence

$$w_n = (1/n + p)^{-1} x,$$

where $p = (xx^*)^{1/2}$. Note that p commutes with its range projection $[p]$ and so $x = [x]x = [p]x$, from which it follows that $[p]w_n = w_n$ for all n. By property (iv) above, $(1/n + p)^{-1} \in N \overline{\otimes} A$ for all n, so by (i) we have $w_n \in X_g$ for all n. We will show that w_n converges to a partial isometry $w \in X_g \cap (N' \cap M)$ such that $x = pw$ and $ww^* = [p] = [x]$. Note that for all m and n

$$(w_m - w_n)(w_m - w_n)^* = ((1/m + p)^{-1} - (1/n + p)^{-1})^2,$$

which converges weakly to zero by functional calculus. Then the sequence w_n^* converges strongly, so w_n converges weakly to an element $w \in X_g \cap (N' \cap M)$. Now the sequence

$$pw_n = p(1/n + p)^{-1}x$$

converges weakly to x (again, by functional calculus), so we have $x = pw$. Then also $xx^* = pww^*p$, which implies that $ww^* \geq [p]$. Then $ww^* = [p]$, i.e., $w \in X_g$ is a partial isometry in the relative commutant.

Now let $\mathcal{W} = \{w_i\}_{i \in J}$ be a maximal collection of partial isometries from the first step satisfying $w_i w_j^* = 0$ for $i \neq j$. This condition is equivalent to the condition that the w_i have mutually orthogonal initial projections, so such a collection exists by Zorn's Lemma. We claim that this set spans X_g over $N\overline{\otimes}A$, i.e. each $y \in X_g$ can be expressed as a weakly convergent sum of the form $\sum_{j \in J} n_j w_j$, for some elements $n_j \in N\overline{\otimes}A$. We will then have a decomposition of X_g as

$$\sum_{j \in J}(N\overline{\otimes}A)w_j.$$

This will prove the first statement in our theorem, since A is generated by $\mathcal{U}(A) \subseteq \mathcal{N}(N)$ and $w_j \in \mathcal{SN}(N)$ for all j, since each w_j commutes with N.

We claim that if $y \in X_g$ satisfies $yw_j^* = 0$ for all j, then $y = 0$. For any such y, if $\mathbb{E}_{N' \cap M}(y) = 0$, then y is orthogonal to $N' \cap M$, from which it follows that $y \in N$. But now also $y^*yw_j^*w_j = 0$ for all j, since y commutes with the w_j. But now y^*y and $w_j^*w_j$ are nonzero positive operators and

$$\tau(y^*y)\tau(w_j^*w_j) = \tau(y^*yw_j^*w_j) = 0,$$

a contradiction. Thus, we may assume that $z = \mathbb{E}_{N' \cap M}(y)$ is nonzero.

Now, if $y \in X_g$ satisfies $yw_j^* = 0$ for all j then z satisfies the same property. Write $z = pw$ as above with $w \in X_g$ a partial isometry and $p = (zz^*)^{1/2}$. Then $pww_i^* = 0$ for all $i \in J$. Since p is self-adjoint, we have $Ran(p) \subseteq ker(p)^{\perp}$ and by construction, the final projection of w is $[p]$. Thus, w maps into $ker(p)^{\perp}$. But then zz_j^* maps into $ker(p) \cap ker(p)^{\perp} = 0$, so $ww_j^* = 0$, for all $j \in J$. But $\mathcal{W}$ was chosen to be maximal with respect to this property, so $w = 0$. It follows that $z = 0$, a contradiction. Thus if $y \in X_g$, and $yw_j^* = 0$ for all j then $y = 0$. Next, consider the series

$$\sum_j w_j^* w_j.$$

Each $w_j^* w_j$ is a projection in the relative commutant of N, and these are orthogonal for $i \neq j$. Then the partial sums of this series are an increasing sequence of positive operators in the unit ball of $N' \cap M$, so converge strongly to some positive element $t \in N' \cap M$. If $y \in X_g$, then yt is the weak limit of the sums

$$\sum_{j=1}^{n} (yw_i^*) w_i,$$

which are in X_g by properties (i) and (iii) above. Then also $yt \in X_g$. Fix $j \in J$. We wish to show that

$$(y - yt)w_j^* = 0.$$

Indeed, since the left hand side of this equation is the weak limit of

$$(y - \sum_{i=1}^{n} (yw_i^*) w_i) w_j^*$$

we get $(y - yt)w_j^* = yw_j^* - yw_j^* = 0$. Since j was arbitrary, by our first argument we have

$$y = yt = \sum_{j \in J} (yw_j^*) w_j.$$

This is an expression for y in the span of the w_j over $N \overline{\otimes} A$. Thus, we have shown that X_g is generated by $\{w_j\}$ as a left $N \overline{\otimes} A$- module.

We now use averaging techniques similar to those in the previous chapter to show that X_g is singly-generated. Let $\{v_i\}$ be a countable set of partial isometries in N

with orthogonal initial and final projections, possible because N is diffuse. Denote by z_n the partial sum $\sum_{i\leq n} v_i w_i$. This is a partial isometry by orthogonality of the v_i. Then also the sums

$$e_n = z_n^* z_n = \sum_{i\leq n} v_i^* v_i w_i^* w_i, \text{ and } f_n = z_n z_n^* = \sum_{i\leq n} v_i v_i^* w_i w_i^*$$

are projections in the von Neumann algebra Q generated by N and $N' \cap M$. Moreover, these partial sums are increasing sequences of positive operators in the unit ball of Q, so converge strongly to projections e and f in Q, respectively. These projections are clearly equivalent in Q, so determine a partial isometry $w_g \in Q$. It is then easily seen that the partial sums z_n must converge strongly to w_g.

Now consider the weakly closed module $(N\overline{\otimes}A)w_g N$. For any $j \geq 1$, we have that the operator

$$v_j^* w_g = \sum_{i\geq 1} v_j^* v_i w_i = v_j^* v_j w_j$$

lives in $(N\overline{\otimes}A)w_g N$. Then by averaging over the unitary group of N as above, we note that $\mathbb{E}_{N'\cap M}(v_j^* v_j w_j) = \tau(v_j^* v_j) w_j$ is also in $(N\overline{\otimes}A)w_g N$, so $w_j \in (N\overline{\otimes}A)w_g N$, for all $j \geq 1$.

It now follows easily that

$$X_g = \sum_{j\geq 1} (N\overline{\otimes}A)w_j = (N\overline{\otimes}A)w_g N.$$

Of course, $g \in G$ was arbitrary, and it easy to see that the partial isometries $v_{g,i} = w_i u_g$ are in $\mathcal{SN}(N)$ for all $g \in G$ and $i \geq 1$. For each $g \in G$, the desired partial isometry is the weakly convergent sum $\sum v_i w_i u_g = w_g u_g$. This proves the second part of our theorem. $\square$

The reader will surely notice that the decomposition of intermediate subalgebras in part (ii) of the above theorem depends on the choice of countable group G in the construction earlier in the chapter. However, we will find in the next chapter a coset decomposition of the group $\mathcal{N}(N)$, and a main result of that chapter will show that any choice of countable group produces the same coset structure. Thus, our choice of G is essentially unique.

It should also be noted that the decomposition of an intermediate subalgebra as

$$P = \sum_{g \in G} (N \overline{\otimes} A) v_g N$$

does not necessarily reflect the structure of P as a von Neumann subalgebra. The object on the right hand side of this equality is, in general, just a weakly closed left module over $N \overline{\otimes} A$. A few natural questions follow from this observation. First, is Theorem $V.9$ or a generalization true for general weakly closed modules in the normalizer of N ? We expect that this is a difficult question which would certainly involve some techniques from non self-adjoint operator algebras, beyond what we have developed here. However, it would be interesting to know, for instance, the structure of the weakly closed triangular subalgebras of the normalizer.

Second, in the case where P is a von Neumann subalgebra, what additional assumptions on the partial isometries v_g will guarantee that the weakly closed module

$$\sum_{g \in G} (N \overline{\otimes} A) v_g N$$

is a von Neumann algebra? In the spirit of Galois theory of subfactors, it is also natural to ask whether the sequences of partial isometries reflect the lattice structure of the intermediate subalgebras between N and $\mathcal{N}_M(N)''$. That is, suppose we have an inclusion of von Neumann algebras $N \subseteq P_2 \subseteq P_1 \subseteq \mathcal{N}(N)''$, and decompositions $P_1 = \sum (N \overline{\otimes} A_1) v_g N$ and $P_2 = (N \overline{\otimes} A_2) w_g N$. What then can be said about the relationship between A_1 and A_2 and, respectively, between the sets $\{v_g\}$ and $\{w_g\}$?

We answer these latter questions in the remainder of this chapter. The following proposition shows that the pair $(A, \{w_g\})$ associated to an intermediate subalgebra P reflects the self-adjointness and algebra structure of P.

Proposition V.10. *Suppose that P is weakly closed and $N \subseteq P \subseteq \mathcal{N}(N)''$. In the decomposition*

$$P = \sum_{g\in G} (N\overline{\otimes}A)v_g N = \sum_{g\in G} X_g u_g,$$

we have

1. *P is self-adjoint if and only if $Adu_{g^{-1}}(X_g^*) \subseteq X_{g^{-1}}$, for all $g \in G$.*
2. *P is an algebra if and only if $X_g Adu_g(X_h)\omega(g,h) \subseteq X_{gh}$, for all $g, h \in G$.*

Proof. To prove the first statement, let $x = \sum_g x_g u_g \in P$. Then

$$x^* = \sum_{g\in G} u_g^* x_g^* = \sum_{g\in G} u_g^{-1} x_g^* = \sum_{g\in G} Adu_{g^{-1}}(x_g^*)u_{g^{-1}},$$

and this last operator is in P if and only if $Adu_{g^{-1}}(X_g^*) \subseteq X_{g^{-1}}$. This proves (1). We obtain the second statement by noting that if $x = \sum x_g u_g$, $y = \sum y_h u_h \in P$ then

$$xy = \sum_{g,h} x_g Adu_g(y_h)\omega(g,h)u_{gh}$$

is in P if and only if $X_g Adu_g(X_h)\omega(g,h) \subseteq X_{gh}$, for all $g, h \in G$. □

The final result of this chapter concerns the lattice of intermediate subalgebras $N \subseteq P \subseteq \mathcal{N}(N)''$. We show that the pairs $(A, \{w_g\})$ associated to intermediate subalgebras reflect completely the partial ordering of subalgebras by inclusion. To this end, we define a partial order $\prec$ on the pairs by $(A_2, \{v_g\}) \prec (A_1, \{w_g\})$ if and only if the following three conditions hold:

1. $A_2 \subseteq A_1$

2. $v_g = v_g(w_g^* w_g)$, for all $g \in G$.

3. $v_g \in A_1 w_g$, for all $g \in G$.

Proposition V.11. *Let* $P_1 = \sum (N \overline{\otimes} A_1) w_g u_g N$ *and* $P_2 = \sum (N \overline{\otimes} A_2) v_g u_g N$. *Then* $P_2 \subseteq P_1$ *if and only if* $(A_2, \{v_g\}) \prec (A_1, \{w_g\})$.

Proof. The sufficiency of the second condition is obvious. Suppose that $P_2 \subseteq P_1$. For $i = 1, 2$ and $g \in G$ we write

$$X_{gi} = \{x \in Q : x u_g \in P_i\}.$$

Then it is easy to see that $X_{g2} \subseteq X_{g1}$, for all $g \in G$. It follows immediately that $(N \overline{\otimes} A_2) \subseteq (N \overline{\otimes} A_1)$, hence also $A_2 \subseteq A_1$. Then also $v_g \in X_{g1} = (N \overline{\otimes} A_1) w_g N$, for all g. Then for arbitrary fixed $g \in G$, write

$$v_g = v_g(w_g^* w_g) + v_g(1 - w_g^* w_g).$$

Observe that $v_g w_g^* w_g \in X_{g1}$ by the module property, so also $v_g(1 - w_g^* w_g) \in X_{g1}$, and is orthogonal to $w_g^* w_g$. That is, there exists a nonzero element $y \in X_{g2} \cap (N' \cap M)$ satisfying $y w_g^* = 0$. We deduce a contradiction.

If such a y exists, then also the left $N \overline{\otimes} A_2$ - module $(N \overline{\otimes} A_2) y$ is orthogonal to w_g. By the proof of Theorem V.9, the partial isometry $\tilde{w}$ from the polar decomposition of y also lives in $(N \overline{\otimes} A_2) y \cap (N' \cap M)$. We may also conclude that $\tilde{w} w_g^* = 0$, again by the proof of Theorem V.9. Recall from the construction of w_g that we have

$$w_g = \sum_{i \geq 1} v_i w_i,$$

where the set $\{w_i\} \subseteq X_{g1} \cap (N' \cap M)$ was chosen to be maximal with respect to orthogonality of the initial projections of the partial isometries. But now we have that

$$0 = \tilde{w}w_g^* = \sum_i v_i^* \tilde{w} w_i^*.$$

Multiplying successively on the left by the v_i, and averaging over the unitary group of N, we see that $\tilde{w}w_i^* = 0$ for all i. That is, the initial projection of $\tilde{w}$ is orthogonal to the initial projections of the w_i. Then $\tilde{w} = 0$, a contradiction to the supposition that y was nonzero. It follows that $v_g = v_g(w_g^* w_g)$, as desired. $\square$

CHAPTER VI

FURTHER RESULTS AND APPLICATIONS

In this chapter, we apply the techniques above to characterize the normalizing unitaries and seminormalizing partial isometries of a subfactor. The following results show that the normalizing algebra of a II_1 factor is remarkably rigid in that all normalizing unitaries of the subfactor come from products of elements of a certain countable group G with certain "trivial" normalizing unitaries, coming from the subfactor and its relative commutant.

Our methods apply to a general inclusion of II_1 factors, so are applicable, for instance, to inclusions arising in the group von Neuman algebra, crossed product, and tensor product settings. We present generalizations of recent results of Smith, White, and Wiggins in [43]. We emphasize, however, that our techniques do not seem to address one-sided normalizers, as do theirs.

Lemma VI.1. *Let $N \subseteq M$ be an inclusion of II_1 factors. Let $\mathcal{N}(N)'' = Q \rtimes_{\alpha}^{\omega} G$ be the decomposition of the normalizer above. Suppose that $\phi = Adg$ for some $g \in G$ and $\psi = Adv$ for some unitary $v \in \mathcal{N}(N)$. If $\psi \circ \phi$ is an inner automorphism of N, then there exist $u \in \mathcal{U}(N)$ and $w \in \mathcal{U}(N' \cap M)$ such that $v = uwg^{-1}$.*

Proof. If $\psi \circ \phi$ is inner then there exists a unitary $u \in \mathcal{U}(N)$ such that

$$vgxg^{-1}v^* = uxu^*,$$

for all $x \in N$. Then u^*vg is a unitary $w \in N' \cap M$, from which the result follows immediately. $\square$

Theorem VI.2. *Let $N \subseteq M$ be an inclusion of II_1 factors. Let Q and G be as above.*

(i) If $v \in \mathcal{N}(N)$, then there exists $g \in G$, $u \in \mathcal{U}(N)$, and $w \in \mathcal{U}(N' \cap M)$ such that

$v = uwg$.

(ii) If $v \in \mathcal{SN}(N)$, then there exists a unitary $u \in \mathcal{N}(N)$ and a projection $e \in N' \cap M$ such that $v = ue$.

Proof. Let us first prove statement (i). If $v \in \mathcal{N}(N)$, then by the remark following Theorem V.6, we can write v as a Fourier series over G

$$v = \sum x_i g_i$$

with a $\|\cdot\|_2$-summable sequence of coefficents in Q. Let α denote conjugation by v. Recall from our previous computations that α also defines automorphisms of $N' \cap M$ and Q, respectively. By the normalizing condition, we get that

$$(\sum x_i g_i)y = \alpha(y)(\sum x_i g_i),$$

for all $y \in Q$. We now obtain a countable set of equations

$$x_i \alpha_{g_i}(y) = \alpha(y) x_i,$$

and replacing y by $\alpha_{g_i^{-1}}(y)$ in each equation gives the set

$$x_i y = \theta_i(y) x_i, \tag{VI.1}$$

for all $y \in Q$, where $\theta_i = \alpha \circ \alpha_{g_i-1}$. Now if θ_i is inner on N for some i, then we are done by the previous lemma. Thus, assume that θ_i acts outerly on N for all i. Equivalently, θ_i acts freely on N, so also θ_i acts freely on Q by Lemma V.2. Thus, if θ_i is a nontrivial automorphism of Q, freeness and the last equation above imply that $x_i = 0$. On the other hand, if $\theta_i = 1$, then $\alpha = \alpha_{g_i}$ on Q, so by the previous lemma we have that $v = wg_i$, for some unitary $w \in N' \cap M$. This proves (i).

We now wish to characterize the seminormalizing partial isometries of N in terms of the group G. If we let w be such a partial isometry, recall that w partially implements an automorphism of N in that there exists $u \in \mathcal{N}(N)$ such that

$$wxw^* = Adu(x)ww^*,$$

for all $x \in N$. Let From (i) we know that u has the form $u = u_0 g$, where $g \in G$ and u_0 is a normalizing unitary in Q. Now, denote the source and range projections of w by p_s and p_r, respectively. Write v for the partial isometry $p_r u = u\phi^{-1}(p_r)$, where $\phi = Adu$. We then have

$$u\phi^{-1}(p_r)x = \phi(x)u\phi^{-1}(p_r),$$

for all $x \in N$, since $\phi^{-1}(p_r)$ lies in the relative commutant of N. It follows that for all $x \in N$,

$$\begin{aligned} vxv^* &= u\phi^{-1}(p_r)x\phi^{-1}(p_r)u^* \\ &= \phi(\phi^{-1}(p_r)x\phi^{-1}(p_r)) \\ &= p_r\phi(x)p_r \\ &= \phi(x)p_r = wxw^*. \end{aligned}$$

Moreover, a few simple computations show that v^*w is a partial isometry with source projection p_s and range projection $\phi^{-1}(p_r)$. From the equation

$$vxv^* = wxw^*,$$

for $x \in N$, we show that $v^*w \in N' \cap M$, as follows. Multiplying on the left by v^* and noting that v is a partial isometry whose source projection is in N', we obtain

$$xv^* = v^*wxw^*,$$

from which it follows that

$$xv^*w = v^*wx,$$

for all $x \in N$, by multiplying the previous equation on the right by w and proceeding similarly. One can also see that

$$v^*w = (u\phi^{-1}(p_r))^*w = \phi^{-1}(p_r)u^*w = u^*w,$$

so that $w = uv^*w = \phi(v^*w)u = \phi(v^*w)u_0g$. We have shown that v^*w is a partial isometry, so $v_0 = \phi(v^*w)u_0$ is also a partial isometry, with initial and final projections in the relative commutant of N. In fact, in part (i) we showed that u_0 is of the form $u_0 = v_0w_0$, where $v_0 \in \mathcal{U}(N)$ and $w_0 \in \mathcal{U}(N' \cap M)$. Recall also [18] that any partial isometry in a finite von Neumann algebra extends to a unitary, so that we can write $\phi(v^*w) = u_1e$, for some unitary $u_1 \in N' \cap M$ and projection $e \in N' \cap M$. We then have

$$w = \phi(v^*w)u = u_1ev_0w_0g = Adu_1(e)u_1v_0w_0g.$$

Clearly the unitary $u_1v_0w_0g$ is in $\mathcal{N}(N)$ and $Adu_1(e)$ is a projection in $N' \cap M$, so this proves the second result. $\square$

As we mentioned above in the remark following Theorem V.9, the above result shows that the group $\mathcal{N}_M(N)$ admits a partition into a countable collection of cosets $\mathcal{U}(N)\mathcal{U}(N' \cap M)u_g$, and that any choice of countable subgroup of $\mathcal{N}_M(N)$ from the construction in Chapter V will produce the same coset structure.

The corollary below is a new proof of a recent result of Wiggins [47] . Recall that in a crossed product of a factor by a group of outer automorphisms $N \rtimes_\theta G$, the intermediate subalgebras $N \subseteq P \subseteq N \rtimes_\theta G$ are all factors of the form $P = N \rtimes_\theta H$, for some subgroup $H \subseteq G$. Thus, we have a characterization below of the normalizing unitaries for all intermediate subalgebras in a crossed product factor. We write $\mathcal{N}_G(H)$

for the algebraic normalizer of H in G, the group elements of G that fix H under conjugation.

Corollary VI.3. *Let N be a II_1 factor and let $\theta : G \to Aut(N)$ be an action of a discrete group G on N by outer automorphisms. Let H be a subgroup of G. If $u \in N \rtimes_\theta G$ is a normalizing unitary of $N \rtimes_\theta H$, then there exists a $u_0 \in \mathcal{U}(N \rtimes_\theta H)$, and a $g \in \mathcal{N}_G(H)$ such that $u = u_0 g$.*

Proof. The normalizer of $N \rtimes H$ is a von Neumann subalgebra between $N \rtimes H$ and $N \rtimes G$, so by [2] it is equal to $N \rtimes K$ for some group K between H and G. Thus, we may assume that $N \rtimes H$ is regular in $N \rtimes G$. Write $P = N \rtimes H$, and let $\mathbb{E}_P : N \rtimes G \to P$ denote the conditional expectation. Note that P is an irreducible subfactor of $M = N \rtimes G$, so by the results of Chapter V, there exists a countable, discrete group K, and a cocycle action β of K on P such that

$$M = (N \rtimes H) \rtimes^{\mu}_{\beta} K.$$

We show that H is actually a normal subgroup of G. By the above crossed product decomposition, we can write any $g \in G$ as $g = \sum_{k \in K} m_k k$, where each $m_k \in N \rtimes H$, with convergence in the $\|\cdot\|_2$-norm on M. Moreover, it is easily seen that $m_k = \mathbb{E}_P(gk^{-1})$, for all k. Given $g \in G$, choose $k_0 \in K$ from this expansion such that $\mathbb{E}_P(gk^{-1})$ does not vanish. Let $u = gk^{-1}$. Then $\psi = AdU^*$ defines an inner automorphism of M such that $\psi(N) \subseteq N \rtimes H$. Then if $x \in N$, the equation

$$U^* x = \psi(x) U^*$$

implies that $\mathbb{E}_P(U)^* x = \psi(x)\mathbb{E}_P(U)^*$, for all $x \in N$. It follows easily that $\mathbb{E}_P(U)^* U \in N' \cap M = \mathbb{C}1$. Then $\mathbb{E}_P(U)^*$ is invertible and $g = (\mathbb{E}_P(U)^*)^{-1} k$. Then for any $h \in H$, we have

$$ghg^{-1} = (\mathbb{E}_P(U)^*)^{-1}khk^{-1}\mathbb{E}_P(U)^* \in N \rtimes H.$$

But also ghg^{-1} is in G and normalizes N, so in fact $ghg^{-1} \in H$, by Theorem $VI.2$. Thus, H is a normal subgroup of G.

Since H is normal in G, we may now follow the twisted crossed product construction from Chapter V to obtain a decomposition

$$M = (N \rtimes H) \rtimes_\alpha^\omega L,$$

where L is the quotient group G/H. The result now follows by the methods of Theorem VI.2. $\square$

In the case of an inclusion $LH \subseteq LK$, where H is a subgroup of K, we can use the above techniques to obtain more information about normalizing unitaries in $\mathcal{N}(LH)$. In a manner similar to crossed products, the normalizers and seminormalizers of LH are parameterized by the group normalizer of H in K. The following result generalizes Theorem 6.2 of [43], in which a similar result was obtained for the case of an irreducible inclusion of group von Neumann algebras. Note that our techniques are substantially different from those used in that work.

Theorem VI.4. *Let $H \subseteq G$ be an inclusion of countable, discrete i.c.c. groups. Then*

(i) If $v \in \mathcal{N}(LH)$, then there exists $k \in \mathcal{N}_G(H)$, $u \in LH$, and $w \in \mathcal{U}(LH' \cap LG)$ such that $v = uwk$.

(ii) If $v \in \mathcal{SN}(LH)$, then there exists a $k \in N_G(H)$ and a seminormalizing partial isometry $w \in Q$ such that $v = wk$.

Proof. Let M denote $\mathcal{N}(LH)''$. Then as in Chapter V there exists a countable group

Γ, and a free action of Γ on the von Neumann algebra Q generated by LH and $LH' \cap LG$ such that M is isomorphic to $Q \rtimes_{\alpha}^{\omega} \Gamma$. This induces a decomposition of M into a direct sum

$$M = \sum_{\gamma \in \Gamma} Q\gamma.$$

We have a similar direct sum decomposition of LG over LH. Indeed, let $\{Hg_i\}_{i \geq 1}$ be a listing of the right H cosets in G. Since the cosets Hg_i are disjoint, the subspaces $\ell^2(H)u_{g_i}$ are orthogonal in $\ell^2(G)$ in the left regular representation, and fill up $\ell^2(G)$. It follows that any $x \in LG$ can be written as a $\|\cdot\|_2$- convergent series $\sum_{i \geq 1} x_i u_{g_i}$, where $x_i = \mathbb{E}(x u_{g_i}^*) \in LH$.

Now write K for $\mathcal{N}_G(H)$. We will show that there exists some $k \in K$ in each $Q\gamma$ coset in M. Since, by Theorem VI.2, every normalizer $u \in \mathcal{N}(LH)$ lies in one of these cosets, statement (i) above will follow. Fix $\gamma \in \Gamma$. By the decomposition of LG given above, we may write u_γ as a convergent sum $\sum x_i u_{g_i}$. The normalizing condition applied to u_γ implies that

$$(\sum x_i u_{g_i})x = Adu_\gamma(x)(\sum x_i u_{g_i}),$$

for all $x \in LH$. From this equation, and the substitution $x \mapsto Adu_\gamma^{-1}(x)$ we obtain a countable set of equations

$$x_i \phi_i(x) = x x_i,$$

for all $x \in LH$, where ϕ_i denotes $Adu_{g_i} Adu_\gamma^{-1}$. We remark that at this point, it is not yet evident that any u_{g_i} normalizes LH, so ϕ_i maps LH to $Adu_{g_i}(LH)$. For fixed $i \geq 1$, we substitute $x \mapsto \phi_i^{-1}(y)$ and take adjoints to obtain

$$y x_i^* = x_i^* \phi_i^{-1}(y),$$

for all $y \in \phi_i(LH)$. It follows that $x_i x_i^* \in LH \cap LH' = \mathbb{C}1$. Taking the polar decomposition of x_i in LH, we obtain a partial isometry $w_i \in LH$ satisfying

$$w_i\phi_i(x) = xw_i,$$

for all $x \in LH$. But now since $x_ix_i^*$ is a scalar, in fact w_i is a unitary in LH, and satisfies

$$w_i^*xw_i = \phi_i(x).$$

That is to say, $Adu_{g_i}Adu_\gamma^{-1}$ is an inner automorphism of LH, so an easy algebraic calculation shows that $u_{g_i} = w_i^*v_iu_\gamma$, for some unitary $v_i \in LH' \cap LG$. It follows that $u_{g_i} \in \mathcal{N}(LH)$, and it follows easily that $g_i \in \mathcal{N}_G(H)$. Thus, in the expression $u_\gamma = \sum x_iu_{g_i}$, each group element u_{g_i} either has zero coefficient x_i or lies in the same Q-coset as γ. Thus if $u \in \mathcal{N}(LH)$ is an arbitrary normalizer, it lies in some Q-coset $Q\gamma$, and we have shown that any such coset contains some $k \in \mathcal{N}_G(H)$, so statement (i) is proved. Statement (ii) is obtained, in turn, by following the proof of Theorem $VI.2$. □

As a final application in this chapter, we consider a tensor product of a pair of subfactors $N_i \subseteq M_i$, for $i = 1, 2$. Clearly, $N_1\overline{\otimes}N_2$ is a subfactor of $M_1\overline{\otimes}M_2$, but it is not obvious that the normalizer of $N = N_1\overline{\otimes}N_2$ can be computed in terms of the individual normalizers $\mathcal{N}_{M_i}(N_i)$. This was shown by Smith, White, and Wiggins to be possible when the inclusions are irreducible. We give a proof for general subfactors using our methods.

Corollary VI.5. *Suppose that $N_i \subseteq M_i$ are regular subfactors. Let $C_i = N_i' \cap M_i$. Then if $u \in \mathcal{N}_{M_1\overline{\otimes}M_2}(N_1\overline{\otimes}N_2)$, then there exist unitaries $u_i \in \mathcal{N}_{M_i}(N_i)$, $v \in \mathcal{U}(N_1\overline{\otimes}N_2)$, and $w \in \mathcal{U}(C_1\overline{\otimes}C_2)$ such that $u = (u_1 \otimes u_2)vw$.*

Proof. By results of Chapters IV and V, we have crossed product decompositions $M_1 = (N_1\overline{\otimes}C_1) \rtimes_\alpha^\omega G_1$, and $M_2 = (N_2\overline{\otimes}C_2) \rtimes_\beta^\mu G_2$. We show that $M_1\overline{\otimes}M_2$ has a similar decomposition. First, we define a twisted action θ of the group $G = G_1 \times G_2$ on $N_1 \overline{\otimes} N_2$ by

$$\theta_{(g,h)} = \alpha_g \otimes \beta_h.$$

As in Chapter V, this induces an action of the same group on the relative commutant of $N_1 \overline{\otimes} N_2$, which is $C_1 \overline{\otimes} C_2$ by Tomita's commutant theorem. Then for $(g,h) \in G_1 \times G_2$, the unitaries $u_g \otimes u_h$ normalize $Q_1 \otimes Q_2$, where $Q_i = N_i \otimes C_i$, i = 1,2. An elementary tensor of $M_1 \overline{\otimes} M_2$ can be written as

$$m_1 \otimes m_2 = (\sum_{g \in G_1} x_g u_g) \otimes (\sum_{h \in G_2} y_h u_h) = \sum_{(g,h) \in G_1 \times G_2} (x_g \otimes y_g)(u_g \otimes u_h),$$

and it follows that $M = M_1 \overline{\otimes} M_2$ is generated by $Q = Q_1 \overline{\otimes} Q_2$ and the normalizing unitaries $u_g \otimes u_h$. Now $G = G_1 \times G_2$ acts freely on $N = N_1 \overline{\otimes} N_2$, so acts freely on Q by Lemma $V.2$. It follows that $\mathbb{E}(u_g \otimes u_h) = 0$ for all $(g,h) \in G$, where $\mathbb{E} : M \to Q$ is the natural (unique) conditional expectation. By Lemma $V.5$, we have a decomposition $M = Q \rtimes_\theta^\sigma G$, for a $\mathcal{U}(Q)$-valued cocycle σ on $G \times G$. By Theorem $VI.2$, we have that any unitary normalizer of $N_1 \overline{\otimes} N_2$ is of the form $u = u_0 u_1 u_s$, for unitaries $u_0 \in N_1 \overline{\otimes} N_2$, $u_1 \in C_1 \overline{\otimes} C_2$, and some $s \in G_1 \times G_2$. Since $u_s = u_g \otimes u_h$, this proves the result. $\square$

CHAPTER VII

APPLICATION: NEW EXAMPLES OF NORMING ALGEBRAS

Norming algebras were first defined by Pop, Sinclair, and Smith in [30], and used as a means for estimating the cb-norm of maps between operator algebras. Various examples of subalgebras of C^*-subalgebras which are norming (and non-norming) were constructed in [30] and [29]. Applications to the Hochschild Cohomology problem, Kadison's similarity problem, and the so-called bounded projection problem have also been explored (see also [28]).

If a von Neumann algebra $M \subseteq B(\mathcal{H})$ is injective, then there exists a conditional expectation $\Phi : B(\mathcal{H}) \to M$, which is by definition a contractive projection. The *bounded projection problem* asks whether the existence of a bounded projection $P : B(\mathcal{H}) \to M$ implies that M is injective. In [30], it was shown that the answer is affirmative when M contains a norming hyperfinite subfactor. As a corollary of that result and our work below, we will obtain a previously unknown instance in which the answer to the problem is affirmative.

We recall the following definition:

Definition VII.1. We say that A norms B if, for each $k \geq 1$ and for each $X \in M_k(B)$,

$$\|X\| = sup\left\{\|RXC\| : R \in Row_k(A), C \in Col_k(A), \|R\|, \|C\| \leq 1\right\}.$$

We say that A *row norms* B (respectively, *column norms* B), if the above equality holds with $\|RXC\|$ replaced by $\|RX\|$ (respectively $\|XC\|$). It is shown in [30] that norming, row norming, and column norming are equivalent for an inclusion $A \subseteq B$ of C^*-algebras. We say that A is λ-*norming* for B if there exists $\lambda \in (0, 1)$ such that

$$\lambda \|X\| \leq sup\left\{\|RXC\| : R \in Row_k(A), C \in Col_k(A), \|R\|, \|C\| \leq 1\right\}.$$

The following are examples of norming and non-norming subalgebras of various C^*-algebras, discovered in [30].

Theorem VII.2. *(Pop, Sinclair, Smith [30])*

(i.) *Any C^*-algebra norms itself.*

(ii.) *The algebra $\mathbb{C}$ norms any abelian C^*-algebra A.*

(iii.) *Let w be a word in $\mathbb{F}_2$. Then the von Neumann subalgebra generated by w is not λ-norming for any λ.*

We recall another important result from that paper.

Lemma VII.3. *[Theorem 3.2 of [30]] Let $A \subseteq B$ be an inclusion of C^*-algebras, where A has no finite-dimensional representations and B is finitely generated as a left A-module. Then A norms B.*

It follows from Lemma VII.3 that whenever N is a von Neumann algebra with no finite-dimensional representations and P is a von Neumann algebra, N norms $N\overline{\otimes}P$ (see [30] for details). We prove here a similar result, requiring a slightly different embedding of N inside $N\overline{\otimes}M_n(P) = M_n(N)\overline{\otimes}P$. In particular, let $n \geq 1$ be an integer and α_1, α_2, $\ldots, \alpha_n$ be outer automorphisms of a type II_1 factor N. With respect to these automorphisms, we define a map $\pi_n : N \to M_n(N)\overline{\otimes}P$ by

$$\pi_n(x) = \sum_{j=1}^{n} \alpha_j(x) \otimes e_{jj} \otimes 1,$$

where e_{ij} denotes the standard matrix unit in $M_n(\mathbb{C})$. We then have the following auxiliary result.

Lemma VII.4. *Let N and P be von Neumann algebras, and suppose that N has no finite dimensional representations. Then for any integer $n \geq 1$, $\pi_n(N)$ norms $M_n(N)\overline{\otimes}P$.*

Proof. Clearly, it is enough to show that $\pi_n(N)$ norms $M_n(N)\overline{\otimes}B(\mathcal{H})$, for any Hilbert space $\mathcal{H}$. To do this, let $k \geq 1$ be an integer and X a matrix in $M_k(M_n(N)\overline{\otimes}B(\mathcal{H}))$ with $\|X\| = 1$, and set $\varepsilon > 0$. Let $\{p_\gamma\}$ be a net of finite rank projections in $B(\mathcal{H})$ converging strongly to the identity. Then there exists γ_0 such that

$$\|(1_k \otimes 1 \otimes p_\gamma)X(1_k \otimes 1 \otimes p_\gamma)\| > 1 - \varepsilon,$$

for all $\gamma \geq \gamma_0$, where $1_k \in M_k$ and $1 \in M_n(N)$ denote the respective units. The operators $(1_k \otimes 1 \otimes p_\gamma)X(1_k \otimes 1 \otimes p_\gamma)$ are in $M_k(M_n(N) \otimes B(p_\gamma\mathcal{H}))$, and since p_γ has finite rank, we are in the situation of an inclusion

$$\pi_n(N) \subseteq M_n(N)\overline{\otimes}M_s(\mathbb{C}).$$

The von Neumann algebra $M_n(N)\overline{\otimes}M_s(\mathbb{C})$ is clearly a finite-dimensional left module over $\pi_n(N) \otimes 1$, so by Lemma $VII.3$, the latter algebra norms the former. We may then choose a row matrix $R \in Row_n(\pi_n(N) \otimes 1)$ so that

$$\|R(1_k \otimes 1 \otimes p_{\gamma_0})X(1_k \otimes 1 \otimes p_{\gamma_0})\| > 1 - \varepsilon.$$

But since the operators $1 \otimes p_\gamma$ all commute with $\pi_n(N) \otimes 1$, this says

$$\|RX\| \geq \|(1_k \otimes 1 \otimes p_{\gamma_0})RX(1_k \otimes 1 \otimes p_{\gamma_0})\| > 1 - \varepsilon.$$

Thus, $\pi_n(N)$ norms $M_n(N)\overline{\otimes}B(\mathcal{H})$. □

In particular, the above lemma is true if N is a finite factor. It was shown in [31] that if N is a II_1 factor and G a discrete group acting on N by outer automorphisms, then N norms the crossed product $N \rtimes G$. Our setting generalizes this result in the type II_1 situation.

We come to the main observation of this chapter.

Theorem VII.5. *Suppose that $N \subseteq M$ is a regular inclusion of* II_1 *factors. Then N norms M.*

Proof. We exploit the structure results from Chapters IV and V as well as Lemma $VII.4$, and follow the idea of the proof of Pop and Smith for the crossed product case. As in previous chapters, write C for $N' \cap M$ and Q for the von Neumann algebra generated by N and C. In Chapter IV we showed that $Q = N\overline{\otimes}C$. By Theorem $V.6$, there exists a representation of M on $L^2(Q)\otimes\ell^2(G)$, where G is the countable discrete group from Chapter V, given by the maps π and u, defined on Q and G (respectively)

$$\pi(x)\xi(g) = \alpha_g^{-1}(x)\xi(g), \text{ where } \alpha_g = Adu_g, \text{ and}$$

$$u_h\xi(g) = \omega(h, h^{-1}g)\xi(h^{-1}g),$$

where ω is the $\mathcal{U}(Q)$-valued cocycle from Chapter V. We also recall that both N and $N' \cap M$ are stable under the action of G. It follows that each $x \in N$ in this representation sits inside M as a "diagonal matrix"

$$x \mapsto \sum_{g\in G}(\alpha_g^{-1}(x) \otimes 1) \otimes e_{g,g},$$

where $e_{g,h}$ denotes the obvious matrix unit in $B(\ell^2(G))$.

For each finite subset $F \subseteq G$, there exists a projection $p_F = span\,\{\delta_g : g \in F\}$ in $B(\ell^2(G))$. Then $(1 \otimes p_F)M(1 \otimes p_F)$ embeds in $M_n(N\overline{\otimes}C)$, where $n = |F|$. Now let $k \geq 1$ and suppose that $X \in M_k(Q\overline{\otimes}B(\ell^2(G)))$. The net of projections $\{1 \otimes p_F \otimes 1_k\}_F$ commutes with $\pi(N \otimes 1) \otimes 1_k$ and converges strongly to the identity. Given $\varepsilon > 0$, choose F so that

$$\|(1 \otimes p_F \otimes 1_k)X(1 \otimes p_F \otimes 1_k)\| > \|X\| - \varepsilon.$$

We showed in Lemma $VII.4$ that $\pi(N\otimes 1)$ norms $M_n(N\overline{\otimes}C)$, and $1\otimes p_F \otimes 1_k)X(1\otimes$

$p_F \otimes 1_k) \in M_k(M_n(N\overline{\otimes}C))$, so there exists $R \in Row_k((1 \otimes p_F)\pi(N \otimes 1))$ such that

$$\|R(1 \otimes p_F \otimes 1_k)X(1 \otimes p_F \otimes 1_k)\| > \|X\| - \varepsilon, \text{ that is,}$$

$$\|(1 \otimes p_F)RX(1 \otimes p_F \otimes 1_k)\| > \|X\| - \varepsilon.$$

Passing to the strong limit in F of this last inequality shows that N norms M. $\square$

As an immediate corollary of this result, and Theorem 6.5 of [30], we have

Corollary VII.6. *Let $M \subseteq B(\mathcal{H})$ be a separable II_1 factor with a regular hyperfinite subfactor. If there exists a bounded projection $\phi : B(\mathcal{H}) \to M$, then M is injective.*

CHAPTER VIII

NORMALIZERS IN NONSEPARABLE II_1 FACTORS AND COHOMOLOGY*

A. Notation and background

In this section we collect the basic definitions and results of Hochschild cohomology that will be used in the next section. The reader may wish to consult [40] for a more detailed exposition. Let M be a von Neumann algebra and let X be a Banach M-bimodule (In the next section, we will restrict attention to the case $X = M$). Let $\mathcal{L}^n(M, X)$ denote the vector space of n-linear bounded maps $\phi : M^n \to X$. Define the coboundary map $\partial : \mathcal{L}^n(M, X) \to \mathcal{L}^{n+1}(M, X)$ by

$$\begin{aligned}\partial\phi(x_1, \dots, x_{n+1}) = {} & x_1\phi(x_2, \dots, x_{n+1}) \\ & + \sum_{j=1}^{n}(-1)^j\phi(x_1, \dots, x_{j-1}, x_jx_{j+1}, \dots, x_{n+1}) \\ & + (-1)^{n+1}\phi(x_1, \dots, x_n)x_{n+1}.\end{aligned}$$

An algebraic computation shows that $\partial^2 = 0$. We thus obtain the Hochschild complex and define the nth continuous Hochschild cohomology group to be

$$H^n(M, X) = \frac{ker\partial}{Im\partial}.$$

The maps $\phi \in ker\partial$ are called $(n+1)$-cocycles, and the maps in $Im\partial$ are called n-coboundaries. It is easy to check that the 2-cocycles are precisely the bounded derivations from M into X.

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Various extension and averaging results are of central importance in computing cohomology groups. The most basic extension theorem states that normal cohomology and continuous cohomology are equal when the space X is assumed to be a dual normal M-module (see [38, Theorem 3.3]). In that result, it is shown that a continuous cocycle can be modified by a coboundary so that the resulting cocycle is separately normal in each variable. Thus, we will assume in what follows that all n-cocycles are separately normal in each variable. We denote the vector space of bounded, separately normal n-linear maps from M to X by $\mathcal{L}_w^n(M, X)$. The general strategy of averaging arguments in cohomology is to replace a continuous n-cocycle ϕ with a modified cocyle $\phi + \partial\psi$ which has more desirable continuity and modularity properties. The coboundary $\partial\psi$ is obtained by an averaging process over a suitable group of unitaries in the underlying von Neumann algebra. In the present work, it will be essential that modifications to a given cocycle are made while preserving certain norm estimates on the original cocycle. The averaging lemma we need is the following, which can be found in [40]:

Lemma VIII.1. *Let R be a hyperfinite von Neumann subalgebra of a von Neumann algebra M. Then there is a bounded linear map $L_n : \mathcal{L}_w^n(M, M) \to \mathcal{L}_w^{n-1}(M, M)$ such that $\phi + \partial L_n\phi$ is a separately normal R-module map for any n-cocycle ϕ. Moreover, $\|L_n\| \leq \ell(n)$, a constant depending only on n.*

We will also need two extension results, the first of which is essentially Lemma 3.3.3 of [40].

Lemma VIII.2. *Let A be a C^*-algebra on a Hilbert space $\mathcal{H}$ with weak closure $\overline{A}$ and let X be a dual Banach space. If $\phi : A^n \to X$ is bounded, n-linear, and separately normal in each variable then ϕ extends uniquely, without changing the norm, to a bounded, separately normal, n-linear map $\overline{\phi} : (\overline{A})^n \to X$.*

Lemma VIII.3. *Let A be a C^*-algebra, and denote its weak closure by $\overline{A}$. Then for each n there exists a linear map $V_n : \mathcal{L}_w^n(A, X) \to \mathcal{L}_w^n(\overline{A}, X)$ such that when ϕ is as in Lemma VIII.2, then $V_n\phi = \overline{\phi}$. Moreover, $\|V_n\| \leq 1$ and $\partial V_n = V_{n+1}\partial$, for all $n \geq 1$.*

Proof. For ϕ in $\mathcal{L}_w^n(A, X)$, we define $V_n\phi = \overline{\phi}$. This is well-defined and linear by the uniqueness in Lemma VIII.2. Moreover, since the extension in Lemma VIII.2 is norm-preserving, we have $\|V_n\| \leq 1$. Now for any $x_1, x_2, \ldots x_{n+1} \in A$, we have

$$\partial V_n\phi(x_1, \ldots x_{n+1}) = \partial\phi(x_1, \ldots x_{n+1}) = V_{n+1}\partial\phi(x_1, \ldots, x_{n+1}).$$

Both of the maps $\partial V_n\phi$ and $V_{n+1}\partial\phi$ are separately normal, so are uniquely defined on $(\overline{A})^n$. Thus, we will have

$$\partial V_n\phi(x_1, \ldots x_{n+1}) = V_{n+1}\partial\phi(x_1, \ldots, x_{n+1}),$$

for all $x_1, \ldots, x_{n+1}$ in $\overline{A}$. This completes the proof. □

B. The main theorem

A corollary of the following result is that the study of Cartan masas can, in many instances, be reduced to the separable case. The techniques of the proof come from [42, Theorem 2.5], in which a similar result is proved for singular masas in II_1 factors.

Proposition VIII.4. *Let N be a II_1 factor with Cartan masa A, and let M_0 be a separable von Neumann subalgebra of N. Let $\phi : N^n \to N$ be a separately normal n-cocycle. Then there exists a separable subfactor M such that $M_0 \subseteq M \subseteq N$, $M \cap A$ is a Cartan masa in M, and ϕ maps M^n into M.*

Proof. For a von Neumann algebra Q, and $x \in Q$, denote by $K_Q^n(x)$ and $K_Q^w(x)$, respectively, the operator and $\|\cdot\|_2$-norm closures of the set $K_Q(x)$. Recall that when Q is a von Neumann subalgebra of M, for any $x \in M$, $\mathbb{E}_{Q' \cap M}(x)$ picks out the element of minimal $\|\cdot\|_2$-norm in $K_Q^w(x)$. We will construct, inductively, a sequence of separable von Neumann algebras

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq N$$

and abelian subalgebras $B_k \subseteq M_k$ so that $M = (\bigcup_{k=1}^{\infty} M_k)''$ has the required properties. The von Neumann algebra $B = (\bigcup_{n=0}^{\infty} B_n)''$ will be a masa in M, and thus equal to $M \cap A$. The inductive hypothesis is as follows: We assume that sets $M_1, \ldots, M_k$ and $B_j = M_j \cap A$, satisfy: for $1 \leq j \leq k-1$ and fixed sequences $\{y_{j,r}\}_{r=1}^{\infty}$, $\|\cdot\|_2$-norm dense in the $\|\cdot\|_2$-closed unit ball of the separable von Neumann algebra M_j,

(i) $\mathbb{E}_A(y_{j,r}) \in B_j \cap K_{B_{j+1}}^w(y_{j,r})$, for $r \geq 1$,

(ii) $K_{M_{j+1}}^n(y_{j,r}) \cap \mathbb{C}1$ is nonempty for all $r \geq 1$;

(iii) For each j, there is a countable set of unitaries $\mathcal{U}_{j+1} \subseteq N(A) \cap M_{j+1}$ such that $\mathcal{U}_{j+1}'' \supseteq M_j$;

(iv) The cocycle ϕ maps $(M_j)^n$ into M_{j+1}.

In the inductive step we will construct M_{k+1} so that the sets $M_1, \ldots, M_{k+1}$ and $B_1, \ldots, B_k, B_{k+1} = M_{k+1} \cap A$ satisfy properties (i)-(iv).

We first prove that this sequence of algebras gives the desired result. The von Neumann algebra $M = (\bigcup_{n=0}^{\infty} M_n)''$ will be a separable subalgebra of N, by our construction. We show M is a factor.

For any $x \in M$, $K_M^w(x) \cap \mathbb{C}1$ is nonempty, by the following approximation argument. First suppose that $x \in M_k$, for some $k \geq 1$ and $\|x\| \leq 1$. Let $\varepsilon > 0$ be given. Choose an element $y_{k,r} \in M_k$ as above with $\|y_{k,r} - x\|_2 < \varepsilon$. By condition (ii) we can choose an element

$$a_{k,r} = \sum_{i=1}^{m} \lambda_i u_i y_{k,r} u_i^* \in K_{M_{k+1}}(y_{k,r})$$

whose $\|\cdot\|_2$-norm distance to $\mathbb{C}1$, which we denote $\operatorname{dist}_2(a_{k,r}, \mathbb{C}1)$, is less than ε. Then $a = \sum_{i=1}^m \lambda_i u_i x u_i^*$ is an element of $K_{M_{k+1}}(x)$ with $\operatorname{dist}_2(a, \mathbb{C}1) < 2\varepsilon$. It follows that $K^w_{M_{k+1}}(x) \cap \mathbb{C}1$ is nonempty.

Now let $x \in M$, $\|x\| \leq 1$, and fix $\varepsilon > 0$. By the Kaplansky density theorem, there exists a $k \geq 1$, and an element $x_\alpha \in M_k$ of norm at most 1 such that $\|x_\alpha - x\|_2 < \varepsilon$. Then by what we did above, $K^w_{M_{k+1}}(x_\alpha) \cap \mathbb{C}1$ is nonempty. It follows, by similar argument to the one above, that there exists an element $a \in K_M(x)$ with $\operatorname{dist}_2(a, \mathbb{C}1) < 2\varepsilon$. Thus, $K^w_M(x)$ has nonempty intersection with $\mathbb{C}1$. By scaling, this result is true for $x \in M$ of arbitrary norm. To see that M is a factor, note that if x is central in M then $K^w_M(x) = \{x\}$, and since this set meets $\mathbb{C}1$, x must be a multiple of the identity.

We now prove that $B = (\bigcup_{n=0}^\infty B_n)''$ is a masa in M. First, condition (i) implies that $\mathbb{E}_A(x) \in B \cap K^w_B(x) \subseteq B \cap K^w_A(x)$, for all $x \in M$. This follows from an approximation argument similar to the one above, in which we prove the claim first for all $x \in M_k$ with $\|x\| \leq 1$ and then extend to all of M. Since $\mathbb{E}_A(x)$ is the element of minimal $\|\cdot\|_2$-norm in $K^w_A(x)$, it also has this property in $K^w_B(x)$. But then $\mathbb{E}_A(x) = \mathbb{E}_{B'\cap M}(x)$. Since for all $x \in M$, $\mathbb{E}_A(x) = \mathbb{E}_B(x)$, one has

$$x = \mathbb{E}_{B'\cap M}(x) = \mathbb{E}_A(x) = \mathbb{E}_B(x),$$

for all $x \in B' \cap M$. Thus, $B' \cap M \subseteq B$. Since B is abelian, the opposite inclusion also holds. Thus B is a masa in M, and $B = M \cap A$. We show B is Cartan. By condition (iii), we will have $M = (\bigcup_{k=0}^\infty \mathcal{U}''_{k+1})''$. We claim that this last set is precisely $\mathcal{N}(B, M)''$, and hence that B is Cartan in M. Fix $k \geq 0$ and let $u \in \mathcal{U}_{k+1}$. Then since $u \in \mathcal{N}(A) \cap M_{k+1}$, for any $j \leq k+1$ we have

$$uB_ju^* = u(A \cap M_j)u^* \subseteq A \cap M_{k+1} = B_{k+1} \subseteq B.$$

If $j > k+1$, since $u \in M_{k+1} \subseteq M_j$, we have

$$uB_ju^* = u(A \cap M_j)u^* = A \cap M_j = B_j \subseteq B.$$

Then $u(\bigcup_{j=0}^{\infty} B_j)u^* \subseteq B$, and $u \in N(B, M)$. Then also $\mathcal{U}_{k+1}'' \subseteq \mathcal{N}(B, M)''$. The claim follows. Then $M \subseteq \mathcal{N}(B, M)''$, and since the other containment holds trivially, $M = \mathcal{N}(B, M)''$. That is, B is a Cartan masa in M. Finally, by our construction of M, condition (iv) will imply that ϕ maps M^n into M, since ϕ is separately normal in each variable.

We proceed to construct the algebras M_k, B_k. Put $B_0 = M_0 \cap A$. Assume that $M_1, \ldots, M_k$, and $B_1, \ldots, B_k$ have been constructed, satisfying conditions (i)-(iv), specified above. By Dixmier's approximation theorem, [18, Theorem 8.3.5], each sequence $\{y_{k,r}\}_{r=1}^{\infty}$ is inside a von Neumann algebra $Q_0 \subseteq N$ (generated by a countable set of unitaries) such that $K_{Q_0}^n(y_{k,r}) \cap \mathbb{C}1$ is nonempty. To get condition (iii), observe that since $\mathcal{N}(A)'' = N$, Lemma II.9 applies and there is a countable set of unitaries $\mathcal{U}_{k+1}$ satisfying the desired condition. Since ϕ is normal in each variable, by [38, Theorem 4.4], ϕ is jointly $\|\cdot\|_2$-norm continuous when restricted to bounded balls in $(M_k)^n$. Since M_k is separable, it follows that $\phi((M_k)^n)$ generates a separable von Neumann algebra Q_1. Finally, since for all $r \in \mathbb{N}, \mathbb{E}_A(y_{k,r}) \in K_A^w(y_{k,r}) \cap B_k$, there exists a set of unitaries $\{u_m\}_{m=1}^{\infty}$ generating a von Neumann algebra $Q_2 \subseteq A$ such that $\mathbb{E}_A(y_{k,r}) \in K_{Q_2}^w(y_{k,r})$, for all r. This will give condition (i). We complete the construction by letting M_{k+1} be the von Neumann algebra generated by $Q_0, Q_1, Q_2, \mathcal{U}_{k+1}, M_k$, and $\mathbb{E}_A(M_k)$. $\square$

We are now in a position to compute the cohomology groups $H^n(N, N)$, where N is a general type II_1 factor with a Cartan subalgebra. Note that for any such N, associated to each finite set $F \subseteq N$ is the separable von Neumann algebra $M_F \subseteq N$

which it generates. Thus every such II_1 factor N satisfies the hypothesis of Proposition $VIII.4$. This leads to our main theorem.

Theorem VIII.5. *Let N be a type II_1 factor with a Cartan subalgebra A. Then $H^n(N, N) = 0$ for all $n \geq 1$.*

Proof. Since the case $n = 1$ is the Kadison-Sakai result, we assume n is at least 2. We refer the reader to the proof of the separable case in [41]. Let $\phi : N^n \to N$ be a cocycle, which we may assume to be separately normal. By Proposition $VIII.4$, for each finite set $F \subseteq N$, there exists a separable subfactor N_F such that $M_F \subseteq N_F \subseteq N$, $N_F \cap A$ is Cartan in N_F, and ϕ maps $(N_F)^n$ into N_F. Denote the restriction of ϕ to N_F by ϕ_F. By the separable case, ϕ_F is a coboundary, i.e., there exists an $(n-1)$-linear and bounded map $\psi : (N_F)^{n-1} \to N_F$ such that $\phi_F = \partial\psi_F$. Moreover, there is a uniform bound on $\|\psi_F\|$; we confine this argument to the end of the proof. Let $\mathbb{E}_F : N \to N_F$ be the conditional expectation. Define $\theta_F : N^{n-1} \to N$ by $\theta_F = \psi_F \circ (\mathbb{E}_F)^{n-1}$. Order the finite subsets of M by inclusion. Because $\|\psi_F\|$ is uniformly bounded, so is $\|\theta_F\|$. Now, for any $(n-1)$-tuple $(x_1, \ldots, x_{n-1}) \in N^{n-1}$, $\{\theta_F(x_1, \ldots, x_{n-1})\}_F$ is a bounded net as F ranges over all finite sets containing $x_1, \ldots, x_{n-1}$. This has an ultraweakly convergent subnet (which we also denote by $\{\theta_F(x_1, \ldots, x_{n-1})\}_F$), by ultraweak compactness of bounded subsets of N ([18, Chapter 7]). Define $\theta : N^{n-1} \to N$ by

$$\theta(x_1, \ldots, x_{n-1}) = \lim_F \theta_F(x_1, \ldots, x_{n-1}).$$

Then θ is clearly $(n-1)$-linear, and bounded by the uniform bound on $\|\theta_F\|$. We claim $\phi = \partial\theta$. Let $(x_1, \ldots, x_n) \in M^n$. Then

$$\phi(x_1, \ldots, x_n) = \phi_F(x_1, \ldots, x_n) = \partial\theta_F(x_1, \ldots, x_n),$$

for all finite sets F containing $x_1, \ldots, x_n$. Ordering these sets by inclusion, we will obtain a subnet $\{\theta_F\}$ such that $\partial\theta_F(x_1,\dots,x_n)$ converges weakly to $\partial\theta(x_1,\dots,x_n)$. Passing to limits in the above equality gives $\phi(x_1,\dots,x_n) = \partial\theta(x_1,\dots,x_n)$. This proves the claim, and the result follows.

It remains to be shown that there is a uniform bound on the norms of the maps ψ_F, constructed above. It suffices to obtain an estimate for each of these maps in terms of n and $\|\phi_F\|$, since this last quantity is dominated by $\|\phi\|$ for all F. We drop the index F, since it plays no further role in the proof.

Now M will denote a separable II_1 factor with Cartan masa A. By [41, Theorem 2.2], there is a hyperfinite factor R such that

$$A \subseteq R \subseteq M \text{ and } R' \cap M = \mathbb{C}1.$$

Let $\phi : M^n \to M$ be a separately normal cocycle. We will show that ϕ is the image under ∂ of an $(n-1)$-linear map whose norm is at most $K\|\phi\|$, where K is a constant depending only on n. By Lemma VIII.1 there exists a map L_n such that $\theta = \phi - \partial L_n\phi$ is a separately normal R-module map, and $\|\theta\| \le l(n)\|\phi\|$, where $l(n)$ is a constant depending only on n. Let $\mathcal{U}$ be a generating set of unitaries for M. Then the weak closure of $C^*(\mathcal{U})$ is M. Let $\phi_C : C^*(\mathcal{U})^n \to M$ denote the restriction of ϕ. The proof of the separable case gives $\theta = \partial\alpha$, where $\alpha : C^*(\mathcal{U})^{n-1} \to M$ is $(n-1)$-linear and has norm at most $\sqrt{2}\|\theta\|$. Then

$$\phi_C = \theta + \partial\zeta = \partial(\alpha + \zeta),$$

where ζ denotes the restriction of $L_n\phi$ to $C^*(\mathcal{U})^{n-1}$. Write ψ for $\alpha+\zeta$. Then by what we have done above, there exists a constant $K(n)$ such that

$$\|\psi\| \le K(n)\|\phi\|.$$

We now wish to extend the equality $\phi_C = \partial\psi$ to one involving maps defined on M, while preserving this last norm estimate. Applying the map V_n from Lemma VIII.3 to both sides of this last equality, we get

$$V_n\phi_C = V_n(\partial\psi) = \partial(V_{n-1}\psi).$$

Because the separately normal map ϕ_C extends uniquely to ϕ on M, this says $\phi = \partial(V_{n-1}\psi)$. The map $V_{n-1}\psi$ is in $\mathcal{L}_w^{n-1}(M, M)$, and $\|V_{n-1}\psi\| \leq K(n)\|\phi\|$. This gives the required norm estimate, completing the proof of the theorem. □

Using a very similar argument to the one above, we can also generalize the result of [39] to the nonseparable case. In particular, we have the following result.

Theorem VIII.6. *Suppose that $N \subseteq M$ is an inclusion of type* II_1 *factors of finite Jones index such that N has a Cartan subalgebra. then $H^n(N, M) = 0$, for all $n \geq 1$.*

This result states that every cocycle on N which takes values in a factor $M \supseteq N$ satisfying $[M : N] < \infty$ is a coboundary. The key additional element in the proof is to write elements of M in terms of a Pimsner-Popa basis of M over N. We refer the reader to [27] for the result of Pimsner and Popa, which states essentially that M is finitely generated as a left and right module over N.

C. Further results

In this section we extend proposition $VIII.4$ to include the case of general masas. Given an arbitrary type II_1 factor with a masa A, we show that any separable subalgebra of N is contained in a separable subfactor which has a masa by intersection with A. We will make use of a recent result of Chifan [1] to characterize exactly the normalizing algebra of the masa in the separable subfactor.

Definition VIII.7. Let M be a II_1 factor and $B \subseteq N \subseteq M$ von Neumann subalgebras of M. We say that the inclusion $B \subseteq N \subseteq M$ has the relative weak asymptotic homomorphism property (rWAHP) if, for all $x_1, \ldots, x_n \in M$ and $\varepsilon > 0$, there exists a unitary $u \in B$ with

$$\left\| \mathbb{E}_B(x_i u x_j^* - \mathbb{E}_N(x_i) u \mathbb{E}_N(x_j)^*) \right\|_2 < \varepsilon,$$

for all $1 \leq i, j \leq n$.

We have the following theorem.

Theorem VIII.8. *(Chifan, [1]) Let A be a masa in a II_1 factor M, and let $N = \mathcal{N}_M(A)''$. Then the inclusion $A \subseteq N \subseteq M$ has the relative WAHP.*

A partial converse to this theorem is known. In particular, if an inclusion $A \subseteq N \subseteq M$ has the rWAHP, then the normalizer $\mathcal{N}_M(A)''$ is contained in N. See [45] for details. The following result generalizes both proposition $VIII.4$ of the present note and theorem 2.5 of [42].

Theorem VIII.9. *Let N be a II_1 factor with a masa A, and let M_0 be a separable subalgebra of N. Then there exists a separable factor M such that $M_0 \subseteq M \subseteq N$, and $M \cap A$ is a masa in M satisfying*

$$\mathcal{N}_M(M \cap A)'' = (M \cap \mathcal{N}(A))''.$$

Proof. The proof of will be identical to that of proposition $VIII.4$ in some respects. The strategy is the same: given a separable subalgebra $M_0 \subseteq N$, we will construct a tower of separable subalgebras $M_0 \subseteq M_1 \subseteq \ldots$ and abelian algebras $B_k = M_k \cap A$ so that $M = (\bigcup_{k \geq 0} M_k)''$ has the required properties. As before, the von Neumann algebra $B = (\bigcup_{\geq 0} B_k)''$ will be a masa in M, and thus equal to $M \cap A$. The inductive hypothesis is as follows: We assume that sets $M_1, \ldots, M_k$ and $B_j = M_j \cap A$, satisfy: for $1 \leq j \leq k-1$ and fixed sequences $\{y_{j,r}\}_{r=1}^{\infty}$, $\|\cdot\|_2$-norm dense in the $\|\cdot\|_2$-closed unit ball of the separable von Neumann algebra M_j,

(i) $\mathbb{E}_A(y_{j,r}) \in B_j \cap K^w_{B_{j+1}}(y_{j,r})$, for $r \geq 1$,

(ii) $K^n_{M_{j+1}}(y_{j,r}) \cap \mathbb{C}1$ is nonempty for all $r \geq 1$;

(iii) $\mathbb{E}_{\mathcal{N}(A)''}(y_{j,r}) \in (M_{j+1} \cap \mathcal{N}(A))''$, for all $r \geq 1$.

(iv) For all $N \geq 1$ and for all $\varepsilon > 0$ there exists a unitary $u \in B_{j+1}$ such that

$$\left\| \mathbb{E}_A(y_{j,m} u y^*_{j,n} - \mathbb{E}_{\mathcal{N}(A)''}(y_{j,m}) u \mathbb{E}_{\mathcal{N}(A)''}(y_{j,n})^*) \right\|_2 < \varepsilon,$$

for all $1 \leq m, n \leq N$.

In the inductive step we construct M_{k+1} so that the sets $M_1, \ldots, M_{k+1}$ and $B_1, \ldots, B_k$, $B_{k+1} = M_{k+1} \cap A$ satisfy properties (i)-(iv). Properties (i) and (ii) are obtained for M_{k+1} by the same procedure as in proposition $VIII.4$. It is easy to see that condition (iv) is obtained using theorem $VIII.8$. To get (iii), note that by lemma $II.9$, there exists a countable set F of normalizing unitaries such that

$$F'' \supseteq \left\{ \mathbb{E}_{\mathcal{N}(A)''}(y_{k,r}) : r \geq 1 \right\}.$$

Letting F be part of a generating set for M_{k+1} will then yield property (iii) for M_{k+1}.

It remains to show that the constructed sequence of algebras gives the desired result. That M is a factor and that $B = M \cap A$ is a masa in M follow from properties (i) and (ii) as in proposition $VIII.4$. From condition (iv) and a "3ε-type" approximation argument, we obtain, given $\varepsilon > 0$ and a finite set $x_1, \ldots, x_n \in M$, a unitary $u \in B$ such that

$$\left\| \mathbb{E}_A(x_i u x^*_j - \mathbb{E}_{\mathcal{N}(A)''}(x_i) u \mathbb{E}_{\mathcal{N}(A)''}(x_j)^*) \right\|_2 < \varepsilon, \tag{VIII.1}$$

for all $1 \leq i, j \leq n$. We will use conditions (iii) and (iv) to determine the normalizing algebra of B. We claim that $\mathbb{E}_{N(A)''}(x) \in (M \cap \mathcal{N}(A))''$, for all $x \in M$. Let $\varepsilon > 0$, and let $x \in M$. By scaling, we may assume that $\|x\| \leq 1$. Then by the Kaplansky density theorem there exists a net $\{x_\alpha\} \in \bigcup_{k\geq 0} M_k$ such that x_α converges to x in $\|\cdot\|_2$, and $\|x_\alpha\| \leq 1$ for all α. Choose α_0 so that

$$\|x_\alpha - x\|_2 < \varepsilon,$$

for all $\alpha \geq \alpha_0$. There exists a $k \geq 1$ and $y_{k,r}$ in the unit ball of M_k such that

$$\|y_{k,r} - x_{\alpha_0}\|_2 < \varepsilon.$$

Then $\|y_{k,r} - x\|_2 < 2\varepsilon$. Moreover, we have shown by our inductive argument that

$$\mathbb{E}_{\mathcal{N}(A)''}(y_{k,r}) \in (M_{k+1} \cap \mathcal{N}(A))'' \subseteq (M \cap \mathcal{N}(A))'',$$

and by contractivity of the conditional expectation we have

$$\left\|\mathbb{E}_{\mathcal{N}(A)''}(x) - \mathbb{E}_{\mathcal{N}(A)''}(y_{k,r})\right\|_2 < 2\varepsilon.$$

It follows that $dist_2(\mathbb{E}_{\mathcal{N}(A)''}(x), (M \cap \mathcal{N}(A))'') < 2\varepsilon$. Then

$$\mathbb{E}_{\mathcal{N}(A)''}(x) \in L^2((M \cap \mathcal{N}(A))'') \cap N,$$

so

$$\mathbb{E}_{\mathcal{N}(A)''}(x) = e_{(M\cap\mathcal{N}(A))''}(\mathbb{E}_{\mathcal{N}(A)''}(x)) = \mathbb{E}_{(M\cap\mathcal{N}(A))''}(\mathbb{E}_{\mathcal{N}(A)''}(x)),$$

that is, $\mathbb{E}_{\mathcal{N}(A)''}(x) \in (M \cap \mathcal{N}(A))''$. It now follows that

$$\mathbb{E}_{\mathcal{N}(A)''}(x) = \mathbb{E}_{(M\cap\mathcal{N}(A))''}(\mathbb{E}_{\mathcal{N}(A)''}(x)) = \mathbb{E}_{(M\cap\mathcal{N}(A))''}(x),$$

for all $x \in M$. Note also that for all $x \in M$, we have $\mathbb{E}_A(x) = \mathbb{E}_{M\cap A}(x)$ by (ii) and the same argument given in proposition $VIII.4$. Thus, inequality (1) above becomes as follows. Given $\varepsilon > 0$ and $x_1, \ldots, x_n \in M$, there exists a unitary $u \in B$ such that

$$\left\|\mathbb{E}_{M\cap A}(x_i u x_j^* - \mathbb{E}_{(M\cap\mathcal{N}(A))''}(x_i) u \mathbb{E}_{(M\cap\mathcal{N}(A))''}(x_j)^*)\right\|_2 < \varepsilon, \tag{VIII.2}$$

for all $1 \leq i, j \leq n$. By the remark following theorem $VIII.8$, this says that

$$\mathcal{N}(M \cap A)'' \subseteq (M \cap \mathcal{N}(A))''.$$

The opposite inclusion follows directly from the definitions. This completes the proof.

$\square$

CHAPTER IX

CONCLUDING REMARKS

In this dissertation, we have developed further the study of normalizers of finite von Neumann algebras begun in [8], [11], and [12] and continued more recently in [42], [43], [46] and [47], among other works. In this short conclusion, we discuss some subjects of ongoing work for the author, as well as some possible future directions for research on structure theory of normalizers and its application to other areas.

A. Seminormalizers

In Theorem VI.2 part (ii), we proved that every seminormalizing partial isometry v of a subfactor $N \subseteq M$ is a cutdown $v = up$ of a normalizing unitary $u \in \mathcal{N}_M(N)$ by a projection $p \in N' \cap M$. When N is a masa, the same result is known, by Dye's theorem, to be true, since $\mathcal{SN}(N) = \mathcal{GN}(N)$ when N is a masa. In current work, the author has shown that this same result is true for an inclusion of finite von Neumann algebras $N \subseteq M$ satisfying $N' \cap M \subseteq N$. We conjecture further that the result is true for a general inclusion $N \subseteq M$ of finite von Neumann algebras. This is a subject of ongoing investigation by the author.

A second possible line of investigation involves tensor products. The tensor product result in Chapter VI is somewhat unsatisfying in that it assumes regularity of the inclusions $N_i \subseteq M_i$, $i = 1, 2$. A general result for tensor products of seminormalizers would most likely require an extension of the techniques in [43].

B. Singly generated II_1 factors

The generator problem asks whether every separable, finite von Neumann algebra is generated by a single element. As an application of our work in this dissertation, we hope to find new examples of separable finite von Neumann algebras that are singly-generated. The proof of the following result, which is rooted in methods of Shen [36], can be found in [37]. We recall the notation $\mathcal{G}(N)$ for the so-called Shen invariant of a II_1 factor, which plays a role in determining whether a II_1 factor is singly generated.

Theorem IX.1. *Let N be a diffuse, regular von Neumann subalgebra of a II_1 factor M such that $\mathcal{G}(N) = 0$. Then $\mathcal{G}(M) = 0$ and M is singly generated.*

In particular, Theorem IX.1 is true when N is a Cartan maximal abelian subalgebra. For such inclusions, we know by Dye's theorem that N is Cartan in any intermediate subalgebra $N \subseteq P \subseteq M$. We have shown, however, that Dye's theorem fails even in the case when both N and M are II_1 factors. This leaves open the question of whether intermediate subalgebras of such inclusions $N \subseteq M$ are also singly generated. Our methods from Chapter III will be relevant to settling this question.

C. Normalizers and mixing phenomena in von Neumann algebras

The author, in current joint work with Junsheng Fang and Kunal Mukherjee, has made the following two definitions. The first definition generalizes the concept of strong singularity for subalgebras (hence, also, singularity for the case of masas). The second definition, a generalization of a definition in [15] represents a condition which is clearly stronger than that of strong singularity.

Definition IX.2. Let $B \subseteq A \subseteq N$ be an inclusion of finite von Neumann algebras. We say that $B \subseteq A$ has the *relative weak mixing property* if for every $\varepsilon > 0$ and

$x_1, x_2, \ldots, x_k \in N$ there exists a unitary operator $u \in B$ such that

$$\left\|\mathbb{E}_A(x_i u x_j^*) - \mathbb{E}_A(x_i) u \mathbb{E}_A(x_j^*)\right\|_2 < \varepsilon.$$

Definition IX.3. Let $B \subseteq N$ be an inclusion of finite von Neumann algebras. We say that B is a *strongly mixing* subalgebra if for any $x, y \in N$ such that $\mathbb{E}_B(x) = \mathbb{E}_B(y) = 0$ and any sequence $\{u_n\}$ of unitaries in B converging weakly to zero, one has $\|\mathbb{E}_B(x u_n y)\|_2 \to 0$.

As a joint effort with Fang and Mukherjee, the author has proved that if $B \subseteq N$ is a strongly mixing inclusion of finite von Neumann algebras, then $B_0 \subseteq B$ is relatively weakly mixing in N for every diffuse subalgebra B_0. A characterization of relative weak mixing, given in the same paper, is the following.

Theorem IX.4. *Suppose that A is a masa in a II_1 factor N, and let $B \subseteq A$ be a diffuse subalgebra. Then the following are equivalent.*

(i) $A \subseteq B$ *has the relative weak mixing property in* N*.*

(ii) $\|(1 - \mathbb{E}_A)\mathbb{E}_{vBv^*}\|_{\infty,2} = \|\mathbb{E}_A(v^*v)\|_2$*, for all partial isometries* $v \in N$ *such that* $vv^* \leq p$ *and* $v^*v \leq 1 - p$*, for some projection* $p \in A$*.*

(iii) If there exists $x \in N$ *such that* $xBx^* \subseteq A$*, then* $x \in A$*.*

A possibly fruitful direction of research, then, would be to explore further the connections between the mixing properties of von Neumann subalgebras given above and various characterizations of normalizers.

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