

MASAS AND BIMODULE DECOMPOSITIONS OF
 II_1 FACTORS

A Dissertation

by

KUNAL KRISHNA MUKHERJEE

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2009

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Chair of Committee,	Kenneth Dykema
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ABSTRACT

Masas and Bimodule Decompositions of

II_1 Factors. (August 2009)

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The *measure-multiplicity-invariant* for masas in II_1 factors was introduced by Dykema, Smith and Sinclair to distinguish masas that have the same *Pukánszky invariant*. In this dissertation, the measure class (*left-right-measure*) in the *measure-multiplicity-invariant* is studied, which equivalent to studying the structure of the standard Hilbert space as an associated bimodule. The focal point of this analysis is: To what extent the associated bimodule remembers properties of the masa. The structure of normaliser of any masa is characterized depending on this measure class, by using Baire category methods (Selection principle of Jankov and von Neumann). Measure theoretic proofs of Chifan's normaliser formula and the equivalence of *weak asymptotic homomorphism property (WAHP)* and singularity is presented. Stronger notions of singularity is also investigated. Analytical conditions based on Fourier coefficients of certain measures are discussed, that partially characterize strongly mixing masas and masas with nontrivial centralizing sequences. The analysis also provide conditions in terms of operators and L^2 vectors that characterize masas whose *left-right-measure* belongs to the class of product measure. An example of a simple masa in the hyperfinite II_1 factor whose *left-right-measure* is the class of product measure is exhibited. An example of a masa in the hyperfinite II_1 factor whose *left-right-measure* is singular to the product measure is also presented. Unitary conjugacy

of masas is studied by providing examples of non unitary conjugate masas. Finally, it is shown that for $k \geq 2$ and for each subset $S \subset \mathbb{N}$, there exist uncountably many non conjugate singular masas in $L(\mathbb{F}_k)$ whose Pukánszky invariant is $S \cup \{\infty\}$.

To Shrimat Bibhabananda Brahmachari

ACKNOWLEDGMENTS

This dissertation arose out of years of research that has been done since I came to Texas A&M University in the year 2003. Since that time, I have been helped by a number of people who contributed in assorted ways to my research. It is a pleasure to convey my gratitude to them in this humble acknowledgment.

Firstly, I would like to record my profound gratitude to my advisor, Ken Dykema, for his supervision, advice, and guidance from the very early stage of this research, and for giving me all essential suggestions throughout the work. This work would not have been possible without his continuous help.

I gratefully thank Roger Smith and David Kerr for giving their precious time and providing their constructive and valuable comments on this research. I have also benefited by the advice and guidance of Stuart White, who always granted me his time, whenever I needed.

I am thankful to Ron Douglas and Fred Dahm, along with Roger Smith, all of whom, in the midst of all their activities, accepted to be members of my dissertation committee.

I have also benefited by the outstanding work of Roger Smith and Allan Sinclair in their book 'Finite Von-Neumann Algebras and Masas', which they made available to me for my reference, long before it reached the bookstores.

Collective and individual acknowledgments are also owed to my friends namely, Junsheng Fang, Jan Cameron, Gabriel Tucci and Alan Wiggins for providing me with their ideas and valuable suggestions throughout my work, and thus creating a very friendly atmosphere for research. I would also like to thank Francisco Torres-Ayala and Daniel Redelmeier for helping me with some of my teaching duties and

responsibilities.

I would take this opportunity to express my appreciation for the wonderful library and perfect administration here at Texas A&M, which proved very valuable in the completion of this work.

My parents deserve special mention for their constant support and prayers. My father, Ashok Mukherjee, who from very beginning, is the person who developed my learning character, by showing me the joy of intellectual pursuit ever since I was a child. My mother, Malabika Mukherjee, is the one who sincerely brought me up with her supportive care and gentle love.

I would like to acknowledge my parents-in-law, Bishwanath and Sukla Bhattacharya for their help and support during the final phase of my research.

Words are not enough to express my appreciation for my wife, Gargi, and daughter Shrividya, whose patience, dedication, love and persistent confidence in me, has reflected their thoughtful support throughout this study.

For reasons uncountable to enumerate, I would like to express my sincere gratitude to my guru, Kaulacharya Bibhabanandanath, to whom this dissertation is fondly dedicated. His inspiration and blessings guided me with an immense strength through all tough times in the six long years of this study. I am indebted to him for having provided me with everything I needed.

Finally, I would like to thank everybody who was important for the successful realization of this dissertation, and express my apologies that I could not mention each personally, one by one.

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CHAPTER I

INTRODUCTION

The moral of this dissertation is: ‘*The phenomena - regularity, semiregularity, singularity, weak asymptotic homomorphism property (WAHP), asymptotic homomorphism property (AHP) and strong mixing of masas in finite von Neumann algebras, can all be explained by measure theory*’. Throughout the entire manuscript \mathcal{M} will always denote a separable II_1 factor. Let $A \subset \mathcal{M}$ be a maximal abelian self-adjoint subalgebra, henceforth abbreviated as a *masa*. It is a theorem of von Neumann that A is isomorphic to $L^\infty([0, 1], dx)$. So the study of masas in type II_1 factors is understanding its position up to automorphisms of the ambient von Neumann algebra. For a masa $A \subset \mathcal{M}$, Dixmier in [7] defined the group of *normalizing unitaries* (or *normaliser*) of A to be the set

$$N(A) = \{u \in \mathcal{U}(\mathcal{M}) : uAu^* = A\},$$

where $\mathcal{U}(\mathcal{M})$ denotes the unitary group of \mathcal{M} . He called

- (i) A to be *regular* (also *Cartan*) if $N(A)'' = \mathcal{M}$,
- (ii) A to be *semiregular* if $N(A)''$ is a subfactor of \mathcal{M} ,
- (iii) A to be *singular* if $N(A) \subset A$.

He also exhibited the presence of all three kinds of masas in the hyperfinite II_1 factor.

Two masas A, B of \mathcal{M} are said to be *conjugate* if there is an automorphism θ of \mathcal{M} such that $\theta(A) = B$. If there is an unitary $u \in \mathcal{M}$ such that $uAu^* = B$ then A and B are called *unitarily (inner) conjugate*.

Feldman and Moore in [12], [13] characterized pairs $A \subset \mathcal{M}$, where A is a Car-

The journal model is *Journal of Functional Analysis*.

tan subalgebra, as those coming from r -discrete transitive measured groupoids with a finite measure space X as base. Given such a groupoid, the algebra A is $L^\infty(X)$ and the group of bisections of the groupoid injects into $\mathcal{U}(\mathcal{M})$ as the normaliser of A . It is a remarkable achievement of Connes, Feldman and Weiss [5] that any countable amenable measured equivalence relation is generated by a single transformation of the underlying space. When translated into the language of operator algebras via the Feldman-Moore construction, this theorem together with a theorem of Krieger [22] says that, *if \mathcal{M} is injective then any two Cartan subalgebras are conjugate by an automorphism of \mathcal{M}* . However it follows from their theorem that, there are uncountably many equivalence classes of Cartan masas up to unitary conjugacy in the hyperfinite II_1 factor. See [28] for more examples. There exist II_1 factors with non conjugate Cartan masas (see [6]). These masas were distinguished with the presence or absence of nontrivial centralizing sequences. Recently Ozawa and Popa have exhibited examples of II_1 factors with no or at most one Cartan masa up to unitary conjugacy (see [27]).

The absence of Cartan masas in II_1 factors was first due to Voiculescu in [46]. In fact, it was his amazing discovery that, for any diffuse abelian algebra $A \subset L(\mathbb{F}_n)$, the standard Hilbert space $l^2(\mathbb{F}_n)$ as a A , A -bimodule contains a copy of $L^2(A) \otimes L^2(A)$. His result was improved by Dykema in [10] to rule out the presence of masas in free group factors with finite multiplicity.

Getting back to singular masas, in 1960 Pukánszky showed in [37] that there are countable non conjugate singular masas in the hyperfinite II_1 factor by introducing an algebraic invariant for masas in II_1 factors, today known as the *Pukánszky invariant*.

In 1983 Popa [33] succeeded in showing that all separable continuous semifinite von Neumann algebras and all separable factors of type III_λ , $0 \leq \lambda < 1$ have singular masas. Although they exist, citing explicit examples is a very hard job. In this

direction, Smith and Sinclair in [42] have given concrete examples of uncountably many non conjugate singular masas in the hyperfinite II_1 factor. Their examples come from analyzing the double coset structure of abelian subgroups of amenable icc groups. White and Sinclair [41] have given explicit examples of a continuous path of non conjugate singular masas (Tauer masas) in the hyperfinite II_1 factor. All the masas in this path have the same algebraic invariant of Pukánszky. They were once again distinguished by the presence or absence of nontrivial centralizing sequences in appropriate compressions. Subsequently, White in [49] proved that, any possible value of the Pukánszky invariant can be realized in the hyperfinite II_1 factor, and any McDuff factor which contains a masa of Pukánszky invariant $\{1\}$, contains masas of any arbitrary Pukánszky invariant.

Singularity is often quite hard to check (see [38]). In order to check if a masa is singular, analytical properties ‘*asymptotic homomorphism property*’ (AHP) and ‘*weak asymptotic homomorphism property*’ (WAHP) were discovered in [39], [40]. Subsequently Smith, Sinclair, White and Wiggins in [44] characterized pairs $A \subset \mathcal{M}$, where A is a singular masa in a II_1 factor \mathcal{M} to be precisely those for which A satisfies ‘*WAHP*’. All the theories that we have outlined have a common theme namely, ‘*What is the structure of the standard Hilbert space as a w^* A , A -bimodule?*’

Although many invariants of masas in II_1 are known, the first successful attempt to distinguish masas with a natural invariant, which have the same Pukánszky invariant, was due to Dykema, Smith and Sinclair in [11]. We call this the *measure-multiplicity-invariant*. This invariant has two main components, a measure class and a multiplicity function. This invariant is not a new one and has existed in the literature for quite some time. For Cartan masas this invariant has very deep meaning and it is very hard to distinguish Cartan masas with this invariant. The term *multiplicity* in the *measure-multiplicity-invariant* is actually the *Pukánszky* invariant of the

masa, making it a stronger invariant. A slightly different invariant was considered by Neshveyev and Størmer in [24].

Our intention is to study singular masas and distinguish them. In order to do so, it is necessary to think of singularity from a different point of view. The theory of Cartan masas and singular masas has so far been viewed from two different angles. While Cartan masas fit to the theory of orbit equivalence on one hand [13], singular masas fit to the intertwining techniques of Popa on the other [44]. But we would like to have an unique approach that explains all these phenomena. This is the primary goal of this study. In this work, we characterize masas by studying the structure of the standard Hilbert space as their associated bimodule.

Our second goal is to investigate that, after such a theory is outlined, whether it is possible to obtain proofs of important theorems regarding masas, that were obtained by a number of researchers using different ideas. Many old theorems can indeed be proved but we will mainly prove Chifan's result on tensor products [3] and the equivalence of WAHP and singularity [44]. In fact, it seems that studying the bimodule is the most natural way to approach these problems, as one can exploit a lot of results from Real Analysis.

In order to distinguish singular masas which have the same multiplicity, understanding the measure in the *measure-multiplicity-invariant* is the most important task. So we study this invariant thoroughly throughout this manuscript.

We have learnt later that Popa and Shylakhtenko in [35] have results of similar flavor in this direction. However our approach is completely different. We think that what is really involved in understanding the types of masas are the *measurable selection principle* of Jankov and von Neumann and a generalized version of Dye's theorem on groupoid normalisers. This is evident from [5], [12] and [13]. We present completely measure theoretic proofs based upon Baire category methods (selection

principle). As an outcome of our approach, many theorems related to structure theory of masas proved by different techniques just follow easily from our technique.

Singular masas are often constructed by considering weakly or strongly mixing actions of infinite abelian groups on finite von Neumann algebras. We will show that the definition of WAHP can be strengthened by considering Haar unitaries and Cesàro sums which exactly resemble the definition of weakly mixing actions. Weakly mixing actions are characterized by null sets of certain measures. The story of singular masas is also similar.

Feldman and Moore constructed von Neumann algebras from Borel or measurable equivalence relations with countably many elements in each equivalence class, on standard Borel spaces. So, firstly, it is important to understand, how an equivalence relation is related to a masa or why is it so natural to consider any kind of Borel equivalence relations. Most papers on singular masas will have the following statement: ‘Singular masas are hard to construct’. We give the reasons for it in this work.

The moral is, while *Cartan masas* correspond to *countable equivalence relations*, *singular masas* correspond to *continuous equivalence relations*. This is the precise reason, singular masas are so hard to construct, as there is no counterpart of Feldman-Moore theory for continuous equivalence relations.

There are stronger notions for singularity of masas. One such phenomenon is called *strongly mixing* which appeared in [18]. An even more stronger notion exists, which we call *uniformly mixing*. Mixing is a term that has its roots in measure theory and we will justify that the study of strongly mixing or uniformly mixing masas can be linked to that of Fourier coefficients of certain measures. Thus, our analysis will show that masas which do not arise from dynamical systems possess properties similar to the ones arising from ergodic theory.

We will also present measure theoretic properties of masas that possess nontrivial centralizing sequences. We will discuss conditions that prevent a masa to possess nontrivial centralizing sequences. In this direction, the examples in [6] suggest that through our approach, a complete characterization is not possible.

It is a long standing open problem in Ergodic Theory and Dynamical Systems that :‘*Is there a simple W^* -dynamical system with pure Lebesgue spectrum*’? Translated to the theory of von Neumann algebras, it asks :‘Is there a Lebesgue probability space (X, μ) and an infinite discrete abelian group G that implements a (free) weakly mixing action on X by measure preserving transformations so that the group von Neumann algebra $L(G)$ inside the crossed product factor has multiplicity 1 (equivalently the Pukánszky invariant of $L(G) = \{1\}$) and the spectrum is Lebesgue measure on $[0, 1] \times [0, 1]$ ’? An example of a masa with the properties ‘simple and Lebesgue spectrum’ is not known. We will exhibit that the hyperfinite II_1 factor has such a masa. In fact, it a masa constructed by White and Sinclair [41]. Thus there is a chance that the conjecture of simple dynamical system with pure Lebesgue spectrum might have an affirmative answer.

The question of unitary conjugacy of masas and subalgebras has been investigated by a number of experts. Although we can provide independent proofs of similar conditions through our techniques, i.e, Baire category methods, we refrain from doing so. Instead, we will use results from [36] to produce uncountably many non inner conjugate singular masas in the hyperfinite II_1 factor with identical bimodule structure.

Finally, borrowing ideas from [11], [41] we will show that, given any subset S of \mathbb{N} , there exist uncountably many non conjugate singular masas in the free group factors with Pukánszky invariant $S \cup \{\infty\}$.

Experts have tried to distinguish singular masas by a number of invariants. Our work will show that, the *measure-multiplicity-invariant* understands most of

the invariants that are available in the literature. However, the *measure-multiplicity-invariant* is far from being a complete invariant. Nevertheless, it is a complete invariant as far as unitary conjugacy is concerned.

CHAPTER II

MEASURE MULTIPLICITY INVARIANT

This work relies on the theory of direct integrals and measure theory. So we have divided this chapter into four sections. In Section A we give some well known results about direct integrals of Hilbert spaces with respect to an abelian von Neumann algebra. In Section B we give some preliminaries about masas in II_1 factors and in Section C we will define the *measure-multiplicity-invariant* of masas in II_1 factors. Section C has a subsection in which we discuss some facts on disintegration of measures. Section D contains some technical results about measurable functions.

Notation: Throughout the entire manuscript \mathbb{N}_∞ will denote the set $\mathbb{N} \cup \{\infty\}$.

A. Direct Integrals

Let a separable Hilbert space \mathcal{H} be the direct integral of a μ -measurable field of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$ over the base space (X, μ) where X is a σ -compact space and μ is a positive, complete Borel measure.

Definition II.1. An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be *decomposable* relative to the decomposition $\mathcal{H} \cong \int_X^\oplus \mathcal{H}_x d\mu(x)$ if there exists a μ -measurable field of operators $T_x \in \mathbf{B}(\mathcal{H}_x)$, such that $x \mapsto \|T_x\| \in L^\infty(X, \mu)$ and $T = \int_X^\oplus T_x d\mu(x)$.

If $T_x = c(x)I_{\mathcal{H}_x}$, where $c(x) \in \mathbb{C}$ for almost all x , then T is said to be *diagonalizable*.

It is easy to see that the fibres of a *decomposable* operator are uniquely determined up to an almost sure equivalence. The collection of *diagonalizable* and *decomposable* operators both form von Neumann subalgebras of $\mathbf{B}(\mathcal{H})$, with the later being the commutant of the former. Whenever there is no danger of confusion we will use the term measurable instead of μ -measurable.

Theorem II.2. *Let $A \subset \mathbf{B}(\mathcal{H})$ be a diffuse abelian von Neumann algebra on a separable Hilbert space \mathcal{H} . Then there exists a measure space (X, μ) , where X is a σ -compact space, μ is a positive, Borel, non-atomic, complete measure on X and a measurable field of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$, such that \mathcal{H} is unitarily equivalent to,*

$$\mathcal{H} \cong \int_X^\oplus \mathcal{H}_x d\mu(x) \quad (\text{A.1})$$

and A is (unitarily equivalent to) the algebra of diagonalizable operators on $\int_X^\oplus \mathcal{H}_x d\mu(x)$ with respect to this decomposition.

The dimension function of the decomposition in Thm. II.2 is defined as

$$m : X \mapsto \mathbb{N}_\infty \text{ by, } m(x) = \dim(\mathcal{H}_x).$$

The dimension function m is μ -measurable. Such results are known in greater generality. For a measure space (X, μ) we denote by $[\mu]$ the equivalence class of measures on X that are mutually absolutely continuous with respect to μ . There is also an uniqueness of the decomposition in Thm. II.2.

Theorem II.3. (*Uniqueness*) *Let $A \subset \mathbf{B}(\mathcal{H})$ be a diffuse abelian von Neumann algebra on a separable Hilbert space \mathcal{H} . If (X, μ) and (Y, ν) are Borel measure spaces where X, Y are σ -compact spaces, μ, ν are positive, Borel, non-atomic, complete measures on X, Y respectively, such that \mathcal{H} abstractly decomposes (unitarily equivalent to) into,*

$$\mathcal{H} \cong \int_X^\oplus \mathcal{H}_x d\mu(x) \cong \int_Y^\oplus \mathcal{H}'_y d\nu(y) \quad (\text{A.2})$$

with multiplicity functions m_X, m_Y respectively, for measurable fields of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}, \{\mathcal{H}'_y\}_{y \in Y}$ and A is (unitarily equivalent to) the algebra of diagonalizable operators with respect to both these decompositions, then there exists a Borel isomor-

phism $T : X \mapsto Y$ such that

$$[T_*\mu] = [\nu] \text{ and } m_X \circ T^{-1} = m_Y, \nu \text{ a.e.} \quad (\text{A.3})$$

We will be always working with finite measures. Since direct integrals of Hilbert spaces does not change when the measures are scaled, we will most of the time assume that the measures have total mass 1. Details of these facts can be found in [19], [25].

B. Basics on Masas in II_1 Factors

Definition II.4. Given a type I von Neumann algebra \mathcal{B} we shall write $\text{Type}(\mathcal{B})$ for the set of all those $n \in \mathbb{N}_\infty$ such that \mathcal{B} has a nonzero component of type I_n .

Let \mathcal{M} be a separable II_1 factor with the faithful, normal, tracial state τ . This trace induces the two-norm $\|x\|_2 = \tau(x^*x)^{1/2}$ on \mathcal{M} and we write $L^2(\mathcal{M})$ for the Hilbert space completion of \mathcal{M} with respect to this norm. Let \mathcal{M} act on $L^2(\mathcal{M})$ via left multiplication. Let J denote the anti-unitary conjugation operator on $L^2(\mathcal{M})$ obtained by extending the densely defined map $J(x) = x^*$. Inclusions of von Neumann algebras will always be assumed to be unital until further notice.

Given a von Neumann subalgebra \mathcal{N} of \mathcal{M} , let $\mathbb{E}_{\mathcal{N}}$ be the unique trace preserving conditional expectation from \mathcal{M} onto \mathcal{N} . This conditional expectation is obtained by restricting the orthogonal projection $e_{\mathcal{N}}$ from $L^2(\mathcal{M})$ onto $L^2(\mathcal{N})$ to \mathcal{M} .

Let $A \subset \mathcal{M}$ be a masa. Then the augmented algebra $\mathcal{A} = (A \cup JAJ)''$ is an abelian algebra, with a type I commutant, the commutant being taken in $\mathbf{B}(L^2(\mathcal{M}))$ and the center of \mathcal{A}' is \mathcal{A} . The Jones projection e_A onto $L^2(A)$ lies in \mathcal{A} [43]. Hence, $\mathcal{A}'(1 - e_A)$ decomposes as,

$$\mathcal{A}'(1 - e_A) = \bigoplus_{n \in \mathbb{N}_\infty} \mathcal{A}'P_n \quad (\text{B.1})$$

where $P_n \in \mathcal{A}$ are orthogonal projections summing up to $1 - e_A$ and $\mathcal{A}'P_n$ is homogeneous algebra of type n whenever $P_n \neq 0$.

Lemma II.5. *If $A \subset \mathcal{M}$ be a masa and $B \subseteq \mathcal{M}$ be any subalgebra, then $(A \cup JBJ)''$ is diffuse.*

Proof. If not, let $p \neq 0$ be a minimal projection in $(A \cup JBJ)''$. Then $Ap \cong \mathbb{C}$. Since $A \cong L^\infty([0, 1], dx)$ so $f \mapsto fp$, $f \in A$ is a one dimensional normal representation of A . Therefore there exists a L^1 function $g \geq 0$ so that $f \mapsto \int_0^1 fg dx$ implements this representation, which is not a homomorphism unless $g = 0$. \square

Definition II.6. The *Pukánszky invariant* of a masa A in a II_1 factor \mathcal{M} , denoted by $Puk(A)$ (or $Puk_{\mathcal{M}}(A)$ when the containing factor is ambiguous) is $\{n \in \mathbb{N}_\infty : P_n \neq 0\}$ which is precisely $\text{Type}(\mathcal{A}'(1 - e_A))$.

Definition II.7. If A is an abelian von Neumann subalgebra of \mathcal{M} , let $\mathcal{GN}(A)$ or $\mathcal{GN}(A, \mathcal{M})$ be the *normalising groupoid*, consisting of those partial isometries $v \in \mathcal{M}$ that satisfy $v^*v, vv^* \in A$ and $vAv^* = Avv^* = vv^*A$.

A theorem of Dye [8] says that, a partial isometry $v \in \mathcal{GN}(A)$ if and only if there is an unitary $u \in N(A)$ and a projection $p \in A$ such that $v = up = (upu^*)u$. Thus $\mathcal{GN}(A)'' = N(A)''$. Popa in [34] connected the *Pukánszky invariant* to the type of a masa showing that if $1 \notin Puk(A)$, then A is singular and that the *Pukánszky invariant* of a Cartan masa is $\{1\}$.

Singularity is difficult to verify. The following two conditions were introduced in [40], [39] and [44] as they imply singularity and are often easier to verify in explicit situations.

Definition II.8. (Smith, Sinclair) Let A be a masa in a II_1 factor \mathcal{M} .

(i) A is said to have the *asymptotic homomorphism property* (AHP) if there exists an

unitary $v \in A$ such that

$$\lim_{|n| \rightarrow \infty} \|\mathbb{E}_A(xv^n y) - \mathbb{E}_A(x)v^n \mathbb{E}_A(y)\|_2 = 0 \text{ for all } x, y \in \mathcal{M}.$$

(ii) A has the *weak asymptotic homomorphism property* (WAHP) if, for each $\epsilon > 0$ and each finite subset $x_1, \dots, x_n \in \mathcal{M}$ there is an unitary $u \in A$ such that

$$\|\mathbb{E}_A(x_i u x_j^*) - \mathbb{E}_A(x_i) u \mathbb{E}_A(x_j^*)\|_2 < \epsilon \text{ for } 1 \leq i, j \leq n.$$

In [44] it was shown that singularity is equivalent to WAHP. We will prove that WAHP is indeed the most natural property. The next proposition is well known, we state it for completeness.

Proposition II.9. *Let $\mathcal{N} \subseteq \mathbf{B}(\mathcal{H})$ be a von Neumann algebra. Let $x_{i,j} \in \mathcal{N}$ and $x'_{i,j} \in \mathcal{N}'$ for $i, j = 1, 2, \dots, n$. Then the following conditions are equivalent:*

- (i) $\sum_{k=1}^n x_{i,k} x'_{k,j} = 0$ for all $1 \leq i, j \leq n$.
- (ii) *There exist elements $z_{i,j} \in \mathbf{Z}(\mathcal{N})$, $i, j = 1, 2, \dots, n$ such that for all i, j*

$$\sum_{k=1}^n x_{i,k} z_{k,j} = 0, \quad \sum_{k=1}^n z_{i,k} x'_{k,j} = x'_{i,j}.$$

C. The Invariant

We consider the *conjugacy invariant* for a masa A in a II_1 factor \mathcal{M} derived from writing the direct integral decomposition of its left-right action. More precisely, we choose a compact Hausdorff space Y such that $C(Y) \subset A$, is a norm separable unital C^* subalgebra and $C(Y)$ is *w.o.t* dense in A . The trace τ restricted to $C(Y)$ gives rise to a probability measure ν on Y so that A is isomorphic to $L^\infty(Y, \bar{\nu})$, with $\bar{\nu}$ a completion of ν . For simplicity of notation we will use the same symbol ν to denote its completion. Now $a \otimes b \mapsto aJb^*J$, $a, b \in C(Y)$ extends to an injective

*-homomorphism π of $C(Y) \otimes C(Y)$ in $L^2(\mathcal{M})$. Indeed, as \mathcal{M} is a factor so the map,

$$\sum_{i=1}^n a_i \otimes b_i \mapsto \sum_{i=1}^n a_i J b_i^* J$$

is injective by Prop. II.9. Hence it induces a norm on $C(Y) \otimes_{alg} C(Y)$. Since abelian C^* algebras are nuclear, this norm must be the min norm, and therefore $a \otimes b \mapsto a J b^* J$ extends to an injective representation of $C(Y) \otimes C(Y)$ in $L^2(\mathcal{M})$. Therefore $C(Y \times Y)$ is a *w.o.t* dense unital subalgebra of \mathcal{A} , so that \mathcal{A} is isomorphic to $L^\infty(Y \times Y, \eta_{Y \times Y})$ for a complete, positive, Borel measure $\eta_{Y \times Y}$. By Lemma II.5, $\eta_{Y \times Y}$ is non-atomic.

Remark II.10. In general, if we allow \mathcal{M} to be a finite von Neumann algebra that is not a factor, then measure will be supported on smaller sets. This is the reason we consider factors, although most results of this manuscript go through even for finite von Neumann algebras.

Thus in view of the uniqueness of direct integrals with respect to an abelian algebra (see Thm. II.2, II.3), $L^2(\mathcal{M})$ admits a direct integral decomposition $\{\mathcal{H}_{x,y}\}$ over the base space $(Y \times Y, \eta_{Y \times Y})$ so that $\mathcal{A} \cong L^\infty(Y \times Y, \eta_{Y \times Y})$ is the algebra of *diagonalizable operators* with respect to this decomposition. Let m_Y denote the multiplicity function of the above decomposition. It is clear from the direct integral decomposition that, the *Pukánszky invariant* of A is the set of *essential values* of m_Y (also check Cor. 3.2, [24]). We will call $[\eta_{Y \times Y}]$ the *left-right-measure* of A . For reasons that will become clear, we will in most situation use the same terminology for the class of the measure $\eta_{Y \times Y}$ when restricted to the *off diagonal*. This will be clear from the context and will cause no confusion. A related invariant was considered by Neshveyev and Størmer in [24], which was a complete invariant for the pair (A, J) .

Although the existence of such a measure is guaranteed, we need an algorithm to figure out the *left-right-measure*. In order to do so fix a nonzero vector $\xi \in L^2(\mathcal{M})$.

The cyclic projection P_ξ with range $[\mathcal{A}\xi]$ is in \mathcal{A}' and hence *decomposable*. For $f, g \in C(Y)$, there exists a complete positive measure μ_ξ (we complete it if necessary) on $Y \times Y$ such that

$$\langle fJg^*J\xi, \xi \rangle_{L^2(\mathcal{M})} = \int_{Y \times Y} f(t)g(s)d\mu_\xi(t, s). \quad (\text{C.1})$$

$\mathcal{A}P_\xi$ is a diffuse abelian algebra in $\mathbf{B}(P_\xi(L^2(\mathcal{M})))$ with a cyclic vector, so is maximal abelian. Thanks to von Neumann, we have only one. Therefore,

$$P_\xi(L^2(\mathcal{M})) \cong \int_{Y \times Y}^{\oplus} \mathbb{C}_{t,s} d\mu_\xi(t, s) \text{ where } \mathbb{C}_{t,s} = \mathbb{C}. \quad (\text{C.2})$$

Moreover $\mathcal{A}P_\xi$ is the *diagonalizable algebra* with respect to the decomposition in Eq. (C.2).

Two orthogonal cyclic subspaces $[\mathcal{A}\xi_1]$ and $[\mathcal{A}\xi_2]$ with cyclic vectors ξ_1, ξ_2 does not necessarily keep the fibres of its associated projections P_{ξ_1} and P_{ξ_2} orthogonal, neither does assert that they are direct integrals over disjoint subsets of $Y \times Y$. However, using the ‘gluing lemma’ (Lemma 5.7, [11]) we single out a measure μ_{ξ_1, ξ_2} so that $(P_{\xi_1} + P_{\xi_2})(L^2(\mathcal{M}))$ has a direct integral decomposition with respect to $(Y \times Y, \eta_{\xi_1, \xi_2})$ and $\mathcal{A}(P_{\xi_1} + P_{\xi_2})$ is the *diagonalizable algebra* respecting that decomposition. This is the step where one will see the possible updates of the *multiplicity function*. Since we are working on a separable Hilbert space, after at most a countable infinite iterations of this procedure we will finally find a measure μ on $Y \times Y$ so that

$$L^2(\mathcal{M}) \cong \int_{Y \times Y}^{\oplus} \mathcal{H}'_x d\mu(x) \quad (\text{C.3})$$

and \mathcal{A} is *diagonalizable* with respect to the decomposition in Eq. (C.3). Modulo the uniqueness of direct integrals we have found the measure. Needless to say, different choices of cyclic subspaces will produce same measure modulo the uniqueness. However for purpose of explicit computation to distinguish masas one learns, that nice

choices of cyclic projections (vectors) is perhaps a little too costly.

For a set X we denote by $\Delta(X)$ the set $\{(x, y) \in X \times X : x = y\}$. The restriction of τ to $C(Y) \subset A$ gives rise to a Borel probability measure whose completion is denoted by ν_Y .

Lemma II.11. *The measure $\eta_{Y \times Y}$ has the following properties:*

(i) $[\eta_{Y \times Y}]$ is invariant under the flip map $\theta : (s, t) \mapsto (t, s)$ on $Y \times Y$.

(ii) If π_1 and π_2 denote the coordinate projections from $Y \times Y$ onto Y then,

$$[(\pi_i)_* \eta_{Y \times Y}] = [\nu_Y] \text{ for } i = 1, 2. \quad (\text{C.4})$$

(iii) The subspace $\int_{\Delta(Y)}^{\oplus} \mathcal{H}_{t,s} d\eta_{Y \times Y}(t, s)$ is identified with $L^2(A)$ and $m_Y(t, t) = 1$, $\eta_{Y \times Y}$ a.e. on $\Delta(Y)$.

(iv) The topological (closed) support of $\eta_{Y \times Y}$ is $Y \times Y$.

(v) If E and F are measurable subsets of Y with $\nu_Y(E) > 0$ and $\nu_Y(F) > 0$ then $\eta_{Y \times Y}(E \times F) > 0$.

The multiplicity function m_Y has the property that

$$m_Y(s, t) = m_Y(t, s),$$

$\eta_{Y \times Y}$ almost everywhere.

Proof. Most of Lemma II.11 is known so we only prove (v). The sets E and F correspond to nonzero projections p and q respectively in A . If $\eta_{Y \times Y}(E \times F) = 0$ then $p\zeta q = 0$ for all $\zeta \in L^2(\mathcal{M})$. Thus $pxq = 0$ for all $x \in \mathcal{M}$. Thus we have two nonzero projections in A whose central carriers are orthogonal. This violates that \mathcal{M} is a factor. \square

We are now almost ready to give the definition of the *measure-multiplicity-invariant* of a masa in a separable II_1 factor. Let A be a masa in \mathcal{M} . Let Y be any

compact Hausdorff space such that the unital inclusion of $C(Y)$ in A is *w.o.t* dense and $C(Y)$ is norm separable. To each such Y , we associate a quadruple $(Y, \nu_Y, [\eta_{Y \times Y}], m_Y)$.

Define an equivalence relation on the quadruples $(Y, \nu_Y, [\eta_{Y \times Y}], m_Y)$ by

$(Y, \nu_Y, [\eta_{Y \times Y}], m_Y) \sim_{m.m.} (Y', \nu_{Y'}, [\eta_{Y' \times Y'}], m_{Y'})$ if and only if there exists a Borel isomorphism $F : Y \mapsto Y'$ such that,

$$\begin{aligned} F_* \nu_Y &= \nu_{Y'}, \\ (F \times F)_* [\eta_{Y \times Y}] &= [\eta_{Y' \times Y'}] \text{ and} \\ m_Y \circ (F \times F)^{-1} &= m_{Y'}, \eta_{Y' \times Y'} \text{ a.e.} \end{aligned} \tag{C.5}$$

We also have, $[\eta_{Y \times Y}] = [\eta_{|\Delta(Y)}] + [\eta_{|\Delta(Y)^c}]$.

Therefore if $(Y, \nu_Y, [\eta_{Y \times Y}], m_Y) \sim_{m.m.} (Y', \nu_{Y'}, [\eta_{Y' \times Y'}], m_{Y'})$ then,

$$\begin{aligned} (F \times F)_* [\eta_{|\Delta(Y)^c}] &= [\eta_{|\Delta(Y')^c}], \\ m_{|\Delta(Y)^c} \circ (F \times F)^{-1} &= m_{|\Delta(Y')^c}, \eta_{|\Delta(Y')^c} \text{ a.e.} \end{aligned} \tag{C.6}$$

Lemma II.12. *If $C(Y_1) \subseteq C(Y_2) \subset A \subset \mathcal{M}$ be two w.o.t dense, unital, norm separable C^* subalgebras of A then $(Y_1, \nu_{Y_1}, [\eta_{Y_1 \times Y_1}], m_{Y_1}) \sim_{m.m.} (Y_2, \nu_{Y_2}, [\eta_{Y_2 \times Y_2}], m_{Y_2})$.*

Proof. The inclusion $i : C(Y_1) \hookrightarrow C(Y_2)$ results from a continuous surjection $\theta : Y_2 \mapsto Y_1$. Therefore for all $f \in C(Y_1)$,

$$\tau(f) = \int_{Y_1} f d\nu_{Y_1} = \int_{Y_2} i(f) d\nu_{Y_2} = \int_{Y_2} (f \circ \theta) d\nu_{Y_2} = \int_{Y_1} f d(\theta_* \nu_{Y_2}).$$

Therefore, $\theta_* \nu_{Y_2} = \nu_{Y_1}$.

The inclusion i preserve least upper bounds at the level of continuous functions. So i extends to a surjective $*$ -homomorphism \tilde{i} between $L^\infty(Y_1, \nu_{Y_1})$ and $L^\infty(Y_2, \nu_{Y_2})$ which is normal (Lemma 10.1.10 [19]). It is easy to see that \tilde{i} is also implemented by θ . That \tilde{i} is injective is obvious. So θ is a Borel isomorphism between the underlying

measure spaces.

Arguing similarly it is easy to see that $\theta \times \theta : Y_2 \times Y_2 \mapsto Y_1 \times Y_1$ implements an isomorphism between $L^\infty(Y_1 \times Y_1, \eta_{Y_1 \times Y_1})$ and $L^\infty(Y_2 \times Y_2, \eta_{Y_2 \times Y_2})$. The statements regarding the measure classes now follows easily.

The statement about the multiplicity function is obvious from the uniqueness of direct integrals in Thm. II.2, II.3 and the fact $L^\infty(Y_1 \times Y_1, \eta_{Y_1 \times Y_1}) \cong L^\infty(Y_2 \times Y_2, \eta_{Y_2 \times Y_2}) \cong \mathcal{A}$. \square

Proposition II.13. *Let $A \subset \mathcal{M}$ be a masa. The collection of quadruples $(Y, \nu_Y, [\eta_{Y \times Y}], m_Y)$ for Y a compact Hausdorff space such that $C(Y) \subset A$ is unital, norm separable and *w.o.t* dense in A , under the equivalence relation $\sim_{m.m}$ has exactly one equivalence class.*

Proof. If $C(Y_1), C(Y_2) \subset A$ be two *w.o.t* dense, unital, norm separable subalgebras of A then $C^*(C(Y_1) \cup C(Y_2)) \cong C(Y_3)$ for a compact Hausdorff space Y_3 , and $C(Y_3)$ is unital, norm separable and *w.o.t* dense in A . Therefore by Lemma II.12, $(Y_3, \nu_{Y_3}, [\eta_{Y_3 \times Y_3}], m_{Y_3}) \sim_{m.m} (Y_i, \nu_{Y_i}, [\eta_{Y_i \times Y_i}], m_{Y_i})$ for $i = 1, 2$. \square

Definition II.14. Let $A \subset \mathcal{M}$ be a masa. We define the *measure-multiplicity-invariant* of A as the *equivalence class* of the quadruples $(Y, \nu_Y, [\eta_{|\Delta(Y)^c}], m_{|\Delta(Y)^c})$ under $\sim_{m.m}$ where,

(i) Y is a compact Hausdorff space such that $C(Y)$ is an unital, norm separable and *w.o.t* dense subalgebra of A .

(ii) ν_Y is the completion of the probability measure obtained from restricting τ on $C(Y)$.

(iii) $[\eta_{|\Delta(Y)^c}]$ is the equivalence class of the measure $\eta_{Y \times Y}$ restricted to $\Delta(Y)^c$,

(iv) $m_{|\Delta(Y)^c}$ is the multiplicity function restricted to $\Delta(Y)^c$,

obtained from the *direct integral decomposition* of $L^2(\mathcal{M})$ over the base space $(Y$

$\times Y, \eta_{Y \times Y}$) so that \mathcal{A} is the algebra of *diagonalizable operators* with respect to this decomposition.

The *measure-multiplicity-invariant* is an *invariant* for masas in the following sense. If $A \subset \mathcal{M}$ and $B \subset \mathcal{N}$ are masas in II_1 factors \mathcal{M}, \mathcal{N} respectively, and there is an unitary $U : L^2(\mathcal{M}) \mapsto L^2(\mathcal{N})$ such that, $UAU^* = B$ and $UJ_{\mathcal{M}}AJ_{\mathcal{M}}U^* = J_{\mathcal{N}}BJ_{\mathcal{N}}$ then for any choice of compact Hausdorff spaces Y_A, Y_B with $\overline{C(Y_A)}^{s.o.t} = A$ and $\overline{C(Y_B)}^{s.o.t} = B$, $1_{\mathcal{M}} \in C(Y_A)$, $1_{\mathcal{N}} \in C(Y_B)$ and $C(Y_A), C(Y_B)$ norm separable, there exists a Borel isomorphism

$$F_{Y_A, Y_B} : (Y_A, \nu_{Y_A}) \mapsto (Y_B, \nu_{Y_B}) \text{ such that,}$$

$$\begin{aligned} (F_{Y_A, Y_B})_* \nu_{Y_A} &= \nu_{Y_B}, \\ (F_{Y_A, Y_B} \times F_{Y_A, Y_B})_* [\eta_{|\Delta(Y_A)^c}] &= [\eta_{|\Delta(Y_B)^c}] \text{ and} \\ m_{|\Delta(Y_A)^c} \circ (F_{Y_A, Y_B} \times F_{Y_A, Y_B})^{-1} &= m_{|\Delta(Y_B)^c}, \quad \eta_{|\Delta(Y_B)^c} \text{ a.e.} \end{aligned} \tag{C.7}$$

We will denote the *measure-multiplicity-invariant* of a masa A by $m.m(A)$ (or $m.m_{\mathcal{M}}(A)$ when the containing factor is ambiguous).

1. Conditional Measures and Masas

As we will see later, the *measure-multiplicity-invariant* contains substantial information about the masa. In order to extract more information we need to establish some house keeping results in measure theory.

Disintegration of measures is a very useful tool in ergodic theory, in the study of conditional probabilities and descriptive set theory. *Measurable selection principle* is a term closely linked to disintegration of measures and has been studied by a number of mathematicians in the last century. A detailed exposition of the existence

of disintegration can be found in [2].

For the general definition of disintegration of measures we will restrict to the following set up. Let T be a measurable map from (X, σ_X) to (Y, σ_Y) where σ_X, σ_Y are σ -algebras of subsets of X, Y respectively. Let λ be a σ -finite measure on σ_X and μ a σ -finite measure on σ_Y . Here λ is the measure to be disintegrated and μ is often the push forward measure $T_*\lambda$, although other possibilities for μ is allowed.

Definition II.15. We say that λ has a disintegration $\{\lambda_t\}_{t \in Y}$ with respect to T and μ or a (T, μ) disintegration if:

(i) λ_t is a σ -finite measure on σ_X concentrated on $\{T = t\}$ (or $T^{-1}\{t\}$), i.e, $\lambda_t(\{T \neq t\}) = 0$, for μ -almost all t ,

and for each nonnegative measurable function f on X

(ii) $t \mapsto \lambda_t(f)$ is measurable.

(iii) $\lambda(f) = \mu^t(\lambda_t(f)) \stackrel{\text{defn}}{=} \int_Y \lambda_t(f) d\mu(t)$.

In probability theory, the measures λ_t are called the disintegrating measures and μ is called the mixing measure. One also writes $\lambda(\cdot | T = t)$ for $\lambda_t(\cdot)$ on occasion.

When λ and almost all λ_t are probability measures, one refers to the disintegrating measures as (regular) conditional distributions and $t \mapsto \lambda_t$ is called the transition kernel.

The reader should be cautious that ‘measurable’ in Defn. II.15 (ii), (iii) means measurable with respect to the σ -algebra of completion of λ .

Theorem II.16. [2](Existence Theorem) *Let λ be a σ -finite Radon measure on a metric space X and T be a measurable map into (Y, σ_Y) . Let μ be a σ -finite measure on σ_Y such that $T_*\lambda \ll \mu$. If σ_Y is countably generated and contains all singleton sets $\{t\}$, then λ has a (T, μ) disintegration. The measures λ_t are uniquely determined up to an almost sure equivalence: if λ_t^* is another (T, μ) disintegration then $\mu(\{t : \lambda_t \neq \lambda_t^*\}) = 0$.*

$$\lambda_t^*) = 0.$$

The condition $T_*\lambda \ll \mu$ in Thm. II.16 is actually necessary for the disintegration to exist. The original version of Thm. II.16 is due to von Neumann.

Proposition II.17. *Let λ be a Radon measure on a compact metric space X and T be a measurable map into (Y, σ_Y) . Let μ be a σ -finite measure on σ_Y such that $T_*\lambda \ll \mu$. Assume that σ_Y is countably generated and contains all singleton sets. Let $t \mapsto \lambda_t$ denote the (T, μ) disintegration of λ . Let X_a denote the set of atoms of $\{\lambda_t\}_{t \in Y}$ i.e.,*

$$X_a = \{x \in X \mid \exists t \in Y : \lambda_t(\{x\}) > 0\}.$$

Then X_a is a measurable set, measurable with respect to the σ -algebra of the completion of λ .

Proof. There is a measurable set $E \subseteq Y$ with $\mu(E^c) = 0$ such that for $t \in E$, λ_t is concentrated on the set $\{T = t\}$. We can assume without loss of generality that $E = Y$. Now for $t \in Y$, the measure λ_t is concentrated on $\{T = t\}$, so

$$\{x \in X \mid \exists t \in Y : \lambda_t(\{x\}) > 0\} = \{x \in X \mid \lambda_{T_x}(\{x\}) > 0\}.$$

Let \mathcal{B} be a countable base for the topology on X . Then

$$\{x \in X \mid \lambda_{T_x}(\{x\}) > 0\} = \cup_{n=1}^{\infty} X_a^{(n)} \text{ with}$$

$$\begin{aligned} X_a^{(n)} &= \left\{ x \in X \mid \forall U \in \mathcal{B} : x \in U \Rightarrow \lambda_{T_x}(U) \geq \frac{1}{n} \right\} \\ &= \bigcap_{U \in \mathcal{B}} \left((X \setminus U) \cup \left\{ x \in U \mid \lambda_{T_x}(U) \geq \frac{1}{n} \right\} \right). \end{aligned}$$

Therefore, $\{x \in X \mid \exists t \in Y : \lambda_t(\{x\}) > 0\}$ is a measurable set by property (ii) of disintegration. \square

The next few lemmas are undoubtedly known to probabilists but we lack the reference. So we record them for convenience. We will omit their proofs as they are easy.

Lemma II.18. *Let λ_1, λ_2 be two Radon measures on a compact metric space X and T be a measurable map into (Y, σ_Y) . Let μ be a σ -finite measure on σ_Y such that $T_*\lambda_1, T_*\lambda_2 \ll \mu$. Assume σ_Y is countably generated and contains all singleton sets $\{t\}$. Let λ_t^1, λ_t^2 be the (T, μ) disintegration of λ_1, λ_2 respectively. Let λ_t^0 be the (T, μ) disintegration of $\lambda_1 + \lambda_2$. Then*

$$\lambda_t^0 = \lambda_t^1 + \lambda_t^2 - \mu \text{ a.e.}$$

Lemma II.19. *Let λ_1, λ_2 be two Radon measures on compact metric spaces X, Y and T, S be measurable maps from X, Y into $(Z, \sigma_Z), (W, \sigma_W)$ respectively. Let μ, ν be σ -finite measures on σ_Z, σ_W respectively such that $T_*\lambda_1 \ll \mu, S_*\lambda_2 \ll \nu$.*

Assume σ_Z, σ_W are countably generated and contains all singleton sets $\{t\}, \{s\}$ respectively. Let λ_t^1, λ_s^2 be the $(T, \mu), (S, \nu)$ disintegration of λ_1, λ_2 respectively. Let $\lambda_{t,s}^0$ be the $(T \otimes S, \mu \otimes \nu)$ disintegration of $\lambda_1 \otimes \lambda_2$. Then

$$\lambda_{t,s}^0 = \lambda_t^1 \otimes \lambda_s^2 - \mu \otimes \nu \text{ a.e.}$$

Lemma II.20. *Let λ_1, λ_2 be two Radon measures on a compact metric space X and T be a measurable map into (Y, σ_Y) . Let μ be a σ -finite measure on σ_Y such that $T_*\lambda_1 \ll \mu$ and $T_*\lambda_2 \ll \mu$. Assume σ_Y is countably generated and contains all singleton sets $\{t\}$. Let λ_t^1, λ_t^2 be the (T, μ) disintegrations of λ_1, λ_2 respectively.*

(i) Assume that $\lambda_1 \ll \lambda_2 \ll \lambda_1$. Then for μ almost all t , $\lambda_t^1 \ll \lambda_t^2 \ll \lambda_t^1$. Moreover,

if $g = \frac{d\lambda_1}{d\lambda_2}$ then $\frac{d\lambda_t^1}{d\lambda_t^2} = g_t$ a.e. μ , where

$$g_t = \begin{cases} g_{\{T=t\}} & \text{on } \{T=t\}, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely if $\lambda_t^1 \ll \lambda_t^2 \ll \lambda_t^1$ for μ almost all t then $\lambda_1 \ll \lambda_2 \ll \lambda_1$.

(ii) If $\lambda_1 \perp \lambda_2$ then $\lambda_t^1 \perp \lambda_t^2$ for μ almost all t .

Lemma II.21. *Let λ be a Radon measure on $X \times X$ where X is a compact metric space. Let μ be a σ -finite measure on X such that $(\pi_i)_* \lambda \ll \mu$ where $\pi_i, i = 1, 2$ are coordinate projections onto X .*

Assume that λ is invariant under the flip of coordinates i.e. $\theta_ \lambda \ll \lambda \ll \theta_* \lambda$, where $\theta : X \times X \mapsto X \times X$ by $\theta(x, y) = (y, x)$. Let λ_s^1, λ_t^2 be the $(\pi_1, \mu), (\pi_2, \mu)$ disintegrations of λ respectively. Then for μ almost all t ,*

$$\lambda_t^1 \ll \theta_* \lambda_t^2 \ll \lambda_t^1.$$

In particular, if for μ almost all t , λ_t^2 has an atom at (s, t) , then λ_t^1 has an atom at (t, s) almost everywhere.

Theorem II.22. *Let $A \subset \mathcal{M}$ and $B \subset \mathcal{N}$ be masas in separable II_1 factors \mathcal{M}, \mathcal{N} . Let $C(X_1) \subset A, C(X_2) \subset B$ be w.o.t dense, norm separable, unital subalgebras of A, B respectively, where X_i are compact metric spaces for $i = 1, 2$. Let ν_{X_i} denote the tracial measures with respect to the w.o.t dense subalgebras on X_i respectively for $i = 1, 2$. Let $[\lambda_1], [\lambda_2]$ denote the left-right-measures of A and B respectively. All the mentioned measures are assumed to be complete. Suppose there is an unitary $U : L^2(\mathcal{M}) \mapsto L^2(\mathcal{N})$ such that $UAU^* = B$ and $UJ_{\mathcal{M}}AJ_{\mathcal{M}}U^* = J_{\mathcal{N}}BJ_{\mathcal{N}}$.*

Then there is a Borel isomorphism $F : X_1 \mapsto X_2$ such that, $F_ \nu_{X_1} = \nu_{X_2}$ and the following is true:*

Denoting by $\lambda_t^{1,X_1}, \lambda_s^{2,X_1}$ the $(\pi_1, \nu_{X_1}), (\pi_2, \nu_{X_1})$ disintegrations of λ_1 respectively and $\lambda_{t'}^{1,X_2}, \lambda_{s'}^{2,X_2}$ the $(\pi_1, \nu_{X_2}), (\pi_2, \nu_{X_2})$ disintegrations of λ_2 respectively, one has

$$\begin{aligned} [\lambda_{t'}^{1,X_2}] &= [(F \times F)_* \lambda_{F^{-1}t'}^{1,X_1}], \nu_{X_2} \text{ almost all } t', \\ [\lambda_{s'}^{2,X_2}] &= [(F \times F)_* \lambda_{F^{-1}s'}^{2,X_1}], \nu_{X_2} \text{ almost all } s', \end{aligned}$$

where π_1, π_2 denotes the projection onto the first and second coordinates respectively.

We need some auxiliary results on convergence of measures in total variation norm. The set of finite signed measures on a measurable space (X, \mathcal{F}) is a Banach space equipped with the total variation norm $\|\cdot\|_{t.v.}$, also called the L_1 -norm, which is defined by $\|\mu\|_{t.v.} = |\mu|(X)$ where $|\mu|$ denotes the variation measure of μ . It is well known that for probability measures P, Q

$$\|P - Q\|_{t.v.} = 2 \sup_{B \in \mathcal{F}} |P(B) - Q(B)| = \int_X |f - g| d\lambda \quad (\text{C.8})$$

where f, g are density functions of P, Q respectively with respect to **any** σ -finite measure λ dominating both P, Q . (see [29]).

Lemma II.23. *Let $\lambda_n, \lambda, \lambda_0$ be Radon measures on a compact metric space X such that, $\lambda_0 \neq 0$, $\lambda_n \ll \lambda$ for $n = 1, 2, \dots$, $\lambda_0 \ll \lambda$ and $\lambda_n \rightarrow \lambda_0$ in $\|\cdot\|_{t.v.}$. Let T be a measurable map into (Y, σ_Y) . Let μ be a σ -finite measure on σ_Y such that $T_*\lambda \ll \mu$. Assume σ_Y is countably generated and contains all singleton sets $\{t\}$. Let $\lambda_t^n, \lambda_t^0, \lambda_t$ be the (T, μ) disintegrations of $\lambda_n, \lambda_0, \lambda$ respectively.*

(i) *Then there is a μ null set E and a subsequence $\{n_k\}$ ($n_k < n_{k+1}$ for all k) such that for all $t \in E^c$,*

$$\sup_{A \subseteq \{T=t\}, A \text{ Borel}} |\lambda_t^{n_k}(A) - \lambda_t^0(A)| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

(ii) Moreover, if for μ almost all t one has λ_t^n is completely atomic (or completely non-atomic) for all n , then so is λ_t^0 almost everywhere.

Proof. For $n = 1, 2, \dots$, the hypothesis guarantees the (T, μ) disintegrations of λ_n as well as λ_0 . Also $\lambda_n(X) \rightarrow \lambda_0(X)$. Then denoting $f_n = \frac{d\lambda_n}{d\lambda}$ and $g = \frac{d\lambda_0}{d\lambda}$ we have from the proof of Lemma II.20, $f_n|_{\{T=t\}} = \frac{d\lambda_t^n}{d\lambda_t}$, $g|_{\{T=t\}} = \frac{d\lambda_t^0}{d\lambda_t}$ for μ almost all t . (The Radon-Nikodym derivatives are zero outside $\{T = t\}$). Dropping to a subsequence if necessary we can assume without loss of generality that $\inf_n \lambda_n(X) \neq 0$. We have $\frac{\lambda_0(X)}{\lambda_n(X)} \rightarrow 1$. An easy triangle inequality argument shows that $\left\| \frac{\lambda_n}{\lambda_n(X)} - \frac{\lambda_0}{\lambda_0(X)} \right\|_{t,v} \rightarrow 0$. Therefore

$$\begin{aligned} \left\| \frac{\lambda_n}{\lambda_n(X)} - \frac{\lambda_0}{\lambda_0(X)} \right\|_{t,v} &= \int_X \left| \frac{f_n}{\lambda_n(X)} - \frac{g}{\lambda_0(X)} \right| d\lambda \\ &= \int_Y \lambda_t \left(\left| \frac{f_n}{\lambda_n(X)} - \frac{g}{\lambda_0(X)} \right| \right) d\mu(t) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Dropping to a further subsequence if necessary one can assume that there is a μ null set $E \subset Y$ such that

$$(i) \lambda_t \left(\left| \frac{f_n}{\lambda_n(X)} - \frac{g}{\lambda_0(X)} \right| \right) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (\text{C.9})$$

$$(ii) \lambda_t^n \neq 0 \text{ and finite for all } n, \quad \lambda_t^0 \neq 0 \text{ and finite}$$

for all $t \in E^c$. Given $\epsilon > 0$ there is a $n_0 \in \mathbb{N}$ such that $\left| 1 - \frac{\lambda_n(X)}{\lambda_0(X)} \right| < \epsilon$ for all $n \geq n_0$.

Fix $t \in E^c$. Then

$$\begin{aligned} \lambda_t(|f_n - g|) &\leq \int_{\{T=t\}} \left| f_n(s) - \frac{g(s)\lambda_n(X)}{\lambda_0(X)} \right| d\lambda_t(s) \\ &\quad + \int_{\{T=t\}} \left| \frac{\lambda_n(X)}{\lambda_0(X)} g(s) - g(s) \right| d\lambda_t(s) \\ &\leq \lambda_n(X) \lambda_t \left(\left| \frac{f_n}{\lambda_n(X)} - \frac{g}{\lambda_0(X)} \right| \right) + \epsilon \|g|_{T=t}\|_{1,\lambda_t} \text{ for all } n \geq n_0. \end{aligned}$$

Therefore $\lambda_t(|f_n - g|) \rightarrow 0$ as $n \rightarrow \infty$.

Now let $A \subseteq \{T = t\}$ be a Borel set. Then

$$\begin{aligned} |\lambda_t^n(A) - \lambda_t^0(A)| &= \left| \int_{\{T=t\} \cap A} (f_n - g) d\lambda_t \right| \\ &\leq \lambda_t(|f_n - g|). \end{aligned}$$

So $\sup_{A \subseteq \{T=t\}, A \text{ Borel}} |\lambda_t^n(A) - \lambda_t^0(A)| \leq \lambda_t(|f_n - g|) \rightarrow 0$ as $n \rightarrow \infty$. This proves the first assertion.

(ii) In this case throwing off another null set if necessary and naming it E as well we assume that in addition to (i), (ii) in Eq. (C.9) one has λ_t^n is completely atomic, and for all n , λ_t^n has at most countably many atoms for all $t \in E^c$. Fix $t \in E^c$.

Since $\lambda_t^n(\{T = t\}) \rightarrow \lambda_t^0(\{T = t\})$, dropping to a subsequence depending on t if necessary one can make sure $\inf_n \lambda_t^n(\{T = t\}) > 0$. An easy triangle inequality shows that

$$\left\| \frac{\lambda_t^n}{\lambda_t^n(\{T = t\})} - \frac{\lambda_t^0}{\lambda_t^0(\{T = t\})} \right\|_{t,v} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (\text{C.10})$$

Suppose there is a measurable subset B of $\{T = t\}$ such that $\lambda_t^0(B) > 0$ and B contains no atoms of λ_t^0 . We can assume without loss of generality that B does not contain any atom of λ_t^n for all n . (There can only be countably many atoms of λ_t^n almost everywhere for each n).

Given $\frac{\lambda_t^0(B)}{\lambda_t^0(\{T=t\})} > \epsilon_1 > 0$ there is a $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

$$\left\| \frac{\lambda_t^n}{\lambda_t^n(\{T = t\})} - \frac{\lambda_t^0}{\lambda_t^0(\{T = t\})} \right\|_{t,v} \leq 2\epsilon_1. \quad (\text{C.11})$$

Then the inequality in Eq. (C.11) is violated by using Eq. (C.8) and the set B . So λ_t^0 is completely atomic. In the case when almost all fibres are completely non-atomic, the conclusion is obvious from Eq. (C.10). \square

For a masa $A \subset \mathcal{M}$, fix a compact Hausdorff space X such that $C(X) \subset A$

is an unital, norm separable and *w.o.t* dense C^* subalgebra. For $\zeta \in L^2(\mathcal{M})$ let $\kappa_\zeta : C(X) \otimes C(X) \mapsto \mathbb{C}$ be the linear functional defined by

$$\kappa_\zeta(a \otimes b) = \langle a\zeta b, \zeta \rangle.$$

Then κ_ζ induces an unique Radon measure η_ζ on $X \times X$ given by

$$\kappa_\zeta(a \otimes b) = \int_{X \times X} a(t)b(s)d\eta_\zeta(t, s) \quad (\text{C.12})$$

and $\|\eta_\zeta\|_{t.v} = \|\kappa_\zeta\|$.

For $\zeta_1, \zeta_2 \in L^2(\mathcal{M})$ let η_{ζ_1, ζ_2} denote the possibly complex measure on $X \times X$ obtained from the vector functional

$$\langle a\zeta_1 b, \zeta_2 \rangle = \int_{X \times X} a(t)b(s)d\eta_{\zeta_1, \zeta_2}(t, s), \quad a, b \in C(X). \quad (\text{C.13})$$

We will write $\eta_{\zeta, \zeta} = \eta_\zeta$. Note that η_ζ is a positive measure for all $\zeta \in L^2(\mathcal{M})$. It is easy to see that the following polarization type identity holds:

$$4\eta_{\zeta_1, \zeta_2} = (\eta_{\zeta_1 + \zeta_2} - \eta_{\zeta_1 - \zeta_2}) + i(\eta_{\zeta_1 + i\zeta_2} - \eta_{\zeta_1 - i\zeta_2}). \quad (\text{C.14})$$

Note that the decomposition of η_{ζ_1, ζ_2} in Eq. (C.14) need not be its Hahn decomposition in general, but

$$4|\eta_{\zeta_1, \zeta_2}| \leq (\eta_{\zeta_1 + \zeta_2} + \eta_{\zeta_1 - \zeta_2}) + (\eta_{\zeta_1 + i\zeta_2} + \eta_{\zeta_1 - i\zeta_2}) = 4(\eta_{\zeta_1} + \eta_{\zeta_2}).$$

So

$$|\eta_{\zeta_1, \zeta_2}| \leq \eta_{\zeta_1} + \eta_{\zeta_2}. \quad (\text{C.15})$$

Lemma II.24. *If $\zeta_n, \zeta \in L^2(\mathcal{M})$ be such that, $\zeta_n \rightarrow \zeta$ in $\|\cdot\|_2$ then*

$$\eta_{\zeta_n} \rightarrow \eta_\zeta \text{ in } \|\cdot\|_{t.v}.$$

Proof. Obvious. □

Proposition II.25. *Let $A \subset \mathcal{M}$ be a masa. Let X be a compact Hausdorff space such that $C(X) \subset A$ is unital, norm separable and w.o.t dense in A and let ν be the tracial measure. Let $0 \neq \zeta \in L^2(N(A)'')$. Then $\eta_{\zeta_t}, \eta_{\zeta_s}$ is completely atomic ν almost all t, s where η_ζ is the measure defined in Eq. (C.12) and $\eta_{\zeta_t}, \eta_{\zeta_s}$ are (π_1, ν) and (π_2, ν) disintegrations of η_ζ respectively.*

Proof. We only prove for the (π_1, ν) disintegration. If $\zeta = u$ where $u \in N(A)$ then the result is obvious as the measure η_u will be concentrated on the automorphism graph. The span of $N(A)$ being s.o.t dense in $N(A)''$ it suffices by Lemma II.24 and II.23 to prove the statement when $\zeta = \sum_{i=1}^n c_i u_i$ where $u_i \in N(A)$ and $c_i \in \mathbb{C}$ for $1 \leq i \leq n$. Now for $a, b \in A$

$$\langle a(\sum_{i=1}^n c_i u_i)b, (\sum_{i=1}^n c_i u_i) \rangle = \sum_{i=1}^n |c_i|^2 \langle a u_i b, u_i \rangle + \sum_{i \neq j=1}^n c_i \bar{c}_j \langle a u_i b, u_j \rangle.$$

The measures given by $a \otimes b \mapsto |c_i|^2 \langle a u_i b, u_i \rangle$, $a, b \in C(X)$ are concentrated on the automorphism graphs implemented by u_i and hence definitely disintegrates as atomic measures and so does their sum from Lemma II.18. The measures given by $a \otimes b \mapsto c_i \bar{c}_j \langle a u_i b, u_j \rangle$, $a, b \in C(X)$ for $i \neq j$ are possibly complex measures. However Eq. (C.15) forces that these measures are also concentrated on the union of the automorphism graphs implemented by u_i and u_j . Thus $\eta_{\sum_{i=1}^n c_i u_i}$ is concentrated on the union of the automorphism graphs implemented by u_i , $1 \leq i \leq n$. Hence the result follows. □

D. Some Technical Results on Measurable Functions

Structure theorems of continuous and measurable functions are what that will come into play, when we attempt to use the measurable selection principle of Jankov

and von Neumann. So we develop some technical lemmas in this section.

Definition II.26. Let $f : [0, 1] \mapsto \mathbb{R}$ be a function and E be a subset of $[0, 1]$. Then f is said to satisfy condition (N) or null condition of Lusin relative to E if $f(A)$ is a set of measure 0 whenever $A \subset E$ is a set of measure 0.

The definition implicitly assumes that there are two measures on $[0, 1]$ and \mathbb{R} . For our purpose these measures will always be the Lebesgue measure, which we will denote by λ .

Proposition II.27. (*Tietze's Extension Type*) Let $E \subsetneq [0, 1]$ be closed and let $f : E \mapsto [0, 1]$ be a continuous function that satisfy the property that for a measurable set $A \subset E$, $\lambda(A) = 0$ if and only if $\lambda(f(A)) = 0$. Then there exists a continuous function $F : [0, 1] \mapsto [0, 1]$ such that

(i) $F|_E = f$,

(ii) F satisfies the property that for a measurable set $A \subset [0, 1]$, $\lambda(A) = 0$ if and only if $\lambda(F(A)) = 0$.

Proof. Since E is closed it is a compact subset of $[0, 1]$. Therefore E has greatest and least members m and M respectively. If $m \neq 0$ or $M \neq 1$ then extend f to a function h on $E_1 = E \cup \{0\} \cup \{1\}$ by assigning the values $f(m)$ and $f(M)$ at the points 0 and 1 respectively. The function h is continuous on E_1 and satisfies the same condition as f relative to E_1 . So without loss of generality we can assume $0, 1 \in E$.

The complement of E is a open set in $[0, 1]$ and $E^c \subset (0, 1)$. Then E^c can be written as a countable disjoint union of intervals $\bigcup_{i=1}^{\infty} (a_i, b_i)$. Then note that $a_i, b_i \in E$ for all i .

So we only have to define an extension on (a_i, b_i) . Define

$$F(x) = \begin{cases} f(x) & \text{if } x \in E, \\ \lambda f(a_i) + (1 - \lambda)f(b_i) & \text{if } x = \lambda a_i + (1 - \lambda)b_i \in (a_i, b_i), \\ & 0 < \lambda < 1 \text{ and } f(a_i) \neq f(b_i), \\ \frac{2(1-f(a_i))}{b_i-a_i}(x - a_i) + f(a_i) & \text{if } a_i < x \leq \frac{a_i+b_i}{2} \text{ and} \\ & f(a_i) = f(b_i) < 1, \\ \frac{2(1-f(b_i))}{a_i-b_i}(x - b_i) + f(b_i) & \text{if } \frac{a_i+b_i}{2} \leq x < b_i \text{ and} \\ & f(a_i) = f(b_i) < 1, \\ \frac{2(x-a_i)}{a_i-b_i} + 1 & \text{if } a_i < x \leq \frac{a_i+b_i}{2} \text{ and} \\ & f(a_i) = f(b_i) = 1, \\ \frac{2(x-b_i)}{b_i-a_i} + 1 & \text{if } \frac{a_i+b_i}{2} \leq x < b_i \text{ and} \\ & f(a_i) = f(b_i) = 1. \end{cases}$$

The function F is now continuous, as it is a linear interpolation obtained from f and the construction satisfy the required conditions. \square

Theorem II.28. (Foran, [14]) *A necessary and sufficient condition for a continuous function $F : [0, 1] \mapsto [0, 1]$ to satisfy condition (N) relative to $[0, 1]$ is that there exist a sequence of measurable sets $E_n \subseteq [0, 1]$, $n = 0, 1, \dots$, such that the following properties are true:*

- (i) $[0, 1] = \bigcup_{n=0}^{\infty} E_n$,
- (ii) $\lambda(F(E_n)) \leq n\lambda(E_n)$ for all $n \geq 0$,
- (iii) for each $n > 0$, F is one to one on E_n .

Proposition II.29. *Let $F : [0, 1] \mapsto [0, 1]$ be a measurable function such that for any measurable set $A \subset [0, 1]$, $\lambda(A) = 0$ if and only if $\lambda(F(A)) = 0$. Then there exists a*

measurable set $E \subseteq [0, 1]$ such that $\lambda(E) > 0$ and F is one to one on E .

Moreover, if $Y_0 \subseteq [0, 1]$ is such that $\lambda(Y_0) > 0$, then there exists $Y_1 \subseteq Y_0$ with $\lambda(Y_1) > 0$ such that F is one to one on Y_1 .

Proof. Let $\epsilon > 0$. By Lusin's theorem, choose a closed set $H \subset [0, 1]$ such that $\lambda([0, 1] \setminus H) < \epsilon$ and $F|_H$ is continuous relative to H . Clearly, $F|_H$ satisfy the property that $A \subset H$, $\lambda(A) = 0$ if and only if $\lambda(F|_H(A)) = 0$. By Prop. II.27, extend F to a continuous function $\tilde{F} : [0, 1] \mapsto [0, 1]$ such that \tilde{F} has the property that for $A \subset [0, 1]$, $\lambda(A) = 0$ if and only if $\lambda(\tilde{F}(A)) = 0$.

Now by Thm. II.28, choose measurable subsets $E_n \subseteq [0, 1]$ such that $[0, 1] = \bigcup_{n=0}^{\infty} E_n$, $\lambda(\tilde{F}(E_n)) \leq n\lambda(E_n)$ for all $n = 0, 1, \dots$, and for each $n > 0$, \tilde{F} is one to one on E_n .

Since $\lambda(\tilde{F}(E_0)) = 0$ so $\lambda(E_0) = 0$. If $\lambda(E_n \cap H) = 0$ for all $n > 0$ then $\lambda(H) = 0$, which is not the case. Therefore there is a $n_0 > 0$ such that $\lambda(E_{n_0} \cap H) > 0$. But $\tilde{F}|_{E_{n_0} \cap H} = F|_{E_{n_0} \cap H}$ and clearly F is one to one on $E_{n_0} \cap H$. Rename $E = E_{n_0} \cap H$. This proves the first assertion.

Suppose $\lambda(Y_0) > 0$. By choosing $\epsilon > 0$ small enough one can make sure that the closed set H in the first part of the proof satisfies $\lambda(Y_0 \cap H) > 0$. The same argument as the first part applies, and there exists a $n_0 > 0$ such that F is one to one on $Y_1 = Y_0 \cap H \cap E_{n_0}$ and $\lambda(Y_1) > 0$. \square

CHAPTER III

CHARACTERIZATION BY BAIRE CATEGORY METHODS

The study of Cartan masas in II_1 factors has received special attention by many experts. Our approach of studying *measure-multiplicity-invariant* was also considered implicitly by Popa and Shlyakhtenko in [36]. In this chapter we will use an alternative approach to characterize masas by their *left-right-measure*. This chapter has four sections. Section A uses operator algebraic tools to generalize Dye's theorem on groupoid normalisers. Section B is very technical and contains measure theoretic details to analyze bimodules. In Section C we provide a proof of Chifan's normaliser formula. Section D presents a direct proof of the equivalence of *WAHP* and singularity using measure theory.

A. Fundamental Set and Generalized Dye's Theorem

This section is intended to characterize some operators in the normalizing algebra of a masa. Throughout this section \mathcal{N} will denote a finite von Neumann algebra gifted with a faithful, normal, normalized trace τ . $B \subset \mathcal{N}$ will denote a von Neumann subalgebra of \mathcal{N} .

As usual \mathcal{N} will be assumed to be acting on $L^2(\mathcal{N}, \tau)$ by left multipliers. $L^2(\mathcal{N}, \tau)$ is a B - B Hilbert w^* -bimodule for any von Neumann subalgebra $B \subset \mathcal{N}$. We know if \mathbb{E}_B denotes the unique trace preserving conditional expectation onto B , then \mathbb{E}_B is given by the Jones projection e_B associated to B via the formula $\mathbb{E}_B(x)\hat{1} = e_B(x\hat{1})$. For $b_1, b_2 \in B$ and $\zeta \in L^2(\mathcal{N}, \tau)$ one has

$$e_B(b_1\zeta b_2) = b_1 e_B(\zeta) b_2. \tag{A.1}$$

We will interchangeably use the symbols \mathbb{E}_B and e_B .

Definition III.1. For a subalgebra $B \subset \mathcal{N}$ define the *fundamental set* of B to be

$$N^f(B) = \{x \in \mathcal{N} : Bx = xB\}.$$

Note that $x \in N^f(B)$ implies $x^* \in N^f(B)$.

Definition III.2. For a subalgebra $B \subset \mathcal{N}$ define the *weak-fundamental set* of B to be

$$N_2^f(B) = \{\zeta \in L^2(\mathcal{N}, \tau) : B\zeta = \zeta B\}.$$

Note that $\zeta \in N_2^f(B)$ implies $\zeta^* \in N_2^f(B)$ and $N^f(B) \subset N_2^f(B)$. When B is a masa, $\zeta \in N_2^f(B)$ implies $a\zeta, \zeta a \in N_2^f(B)$ for all $a \in B$.

To understand the normaliser of a masa the set $N_2^f(B)$ will naturally arise into the scene. However working with vectors in $L^2(\mathcal{N}, \tau)$ is always a technical issue. Polar decomposition of vectors and the theory of L^1 spaces are the tools we need, for which we will give a short exposition. For details check Appendix B of [43]. To keep it short we will omit most proofs. It is here, where one usually encounters unbounded operators. For results proved in this section we have borrowed ideas from Roger Smith and Stuart White.

The positive cone $L^2(\mathcal{N}, \tau)^+$ in $L^2(\mathcal{N}, \tau)$ is defined to be $\overline{\mathcal{N}^+}^{\|\cdot\|_2}$ i.e. the closure of the positive elements of \mathcal{N} in $L^2(\mathcal{N}, \tau)$. It can be shown that $L^2(\mathcal{N}, \tau)$ is the algebraic span of $L^2(\mathcal{N}, \tau)^+$. For $x \in \mathcal{N}$ the equation $\|x\|_1 = \tau(|x|)$ defines a norm on \mathcal{N} . The completion of \mathcal{N} with respect to $\|\cdot\|_1$ is denoted by $L^1(\mathcal{N}, \tau)$. It can be shown that

$$\|x\|_1 = \sup\{|\tau(xy)| : y \in \mathcal{N}, \|y\| \leq 1\}. \quad (\text{A.2})$$

So $|\tau(x)| \leq \|x\|_1$. Thus by density of \mathcal{N} in $L^1(\mathcal{N}, \tau)$, τ extends to a bounded linear

functional on $L^1(\mathcal{N}, \tau)$ which will also be denoted by τ . One can analogously define the positive cone of $L^1(\mathcal{N}, \tau)$ which we denote by $L^1(\mathcal{N}, \tau)^+$. Clearly $\|x\|_1 = \|x^*\|_1$. Consequently, the Tomita operator J extends to a surjective anti-linear isometry to $L^1(\mathcal{N}, \tau)$ which will also be denoted by J . Moreover $J^2 = 1$. We will interchangeably use the notations $J\zeta$ and ζ^* for $\zeta \in L^1(\mathcal{N}, \tau)$.

Both the spaces $L^1(\mathcal{N}, \tau)$ and $L^2(\mathcal{N}, \tau)$ are unitary \mathcal{N} - \mathcal{N} bimodules. The space $L^1(\mathcal{N}, \tau)$ can be identified with the predual of \mathcal{N} and $L^2(\mathcal{N}, \tau)$ is dense in $L^1(\mathcal{N}, \tau)$. One also has $\tau(x\zeta) = \tau(\zeta x)$ for $x \in \mathcal{N}$ and $\zeta \in L^1(\mathcal{N}, \tau)$. Note that \mathbb{E}_B is a contraction from \mathcal{N} onto B . It can be shown that for $x \in \mathcal{N}$,

$$\|\mathbb{E}_B(x)\|_1 \leq \|x\|_1. \quad (\text{A.3})$$

Thus \mathbb{E}_B has an unique bounded extension to a contraction from $L^1(\mathcal{N}, \tau)$ onto $L^1(B, \tau)$, which will as well be denoted by \mathbb{E}_B . This extension preserves the extension of the trace τ , is B modular, positive and faithful. The bilinear map $\Psi : \mathcal{N} \times \mathcal{N} \mapsto \mathcal{N}$ defined by $\Psi(x, y) = xy$ satisfies

$$\|\Psi(x, y)\|_1 \leq \|x\|_2 \|y\|_2 \quad (\text{A.4})$$

by Cauchy-Schwarz inequality. Therefore Ψ lifts to a jointly continuous map from $L^2(\mathcal{N}, \tau) \times L^2(\mathcal{N}, \tau)$ into $L^1(\mathcal{N}, \tau)$. The extension is actually a surjection. Since Ψ is the product map of operators at the level of von Neumann algebra one calls $\Psi(\zeta_1, \zeta_2)$ to be $\zeta_1\zeta_2$, for $\zeta_1, \zeta_2 \in L^2(\mathcal{N}, \tau)$.

Lemma III.3. (*B.5.1, [43]*) *Let $a, b \in \mathcal{N}$ be positives. Then*

$$\left\| a^{\frac{1}{2}} - b^{\frac{1}{2}} \right\|_2^2 \leq 2 \|a - b\|_1. \quad (\text{A.5})$$

Elements of $L^1(\mathcal{N}, \tau)$ and $L^2(\mathcal{N}, \tau)$ can be regarded as unbounded operators on

$L^2(\mathcal{N}, \tau)$. By using the unbounded operator theory for operators affiliated to \mathcal{N} , for each $\zeta \in L^1(\mathcal{N}, \tau)^+$ there exists a unique $0 \leq \zeta_0 \in L^2(\mathcal{N}, \tau)$ such that $\zeta_0^* \zeta_0 = \zeta_0^2 = \zeta$. In this case, ζ_0 is said to be the *square root* of ζ and one writes $\zeta_0 = \sqrt{\zeta} = \zeta^{\frac{1}{2}}$. For $\zeta \in L^2(\mathcal{N}, \tau)$ one has $\zeta^* \zeta \in L^1(\mathcal{N}, \tau)$. From Eq. A.4 and Lemma III.3 it follows that $\zeta^* \zeta \in L^1(\mathcal{N}, \tau)^+$. In particular, $\sqrt{\zeta^* \zeta} \in L^2(\mathcal{N}, \tau)$ for any $\zeta \in L^2(\mathcal{N}, \tau)$ and the square root of any positive in $L^1(\mathcal{N}, \tau)$ is a unique element of $L^2(\mathcal{N}, \tau)$. One also writes $|\zeta| = \sqrt{\zeta^* \zeta}$ for $\zeta \in L^2(\mathcal{N}, \tau)$. If $\zeta \in L^1(\mathcal{N}, \tau)$ be self adjoint i.e, $\zeta = \zeta^*$ then $\zeta = \zeta_+ - \zeta_-$ where $\zeta_{\pm} \in L^1(\mathcal{N}, \tau)^+$ and this decomposition is unique by requiring that $\zeta_+^{\frac{1}{2}} \zeta_-^{\frac{1}{2}} = 0$.

Let $\zeta \in L^2(\mathcal{N}, \tau)$. Consider the projections p, q in $\mathbf{B}(L^2(\mathcal{N}, \tau))$ whose ranges are $\overline{J\mathcal{N}J\sqrt{\zeta^* \zeta}}, \overline{J\mathcal{N}J\zeta}$ respectively. Since the ranges of p, q are invariant subspaces of $J\mathcal{N}J = \mathcal{N}'$ so p, q lies in \mathcal{N} . Using unbounded operators one obtains *polar decomposition* of vectors (Eq. (A.7)) which we formalize below.

Theorem III.4. *There is a unique partial isometry $v \in \mathcal{N}$ with initial projection p and final projection q which satisfy the following condition:*

$$vJx^*J\sqrt{\zeta^* \zeta} = Jx^*J\zeta, \quad x \in \mathcal{N}. \quad (\text{A.6})$$

In particular,

$$v\sqrt{\zeta^* \zeta} = \zeta. \quad (\text{A.7})$$

(i) *Let $B \subset \mathcal{N}$ be a masa, then $\zeta \in L^2(B, \tau)$ imply $p, q \in B$.*

(ii) *For $\zeta \in L^2(\mathcal{N}, \tau)$ if $\zeta^* \zeta \in \mathcal{N}$ then $\zeta \in \mathcal{N}$.*

For $\zeta \in L^2(\mathcal{N}, \tau)$ we define the *left and right kernel* of ζ to be respectively $Ker_l(\zeta) = \{x \in \mathcal{N} : \zeta x = 0\}$ and $Ker_r(\zeta) = \{x \in \mathcal{N} : x\zeta = 0\}$. Then $Ker_l(\cdot), Ker_r(\cdot)$ are subspaces of \mathcal{N} . $Ker_l(\cdot), Ker_r(\cdot)$ are *w.o.t* and *s.o.t* closed.

If $\zeta \in L^1(\mathcal{N}, \tau)$ then *the left and the right kernels* of ζ can be defined analogously. We will denote the *kernels* of the L^1 vectors by $Ker_l(\cdot), Ker_r(\cdot)$ as well. This is slight abuse of notation. In this case, they are norm closed subspaces of \mathcal{N} .

For $\zeta \in L^2(\mathcal{N}, \tau)$ we have

$$Ker_l(\zeta) = Ker_l(\sqrt{\zeta^* \zeta}) = Ker_l(\zeta^* \zeta). \quad (\text{A.8})$$

However the righthand side is defined in L^1 sense. Therefore for $\zeta \in L^2(\mathcal{N}, \tau)$, $Ker_l(\zeta^* \zeta)$ (respectively $Ker_r(\zeta \zeta^*)$) are in fact *w.o.t* closed. Similar statements hold for $Ker_r(\cdot)$ as well.

For $\zeta \in L^2(\mathcal{N}, \tau)$ we define the *left and right ranges* of ζ to be respectively $Ran_l(\zeta) = \{\zeta x : x \in \mathcal{N}\}$ and $Ran_r(\zeta) = \{x \zeta : x \in \mathcal{N}\}$.

Note that for $\zeta \in L^2(\mathcal{N}, \tau)$,

$$\begin{aligned} \{x \in \mathcal{N} : \zeta x = 0\} &= \{x \in \mathcal{N} : \langle \zeta x, y \rangle = 0 \text{ for all } y \in \mathcal{N}\} \\ &= \{x \in \mathcal{N} : \langle x, \zeta^* y \rangle = 0 \text{ for all } y \in \mathcal{N}\} \end{aligned} \quad (\text{A.9})$$

implies $Ker_l(\zeta) = Ran_l(\zeta^*)^\perp \cap \mathcal{N}$.

Proposition III.5. *Let $\zeta \in L^2(\mathcal{N}, \tau)$ and let $\zeta = v \sqrt{\zeta^* \zeta}$ be its polar decomposition. Then $v^* v$ is the projection from $L^2(\mathcal{N}, \tau)$ onto $Ker_l(\zeta)^\perp$ and vv^* is the projection onto $\overline{Ran_l(\zeta)}$.*

Proposition III.6. *Let $\zeta \in L^2(\mathcal{N}, \tau)$ and let $\zeta = v |\zeta|$ be its polar decomposition. Then $|\zeta|^{\frac{1}{2^k}} \rightarrow v^* v$ as $k \rightarrow \infty$ in $\|\cdot\|_2$.*

The proof of Prop. III.6 is a direct application of the monotone convergence theorem.

Lemma III.7. *Let $A \subset \mathcal{N}$ be a masa. Let $\zeta \in L^1(\mathcal{N}, \tau)$ be a nonzero vector such that $a\zeta = \zeta a$ for all $a \in A$. Then $\zeta \in L^1(A, \tau)$.*

Proof. First assume $\zeta \geq 0$. Then use uniqueness of square roots of L^1 vectors. In the general case write ζ as a linear combination of four positives. We omit the details. \square

Proposition III.8. *Let $A \subset \mathcal{N}$ be a masa. Let $0 \neq \zeta \in L^1(\mathcal{N}, \tau)^+$ be such that $A\zeta = \zeta A$. Then $\zeta \in L^1(A, \tau)^+$.*

Proof. Let $\mathcal{I} = \{a \in A : a\zeta = 0\}$. Then \mathcal{I} is a weakly closed ideal (see Eq. (A.8) and related discussion) in A and so has the form $A(1-p)$ for some projection $p \in A$. Then $p\zeta = \zeta$, so $\zeta = \zeta p$ by operating with extended Tomita's involution operator. Thus $Ap\zeta = A\zeta p = \zeta Ap$.

For $a_1, a_2 \in A$ if $\zeta a_1 p = \zeta a_2 p$ then $\zeta(a_1 - a_2)p = 0$, so $p(a_1^* - a_2^*)\zeta = 0$. Hence $p(a_1^* - a_2^*) \in \mathcal{I}$, but $1-p$ is the identity for \mathcal{I} . So $p(a_1^* - a_2^*) = 0$ and hence $a_1 p = a_2 p$. This means there is a well defined map $\psi : Ap \mapsto Ap$ such that

$$ap\zeta = \zeta\psi(ap) \text{ for } a \in A.$$

Taking conditional expectation (see Eq. (A.3) and related discussion) one gets $(ap - \psi(ap))\mathbb{E}_A(\zeta) = 0$ (the left and the right action by elements of A coincides on $L^1(A, \tau)$). Suppose there is an operator $a \in A$ such that $ap - \psi(ap) \neq 0$. Write $ap - \psi(ap) = bp$ for $b \in A$. Then $pb^*bp\mathbb{E}_A(\zeta) = 0$, so $\mathbb{E}_A(pb^*bp\zeta) = 0$. Let $\zeta = \lim_n x_n$ in $\|\cdot\|_1$ where $x_n \in \mathcal{N}^+$. Therefore

$$\lim_n \tau(x_n^{\frac{1}{2}}(bp)^*bp x_n^{\frac{1}{2}}) = \lim_n \tau(pb^*bp x_n) = \lim_n \tau(\mathbb{E}_A(pb^*bp x_n)) = 0.$$

The last statement follows from Eq. (A.2) and Eq. (A.3). So $\lim_n bp x_n^{\frac{1}{2}} = 0$ in $\|\cdot\|_2$ and hence $bp\zeta = \lim_n bp x_n = 0$, in $\|\cdot\|_1$ by Lemma III.3 and Eq. (A.4). Thus $bp \in \mathcal{I}$ so $bp = bp(1-p) = 0$, a contradiction. Thus $\psi(ap) = ap$ for all $a \in A$.

Now $\zeta \in L^1(p\mathcal{N}p, \tau)$ and Ap is a masa in $p\mathcal{N}p$, thus $\zeta \in L^1(Ap, \tau)$ as $ap\zeta = \zeta\psi(ap) = \zeta ap$ for all $a \in A$, from Lemma III.7. \square

Theorem III.9. (*Generalized Dye's theorem- L^2 form*) Let $A \subset \mathcal{N}$ be a masa. Then $\zeta \in N_2^f(A)$ if and only if $\zeta = v\xi$ for some $\xi \in L^2(A, \tau)$ and $v \in \mathcal{GN}(A)$. In particular, $\overline{\text{span}N^f(A)}^{\|\cdot\|_2} = L^2(N(A)'', \tau)$.

Proof. Case 1: Assume $\zeta \in N_2^f(A)$ and $\zeta \geq 0$ i.e. $\zeta \in \overline{\mathcal{N}^+}^{\|\cdot\|_2}$. Then $\zeta \in L^1(\mathcal{N}, \tau)^+$ as well. From Prop. III.8 we get $\zeta \in L^1(A, \tau) \cap L^2(\mathcal{N}, \tau) = L^2(A, \tau)$.

Case 2: Let $\zeta \in N_2^f(A)$. We may without loss of generality assume that $\|\zeta\|_2 = 1$. Then as $A\zeta = \zeta A$ we also have $A\zeta^* = \zeta^*A$. So $A\zeta^*\zeta = \zeta^*A\zeta = \zeta^*\zeta A$. From Prop. III.8,

$$\zeta^*\zeta \in L^1(A, \tau)$$

and similarly we have $\zeta\zeta^* \in L^1(A, \tau)$. Then $\|\zeta^*\zeta\|_1 \leq 1$.

Arguing as in Prop. III.8, there are projections $p_1, p_2 \in A$ such that $J_1 = \{a \in A : a\zeta = 0\} = A(1 - p_1)$ and $J_2 = \{a \in A : \zeta a = 0\} = A(1 - p_2)$. Therefore we have $p_1\zeta = \zeta$ and $\zeta p_2 = \zeta$.

Then there is a well defined map (as explained before) $\psi : Ap_1 \mapsto Ap_2$ such that

$$ap_1\zeta = \zeta\psi(ap_1) \text{ for all } a \in A.$$

Let $\zeta = v\sqrt{\zeta^*\zeta}$ be the polar decomposition of ζ from Thm. III.4. Then v is a partial isometry in \mathcal{N} and the initial space of v is

$\{\sqrt{\zeta^*\zeta}x : x \in \mathcal{N}\}^{-\|\cdot\|_2}$ and the final space is $\{\zeta x : x \in \mathcal{N}\}^{-\|\cdot\|_2}$. Moreover the projections v^*v and vv^* are in A .

Indeed, by Prop. III.5, v^*v is the projection onto $\text{Ker}_l(\zeta)^\perp$ and vv^* onto $\overline{\text{Ran}_l(\zeta)}$. By Prop. III.6, $v^*v \in A$. Replacing ζ by ζ^* and using $\text{Ker}_l(\zeta)^\perp = \overline{\text{Ran}_l(\zeta^*)}$ (see Eq. (A.9)), a similar argument will yield $vv^* \in A$. Clearly $v^*v = p_2$ and $vv^* = p_1$. Then

$$ap_1v\sqrt{\zeta^*\zeta} = v\sqrt{\zeta^*\zeta}\psi(ap_1).$$

Now

$$J_0 = \{b \in A : ap_1vb = vb\psi(ap_1) \text{ for all } a \in A\}$$

is a weakly closed ideal in A and its closure in $\|\cdot\|_2$ is precisely the set

$$J_0^{-\|\cdot\|_2} = \{\xi \in L^2(A, \tau) : ap_1v\xi = v\xi\psi(ap_1) \text{ for all } a \in A\}$$

which contains $\sqrt{\zeta^*\zeta}$.

Since the left and right action of A on $L^2(A, \tau)$ agree, so $\xi_0 \in J_0^{-\|\cdot\|_2}$ and $a \in A$ implies that $\xi_0a, a\xi_0 \in J_0^{-\|\cdot\|_2}$.

Since the *w.o.t* closed ideal J_0 in A is just a cutdown of A by a projection from A , any positive $\zeta_0 \in J_0^{-\|\cdot\|_2}$ is a limit in $\|\cdot\|_2$ of an increasing sequence of positive operators from J_0 . Now it follows that $|\zeta|^{\frac{1}{2^k}} \in J_0^{-\|\cdot\|_2}$ for all $k \in \mathbb{N}$. Therefore by Prop. III.6 it follows that $v^*v = p_2 \in J_0^{-\|\cdot\|_2}$ and hence $p_2 \in J_0 \subseteq A$. Similarly arguing with $\zeta\zeta^*$ one shows $p_1 \in A$. Therefore

$$ap_1vp_2 = vp_2\psi(ap_1) \text{ for all } a \in A.$$

Then

$$v^*av = (vp_2)^*avp_2 = v^*ap_1v = v^*vp_2\psi(ap_1) = \psi(ap_1).$$

Therefore v^* and hence v are groupoid normalisers. So

$$\zeta = v\xi.$$

for $v \in \mathcal{GN}(A)$ and $\xi = |\zeta| \in L^2(A, \tau)^+$. □

B. Analysis of Bimodules through Measure Theory

Let $A = L^\infty(X, \nu_X), B = L^\infty(Y, \nu_Y)$ be two diffuse commutative von Neumann algebras, where ν_X, ν_Y are probability measures. Let $C(A, B)$ denote the set of all A, B -bimodules. This set $C(A, B)$ contains three distinguished subsets.

We will use the variable s to denote the first variable and t to denote the second variable. Following [36] we define:

Definition III.10. A *discrete* (respectively, *diffuse*) A, B -bimodule is a Hilbert space \mathcal{H} so that $\mathcal{H} \cong \bigoplus_{i \in I} L^2(X \times Y, \mu_i)$ where for all i , μ_i disintegrates as $\mu_i(s, t) = \mu_t^{(i)}(s) \nu_Y(t)$ with $\mu_t^{(i)}$ atomic (respectively non-atomic) for ν_Y almost all t . The bimodule \mathcal{H} is mixed if $\mu_t^{(i)}$ contains atoms on a set of positive ν_Y measure for some i , and $\mu_t^{(j)}$ contains a non-atomic part on some set of positive ν_Y measure for some j .

It is to be noted that in view of Lemma II.20, the definition above only cares about the equivalence class of the measures μ_i and not a particular member of the class. The definition forces μ_i to be a non-atomic measure, and the existence of such a disintegration actually forces the push forward of μ_i 's on the space Y to be dominated by ν_Y . We will restrict ourselves to the case I is countable. Let $C_d(A, B), C_{n.a}(A, B), C_m(A, B)$ denote the set of all discrete, diffuse, mixed A, B -bimodules respectively.

Denote by $C_d(A) \subset C_d(A, A) \subset C(A, A)$ the set of those bimodules $\mathcal{H} \in C_d(A, A)$ for which $\bar{\mathcal{H}} \in C_d(A, A)$. Here $\bar{\mathcal{H}}$ is the opposite Hilbert space of \mathcal{H} with left and right actions interchanged. Bimodules in $C_d(A)$ are precisely those for which the associated measures μ_i 's in Defn. III.10 have a completely atomic disintegration along both variables. Similarly define $C_{n.a}(A), C_m(A)$. Note that the spaces $C_d(A), C_{n.a}(A), C_m(A)$ are all closed with respect to taking sub bimodules.

When A, B are masas in a II_1 factor \mathcal{M} the standard Hilbert space $L^2(\mathcal{M})$ is naturally a w^* -continuous A, B bimodule, meaning it carries a pair of mutually commuting normal representations of A and B .

Note that when we deal with the *left-right-measure* of a masa, knowing the disintegration along the second variable enables us to know the disintegration along the first variable as well, by pushing forward the former with the flip map (see Lemma II.21).

Before we proceed to the characterization of masas we will have to make few definitions and statements that are very valuable tools yet not appear in standard measure theory courses. For details see [16], [26].

Definition III.11. Let X be a Polish space. A subset B of X is said to have *Baire property* if there is an open set $\mathcal{O} \subset X$ and a comeager set $A \subset X$ such that $A \cap \mathcal{O} = A \cap B$.

The collection of sets with *Baire property* forms a σ -algebra which includes the *Borel σ -algebra*.

Definition III.12. Let X and Y be Polish spaces. A function $f : X \mapsto Y$ is said to be *Baire measurable* if the inverse image of any *open set* has *Baire property*. The function f is said to be *universally Baire measurable* if given any Borel function g into X the function $f \circ g$ is Baire measurable.

Note that in particular every Borel function is *Baire measurable*.

Definition III.13. A subset E of a Polish space is said to be *universally measurable* if it is measurable with respect to any *complete Borel probability measure*.

Definition III.14. A subset E of a Polish space X is said to be Σ_1^1 or *analytic*, if there is a Polish space Y , a Borel subset B of Y and a Borel function $f : Y \mapsto X$ such that $f(B) = E$. In other words, Σ_1^1 sets are Borel images of Borel sets.

Remark III.15. The above definition of analytic sets is as per [16]. However in, [19] continuous images rather than Borel images are used. The two definitions are in fact equivalent.

A very nontrivial theorem of Lusin says the following.

Theorem III.16. (*Lusin*) *Every Σ_1^1 set has Baire property. Every $\tilde{\Sigma}_1^1$ set is universally measurable.*

For a function $f : Y \mapsto X$, the graph of f will be denoted by $\Gamma(f) = \{(f(y), y) : y \in Y\}$. The next theorem is very crucial in all our analysis.

Theorem III.17. (*Selection Principle - Jankov, von Neumann*) *Let X, Y be Polish spaces and let $E \subset X \times Y$ be in $\tilde{\Sigma}_1^1$. Then E can be uniformized by a function that is both Baire and universally measurable, in the sense that for some $h : Y \mapsto X$ we have*

$$\Gamma(h|_{\pi_Y(E)}) \subseteq E$$

with the property that $h^{-1}(U)$ has the Baire property and is measurable with respect to any Borel probability measure for all open $U \subseteq X$.

Remark III.18. Let ν_X and ν_Y be any two Borel probability measures on X, Y respectively. Let σ_{ν_X} and σ_{ν_Y} be the σ -algebras associated to the measures ν_X, ν_Y respectively. If h is the function in Thm. III.17, then the inverse image of any Borel set in X under h will lie in σ_{ν_Y} , because the collection of subsets of X whose inverse images fall in σ_{ν_Y} is a σ -algebra and contains all open sets. If in addition, h satisfies the property that $\nu_X(h(F)) = 0$ if and only if $\nu_Y(F) = 0$, then h is $(\sigma_{\nu_Y}, \sigma_{\nu_X})$ measurable.

Let $A \subset \mathcal{M}$ be a masa. Without loss of generality we assume that $A = L^\infty([0, 1], \lambda)$ where λ is the Lebesgue measure on $[0, 1]$. Let $[\eta_{[0,1] \times [0,1]}]$ denote the

left-right-measure of A . We are including the diagonal. Fix any member $\eta_{[0,1] \times [0,1]}$ from the equivalence class. Since our base space is now fixed we will rename $\eta_{[0,1] \times [0,1]}$ by η to reduce the notation. We assume that η is a finite measure.

Consider the set $S_a = ([0, 1] \times [0, 1])_a$ as defined in Prop. II.17 with respect to the disintegration along the y -axis i.e. the t variable. Then by Prop. II.17, S_a is a $[\eta]$ -measurable set, i.e. measurable with respect to the completion σ -algebra associated to η . Define measures

$$\eta_a = \eta|_{(S_a \setminus \Delta([0,1]))} \text{ and } \eta_{n.a} = \eta|_{(S_a^c \setminus \Delta([0,1]))}.$$

Then

(i) $\eta_{\Delta([0,1])^c} = \eta_a + \eta_{n.a}$, $\eta_a \perp \eta_{n.a}$.

(ii) Both $\eta_a, \eta_{n.a}$ have disintegrations along the x, y axes with respect to λ .

Note that the disintegration of the measure η_a along the x and y -axes must have at most countably many atoms almost all fibres (see Lemma II.21), otherwise η is an infinite measure. Since changing the measure η_a or η on a set of measure 0 does not change the measure class of η_a or η , we can as well assume without loss of generality that, the disintegration of the measure η_a along y -axis (second variable) has at most countable number of atoms for all fibres. With this as set up we formalize the characterization theorem of masas. Thm. III.19 will be proved latter in this section.

Theorem III.19. (*Classification of Types*) *A masa $A \subset \mathcal{M}$ is*

(i) *Cartan if and only if $\eta_{n.a} = 0$ equivalently $L^2(A)^\perp \in C_d(A)$,*

(ii) *singular if and only if $\eta_a = 0$ equivalently $L^2(A)^\perp \in C_{n.a}(A)$,*

(iii) *$A \not\subseteq N(A)'' \not\subseteq \mathcal{M}$ if and only if $\eta_a \neq 0, \eta_{n.a} \neq 0$ equivalently*

$$L^2(A)^\perp \in C_m(A).$$

(iv) A is semiregular if and only if $\eta_a(E \times F) > 0$ whenever $\lambda(E) > 0, \lambda(F) > 0$ for measurable sets $E, F \subset [0, 1]$.

Remark III.20. First of all, in view of Lemma II.18 and II.20, the characterization does not depend on any particular member of the *left-right-measure*.

Secondly, $L^2(A)$ is always included in $C_d(A)$, the disintegration having one atom at each point of the diagonal. This is the reason one excludes $L^2(A)$ from statements in Thm. III.19.

Finally, from our discussion on direct integrals, it follows that $L^2(A)^\perp$ is the direct integral over $[0, 1] \times [0, 1]$ with respect to the measure $\eta|_{(\Delta[0,1])^c}$, the measurable field of Hilbert spaces depending on $m_{[0,1]}$ or the *Pukánszky invariant*. So the equivalent statements regarding the type of bimodules and measure in Thm. III.19 are obvious statements.

The next technical lemma is the key to characterization of masas. There are several measures involved in its statement and proof. Since there is danger of confusion with measurability of objects involved we will always use phrases like ‘ μ -measurable’.

Lemma III.21. *Let $\eta_a \neq 0$. Let $Y \subseteq (\Delta[0, 1])^c$ be a η -measurable set of strictly positive η_a -measure. There exists a λ -measurable set $E^Y \subseteq [0, 1]$ with $\lambda(E^Y) > 0$ and a function $h_Y : [0, 1] \mapsto [0, 1]$ such that*

- (i) h_Y is λ -measurable,
- (ii) $\Gamma(h_Y)$ is a η -measurable set,
- (iii) $\eta(\Gamma(h_Y)) > 0$ and $(h_Y(t), t) \in Y \cap S_a$ for $t \in E^Y$,
- (iv) for $E \subset [0, 1]$, $\lambda(E) = 0$ if and only if $\lambda(h_Y(E)) = 0$.

Proof. We have

$$\eta(S_a \cap Y) = \eta_a(S_a \cap Y) = \eta_a(Y) > 0.$$

Consider the disintegration of $\eta|_Y$ along the y -axis. There is a set $F^Y \subseteq [0, 1]$ such that $\lambda(F^Y) > 0$ and for each $t \in F^Y$ the measure $(\eta|_Y)_t$ has atoms with at most countable number of atoms and for $t \notin F^Y$ the same disintegration has no atoms. This is true because η is a finite measure, the set F^Y being $\pi_y(S_a \cap Y)$, π_y denoting the projection on to the y -axis. The set $S_a \cap Y$ is η -measurable, so $S_a \cap Y = B \cup N$ where B is a Borel set in $[0, 1] \times [0, 1]$ and N is a η -null set. The set B is a continuous image of a Polish space by Thm. 14.3.5 of [19] and so is $\pi_y(B)$. By Defn II.15, $\lambda(\pi_y(N)) = 0$. So F^Y is λ -measurable set by Thm. III.16. Throwing off another λ -null set from F^Y if necessary we can as well assume without loss of generality that F^Y is a Borel set.

Let $F_a^Y = ((Y \cap S_a) \cap ([0, 1] \times F^Y))$ which is η -measurable. Write $F_a^Y = E_a^Y \cup N_1$ where N_1 is a η -null set and E_a^Y is a Borel set. Then by Thm. 14.3.5 of [19], E_a^Y is in Σ_1^1 , in fact it is the continuous image of a Polish space. The hypothesis guarantees $\eta(E_a^Y) > 0$.

Let $E^Y = \pi_y(E_a^Y)$. Then E^Y is in Σ_1^1 and hence E^Y is λ -measurable by Thm. III.16. Therefore by Def II.15, $\lambda(E^Y) > 0$. By Thm. III.17 applied to E_a^Y , there exists a function $h_Y : [0, 1] \mapsto [0, 1]$ that is both *Baire* and *universally measurable* in the sense of Thm. III.17, such that $\Gamma(h_Y|_{E^Y}) \subseteq E_a^Y$.

The inverse image under h_Y of any Borel subset of $[0, 1]$ belongs to σ_λ . Therefore given $\epsilon > 0$, by Lusin's theorem there is a closed subset $G^Y \subseteq E^Y$ such that $\lambda(E^Y \setminus G^Y) < \epsilon$ and $h_Y|_{G^Y}$ is continuous. Then $h_Y|_{G^Y}$ is Borel measurable. So by Cor. 2.11 of [25], $\Gamma(h_Y|_{G^Y})$ is Borel measurable and hence η -measurable.

The disintegration along the y -axis of the measure $\eta|_{\Gamma(h_Y|_{G^Y})}$ is precisely the atom at the point $(h_Y(t), t)$ for each $t \in G^Y$ of the measure η_t . Outside G^Y we don't care.

If $\eta(\Gamma(h_{Y|G^Y})) = 0$ then by definition of disintegration

$$0 = \int_{G^Y} \eta_t(\Gamma(h_Y)) d\lambda(t)$$

which implies that for λ almost all $t \in G^Y$ the point $(h_Y(t), t)$ is not an atom of η_t and hence cannot be in S_a . So $\eta(\Gamma(h_{Y|G^Y})) > 0$.

Clearly, $h_{Y|G^Y}$ satisfies the property that for any $E \subset G^Y$, $\lambda(E) = 0$ if and only if $\lambda(h_Y(E)) = 0$. Therefore by Thm. II.27, extend $h_{Y|G^Y}$ to a continuous function \tilde{h}_Y which satisfies the property that for any $E \subset [0, 1]$, $\lambda(E) = 0$ if and only if $\lambda(\tilde{h}_Y(E)) = 0$. So by Rem III.18, \tilde{h}_Y is $(\sigma_\lambda, \sigma_\lambda)$ measurable. Rename \tilde{h}_Y to h_Y and G^Y to E^Y . The rest is clear from construction. \square

Lemma III.22. *Let $\eta_a \neq 0$. Let $Y \subseteq (\Delta[0, 1])^c$ be a η -measurable set of strictly positive η_a -measure. Then $\mathcal{U}(A) \subsetneq N(A)$, where $\mathcal{U}(A)$ denotes the unitary group of A . More precisely, there exists a subset F^Y of $[0, 1]$ such that $\lambda(F^Y) > 0$, a invertible map $h_Y : F^Y \mapsto h_Y(F^Y)$ and a nonzero vector $\zeta_Y \in L^2(N(A)'') \ominus L^2(A)$ such that*

- (i) $\zeta_Y = v_Y \rho_Y$ with $v_Y \in \mathcal{GN}(A)$, $\rho_Y \in L^2(A)^+$
- (ii) $\overline{A\zeta_Y A}^{\|\cdot\|_2} \cong \int_{\Gamma(h_Y)}^{\oplus} \mathbb{C}_{s,t} d\eta(s, t)$, where $\mathbb{C}_{s,t} = \mathbb{C}$,
- (iii) $\Gamma(h_Y) \subseteq Y \cap S_a$,
- (iv) $\eta(\Gamma(h_Y)) > 0$,
- (v) $1 \in Puk(A)$.

Proof. Using Lemma III.21, choose the function h_Y that satisfies the conclusion of that Lemma. Note that h_Y satisfies the conditions of Prop. II.29. Apply Prop. II.29 to the function h_Y and the set E^Y to extract a set $F^Y \subseteq E^Y$ such that $\lambda(F^Y) > 0$

and h_Y is one to one on F^Y . So

$$h_Y : F^Y \mapsto h_Y(F^Y) \text{ is invertible.}$$

Note that as $\lambda(F^Y) > 0$ so $\eta(\Gamma(h_{Y|F^Y})) > 0$. There is no information of the *Pukánszky invariant* yet. So assume that $Puk(A) = \{n_i : n_i \in \mathbb{N}_\infty, i \in I\}$, where the indexing set I could be finite or countable. Let

$$E_{n_i} = \{(s, t) \in \Delta([0, 1])^c : m_{[0,1]}(s, t) = n_i\},$$

where $m_{[0,1]}$ denotes the multiplicity function of the direct integral decomposition of $L^2(\mathcal{M})$ over $[0, 1] \times [0, 1]$ with respect to the measure η . Then for each $i \in I$ it is well known that E_{n_i} are η -measurable sets. Also

$$\begin{aligned} \int_{E_{n_i}}^{\oplus} \mathbb{C}_{s,t}^{n_i} d\eta(s, t) &\cong L^2(E_{n_i}, \eta|_{E_{n_i}}) \otimes \mathbb{C}^{n_i} \text{ where } \mathbb{C}_{s,t}^{n_i} = \mathbb{C}^{n_i}, \text{ and} \\ \bigoplus_{i \in I} L^2(E_{n_i}, \eta|_{E_{n_i}}) \otimes \mathbb{C}^{n_i} &\cong L^2(\mathcal{M}) \ominus L^2(A). \end{aligned}$$

In the above equation \mathbb{C}^∞ stands for $l^2(\mathbb{N})$. Fix orthonormal bases $\{e_j^{(n_i)}\}_{1 \leq j \leq n_i}$ of \mathbb{C}^{n_i} for all $i \in I$. Then

$$\sum_{i \in I} \chi_{\Gamma(h_{Y|F^Y}) \cap E_{n_i}} \otimes e_1^{(n_i)}$$

where χ denotes the indicator function, can be identified with a vector $\zeta_Y \in (1 - e_A)(L^2(\mathcal{M}))$ such that

$$A\zeta_Y A = A\zeta_Y = \zeta_Y A. \tag{B.1}$$

Eq. (B.1) is easy to check, in fact one only uses that fact that h_Y is locally one to one and onto. That $\zeta_Y \neq 0$ is due to the fact $\eta(\Gamma(h_{Y|F^Y})) > 0$. Then from Theorem III.9, it follows that $\zeta_Y = v_Y \rho_Y$ where $\rho_Y = (\zeta_Y^* \zeta_Y)^{\frac{1}{2}} \in L^2(A)^+$ and $v_Y \in \mathcal{GN}(A)$. Clearly, $v_Y \notin A$, as otherwise $\overline{A\zeta_Y A}^{\|\cdot\|_2} \subseteq L^2(A)$ would become the direct integral of

complex numbers over some subset of the diagonal with respect to the measure $\Delta_*\lambda$, where $\Delta : [0, 1] \mapsto [0, 1] \times [0, 1]$ is the map $\Delta(x) = (x, x)$.

Thus $\zeta_Y \in L^2(N(A)''')$ and hence $\overline{A\zeta_Y A}^{\|\cdot\|_2} \subseteq L^2(N(A)''')$. Clearly,

$$\overline{A\zeta_Y A}^{\|\cdot\|_2} \cong \int_{\Gamma(h_Y|_{F^Y})}^{\oplus} \mathbb{C}_{s,t} d\eta(s, t), \text{ where } \mathbb{C}_{s,t} = \mathbb{C}. \quad (\text{B.2})$$

So $\overline{A\zeta_Y A}^{\|\cdot\|_2} \perp L^2(A)$ and $\overline{A\zeta_Y A}^{\|\cdot\|_2} \in C_d(A)$. Since $\overline{A\zeta_Y A}^{\|\cdot\|_2} \subseteq L^2(N(A)''')$ so $\eta(\Gamma(h_Y|_{F^Y}) \cap E_{n_i}) = 0$ if $n_i \geq 2$ from a result of Popa [34]. Thus $1 \in \text{Puk}(A)$. \square

Each partial isometry $0 \neq v \in \mathcal{GN}(A)$ implements a measure preserving local isomorphism $T : ([0, 1], \lambda) \mapsto ([0, 1], \lambda)$ such that $vav^* = a \circ T^{-1}$ for all $a \in A$. With abuse of notation we will write $v = T$. Then $\Gamma(v) = \{(T(t), t) : t \in \text{Dom}(T)\}$, $\text{Dom}(T)$ denoting the domain of T .

Lemma III.23. *Let $\eta_a \neq 0$. Let $Y \subseteq (\Delta[0, 1])^c$ be a η -measurable set of strictly positive η_a -measure. Then there is a nonzero partial isometry $v \in \mathcal{GN}(A)$ such that $\Gamma(v) \subseteq Y$.*

Proof. By Lemma III.22, there exists a subset F^Y of $[0, 1]$ such that $\lambda(F^Y) > 0$, a invertible map $h_Y : F^Y \mapsto h_Y(F^Y)$ and a nonzero vector $\zeta_Y \in L^2(N(A)''') \ominus L^2(A)$ such that $\zeta_Y = v_Y \rho_Y$ with $v_Y \in \mathcal{GN}(A)$, $\rho_Y \in L^2(A)^+$ and satisfying property (ii), (iii), (iv) of Lemma III.22.

Let $\eta_{\zeta_Y}, \eta_{v_Y}$ be the measures on $[0, 1] \times [0, 1]$ defined in Eq. (C.12). Let $q_Y = v_Y v_Y^* \in A$. With abuse of notation we will regard q_Y as a measurable subset of $[0, 1]$ as well. We claim that, $\eta_{\zeta_Y} \ll \eta_{v_Y} \ll \eta_{\zeta_Y}$. Indeed for $a, b \in C[0, 1]$,

$$\begin{aligned} \int_{[0,1] \times [0,1]} a(s)b(t) d\eta_{\zeta_Y}(s, t) &= \int_{\Gamma(h_Y)} a(s)b(t) d\eta_{\zeta_Y}(s, t) \\ &= \tau(\rho_Y^* v_Y^* a v_Y \rho_Y b) \\ &= \tau(\rho_Y^* v_Y^* a v_Y b \rho_Y) \text{ (as } \rho_Y b = b \rho_Y) \end{aligned}$$

$$\begin{aligned}
&= \tau(v_Y^* a v_Y b \rho_Y \rho_Y^*) \\
&= \tau(v_Y^* a v_Y b \rho_Y^* \rho_Y) \\
&= \tau(v_Y^* a v_Y \rho_Y^* \rho_Y b) \\
&= \int_{q_Y} a(v_Y(t)) b(t) |\rho_Y(t)|^2 d\lambda(t) \\
&= \int_{\Gamma(v_Y)} a(s) b(t) |\rho_Y(t)|^2 d\eta_{v_Y}(s, t) \\
&= \int_{[0,1] \times [0,1]} a(s) b(t) |\rho_Y(t)|^2 d\eta_{v_Y}(s, t).
\end{aligned}$$

In the above string of equalities we have used the facts that τ extends to a trace like functional on $L^1(A)$ and the left and right actions of A on $L^2(A)$, $L^1(A)$ coincides. Using Thm. III.9, by standard arguments it follows that $\eta_{\zeta_Y} \ll \eta_{v_Y} \ll \eta_{\zeta_Y}$. Thus the result follows with $v = v_Y$. \square

Suppose $\{v_j\}_{j \in J}$ is a family of partial isometries in $\mathcal{GN}(A)$ such that $Av_j \perp Av_{j'}$ whenever $j \neq j'$. Denote by [34]

$$\sum_{j \in J} Av_j = \left\{ x \in \mathcal{M} : x = \sum_{j \in J} a_j v_j, \text{ for } a_j \in A \text{ with } \sum_{j \in J} \|a_j v_j\|_2^2 < \infty \right\}.$$

Theorem III.24. (Compare Cor. 2.5 [34]) *Let $\eta_a \neq 0$. Then $A \subsetneq N(A)''$. Moreover, there is a sequence $\{v_n\}_{n=0}^\infty \subset \mathcal{GN}(A)$ of nonzero partial isometries (with possibility that the sequence could be finite) with $v_0 = 1$ such that,*

- (i) $\Gamma(v_n) \cap \Gamma(v_m) = \emptyset$ for $n \neq m$,
- (ii) $\eta_a([0, 1] \times [0, 1]) = \sum_{n=1}^\infty \eta_a(\Gamma(v_n))$,
- (iii) $\bigoplus_{n=0}^\infty \overline{Av_n}^{\|\cdot\|_2} \cong \int_{\bigcup_{n=0}^\infty \Gamma(v_n)}^\oplus \mathbb{C}_{s,t} d(\eta_a + \Delta_* \lambda)(s, t) \cong L^2(N(A)'')$,

(where $\mathbb{C}_{s,t} = \mathbb{C}$ and $\Delta : [0, 1] \mapsto [0, 1] \times [0, 1]$ by $\Delta(x) = (x, x)$)

$$(iv) \quad N(A)'' = \sum_{n=0}^{\infty} Av_n,$$

and \mathcal{A} restricted to $\bigoplus_{n=0}^{\infty} \overline{Av_n}^{\|\cdot\|_2}$ is diagonalizable with respect to the decomposition in (iii).

Proof. First of all assuming that (i) in the statement is true it follows that $Av_n \perp Av_m$ whenever $n \neq m$. Indeed, $\overline{Av_n}^{\|\cdot\|_2} \subseteq L^2(N(A)'')$. Now \mathcal{A} restricted to $\overline{Av_n}^{\|\cdot\|_2}$ is an abelian algebra with a cyclic vector, so it is maximal abelian. The projection e_{Av_n} onto $\overline{Av_n}^{\|\cdot\|_2}$ is in \mathcal{A} . So $\overline{Av_n}^{\|\cdot\|_2}$ is the direct integral of complex numbers over a subset X_n of $[0, 1] \times [0, 1]$ with respect to the measure η and \mathcal{A} restricted to $\overline{Av_n}^{\|\cdot\|_2}$ is diagonalizable with respect to this decomposition. But $\eta(X_n \Delta \Gamma(v_n)) = 0$. Again $\eta_{n,a}(\Gamma(v_n)) = 0$. So the direct integral as stated above is actually with respect to the measure $\eta_a + \Delta_* \lambda$. The graphs being disjoint for $n \neq m$ forces the orthogonality of Av_n and Av_m whenever $n \neq m$. The sum in (iii) therefore makes sense.

Using Lemma III.23, choose a maximal family $\{v_\alpha\}_{\alpha \in \Lambda} \subset \mathcal{GN}(A)$, for some indexing set Λ , such that $\Gamma(v_\alpha) \subset \Delta([0, 1])^c$ for all $\alpha \in \Lambda$ and $\Gamma(v_\alpha) \cap \Gamma(v_\beta) = \emptyset$ whenever $\alpha \neq \beta$. Since $Av_\alpha \perp Av_\beta$ whenever $\alpha \neq \beta$ (by similar argument as above) so the indexing set must be countable by the separability assumption of $L^2(\mathcal{M})$. So we index this maximal family by $\{v_n\}_{n=1}^{\infty}$. Let $v_0 = 1$. So (i) follows by construction.

If $\eta_a([0, 1] \times [0, 1]) > \sum_{n=1}^{\infty} \eta_a(\Gamma(v_n))$ then $S_a \setminus \bigcup_{n=1}^{\infty} \Gamma(v_n)$ is a set of strictly positive η_a measure. A further application of Lemma III.23 violates the maximality of $\{v_n\}_{n=1}^{\infty}$. This proves (ii).

By the argument of the first paragraph and Lemma 5.7 [11],

$$\bigoplus_{n=1}^{\infty} \overline{Av_n}^{\|\cdot\|_2} \cong \int_{\bigcup_{n=1}^{\infty} \Gamma(v_n)}^{\oplus} \mathbb{C}_{s,t} d\eta_a(s, t) \subseteq L^2(N(A)'') \ominus L^2(A) \quad (\text{B.3})$$

and \mathcal{A} restricted to $\bigoplus_{n=1}^{\infty} \overline{Av_n}^{\|\cdot\|_2}$ is diagonalizable with respect to the decomposition

in Eq. (B.3).

If $0 \neq \zeta = \zeta^* \in L^2(N(A)'') \ominus L^2(A)$ is such that $\zeta \perp Av_n$ for all $n \geq 1$ then $A\zeta A \perp Av_n$ for all $n \geq 0$. By arguments similar to the first paragraph, $\overline{A\zeta A}^{\|\cdot\|_2}$ is the direct integral over a η -measurable set X_ζ , of complex numbers with respect to the measure η and \mathcal{A} restricted to $\overline{A\zeta A}^{\|\cdot\|_2}$ is diagonalizable respecting this decomposition. If ζ as a L^2 function stays nonzero on a set of positive $\Delta_*\lambda$ -measure then ζ cannot be perpendicular to $L^2(A)$. By Prop. II.25, $\overline{A\zeta A}^{\|\cdot\|_2} \in C_d(A)$ and hence by Theorem II.22 and Lemma 5.7 [11], we can assume $X_\zeta \subset S_a \setminus \Delta([0, 1])$. Since $\zeta \neq 0$ so $\eta(X_\zeta) = \eta_a(X_\zeta) > 0$. Since η_a is concentrated on $\cup_{n=1}^\infty \Gamma(v_n)$, so $X_\zeta \cap \Gamma(v_n)$ has strictly positive η_a and hence η measure for some $n \geq 1$. Note that $e_{N(A)''} \in \mathcal{A}$ and $\mathcal{A}e_{N(A)''} = \mathcal{A}'e_{N(A)''}$ from [34]. On the other hand, by Lemma 5.7 [11], $L^2(N(A)'') \ominus L^2(A)$ will be expressed as a direct integral over some subset of $[0, 1] \times [0, 1]$ with respect to η , with multiplicity strictly bigger than 1 on a set of positive η -measure. This contradicts $\mathcal{A}e_{N(A)''}$ is maximal abelian. Thus

$$\oplus_{n=0}^\infty \overline{Av_n}^{\|\cdot\|_2} \cong \int_{\cup_{n=0}^\infty \Gamma(v_n)}^\oplus \mathbb{C}_{s,t} d(\eta_a + \Delta_*\lambda)(s, t) \cong L^2(N(A)''),$$

with associated statements about diagonalizability of \mathcal{A} . Finally

$$\sum_{n=0}^\infty Av_n = \overline{\sum_{n=0}^\infty Av_n}^{\|\cdot\|_2} \cap \mathcal{M} = \left(\oplus_{n=0}^\infty \overline{Av_n}^{\|\cdot\|_2} \right) \cap \mathcal{M} = L^2(N(A)'') \cap \mathcal{M} = N(A)'' .$$

□

Remark III.25. Thm. III.24 generalizes Cor. 2.5 of [34]. In general we cannot hope to find unitaries as was the case in Cor. 2.5 [34]. The situation in Cor. 2.5 of [34] was completely different, where the assumption was that, the masa is Cartan. Assuming the masa is Cartan, forces the disintegration of the measure η_a to have at least one atom off the diagonal in almost every fibre. Such an assumption cannot be made for a

general masa. For example consider the following situation. Let $C \subset \mathcal{R}$ be a Cartan masa and let $S \subset \mathcal{R}$ be a singular masa, where \mathcal{R} denotes the hyperfinite II_1 factor. Then $C \oplus S \subset \mathcal{R} \oplus \mathcal{R}$ is a masa, where the trace on $\mathcal{R} \oplus \mathcal{R}$ is $\frac{1}{2}\tau_{\mathcal{R}} \oplus \frac{1}{2}\tau_{\mathcal{R}}$, $\tau_{\mathcal{R}}$ denoting the unique, normal, faithful tracial state of \mathcal{R} . Then $C \oplus S \subset (\mathcal{R} \oplus \mathcal{R}) * \mathcal{R} \cong L(\mathbb{F}_2)$ (from [9]) is a masa for which such an assumption will fail from Prop. 5.10 [11].

We will now present the proof of Thm III.19.

Proof of III.19. Case (i). The necessary and sufficient condition for Cartan masas follows directly from Thm. III.24.

Case (ii). The result for singular masas also follows from Thm. III.24.

Case (iii). Let $A \subsetneq N(A)'' \subsetneq \mathcal{M}$. If $\eta_a = 0$ then, by conclusion of (ii), A would become singular. Therefore $\eta_a \neq 0$. If $\eta_{n.a} = 0$ then by conclusion of part (i), A would be Cartan. Therefore $\eta_{n.a} \neq 0$ as well.

Conversely, if $\eta_{n.a} \neq 0$ and $\eta_a \neq 0$, then by Theorem III.24, $A \subsetneq N(A)'' \subsetneq \mathcal{M}$.

Case (iv). First assume that $N(A)''$ is a factor. From Thm. III.24 it follows that

$$L^2(N(A)'') \cong \int_{[0,1] \times [0,1]}^{\oplus} \mathbb{C}_{s,t} d(\eta_a + \tilde{\Delta}_* \lambda)(s, t), \text{ where } \mathbb{C}_{s,t} = \mathbb{C}$$

and \mathcal{A} restricted to this subspace is diagonalizable, where $\tilde{\Delta} : [0, 1] \mapsto [0, 1] \times [0, 1]$ is defined by $\tilde{\Delta}(x) = (x, x)$. Therefore $[\eta_a + \tilde{\Delta}_* \lambda]$ is the *left-right-measure* of the inclusion $A \subset N(A)''$. Suppose there are measurable sets $E, F \subset [0, 1]$ such that $\lambda(E) > 0$, $\lambda(F) > 0$ and $\eta_a(E \times F) = 0$. Now E and F corresponds to nonzero projections p, q respectively in A . Thus $p\zeta q = 0$ for all $\zeta \in L^2(N(A)'')$ and hence $pxq = 0$ for all $x \in N(A)''$. Thus $C_p C_q = 0$ where C_p, C_q denotes the central carriers of p and q respectively in $N(A)''$. So $N(A)''$ cannot be a factor.

Conversely assume $N(A)''$ has a nontrivial center. Let $p \in \mathbf{Z}(N(A)'')$ be a

projection which is different from 0 and 1. Then

$$N(A)'' = N(A)''p \oplus N(A)''(1-p).$$

So $p \in A' \cap N(A)''$ and hence $p \in A$. So

$$A = Ap \oplus A(1-p). \tag{B.4}$$

It follows that there exists λ -measurable sets $F_1, F_2 \subset [0, 1]$ such that, $\lambda(F_i) > 0$ for $i = 1, 2$, where F_1 corresponds to p and F_2 corresponds to $(1-p)$.

With respect to the Eq. (B.4) let $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$ be the decompositions of $a, b \in C([0, 1])$. For $\zeta \in L^2(N(A)'')$ one has an analogous decomposition $\zeta = \zeta_1 \oplus \zeta_2$ with $\zeta_1 = p\zeta p$ and $\zeta_2 = (1-p)\zeta(1-p)$. The equation

$$\langle a\zeta b, \zeta \rangle = \langle (a_1 \oplus a_2)(\zeta_1 \oplus \zeta_2)(b_1 \oplus b_2), (\zeta_1 \oplus \zeta_2) \rangle = \langle a_1\zeta_1 b_1, \zeta_1 \rangle + \langle a_2\zeta_2 b_2, \zeta_2 \rangle$$

shows that the *left-right-measure* of the inclusion $A \subset N(A)''$ will be concentrated on $F_1 \times F_1 \cup F_2 \times F_2$. This completes the proof. \square

C. Consequences of the Characterization, Chifan's Normaliser Formula

The following results about masas that were proved by experts in different ways, are just easy consequences of the *measurable selection principle* as we have described in the previous section.

Corollary III.26. *If $A \subset \mathcal{M}$ is a Cartan masa then $A \subset B$ is a Cartan masa for all von Neumann subalgebra $A \subsetneq B \subsetneq \mathcal{M}$.*

Proof. By Lemma 5.7 of [11] the *left-right-measure* of the inclusion $A \subset \mathcal{M}$ is $[\eta_B + \eta_{B^\perp}]$ where $[\eta_B]$ is the *left-right-measure* of the inclusion $A \subset B$ and $\eta_B \perp \eta_{B^\perp}$. It follows that η_B has atomic disintegration along both axes. The result is then

immediate from Thm. III.19 and Thm III.24. \square

Corollary III.27. *Let $A \subset \mathcal{M}$ be a masa and let Q be a finite von Neumann algebra such that $\dim(Q) \geq 2$. Then $N_{\mathcal{M}*Q}(A) = N_{\mathcal{M}}(A)$.*

Proof. In this proof we consider *left-right-measures* restricted to the off diagonal. First of all it well known that $A \subset \mathcal{M} * Q$ is a masa. Let $[\eta_{\mathcal{M}}]$ denote the *left-right-measure* of the inclusion $A \subset \mathcal{M}$. Write $\eta_{\mathcal{M}} = \eta_1 + \eta_2$ where $\eta_1 \ll \lambda \otimes \lambda$ and $\eta_2 \perp \lambda \otimes \lambda$. Using Prop. 5.10 and Lemma 5.7 [11] it follows that the *left-right-measure* $[\eta_{\mathcal{M}*Q}]$ of the inclusion $A \subset \mathcal{M} * Q$ is given by

$$\eta_{\mathcal{M}*Q} = \begin{cases} \eta_{\mathcal{M}} + \lambda \otimes \lambda & \text{if } \eta_1 = 0, \\ \eta_2 + \lambda \otimes \lambda & \text{if } \eta_1 \neq 0. \end{cases}$$

The rest is obvious from Thm. III.19 and Thm. III.24. \square

Corollary III.28. *Let $A \subset \mathcal{M}$ be a Cartan masa and let $A \subset B \subsetneq \mathcal{M}$ be an intermediate subalgebra. Then there is a $v \in \mathcal{GN}(A)$ such that $v \perp B$.*

Proof. By Lemma 5.7 [11], the *left-right-measure* of the inclusion $A \subset \mathcal{M}$ is $[\eta_B + \eta_{B^\perp}]$ where $\eta_B \perp \eta_{B^\perp}$ and $[\eta_B]$ is the *left-right-measure* of the inclusion $A \subset B$. Note $\eta_{B^\perp} \neq 0$. Apply Lemma III.23. \square

We prove the next theorem in the context of II_1 factors. But it can be easily generalized to finite von Neumann algebras. Let \mathcal{M}_i , $i = 1, 2$ be separably acting II_1 factors with normal, faithful tracial states τ_i respectively. Let \mathcal{M}_i act on $L^2(\mathcal{M}_i, \tau_i)$ by left multiplication. Let $A \subset \mathcal{M}_1$ and $B \subset \mathcal{M}_2$ be masas. Fix compact Polish spaces X, Y such that $C(X) \subset A$ and $C(Y) \subset B$ are unital, norm separable and *w.o.t* dense. Let ν_X and ν_Y denote the tracial measures for A, B respectively, which will be assumed to be complete. Let the *left-right-measure* of A on $X \times X$ be $[\sigma_1]$

and that of B on $Y \times Y$ be $[\sigma_2]$. Here we are allowing the diagonals, i.e, we are assuming $\sigma_{1|\Delta(X)} = (\tilde{\Delta}_X)_*\nu_X$ and $\sigma_{2|\Delta(Y)} = (\tilde{\Delta}_Y)_*\nu_Y$ where $\tilde{\Delta}_X : X \mapsto X \times X$ by $\tilde{\Delta}_X(x) = (x, x)$ and $\tilde{\Delta}_Y : Y \mapsto Y \times Y$ by $\tilde{\Delta}_Y(y) = (y, y)$.

By Tomita's theorem on commutants $A\overline{\otimes}B$ is a masa in $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$. The space $X \times Y$ is compact and Polish, and $C(X \times Y)$ is unital, norm separable and *w.o.t* dense in $A\overline{\otimes}B$. The standard Hilbert space and the Tomita's involution operator for $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$ are $L^2(\mathcal{M}_1, \tau_1) \otimes L^2(\mathcal{M}_2, \tau_2)$ and $J_{\mathcal{M}_1} \otimes J_{\mathcal{M}_2}$ respectively. The tracial measure for $A\overline{\otimes}B$ on $X \times Y$ is clearly $\nu_X \otimes \nu_Y$. With this as set up we formulate the next theorem which appeared in [3]. The same proof actually generalizes to infinite tensor products.

Theorem III.29. (*Chifan's Normaliser Formula*) *Let $A \subset \mathcal{M}_1$ and $B \subset \mathcal{M}_2$ be masas in separably acting II_1 factors \mathcal{M}_1 and \mathcal{M}_2 . Then*

$$N(A\overline{\otimes}B)'' = N(A)''\overline{\otimes}N(B)''.$$

Proof. Fix σ_1 and σ_2 from the aforesaid class of *left-right-measures*. The *left-right-measure* of $A\overline{\otimes}B$ on $(X \times Y) \times (X \times Y)$ which is denoted by $[\beta]$ is given by

$$d\beta(s_X, s_Y, t_X, t_Y) = d\sigma_1(s_X, t_X)d\sigma_2(s_Y, t_Y)$$

from Prop. 5.2 [11]. Here s is the variable running along the first coordinate (horizontal direction) and t along the second coordinate (vertical direction). Then from Lemma II.19 it follows that the disintegration of β along the t variable (vertical direction) is given by

$$\beta_{t_{X \times Y}} = \sigma_{1t_X} \otimes \sigma_{2t_Y}, (t_X, t_Y) - \text{a.e } \nu_X \otimes \nu_Y, \text{ where } t_{X \times Y} = (t_X, t_Y).$$

For fixed $t_{X \times Y} = (t_X, t_Y) \in X \times Y$ the measure $\beta_{t_{X \times Y}}$ has an atom at the point

(s_X, s_Y, t_X, t_Y) if and only if σ_{1t_X} has an atom at (s_X, t_X) and σ_{2t_Y} has an atom at (s_Y, t_Y) . Therefore

$$((X \times Y) \times (X \times Y))_a = S_{2,3}((X \times X)_a \times (Y \times Y)_a),$$

where $S_{2,3}$ denotes the permutation $(2, 3)$ on four symbols (see Prop. II.17). Therefore,

$$\beta_{|((X \times Y) \times (X \times Y))_a} = \sigma_{1|(X \times X)_a} \otimes \sigma_{2|(Y \times Y)_a}.$$

Hence denoting $\mathbb{C}_{s_X, s_Y, t_X, t_Y} = \mathbb{C}$, $\mathbb{C}_{s_X, t_X} = \mathbb{C} = \mathbb{C}_{s_Y, t_Y}$ we have

$$\begin{aligned} L^2(N(A \overline{\otimes} B)'') &\cong \int_{((X \times Y) \times (X \times Y))_a}^{\oplus} \mathbb{C}_{s_X, s_Y, t_X, t_Y} d\beta(s_X, s_Y, t_X, t_Y) \\ &\cong \int_{(X \times X)_a}^{\oplus} \mathbb{C}_{s_X, t_X} d\sigma_1(s_X, t_X) \otimes \int_{(Y \times Y)_a}^{\oplus} \mathbb{C}_{s_Y, t_Y} d\sigma_2(s_Y, t_Y) \\ &\cong L^2(N(A)'') \otimes L^2(N(B)'') \text{ from Thm. III.24.} \end{aligned}$$

Since the containment $N(A)'' \overline{\otimes} N(B)'' \subseteq N(A \overline{\otimes} B)''$ is obvious we are done. \square

As a corollary we obtain the following result that was proved in [44].

Corollary III.30. *Let $A \subset \mathcal{M}_1$ and $B \subset \mathcal{M}_2$ be singular masas in separably acting II_1 factors \mathcal{M}_1 and \mathcal{M}_2 . Then $A \overline{\otimes} B$ is singular in $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$.*

D. Asymptotic Homomorphism and Measure Theory

The equivalence of WAHP and singularity is a nontrivial theorem [44]. In this section we will give a direct proof of the equivalence of WAHP and singularity by using measure theoretic tools. We will also present partial results about AHP. In order to do so we will first have to relate certain norms to the *left-right-measure*. The measure theoretic tools described in this section will be used in subsequent chapters for explicit calculation of *left-right-measures*.

Let $A \subset \mathcal{M}$ be a masa. Let λ denote the Lebesgue measure on $[0, 1]$ so that $A \cong L^\infty([0, 1], \lambda)$. Then λ is the tracial measure. Let $[\eta]$ denote the *left-right-measure* of A . We assume that η is a probability measure on $[0, 1] \times [0, 1]$ and $\eta(\Delta([0, 1])) = 0$. Let $B[0, 1]$ denote the collection of all bounded measurable functions on $[0, 1]$.

Notation: The disintegrated measures are usually written with a subscript $t \mapsto \eta_t$ in the literature. But in this section we will use the superscript notation $t \mapsto \eta^t$ to denote them. The (π_1, λ) disintegration of measures will be indexed by the variable t and the (π_2, λ) disintegration will be indexed by the variable s .

In all the following results that uses disintegration of measures, we will only state or prove the result with respect to the (π_1, λ) disintegration. Statements about the (π_2, λ) disintegration are analogous.

Lemma III.31. *Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. Let η_x denote the measure on $[0, 1] \times [0, 1]$ defined in Eq. (C.12) of chapter II. Then η_x admits (π_i, λ) disintegrations $[0, 1] \ni t \mapsto \eta_x^t$ and $[0, 1] \ni s \mapsto \eta_x^s$, where π_i , $i = 1, 2$ denotes the coordinate projections. Moreover,*

$$\eta_x^t([0, 1] \times [0, 1]) = \mathbb{E}_A(xx^*)(t), \quad \lambda \text{ a.e.}$$

Proof. From Lemma 5.7 of [11] it follows that there is a measure η_0 such that (i) $\eta_0 \perp \eta_x$, (ii) $[\eta_x + \eta_0]$ is the *left-right-measure* of A . Therefore $[(\pi_i)_*(\eta_0 + \eta_x)] = [\lambda]$ by Lemma II.11 and hence $(\pi_i)_*(\eta_x) \ll \lambda$ for $i = 1, 2$. Consequently from Thm. II.16, η_x admits (π_i, λ) disintegrations for $i = 1, 2$.

Note that $\eta_x([0, 1] \times [0, 1]) = \tau(xx^*) = \tau(\mathbb{E}_A(xx^*))$. From (ii) of Defn. II.15 it follows that $[0, 1] \ni t \mapsto \eta_x^t([0, 1] \times [0, 1])$ is measurable. Let $E \subseteq [0, 1]$ be any Borel set. Then there exists a sequence of functions $f_n \in C[0, 1]$ such that $0 \leq f_n \leq 1$ and $f_n \rightarrow \chi_E$ pointwise. By dominated convergence theorem we have $\eta_x(f_n \otimes 1) \rightarrow$

$\eta_x(\chi_E \otimes 1)$. On the other hand,

$$\begin{aligned} \eta_x(f_n \otimes 1) &= \langle f_n x, x \rangle = \tau(f_n x x^*) = \tau(f_n \mathbb{E}_A(x x^*)) = \int_0^1 f_n(t) \mathbb{E}_A(x x^*)(t) d\lambda(t) \\ &\rightarrow \int_0^1 \chi_E(t) \mathbb{E}_A(x x^*)(t) d\lambda(t), \text{ as } n \rightarrow \infty, \\ &= \int_E \mathbb{E}_A(x x^*)(t) d\lambda(t). \end{aligned}$$

From Defn. II.15 again we have

$$\eta_x(\chi_E \otimes 1) = \int_0^1 \eta_x^t(\chi_E \otimes 1) d\lambda(t) = \int_E \eta_x^t([0, 1] \times [0, 1]) d\lambda(t).$$

Therefore for all Borel sets $E \subseteq [0, 1]$ we have

$$\int_E \eta_x^t([0, 1] \times [0, 1]) d\lambda(t) = \int_E \mathbb{E}_A(x x^*)(t) d\lambda(t).$$

Thus, $\eta_x^t([0, 1] \times [0, 1]) = \mathbb{E}_A(x x^*)(t)$ for λ almost all t . \square

Lemma III.32. *Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. Let $f \in B[0, 1]$. Then the functions $[0, 1] \ni t \mapsto \eta_x^t(1 \otimes f)$, $[0, 1] \ni s \mapsto \eta_x^s(f \otimes 1)$ are in $L^\infty([0, 1], \lambda)$.*

Proof. We will only prove for the (π_1, λ) disintegration. From Lemma III.31 we know that η_x admits a (π_1, λ) disintegration. From Defn. II.15 we also know that $[0, 1] \ni t \mapsto \eta_x^t(1 \otimes f)$ is measurable. Now if $0 \leq t \leq 1$ then

$$|\eta_x^t(1 \otimes f)| \leq \|f\| \eta_x^t([0, 1] \times [0, 1]).$$

Now use Lemma III.31. \square

Lemma III.33. *Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. Let $b, w \in B[0, 1]$. Then*

$$\|\mathbb{E}_A(b x w x^*)\|_2^2 = \int_0^1 |b(t)|^2 |\eta_x^t(1 \otimes w)|^2 d\lambda(t).$$

Proof. We have noted before that η_x admits (π_i, λ) disintegrations for $i = 1, 2$. Sec-

only, as $b, w \in B[0, 1]$, so $[0, 1] \ni t \mapsto b(t)\eta_x^t(1 \otimes w)$ is in $L^\infty([0, 1], \lambda)$ from Lemma III.32. Now

$$\begin{aligned}
\|\mathbb{E}_A(bxwx^*)\|_2^2 &= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \leq 1}} |\langle a, \mathbb{E}_A(bxwx^*) \rangle|^2 \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \leq 1}} |\tau(a\mathbb{E}_A(bxwx^*))|^2 \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \leq 1}} |\tau(\mathbb{E}_A(abxwx^*))|^2 \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \leq 1}} |\tau(abxwx^*)|^2 \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \leq 1}} \left| \int_{[0,1] \times [0,1]} a(t)b(t)w(s)d\eta_x(t, s) \right|^2 \\
&\qquad\qquad\qquad (\text{ from Eq. (C.12) of chapter II}) \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \leq 1}} \left| \int_0^1 a(t)b(t)\eta_x^t(1 \otimes w)d\lambda(t) \right|^2 \quad (\text{from Defn. II.15}) \\
&= \int_0^1 |b(t)|^2 |\eta_x^t(1 \otimes w)|^2 d\lambda(t) \quad (\text{from Lemma III.32}).
\end{aligned}$$

□

The following facts are well known, we just record them for completeness. For details we refer the reader to [20]. Recall that a subset $S \subseteq \mathbb{Z}$ is said to be of *full density* if

$$\lim_n \frac{\#(S \cap [-n, n])}{2n + 1} = 1.$$

Definition III.34. A measure μ on $[0, 1]$ is called *mixing* (or sometimes *Rajchman*) if its Fourier coefficients $\hat{\mu}_n = \int_0^1 e^{2\pi int} d\mu(t)$ converge to 0 as $|n| \rightarrow \infty$.

By the Riemann-Lebesgue lemma any absolutely continuous measure is mixing. However there are many mixing singular measures as well. Atomic measures can never be mixing. The next proposition justifies why non-atomic measures are called *weak* (or *weakly*) *mixing measures*.

Proposition III.35. (*Wiener*) *A measure μ on $[0, 1]$ is non-atomic (diffuse) if and only if for a set $S \subseteq \mathbb{Z}$ of full density*

$$\lim_{n \in S, |n| \rightarrow \infty} \hat{\mu}_n = 0.$$

From Prop. 2.5 and Prop. 2.19 of [20], *mixing* and *weakly mixing* are just not properties of measures, they are in fact properties of equivalence class of measures.

We need the following fact from the calculus course. A bounded sequence of complex numbers $\{a_n\}_{n \in \mathbb{Z}}$ converges to 0 strongly in the sense of Cesàro i.e,

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |a_n| = 0 \quad (\text{D.1})$$

if and only if there is a set $S \subseteq \mathbb{Z}$ of full density such that

$$\lim_{n \in S, |n| \rightarrow \infty} |a_n| = 0. \quad (\text{D.2})$$

Let $x, y \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = \mathbb{E}_A(y) = 0$. Let $a \in A$. Then the following polarization identity holds:

$$\begin{aligned} 4 \mathbb{E}_A(xay^*) &= \mathbb{E}_A((x+y)a(x+y)^*) - \mathbb{E}_A((x-y)a(x-y)^*) \\ &\quad + i \mathbb{E}_A((x+iy)a(x+iy)^*) - i \mathbb{E}_A((x-iy)a(x-iy)^*). \end{aligned} \quad (\text{D.3})$$

Thus WAHP for a masa is equivalent to the following. For each finite set $\{x_i\}_{i=1}^n \subset \mathcal{M}$ with $\mathbb{E}_A(x_i) = 0$ for all $1 \leq i \leq n$ and $\epsilon > 0$, there exists an uni-

tary $u \in A$ such that

$$\|\mathbb{E}_A(x_i u x_i^*)\|_2 \leq \epsilon \text{ for all } 1 \leq i \leq n.$$

We will only prove the harder part of the equivalence of singularity and WAHP.

Theorem III.36. *Let $A \subset \mathcal{M}$ be a masa such that $L^2(A)^\perp \in C_{n,a}(A)$. Then A has WAHP.*

Proof. Suppose to the contrary A does not have WAHP. Then there is a $\epsilon > 0$ and operators $0 \neq x_i \in \mathcal{M}$, $1 \leq i \leq n$ with $\mathbb{E}_A(x_i) = 0$ for all i , such that

$$\inf_{u \in \mathcal{U}(A)} \sum_{i=1}^n \|\mathbb{E}_A(x_i u x_i^*)\|_2^2 \geq \epsilon,$$

where $\mathcal{U}(A)$ denotes the unitary group of A . Note that for all $1 \leq i \leq n$, $\overline{Ax_i A}^{\|\cdot\|_2} \in C_{n,a}(A)$ by Lemma 5.7 of [11] and Thm. III.19. Equivalently, if $t \mapsto \eta_{x_i}^t$ and $s \mapsto \eta_{x_i}^s$ denote the (π_1, λ) and (π_2, λ) disintegrations respectively of η_{x_i} , then for λ almost all t , the measure $\eta_{x_i}^t$ is completely non-atomic and similar statements hold for $\eta_{x_i}^s$.

Let $v \in A$ be the Haar unitary corresponding to the function $t \mapsto e^{2\pi i t}$. Then v generates A . Now from Lemma III.33 we have

$$\sum_{i=1}^n \|\mathbb{E}_A(x_i v^k x_i^*)\|_2^2 = \int_0^1 \sum_{i=1}^n |\eta_{x_i}^t(1 \otimes v^k)|^2 d\lambda(t) \geq \epsilon \text{ for all } k \in \mathbb{Z}. \quad (\text{D.4})$$

Throwing off a λ -null set F we assume that for $t \in F^c$ the measures $\eta_{x_i}^t$ are completely non-atomic, finite, concentrated on $\{t\} \times [0, 1]$ and $\eta_{x_i}^t([0, 1] \times [0, 1]) = \mathbb{E}_A(x_i x_i^*)(t)$ for all $1 \leq i \leq n$ (see Lemma III.31). Let

$$a_k(t) = \sum_{i=1}^n |\eta_{x_i}^t(1 \otimes v^k)|^2, \quad k \in \mathbb{Z}, t \in [0, 1].$$

Then a_k is measurable for all $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$ and $t \in F^c$ we have

$$a_k(t) = \sum_{i=1}^n \left| \int_{[0,1] \times [0,1]} e^{2\pi i k s} d\eta_{x_i}^t(t, s) \right|^2 \leq \sum_{i=1}^n (\eta_{x_i}^t([0,1] \times [0,1]))^2.$$

Then by Lemma III.31, $a_k(t) \leq \sum_{i=1}^n |\mathbb{E}_A(x_i x_i^*)(t)|^2 < \infty$, for all $t \in F^c$ and for all $k \in \mathbb{Z}$. Define

$$s_N(t) = \frac{1}{2N+1} \sum_{k=-N}^N a_k(t), N \in \mathbb{N}.$$

Therefore, s_N is measurable for all $N \in \mathbb{N}$. Since $\eta_{x_i}^t$ is completely non-atomic for all $1 \leq i \leq n$ and $t \in F^c$ so

$s_N(t) \rightarrow 0$ as $N \rightarrow \infty$ for all $t \in F^c$ from Eq. (D.1), (D.2) and Prop III.35.

Again since $s_N(t) \leq \sum_{i=1}^n |\mathbb{E}_A(x_i x_i^*)(t)|^2$ for $t \in F^c$ (from Lemma III.31), so by dominated convergence theorem

$$\int_0^1 s_N(t) d\lambda(t) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Therefore,

$$\begin{aligned} \int_0^1 s_N(t) d\lambda(t) &= \frac{1}{2N+1} \sum_{k=-N}^N \int_0^1 \sum_{i=1}^n |\eta_{x_i}^t(1 \otimes v^k)|^2 d\lambda(t) \\ &= \frac{1}{2N+1} \sum_{k=-N}^N \left(\sum_{i=1}^n \|\mathbb{E}_A(x_i v^k x_i^*)\|_2^2 \right) \rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

Consequently from Eq. (D.2) there is a set $S \subseteq \mathbb{Z}$ of full density such that

$$\lim_{k \in S, |k| \rightarrow \infty} \sum_{i=1}^n \|\mathbb{E}_A(x_i v^k x_i^*)\|_2^2 = 0.$$

This is a contradiction to Eq. (D.4). So A must have WAHP. \square

The proof of Thm. III.36 yields the following result.

Theorem III.37. *Let $A \subset \mathcal{M}$ be a singular masa. Then given any finite set $\{x_i\}_{i=1}^n \subset \mathcal{M}$ with $\mathbb{E}_A(x_i) = 0$ for all i ,*

$$\frac{1}{2N+1} \sum_{k=-N}^N \left(\sum_{i=1}^n \|\mathbb{E}_A(x_i v^k x_i^*)\|_2^2 \right) \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (\text{D.5})$$

where v is a Haar unitary generator of A .

Remark III.38. Thus the unitary in the definition of WAHP can always be chosen to be v^k where k is a large integer and v is a Haar unitary generator of the masa. This strengthens the definition of WAHP. Note that Eq. (D.5) is very closely related to definition of weakly mixing actions of abelian groups on finite von Neumann algebras.

The measures η_x^t, η^t are concentrated on $\{t\} \times [0, 1]$ for λ almost all t . We will denote by $\tilde{\eta}_x^t, \tilde{\eta}^t$ the restriction of the measures η_x^t and η^t respectively on $\{t\} \times [0, 1]$. Thus $\tilde{\eta}_x^t, \tilde{\eta}^t$ can be regarded as measures on $[0, 1]$.

Theorem III.39. *Let $A \subset \mathcal{M}$ be a masa. Let $[\eta]$ denote the left-right-measure of A . If for λ almost all t the measures $\tilde{\eta}^t$ are mixing, then A has AHP with respect to a Haar unitary generator of A .*

Proof. From Prop. 2.5 of [20] it follows that for λ almost all t , any measure in the equivalence class $[\tilde{\eta}^t]$ is mixing. In view of Eq. (D.3), it is enough to show that for all $x \in \mathcal{M}$ with $\mathbb{E}_A(x) = 0$,

$$\|\mathbb{E}_A(xv^n x^*)\|_2 \rightarrow 0 \text{ as } |n| \rightarrow \infty,$$

where $v \in A$ is a Haar unitary generator of A . Let $v \in A$ correspond to the function $s \mapsto e^{2\pi i s}$. By Lemma III.33

$$\|\mathbb{E}_A(xv^n x^*)\|_2^2 = \int_0^1 |\eta_x^t(1 \otimes v^n)|^2 d\lambda(t).$$

From Lemma 5.7 [11] we know that $\eta_x \ll \eta$ and hence for λ almost all t , $\eta_x^t \ll \eta^t$ from Lemma II.20. So $\tilde{\eta}_x^t \ll \tilde{\eta}^t$ for λ almost all t . Thus $\tilde{\eta}_x^t$ is mixing measure from Prop. 2.5 of [20] for λ almost all t . Also from Lemma III.31, the measures η_x^t are finite for λ almost all t . Use Lemma III.31 and apply dominated convergence theorem to finish the proof. □

CHAPTER IV

STRONGER NOTIONS OF SINGULARITY

In this chapter we study singular masas in II_1 factors that possess special properties. This chapter has four sections. First we make a common setup in Section A that will be in force in all the remaining three sections. Notations and facts from Section A will be used left and right in Sections B, C and D. Section B deals with masas for which the *left-right-measure* is the class of product measure. In Section C we study strongly mixing masas and its *left-right-measure*. Section D deals with presence or absence of nontrivial centralizing sequences in masas.

A. The Common Setup

Let $A \subset \mathcal{M}$ be a masa. Let λ denote the Lebesgue measure on $[0, 1]$ so that $A \cong L^\infty([0, 1], \lambda)$. Then λ is the tracial measure. Let $[\eta]$ denote the *left-right-measure* of A . We assume that η is a probability measure on $[0, 1] \times [0, 1]$ and $\eta(\Delta([0, 1])) = 0$. Let $B[0, 1]$ denote the collection of all bounded measurable functions on $[0, 1]$.

To understand the relation between properties of masas and its *left-right-measure*, disintegration of measures will be used. For disintegration of measures we refer the reader to the subsection ‘Conditional Measures and Masas’ of chapter II. If β in Defn. II.15 is a complex measure then the disintegration of β is obtained by decomposing it into a linear combination of four positive measures, using the Hahn decomposition of its real and imaginary parts.

Notation: The disintegrated measures are usually written with a subscript $t \mapsto \beta_t$ in the literature. But in this paper we will use the superscript notation $t \mapsto \beta^t$ to denote them. The (π_1, λ) disintegration of measures on $[0, 1] \times [0, 1]$ will be indexed by the variable t (points of the x axis) and the (π_2, λ) disintegration will be indexed

by the variable s (points of the y axis), where π_i are coordinate projections. We will only consider the (π_i, λ) disintegration of the measures $\eta_\zeta, \eta_{\zeta_1, \zeta_2}$ defined in Eq. (C.12), Eq. (C.13) of chapter II. These disintegrations exist from Thm. II.16 (see [2]). The measures η_ζ^t, η^t are concentrated on $\{t\} \times [0, 1]$ and η_ζ^s, η^s are concentrated on $[0, 1] \times \{s\}$ for λ almost all t, s . We will denote by $\tilde{\eta}_\zeta^t, \tilde{\eta}^t$ the restriction of the measures η_ζ^t and η^t respectively on $\{t\} \times [0, 1]$. Similarly define $\tilde{\eta}_\zeta^s$ and $\tilde{\eta}^s$. Thus $\tilde{\eta}_\zeta^t, \tilde{\eta}^t, \tilde{\eta}_\zeta^s, \tilde{\eta}^s$ can be regarded as measures on $[0, 1]$. This notation will be used in the subsequent sections.

The *left-right-measure* $[\eta]$ of A has the following property. If $\theta : [0, 1] \times [0, 1] \mapsto [0, 1] \times [0, 1]$ is the flip map i.e, $\theta(t, s) = (s, t)$ then $\theta_*\eta \ll \eta \ll \theta_*\eta$ (see Lemma II.11). In fact, it is possible to obtain a choice of η for which $\theta_*\eta = \eta$. So in most of our analysis we will only state theorems with respect to the (π_1, λ) disintegration. An analogous statement with respect to the (π_2, λ) disintegration is also possible, which we won't bother to state.

The discussion that follows will be used in our theorems. Let $\{a_n\}_{n \in \mathbb{N}_\infty}$ be such that

$$\begin{aligned} a_n &> 0 \text{ if } n \in \text{Puk}(A) \\ a_n &= 0 \text{ if } n \notin \text{Puk}(A) \end{aligned}$$

and $\sum_{n \in \mathbb{N}_\infty} a_n = 1$. For $\mathbb{N}_\infty \ni n \in \text{Puk}(A)$, let $E_n \subseteq [0, 1] \times [0, 1] \setminus \Delta([0, 1])$ denote the set where the multiplicity function in $m.m(A)$ takes the value n . It is well known that E_n is η measurable. Then

$$\begin{aligned} L^2(\mathcal{M}) \ominus L^2(A) &\cong \bigoplus_{n \in \text{Puk}(A)} L^2(E_n, \eta|_{E_n}) \otimes \mathbb{C}^n & (A.1) \\ &\cong \bigoplus_{n \in \text{Puk}(A)} \int_{[0,1] \times [0,1]}^\oplus \mathbb{C}_{t,s}^n d\eta|_{E_n}(t, s) \end{aligned}$$

where $\mathbb{C}_{t,s}^n = \mathbb{C}^n$ for $(t, s) \in E_n$ when $n < \infty$, and $\mathbb{C}^\infty = l_2(\mathbb{N})$. Under this decomposition one has

$$\mathcal{A}'(1 - e_A) \cong \bigoplus_{n \in Puk(A)} L^\infty(E_n, \eta|_{E_n}) \otimes \mathcal{M}_n(\mathbb{C}),$$

where $\mathcal{M}_\infty(\mathbb{C})$ is to be interpreted as $\mathbf{B}(l_2(\mathbb{N}))$.

Consequently it follows that for $\mathbb{N}_\infty \ni n \in Puk(A)$ the projections $\chi_{E_n} \otimes 1_n$ lie in $\mathbf{Z}(\mathcal{A}') = \mathcal{A}$, where 1_n denotes the identity of $\mathcal{M}_n(\mathbb{C})$ if $n < \infty$ and $1_\infty = 1_{\mathbf{B}(l_2(\mathbb{N}))}$. For $n \in Puk(A)$ choose vectors $\zeta_i^{(n)}$, $1 \leq i \leq n$ so that the projections $P_i^{(n)} : L^2(\mathcal{M}) \mapsto \overline{A\zeta_i^{(n)}A}^{\|\cdot\|_2}$ are mutually orthogonal, equivalent in \mathcal{A}' , $\sum_{i=1}^n P_i^{(n)} = \chi_{E_n} \otimes 1_n$ and for all $a, b \in C[0, 1] \subset A$

$$\langle a\zeta_i^{(n)}b, \zeta_i^{(n)} \rangle = \begin{cases} \frac{a_n}{n} \int_{E_n} a(t)b(s)d\eta(t, s), & \text{if } n < \infty \\ \frac{a_\infty}{2^i} \int_{E_\infty} a(t)b(s)d\eta(t, s), & \text{if } n = \infty. \end{cases}$$

It follows that for all $a, b \in C[0, 1]$

$$\left\langle a \left(\bigoplus_{n \in Puk(A)} \left(\bigoplus_{i=1}^n \zeta_i^{(n)} \right) \right) b, \bigoplus_{n \in Puk(A)} \left(\bigoplus_{i=1}^n \zeta_i^{(n)} \right) \right\rangle = \int_{[0,1] \times [0,1]} a(t)b(s)d\eta(t, s). \quad (\text{A.2})$$

With abuse of notation we will write $e_A(\zeta) = \mathbb{E}_A(\zeta)$ for L^1 and L^2 vectors.

B. Uniformly Mixing Masas

It is not always easy to describe properties of a singular masa based on its *left-right-measure*. However, we can write nice properties of masas when the *left-right-measure* is the class of product measure. Examples of such masas are easy to give in many situations and many known masas, for example, the single generator masa, the radial masa in the free group factors, the masas that arise out of Bernoulli shift actions of abelian groups belong to this class. In this section we will give analytical

conditions for the *left-right-measure* of a masa in any II_1 factor to be the class of product measure.

The *left-right-measure* of any masa in the interpolated free group factors contains $\lambda \otimes \lambda$ as a summand. This statement of Voiculescu [46] is one of the most important theorem in the subject. This is the precise reason for absence of Cartan subalgebras in the interpolated free group factors. Recently, the authors of [17] distinguished a II_1 factor \mathcal{N} from the interpolated free group factors, by showing \mathcal{N} contains an exotic masa whose *left-right-measure* is singular with respect to $\lambda \otimes \lambda$, yet very close to $\lambda \otimes \lambda$. However, the factor \mathcal{N} in their example resembles almost like the free group factors. In many cases, the *left-right-measures* are hard to calculate. So we need conditions in terms of operators that characterize the Lebesgue class.

Definition IV.1. A masa $A \subset \mathcal{M}$ is said to satisfy the *uniformly mixing* condition if there exists a set $S \subset M$ such that $\mathbb{E}_A(x) = 0$ for all $x \in S$ and

- (i) the linear span of S is dense in $L^2(\mathcal{M}) \ominus L^2(A)$,
- (ii) there is an orthonormal basis $\{v_n\}_{n=1}^\infty \subset A$ of $L^2(A)$ such that

$$\sum_{n=1}^{\infty} \|\mathbb{E}_A(xv_nx^*)\|_2^2 < \infty$$

for all $x \in S$,

- (iii) there is a nonzero vector $\zeta \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\mathbb{E}_A(\zeta u^n \zeta^*) = 0$ for all $n \neq 0$, where u is a Haar unitary generator of A .

In this section we will prove that the uniformly mixing condition is sufficient for the *left-right-measure* of A to be the class of product measure. We do not know whether the same condition is necessary for the *left-right-measure* of any masa to be of the product class. However, in Thm. IV.9 we provide an analogous condition which is necessary for the *left-right-measure* to be of the product class. Note that

the sum in Defn. IV.1 is independent of the choice of the orthonormal basis. This just follows by expanding elements of one orthonormal basis with respect to another. We suspect that condition (iii) in Defn. IV.1 is redundant and follows from the first two conditions. Conditions (i) and (ii) in Defn. IV.1 forces that $\eta \ll \lambda \otimes \lambda$. To assure $\lambda \otimes \lambda \ll \eta$ we need condition (iii). To prove that uniformly mixing condition implies the *left-right-measure* to be the class of product measure, we need to prove some auxiliary lemmas. In Section D of chapter III we proved a result similar to the following.

Lemma IV.2. *Let $\zeta_1, \zeta_2 \in L^2(\mathcal{M})$ be such that $\mathbb{E}_A(\zeta_1) = 0 = \mathbb{E}_A(\zeta_2)$. Let η_{ζ_1, ζ_2} denote the measure on $[0, 1] \times [0, 1]$ defined in Eq. C.13 of chapter II.*

1°. *Then η_{ζ_1, ζ_2} admits (π_i, λ) disintegrations $[0, 1] \ni t \mapsto \eta_{\zeta_1, \zeta_2}^t$ and $[0, 1] \ni s \mapsto \eta_{\zeta_1, \zeta_2}^s$, where $\pi_i, i = 1, 2$ denotes the coordinate projections. Moreover,*

$$\eta_{\zeta_1, \zeta_2}^t([0, 1] \times [0, 1]) = \mathbb{E}_A(\zeta_1 \zeta_2^*)(t), \quad \lambda \text{ a.e.}$$

2°. *Let $f \in C[0, 1]$. Then the functions $[0, 1] \ni t \mapsto \eta_{\zeta_1, \zeta_2}^t(1 \otimes f), [0, 1] \ni s \mapsto \eta_{\zeta_1, \zeta_2}^s(f \otimes 1)$ are in $L^1([0, 1], \lambda)$.*

If $\zeta_i \in \mathcal{M}$ for $i = 1, 2$ then $[0, 1] \ni t \mapsto \eta_{\zeta_1, \zeta_2}^t(1 \otimes f), [0, 1] \ni s \mapsto \eta_{\zeta_1, \zeta_2}^s(f \otimes 1)$ are in $L^\infty([0, 1], \lambda)$.

3°. *Let $b, w \in C[0, 1]$. If $\mathbb{E}_A(\zeta_1 w \zeta_2^*) \in L^2(A)$ then*

$$\|\mathbb{E}_A(b \zeta_1 w \zeta_2^*)\|_2^2 = \int_0^1 |b(t)|^2 |\eta_{\zeta_1, \zeta_2}^t(1 \otimes w)|^2 d\lambda(t).$$

Proof. 1°. That η_{ζ_1, ζ_2} admits the stated disintegrations follows from Eq. (C.14) of chapter II, Lemma 5.7 [11] and Lemma II.11. The next statement in 1° follows from an argument similar to the proof of Lemma III.32.

2°. From Eq. C.15 of chapter II, $|\eta_{\zeta_1, \zeta_2}|$ admits (π_i, λ) disintegrations. Use Hahn

decomposition of measures and Lemma II.20 to see that $|\eta_{\zeta_1, \zeta_2}^t|^t = |\eta_{\zeta_1, \zeta_2}^t|$ for λ almost all t . The function $t \mapsto \eta_{\zeta_1, \zeta_2}^t(1 \otimes f)$ is clearly measurable from Defn. II.15 and from Eq. (C.15) of chapter II,

$$\begin{aligned} & \int_0^1 |\eta_{\zeta_1, \zeta_2}^t(1 \otimes f)| d\lambda(t) \\ & \leq \|f\| \int_0^1 |\eta_{\zeta_1, \zeta_2}^t| ([0, 1] \times [0, 1]) d\lambda(t) \\ & \leq \|f\| \left(\int_0^1 \eta_{\zeta_1}^t([0, 1] \times [0, 1]) d\lambda(t) + \int_0^1 \eta_{\zeta_2}^t([0, 1] \times [0, 1]) d\lambda(t) \right) \\ & = \|f\| (\|\mathbb{E}_A(\zeta_1 \zeta_1^*)\|_1 + \|\mathbb{E}_A(\zeta_2 \zeta_2^*)\|_1) < \infty. \end{aligned}$$

When $\zeta_i \in \mathcal{M}$ a similar argument shows the stated functions are in $L^\infty([0, 1], \lambda)$.

3° Since

$$\begin{aligned} \infty & > \sup_{a \in C[0,1], \|a\|_2 \leq 1} \left| \int_0^1 a(t)b(t)\mathbb{E}_A(\zeta_1 w \zeta_2^*)(t) d\lambda(t) \right| \\ & = \sup_{a \in C[0,1], \|a\|_2 \leq 1} |\tau(ab\mathbb{E}_A(\zeta_1 w \zeta_2^*))| \\ & = \sup_{a \in C[0,1], \|a\|_2 \leq 1} |\tau(ab\zeta_1 w \zeta_2^*)| \\ & = \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \leq 1}} \left| \int_0^1 a(t)b(t)\eta_{\zeta_1, \zeta_2}^t(1 \otimes w) d\lambda(t) \right| \end{aligned}$$

and $t \xrightarrow{g} b(t)\eta_{\zeta_1, \zeta_2}^t(1 \otimes w)$ is in $L^1(\lambda)$ so g is in $L^2(\lambda)$ and

$$\|\mathbb{E}_A(b\zeta_1 w \zeta_2^*)\|_2^2 = \int_0^1 |b(t)|^2 |\eta_{\zeta_1, \zeta_2}^t(1 \otimes w)|^2 d\lambda(t).$$

□

Let $w := \{w_n\}_{n=1}^\infty \subset C[0, 1]$ be an orthonormal basis of $L^2(A)$.

Proposition IV.3. *Let $x_i \in L^2(\mathcal{M})$ for $i = 1, 2$ be such that $\mathbb{E}_A(x_i) = 0$. Let us*

suppose that

$$\sum_{n=1}^{\infty} \|\mathbb{E}_A(x_1 w_n x_2^*)\|_2^2 < \infty.$$

If $w' := \{w'_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $L^2(A)$ with $w'_n \in C[0, 1]$ for all n , then there is a set $F(w, w') \subset [0, 1]$ which depends on w, w' such that $\lambda(F(w, w')) = 0$ and for all $t \in F(w, w')^c$

$$\sum_{n=1}^{\infty} |\eta_{x_1, x_2}^t(1 \otimes w'_n)|^2 \leq \sum_{n=1}^{\infty} |\eta_{x_1, x_2}^t(1 \otimes w_n)|^2 < \infty.$$

Proof. Note that the hypothesis implies that for any $a \in C[0, 1]$,

$$\sum_{n=1}^{\infty} \|\mathbb{E}_A(ax_1 w_n x_2^*)\|_2^2 < \infty$$

and this sum is independent of the choice of the orthonormal basis. Therefore for all $a \in C[0, 1]$

$$\sum_{n=1}^{\infty} \|\mathbb{E}_A(ax_1 w'_n x_2^*)\|_2^2 \leq \sum_{n=1}^{\infty} \|\mathbb{E}_A(ax_1 w_n x_2^*)\|_2^2.$$

Let $r \in A$ be a nonzero projection. Identify r with a measurable subset E_r of $[0, 1]$. We can assume E_r is a Borel set. We claim that

$$\int_{E_r} \sum_{n=1}^{\infty} |\eta_{x_1, x_2}^t(1 \otimes w'_n)|^2 d\lambda(t) \leq \int_{E_r} \sum_{n=1}^{\infty} |\eta_{x_1, x_2}^t(1 \otimes w_n)|^2 d\lambda(t). \quad (\text{B.1})$$

If the claim is true then by standard measure theory arguments we are done.

First assume E_r is a compact set. Choose a sequence of continuous functions f_l such that $0 \leq f_l \leq 1$ and $f_l \downarrow \chi_{E_r}$ pointwise as $l \rightarrow \infty$. Therefore by Lemma IV.2 and monotone convergence theorem, for all l we have,

$$\int_0^1 f_l^2(t) \sum_{n=1}^{\infty} |\eta_{x_1, x_2}^t(1 \otimes w'_n)|^2 d\lambda(t) = \sum_{n=1}^{\infty} \|\mathbb{E}_A(f_l x_1 w'_n x_2^*)\|_2^2$$

$$\begin{aligned}
&\leq \sum_{n=1}^{\infty} \|\mathbb{E}_A(f_1 x_1 w_n x_2^*)\|_2^2 \\
&= \int_0^1 f_1^2(t) \sum_{n=1}^{\infty} |\eta_{x_1, x_2}^t(1 \otimes w_n)|^2 d\lambda(t).
\end{aligned}$$

Passing to limits we see that Eq. (B.1) is true whenever E_r is compact. Now use regularity of λ so see that Eq. (B.1) is true for all Borel sets of positive measure. \square

Let $X = \{\underline{f} = (f_1, f_2, f_3, \dots) : f_k \in C[0, 1] \forall k \in \mathbb{N}\}$. Equip X with the metric d given by

$$d(\underline{f}, \underline{g}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f_k - g_k\|_{\infty}}{1 + \|f_k - g_k\|_{\infty}}. \quad (\text{B.2})$$

Then (X, d) is a separable metric space. Also for a sequence $\underline{f}^{(n)} \in X$, $\underline{f}^{(n)} \xrightarrow{d} \underline{f}$ as $n \rightarrow \infty$ implies that $f_k^{(n)} \rightarrow f_k$ in $\|\cdot\|_{\infty}$ for all $k \in \mathbb{N}$.

Let $\mathcal{O} = \{\underline{f} \in X : \{f_k\}_{k=1}^{\infty} \text{ be an orthonormal sequence in } L^2([0, 1], \lambda)\}$. Then $\mathcal{O} \subset (X, d)$ is a closed set. Note that (\mathcal{O}, d) is separable.

Proposition IV.4. *Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. Let us suppose that*

$$\sum_{k=1}^{\infty} \|\mathbb{E}_A(x w_k x^*)\|_2^2 < \infty.$$

Then $\eta_x \ll \lambda \otimes \lambda$.

Proof. Let $\{\underline{w}^{(m)}\}_{m=1}^{\infty} \subset (\mathcal{O}, d)$ be any countable dense set. From Prop. IV.3 and Lemma IV.2, it follows that there is a set $F \subset [0, 1]$ with $\lambda(F) = 0$ such that for $t \in F^c$, η_x^t is a finite measure and

$$\sum_{k=1}^{\infty} \left| \eta_x^t(1 \otimes w_k^{(m)}) \right|^2 \leq \sum_{k=1}^{\infty} |\eta_x^t(1 \otimes w_k)|^2 < \infty$$

for all $m \in \mathbb{N}$. Let $\underline{v} = \{v_k\}_{k=1}^{\infty} \in \mathcal{O}$. There exists a subsequence $\{\underline{w}^{(m_j)}\}_{j=1}^{\infty}$ such

that $d(\underline{w}^{(m_j)}, \underline{v}) \rightarrow 0$ as $j \rightarrow \infty$. Therefore for $t \in F^c$

$$\begin{aligned}
\sum_{k=1}^{\infty} |\eta_x^t(1 \otimes v_k)|^2 &= \sum_{k=1}^{\infty} \lim_j \left| \eta_x^t(1 \otimes w_k^{(m_j)}) \right|^2 \quad (\text{by Dominated convergence}) \\
&= \sum_{k=1}^{\infty} \liminf_j \left| \eta_x^t(1 \otimes w_k^{(m_j)}) \right|^2 \\
&\leq \liminf_j \sum_{k=1}^{\infty} \left| \eta_x^t(1 \otimes w_k^{(m_j)}) \right|^2 \quad (\text{by Fatou's Lemma}) \\
&\leq \sum_{k=1}^{\infty} |\eta_x^t(1 \otimes w_k)|^2 < \infty \quad (\text{as } t \in F^c).
\end{aligned}$$

Therefore for each $t \in F^c$,

$$\sup_{f \in \mathcal{O}} \sum_{k=1}^{\infty} |\eta_x^t(1 \otimes f_k)|^2 \leq \sum_{k=1}^{\infty} |\eta_x^t(1 \otimes w_k)|^2 < \infty. \quad (\text{B.3})$$

Fix $t \in F^c$. If $\tilde{\eta}_x^t$ contains a part which is singular with respect to λ then the supremum on the left hand side of Eq. (B.3) is infinite. Indeed, for simplicity assume $\tilde{\eta}_x^t \perp \lambda$. Choose a compact set $K \subset [0, 1]$ of almost full $\tilde{\eta}_x^t$ measure such that $\lambda(K) = 0$. Fix a large positive number N . By regularity of λ , there is a open set U containing K such that $\lambda(U) < \frac{1}{N^8}$. Using compactness of K we can find a finite number of open intervals (a_i, b_i) and small positive numbers δ_i for $i = 1, 2, \dots, m$ such that the open intervals $\{(a_i - \delta_i, b_i + \delta_i)\}_{i=1}^m$ are disjoint and $K \subset \cup_{i=1}^m (a_i, b_i) \subset \cup_{i=1}^m (a_i - \delta_i, b_i + \delta_i) \subset U$. Define

$$f_i(s) = \begin{cases} N & \text{if } a_i \leq s \leq b_i \\ \frac{N}{\delta_i}(s - a_i) + N & \text{if } a_i - \delta_i \leq s \leq a_i \\ -\frac{N}{\delta_i}(s - b_i) + N & \text{if } b_i \leq s \leq b_i + \delta_i \\ 0 & \text{otherwise.} \end{cases}$$

Then $f = \sum_{i=1}^m f_i$ is continuous and $\|f\|_{2,\lambda} = O(\frac{1}{N^3})$. Now consider $g = \frac{f}{\|f\|_{2,\lambda}}$.

Inductively construct an orthonormal sequence in $C[0, 1]$ with the first function as g , orthogonal with respect to λ measure. It is now clear that in this way the supremum in Eq. (B.3) can be made to exceed any large number.

Consequently, it follows that for all $t \in F^c$

$$\tilde{\eta}_x^t \ll \lambda. \quad (\text{B.4})$$

Finally from Lemma II.20 it follows that $\eta_x \ll \lambda \otimes \lambda$. \square

Remark IV.5. Note that the proof of Lemma IV.4 actually shows that $\tilde{\eta}_x^t \ll \lambda$ with $\frac{d\tilde{\eta}_x^t}{d\lambda} \in L^2([0, 1], \lambda)$ for λ almost all t .

The set of finite signed measures on the measurable space (X, σ_X) is a Banach space equipped with the total variation norm $\|\cdot\|_{t.v.}$, also called the L_1 -norm, which is defined by $\|\mu\|_{t.v.} = |\mu|(X)$ where $|\mu|$ denotes the variation measure of μ . It is well known that for probability measures μ, ν

$$\|\mu - \nu\|_{t.v.} = 2 \sup_{B \in \sigma_X} |\mu(B) - \nu(B)| = \int_X |f - g| d\gamma \quad (\text{B.5})$$

where f, g are density functions of μ, ν respectively with respect to **any** σ -finite measure γ dominating both μ, ν (see [29]).

Theorem IV.6. *Let $A \subset \mathcal{M}$ be a masa satisfying the uniformly mixing condition. Then the left-right-measure of A is the class of product measure.*

Proof. Fix a set $S \subset \mathcal{M}$ such that $\mathbb{E}_A(x) = 0$ for all $x \in S$, $\overline{\text{span } S}^{\|\cdot\|_2} = L^2(A)^\perp$ and

$$\sum_{k=1}^{\infty} \|\mathbb{E}_A(xu_kx^*)\|_2^2 < \infty$$

for all $x \in S$, where $\{u_k\}_{k=1}^{\infty} \subset C[0, 1]$ is an orthonormal basis of $L^2(A)$. From Eq. (A.2) there is a vector $\zeta \in L^2(A)^\perp$ such that $\|\zeta\|_2 = 1$ and $\eta_\zeta = \eta$. Choose a sequence $x_n \in \text{span } S$ such that $\|x_n\|_2 = 1$ and $x_n \rightarrow \zeta$ in $\|\cdot\|_2$ as $n \rightarrow \infty$. Then (Lemma

II.24), we have $\eta_{x_n} \rightarrow \eta_\zeta = \eta$ in $\|\cdot\|_{t,v}$. Write $x_n = \sum_{i=1}^{k_n} c_{i,n} y_{i,n}$ with $y_{i,n} \in S$, $c_{i,n} \in \mathbb{C}$ for all $1 \leq i \leq k_n$ and $n \in \mathbb{N}$. As $y_{i,n} \in S$, so for all $n \in \mathbb{N}$, $1 \leq i \leq k_n$

$$\sum_{k=1}^{\infty} \|\mathbb{E}_A(y_{i,n} u_k y_{i,n}^*)\|_2^2 < \infty.$$

From Prop. IV.4 we have $\eta_{y_{i,n}} \ll \lambda \otimes \lambda$. But

$$\eta_{x_n} = \sum_{i=1}^{k_n} |c_{i,n}|^2 \eta_{y_{i,n}} + \sum_{i \neq j=1}^{k_n} c_{i,n} \bar{c}_{j,n} \eta_{y_{i,n}, y_{j,n}}.$$

For $1 \leq i \neq j \leq k_n$ the measures $\eta_{y_{i,n}, y_{j,n}}$ are possibly complex measures but from Eq. C.15 of chapter II, $|\eta_{y_{i,n}, y_{j,n}}| \leq \eta_{y_{i,n}} + \eta_{y_{j,n}} \ll \lambda \otimes \lambda$. Therefore $\eta_{x_n} \ll \lambda \otimes \lambda$. Since η_{x_n} are probability measures so from Eq. (B.5)

$$\frac{1}{2} \|\eta_{x_n} - \eta_{x_m}\|_{t,v} = \int_{[0,1] \times [0,1]} |f_n(t, s) - f_m(t, s)| d(\lambda \otimes \lambda)(t, s) \rightarrow 0$$

as $n, m \rightarrow \infty$, where $f_n = \frac{d\eta_{x_n}}{d(\lambda \otimes \lambda)}$. Thus there is a function $f \in L^1([0, 1] \times [0, 1], \lambda \otimes \lambda)$ such that

$$\int_{[0,1] \times [0,1]} |f_n(t, s) - f(t, s)| d(\lambda \otimes \lambda)(t, s)$$

as $n \rightarrow \infty$. As η_{x_n} is a probability measure for each n so $\|f_n\|_{L^1(\lambda \otimes \lambda)} = 1$ for all n . Therefore $\|f\|_{L^1(\lambda \otimes \lambda)} = 1$ and $\eta_{x_n} \rightarrow f d(\lambda \otimes \lambda)$ in $\|\cdot\|_{t,v}$. By uniqueness of limits $\eta = f d(\lambda \otimes \lambda)$.

We will now use condition (iii) of Defn. IV.1 to show that $\lambda \otimes \lambda \ll \eta$. Let $v \in A$ be the Haar unitary corresponding to the function $t \mapsto e^{2\pi i t}$. Suppose $\xi \in L^2(\mathcal{M}) \ominus L^2(A)$ is a nonzero vector such that $\mathbb{E}_A(\xi v^n \xi^*) = 0$ for all $n \neq 0$. Then by 3° of Lemma IV.2 we have

$$\eta_\xi^t(1 \otimes v^n) = 0 \text{ for all } n \neq 0 \text{ and for } \lambda \text{ almost all } t.$$

By standard theorems in Fourier analysis it follows that $\tilde{\eta}_\xi^t$ is equal to λ for λ almost all t . Finally by Lemma 5.7 [11] we have $[\eta] = [\lambda \otimes \lambda]$. \square

The proof of the last part of Thm. IV.6 can be summarized in the corollary.

Corollary IV.7. *Let $A \subset \mathcal{M}$ be a masa such that the left-right-measure of A contains the product class as a summand. Then there is a nonzero $\xi \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\mathbb{E}_A(\xi v^n \xi^*) = 0$ for all $n \neq 0$ where v is a Haar unitary generator of A .*

The next result is strengthening the *uniformly mixing* condition. The idea of its proof is hidden in the proof of Thm. IV.6.

Theorem IV.8. *Let $A \subset \mathcal{M}$ be a masa. Then the following are equivalent.*

- (1) *A satisfies (i) and (ii) of the uniformly mixing condition.*
- (2) *There exists a set $D \subset \mathcal{M}$ such that $\mathbb{E}_A(x) = 0$ for all $x \in D$, D is dense in $L^2(A)^\perp$ and*

$$\sum_{k=1}^{\infty} \|\mathbb{E}_A(x_1 v_k x_2^*)\|_2^2 < \infty \text{ for all } x_1, x_2 \in D$$

for some orthonormal basis $\{v_k\} \subset C[0, 1]$ of $L^2(A)$.

Proof. (2) \Rightarrow (1) is obvious.

(1) \Rightarrow (2). Let $v \in A$ be the Haar unitary corresponding to the function $t \mapsto e^{2\pi i t}$. From Thm. IV.6 we get that the *left-right-measure* of A is dominated by $\lambda \otimes \lambda$. Let $x_1 = \sum_{i=1}^n c_i y_i^1$, $x_2 = \sum_{j=1}^m d_j y_j^2$ with $y_i^1, y_j^2 \in S$, $c_i, d_j \in \mathbb{C}$ for all $1 \leq i \leq n$, $1 \leq j \leq m$. Note that for all i, j

$$\sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(y_i^1 v^k y_i^{1*})\|_2^2 < \infty, \sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(y_j^2 v^k y_j^{2*})\|_2^2 < \infty.$$

Then $\eta_{y_i^1}, \eta_{y_j^2} \ll \lambda \otimes \lambda$ with $\frac{d\eta_{y_i^1}}{d(\lambda \otimes \lambda)}, \frac{d\eta_{y_j^2}}{d(\lambda \otimes \lambda)} \in L^2(\lambda \otimes \lambda)$ (see Rem. IV.5 and Lemma II.20). As argued in the proof of Thm. IV.6, $\eta_{y_i^1, y_j^2} \ll \lambda \otimes \lambda$. But because

$|\eta_{y_i^1, y_j^2}| \leq \eta_{y_i^1} + \eta_{y_j^2}$ so we conclude that $f_{x_1, x_2} = \frac{d\eta_{x_1, x_2}}{d(\lambda \otimes \lambda)} \in L^2(\lambda \otimes \lambda)$. Therefore

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} \|\mathbb{E}_A(x_1 v^k x_2^*)\|_2^2 &= \sum_{k \in \mathbb{Z}} \int_0^1 |\eta_{x_1, x_2}^t(1 \otimes v^k)|^2 d\lambda(t) \\
&= \int_0^1 \sum_{k \in \mathbb{Z}} |\eta_{x_1, x_2}^t(1 \otimes v^k)|^2 d\lambda(t) \\
&= \int_0^1 \sum_{k \in \mathbb{Z}} \left| \int_0^1 v^k(s) d\eta_{x_1, x_2}^t(s) \right|^2 d\lambda(t) \\
&= \int_0^1 \sum_{k \in \mathbb{Z}} \left| \int_0^1 f_{x_1, x_2}(t, s) v^k(s) d\lambda(s) \right|^2 d\lambda(t) \text{ (Lemma II.20)} \\
&= \int_0^1 \|f_{x_1, x_2}(t, \cdot)\|_{L^2(\lambda)}^2 d\lambda(t) \\
&= \int_{[0,1] \times [0,1]} |f_{x_1, x_2}(t, s)|^2 d(\lambda \otimes \lambda)(t, s) < \infty.
\end{aligned} \tag{B.6}$$

Finally, let $D = \text{span } S$. □

When the *left-right-measure* of a masa belongs to the class of product measure, the masa satisfies a condition very close to the uniformly mixing condition. This is the content of the next theorem.

Theorem IV.9. *Let $A \subset \mathcal{M}$ be a masa. Let the left-right-measure of A be the class of product measure. Then there is a set $S \subset L^2(\mathcal{M}) \ominus L^2(A)$ such that $\text{span } S$ is dense in $L^2(A)^\perp$,*

$$\sum_{n=1}^{\infty} \|\mathbb{E}_A(\zeta w_n \zeta^*)\|_2^2 < \infty \text{ for all } \zeta \in S,$$

for some orthonormal basis $\{w_n\}_{n=1}^{\infty} \subset A$ of $L^2(A)$ and there is a nonzero $\xi \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\mathbb{E}_A(\xi v^n \xi^*) = 0$ for all $n \neq 0$ where v is a Haar unitary generator of A .

Proof. We will first consider the case $\text{Puk}(A) = \{1\}$. In this case

$$L^2(\mathcal{M}) \ominus L^2(A) \cong L^2([0, 1] \times [0, 1] \setminus \Delta([0, 1]), \lambda \otimes \lambda),$$

the left and the right actions of A being given by

$$(af)(t, s) = a(t)f(t, s), (fb)(t, s) = b(s)f(t, s)$$

where $f \in L^2(A)^\perp$ and $a, b \in A$.

Let $0 \neq \zeta \in L^2(A)^\perp$ be a continuous function. Then for $a, b \in C[0, 1]$

$$\begin{aligned} \langle a\zeta b, \zeta \rangle_{L^2(\mathcal{M})} &= \langle a\zeta b, \zeta \rangle_{L^2(\lambda \otimes \lambda)} \\ &= \int_{[0,1] \times [0,1]} a(t)b(s)\zeta(t, s)\overline{\zeta(t, s)}d\lambda(t)d\lambda(s) \\ &= \int_{[0,1] \times [0,1]} a(t)b(s)|\zeta(t, s)|^2 d\lambda(t)d\lambda(s). \end{aligned}$$

Therefore $\frac{d\eta_\zeta}{d(\lambda \otimes \lambda)} = |\zeta|^2$ which is bounded, in particular in $L^2(\lambda \otimes \lambda)$. We claim that $\mathbb{E}_A(\zeta b \zeta^*) \in L^2(A)$ for any $b \in C[0, 1]$. Fix $a \in C[0, 1]$. Then

$$\begin{aligned} \int_0^1 a(t)\mathbb{E}_A(\zeta b \zeta^*)(t)d\lambda(t) &= \tau(a\mathbb{E}_A(\zeta b \zeta^*)) \quad (\tau \text{ extends to } L^1) & (B.7) \\ &= \tau(a\zeta b \zeta^*) \\ &= \int_{[0,1] \times [0,1]} a(t)b(s)d\eta_\zeta(t, s) \\ &= \int_{[0,1] \times [0,1]} a(t)b(s)|\zeta|^2(t, s)d\lambda(t)d\lambda(s) \\ &= \int_0^1 a(t)\lambda(|\zeta|^2(t, \cdot)b)d\lambda(t). \end{aligned}$$

Now consider the function $[0, 1] \ni t \xrightarrow{g} \lambda(|\zeta|^2(t, \cdot)b)$. It is clearly λ -measurable and

$$\int_0^1 |\lambda(|\zeta|^2(t, \cdot)b)|^2 d\lambda(t) = \int_0^1 \left(\int_0^1 |\zeta|^2(t, s)b(s)d\lambda(s) \right)^2 d\lambda(t)$$

$$\begin{aligned}
&\leq \|b\|^2 \int_0^1 \left(\int_0^1 |\zeta|^2(t, s) d\lambda(s) \right)^2 d\lambda(t) \\
&\leq \|b\|^2 \int_0^1 \int_0^1 |\zeta|^4(t, s) d\lambda(t) d\lambda(s) < \infty.
\end{aligned}$$

Therefore from Eq. (B.7) we get,

$$\begin{aligned}
\sup_{a \in C[0,1], \|a\|_2 \leq 1} \left| \int_0^1 a(t) \mathbb{E}_A(\zeta b \zeta^*)(t) d\lambda(t) \right| &= \sup_{a \in C[0,1], \|a\|_2 \leq 1} \left| \int_0^1 a(t) \lambda(|\zeta|^2(t, \cdot) b) d\lambda(t) \right| \\
&= \left(\int_0^1 |\lambda(|\zeta|^2(t, \cdot) b)|^2 d\lambda(t) \right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Consequently it follows that $\mathbb{E}_A(\zeta b \zeta^*) \in L^2(A)$ and

$$\|\mathbb{E}_A(\zeta b \zeta^*)\|_2^2 = \int_0^1 |\lambda(|\zeta|^2(t, \cdot) b)|^2 d\lambda(t).$$

Let $v \in A$ be the Haar unitary corresponding to the function $t \mapsto e^{2\pi i t}$. Then $\{v^n\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $L^2(A)$ and by Plancherel's theorem,

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^n \zeta^*)\|_2^2 &= \sum_{n \in \mathbb{Z}} \int_0^1 |\lambda(|\zeta|^2(t, \cdot) v^n)|^2 d\lambda(t) \\
&= \int_0^1 \sum_{n \in \mathbb{Z}} |\lambda(|\zeta|^2(t, \cdot) v^n)|^2 d\lambda(t) \\
&= \int_0^1 \int_0^1 |\zeta|^4(t, s) d\lambda(s) d\lambda(t) < \infty.
\end{aligned}$$

Thus $\{\zeta \in L^2(A)^\perp : \sum_{n \in \mathbb{Z}} \|\mathbb{E}_A(\zeta v^n \zeta^*)\|_2^2 < \infty\}$ is dense in $L^2(A)^\perp$.

In the general case, write

$$L^2(\mathcal{M}) \ominus L^2(A) = \bigoplus_{n \in Puk(A)} \left(\overline{\bigoplus_{i=1}^n A \zeta_i^{(n)} A} \right)^{\|\cdot\|_2}$$

where $\zeta_i^{(n)}$ are vectors defined in Sec. 1. For each $n \in Puk(A)$ and $1 \leq i \leq n$ we consider the left and right actions of A on $\overline{A \zeta_i^{(n)} A}^{\|\cdot\|_2}$ to reduce the problem to a case similar to having one bicyclic vector.

Finally, let $\zeta \in L^2(\mathcal{M})$ correspond to the function $\chi_{\Delta([0,1]^c)}$. Then $\eta_\zeta = \lambda \otimes \lambda$. By arguments exactly similar to the first part of the proof conclude that $\mathbb{E}_A(\zeta a \zeta^*) \in L^2(A)$ for all $a \in A$. But by 3° of Lemma IV.2 we get,

$$\|\mathbb{E}_A(\zeta v^n \zeta^*)\|_2^2 = \int_0^1 |\eta_\zeta^t(1 \otimes v^n)|^2 d\lambda(t) = 0 \text{ for all } n \neq 0.$$

□

C. Strongly Mixing Masas

The study of strongly mixing masas was initiated by Jolissaint and Stalder in [18]. Their study was motivated from an algebraic point of view namely, inclusion of groups and dynamical systems. Their approach is a way to generate nice examples of such masas. In this section we study the same from a measure theoretic point of view.

Recall from [18] that a subset $E \subset \mathcal{U}(\mathcal{M})$ is said to be *almost orthonormal* if for every $\epsilon > 0$ and $\phi \in \mathcal{M}_*$ there is a finite subset $F \subset E$ such that $|\phi(u)| < \epsilon$ for all $u \in E \setminus F$. An *almost orthonormal* subset is necessarily countable since \mathcal{M} is separable. As the definition says, such sets are *weakly null* i.e, elements of these sets converge in *w.o.t* to 0. See Prop. 2.4 of [18] for more information.

Definition IV.10. [18] Let $A \subset \mathcal{M}$ be a diffuse abelian subalgebra. Then A is said to be *strongly mixing* in \mathcal{M} if for all *almost orthonormal* subgroups $G \subset \mathcal{U}(A)$,

$$\lim_{u \rightarrow \infty, u \in G} \|\mathbb{E}_A(uxu^*y) - \mathbb{E}_A(x)\mathbb{E}_A(y)\|_2 = 0$$

for all $x, y \in \mathcal{M}$.

In [18] it was shown that such an abelian algebra is automatically a singular masa.

Theorem IV.11. *Let $A \subset \mathcal{M}$ be a diffuse abelian subalgebra that satisfies the uniformly mixing condition. Then for any almost orthonormal sequence $\{u_n\}_{n=1}^\infty \subset A$ and $x, y \in \mathcal{M}$ such that $\mathbb{E}_A(x) = \mathbb{E}_A(y) = 0$,*

$$\|\mathbb{E}_A(u_n^* x u_n y)\|_2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In particular, A is a strongly mixing masa.

A slightly different version of this statement is proved in Sec 11.4 of [43], so we skip its proof. We will show that masas for which the *left-right-measure* is the class of product measure are strongly mixing. This will give a second proof of the fact that the radial (laplacian) masa in $L(\mathbb{F}_k)$, $2 \leq k < \infty$ is *strongly mixing* as its *left-right-measure* is the class of product measure. The calculation of the *left-right-measure* of the radial masa follows directly from Rădulescu's calculation (Lemma 3, [38]).

Theorem IV.12. *Let Γ be an icc group and let \mathbb{Z} be an infinite abelian subgroup of Γ . Suppose $L(\mathbb{Z}) \subset L(\Gamma)$ is a strongly mixing masa. Then the left-right-measure of $L(\mathbb{Z})$ is the class of product measure.*

In particular, if \mathbb{Z} is a malnormal subgroup of Γ then the left-right-measure of $L(\mathbb{Z})$ is the class of product measure.

Proof. In [18] it was shown that the hypothesis is equivalent to the following condition:

(ST) For every finite subset $F \subset \Gamma \setminus \mathbb{Z}$, there exists a finite subset E of \mathbb{Z} such that $gg_0h \notin \mathbb{Z}$ for all $g_0 \in \mathbb{Z} \setminus E$ and all $g, h \in F$.

Now S^1 is the character group of \mathbb{Z} . Let μ be the normalized Haar measure on S^1 . The *left-right-measure* of $L(\mathbb{Z})$ is naturally supported on $S^1 \times S^1$. Let $u_g \in L(\Gamma)$ be the unitary operator corresponding to the group element $g \in \Gamma$. Fix $g \in \Gamma \setminus \mathbb{Z}$. Then taking $F = \{g, g^{-1}\}$, there is a finite subset E of \mathbb{Z} such that $\mathbb{E}_{L(\mathbb{Z})}(u_g u_h u_g^*) = 0$

for all $h \in \mathbb{Z} \setminus E$. Therefore $\eta_{u_g}^t(1 \otimes \check{h}) = 0$ for μ almost all $t \in S^1$ and $h \in \mathbb{Z} \setminus E$, where \check{h} is the canonical image of h in $C(S^1)$. By a theorem of F. Riesz and M. Riesz (see for instance [15]), it follows that $\tilde{\eta}_{u_g}^t \ll \mu$ for μ almost all t . But note that $\frac{\tilde{\eta}_{u_g}^t}{d\mu}$ is a trigonometric polynomial for μ almost all t . Thus for $\mu(\{s \in S^1 : \frac{\tilde{\eta}_{u_g}^t}{d\mu}(s) = 0\}) = 0$ for μ almost all t . Thus $\mu \ll \tilde{\eta}_{u_g}^t$ for μ almost all t . By Lemma II.20 we have $\eta_{u_g} \ll \mu \otimes \mu \ll \eta_{u_g}$ for each $g \in \Gamma \setminus \mathbb{Z}$.

Now $\text{span}\{u_g : g \in \Gamma \setminus \mathbb{Z}\}$ is dense in $L^2(L(\mathbb{Z}))^\perp$ in $\|\cdot\|_2$. By approximation arguments it follows that the *left-right-measure* of $L(\mathbb{Z})$ is mutually absolutely continuous with respect to $\mu \otimes \mu$.

The last statement follows by observing that $\mathbb{E}_{L(\mathbb{Z})}(u_g u_h u_g^*) = 0$ for all $h \in \mathbb{Z} \setminus \{0\}$ and $g \in \Gamma \setminus \mathbb{Z}$. \square

The next theorem is a measure theoretic formulation the main results of [18]. Recall from chapter III, a finite measure μ on the $[0, 1]$ is called *mixing* (or sometimes *Rajchman*) if its Fourier coefficients $\hat{\mu}_n = \int_0^1 e^{2\pi i n t} d\mu(t)$ converge to 0 as $n \rightarrow \pm\infty$.

There is also an analogous definition of mixing measures on S^1 . By the Riemann-Lebesgue lemma any measure absolutely continuous with respect to Lebesgue measure is mixing. However there are many mixing singular measures as well. Any measure absolutely continuous with respect to a mixing measure is mixing. Thus mixing is a property of equivalence class of measures. Mixing measures can be characterized in a geometric way as being asymptotically uniformly distributed.

Let N be a diffuse, separable, finite von Neumann algebra equipped with a faithful, normal tracial state τ . Let Γ be a countable discrete group which acts on N by a τ -preserving action α . The action α is said to be strongly mixing [18] if, given $\epsilon > 0$ there exists a finite set $E \subset \Gamma$ such that

$$|\tau(\alpha_g(x)y) - \tau(x)\tau(y)| < \epsilon \text{ for all } g \in \Gamma \setminus E \text{ and for all } x, y \in N.$$

In [18] it was shown that, if Γ is abelian then, $L(\Gamma) \subset N \rtimes_{\alpha} \Gamma$ is a strongly mixing masa if and only if the action α is strongly mixing. In the next theorem, we relate strongly mixing actions of countable discrete abelian groups to Fourier coefficients of the *left-right-measure* of $L(\Gamma)$. Since we are not very familiar with abstract Harmonic Analysis on groups, we will assume $\Gamma = \mathbb{Z}$.

Theorem IV.13. *Let α be a free strongly mixing action of \mathbb{Z} on a diffuse, separable, finite von Neumann algebra N , preserving a faithful, normal tracial state τ . If $[\eta]$ is the left-right-measure of $L(\mathbb{Z}) \subset N \rtimes_{\alpha} \mathbb{Z}$ then, $\tilde{\eta}^t$ is a mixing measure for λ almost all t .*

Proof. Let $\mathcal{M} = N \rtimes_{\alpha} \mathbb{Z}$. The tracial state on \mathcal{M} will be denoted by τ as well. The elements of \mathcal{M} has a Fourier expansion of the form $x = \sum_{n \in \mathbb{Z}} x_n u_n$ where $x_n \in N$ and $u_n \in L(\mathbb{Z})$ are the canonical unitaries implementing the action. The Fourier expansion of x converges in $\|\cdot\|_2$. Suppose $x \in N$ and $n, n_1, n_2 \in \mathbb{Z}$. Then the equation

$$\begin{aligned} \langle u_{n_1} x u_n u_{n_2}, x u_n \rangle &= \tau(u_{n_1} x u_n u_{n_2} u_{-n} x^*) \\ &= \tau(u_{n_1} x u_{n_2} x^*) \end{aligned} \tag{C.1}$$

implies that $\eta_{x u_n} = \eta_x$ for all $x \in N$ and all $n \in \mathbb{Z}$. Again for $n_1, n_2 \in \mathbb{Z}$ and $x \in N$,

$$\begin{aligned} \langle u_{n_1} x u_{n_2}, x \rangle &= \tau(u_{n_1} x u_{n_2} x^*) \\ &= \tau(\alpha_{n_1}(x) u_{n_1+n_2} x^*) \\ &= \tau(x^* \alpha_{n_1}(x) u_{n_1+n_2}) \\ &= \tau(x^* \alpha_{n_1}(x)) \tau(u_{n_1+n_2}) \text{ by orthogonality.} \end{aligned} \tag{C.2}$$

Note that the *left-right-measure* of $L(\mathbb{Z}) \subset \mathcal{M}$ is naturally supported on $\hat{\mathbb{Z}} = S^1$, where $\hat{\mathbb{Z}}$ is the character group of \mathbb{Z} . Identify $L(\mathbb{Z}) = L^{\infty}(S^1, \lambda_0)$, where λ_0 is the

normalized Haar measure on S^1 , via the standard identification, which sends u_n to the function $e_n(t) = t^n$, $t \in S^1$, $n \in \mathbb{Z}$. Now for $m \in \mathbb{Z}$,

$$\begin{aligned}
\mathbb{E}_{L(\mathbb{Z})}(xu_mx^*) &= \sum_{n \in \mathbb{Z}} \langle \mathbb{E}_{L(\mathbb{Z})}(xu_mx^*), u_n \rangle u_n \\
&= \sum_{n \in \mathbb{Z}} \tau(\mathbb{E}_{L(\mathbb{Z})}(xu_mx^*)u_{-n})u_n \\
&= \sum_{n \in \mathbb{Z}} \tau(xu_mx^*u_{-n})u_n \\
&= \sum_{n \in \mathbb{Z}} \tau(u_{-n}xu_mx^*)u_n \\
&= \sum_{n \in \mathbb{Z}} \tau(x^*\alpha_{-n}(x))\tau(u_{m-n})u_n \text{ (from Eq. C.2)} \\
&= \tau(x^*\alpha_{-m}(x))u_m.
\end{aligned}$$

Therefore, $\eta_x^t(1 \otimes e_m) = \tau(x^*\alpha_{-m}(x))e_m(t)$ for λ_0 almost all $t \in S^1$. Since the action α is strongly mixing so $\tilde{\eta}_x^t$ is a mixing measure for λ_0 almost all t , whenever $\tau(x) = 0$.

Let $x = \sum_{i=1}^n x_i u_{k_i} \in \mathcal{M}$ be such that $\mathbb{E}_{L(\mathbb{Z})}(x) = 0$. Therefore, $\langle x, u_{k_i} \rangle = 0$ for all $1 \leq i \leq n$ and hence $\tau(x_i) = 0$. Now from Eq. C.1 we get,

$$\eta_x = \sum_{i=1}^n \eta_{x_i} + \sum_{i \neq j=1}^n \eta_{x_i u_{k_i}, x_j u_{k_j}}.$$

It is easy to see that $\eta_{x_i u_{k_i}, x_j u_{k_j}} = (1 \otimes e_{k_i - k_j})d\eta_{x_i, x_j}$ for all $i \neq j$. Thus from Eq. C.15 of chapter II, Lemma II.20 it follows that,

$$\int_{S^1} s^m d\tilde{\eta}_{x_i u_{k_i}, x_j u_{k_j}}^t(s) = \int_{S^1} s^m s^{k_i - k_j} d\tilde{\eta}_{x_i, x_j}^t(s) \rightarrow 0 \text{ as } m \rightarrow \infty$$

for λ_0 almost all t . This shows that η_x^t is a mixing measure for λ_0 almost all t .

Let $[\eta]$ denote the *left-right-measure* of $L(\mathbb{Z})$. We assume $\eta(\Delta(S^1)) = 0$ and η is a probability measure. From Eq. A.2, we know that there is a nonzero vector

$\zeta \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\eta = \eta_\zeta$. Let

$$x_n = \sum_{i=1}^{k_n} x_i^{(n)} u_{k_i}^{(n)} \in \mathcal{M} \text{ with } x_i^{(n)} \in N$$

be such that $\mathbb{E}_A(x_n) = 0$, $\|x_n\|_2 \leq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow \zeta$ as $n \rightarrow \infty$ in $\|\cdot\|_2$. Then $\eta_{x_n} \rightarrow \eta_\zeta = \eta$ in $\|\cdot\|_{t,v}$ from Lemma II.24. Then from Lemma II.23, there is a subsequence n_k with $n_k < n_{k+1}$ for all k and a set $E \subset S^1$ with $\lambda_0(E) = 0$, such that for all $t \in E^c$,

$$\sup_{A \subset S^1, A \text{ Borel}} \left| \tilde{\eta}_{x_{n_k}}^t(A) - \tilde{\eta}^t(A) \right| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Note that $\tilde{\eta}_{x_{n_k}}^t$ are mixing measures for all k and for λ_0 almost all t . From standard approximation arguments it follows $\tilde{\eta}^t$ is a mixing for λ_0 almost all t . \square

Theorem IV.14. *Let $A \subset \mathcal{M}$ be a masa. Suppose the left-right-measure of A is the class of product measure. Let $x, y \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0 = \mathbb{E}_A(y)$. If $u_n \in A$ is a bounded sequence that goes to zero in w.o.t then, $\mathbb{E}_A(xu_ny^*)$ goes to zero, λ almost everywhere.*

Before we prove Thm. IV.14, we need to make an observation. Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. In all our results that involved disintegration of measures, we have always worked with functions of the form $[0, 1] \ni t \mapsto \eta_x^t(1 \otimes a)$ where $a \in C[0, 1] \subset A$. The reason we chose $a \in C[0, 1]$, was to assure that the function $[0, 1] \ni t \mapsto \eta_x^t(1 \otimes a)$ is λ -measurable. However, if $[\eta] = [\lambda \otimes \lambda]$ then we can allow a to be in $L^\infty([0, 1], \lambda)$. In this case, measurability is not an issue. Now we prove Thm. IV.14.

Proof. First, fix $x \in \mathcal{M}$ with $\mathbb{E}_A(x) = 0$. Note that $\eta_x \ll \lambda \otimes \lambda$. Let $g = \frac{d\eta_x}{d(\lambda \otimes \lambda)}$. Then $g \in L^1(\lambda \otimes \lambda)$. From Lemma II.20, $\tilde{\eta}_x^t \ll \lambda$ and $\frac{d\tilde{\eta}_x^t}{d\lambda} = g_t$ for λ almost all t , where

$$g_t = g(t, \cdot) \text{ on } \{(t, s) : s \in [0, 1]\}$$

It is easy to verify that, $[0, 1] \ni t \mapsto \eta_x^t(1 \otimes u_n)$ is in $L^\infty([0, 1], \lambda)$ for all n . For $a \in A$ the equation

$$\begin{aligned} \langle a^*, \mathbb{E}_A(xu_nx^*) \rangle &= \tau(a\mathbb{E}_A(xu_nx^*)) \\ &= \tau(axu_nx^*) \\ &= \int_0^1 a(t)\eta_x^t(1 \otimes u_n)d\lambda(t) \end{aligned}$$

implies that, $\mathbb{E}_A(xu_nx^*)(t) = \eta_x^t(1 \otimes u_n)$ for λ almost all t . Thus for λ almost all t we have,

$$\begin{aligned} \mathbb{E}_A(xu_nx^*)(t) &= \eta_x^t(1 \otimes u_n) \\ &= \int_0^1 u_n(s)g_t(s)d\lambda(s) \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

as $\{u_n\}$ is bounded and converges to zero in w.o.t. Now use Eq. D.3 of chapter III to finish the proof. \square

Definition IV.15. [21] Let U be an isometry on a Hilbert space \mathcal{H} . A vector $\zeta \in \mathcal{H}$ is said to be a wandering vector for U if $\langle U^n\zeta, U^m\zeta \rangle = 0$ for all $n \neq m \in \mathbb{Z}$.

In Defn. IV.15, we interpret $U^{-n} = U^{*n}$ for $n > 0$.

Theorem IV.16. *Let $A \subset \mathcal{M}$ be a uniformly mixing masa. Let $v \in A$ be a Haar unitary generator of A . Then the span of the wandering vectors for v which are orthogonal to A is dense in $L^2(A)^\perp$.*

Proof. In this proof we will borrow notations and ideas explained in Eq. (A.1), Eq. (A.2) and related discussions following Defn. IV.1.

Let $\mathcal{E} = \{\underline{\epsilon} = \{\epsilon_{i,n}\}_{1 \leq i \leq n, n \in Puk(A)} : \epsilon_{i,n} = \pm 1\}$. Let v correspond to the function $t \mapsto e^{2\pi it}$. Since A is *uniformly mixing* so from our results in section B, the *left-right-measure* of A is $[\lambda \otimes \lambda]$. For $\mathbb{N}_\infty \ni n \in Puk(A)$ there exists vectors $\zeta_i^{(n)}$, $1 \leq i \leq n$ so

that the projections $P_i^{(n)} : L^2(\mathcal{M}) \mapsto \overline{A\zeta_i^{(n)}A}^{\|\cdot\|_2}$ are mutually orthogonal, equivalent in \mathcal{A}' , $\overline{A\zeta_i^{(n)}A}^{\|\cdot\|_2} \perp L^2(A)$, $\mathcal{A}'(\sum_{i=1}^n P_i^{(n)})$ is type I_n and for $a, b \in C[0, 1]$, and for all $\underline{\epsilon} \in \mathcal{E}$,

$$\left\langle a \left(\bigoplus_{n \in \text{Puk}(A)} \left(\bigoplus_{i=1}^n \epsilon_{i,n} \zeta_i^{(n)} \right) \right) b, \bigoplus_{n \in \text{Puk}(A)} \left(\bigoplus_{i=1}^n \epsilon_{i,n} \zeta_i^{(n)} \right) \right\rangle = \int_{[0,1] \times [0,1]} a(t)b(s)d\lambda(t)d\lambda(s).$$

Fix $\underline{\epsilon} \in \mathcal{E}$ and let $\zeta_{\underline{\epsilon}} = \bigoplus_{n \in \text{Puk}(A)} \bigoplus_{i=1}^n \epsilon_{i,n} \zeta_i^{(n)}$. By Lemma IV.24 we find

$$\|\mathbb{E}_A(\zeta_{\underline{\epsilon}} v^n \zeta_{\underline{\epsilon}}^*)\|_1 = \int_0^1 |\lambda(1 \otimes v^n)| d\lambda(t) = 0, \text{ for all } n \neq 0. \quad (\text{C.3})$$

Therefore for $n \neq m$

$$|\langle v^n \zeta_{\underline{\epsilon}}^*, v^m \zeta_{\underline{\epsilon}}^* \rangle| = |\tau(\zeta_{\underline{\epsilon}} v^{n-m} \zeta_{\underline{\epsilon}}^*)| \leq \|\mathbb{E}_A(\zeta_{\underline{\epsilon}} v^{n-m} \zeta_{\underline{\epsilon}}^*)\|_1 = 0.$$

This establishes the existence of a wandering vector $\zeta_{\underline{\epsilon}}^*$ for v . Note that $\zeta_{\underline{\epsilon}}^* \perp L^2(A)$.

For $u \in \mathcal{U}(A)$ and $b, c \in A$, from Eq. (C.3) we have

$$\mathbb{E}_A((b\zeta_{\underline{\epsilon}}u)v^n(c\zeta_{\underline{\epsilon}}u)^*) = b\mathbb{E}_A(\zeta_{\underline{\epsilon}}v^n\zeta_{\underline{\epsilon}}^*)c^* = 0 \text{ for all } n \neq 0.$$

In particular, $b\zeta_{\underline{\epsilon}}u$ is a wandering vector for v . Let

$$W = \text{span} \{b\zeta_{\underline{\epsilon}}u : u \in \mathcal{U}(A), b \in A, \underline{\epsilon} \in \mathcal{E}\}.$$

It is easy to check that W is dense in $L^2(A)^\perp$. □

Theorem IV.17. *Let $A \subset \mathcal{M}$ be a strongly mixing masa. Then $\mathcal{N}_{\mathcal{M}}(B) = A$ for any diffuse subalgebra $B \subseteq A$ where $\mathcal{N}_{\mathcal{M}}(B) = \{u \in \mathcal{U}(\mathcal{M}) : uBu^* = B\}$.*

Proof. Fix a diffuse subalgebra $B \subset A$. Let $v \in B$ be a Haar unitary generator of B . Then $v^k \xrightarrow{w.o.t} 0$ as $|k| \rightarrow \infty$. For any $x \in \mathcal{M}$ with $\mathbb{E}_A(x) = 0$ we have $\|\mathbb{E}_A(xv^kx^*)\|_2 \rightarrow 0$ as $|k| \rightarrow \infty$. Since $\|\mathbb{E}_B(y)\|_2 \leq \|\mathbb{E}_A(y)\|_2$ for any $y \in \mathcal{M}$ so $\|\mathbb{E}_B(xv^kx^*)\|_2 \rightarrow 0$ as $|k| \rightarrow \infty$. Thus by arguments similar to the proof of Thm III.36,

Thm. III.37 of we get $\overline{Bx\bar{B}}^{\|\cdot\|_2} \in C_{n.a}(B)$. Thus $L^2(A)^\perp \in C_{n.a}(B)$ from Lemma 5.7 [11] and Lemma II.24, II.23. Now if $\zeta \in L^2(\mathcal{M})$ is such that $\overline{B\zeta\bar{B}}^{\|\cdot\|_2} \in C_d(B)$ then the decomposition

$$\overline{B\zeta\bar{B}}^{\|\cdot\|_2} = \overline{B\mathbb{E}_A(\zeta)\bar{B}}^{\|\cdot\|_2} \oplus \overline{B(1 - \mathbb{E}_A)(\zeta)\bar{B}}^{\|\cdot\|_2}$$

shows that $\overline{B(1 - \mathbb{E}_A)(\zeta)\bar{B}}^{\|\cdot\|_2} \in C_d(B)$, which is true only when $(1 - \mathbb{E}_A)(\zeta) = 0$. But by using arguments similar to the proof of Thm. III.19, $\zeta \in L^2(N(B)'')$ if and only if $\overline{Bx\bar{B}}^{\|\cdot\|_2} \in C_d(B)$. Thus we are done. \square

Remark IV.18. The conclusion of Thm. IV.17 is false if A is just assumed to be singular. There exists a singular masa A in the hyperfinite II_1 factor \mathcal{R} that contains a diffuse subalgebra B , so that B is a Cartan masa inside an infinite index subfactor of \mathcal{R} [41].

D. Γ and Non Γ Masas

In this section we study properties of the *left-right-measures* of masas that possess nontrivial centralizing sequences of the factor. We also study properties of the *left-right-measure* that prevents a masa to possess nontrivial centralizing sequences.

Definition IV.19. A centralizing sequence in \mathcal{M} is a bounded sequence $\{x_n\} \subset \mathcal{M}$ such that $\|x_n y - y x_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$ for all $y \in \mathcal{M}$.

The centralizing sequence $\{x_n\}$ is trivial if there exists a sequence $\lambda_n \in \mathbb{C}$ so that $\|x_n - \lambda_n\|_2 \rightarrow 0$.

Definition IV.20. [41] Let $A \subset \mathcal{M}$ be a masa. Define

$$\Gamma(A) = \sup\{\tau(p) : p \in A \text{ is a projection and}$$

Ap contains nontrivial centralizing sequences of $p\mathcal{M}p\}$.

It is immediate that $\Gamma(A) = \Gamma(\theta(A))$ where θ is an automorphism of \mathcal{M} .

Theorem IV.21. *Let $A \subset \mathcal{M}$ be a masa. Let the left-right-measure of A be $[(\lambda \otimes \lambda) + \mu]$ where $\mu \perp \lambda \otimes \lambda$. Then A cannot contain non trivial centralizing sequences of \mathcal{M} . Moreover, $\Gamma(A) = 0$.*

Proof. From Eq. (A.2) there exists a vector $\zeta \in L^2(\mathcal{M}) \ominus L^2(A)$ such that for $a, b \in C[0, 1]$

$$\langle aJb^*J\zeta, \zeta \rangle = \lambda(a)\lambda(b) + \mu(a \otimes b).$$

Therefore

$$\langle a\zeta, \zeta \rangle = \lambda(a) + \mu(a \otimes 1) \text{ and } \langle \zeta a, \zeta \rangle = \lambda(a) + \mu(1 \otimes a) \text{ for } a \in C[0, 1]. \quad (\text{D.1})$$

If possible, let $a_n \in A$ denote a non trivial centralizing sequence such that $\tau(a_n) = 0$ for all n . By making a density argument we can assume that $a_n \in C[0, 1]$ for all n . Assume that $\limsup_n \|a_n\|_2 = \alpha > 0$. A triangle inequality argument shows that $\|a_n\zeta - \zeta a_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$. However

$$\begin{aligned} \|a_n\zeta - \zeta a_n\|_2^2 &= \langle a_n\zeta, a_n\zeta \rangle - \langle \zeta a_n, a_n\zeta \rangle - \langle a_n\zeta, \zeta a_n \rangle + \langle \zeta a_n, \zeta a_n \rangle \\ &= 2\lambda(a_n^*a_n) + \mu(a_n^*a_n \otimes 1) + \mu(1 \otimes a_n^*a_n) - \mu(a_n \otimes a_n^*) - \mu(a_n^* \otimes a_n). \end{aligned} \quad (\text{D.2})$$

But by Cauchy-Schwartz inequality,

$$\begin{aligned} &\mu(a_n^*a_n \otimes 1) + \mu(1 \otimes a_n^*a_n) - \mu(a_n \otimes a_n^*) - \mu(a_n^* \otimes a_n) \\ &= \int_{\Delta([0,1])^c} |a_n(t)|^2 d\mu(t, s) - \int_{\Delta([0,1])^c} a_n(t)\overline{a_n(s)} d\mu(t, s) \\ &\quad + \int_{\Delta([0,1])^c} |a_n(s)|^2 d\mu(t, s) - \int_{\Delta([0,1])^c} \overline{a_n(t)}a_n(s) d\mu(t, s) \\ &\geq \int_{\Delta([0,1])^c} |a_n(t)|^2 d\mu(t, s) \end{aligned}$$

$$\begin{aligned}
& - \left(\int_{\Delta([0,1])^c} |a_n(t)|^2 d\mu(t, s) \right)^{\frac{1}{2}} \left(\int_{\Delta([0,1])^c} |a_n(s)|^2 d\mu(t, s) \right)^{\frac{1}{2}} \\
& + \int_{\Delta([0,1])^c} |a_n(s)|^2 d\mu(t, s) \\
& - \left(\int_{\Delta([0,1])^c} |a_n(t)|^2 d\mu(t, s) \right)^{\frac{1}{2}} \left(\int_{\Delta([0,1])^c} |a_n(s)|^2 d\mu(t, s) \right)^{\frac{1}{2}} \\
& = \left\{ \left(\int_{\Delta([0,1])^c} |a_n(s)|^2 d\mu(s, t) \right)^{\frac{1}{2}} - \left(\int_{\Delta([0,1])^c} |a_n(t)|^2 d\mu(s, t) \right)^{\frac{1}{2}} \right\}^2 \geq 0.
\end{aligned}$$

This shows from Eq. (D.2) that $\|a_n\zeta - \zeta a_n\|_2^2 \not\rightarrow 0$ as $n \rightarrow \infty$, a contradiction.

The last statement follows from the above argument by considering compressions of \mathcal{M} by projections in A because, for any nonzero projection $p \in A$, identifying p as the indicator of a measurable set E_p it follows that the *left-right-measure* of the inclusion $Ap \subset p\mathcal{M}p$ will be the class of the restriction of $\lambda \otimes \lambda + \mu$ to $E_p \times E_p$. \square

We state the next result without proof as its proof is similar to the proof of Thm. IV.21.

Proposition IV.22. *Let $A \subset \mathcal{M}$ be a masa. Let the left-right-measure of A restricted to the projection $pJqJ$ be the class of product tracial measure, where p, q are nonzero projections in A . Then*

- (i) $\Gamma(A) < 1$.
- (ii) *If $r \geq p, q$ is any projection in A then Ar cannot contain nontrivial centralizing sequences for $r\mathcal{M}r$.*

Recall that a von Neumann algebra \mathcal{N} is called full if $Int(\mathcal{N})$ is closed in $Aut(\mathcal{N})$.

Corollary IV.23. (i) *Let \mathcal{M} be a II_1 be factor such that for all masas $A \subset \mathcal{M}$ there are nonzero projections $p, q \in A$ so that the left-right-measure of A contains the product measure restricted to the projection $pJqJ$ as a summand. Then \mathcal{M} is a full factor.*

(ii) If $\overline{\text{Int}(\mathcal{M})} = \text{Aut}(\mathcal{M})$, then there is a masa A in \mathcal{M} whose left-right-measure is singular with respect to the product measure.

(iii) Every strongly stable (McDuff) factor contains singular masas whose left-right-measure is singular with respect to the product class.

Proof. (i) and (ii) follows from two theorems. A factor \mathcal{M} is full if and only if every centralizing sequence in \mathcal{M} is trivial. Secondly, if a factor \mathcal{M} possess nontrivial centralizing sequences then there exists a masa $A \subset \mathcal{M}$ such that A contains nontrivial centralizing sequences for \mathcal{M} [4]. Finally use Prop. IV.22.

(iii) follows by tensoring any singular masa in a McDuff factor by the alternating Tauer masa in \mathcal{R} (see the section on Tauer masas). \square

Making the appropriate changes to the proof of Lemma IV.2 we get the following result. Its proof uses basic facts about L^1 spaces associated to finite von Neumann algebras.

Lemma IV.24. *Let $\zeta \in L^2(\mathcal{M})$ be such that $\mathbb{E}_A(\zeta) = 0$. Let η_ζ denote the measure on $[0, 1] \times [0, 1]$ defined in Eq. (C.12) of chapter II. Let $b, w \in C[0, 1]$. Then*

$$\|\mathbb{E}_A(b\zeta w\zeta^*)\|_1 = \int_0^1 |b(t)| |\eta_\zeta^t(1 \otimes w)| d\lambda(t).$$

Proof. We have

$$\begin{aligned} \|\mathbb{E}_A(b\zeta w\zeta^*)\|_1 &= \sup_{a \in B[0,1], \|a\| \leq 1} |\langle \mathbb{E}_A(b\zeta w\zeta^*), a \rangle| \\ &= \sup_{a \in B[0,1], \|a\| \leq 1} |\tau(a\mathbb{E}_A(b\zeta w\zeta^*))| \\ &= \sup_{a \in B[0,1], \|a\| \leq 1} |\tau(\mathbb{E}_A(ab\zeta w\zeta^*))| \\ &= \sup_{a \in B[0,1], \|a\| \leq 1} |\tau(ab\zeta w\zeta^*)| \end{aligned}$$

$$\begin{aligned}
&= \sup_{a \in B[0,1], \|a\| \leq 1} \left| \int_{[0,1] \times [0,1]} a(t)b(t)w(s)d\eta_\zeta(t,s) \right| \\
&= \sup_{a \in B[0,1], \|a\| \leq 1} \left| \int_0^1 a(t)b(t)\eta_\zeta^t(1 \otimes w)d\lambda(t) \right| \quad (\text{from Defn. (II.15)}) \\
&= \int_0^1 |b(t)\eta_\zeta^t(1 \otimes w)| d\lambda(t) \quad (\text{from 2}^\circ \text{ of Lemma IV.2}).
\end{aligned}$$

□

Definition IV.25. A finite measure μ on $[0, 1]$ is called α -rigid for $|\alpha| = 1$, if and only if there is a subsequence $\hat{\mu}_{n_k}$ of $\hat{\mu}_n = \int_0^1 e^{2\pi i n t} d\mu(t)$ that converges to $\alpha\mu([0, 1])$ as $k \rightarrow \infty$. A 1-rigid measure is called rigid or a Dirichlet measure.

We now recall some properties of α -rigid measures. For details check [23]. Let μ be a α -rigid measure on $[0, 1]$. Any sequence n_k along which $\hat{\mu}_{n_k}$ converges to $\alpha\mu([0, 1])$ is said to be a sequence associated with μ . It is easy to see that, μ is α -rigid if and only if the sequence of functions $[0, 1] \ni t \mapsto e^{2\pi i n_k t}$ converges to α in μ -measure. Thus ν is α -rigid with associated sequence n_k for any $\nu \ll \mu$. So α -rigidity is a property of equivalence class of measures and hence can be thought of as a property of unitary operators by considering appropriate Koopman operators. Atomic measures are always rigid.

Theorem IV.26. *Let $A \subset \mathcal{M}$ be a (singular) masa. Let $v \in A$ be a Haar unitary generator of A . Suppose there exists a subsequence n_k ($n_k < n_{k+1}$ for all k) such that for all $y \in \mathcal{M}$*

$$\|v^{n_k}y - yv^{n_k}\|_2 \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Then the measures $\tilde{\eta}^t$ are β_t -rigid for some complex number $\beta_t \in S^1$, λ almost all t .

Proof. We can assume that v corresponds to the function $[0, 1] \ni t \mapsto e^{2\pi i t}$. Standard

density arguments show that if $\xi \in L^2(\mathcal{M})$ then

$$\|v^{n_k}\xi - \xi v^{n_k}\|_2 \rightarrow 0 \text{ as } k \rightarrow \infty.$$

From Eq. (A.2) we know that there is a nonzero vector $\zeta \in L^2(\mathcal{M}) \ominus L^2(A)$ such that $\eta = \eta_\zeta$. Therefore we have $\|\mathbb{E}_A(v^{-n_k}\zeta v^{n_k}\zeta^*) - \mathbb{E}_A(\zeta\zeta^*)\|_1 \rightarrow 0$ as $k \rightarrow \infty$. Consequently using similar arguments as in proof of Lemma IV.24 we have,

$$\|\mathbb{E}_A(v^{-n_k}\zeta v^{n_k}\zeta^*) - \mathbb{E}_A(\zeta\zeta^*)\|_1 = \int_0^1 |e^{-2\pi i n_k t} \eta^t(1 \otimes v^{n_k}) - \mathbb{E}_A(\zeta\zeta^*)(t)| d\lambda(t) \rightarrow 0$$

as $k \rightarrow \infty$. Hence there exists a further subsequence n_{k_l} and a subset $E \subset [0, 1]$ such that $\lambda(E) = 0$ and for $t \in E^c$,

$$e^{-2\pi i n_{k_l} t} \eta^t(1 \otimes v^{n_{k_l}}) - \mathbb{E}_A(\zeta\zeta^*)(t) \rightarrow 0 \text{ as } l \rightarrow \infty \quad (\text{D.3})$$

and $\mathbb{E}_A(\zeta\zeta^*)(t) = \tilde{\eta}^t([0, 1]) < \infty$ (Lemma IV.2).

Fix $t \in E^c$. Dropping to a subsequence if necessary which depends on t , we assume that $e^{-2\pi i n_{k_l} t} \rightarrow \overline{\beta_t}$ as $l \rightarrow \infty$ for some complex number $\beta_t \in S^1$. Then $\tilde{\eta}^t$ is β_t -rigid. \square

Remark IV.27. Examples of singular masas in the hyperfinite II_1 factor can be constructed that satisfies the hypothesis of Thm. IV.26. There exist weakly mixing actions of a stationary Gaussian process that has the desired properties [47].

Remark IV.28. Suppose a masa $A \subset \mathcal{M}$ possesses nontrivial centralizing sequences of \mathcal{M} . Then identify $A = L^\infty(S^1, \lambda)$, where λ is the normalized Haar measure on S^1 . By Stone-Weierstrass theorem, we can always find a nontrivial centralizing sequence of \mathcal{M} consisting of trigonometric polynomials. We suspect that it is even possible to extract a nontrivial centralizing sequence that consists of convex combinations of $f_n(z) = z^n$ for $n \in \mathbb{Z}$. If the last statement is true, then we can show that $\tilde{\eta}^t$ is

β_t -rigid for some complex number $\beta_t \in S^1$, for λ almost all t .

CHAPTER V

EXAMPLES AND CONCLUSIONS

In this chapter we calculate the *measure-multiplicity-invariant* of some masas in the hyperfinite II_1 factor and free group factors. For results proved in this chapter we are indebted to Stuart White.

A. Tauer Masas in the Hyperfinite II_1 Factor

In this section we will calculate the *left-right-measure* of certain Tauer masas in the hyperfinite II_1 factor \mathcal{R} . The examples of the Tauer masas that we are interested in are directly taken from [41].

Definition V.1. (White) A masa A in \mathcal{R} is said to be a *Tauer masa* if there exists a sequence of finite type I subfactors $\{\mathcal{N}_n\}_{n=1}^\infty$ such that,

- (i) $\mathcal{N}_n \subset \mathcal{N}_{n+1}$ for all n ,
- (ii) $(\cup_{n=1}^\infty \mathcal{N}_n)'' = \mathcal{R}$,
- (iii) $A \cap \mathcal{N}_n$ is a masa in \mathcal{N}_n for every n .

This allows us to write structure of every Tauer masa A in \mathcal{R} with respect to the chain $\{\mathcal{N}_n\}_{n=1}^\infty$ as follows. Switching to the notation of tensor products the above definition means that we can find finite type I subfactors $\{\mathcal{M}_n\}_{n=1}^\infty$ such that, $\mathcal{N}_n = \bigotimes_{r=1}^n \mathcal{M}_r$ for every n . For $m > n$ the m -th finite dimensional approximation of A can be written in terms of the n -th one as,

$$A_m = \bigoplus_{e \in \mathcal{P}(A_n)} e \otimes A_{m,n}^{(e)} \quad (\text{A.1})$$

where the direct sum is over the set of minimal projections $\mathcal{P}(A_n)$ in A_n and $A_{m,n}^{(e)}$ is a masa in $\bigotimes_{r=n+1}^m \mathcal{M}_r$. Note that the Cartan masa arising as the infinite tensor product

of diagonal matrices inside the hyperfinite II_1 factor is a Tauer masa. In [48], White has shown that the Pukánszky invariant of every Tauer masa is $\{1\}$. In fact, it follows from his proof that the bicyclic vector for any Tauer masa can be chosen to be an operator from \mathcal{R} itself.

Sinclair and White [41] has exhibited a continuous path of singular masas in \mathcal{R} no two of which can be connected by automorphisms of \mathcal{R} . We are interested in two masas that correspond to the end points of this path. For all Tauer masas it is clear that the Cantor set is the natural space where we have to build the measures. For ease of calculation we need to index the minimal projections in the approximating stages in a different fashion than that appeared in [41]. It is now time to introduce some notation.

1° **Notation** : If $\mathcal{N}_n = \bigotimes_{r=1}^n \mathcal{M}_{k_r}(\mathbb{C})$, then the minimal projections of A_n will be denoted by ${}^{(n)}f_{\underline{t}^{(n)}}$, where $\underline{t}^{(n)} = (t_1, t_2, \dots, t_n)$ with $1 \leq t_i \leq k_i$, $1 \leq i \leq n$.

The convention that we follow is

$${}^{(n)}f_{t_1, t_2, \dots, t_n} = {}^{(n-1)}f_{t_1, t_2, \dots, t_{n-1}} \otimes {}^{(n)}e_{t_n}^{(t_1, t_2, \dots, t_{n-1})},$$

where ${}^{(n)}e_{t_n}^{(t_1, t_2, \dots, t_{n-1})}$ are the minimal projections of the algebra $A_{n, n-1}^{(t_1, t_2, \dots, t_{n-1})}$, in accordance with Eq. (A.1). The matrix units corresponding to these family of minimal projections will be denoted by ${}^{(n)}f_{\underline{t}^{(n)}, \underline{s}^{(n)}}$ and we will understand ${}^{(n)}f_{\underline{t}^{(n)}, \underline{t}^{(n)}} = {}^{(n)}f_{\underline{t}^{(n)}}$. For two tuples (t_1, t_2, \dots, t_n) and (s_1, s_2, \dots, s_n) such that $t_i = s_i$ for $1 \leq i \leq n-1$ and $t_n \neq s_n$ we will write ${}^{(n)}f_{\underline{t}^{(n)}, \underline{s}^{(n)}} = {}^{(n)}f_{(\cdot, t_n), (\cdot, s_n)}$.

2° **Notation**: For any two subsets $S, T \subseteq \mathcal{M}$, we will denote by $S \cdot T$ the set $\text{span}\{ab : a \in S, b \in T\}$. The normalized trace of $\mathcal{M}_n(\mathbb{C})$ will be denoted by tr_n . The unique normal tracial state of the hyperfinite factor \mathcal{R} will be denoted by $\tau_{\mathcal{R}}$. This

trace $\tau_{\mathcal{R}}$ when restricted to A gives rise to a measure on a Cantor set which will also be denoted by $\tau_{\mathcal{R}}$. For a measure μ on a space X and $f \in L^1(\mu)$ we will denote by $f\mu$ the measure on X obtained as $f\mu(E) = \int_E f d\mu$.

The next lemma is very well known but we record it for completeness.

Lemma V.2. *If A, B are two masas in $\mathcal{M}_n(\mathbb{C})$ orthogonal with respect to the normalized trace tr_n then $A \cdot B = \mathcal{M}_n(\mathbb{C})$.*

1. Tauer Masa of Product Class

Following Sinclair and White [41] we are going to calculate the *measure-multiplicity invariant* of a Tauer masa A whose description we are going to elaborate now. This Tauer masa has the Γ invariant 0 (A is totally non- Γ). We will show that its *left-right-measure* belongs to the product class. This example is important as this is the first example of a masa in \mathcal{R} with simple multiplicity whose *left-right-measure* is the class of product measure.

Let $k_1 = 2$ and for each $r \geq 2$ let k_r be a prime exceeding $k_1 k_2 \cdots k_{r-1}$. Set \mathcal{M}_r to be the algebra of $k_r \times k_r$ matrices. By Thm. 3.2 [32] there is a family $\{^{(r)}D^{\underline{t}(r-1)}\}_{\underline{t}(r-1)}$ of pairwise orthogonal masas in \mathcal{M}_r . Let $\mathcal{N}_n = \bigotimes_{r=1}^n \mathcal{M}_r$. There is a natural inclusion $x \mapsto x \otimes 1$ of \mathcal{N}_n inside \mathcal{N}_{n+1} and one works in the hyperfinite II_1 factor \mathcal{R} obtained as a direct limit of these \mathcal{N}_n with respect to the normalized trace. With respect to the chain $\{\mathcal{N}_n\}_{n=1}^{\infty}$ of finite type I subfactors of \mathcal{R} , A is constructed as follows.

Let $A_1 = D_2(\mathbb{C}) \subset \mathcal{M}_1$ be the diagonal masa. Having constructed A_n one constructs A_{n+1} as,

$$A_{n+1} = \bigoplus_{\underline{t}(n)}^{(n)} f_{\underline{t}(n)} \otimes^{(n+1)} D^{\underline{t}(n)}. \quad (\text{A.2})$$

That $(\cup_{n=1}^{\infty} A_n)''$ is a masa in \mathcal{R} follows from a theorem of Tauer [45]. This Tauer masa is singular from Prop. 2.1 [41].

We denote by $P_{\underline{t}(n), \underline{s}(n)}^{(n)}$ the orthogonal projection from $L^2(\mathcal{R})$ onto the subspace ${}^{(n)}f_{\underline{t}(n)}L^2(\mathcal{R}){}^{(n)}f_{\underline{s}(n)}$ and let,

$$P = \sum_{n=1}^{\infty} \sum_{t_1=1}^{k_1} \sum_{t_2=1}^{k_2} \cdots \sum_{t_{n-1}=1}^{k_{n-1}} \sum_{t_n \neq s_n=1}^{k_n} P_{(\cdot, t_n), (\cdot, s_n)}^{(n)}. \quad (\text{A.3})$$

Clearly, $P_{\underline{t}(n), \underline{s}(n)}^{(n)} = {}^{(n)}f_{\underline{t}(n)}J^{(n)}f_{\underline{s}(n)}J$ and is in \mathcal{A} . At the first sight it might not be clear that the sum in Eq. (A.3) makes sense but we will show that projections involved in the sum are orthogonal and sums to $1 - e_A$.

The following lemma, part of which is recorded by Sinclair and White [41] will be crucial for our calculations.

Lemma V.3. *For each $n \in \mathbb{N}$ let $\mathcal{R} = \mathcal{N}_n \otimes \mathcal{R}_n$ where $\mathcal{R}_n = (\bigotimes_{r=n+1}^{\infty} \mathcal{M}_{k_r}(\mathbb{C}))''$, then*

$$A = \bigoplus_{\underline{t}(n)} {}^{(n)}f_{\underline{t}(n)} \otimes A_{\infty, n+1}^{\underline{t}(n)} \quad \text{where} \quad (\text{A.4})$$

$A_{\infty, n+1}^{\underline{t}(n)}$ are Tauer masas in \mathcal{R}_n and whenever $\underline{t}(n) \neq \underline{s}(n)$ we have

(i) $A_{\infty, n+1}^{\underline{t}(n)}$ and $A_{\infty, n+1}^{\underline{s}(n)}$ are orthogonal in \mathcal{R}_n ,

(ii) $(A_{\infty, n+1}^{\underline{t}(n)} \cdot A_{\infty, n+1}^{\underline{s}(n)})^{-\|\cdot\|_2} = L^2(\mathcal{R}_n)$.

Moreover, for each $\underline{t}(n)$ if $\{A_{m, n+1}^{\underline{t}(n)}\}_{m=1}^{\infty}$ denotes the m -th approximation of $A_{\infty, n+1}^{\underline{t}(n)}$ in \mathcal{R}_n then,

$$A_{1, n+1}^{\underline{t}(n)} = {}^{(n+1)}D^{\underline{t}(n)} \quad \text{and} \quad (\text{A.5})$$

$$A_{m+1, n+1}^{\underline{t}(n)} = \bigoplus_{e \in \mathcal{P}(A_{m, n+1}^{\underline{t}(n)})} e \otimes {}^{(m+1)}D_{e, n+1}^{\underline{t}(n)} \quad (\text{A.6})$$

where for each fixed m and $\underline{t}(n)$, the family $\{{}^{(m+1)}D_{e, n+1}^{\underline{t}(n)}\}_e$ are pairwise orthogonal masas in $\mathcal{M}_{k_{n+m+1}}(\mathbb{C})$.

Proof. We only have to prove (ii). The rest of the statements are just rephrasing

Lemma 5.6 of [41].

Use Lemma V.2, (i) and Eq. (A.5) to conclude

$$\mathcal{M}_{k_{n+1}} \subseteq (A_{\infty, n+1}^{\underline{t}(n)} \cdot A_{\infty, n+1}^{\underline{s}(n)})^{-\|\cdot\|_2}.$$

Since $A_{\infty, n+1}^{\underline{t}(n)}$ and $A_{\infty, n+1}^{\underline{s}(n)}$ are orthogonal so is $A_{m, n+1}^{\underline{t}(n)}$ and $A_{m, n+1}^{\underline{s}(n)}$ for all $m \geq n+1$. Use Lemma V.2 to conclude that $\bigotimes_{r=n+1}^m \mathcal{M}_{k_r}(\mathbb{C}) \subseteq (A_{\infty, n+1}^{\underline{t}(n)} \cdot A_{\infty, n+1}^{\underline{s}(n)})^{-\|\cdot\|_2}$ for all $m \geq n+1$. Hence by density of the algebraic tensor product of matrix algebras in $L^2(\mathcal{R}_n)$ we finish the proof. \square

For each n , let $X_n = \{x_1^{(n)}, x_2^{(n)}, \dots, x_{k_n}^{(n)}\}$ denote a set of k_n points. Let $Y^{(n)} = \prod_{k=1}^n X_k$, $X^{(n)} = \prod_{k=n+1}^{\infty} X_k$ and $X = \prod_{k=1}^{\infty} X_k$, so that for each n , $X = Y^{(n)} \times X^{(n)}$. For $n = 1$, $A_1 = D_2(\mathbb{C}) \cong C(Y^{(1)})$. Having identified A_1, A_2, \dots, A_n with $C(Y^{(1)}), C(Y^{(2)}), \dots, C(Y^{(n)})$ respectively, we identify A_{n+1} with $C(Y^{(n+1)})$ as follows.

For each $1 \leq t_i \leq k_i$, $1 \leq i \leq n$, $\underline{t}(n) = (t_1, \dots, t_n)$, ${}^{(n+1)}D^{\underline{t}(n)} \cong C(X_{n+1})$. Now the projection ${}^{(n)}f_{\underline{t}(n)} \in C(Y^{(n)})$ corresponds to the indicator of a set $\{x_{t_1}^{(1)}, x_{t_2}^{(2)}, \dots, x_{t_n}^{(n)}\} \subseteq Y^{(n)}$. Therefore identify, ${}^{(n)}f_{\underline{t}(n)} \otimes {}^{(n+1)}D^{\underline{t}(n)}$ with ${}^{(n)}f_{\underline{t}(n)} \otimes C(X_{n+1})$. Hence $A_{n+1} \cong C(Y^{(n+1)})$. Therefore, $C(Y^{(1)}) \subset C(Y^{(2)}) \subset \dots \subset C(Y^{(n)}) \subset C(Y^{(n+1)}) \subset \dots \subset C(X) \subset A$ where

$$X = \lim_{\infty \leftarrow n} Y^{(n)}$$

and $C(X)$ is norm separable and *w.o.t* dense in A . Write $B = C(X)$. Therefore

$$B = \bigoplus_{\underline{t}(n)} {}^{(n)}f_{\underline{t}(n)} \otimes B_{\infty, n+1}^{\underline{t}(n)} \tag{A.7}$$

and $B_{\infty, n+1}^{\underline{t}(n)} \cong C(X^{(n+1)})$ (by a similar argument and using Lemma V.3) and is *w.o.t* dense, norm separable C^* subalgebra of $A_{\infty, n+1}^{\underline{t}(n)}$.

Lemma V.4. For each n and $\underline{t}(n) \neq \underline{s}(n)$,

$$(i) (A^{(n)} f_{\underline{t}(n), \underline{s}(n)} A)^{-\|\cdot\|_2} = {}^{(n)}f_{\underline{t}(n)} L^2(\mathcal{R}) {}^{(n)}f_{\underline{s}(n)}.$$

(ii) For $a, b \in B$,

$$\begin{aligned} & \langle a^{(n)} f_{\underline{t}(n), \underline{s}(n)} b, {}^{(n)} f_{\underline{t}(n), \underline{s}(n)} \rangle_{\tau_{\mathcal{R}}} \\ &= k_1 k_2 \cdots k_n \int_X \int_X {}^{(n)} f_{\underline{t}(n)}(t) {}^{(n)} f_{\underline{s}(n)}(s) a(t) \overline{b(s)} d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s). \end{aligned}$$

(iii) $(A^{(n)} f_{\underline{t}(n), \underline{s}(n)} A)^{-\|\cdot\|_2}$ is orthogonal to $(A^{(n)} f_{\underline{t}'(n), \underline{s}'(n)} A)^{-\|\cdot\|_2}$ whenever $\underline{t}(n) \neq \underline{s}(n)$, $\underline{t}'(n) \neq \underline{s}'(n)$ and $(\underline{t}(n), \underline{s}(n)) \neq (\underline{t}'(n), \underline{s}'(n))$.

Proof. For $a, b \in A$, using Eq. (A.4) write

$$a = \bigoplus_{\underline{q}(n)} {}^{(n)} f_{\underline{q}(n)} \otimes a_{\underline{q}(n)} \quad \text{and} \quad b = \bigoplus_{\underline{p}(n)} {}^{(n)} f_{\underline{p}(n)} \otimes b_{\underline{p}(n)}$$

for $a_{\underline{q}(n)} \in A_{\infty, n+1}^{\underline{q}(n)}$, and $b_{\underline{p}(n)} \in A_{\infty, n+1}^{\underline{p}(n)}$. By direct multiplication we get

$$a^{(n)} f_{\underline{t}(n), \underline{s}(n)} \otimes 1_{\mathcal{R}_n} b = {}^{(n)} f_{\underline{t}(n), \underline{s}(n)} \otimes a_{\underline{t}(n)} b_{\underline{s}(n)}.$$

Therefore (i) follows from Lemma V.3 (ii). Moreover for $a, b \in B$

$$\begin{aligned} & \langle a^{(n)} f_{\underline{t}(n), \underline{s}(n)} \otimes 1_{\mathcal{R}_n} b, {}^{(n)} f_{\underline{t}(n), \underline{s}(n)} \otimes 1_{\mathcal{R}_n} \rangle_{\tau_{\mathcal{R}}} \\ &= \text{tr}_{\mathcal{N}_n} ({}^{(n)} f_{\underline{t}(n)}) \tau_{\mathcal{R}_n} (a_{\underline{t}(n)} b_{\underline{s}(n)}) \\ &= \frac{1}{k_1 k_2 \cdots k_n} \tau_{\mathcal{R}_n} (a_{\underline{t}(n)} b_{\underline{s}(n)}) \\ &= k_1 k_2 \cdots k_n \tau_{\mathcal{R}} (a({}^{(n)} f_{\underline{t}(n)} \otimes 1)) \tau_{\mathcal{R}} (b({}^{(n)} f_{\underline{s}(n)} \otimes 1)) \\ & \hspace{15em} \text{(by orthogonality, Lemma V.3(ii))} \\ &= k_1 k_2 \cdots k_n \int_X \int_X a({}^{(n)} f_{\underline{t}(n)} \otimes 1)(t) b({}^{(n)} f_{\underline{s}(n)} \otimes 1)(s) d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s) \\ &= k_1 k_2 \cdots k_n \int_{x_{t_1}^{(1)} \times \cdots \times x_{t_n}^{(n)} \times X^{(n)}} \int_{x_{s_1}^{(1)} \times \cdots \times x_{s_n}^{(n)} \times X^{(n)}} a_{\underline{t}(n)}(t) b_{\underline{s}(n)}(s) d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s) \\ &= k_1 k_2 \cdots k_n \int_{X \times X} {}^{(n)} f_{\underline{t}(n)}(t) {}^{(n)} f_{\underline{s}(n)}(s) a(t) \overline{b(s)} d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s). \end{aligned}$$

This proves (ii). Clearly (iii) follows from (i) and the fact that ${}^{(n)} f_{\underline{t}(n)} J^{(n)} f_{\underline{s}(n)} J$ and ${}^{(n)} f_{\underline{t}'(n)} J^{(n)} f_{\underline{s}'(n)} J$ are orthogonal projections in $L^2(\mathcal{R})$ if $(\underline{t}(n), \underline{s}(n)) \neq (\underline{t}'(n), \underline{s}'(n))$.

□

Lemma V.5. For $m > n$ the projections $\sum_{t_1=1}^{k_1} \sum_{t_2=1}^{k_2} \cdots \sum_{t_{n-1}=1}^{k_{n-1}} \sum_{t_n \neq s_n=1}^{k_n} P_{(\cdot, t_n), (\cdot, s_n)}^{(n)}$ and $P_{(\cdot, t_m), (\cdot, s_m)}^{(m)}$ with $t_m \neq s_m$ are orthogonal. Moreover

$$\sum_{n=1}^{\infty} \sum_{t_1=1}^{k_1} \sum_{t_2=1}^{k_2} \cdots \sum_{t_{n-1}=1}^{k_{n-1}} \sum_{t_n \neq s_n=1}^{k_n} P_{(\cdot, t_n), (\cdot, s_n)}^{(n)} = 1 - e_A.$$

Proof. It follows from (iii) of Lemma V.4 that

$$P^{(n)} = \sum_{t_1=1}^{k_1} \sum_{t_2=1}^{k_2} \cdots \sum_{t_{n-1}=1}^{k_{n-1}} \sum_{t_n \neq s_n=1}^{k_n} P_{(\cdot, t_n), (\cdot, s_n)}^{(n)}$$

is a projection. The projection $P^{(n)} \in \mathcal{A}'$ and hence is in \mathcal{A} as \mathcal{A} is maximal abelian in $\mathbf{B}(L^2(\mathcal{R}))$. Therefore $P^{(n)}$ is decomposable. Denote

$$E_{(\cdot, t_n), (\cdot, s_n)} = (x_{t_1}^{(1)} \times \cdots \times x_{t_{n-1}}^{(n)} \times x_{t_n}^{(n)} \times X^{(n)}) \times (x_{t_1}^{(1)} \times \cdots \times x_{t_{n-1}}^{(n)} \times x_{s_n}^{(n)} \times X^{(n)}).$$

It is not hard to see from Lemma V.4 that

$$P_{(\cdot, t_n), (\cdot, s_n)}^{(n)}(L^2(\mathcal{R})) \cong \int_{E_{(\cdot, t_n), (\cdot, s_n)}}^{\oplus} \mathbb{C}_{t,s} d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s) \text{ where } \mathbb{C}_{t,s} = \mathbb{C},$$

and $\mathcal{A}P_{(\cdot, t_n), (\cdot, s_n)}^{(n)}$ is the diagonalizable algebra with respect to this decomposition. For $\underline{t}(n-1) \neq \underline{t}'(n-1)$ the direct integrals for $P_{(\underline{t}(n-1), t_n), (\underline{t}(n-1), s_n)}^{(n)}$ and $P_{(\underline{t}'(n-1), t_n), (\underline{t}'(n-1), s_n)}^{(n)}$ with $t_n \neq s_n$ and $t_n' \neq s_n'$ rests over disjoint subsets of $X \times X$. Therefore

$$P^{(n)}(L^2(\mathcal{R})) \cong \int_{E_n}^{\oplus} \mathbb{C}_{t,s} d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s), \text{ where } \mathbb{C}_{t,s} = \mathbb{C} \text{ and}$$

$$E_n = \cup_{t_1=1}^{k_1} \cdots \cup_{t_{n-1}=1}^{k_{n-1}} \cup_{t_n \neq s_n=1}^{k_n} E_{(\cdot, t_n), (\cdot, s_n)} = \Delta(Y^{(n-1)}) \times \Delta(X_n)^c \times X^{(n)} \times X^{(n)},$$

and $\mathcal{A}P^{(n)}$ is the diagonalizable algebra with respect to this decomposition. With $m > n$ and $t_m \neq s_m$, $P_{(\cdot, t_m), (\cdot, s_m)}^{(m)}$ will be direct integral over the set $(x_{t_1}^{(1)} \times \cdots \times$

$x_{t_{m-1}}^{(m-1)} \times x_{t_m}^{(m)} \times X^{(m)} \times (x_{t_1}^{(1)} \times \cdots \times x_{t_{m-1}}^{(m-1)} \times x_{s_m}^{(m)} \times X^{(m)})$ which is disjoint from $\Delta(Y^{(n-1)}) \times \Delta(X_n)^c \times X^{(n)} \times X^{(n)}$. Therefore by direct integral theory $P^{(n)}$ and $P_{(\cdot, t_m), (\cdot, s_m)}^{(m)}$ are orthogonal. Now for $x \in \mathcal{R}$ we have for all positive integer N

$$\sum_{n=1}^N P^{(n)} x = \sum_{\underline{t}^{(N)} \neq \underline{s}^{(N)}} P_{\underline{t}^{(N)}, \underline{s}^{(N)}}^{(N)} x = x - \sum_{\underline{t}^{(N)}} P_{\underline{t}^{(N)}, \underline{t}^{(N)}}^{(N)} x = (1 - \mathbb{E}_{A_N})(x).$$

Since by a result of Popa [30] we have $\mathbb{E}_{A_N}(\cdot) \rightarrow \mathbb{E}_A(\cdot)$ pointwise in $\|\cdot\|_2$ as $N \rightarrow \infty$, by density of \mathcal{R} in $L^2(\mathcal{R})$ we are done. \square

Let $c_n = \prod_{r=1}^n k_r$ for $n \geq 1$ and $c_0 = 1$.

Proposition V.6. *The vector $\sum_{n=1}^{\infty} \sum_{t_1=1}^{k_1} \sum_{t_2=1}^{k_2} \cdots \sum_{t_{n-1}=1}^{k_{n-1}} \sum_{t_n \neq s_n=1}^{k_n} \frac{1}{\sqrt{c_n}} f_{(\cdot, t_n), (\cdot, s_n)}^{(n)}$ is a cyclic vector for $\mathcal{A}(1 - e_A)$ and*

$$(1 - e_A)(L^2(\mathcal{R})) \cong \int_{X \times X}^{\oplus} \mathbb{C}_{t,s} d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s), \text{ where } \mathbb{C}_{t,s} = \mathbb{C}.$$

Moreover $\mathcal{A}(1 - e_A)$ is the algebra of diagonalizable operators with respect to this decomposition.

Proof. Fix $n \in \mathbb{N}$. For each $1 \leq t_i \leq k_i$, $1 \leq i \leq n-1$, and $1 \leq t_n \neq s_n \leq k_n$, working on with vectors $\frac{1}{\sqrt{c_n}} f_{(\cdot, t_n), (\cdot, s_n)}^{(n)}$ we find using Lemma V.4 a positive measure

$\eta_{(\cdot, t_n), (\cdot, s_n)}^{(n)}$ supported on

$(x_{t_1}^{(1)} \times \cdots \times x_{t_{n-1}}^{(n-1)} \times x_{t_n}^{(n)} \times X^{(n)}) \times (x_{t_1}^{(1)} \times \cdots \times x_{t_{n-1}}^{(n-1)} \times x_{s_n}^{(n)} \times X^{(n)})$, such that

$$d\eta_{(\cdot, t_n), (\cdot, s_n)}^{(n)} = {}^{(n)}f_{(\cdot, t_n)}(t) {}^{(n)}f_{(\cdot, s_n)}(s) d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s).$$

By making arguments similar as in Lemma V.5 we find for each n a positive measure $\eta^{(n)}$ on $E_n = \Delta(Y^{(n-1)}) \times \Delta(X_n)^c \times X^{(n)} \times X^{(n)}$ given by $\eta^{(n)} = \chi_{E_n} d(\tau_{\mathcal{R}} \otimes$

$\tau_{\mathcal{R}}$) such that

$$P^{(n)}(L^2(\mathcal{R})) = \sum_{t_1=1}^{k_1} \sum_{t_2=1}^{k_2} \cdots \sum_{t_{n-1}=1}^{k_{n-1}} \sum_{t_n \neq s_n=1}^{k_n} P_{(\cdot, t_n), (\cdot, s_n)}^{(n)}(L^2(\mathcal{R})) = \int_{X \times X}^{\oplus} \mathbb{C}_{t,s} d\eta^{(n)}(t, s)$$

where $\mathbb{C}_{t,s} = \mathbb{C}$ and $\mathcal{A}P^{(n)}$ is diagonalizable with respect to this decomposition. Note that

$$\eta^{(n)}(X \times X) = \frac{c_{n-1}(k_n^2 - k_n)}{c_n^2} = \frac{1}{c_{n-1}} - \frac{1}{c_n}. \quad (\text{A.8})$$

The proof of Lemma V.5 shows that the measures $\eta^{(n)}$ are supported on disjoint sets. Hence by Lemma V.5 and Lemma 5.7 [11]

$$(1 - e_A)(L^2(\mathcal{R})) \cong \int_{X \times X}^{\oplus} \mathbb{C}_{t,s} d\eta(s, t), \text{ where } \mathbb{C}_{t,s} = \mathbb{C}, \eta = \sum_{n=1}^{\infty} \eta^{(n)}. \quad (\text{A.9})$$

Moreover $\mathcal{A}(1 - e_A)$ is diagonalizable with respect to the decomposition in Eq. A.9. Clearly

$$\eta(X \times X) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \eta^{(n)}(X \times X) = \lim_{N \rightarrow \infty} c_0 - c_N = 1.$$

Finally $\eta = \tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}$. Indeed for $a, b \in C(X)$,

$$\begin{aligned} \int_{X \times X} a(t)b(s) d\eta(t, s) &= \sum_{n=1}^{\infty} \int_{X \times X} a(t)b(s) d\eta^{(n)}(t, s) \\ &= \sum_{n=1}^{\infty} \sum_{t_1=1}^{k_1} \cdots \sum_{t_{n-1}=1}^{k_{n-1}} \sum_{t_n \neq s_n=1}^{k_n} \int_{X \times X} a(t)b(s) d\eta_{(\cdot, t_n), (\cdot, s_n)}^{(n)}(t, s) \\ &= \sum_{n=1}^{\infty} \sum_{t_1=1}^{k_1} \cdots \sum_{t_{n-1}=1}^{k_{n-1}} \sum_{t_n \neq s_n=1}^{k_n} \int_{X \times X} a(t)b(s)^{(n)} f_{(\cdot, t_n)}^{(n)}(t)^{(n)} f_{(\cdot, s_n)}^{(n)}(s) \\ &\qquad\qquad\qquad d(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(t, s). \end{aligned}$$

But $\sum_{n=1}^{\infty} \sum_{t_1=1}^{k_1} \cdots \sum_{t_{n-1}=1}^{k_{n-1}} \sum_{t_n \neq s_n=1}^{k_n} {}^{(n)}f_{(\cdot, t_n)}(t)^{(n)} f_{(\cdot, s_n)}(s) \uparrow \chi_{\Delta(X)^c}$ pointwise almost everywhere $\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}$. Use dominated convergence theorem and the fact $(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}})(\Delta X) = 0$ to conclude $\eta = \tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}$. This completes the proof. \square

Since the Pukánszky invariant of every *Tauer masa* is $\{1\}$ we have thus computed the *measure-multiplicity-invariant* of A . The above Tauer masa was denoted by $A(0)$ in [41]. There is a Tauer masa of exact opposite flavor than $A(0)$ which we call as the *alternating Tauer masa*.

2. Alternating Tauer Masa

The *alternating Tauer masa* $A(1)$ is a singular Tauer masa in the hyperfinite II_1 factor \mathcal{R} constructed by White and Sinclair [41]. It contains nontrivial centralizing sequences of \mathcal{R} . In fact its Γ -invariant is 1.

The chain for this masa is exactly similar as the masa of the product class described before. Let $A(1)_1 = D_2(\mathbb{C}) \subset \mathcal{M}_1$ be the diagonal masa. Having constructed $A(1)_n \subset \mathcal{N}_n$ we construct $A(1)_{n+1}$ as,

$$A(1)_{n+1} = \begin{cases} A(1)_n \otimes {}^{(n+1)}D_{n+1}, & n \text{ even, } {}^{(n+1)}D_{n+1} \text{ the diagonal masa in } M_{k_{n+1}}(\mathbb{C}), \\ \bigoplus_{\underline{t}^{(n)}} {}^{(n)}f_{\underline{t}^{(n)}} \otimes {}^{(n+1)}D^{\underline{t}^{(n)}}, & n \text{ odd, } {}^{(n+1)}D^{\underline{t}^{(n)}} \text{ pairwise orthogonal.} \end{cases} \quad (\text{A.10})$$

We will prove that the *left-right-measure* of $A(1)$ is singular with respect to the product measure. With the *left-right-measure* of $A(0)$ and $A(1)$ at our disposal we can calculate the same for the entire path of masas exhibited in [41].

Write $\mathcal{R} = \mathcal{R}_{\text{even}} \overline{\otimes} \mathcal{R}_{\text{odd}}$ where $\mathcal{R}_{\text{odd}} = \overline{\otimes}_{r=1}^{\infty} \mathcal{M}_{2r-1}$ and $\mathcal{R}_{\text{even}} = \overline{\otimes}_{r=1}^{\infty} \mathcal{M}_{2r}$. Also denote $\mathcal{N}_{n,\text{even}} = \bigotimes_{r=1,2|r}^n \mathcal{M}_r$ and $\mathcal{N}_{n,\text{odd}} = \bigotimes_{r=1,r \equiv 1 \pmod{2}}^n \mathcal{M}_r$. Then the subfactor \mathcal{R}_{odd} contains a Cartan masa B so that $1 \otimes B \subset A(1)$. By construction B is a infinite tensor product of diagonal masas in the associated matrix algebras. Using similar arguments like the product class Tauer masa we can identify $C(X)$ as a *w.o.t* dense subalgebra of A where $X = \prod_{r=1}^{\infty} X_r$ and X_r is a set of k_r points.

Lemma V.7. *Let M_i for $i = 1, 2$ be II_1 factors with unique tracial states τ_i respectively. Let $B \subset M_2$ and $A \subset M_1 \overline{\otimes} M_2$ be masas so that $1 \otimes B \subset A$. Let $\text{Puk}_{M_1 \overline{\otimes} M_2}(A) = \{1\}$, $\text{Puk}_{M_2}(B) = \{1\}$. Let $C(X) \subset A$ and $C(Y) \subset B$ be w.o.t dense subalgebras with $1 \otimes C(Y) \subset C(X)$ and $[\eta_{X \times X}]$, $[\eta_{Y \times Y}]$ denote the left-right-measures for A and B respectively. If $q : X \mapsto Y$ denotes the continuous surjection associated with the inclusion of $C(Y)$ in $C(X)$ then $[\eta_{Y \times Y}] = [(q \times q)_* \eta_{X \times X}]$.*

Proof. With abuse of notation we will denote $\tau_1 \otimes \tau_2$ to be the tracial measure on X and τ_2 that on Y of A, B respectively. Then $q_*(\tau_1 \otimes \tau_2) = \tau_2$. Note that $f \mapsto f \circ (q \times q)$ for $f \in C(Y \times Y)$ is a injective $*$ -homomorphism from $C(Y \times Y)$ into $C(X \times X)$ which preserves least upper bounds at the level of continuous functions. Therefore i extends to a injective $*$ -homomorphism $\tilde{i} : L^\infty(Y \times Y, \eta_{Y \times Y}) \mapsto L^\infty(X \times X, \eta_{X \times X})$ which is normal (see Lemma 10.1.10, [19]). By Theorem 4.6 [25], $[(q \times q)_* \eta_{X \times X}] = [\eta_{Y \times Y}]$. The hypothesis guarantees that

$$L^2(M_1 \overline{\otimes} M_2) = L^2(X \times X, \eta_{X \times X}) = \int_{X \times X}^{\oplus} \mathbb{C}_{t,s} d\eta_{X \times X}(t, s),$$

$$L^2(M_2) = L^2(Y \times Y, \eta_{Y \times Y}) = \int_{Y \times Y}^{\oplus} \mathbb{C}_{t,s} d\eta_{Y \times Y}(t, s)$$

with respect to which $(A \cup J_{M_1 \overline{\otimes} M_2} A J_{M_1 \overline{\otimes} M_2})''$ and $(B \cup J_{M_2} B J_{M_2})''$ are respectively diagonalizable. Therefore $(q \times q)_* \eta_{X \times X}$ qualifies to be the *left-right-measure* of $B \subset M_2$. \square

Theorem V.8. *The left-right-measure $[\eta]$ of $A(1)$ is singular with respect to $[\tau_{A(1)} \otimes \tau_{A(1)}]$ where $\tau_{A(1)}$ is the tracial measure for $A(1)$.*

Proof. Fix a member η from the equivalence class. Let $C(\prod_{n=1}^{\infty} X_n)$ be w.o.t dense in $A(1)$, then by virtue of construction $C(\prod_{n=1}^{\infty} X_{2n-1})$ is w.o.t dense in B . Let $q : \prod_{n=1}^{\infty} X_n \mapsto \prod_{n=1}^{\infty} X_{2n-1}$ be the continuous surjection associated to the inclusion $1 \otimes B \subset A(1)$. Let

$\eta = f(\tau_{A(1)} \otimes \tau_{A(1)}) + \gamma$ with $\tau_{A(1)} \otimes \tau_{A(1)} \perp \gamma$ and $f \in L^1(\tau_{A(1)} \otimes \tau_{A(1)})$ be the decomposition of η with respect to $\tau_{A(1)} \otimes \tau_{A(1)}$ in the Lebesgue-Radon-Nikodym theorem. Note that if τ_B denotes the tracial measure for B with respect to the subfactor \mathcal{R}_{odd} then $\tau_B = q_*\tau_{A(1)}$ (as the inclusion of B in A arises from q). Now $E \subseteq \prod_{n=1}^{\infty} X_{2n-1} \times \prod_{n=1}^{\infty} X_{2n-1}$ is a null set for $\tau_B \otimes \tau_B$, implies

$$(\tau_{A(1)} \otimes \tau_{A(1)})((q \times q)^{-1}E) = 0,$$

which in turn implies $f(\tau_{A(1)} \otimes \tau_{A(1)})((q \times q)^{-1}E) = 0$.

Therefore $(q \times q)_*(f(\tau_{A(1)} \otimes \tau_{A(1)})) \ll \tau_B \otimes \tau_B$. Also $(q \times q)_*(f(\tau_{A(1)} \otimes \tau_{A(1)})) \neq 0$ if $f \neq 0$. We have

$$(q \times q)_*\eta = (q \times q)_*(f(\tau_{A(1)} \otimes \tau_{A(1)})) + (q \times q)_*\gamma.$$

Decompose $(q \times q)_*\gamma = g(\tau_B \otimes \tau_B) + \beta$ where $g \in L^1(\tau_B \otimes \tau_B)$ and $\beta \perp \tau_B \otimes \tau_B$. It follows $(q \times q)_*\eta$ cannot be singular with $\tau_B \otimes \tau_B$ if $f \neq 0$ or $g \neq 0$. But $[(q \times q)_*\eta]$ is the *left-right-measure* of B with respect to the subfactor \mathcal{R}_{odd} (Lemma V.7), which must be singular with $\tau_B \otimes \tau_B$ as B is Cartan in \mathcal{R}_{odd} . Therefore $f = 0, g = 0, \eta \perp \tau_{A(1)} \otimes \tau_{A(1)}$ and $(q \times q)_*\eta \perp \tau_B \otimes \tau_B$. \square

The fact that the *left-right-measure* of the *alternating Tauer masa* is singular with respect to the product measure will play a huge role in generating masas in free group factors. The alternating masa will be denoted by $A(1)$ in the subsequent sections.

B. Examples of Singular Masas in the Free Group Factors

In this section we show that given any subset S of \mathbb{N} there are uncountably many non conjugate singular masas in $L(\mathbb{F}_k)$, $k \geq 2$ with Pukánszky invariant $S \cup \{\infty\}$. All

examples constructed in this section are constructed from examples appearing in [11], [42]. For any masa A considered in this section, we assume $A = L^\infty([0, 1], \lambda)$, where λ is the Lebesgue measure. If $A \subset \mathcal{M}$ is a masa and $[\eta]$ be its *left-right-measure*, then we will most of the time assume $\eta(\Delta[0, 1]) = 0$. There are few exceptions to this assumption in this section, in which case we will notify.

The next two corollaries are direct applications of results in [11]. They can be generalized to give analogous statements about masas in the interpolated free group factors.

Corollary V.9. *Let $k \in \mathbb{N}_\infty$ and $k \geq 2$. Let $A \subset L(\mathbb{F}_k)$ be a masa. If A is freely complemented then $Puk(A) = \{\infty\}$ and its left-right-measure is the class of product measure. In particular, A is singular.*

Proof. Follows directly from Lemma 5.7 and Prop. 5.10 [11]. Singularity follows from the characterization theorem in chapter II. \square

Corollary V.10. *Let $k \in \mathbb{N}_\infty$ and $k \geq 2$, let $A \subset L(\mathbb{F}_k)$ be a masa. Let $A \subsetneq B \subsetneq L(\mathbb{F}_k)$ where B is a subalgebra and B is freely complemented.*

(i) *If the left-right-measure $[\eta_B]$ of the inclusion $A \subset B$ is singular with respect to $\lambda \otimes \lambda$ then $Puk_{L(\mathbb{F}_k)}(A) = Puk_B(A) \cup \{\infty\}$ and the left-right-measure of the inclusion $A \subset L(\mathbb{F}_k)$ is $[\eta_B + \lambda \otimes \lambda]$.*

(ii) *If the left-right-measure $[\eta_B]$ of the inclusion $A \subset B$ is $[\lambda \otimes \lambda]$, then $Puk_{L(\mathbb{F}_k)}(A) = \{\infty\}$ and the left-right-measure of the inclusion $A \subset L(\mathbb{F}_k)$ is $[\lambda \otimes \lambda]$.*

Proof. Easy. Consult Lemma 5.7 and Prop 5.10 [11]. \square

Let S be a nonempty subset of \mathbb{N} . Let $S = \{n_k\}$ with $n_1 < n_2 < \dots$. Define

$$\mathbb{P}_S = \left\{ \alpha = \{\alpha_{n_k}\}_{k=1}^{|S|} : \alpha_{n_k} > \alpha_{n_{k+1}}, 0 < \alpha_{n_k} < 1 \text{ for all } k, \sum_{k=1}^{|S|} \alpha_{n_k} = 1 \right\}. \quad (\text{B.1})$$

For $\alpha, \beta \in \mathbb{P}_S$, we say $\alpha \neq \beta$ if $\alpha_{n_k} \neq \beta_{n_k}$ for some k .

Case: $\{1, \infty\}$

Fix a sequence $\alpha \in \mathbb{P}_\mathbb{N}$. Consider the hyperfinite II₁ factor \mathcal{R} with the *alternating Tauer masa* $A(1)$. Let $\mathcal{R}_\alpha = \bigoplus_{n=1}^\infty \mathcal{R}$. Equip \mathcal{R}_α with the faithful trace

$$\tau_{\mathcal{R}_\alpha}(\cdot) = \sum_{n=1}^{\infty} \alpha_n \tau_{\mathcal{R}}(\cdot).$$

Then $A_\alpha = \bigoplus_{n=1}^\infty A(1)$ is a singular masa in the hyperfinite algebra \mathcal{R}_α which is still separable. It is still true that $\text{Puk}_{\mathcal{R}_\alpha}(A_\alpha) = \{1\}$. The projections $(0 \oplus \cdots \oplus 0 \oplus 1 \oplus 0 \cdots)$ where 1 appears at the n -th coordinate, is a central projection p_n of \mathcal{R}_α and it belongs to A_α . The projections p_n correspond to indicator of measurable subsets $F_n \subset ([0, 1], \lambda)$, so that $F_n \cap F_m$ is a set of λ measure 0 for all $n \neq m$. Upon applying appropriate transformations the *left-right-measure* of $A(1) \subset \mathcal{R}$ can be transported on each $F_n \times F_n$, which we denote by $[\eta_n]$. It follows from Thm V.8 that η_n is singular with respect to $\lambda \otimes \lambda$. We also assume $\eta_n(F_n \times F_n) = 1$ for all n . The *left-right-measure* of the inclusion $A_\alpha \subset \mathcal{R}_\alpha$ is then the class of

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \eta_n$$

from Prop. 5.2 [11]. Consider $(\mathcal{M}, \tau_{\mathcal{M}}) = (\mathcal{R}_\alpha, \tau_{\mathcal{R}_\alpha}) * (\mathcal{R}, \tau_{\mathcal{R}})$. Then \mathcal{M} is isomorphic to $L(\mathbb{F}_2)$ by a well known theorem of Dykema [9]. Then $A_\alpha \subset L(\mathbb{F}_2)$ is a singular masa by Theorem 2.3 [11]. The *left-right-measure* of the inclusion $A_\alpha \subset L(\mathbb{F}_2)$ is

$$[\lambda \otimes \lambda + \sum_{n=1}^{\infty} \frac{1}{2^n} \eta_n]$$

and $\text{Puk}_{L(\mathbb{F}_2)}(A_\alpha) = \{1, \infty\}$ from Cor. V.10.

Theorem V.11. *For each $\alpha \in \mathbb{P}_\mathbb{N}$, there exists a singular masa $A_\alpha \subset L(\mathbb{F}_2)$ with*

$Puk(A_\alpha) = \{1, \infty\}$. If $\alpha \neq \beta$ are any two elements of $\mathbb{P}_\mathbb{N}$ then A_α and A_β are not conjugate.

In particular, for any countable discrete group Γ there exist uncountably many non conjugate singular masas in $L(\mathbb{F}_2 * \Gamma)$ each having multiplicity $\{1, \infty\}$.

Proof. The construction of the masa $A_\alpha \subset L(\mathbb{F}_2)$ is established in the discussion above. Since automorphisms of II_1 factors preserve traces of projections and orthogonal projections the non conjugacy of A_α and A_β follows by considering the *left-right-measures*.

The final statement follows from the fact [9] that

$$\mathcal{R}_\alpha * (\mathcal{R} * L(\Gamma)) \cong (\mathcal{R}_\alpha * \mathcal{R}) * L(\Gamma) \cong L(\mathbb{F}_2) * L(\Gamma) \cong L(\mathbb{F}_2 * \Gamma)$$

and Cor. V.10. □

Case: $\{\infty\}$

Fix a sequence $\alpha \in \mathbb{P}_\mathbb{N}$ as before. Consider the hyperfinite II_1 factor \mathcal{R} with a singular masa A such that $Puk_{\mathcal{R}}(A) = \{\infty\}$. Consider the inclusion $B = \overline{\bigotimes_{n=1}^\infty A} \subset \overline{\bigotimes_{n=1}^\infty \mathcal{R}}$. Since up to isomorphism there is one hyperfinite II_1 factor so $B \subset \mathcal{R}$ is a singular masa from Chifan's Normaliser formula, $\Gamma(B) = 1$ and $Puk_{\mathcal{R}}(B) = \{\infty\}$ from Lemma 2.4 [49]. The *left-right-measure* of the inclusion $B \subset \mathcal{R}$ clearly contains as a summand (Lemma 5.2 [11]), a measure γ which is singular with respect to $\lambda \otimes \lambda$. In addition, $\gamma_t \neq 0$ for λ almost all t , where $[0, 1] \ni t \mapsto \gamma_t$ denotes the (π_1, λ) disintegration of γ . Now repeat the process described in Case: $\{1, \infty\}$ replacing $A(1)$ by B to conclude:

Theorem V.12. *For each $\alpha \in \mathbb{P}_\mathbb{N}$, there exists a singular masa $A_\alpha \subset L(\mathbb{F}_2)$ with $Puk(A_\alpha) = \{\infty\}$. If $\alpha \neq \beta$ are any two elements of $\mathbb{P}_\mathbb{N}$ then A_α and A_β are not conjugate.*

In particular, for any countable discrete group Γ there exist uncountably many non conjugate singular masas in $L(\mathbb{F}_2 * \Gamma)$ each having multiplicity $\{\infty\}$.

Case: $\{1, n, \infty\}$, $n \neq 1$

Let $1 \neq n \in \mathbb{N}$. Consider the matrix groups

$$G_n = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} \mid f \in P_n, x \in \mathbb{Q} \right\}, \quad H_n = \left\{ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \mid f \in P_n \right\} \subset G_n \text{ where}$$

$$P_\infty = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, p, q \text{ odd} \right\} \text{ and}$$

$$P_n = \{ f 2^{kn} \mid f \in P_\infty, k \in \mathbb{Z} \},$$

of the multiplicative group of nonzero rational numbers. Then $L(G_n)$ is the hyperfinite II_1 factor \mathcal{R} and $L(H_n) \subset L(G_n)$ is a singular masa with *Pukánszky invariant* $\{n\}$. The *left-right-measure* of the inclusion $L(H_n) \subset L(G_n)$ is the class of product Haar measure $\lambda_{\widehat{H}_n} \otimes \lambda_{\widehat{H}_n}$ on $\widehat{H}_n \times \widehat{H}_n$, where \widehat{H}_n denotes the character group of H_n .

As $\mathcal{R} \overline{\otimes} \mathcal{R} \cong \mathcal{R}$ so $L(H_n) \overline{\otimes} A(1)$ is a singular masa in \mathcal{R} from Chifan's Normaliser Formula. The *Pukánszky invariant* of the inclusion $L(H_n) \overline{\otimes} A(1) \subset \mathcal{R}$ is $\{1, n\}$ from Theorem 2.1 [42]. The *left-right-measure* of the inclusion $L(H_n) \overline{\otimes} A(1) \subset \mathcal{R}$ is the class of

$$\lambda_{\widehat{H}_n} \otimes \lambda_{\widehat{H}_n} \otimes \eta + \Delta_* \lambda_{\widehat{H}_n} \otimes \eta + \lambda_{\widehat{H}_n} \otimes \lambda_{\widehat{H}_n} \otimes \Delta_* \lambda$$

where $[\eta]$ is the *left-right-measure* of the *alternating Tauer masa* restricted to the off diagonal and Δ is the map that maps a set to its square by sending $x \mapsto (x, x)$. In this case we need to specify the measures on the diagonals as they are necessary. Given $\alpha \in \mathbb{P}_\mathbb{N}$ replace the role of $A(1)$ in the previous construction (Case: $\{1, \infty\}$) by $L(H_n) \overline{\otimes} A(1)$ to construct a masa $A_{\alpha, n} \subset L(\mathbb{F}_2)$.

Theorem V.13. For each $\alpha \in \mathbb{P}_\mathbb{N}$ and for each $1 \neq n \in \mathbb{N}$ there exists a singular

masa $A_{\alpha,n} \subset L(\mathbb{F}_2)$ with $\text{Puk}(A_{\alpha,n}) = \{1, n, \infty\}$. If $\alpha \neq \beta$ are any two elements of $\mathbb{P}_{\mathbb{N}}$ then A_{α} and A_{β} are not conjugate.

In particular, for any countable discrete group Γ there exist uncountably many non conjugate singular masas in $L(\mathbb{F}_2 * \Gamma)$ each having multiplicity $\{1, n, \infty\}$.

Proof. First use Theorem 3.2 of [11] to see that $\text{Puk}_{L(\mathbb{F}_2)}(A_{\alpha,n}) = \{1, n, \infty\}$. Then use Lemma 5.7, Prop. 5.10 of [11] to see that the *left-right-measure* of the inclusion $A_{\alpha,n} \subset L(\mathbb{F}_2)$ when viewed on $[0, 1] \times [0, 1]$ is of the same form as described in the previous construction (Case: $\{1, \infty\}$). Non conjugacy and the final statement follows for exactly similar reasons as in Theorem V.11. \square

Case: $\{n, \infty\}$, $n \neq 1$

Let $1 \neq n \in \mathbb{N}$. Let $H_n \subset G_n$ and $H_{\infty} \subset G_{\infty}$. Then $L(H_n \times H_{\infty})$ is a singular masa in $L(G_n \times G_{\infty})$ whose *measure-multiplicity-invariant* is the equivalence class of

$$(\widehat{H}_n \times \widehat{H}_{\infty}, [\eta], m)$$

where η is the sum of

- (i) Haar measure on $(\widehat{H}_n \times \widehat{H}_{\infty}) \times (\widehat{H}_n \times \widehat{H}_{\infty})$;
- (ii) Haar measure on the subgroup

$$D_n = \{(\alpha, \beta_1, \alpha, \beta_2) \mid \alpha \in \widehat{H}_n, \beta_1, \beta_2 \in \widehat{H}_{\infty}\};$$

- (iii) Haar measure on the subgroup

$$D_{\infty} = \{(\alpha_1, \beta, \alpha_2, \beta) \mid \alpha_1, \alpha_2 \in \widehat{H}_n, \beta \in \widehat{H}_{\infty}\};$$

and where the multiplicity function is given by

$$m(\gamma) = \begin{cases} 1, & \gamma \in \Delta(\widehat{H}_n \times \widehat{H}_\infty) \\ n, & \gamma \in D_n \setminus \Delta(\widehat{H}_n \times \widehat{H}_\infty) \\ \infty, & \text{otherwise.} \end{cases}$$

This was calculated in [11]. Note that η contains measures singular with respect to product Haar measure as its summand off the diagonal $\Delta(\widehat{H}_n \times \widehat{H}_\infty)$.

For each $\alpha \in \mathbb{P}_\mathbb{N}$, replace the role of $A(1)$ in the previous construction (Case $\{1, \infty\}$) by $L(H_n) \overline{\otimes} L(H_\infty)$ to construct a masa $A_{\alpha,n} \subset L(\mathbb{F}_2)$. Note that $\text{Puk}_{L(\mathbb{F}_2)}(A_{\alpha,n}) = \{n, \infty\}$ from Thm. 3.2 [11]. The *left-right-measure* of the inclusion $A_{\alpha,n} \subset L(\mathbb{F}_2)$ is of the same form as that of A_α as in Case: $\{1, \infty\}$.

Theorem V.14. *For each $\alpha \in \mathbb{P}_\mathbb{N}$ and for each $1 \neq n \in \mathbb{N}$ there exists a singular masa $A_{\alpha,n} \subset L(\mathbb{F}_2)$ with $\text{Puk}(A_{\alpha,n}) = \{n, \infty\}$. If $\alpha \neq \beta$ are any two elements of $\mathbb{P}_\mathbb{N}$ then $A_{\alpha,n}$ and $A_{\beta,n}$ are not conjugate.*

*In particular, for any countable discrete group Γ there exist uncountably many non conjugate singular masas in $L(\mathbb{F}_2 * \Gamma)$ each having multiplicity $\{n, \infty\}$.*

Proof. Clear. We omit the details. □

Theorem V.15. *Let $S = \{n_k : 1 = n_1 < n_2 < \dots\}$ be an arbitrary subset of \mathbb{N} that contains 1 and $|S| > 2$, let $k \in \{2, 3, \dots, \infty\}$ be arbitrary and let Γ be any arbitrary countable discrete group. For each $\alpha \in \mathbb{P}_S$ there exists a singular masa $A_{\alpha,S} \subset L(\mathbb{F}_k * \Gamma)$ whose Pukánszky invariant is $S \cup \{\infty\}$. If $\alpha \neq \beta$ are any two elements of \mathbb{P}_S then $A_{\alpha,S}$ and $A_{\beta,S}$ are not conjugate.*

Proof. We first consider the case when Γ is trivial. Let P_n and P_∞ be the subgroups of multiplicative group of rational numbers as before. Let G_n , $n \geq 1$, be the matrix

group

$$G_n = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix} : x, y \in \mathbb{Q}, f \in P_n, g \in P_\infty \right\}$$

and H_n the subgroup consisting of the diagonal matrices in G_n . Then as noted in [42] G_n is amenable and $L(G_n) \cong \mathcal{R}$. It is also true that $L(H_n)$ is a singular masa in $L(G_n)$ with Pukánszky invariant $\{n, \infty\}$ [11]. Consider $M_n = L(G_n) \overline{\otimes} \mathcal{R} \cong \mathcal{R}$ and consider the masa $A_n = L(H_n) \overline{\otimes} A(1)$. Then $A_n \subset M_n$ is a singular masa with $\text{Puk}_{M_n}(A_n) = \{1, n, \infty\}$ [42]. Now consider

$$\mathcal{M}_\alpha = \bigoplus_{n \in S} M_n \text{ and } A_\alpha = \bigoplus_{n \in S} A_n \text{ where } \tau_{\mathcal{M}_\alpha}(\cdot) = \sum_{n \in S} \alpha_n \tau_{M_n}(\cdot).$$

A_α is a singular masa in \mathcal{M}_α . Then $\mathcal{M}_\alpha * L(\mathbb{Z}) = \mathcal{M}_\alpha * L(\mathbb{F}_1) \cong L(\mathbb{F}_2)$ [9]. Moreover for $3 \leq k \leq \infty$ one has isomorphisms [9]

$$\begin{aligned} \mathcal{M}_\alpha * L(\mathbb{F}_{k-1}) &\cong \mathcal{M}_\alpha * (L(\mathbb{F}_1) * L(\mathbb{F}_{k-2})) \cong (\mathcal{M}_\alpha * L(\mathbb{F}_1)) * L(\mathbb{F}_{k-2}) \\ &\cong L(\mathbb{F}_2) * L(\mathbb{F}_{k-2}) \cong L(\mathbb{F}_k). \end{aligned}$$

Then A_α is singular in both \mathcal{M}_α and $L(\mathbb{F}_k)$ and $\text{Puk}_{L(\mathbb{F}_k)}(A_\alpha) = S \cup \{\infty\}$ from Theorem 3.2 [11]. There exist orthogonal projections $\{p_n\}_{n \in S} \subset A_\alpha$ with the property $\sum_{n \in S} p_n = 1$ and $\tau_{L(\mathbb{F}_k)}(p_n) = \alpha_n$ such that the *left-right-measure* of the inclusion $A_\alpha \subset L(\mathbb{F}_k)$ has $\lambda \otimes \lambda$ as a summand and measures singular with respect to $\lambda \otimes \lambda$ on the squares $p_n \times p_n$ (here by abuse of notation we think of p_n as the measurable set which corresponds to the projection p_n). The singular part on $p_n \times p_n$ has the property that its (π_1, λ) disintegration is non zero almost everywhere on p_n . Non conjugacy of A_α and A_β follows for $\alpha \neq \beta$ follows easily.

The case when Γ is non trivial follows from the fact that

$$\mathcal{M}_\alpha * (L(\mathbb{F}_k) * L(\Gamma)) \cong (\mathcal{M}_\alpha * L(\mathbb{F}_k)) * L(\Gamma) \cong L(\mathbb{F}_k) * L(\Gamma)$$

and Cor. V.10. □

Theorem V.16. *Let $S = \{n_k : n_1 < n_2 < \dots\}$ be an arbitrary subset of \mathbb{N} that does not contain 1 and $|S| \geq 2$, let $k \in \{2, 3, \dots, \infty\}$ be arbitrary and let Γ be any arbitrary countable discrete group. For each $\alpha \in \mathbb{P}_S$, there exists a singular masa $A_{\alpha,S} \subset L(\mathbb{F}_k * \Gamma)$ whose Pukánszky invariant is $S \cup \{\infty\}$. If $\alpha \neq \beta$ are any two elements of \mathbb{P}_S then $A_{\alpha,S}$ and $A_{\beta,S}$ are not conjugate.*

Proof. Once again we first deal with the case when Γ is trivial. Let G_n, H_n for $n \in \mathbb{N}_\infty$ be the groups that were used in Thm. 6.2 [11]. Let $M_n = L(G_n \times G_\infty)$ and $A_n = L(H_n \times H_\infty)$ for $n \in S$. Fix $\alpha \in \mathbb{P}_S$. Let $\mathcal{R}_{\alpha,S} = \bigoplus_{n \in S} M_n$ and $A_{\alpha,S} = \bigoplus_{n \in S} A_n$ where $\mathcal{R}_{\alpha,S}$ is equipped with the trace $\tau_{\mathcal{R}_{\alpha,S}}(\cdot) = \sum_{n \in S} \alpha_n \tau_{M_n}(\cdot)$. Replace the role of the masa $A_{\alpha,n}$ in Theorem V.14 by $A_{\alpha,S}$. We omit the details. The case when Γ is nontrivial follows by standard isomorphism theorems of free products of von Neumann algebras as explained before. □

The only case that is open is the one in which $Puk(A)$ is an infinite subset of \mathbb{N} . The existence of such a masa in the free group factors is unknown. Such a masa cannot be freely complemented by anything from Cor. V.9. We end this section with the observation that the *measure-multiplicity-invariant* for masas in the free group factors is far from being a complete invariant.

Theorem V.17. *There exist singular masas A, B in $L(\mathbb{F}_k)$, $2 \leq k \leq \infty$ with same measure-multiplicity-invariant.*

Proof. Let $\mathcal{R} = \prod_{n \in \mathbb{Z}} (\{0, 1\}, \mu) \rtimes \mathbb{Z}$ where $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$ and the action is Bernoulli shift. Then the copy of \mathbb{Z} gives rise to a masa $A \subset \mathcal{R}$ whose multiplicity

is $\{\infty\}$ and whose *left-right-measure* is the class of product measure. Consequently, for $k \geq 2$, $A \subset \mathcal{R} * \left(\begin{smallmatrix} k-1 \\ * \\ r=1 \end{smallmatrix} \mathcal{R} \right) \cong L(\mathbb{F}_k)$ [9] is a singular masa whose *left-right-measure* is the class of product measure and whose multiplicity function is $m \equiv \infty$. Let B be the single generator masa or radial masa of $L(\mathbb{F}_k)$. From [38] the radial masa has the same *measure-multiplicity-invariant* as A . The same holds for the single generator masas as well. A is not conjugate B as the former is not maximally injective, while the single generator and radial masas are maximally injective [1], [31]. \square

C. Unitary Conjugacy

The question of deciding inner conjugacy of a pair of masa initially started in the works of Feldman and Moore in [13]. In [36] several more equivalent conditions were given that decides the inner conjugacy of masas. In this section we cite examples of non inner conjugate singular masas in \mathcal{R} and free group factors.

Definition V.18. [49] Given a pair of masas A, B in a II_1 factor \mathcal{M} the *mixed-Pukánszky invariant* of A and B denoted by $Puk(A, B)$ (or $Puk_{\mathcal{M}}(A, B)$ if necessary) is $\text{Type}((A \cup JBJ)')$ where the commutant is taken in $\mathbf{B}(L^2(\mathcal{M}))$.

Let A, B be two masas in a II_1 factor \mathcal{M} . Let X, Y be compact Hausdorff spaces such that $C(X) \subset A$ and $C(Y) \subset B$ are unital, norm separable and *w.o.t* dense C^* subalgebras. To each such pair X, Y we associate a tuple $(X, Y, \nu_X, \nu_Y, [\eta_{X \times Y}], m_{X \times Y})$ where $\eta_{X \times Y}$ is the *measure* on $X \times Y$, $m_{X \times Y}$ is the *multiplicity function* that is obtained from the *direct integral decomposition* of $L^2(\mathcal{M})$ over $(X \times Y, \eta_{X \times Y})$ so that $(A \cup JBJ)''$ is unitarily equivalent to the algebra of *diagonalizable operators* with respect to this decomposition, and ν_X, ν_Y are completion of probability measures on X, Y respectively obtained by restricting the trace to $C(X)$ and $C(Y)$. Analogously as before we define an equivalence relation on the collection of tuples

$(X, Y, \nu_X, \nu_Y, [\eta_{X \times Y}], m_{X \times Y})$ indexed by compact Hausdorff spaces X, Y as above by, $(X, Y, \nu_X, \nu_Y, [\eta_{X \times Y}], m_{X \times Y}) \sim_{j.m.m.} (X', Y', \nu_{X'}, \nu_{Y'}, [\eta_{X' \times Y'}], m_{X' \times Y'})$ if and only if there exist Borel isomorphisms,

$$F : (X, \nu_X) \mapsto (X', \nu_{X'}) , G : (Y, \nu_Y) \mapsto (Y', \nu_{Y'}) \text{ such that,}$$

$$F_* \nu_X = \nu_{X'}, G_* \nu_Y = \nu_{Y'},$$

$$(F \times G)_* [\eta_{X \times Y}] = [\eta_{X' \times Y'}] \text{ and}$$

$$m_{X \times Y} \circ (F \times G)^{-1} = m_{X' \times Y'} , \eta_{X' \times Y'} \text{ a.e.}$$

Working as in chapter II one can show that $\sim_{j.m.m.}$ has exactly one equivalence class.

Definition V.19. Let $A, B \subset \mathcal{M}$ be masas in a II_1 factor \mathcal{M} . The *joint-measure-multiplicity invariant* of the pair A and B , is the *equivalence class* of

$$(X, Y, \nu_X, \nu_Y, [\eta_{X \times Y}], m_{X \times Y}) / \sim_{j.m.m.}$$

where X, Y are compact Hausdorff spaces such that $C(X) \subset A, C(Y) \subset B$ are unital, norm separable and *w.o.t* dense subalgebras, ν_X, ν_Y are the complete probability measures obtained from restricting τ on $C(X)$ and $C(Y)$ respectively, $\eta_{X \times Y}$ is the *measure* and $m_{X \times Y}$ is the *multiplicity function*, obtained from the *direct integral decomposition* of $L^2(\mathcal{M})$ over $(X \times Y, \eta_{X \times Y})$ so that $(A \cup JBJ)''$ is the algebra of *diagonalizable operators* with respect to this decomposition.

The measure class $[\eta_{X \times Y}]$ is said to be the *left-right-measure* of the pair (A, B) and the *joint-measure-multiplicity invariant* of a pair (A, B) is denoted by $j.m.m(A, B)$.

Note that the *Pukánszky invariant* and *mixed-Pukánszky invariant* are the essential values of the multiplicity functions in $m.m(A)$ and $j.m.m(A, B)$ respectively.

The *joint-measure-multiplicity invariant* is an invariant for pair of masas in the following sense. If $A, B \subset \mathcal{M}$ and $C, D \subset \mathcal{N}$ are masas in II_1 factors \mathcal{M}, \mathcal{N} respectively, and there is a unitary $U : L^2(\mathcal{M}) \mapsto L^2(\mathcal{N})$ such that, $UAU^* = C, UJ_{\mathcal{M}}BJ_{\mathcal{M}}U^* = J_{\mathcal{N}}DJ_{\mathcal{N}}$ then for any choice of compact Hausdorff spaces Y_A, Y_B, Y_C, Y_D with $C(Y_A), C(Y_B), C(Y_C), C(Y_D)$ unital, norm separable and *w.o.t* dense subalgebras of A, B, C, D respectively, there exist Borel isomorphisms

$F_{Y_A, Y_C} : (Y_A, \nu_{Y_A}) \mapsto (Y_C, \nu_{Y_C})$ and $F_{Y_B, Y_D} : (Y_B, \nu_{Y_B}) \mapsto (Y_D, \nu_{Y_D})$ such that

$$\begin{aligned} (F_{Y_A, Y_C})_* \nu_{Y_A} &= \nu_{Y_C}, (F_{Y_B, Y_D})_* \nu_{Y_B} = \nu_{Y_D}, \\ (F_{Y_A, Y_C} \times F_{Y_B, Y_D})_* [\eta_{Y_A \times Y_B}] &= [\eta_{Y_C \times Y_D}] \text{ and} \\ m_{Y_A \times Y_B} \circ (F_{Y_A, Y_C} \times F_{Y_B, Y_D})^{-1} &= m_{Y_C, Y_D}, \eta_{Y_C \times Y_D} \text{ a.e.} \end{aligned}$$

Also note that $j.m.m(A, A)$ is the same as $m.m(A)$ except for the fact that, the information on the diagonal is absent in $m.m(A)$.

Proposition V.20. *Let A and B be two masas in a II_1 factor \mathcal{M} . Let u, v be unitaries in \mathcal{M} . Then $j.m.m(uAu^*, vBv^*) = j.m.m(A, B)$. In particular, if A and B are inner conjugate then $j.m.m(A, B) = j.m.m(A, A)$.*

Proof. The automorphism $Ad(uJvJ)$ of $\mathbf{B}(L^2(\mathcal{M}))$ takes A to uAu^* and JB to $JvBv^*J$. So the result follows. \square

Let (X, μ) be a Lebesgue probability space with a free, ergodic, measure preserving action T of a countable discrete abelian group G . The crossed product algebra $\mathcal{R}_T = L^\infty(X, \mu) \rtimes_T G$ has two distinguished masas, the image C_T of $L^\infty(X, \mu)$ and the masa S_T generated by the canonical unitaries in \mathcal{R}_T implementing the action. The masa C_T is always *Cartan*.

Corollary V.21. $j.m.m_{\mathcal{R}_T}(C_T, S_T) = (X, \widehat{G}, \mu, \lambda_{\widehat{G}}, [\mu \otimes \lambda_{\widehat{G}}], 1) / \sim j.m.m$ where $\lambda_{\widehat{G}}$ is the normalised Haar measure on \widehat{G} . In particular, if S_T is Cartan (for example irrational rotation of \mathbb{Z}) then C_T, S_T are not inner conjugate.

Proof. The masas C_T and S_T are orthogonal and $\text{span}\{ab : a \in C_T, b \in S_T\}$ is dense in $L^2(\mathcal{R}_T)$. In particular 1 is a cyclic vector for $(C_T \cup JS_TJ)''$. The first statement now follows easily. The *left-right-measure* of C_T is concentrated on the union of automorphism graphs implemented by the group unitaries. The final statement is now obvious. \square

Recently, White has exhibited pairs of Cartan masas (A_n, B_n) , for $n \geq 2$ such that $\text{Puk}(A_n, B_n) = \{n\}$ [49]. So A_n and B_n are not inner conjugate from Prop. V.20.

Let $A, B \subset \mathcal{M}$ be masas. Identify $A \cong L^\infty([0, 1], \lambda)$ and $B \cong L^\infty([0, 1], \lambda)$ where λ is the Lebesgue measure. The bimodules that decide the inner conjugacy of masas A and B are $\overline{A\xi B}^{\|\cdot\|_2}$ and $\overline{B\xi A}^{\|\cdot\|_2}$, where $\xi = \hat{1}$.

Theorem V.22. [36] *Let $A, B \subset \mathcal{M}$ be masas. Then $A = uBu^*$ for some unitary $u \in \mathcal{M}$ if and only if $\overline{A\xi B}^{\|\cdot\|_2} \in C_d(A, B)$ and $\overline{B\xi A}^{\|\cdot\|_2} \in C_d(B, A)$.*

For examples constructed in the next couple of results we refer the reader back to the section on Tauer masas.

Lemma V.23. *Suppose $A, B \subset \mathcal{M}$ are masas. Let $A = \overline{\bigcup_{n=1}^\infty A_n}^{s.o.t}$ and $B = \overline{\bigcup_{n=1}^\infty B_n}^{s.o.t}$ where $A_n \subset A_{n+1}$, $B_n \subset B_{n+1}$ are finite dimensional subspaces for all n . If $\tau(ab) = \tau(a)\tau(b)$ for $a \in A_n$ and $b \in B_n$ and for all n then $\tau(ab) = \tau(a)\tau(b)$ for all $a \in A$, $b \in B$.*

Now we show the presence of two non inner conjugate singular masas in \mathcal{R} which have the same *measure-multiplicity-invariant*. For notations related to indexing

minimal projections we refer the reader to the section on Tauer masas.

Given a rapidly increasing sequence of primes k_r with $k_1 = 2$ one can always ensure $k_r \gg k_1 \cdots k_{r-1}$ for all $r > 1$. This allows one to construct different Tauer masas with respect to the same chain in the following way. Let $A_1 = D_2(\mathbb{C})$ and $B_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$. Having constructed A_n, B_n we construct A_{n+1}, B_{n+1} as,

$$\begin{aligned} A_{n+1} &= \bigoplus_{\underline{t}(n)}^{(n)} f_{\underline{t}(n),A} \otimes {}^{(n+1)}D_A^{\underline{t}(n)} \\ B_{n+1} &= \bigoplus_{\underline{t}(n)}^{(n)} f_{\underline{t}(n),B} \otimes {}^{(n+1)}D_B^{\underline{t}(n)}, \end{aligned} \quad (\text{C.1})$$

where the family $\left\{ {}^{(n+1)}D_A^{\underline{t}(n)}, {}^{(n+1)}D_B^{\underline{t}(n)} \right\}_{\underline{t}(n),A,B}$ are all orthogonal in $\mathcal{M}_{k_{n+1}}(\mathbb{C})$. Let $A = \overline{\bigcup_{n=1}^{\infty} A_n}^{s.o.t}$ and $B = \overline{\bigcup_{n=1}^{\infty} B_n}^{s.o.t}$. Then A, B are Tauer masas in \mathcal{R} for both of which the *left-right-measure* is the class of product measure. We claim that $\tau(ab) = \tau(a)\tau(b)$ for all $a \in A, b \in B$. In view of Lemma V.23 we need to show that $\tau(ab) = \tau(a)\tau(b)$ for $a \in A_n$ and $b \in B_n$. This is definitely true for $n = 1$. Suppose we have proved the assertion for $k = 1, 2, \dots, n$. For $a \in A_{n+1}, b \in B_{n+1}$ decompose a, b with respect to Eq. (C.1) as

$$a = \bigoplus_{\underline{t}(n)}^{(n)} f_{\underline{t}(n),A} \otimes a_{\underline{t}(n),A}, \quad b = \bigoplus_{\underline{s}(n)}^{(n)} f_{\underline{s}(n),B} \otimes b_{\underline{s}(n),B}.$$

Then by using orthogonality,

$$\begin{aligned} \tau(ab) &= \tau\left(\left(\bigoplus_{\underline{t}(n)}^{(n)} f_{\underline{t}(n),A} \otimes a_{\underline{t}(n),A}\right)\left(\bigoplus_{\underline{s}(n)}^{(n)} f_{\underline{s}(n),B} \otimes b_{\underline{s}(n),B}\right)\right) \\ &= \sum_{\underline{t}(n), \underline{s}(n)} \tau\left(\left({}^{(n)}f_{\underline{t}(n),A} {}^{(n)}f_{\underline{s}(n),B}\right) \otimes (a_{\underline{t}(n),A} b_{\underline{s}(n),B})\right) \\ &= \frac{1}{c_n^2} \sum_{\underline{t}(n), \underline{s}(n)} \tau(a_{\underline{t}(n),A}) \tau(b_{\underline{s}(n),B}) \\ &= \tau(a)\tau(b), \end{aligned}$$

where $c_1 = 1$ and for $n > 1$, $c_n = \prod_{r=1}^n k_r$. Therefore induction hypothesis proves the claim. It follows that $\overline{A\xi B}^{\|\cdot\|_2} \notin C_d(A, B)$ and $\overline{B\xi A}^{\|\cdot\|_2} \notin C_d(B, A)$, where $\xi = \hat{1}$. Therefore A, B are *not inner conjugate*.

We will now construct uncountably many non inner conjugate singular masas in \mathcal{R} and the free group factors with same *measure-multiplicity-invariant*. Fix a Cantor set $C = \prod_{n=1}^{\infty} C_n$ where $|C_n| = c_n$ and $c_{n+1} > c_n$ for all n . For each $(r_1, r_2, \dots) \in C$ where $1 \leq r_i \leq c_i$ we will construct a Tauer masa $A_{(r_1, r_2, \dots)}$. The *left-right-measure* of $A_{(r_1, r_2, \dots)}$ will be the class of product measure for each $(r_1, r_2, \dots) \in C$.

Set $k_1 = 2$. The initial approximation to the Tauer masas we are going to construct is $D_2(\mathbb{C})$ i.e,

$$(A_{(r_1, r_2, \dots)})_1 = D_2(\mathbb{C}).$$

Choose a prime $k_2 \gg k_1$ such that $\mathcal{M}_{k_2}(\mathbb{C})$ has at least $k_1 c_1$ orthogonal masas. Fix such a family of $k_1 c_1$ orthogonal masas and name it $\{D_{\underline{t}(1), r_1}\}_{\underline{t}(1), 1 \leq r_1 \leq c_1}$. Now define the second approximations to the Tauer masas as

$$(A_{(r_1, r_2, \dots)})_2 = \bigoplus_{\underline{t}(1)}^{(1)} f_{\underline{t}(1)}^{(r_1)} \otimes D_{\underline{t}(1), r_1}, \quad 1 \leq r_1 \leq c_1,$$

where $^{(1)}f_{\underline{t}(1)}^{(r_1)}$ are the minimal projections of $(A_{(r_1, r_2, \dots)})_1$.

Choose a prime k_3 so large that $k_3 \gg c_1 c_2 k_1 k_2$. So $\mathcal{M}_{k_3}(\mathbb{C})$ has at least $c_1 c_2 k_1 k_2$ orthogonal masas. Fix such a family and name it $\{D_{\underline{t}(2), r_1, r_2}\}_{1 \leq r_1 \leq c_1, 1 \leq r_2 \leq c_2}$. Construct the third approximations of the Tauer masas as

$$(A_{(r_1, r_2, \dots)})_3 = \bigoplus_{\underline{t}(2)}^{(2)} f_{\underline{t}(2)}^{(r_1)} \otimes D_{\underline{t}(2), r_1, r_2}, \quad 1 \leq r_i \leq c_i, i = 1, 2,$$

where $^{(2)}f_{\underline{t}(2)}^{(r_1)}$ are the minimal projections of $(A_{(r_1, r_2, \dots)})_2$.

Continuing inductively we can construct Tauer masas $A_{(r_1, r_2, \dots)}$ where $1 \leq r_i \leq c_i$.

Identify $A_{(r_1, r_2, \dots)} = L^\infty([0, 1], \lambda)$ where λ is the Lebesgue measure and the projections $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in A_{(r_1, r_2, \dots)}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A_{(r_1, r_2, \dots)}$ as indicator of Borel sets E and F respectively. The *left-right-measure* of each $A_{(r_1, r_2, \dots)}$ is the class of $\lambda \otimes \lambda$ from Prop. V.6.

For $c = (r_1, r_2, \dots)$ and $c' = (r'_1, r'_2, \dots)$ with $c \neq c'$ the measure arising out the vector functional $a \otimes b \mapsto \tau(ab)$, $a \in A_c$ and $b \in A_{c'}$ will be $\chi_{E \times E \cup F \times F} d(\lambda \otimes \lambda)$ from Lemma V.3.

Theorem V.24. *Given any Cantor set $C = \prod_{n=1}^{\infty} C_n$ with $|C_{n+1}| > |C_n|$ for all n , there exist a family of non inner conjugate Tauer masas $\{A_c\}_{c \in C}$ in \mathcal{R} the left-right-measure of each of which is the class of product measure.*

In particular, there is an uncountable family of non inner conjugate masas in $L(\mathbb{F}_k)$, $k \geq 2$ of product class and multiplicity $\{1, \infty\}$.

Concluding Remarks: So far, we have not been able to fully characterize strongly mixing masas. This problem is under investigation and we think that a masa $A \subset \mathcal{M}$ is strongly mixing, if and only if, $\tilde{\eta}^t$ is a mixing measure for λ almost all t . In the same direction, we suspect that the converse to Thm. IV.17 is true. Furthermore, we think that the measures $\tilde{\eta}^t$ must be β_t -rigid, λ almost all t , if A contains nontrivial centralizing sequences of \mathcal{M} .

All these questions are technical in nature, and we suspect that sophisticated approximation techniques from Fourier analysis will help to solve these questions. Even though these questions originate in operator algebras but finally they reduce to two technical questions in analysis, namely,

1. Under what conditions, a bounded sequence of measurable functions on $[0, 1]$ converge almost everywhere with respect to λ , and,
2. When can we interchange integrals and limits ?

We have produced uncountably many non conjugate singular masas in the free

group factors, having same multiplicity function. We have also given an example to show that *measure-multiplicity-invariant* is not a complete invariant. Thus the real question regarding the *measure-multiplicity-invariant* is: ‘When are two masas with the same *measure-multiplicity-invariant* conjugate?’ A very similar question was asked by Popa regarding Cartan masas.

We end by asking the following question: Suppose $\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ be the tensor product of two II_1 factors then, does there exist a masa $A \subset \mathcal{M}$, whose *left-right-measure* is singular to $\lambda \otimes \lambda$?

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