ASYMPTOTIC, ALGORITHMIC AND GEOMETRIC ASPECTS OF GROUPS GENERATED BY AUTOMATA

A Dissertation

by

DMYTRO SAVCHUK

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2009

Major Subject: Mathematics

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Approved by:

Co-Chairs of Committee,	Rostislav Grigorchuk
	Volodymyr Nekrashevych
Committee Members,	Sergiy Butenko
	Aleksei Poltoratski
	Zoran Sunik
Head of Department,	Al Boggess

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ABSTRACT

Asymptotic, Algorithmic and Geometric Aspects

of Groups Generated by Automata. (August 2009)

Dmytro Savchuk, B.S., National Taras Shevchenko University of Kyiv, Ukraine;

M.S., National Taras Shevchenko University of Kyiv, Ukraine

Co–Chairs of Advisory Committee: Dr. Rostislav Grigorchuk
Dr. Volodymyr Nekrashevych

This dissertation is devoted to various aspects of groups generated by automata. We study particular classes and examples of such groups from different points of view. It consists of four main parts.

In the first part we study Sushchansky p-groups introduced in 1979 by Sushchansky in "Periodic permutation p-groups and the unrestricted Burnside problem". These groups represent one of the earliest examples of Burnside groups and, at the same time, show the potential of the class of groups generated by automata to contain groups with extraordinary properties. The original definition is translated into the language of automata. The original actions of Sushchansky groups on p-ary tree are not level-transitive and we describe their orbit trees. This allows us to simplify the definition and prove that these groups admit faithful level-transitive actions on the same tree. Certain branch structures in their self-similar closures are established. We provide the connection with so-called G groups introduced by Bartholdi, Grigorchuk and Šunić in "Branch groups" that shows that all Sushchansky groups have intermediate growth and allows us to obtain an upper bound on their period growth functions.

The second part is devoted to the opposite question of realization of known groups as groups generated by automata. We construct a family of automata with

n states, $n \geq 4$, acting on a rooted binary tree and generating the free products of cyclic groups of order 2.

The iterated monodromy group $\mathrm{IMG}(z^2+i)$ of the self-map of the complex plain $z \mapsto z^2 + i$ is the central object of the third part of dissertation. This group acts faithfully on the binary rooted tree and is generated by 4-state automaton. We provide a self-similar measure for this group giving alternative proof of its amenability. We also compute an L-presentation for $\mathrm{IMG}(z^2+i)$ and provide calculations related to the spectrum of the Markov operator on the Schreier graph of the action of $\mathrm{IMG}(z^2 + i)$ on the orbit of a point on the boundary of the binary rooted tree.

Finally, the last part is discussing the package `AutomGrp` for `GAP` system developed jointly by the author and Yevgen Muntyan. This is a very useful tool for studying the groups generated by automata from the computational point of view. Main functionality and applications are provided.

To my wife Olga and daughters Anna and Irina

ACKNOWLEDGMENTS

First of all, I would like to express my warm gratitude to my advisors, Dr. Rostislav Grigorchuk and Dr. Volodymyr Nekrashevych for their support, constant attention and interest to my work, fruitful discussions and help.

Besides, there are really a lot of people to whom I am grateful for their impact on my life. I highly appreciate help of my friends and co-authors, Ievgen Bondarenko, Rostyslav Kravchenko, Yevgen Muntyan, Zoran Šunić, Maria and Yaroslav Vorobets. I am grateful to Vitaliy Sushchansky, my advisor in Kyiv Taras Shevchenko University, for attracting me to the field of group theory and supporting my early interest in groups generated by automata.

I was lucky enough to have many great teachers in my life from whom I learned a lot. I highly acknowledge the efforts of each of them. Particularly, I am greatly indebted to my father for bringing me to the beautiful world of mathematics.

And the last, but not the least, I am really thankful to my immediate family: my wife Olga and daughters Anna and Irina for their help and being a source of immense inspiration.

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CHAPTER I

INTRODUCTION

The first appearance of groups generated by automata goes back to the beginning of 1960's [Glu61, Hořı63]. But it took a while to realize their importance, utility, and, at the same time, complexity. Among the publications from the first decade of the study of automaton groups let us distinguish [Zar64, Zar65] and the book [GP72].

The first substantial results came only in the 1970's and in the beginning of the 1980's when the it was shown in [Ale72, Sus79, Gri80, GS83b] that automaton groups provide examples of finitely generated infinite torsion groups, thus making a contribution to one of the most celebrated problems in algebra — the General Burnside Problem. Initially, this problem was solved by E.S. Golod in [Gol64] using the Golod-Shafarevich theorem.

There are two other version of Burnside problem. The "bounded Burnside problem" (also originally asked by Burnside) asks for the existence of infinite finitely generated group of bounded exponent (a group has bounded exponent n if g^n is trivial for all elements g of the group). This question obviously reduces to the question of finiteness of free Burnside groups $B(m,n)$ with m generators and exponent n. This problem was solved positively in a series of long and very technical papers by Novikov and Adian [NA68, NA68, NA68]. Namely, they have shown that $B(m,n)$ is infinite for any odd $n \geq 4381$ and $m > 1$. Later Adian in [Adi79] improved the bound for n to 665. The question for the even exponent remained open until Ivanov [Iva94] and Lysenok [Lys96] obtained independently proofs for exponents $\geq 2^{48}$ and ≥ 8000 correspondingly.

This dissertation follows the style of *Algebra and Discrete Mathematics.*

The third version of Burnside problem, bearing the name "restricted Burnside problem", asks if there is, for any given m and n, an m generated finite groups with exponent n of maximal order. Of course, if the free Burnside group $B(m,n)$ is finite, this group would be such a maximal group. But, as was discussed above, this is not always the case. For prime exponents this question has been settled in the affirmative by Kostrikin in 1950s. The complete positive solution of the restricted Burnside problem was obtained by Zelmanov [Zel91b, Zel90, Zel91a]. More information on all three versions of the problem can be found in [Adi79, Gol68, Gup89, Kos90, Zel91b, GL02]).

In fact, the possible relation of groups generated by automata to the General Burnside problem for the first time was suggested by Glushkov in [Glu61, p.46]. The methods used to study the properties of the examples from [Ale72, Sus79, Gri80] are very different. The methods used in [Ale72] are typical for the theory of finite automata (in fact, the provided proof was incorrect; the first correct proof appeared in [Mer83] as a combination of the results from [Gri80] and [Mer83], as well as in the third edition of the book [KM82] and in [KAP85]). The exposition in [Sus79] is based on Kaloujnine's tableaux coming from his theory of iterated wreath products of cyclic groups of prime order p. The approach in [Gri80] is based on the ideas of self-similarity and contraction. These ideas are apparent both in the proof of the infiniteness and the torsion property of the group. The self-similarity is apparent from the fact that the set of all states of the automaton is used as a generating set for the group (now it is common to call such groups self-similar). The contraction property here means that the length of the elements contracts by a factor bounded away from 1 when one passes to sections. A principal tool introduced in the beginning of the 1980's was the language of actions on rooted trees suggested by Gupta and Sidki in [GS83b], which helped tremendously in bringing geometric insight to the subject.

A new indication of the importance of automaton groups came when it was shown that some of them provided the first examples of groups of intermediate growth [Gri83, Gri84, Gri85a]. This not only answered the question of J.Milnor [Mil68] about existence of such groups, but also answered a number of other questions in and around group theory, including M. Day's problem [Day57] on existence of amenable but not elementary amenable groups. Basically, even to this day, all known examples of groups of intermediate growth and non-elementary amenable groups are based on automaton groups or groups acting on trees.

Actually, there is no mentioning of automata groups in the original paper by Sushchansky [Sus79] on p-groups. As was mentioned above, V.I. Sushchansky used a different language, namely the language of tableaux, introduced by L. Kaloujnine to study properties of iterated wreath products [Kal48]. For each prime $p > 2$, V.I. Sushchansky constructed a finite family of infinite p-groups generated by two tableaux. Each such a tableau naturally defines an automorphism of a rooted tree and, as was already noticed in [GNS00], can be represented by a finite initial automaton.

After introducing in Chapter II necessary notions, definitions and concepts, Chapter III of this dissertation is devoted to study of Sushchansky groups using the language of automata groups. The results of Chapter III are published in paper [BS07] written jointly with I. Bondarenko. In particular, we construct initial automata generating Sushchansky groups. The associated action on a rooted tree happens to be not level-transitive and we describe its orbit tree and show that there exists a faithful level-transitive action given by finite initial automata. Sushchansky groups are not self-similar, in other words, they are not generated by all states of generating initial automata. But every group generated by initial automata can be naturally embedded into a self-similar group just by adding all states of automata to the generating set. We show that self-similar closures of Sushchansky groups are generated by bounded automata and hence, are amenable. We find elements of infinite order in these groups, show that they are weakly regular branch, and find regular branch subgroups of these groups.

The main result of Chapter III is related to the Milnor problem on growth. It was pointed out in [Gri85a] that all Sushchansky p-groups have intermediate growth, but only the main idea of the proof was given. Here we provide a complete proof of this fact together with new estimates on the growth function, thus contributing to the Milnor question [Mil68]. Also we give an upper bound on the period growth function. The main idea is to use G groups of intermediate growth introduced in [BŠ01] (see also [BGŠ03]). For each Sushchansky p-group we construct a G group of intermediate growth and prove that their growth functions are equivalent.

Investigations in the last two decades [Gri84, Gri85a, GS83b, GS83a, Lys85, Neu86, Sid87a, Sid87b, Gri89, Roz93, Gri98, Gri99, Gri00, BG00a, BG00b, GŻ01, Nek05, GŠ06] show that many automaton groups possess numerous interesting, and sometimes unusual, properties. This includes just infiniteness (the groups constructed in [Gri84, Gri85a] as well as in [GS83a] answer a question from [CM82] on new examples of infinite groups with finite quotients), finiteness of width, or more generally polynomial growth of the dimension of the successive quotients in the lower central series [BG00b] (answering a question of E. Zelmanov on classification of groups of finite width), branch properties [Gri84, Neu86, Gri00] (answering some questions of S. Pride and M. Edjvet [Pri80, EP84]), finiteness of the index of maximal subgroups and presence or absence of the congruence property [Per00, Per02] (related to topics in pro-finite groups), existence of groups with exponential but not uniformly exponential growth [Wil04b, Wil04a, Bar03, Nek07b] (providing an answer to a question of M. Gromov), subgroup separability and conjugacy separability [GW00], further examples of amenable groups but not amenable (or even sub-exponentially amenable) groups [GŻ02a, BV05, GNŠ06], amenability of groups generated by bounded automata [BKN08], and so on.

The above discussion shows that the class of groups generated by automata is interesting and rich as a counterexample-box. On the other hand, all transformations defined by states of finite invertible automata over a fixed alphabet form a group of automatic transformations over this alphabet and an interesting question is the embedding of known groups into this group. For example, Brunner and Sidki proved in [BS98] that $GL_n(\mathbb{Z})$ can be generated by finite automata over the alphabet with 2^n letters. In Chapter IV of this dissertation we address this question in regard to the free products of groups of order 2 (we will often denote the group of order 2 by C_2). The first embedding of such free products into the group of automatic transformations over the 2-letter alphabet was constructed by Olijnyk [Olī99]. We also mention results of C. Gupta, N. Gupta and A. Olijnyk [Oli00, GGO07] who embedded the free product of any finite family of finite groups into a group of automatic transformations over a suitable alphabet.

The above constructions lack the important property of self-similarity. In other words, the group is not generated by all states of a single automaton. The first self-similar example was provided by a 3-state automaton $\mathcal{B}_3$ over 2-letter alphabet whose Moore diagram is depicted in the left half of Figure 1, where $\sigma = (1,2)$ denotes the nontrivial element of $\mathrm{Sym}(\{1,2\})$. This automaton was studied during the summer school in Automata groups held in 2004 at the Autonomous University of Barcelona in Bellaterra. Since then, it is known as the Bellaterra automaton. It was proved by Muntyan (see the proof in [BGK$^+$08] or [Nek05]) that $\mathcal{B}_3$ generates the group isomorphic to the free product of 3 copies of groups of order 2.

Many papers on free groups and free products generated by automata share the same idea of dual automaton. For an automaton $\mathcal{A}$ the dual automaton $\hat{\mathcal{A}}$ is obtained

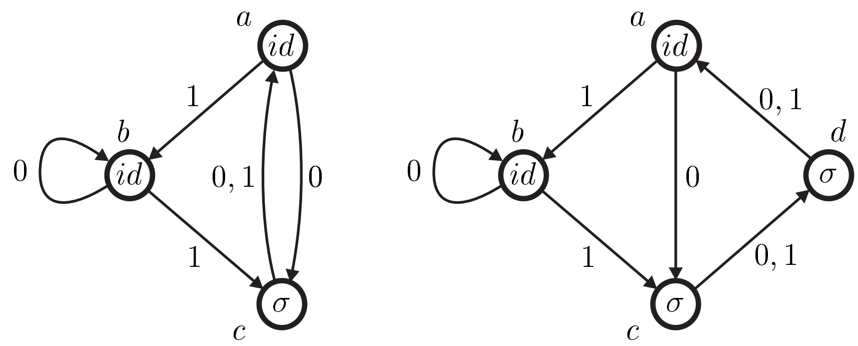


Fig. 1. Bellaterra automata $\mathcal{B}_3$ and $\mathcal{B}_4$

from $\mathcal{A}$ by interchanging the states and the alphabet, and swapping the transition and output functions. For precise definition see Section 1 in Chapter IV. It turns out that the "freeness" properties of the group generated by $\mathcal{A}$ are related to certain transitivity conditions of the action of the group generated by $\hat{\mathcal{A}}$.

The Bellaterra automaton belongs to the class of bireversible automata [Nek05, GM05], which seems to be a natural source for automata generating free groups and free products. An invertible automaton is called bireversible if its dual and the dual to its inverse are also invertible. It is worth mentioning that the Bellaterra automaton was discovered while classifying all bireversible 3-state automata over 2-letter alphabet.

The Bellaterra automaton gives rise to a family of bireversible automata in which all states define involutive transformations. Namely, we modify the automaton $\mathcal{B}_3$ by inserting new states on the path from c to a. More precisely, each automaton in the family is defined by wreath recursion

$$\begin{aligned} a &= (c,b), \\ b &= (b,c), \\ c &= (q_1,q_1)\sigma_0, \\ q_i &= (q_{i+1},q_{i+1})\sigma_i,\ \ i=1,\dots,n-4, \\ q_{n-3} &= (a,a)\sigma_{n-3}, \end{aligned} \tag{1.1}$$

where $\sigma_i \in \mathrm{Sym}(\{0,1\})$ is chosen arbitrarily.

Conjecturally, each automaton in the family for which at least one of the σ_i is nontrivial, generates the free product of groups of order 2. The first result supporting this conjecture was obtained by M. Vorobets and Y. Vorobets [VV06]. It was shown that if the number of states is odd and $\sigma_i = (12)$ for all i, then the conjecture holds. In the subsequent paper by the same authors and B. Steinberg [SVV06] the conjecture was proved for the automata with even number of states and additional condition that the number of nontrivial σ_i is odd.

In Chapter IV we prove that any n-state automaton $\mathcal{B}_n$ from the family (1.1) with $n \geq 4$ satisfying $\sigma_0 = (12)$ and $\sigma_{n-3} = (12)$ generates the free product of groups of order 2. The smallest automaton $\mathcal{B}_4$ in this family is shown in the right half of Figure 1. This result covers the series constructed in [VV06] except one, but the most important case $n = 3$, and partially overlaps with a family constructed in [SVV06]. The results of Chapter IV are presented in paper [SV08] written jointly with Y. Vorobets.

In addition to the formulation of many algebraic properties of groups generated by finite automata, a number of links and applications were discovered during the last decade. This includes asymptotic and spectral properties of the Cayley graphs and Schreier graphs associated to the action on the rooted tree with respect to the set of generators given by the set of states of the automaton. For instance, it is shown in [GŻ01] that the discrete Laplacian on the Cayley graph of the Lamplighter group $\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ has pure point spectrum. This fact was used to answer a question of M. Atiyah on L^2-Betti numbers of closed manifolds [GLSŻ00]. The methods developed in the study of the spectral properties of Schreier graphs of self-similar groups can be used to construct Laplacians on fractal sets and to study their spectral properties (see [GN07, NT08]).

One of the most remarkable discoveries in the recent years, due to, first of all, V. Nekrashevych, is that the so-called iterated monodromy groups (IMG), which can be related to any self-covering map, belong to the class of self-similar groups and that, in the most natural situations, there is an explicit procedure representing them by finite automata.

Even in the case of quadratic maps over $\mathbb{C}$ one gets a rich theory with wonderful applications both to holomorphic dynamics and to group theory [Nek05, BN06].

Already the simplest examples of quadratic polynomials, such as $z^2 - 1$ or $z^2 + i$, show that the corresponding groups can be quite complicated and can have extraordinary properties.

The group $\mathrm{IMG}(z^2 - 1)$ is called the Basilica group after the Julia set of $z^2 - 1$ which (somewhat) resembles the roof of the San Marco Basilica in Venice (the top part of the Julia set is the roof and the bottom part is its reflection in the water). Basilica group is torsion free, of exponential growth, amenable but not elementary (and not even subexponentially) amenable [BV05, GŻ02a], has trivial Poisson boundary, is weakly branch, and has many other interesting properties.

The main object of Chapter V of this dissertation is the group $\mathrm{IMG}(z^2 + i)$, introduced in [BGN03] and later studied by Bux and Pérez [BP06], who proved that $\mathrm{IMG}(z^2 + i)$ has intermediate growth. This is not the first example of a self-similar group of intermediate growth (the first examples were constructed in [Gri80, Gri84]), but it is the first example of a group of intermediate growth that naturally appears in the area of applications of group theory.

The results of Chapter V are published in paper [GSŠ07] written jointly with R. Grigorchuk and Z. Šunić. We give very detailed calculation of the action of $\mathrm{IMG}(z^2+i)$ on the binary rooted tree. Then we show that the group $\mathrm{IMG}(z^2+i)$ is a regular branch group, thus presenting an example of a branch group which naturally appears in holomorphic dynamics. The main body of Chapter V is devoted to the calculation of an L-presentation for $\mathrm{IMG}(z^2+i)$ (i.e., a presentation of a group by generators and relations which involves a finite set of relators and their iterations by substitution). Although it is known that L-presentations are quite common for groups of branch type the number of examples in which explicit computation is performed is rather small. We also note that $\mathrm{IMG}(z^2+i)$ follows in the family of iterated monodromy groups of post-critically finite quadratic polynomials studied in [BN08]. For every group in this class the authors, in particular, give an L-presentation, which, in the case of $\mathrm{IMG}(z^2+i)$, is more complicated than the one obtained in Chapter V.

The presence of L-presentations is important from different points of view. Such presentations are at the first level of complexity after the finite presentations and quite often provide the simplest way to describe a group that is not finitely presented ($\mathrm{IMG}(z^2+i)$ is not finitely presented [Nek07c]). Further, such presentations can be used to embed a group into a finitely presented group in a way that preserves many properties of the original group. We use the obtained L-presentation of $\mathrm{IMG}(z^2+i)$ to embed $\mathrm{IMG}(z^2+i)$ into a finitely presented group with 4 generators and 10 relators, which is amenable but not elementary amenable (this approach has been used for the first time in [Gri98]).

The rest of Chapter V deals with finding a self-affine measure on $\mathrm{IMG}(z^2+i)$.

The notion of a self-affine measure was introduced by Kaimanovich in [Kai05] (under the name of self-similar measure), who extends some ideas (in particular the idea of self-similarity of a random walk) that appeared in the work of Bartholdi and Virág [BV05].

The self-affine measure is closely related to the problem of computation of the spectrum of a Hecke type operator that can be related to any group acting on a rooted tree and to the problem of computation of the spectrum of the discrete Laplace operator (or, what is almost the same, the Markov operator) on the boundary Schreier graph of a group (i.e., the graph of the action of the group on the orbit of a point of the boundary of the tree). A general approach to the spectral problem (which extends the ideas outlined in [BG00a, GŻ99]) is based on a renormalization principle and leads to questions on amenability, multidimensional dynamics and multiparametric self-similarity of operators. Unfortunately, the spectral problem is not solved yet in our situation. What we are able to construct is a rational map on $\mathbb{R}^3$ whose proper invariant set (shaped as a "strange attractor") gives the spectrum of the Markov operator after intersection by a corresponding line. Here we have a situation analogous to the case of Basilica group [GŻ02b]. Further efforts in the description of the shape of the attractor (and hence of the spectrum) are needed.

The Schreier graph in this case, viewed through a macroscope, has a form of a dendrite and this is a reflection of a general fact relating the geometry of Schreier graphs and Julia sets proved by Nekrashevych [Nek05].

In any case, our computations allow us to find a self-affine nondegenerate measure on $\mathrm{IMG}(z^2+i)$, which gives a self-similar random walk on the group. The study of asymptotic properties of such random walks is a promising direction and will be one of our subsequent subjects of investigation.

In many situations automaton groups serve as renorm groups. For instance this happens in the study of classical fractals, in the study of the behavior of dynamical systems [Oli98], and in combinatorics – for example in Hanoi Towers games on k pegs, $k \geq 3$, as observed by Z. Šunić (see [GŠ06, GŠ08]).

There is interest of computer scientists and logicians in automaton groups, since they may be relevant in the solution of important complexity problems (see [RS08] for ideas in this direction). Self-similar groups of intermediate growth are mentioned by Wolfram in [Wol02] as examples of "multiway systems" with complex behavior.

Groups and semigroups generated by automata are extremely interesting from the computational point of view. The word problem can be solved in contracting self-similar groups by using an extremely effective *branch algorithm* [Gri84, Sav03]. The conjugacy problem can also be solved in many cases [WZ97, Roz98, Leo98, GW00, LMU08] (in fact, we do not know of an example of an automaton group with unsolvable conjugacy problem). In some instances, it is even known that the membership problem is solvable [GW03].

On the other hand, major algorithmic problems are unsolved so far in the general case but have solutions in certain special cases. The computations related to these groups are often cumbersome to be performed by hands and the computers may be of a great help here.

There was a strong need in the implementation of the algorithms related to automata groups and semigroups in some computer algebra system. The package `AutomGrp` [MS08] for `GAP` (Groups, Algorithms and Programming) system [GAP08], developed jointly by the author of this dissertation and Yevgen Muntyan to satisfy this need, is discussed in Chapter VI. The package was successfully used in the project of the classification of groups generated by 3-state automata over 2-letter alphabet [BGK+08], as well as by several other authors. Currently the status of the package is "deposited", but we are planning to submit it for refereeing in the nearest future.

Finally, Chapter VII concludes the dissertation by listing main results and stating several open problems and conjectures, as well as possible directions for further investigations.

CHAPTER II

DEFINITIONS AND NOTATIONS

This chapter introduces most of necessary notions used throughout the dissertation.

1 Rooted trees

All groups we consider in this dissertation act on certain rooted trees. We start from defining these basic objects.

Definition 1. A *tree* is a connected graph with no cycles. A *rooted tree* is a tree with a selected vertex called the *root* of the tree.

It is natural to consider a combinatorial distance in the tree (the number of the edges in the shortest path connecting two vertices). The sphere of radius n centered at the root of the tree is called the n-th level of the rooted tree.

Definition 2. A *homogenous (or spherically homogenous) rooted tree* is a rooted tree where the degrees of the vertices of the same level are equal.

Definition 3. A *regular rooted tree of degree d (or d-ary tree)* is an infinite homogenous rooted tree where the degree of a root is d and the degrees of the other vertices are equal to $d+1$.

In a d-ary tree the n-th level has d^n vertices. In the case $d=2$ the tree is called *binary* (see Figure 2).

In order to study the groups acting on trees there is a need to address the vertices of the latter. For regular rooted trees this is done in the following way.

Let X be a finite alphabet of cardinality d. Denote by X^* the set of all finite words over X. The set X^* can be naturally endowed with a structure of a regular

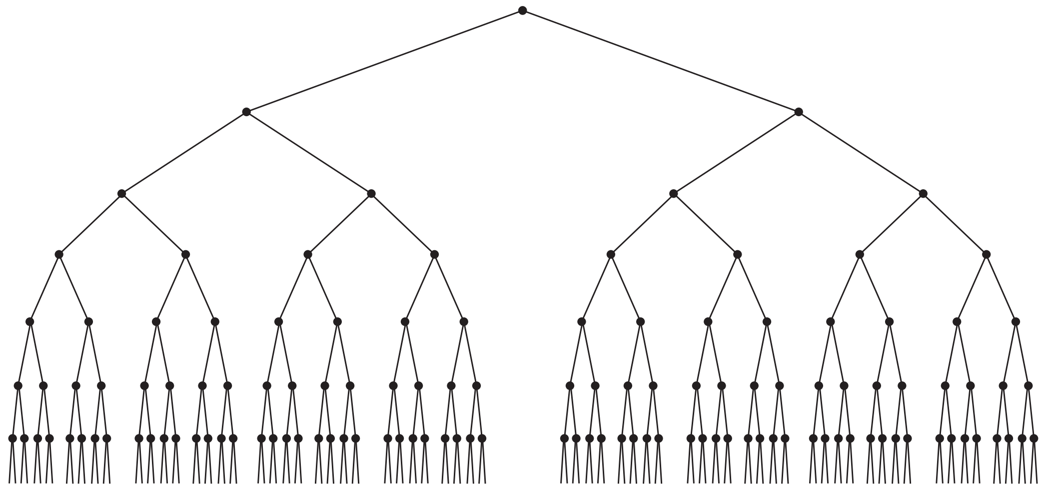


Fig. 2. Binary tree

d-ary rooted tree by declaring that v is connected to vx for each $v \in X^*$ and $x \in X$. The empty word $\emptyset$ plays a role of a root in this tree.

For spherically homogenous rooted trees one can perform similar identification of the vertices of the tree with the finite words over the sequence of alphabets, whose cardinalities agree with the degrees of the vertices on levels.

An important object is the boundary X^ω of tree X^*, consisting of all infinite words over X (if the tree is denoted by T its boundary is usually denoted by ∂T).

The boundary of the tree has a natural ultrametric space structure with respect to the following ultrametric. We define that the distance between words $x_1x_2x_3\ldots \in X^\omega$ and $y_1y_2y_3\ldots \in X^\omega$ to be 2^{-n}, where n is the length of the longest common beginning of these two words. Thus, the words are closer if their common beginning is longer.

Topologically the boundary of a homogenous rooted tree is homeomorphic to the Cantor set.

2 Self-similar groups and automata

Consider the case of a regular d-ary tree X^* (with $X = \{0, 1, 2, \ldots, d-1\}$). The group $\operatorname{Aut} X^*$ of all automorphisms of X^* has the structure of an infinite iterated permutational wreath product $\wr_{i\geq 1} \operatorname{Sym}(d)$ (because $\operatorname{Aut} X^* \cong \operatorname{Aut} X^* \wr_X \operatorname{Sym}(d)$, where $\operatorname{Sym}(d)$ acts naturally on X by permutations). This gives a convenient way to express automorphisms from $\operatorname{Aut} X^*$ in the form

$$g = (g|_0, g|_1, \ldots, g|_{d-1})\sigma_g, \tag{2.1}$$

where $g|_0, g|_1, \ldots, g|_{d-1}$ are automorphisms of the subtrees of X^* with roots at the vertices $0, 1, \ldots, d-1$ (these subtrees are canonically identified with X^*) induced by g, and σ_g is the permutation of X induced by g (i.e., $\sigma_g(x) = g(x)$ – the action of g on $x \in X$). This decomposition is schematically shown in Figure 3.

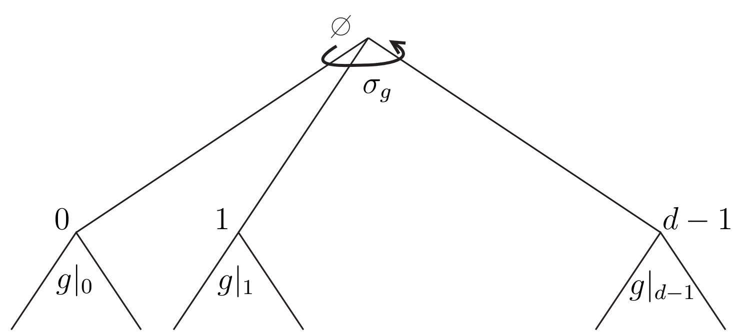


Fig. 3. Decomposition of an automorphism of the tree

More generally, for every $v \in X^*$ we define $g|_v$ to be the automorphism of the subtree of X^* rooted at v (shown in Figure 4) identified with X^* induced by g. The automorphism $g|_v$ is called the *section* of g at v and is uniquely determined by $g(vw) = g(v)g|_v(w)$, for all $w \in X^*$.

Throughout the dissertation we will use the following convention. If g and h are

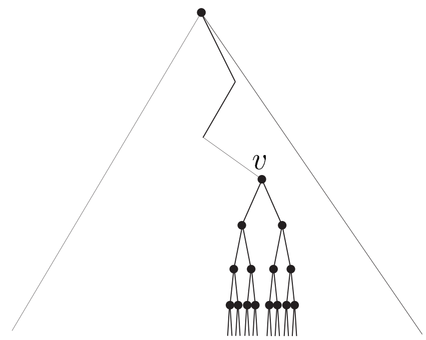


Fig. 4. Subtree rooted at v on which $g|_v$ acts

the elements of some (semi)group acting on set A and $w \in A$, then

$$gh(w) = h\big(g(w)\big). \tag{2.2}$$

Taking into account convention (2.2) one can compute sections of any element of an automaton semigroup as follows. If $g = g_1 g_2 \cdots g_n$ and $v \in X^*$, then

$$g|_v = g_1|_v \cdot g_2|_{g_1(v)} \cdots g_n|_{g_1 g_2 \cdots g_{n-1}(v)}. \tag{2.3}$$

The decomposition (2.1) is particularly important from the computational point of view. The product of automorphisms written in this form is performed in the following way. If $h = (h|_0, h|_1, \ldots, h|_{d-1})\sigma_h$ then

$$gh = (g|_0 h|_{\sigma_g(0)}, \ldots, g|_{d-1} h|_{\sigma_g(d-1)})\sigma_g \sigma_h.$$

Definition 4. A group $G \leq \operatorname{Aut} X^*$ is called *self-similar* if $g|_u \in G$ for all $g \in G$ and $u \in X^*$.

A convenient way to describe a particular finitely generated self-similar group G generated by automorphisms $g_1, g_2, \ldots, g_n$ is through a, so-called, *wreath recursion*.

In this presentation we simply write the action of each g_i in the form

$$g_i = \big(w_1(g_1,\ldots,g_n),\ldots,w_d(g_1,\ldots,g_n)\big)\sigma_{g_i},$$

where w_i, $i=1,\ldots,n$, are words in the free group of rank n.

Another fundamental language which describes self-similar groups is the language of automaton groups (see the survey paper [GNS00] for details).

Definition 5. A *Mealy automaton* is a tuple (Q,X,π,λ), where Q is a set (a set of states), X is a finite alphabet, $\pi\colon Q\times X\to Q$ is a transition function and $\lambda\colon Q\times X\to X$ is an output function. If the set of states Q is finite the automaton is called *finite*.

One can think of an automaton as a sequential machine which, at each moment of time, is in one of its states. Given a word $w\in X^*$ the automaton acts on it as follows. It "eats" the first letter x in w and depending on this letter and on the current state q it "spits out" a new letter $\lambda(q,x)\in X$ and changes its state to $\pi(q,x)$. The new state then handles the rest of word w in the same fashion. Thus the map λ can be extended to $\lambda\colon Q\times X^*\to X^*$ – we just feed the automaton with letters of $u\in X^*$ one by one. Each state q of the automaton defines a map, also denoted by q, from X^* to itself defined by $q(w)=\lambda(q,w)$. In the special case when, for all $q\in Q$, the map $\lambda(q,\cdot)$ is a permutation of X the map $q\colon X^*\to X^*$ is invertible and hence, an automorphism of the tree X^*. In this case the automaton is called *invertible*.

Definition 6. A group of automorphisms of X^* generated by all the states of an invertible automaton $\mathcal{A}$ is called the *automaton group* generated by $\mathcal{A}$.

The class of automaton groups coincides with the class of self-similar groups. Indeed, the action on X^* of every element g of a self-similar group can be encoded by an automaton whose states are the sections of g on the words from X^*, transition and output functions are derived from the representation (2.1). Namely, for each $u \in X^*$, set $\pi(g|_u, x) = g|_{ux}$ and $\lambda(g|_u, x) = g|_u(x)$.

Important subclass of automaton groups consists of groups generated by *finite* automata. For example, we know that for groups in this class the word problem is solvable, though the general algorithm has exponential complexity. Essentially, the general algorithm coincides with the algorithm of minimization of an automaton described in [Eil74].

A standard way to visualize automata is by so-called *Moore diagrams*. Such a diagram is an oriented graph where the set of vertices is Q and for every $q \in Q$, $x \in X$, there is an edge from q to $\pi(q, x)$ labeled by $\big(x, \lambda(q, x)\big)$. In case of invertible automata it is common to label states by the corresponding permutations of X and leave only the first coordinate on the edge labels. An example of a Moore diagram is presented in Figure 21.

3 Amenability

Automata groups have been proven to play an important role in the questions related to amenability. The notion of amenability was introduced by von Neumann in 1929 [vN29] in a relation to Banach-Tarski paradox (see [Wag93]). First, let us give the formal definition of an amenable group.

Definition 7. A group G is called *amenable*, if there is a full measure μ on G, such that:

- $\mu(G) = 1$,
- μ is finitely additive,
- μ is left invariant, i.e. for any subset E of G we have $\mu(E) = \mu(gE)$.

There are a lot of other equivalent definitions of amenability, not only for groups, but also for graphs, different kinds of spaces, C^*-algebras, etc. Below, we provide few of them useful in the context of this dissertation.

Definition 8. Given an infinite graph Γ of bounded degree with the set of vertices V and the set of edges E the *Cheeger isoperimetric constant*

$$h(\Gamma) = \inf_{S\subset V, |S|<\infty} \frac{|\partial S|}{|S|},$$

where ∂S consists of vertices of $V \setminus S$ that have a neighbor in S.

Definition 9. A graph Γ is called *amenable* if $h(\Gamma) = 0$.

Proposition II.1 ([dlAGCS99])**.** *A finitely generated group G is amenable if and only if its Cayley graph with respect to some (every) finite generating set is amenable.*

Another useful criterion for an amenability related to the simple random walk on the Cayley graph of a group was developed by Kesten [Kes59] in the end of 1950s. Let Γ be the Cayley graph of a d-generated group G with the set of vertices V. Consider the Hilbert space $l^2(V)$ of all square-summable functions on V, and the bounded self-adjoint operator T (*Markov operator*) defined on this space by

$$Th(x) = \frac{1}{d}\sum_{y\sim x} h(y)$$

for $h \in l^2(V)$, $x, y \in V$ and $y \sim x$ indicates the summation over all the neighbors of x in Γ. The spectral radius of G is

$$\rho(G) = \sup\big\{|\lambda|\big|\lambda \text{ is in the spectrum of } T\big\}.$$

Theorem II.2 (Kesten criterion for amenability)**.** *A finitely generated group G is amenable if and only if $\rho(G) = 1$.*

Groups generated by automata give a new insight on the spectral problems and analysis on graphs. In particular, using self-similarities of the Schreier graphs of the actions of these groups on the tree, in some cases it is possible to describe completely the spectrum of the Markov on the Cayley graph of the group. The most renown application of this sort at this time is the negative answer to the strong Atiyah conjecture [Ati76] on L^2-Betti numbers obtained by Grigorchuk, Linnel, Schick and Żuk [GLSŻ00]. We address these questions in the last section of Chapter V.

Based on this criterion Grigorchuk in [Gri79] developed another criterion using the notion of cogrowth. This criterion was used by Olshansky in Adian in relation to Tarski monsters and free Burnside groups (see below).

For a survey relating different definitions of amenability and discussing various properties of amenable groups and pseudogroups we refer the reader to [dlAGCS99].

In the original paper [vN29] von Neumann formulated basic properties of the class of amenable groups, listed in the following theorem.

Theorem II.3 ([vN29])**.** *The class of amenable groups AG is closed under the following operations:*

1. *taking subgroups;*
2. *taking factors;*
3. *taking extensions by amenable groups ($N \lhd G$ and G/N are amenable $\Rightarrow G$ is amenable);*
4. *taking direct unions $G_n \in AG, G_n \subset G_{n+1} \Rightarrow \cup_{n\geq 1} G_n \in AG$.*

Definition 10. The minimal class EG closed under operations 1)-4) in Theorem II.3 and containing finite and abelian groups is called the class of *elementary amenable groups.*

Let us also denote by NF the class of groups without free nonabelian subgroups. Here is the direct corollary of Theorem II.3.

Corollary II.4. *The following inclusion holds*

$$EG \subset AG \subset NF.$$

The very natural conjectures posed by Day in [Day57] were asking if Theorem II.3 give the complete description of the class of amenable groups.

Conjecture 1 ([Day57])**.** Is it true that $EG = AG$? $AG = NF$?

Both these conjectures were proved to be wrong. The first question was resolved negatively by Grigorchuk in [Gri83] (see also [Gri84]) by showing that the Grigorchuk group he constructed in [Gri80] has intermediate growth, which is impossible for an elementary amenable group. On the other hand, the Fölner criterion guarantees that any group of subexponential growth is amenable (as a sequence of Fölner sets one can take the balls in the Cayley graph of the group centered at the identity). Even though the Grigorchuk group is not finitely presented, based on the same construction in [Gri98] Grigorchuk constructed a finitely presented example of amenable, but not elementary amenable group.

The examples by Grigorchuk discussed above were based on the subexponential growth of the Grigorchuk group. Similarly to the class EG of elementary amenable groups one can define a class SG of subexponentially amenable groups.

Definition 11. The minimal class SG closed under operations 1)-4) in Theorem II.3 and containing all groups of subexponential growth is called the class of *subexponentially amenable groups.*

In [Gri98] Grigorchuk posed a question in some sense generalizing the Day question on amenability. Namely, is it true that $SG = AG$. The first counterexample to this conjecture is, so-called, Basilica group generated by 3-state automaton. For the first time this group was considered in [GŻ02a] and [GŻ02b] where, in particular, was proved that it does not belong to the class SG. The fact that this group is amenable was proved by Bartholdi and Virág in [BV05] using the random walks on this group. We use similar ideas in Chapter V to show that iterated monodromy group of the complex map $z \mapsto z^2 + i$ is amenable.

The second question of Day was resolved in 1980 negatively by Olshansky in [Ol′80] by constructing a nonamenable group whose all proper subgroups are cyclic. In 1982 Adian [Adi82] proved nonamenability of free Burnside groups $B(m, n)$ for sufficiently large exponents. We also mention that both results of Olshansky and Adian were based on Grigorchuk cogrowth criterion of amenability [Gri79]. Both constructions are not finitely presented. A finitely presented example of a nonamenable group with no nonabelian free subgroups was constructed in [OS02].

Another celebrated finitely presented group solving one of the Day's questions is Thompson's group F first studied by Thompson in 1965 (see the survey paper [CFP96]). It was shown by Brin and Squier in [BS85] that F does not contain a free nonabelian subgroup, and by Cannon, Floyd and Parry [CFP96] that it is not elementary amenable. Thus, the currently open (for 30 years) question of amenability of F is of a great interest and currently is one of the main motivations for studying this group (see, for example, [Bel04, Sav09]).

Figure 5 shows the schematic relation between different classes related to amenability, and the examples discussed above that distinguish these classes one from another.

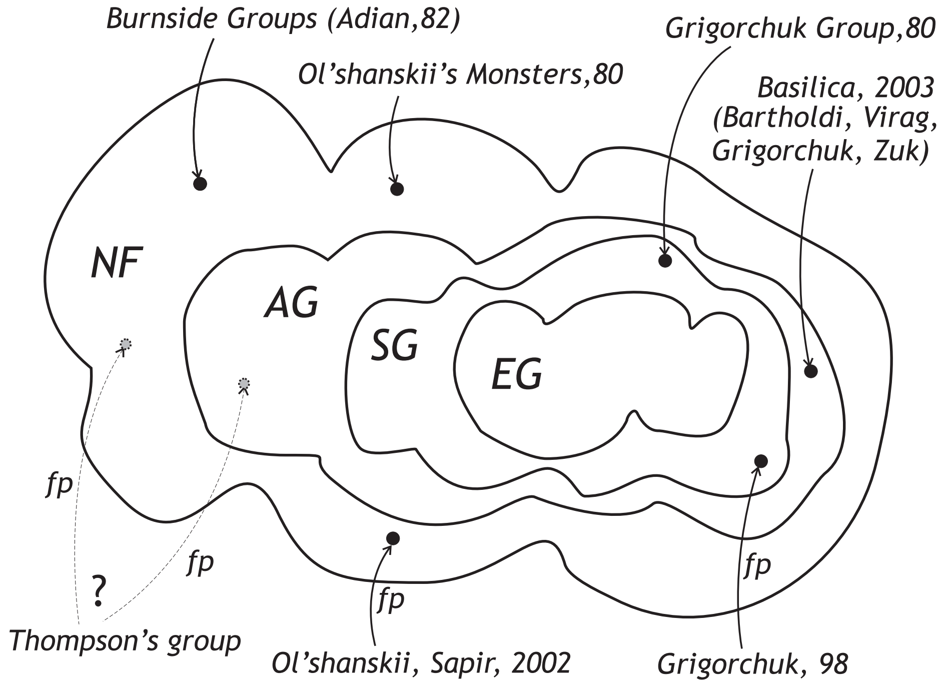


Fig. 5. Classes related to amenability

4 Growth of automata in the sense of Sidki

One of the natural ways to classify automata relies on the cyclic structure of an automaton. This way was originally suggested by S. Sidki in [Sid00a]. Let us first recall the original definition.

Given an automorphism g of tree X^* define $\theta_k(g)$ to be the number of vertices on the k-th level of X^*, such that the sections of g at these vertices act nontrivially on the first level. Also let $\alpha_k(g)$ denote the number of vertices on the k-th level of X^*, such that the sections of g at these vertices are nontrivial (but may act trivially on the first level).

It follows immediately from the definition of θ_k and α_k that $\theta_k(g) \leq \alpha_k(g)$. On the other hand in situations we are interested in there is in some sense converse relation. Namely, the following proposition holds.

Proposition II.5. *Let g be an automorphism of X^* given by finite initial automaton with m states. Then*

$$\alpha_k(g) \leq \theta_k(g) + \theta_{k+1}(g) + \cdots + \theta_{k+m-1}(g)$$

The proof follows from the fact that any nontrivial automorphism of X^* given by finite automaton with m states will act nontrivially on the $(m+1)$-st level of the tree. The above proposition shows that from the asymptotic point of view there is essentially no difference between the sequences $\theta_k(g)$ and $\alpha_k(g)$. More precisely, if one of the sequences is bounded from above by a polynomial of degree n, then the other one is also bounded by the polynomial of degree n (possibly different). At the same time if one of the sequences grows exponentially, then the other one does so.

It is proved in [Sid00a] that there are only two possible behaviors of the sequence $\theta_k(g)$ in the case if g is given by finite automaton.

Proposition II.6 ([Sid00a])**.** *Let g be an automorphism given by a finite initial automaton with m states. Then the sequence $\theta_k(g)$ either grows exponentially or polynomially of degree at most $m-1$.*

The following properties of $\theta_k(g)$ and $\alpha_k(g)$ make these sequences more interesting from the group theoretic point of view

$$\begin{aligned} \theta_k(gh) &\leq \theta_k(g) + \theta_k(h), & \theta_k(g^{-1}) &= \theta_k(g), \\ \alpha_k(gh) &\leq \alpha_k(g) + \alpha_k(h), & \alpha_k(g^{-1}) &= \alpha_k(g) \end{aligned} \tag{2.4}$$

for all automorphisms g, h of X^*.

The properties (2.4) imply that the set $\mathcal{B}_n = \mathcal{B}_n(X)$ of automorphisms g of X^* whose sequence $\theta_k(g)$ (equivalently, $\alpha_k(g)$) is bounded by a polynomial of degree n, forms a subgroup of $\operatorname{Aut} X^*$. It is proved in [Sid00a] that the sequence of groups $\mathcal{B}_n(X)$ is strictly increasing.

Definition 12. An automorphism g of X^* is called *polynomially growing (in the sense of Sidki)* if g belongs to $\mathcal{B}_n$ for some $n \geq 0$.

The minimal number n with this property is called the *degree of growth* (or just the *degree*) of an automorphism g. An automorphism of degree 0 is called *bounded*.

An automorphism, which is not polynomially growing, is called *exponentially growing*.

Definition 13. A noninitial automaton $\mathcal{A}$ over finite alphabet X is called *polynomially growing (in the sense of Sidki)* if all its states define polynomially growing automorphisms of X^*.

The maximal degree of growth of its states is called the *degree of growth* (or just the *degree*) of $\mathcal{A}$.

A noninitial automaton automaton of degree 0 is called *bounded*.

A noninitial automaton, which is not polynomially growing, is called *exponentially growing*.

Fortunately, there is an easy way to check whether the given automaton (initial or noninitial) is of polynomial or exponential growth, and in the first case compute the degree of growth. This information is stored in the cyclic structure of the automaton.

A *cycle* in the automaton $\mathcal{A}$ is the closed simple (does not intersect itself) oriented (all the edged along the path are oriented in the same direction) path in its Moore diagram. A cycle is called *trivial*, if it consists just of one loop and one state, which represents the trivial automorphism of X^*. Two cycles are called *disjoint* if they do not have vertices in common.

The *degree* of a nontrivial cycle is defined inductively. A cycle has degree 0 if all infinite paths leaving the cycle end up in the trivial state. A cycle has degree n if all infinite paths leaving the cycle end up in the cycles of smaller degree or in the trivial state, and there is an infinite path leaving the cycle and remaining in the cycle of degree $n-1$. In other words, a nontrivial cycle has degree n if there is a chain of cycles of length $n+1$ starting from the given cycle, connected by directed paths, and such n is maximal.

Proposition II.7. *[Sid00a]. A finite invertible automaton is of polynomial growth if and only if every two non-trivial cycles in the Moore diagram of the automaton are disjoint. The degree of the automaton in this case is equal to the maximal degree of the cycles in its Moore diagram.*

Corollary II.8. *[Sid00a] A finite invertible automaton is bounded if and only if every two non-trivial cycles in the Moore diagram of the automaton are disjoint and not connected by a directed path.*

The algorithm for determining whether a given automaton is of polynomial growth or not, and finding the degree of growth of polynomially growing automata is implemented in `AutomGrp` package (see Chapter VI and [MS08]).

The classes of polynomially growing automata and bounded automata are important because belonging to these classes imposes strong consequences for the group generated by such an automaton. Below we provide the list of most important properties of groups generated by polynomially growing automata.

It was proved in [Sid04] that finite initial automata of polynomial growth cannot generate a free nonabelian group. This result was generalized in [Nek07a] to arbitrary (not necessary finite state) polynomially growing automorphisms of X^*.

There is a relation between the growth of automaton and amenability of the group generated by this automaton. The first result in this direction is due to Bartholdi and

Virag [BV05] where the amenability of Basilica group, which is generated by bounded automaton, was proved by means of random walks. The consequent works [Kai05, GSŠ07] establishes amenability of other examples of groups generated by bounded automata. Finally, Bartholdi, Kaimanovich and Nekrashevych in [BKN08] established amenability of all groups generated by bounded automata. Recently a stronger result was announced by Amir, Angel and Virag, which claims that all groups generated by linearly growing automata are amenable [AAV09]. The discussion of the methods used in these papers is presented in Chapter V.

It is proved in [BN03] that each group generated by bounded automaton is contracting (see Section 5). This is not true in general for all polynomially growing automata.

Groups generated by bounded automata are also nice from the computational point of view. In particular, there is an algorithm determining the order of a given automorphism defined by bounded automaton.

For further results on bounded automata we refer the reader to the PhD thesis of I. Bondarenko [Bon07].

5 Contracting groups

A very important subclass of the class of self-similar groups is the class of *contracting groups*.

Definition 14. A self-similar group G acting on a finite alphabet X is *contracting* if there exists a finite subset $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n (generally depending on g) such that section $g|_v$ belongs to $\mathcal{N}$ for all words $v \in X^*$ of length at least n. The smallest set $\mathcal{N}$ possessing this property is called the *nucleus* of the group G.

In the case of a group generated by finite automaton there is an equivalent definition of the contracting property, which explains better the term “contracting”.

Definition 15. For a finite invertible automaton $\mathcal{A}$ with the set of states S an automaton group $G = G(\mathcal{A})$ (i.e. G is generated by the set S of states of automaton $\mathcal{A}$) is called *contracting* if there exist constants κ, C, and N, with $0 \leq \kappa < 1$, such that $\big|g|_v\big| \leq \kappa|g| + C$, for all vertices v of length at least N and $g \in G$ (where $|g|$ denotes the length of a shortest word over S representing g in G).

Thus the contracting property just means that the length of sufficiently long words shrinks roughly by a factor of κ as we take their sections at level N. In particular, for all sufficiently long elements g, all sections of g at vertices on level at least N are strictly shorter than g.

This property is a key ingredient in many inductive proofs and algorithms. In many cases it is responsible for a nontrivial torsion, intermediate growth and other exciting properties of some self-similar groups.

Contracting groups are also interesting from algorithmic point of view. As it was mentioned in Section 2 the word problem is solvable in the class of groups generated by finite automata, but the general algorithm has exponential complexity. For contracting groups the situation is much better. There exists an algorithm (first mentioned in [Gri84]), that solves the word problem in a polynomial time. The degree of a polynomial bounding the time required to solve the word problem depends on the degree of the tree and on the size of the nucleus. More precisely, the following theorem was proved in [Sav03].

Theorem II.9. *Let G be a contracting group which acts on the d-ary rooted tree and is generated by n-state automaton whose set of states contains the nucleus of G. Then for any $\varepsilon > 0$ there exists an algorithm of polynomial complexity of degree*

$(n^2-1)\log_2 d+\varepsilon$ *solving the word problem in* G.

The real-world comparison of the polynomial time algorithm solving the word problem in contracting groups and the general exponential algorithm solving the same problem in all groups generated by automata is provided in Section 2 of Chapter VI.

Contracting groups also have rich geometric structure. Each contracting group is the iterated monodromy group of its *limit dynamical system.* This system is an (orbispace) self-covering of the *limit space* of the group. The limit space is a limit of the graphs of the action of G on the levels X^n of the tree X^*. Section 7 provides more information on iterated monodromy groups and their relation to contracting groups.

Unfortunately, there is no known algorithm to determine whether a given self-similar group is contracting or not. But there is a partial answer to this question. Namely, there is an algorithm that for each finitely generated contracting group produces a positive answer to the above question and does not stop otherwise. This algorithm is based on another algorithm determining if a given generating set of a self-similar group contains a nucleus. The last algorithm stops after finitely many steps and either produces a positive answer, or returns an element of a group that needs to be in the nucleus. Thus, iterating this algorithm one can always get that the group is contracting provided that the nucleus was finite.

It is usually harder to show that a self-similar group is *not* contracting. The most common way to do this that works in a lot of cases consists in finding an element of a group G of infinite order that fixes some vertex of the tree and has itself as a section at this vertex. This shows that the nucleus of the group must contain all the powers of this element and, hence, be infinite. Of course, this is based on the problem of finding the order of an element, which does not have a general solution as of now. But there are some situations were it is possible to find the order of an element. For example, for the binary tree and for the elements of the infinite permutational wreath product of C_p there is an algorithm determining if a given element acts spherically transitively on the levels of the tree (see [BGK+08, Ste06]), which is a sufficient condition for the element to have an infinite order. Another wide class of automaton groups where it is possible to compute the order of an element is the class of groups generated by bounded automata [Sid00a].

Note, that the algorithms described above are implemented in the package `AutomGrp` [MS08] (see Chapter VI).

6 Branch groups

Another important class of subgroups of $\operatorname{Aut} X^*$ is the class of branch groups [Gri00, BGŠ03]. There are two nonequivalent definitions of branch groups. In this dissertation we will use the notion of geometrically branch groups. For definition of algebraically branch groups we refer the reader to [BGŠ03].

Let G be a subgroup of $\operatorname{Aut} X^*$. Then for any vertex $v \in X^*$ one can define the subgroup of G consisting of all the elements in G fixing all words in X^* that do not have v as a prefix. This subgroup of G is called the *rigid stabilizer* of v and is denoted by $\operatorname{rist}_G(v)$. Furthermore, the subgroup

$$\operatorname{rist}_G(n) = \left\langle \bigcup_{v \in X^n} \operatorname{rist}_G(v) \right\rangle$$

generated by the union of the rigid stabilizers of vertices at level n, is called the *rigid stabilizer of the n-th level.* Since elements of rigid stabilizers of different vertices on the same level commute we have

$$\operatorname{rist}_G(n) = \prod_{v \in X^n} \operatorname{rist}_G(v).$$

Note that, if G acts transitively on the levels of the tree, then all rigid stabilizers of the vertices on a fixed level are conjugate and, hence, isomorphic.

Definition 16. A group G of tree automorphisms of X^* that acts transitively on the levels of X^* is called a *(geometrically) branch group* if all rigid level stabilizers $\mathrm{rist}_G(n)$, $n \geq 0$, have finite index in G. If all rigid stabilizers are nontrivial then G is called a *weakly branch* group.

The simplest example of a branch group is the full group of automorphisms of the tree. More interesting question is to find finitely generated branch groups. The first example of this sort was constructed by Grigorchuk in [Gri80]. This is the celebrated Grigorchuk 2-group of intermediate growth generated by automaton shown in the left half of Figure 6. Another related example was an infinite p-group constructed by Gupta and Sidki in [GS83b] generated by automaton shown in the right part of Figure 6 (this group acts on a p-ary tree Σ^*, where $\Sigma = \{1, 2, \dots, p\}$, $\sigma = (123\dots p)$ is a long cycle permuting the elements of Σ and e denotes the identity transformation of Σ).

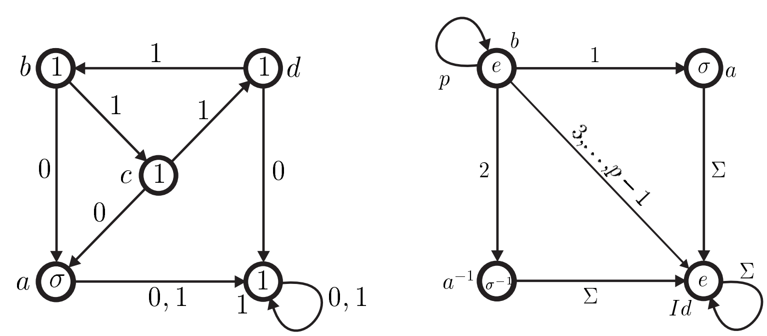


Fig. 6. Automata generating Grigorchuk and Gupta-Sidki groups

It is often easier to prove that a given group belongs to a more narrow class of regular (weakly) branch groups. Consider a self-similar group G and its normal subgroup $\mathrm{St}_G(1)$ consisting of all elements in G that stabilize the first level of X^*.

There is a natural embedding

$$\Psi\colon \operatorname{St}_G(1) \hookrightarrow G \times G \times \cdots \times G$$

given by

$$g \stackrel{\Psi}{\mapsto} (g|_0, g|_1, \dots, g|_{d-1}).$$

Definition 17. Let $K, K_0, \dots, K_{d-1}$ be subgroups of a self-similar group G acting on X^*. We say that K *geometrically contains* $K_0 \times \cdots \times K_{d-1}$ and write

$$K_0 \times \cdots \times K_{d-1} \preceq K$$

if $K_0 \times \cdots \times K_{d-1} \leq \Psi(\operatorname{St}_G(1) \cap K)$.

Definition 18. A group G of tree automorphisms of X^* that acts transitively on the levels of the tree X^* is called a *regular weakly branch group* over its normal subgroup K if

$$K \times \cdots \times K \preceq K.$$

If, in addition, the index of K in G is finite then G is called a *regular branch group* over K.

It can be shown that if G is a regular (weakly) branch group than it is a (weakly) branch group.

The first examples of branch groups appeared as groups with extraordinary properties and as counterexamples to well-known conjectures in group theory. On the other hand, in Section 2 of Chapter V on the example of $\operatorname{IMG}(z^2+i)$ we show that branch groups appear naturally in the connection to holomorphic dynamics.

One more useful notion is the notion of *self-replicating* group (called *fractal* in [BGŠ03]).

Definition 19. A self-similar group G is called *self-replicating* if, for every vertex u in X^*, the map $\varphi_u : G_u \to G$ given by $\varphi_u(g) = g|_u$ is onto (where G_u is the stabilizer of the vertex u in G).

One of the most common applications of this notion is checking the spherical transitivity, required for branchness, of a self-similar group. Namely, the following proposition of folklore type holds.

Proposition II.10. *Let G be a self-replicating self-similar group acting transitively on the first level of the tree. Then G acts spherically-transitively (i.e. G acts transitively on all levels of the tree).*

Proof. Similar to the proof of Lemma III.8 by induction on levels of the tree. □

7 Iterated monodromy groups

The theory of iterated monodromy groups was developed mostly by Nekrashevych. A very detailed exposition can be found in his monograph [Nek05]. Here we give a definition and some basic properties of these groups.

Consider a path connected and locally path connected topological space M. Let M_1 be an open and path connected subset of M and $f\colon M_1 \to M$ be a d-fold covering map. Fix a base point $t \in M$ and let $\pi_1(M, t)$ be the corresponding fundamental group. The set of iterated preimages of t under f has a natural structure of a d-ary rooted tree T. Namely, each point s from this set has exactly d preimages $s_1, \ldots, s_d$ and these preimages are declared to be adjacent to s in T. The nth level of the tree T consists of the d^n points in the set $f^{-n}(t)$. Note that although the intersection of $f^{-n}(t)$ and $f^{-m}(t)$ may be nonempty for $m \neq n$, we formally consider the set of vertices of T to be a disjoint union of the sets $f^{-n}(t)$, $n \geq 0$.

There is a natural action of $\pi_1(M,t)$ on the tree T. Let $\gamma \in \pi_1(M,t)$ be a loop based at t. For any point s of $f^{-n}(t)$, there is a unique preimage $\gamma_{[s]}$ of γ under f^n which starts at s and ends at a point s', which also belongs to $f^{-n}(t)$. We define an action of γ on T by setting $\gamma(s) = s'$ (see Figure 7. This action induces a permutation of $f^{-n}(t)$ because the preimages of γ^{-1} starting at the points of $f^{-n}(t)$ are defined uniquely as well. The group of all permutations of $f^{-n}(t)$ induced by all elements of $\pi_1(M,t)$ is called the *nth monodromy group* of f. If $\gamma(s) = s'$ then $\gamma(f(s)) = f(s')$ since $f(\gamma_{[s]}) = \gamma_{[f(s)]}$, so γ acts on T by a tree automorphism.

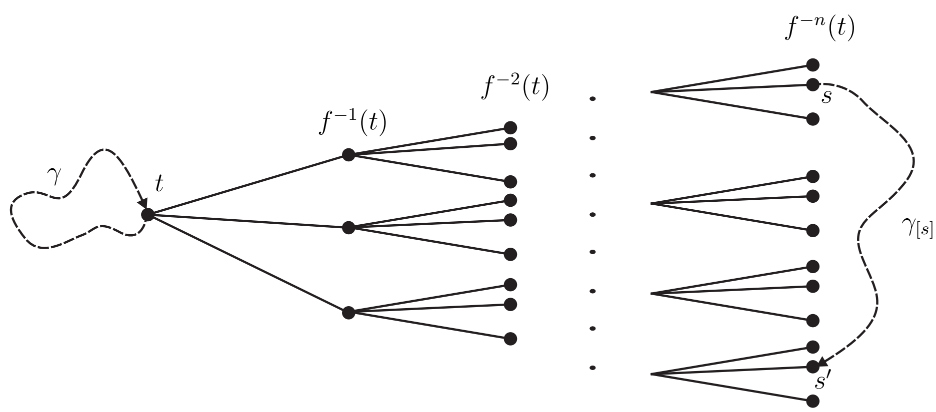


Fig. 7. Action of the fundamental group on the tree of preimages

The action of $\pi_1(M,t)$ on T is not necessary faithful. Let N be the kernel of this action.

Definition 20. The group $\mathrm{IMG}(f) = \pi_1(M,t)/N$ is called the *iterated monodromy group* of f.

It can be shown (see [Nek05] for details) that, up to isomorphism, $\mathrm{IMG}(f)$ does not depend on the choice of the base point t.

In order to describe the automorphisms induced on T by the loops from $\pi_1(M,t)$ we need to come up with a "coordinate system" on T. Let $X = \{0, 1, \dots, d-1\}$ be a standard alphabet of cardinality d. Then the set X^* of all finite words over X also has the structure of a d-ary rooted tree, where v is adjacent to vx, for any $v \in X^*$ and $x \in X$.

We go back now to iterated monodromy groups and construct an isomorphism $\Lambda\colon X^* \to T$ such that the induced action of $\pi_1(M,t)$ on X^* becomes particularly nice (self-similar).

We construct Λ level by level. Set $\Lambda(\varnothing) = t$. For each vertex v in X^n we will construct a path l_v in M joining t to one of its preimages s_v from $f^{-n}(t)$ and define $\Lambda(v) = s_v$. Choose arbitrarily d paths $l_0, \ldots, l_{d-1}$ in M connecting t to its d preimages in $f^{-1}(t)$ and, for $x \in X$, define $\Lambda(x)$ to be the end of the path l_x. Now assume we have already defined $\Lambda(v)$ and corresponding paths l_v for all $v \in X^m$, $m \leq n$ and Λ is an isomorphism between the first n levels of X^* and T such that, for all vertices v on the first n levels, $\Lambda(v)$ is the endpoint of ℓ_v. For any word $xu \in X^{n+1}$ with $x \in X$ and $u \in X^n$ define

$$l_{xu} = l_u f^{-n}_{[\Lambda(u)]}(l_x),$$

where $f^{-n}_{[\Lambda(u)]}(l_x)$ is the unique preimage of the path l_x under f^n starting at the vertex $\Lambda(u)$ (composition of paths is performed from left to right, i.e., the path on the left is traversed first). Define $\Lambda(xu)$ to be the end of the path l_{xu}.

In order to prove that Λ is an isomorphism of trees we need to show that $f\big(\Lambda(xvy)\big) = \Lambda(xv)$, for all $x, y \in X$ and $v \in X^*$. Indeed,

$$f(l_{xvy}) = f(l_{vy}) f(f^{-n}_{[\Lambda(vy)]}(l_x)) = f(l_{vy}) f^{-(n-1)}_{[\Lambda(v)]}(l_x).$$

By definition, $f^{-(n-1)}_{[\Lambda(v)]}(l_x)$ is a path going from $\Lambda(v)$ to $\Lambda(xv)$, so the end $\Lambda(xvy)$ of the path l_{xvy} is mapped to $\Lambda(xv)$ under f. Abusing the notation, we often identify the trees T and X^* and write v for $\Lambda(v)$ (see Figure 8, where solid lines represent edges in the tree T and dashed lines represent paths in M).

Definition 21. The action of $\mathrm{IMG}(f)$ on X^* induced by the isomorphism Λ is called the *standard action* of $\mathrm{IMG}(f)$.

The tree isomorphism Λ allows us to compute iterated monodromy groups using the language of self-similar groups [Nek05]. We provide the details here in order to keep the paper relatively self-contained and to help the understanding of the computations that follow. Recall, that for any loop γ based at t and any $u \in f^{-n}(t)$ we denote by $\gamma_{[u]}$ the unique preimage of γ under f^n that starts at the point u. Similarly, $f_{[u]}^{-n}(l_x)$ denotes the unique preimage of the path l_x starting at u.

Theorem II.11. *The standard action of* $\mathrm{IMG}(f)$ *is self-similar. More precisely, the section* $\gamma|_x$ *of* $\gamma \in \mathrm{IMG}(f)$ *at* $x \in X$ *is given by*

$$\gamma|_x = l_x \gamma_{[x]} (l_{\gamma(x)})^{-1}. \tag{2.5}$$

Proof. Let $v \in X^n$ be an arbitrary word and suppose $\gamma(xv) = yu$, for $y \in X$ and $u \in X^n$. Then vertices v and u are connected by the path

$$p = f_{[v]}^{-n}(l_x) \cdot \gamma_{[xv]} \cdot \left(f_{[u]}^{-n}(l_y)\right)^{-1},$$

which goes through the vertices $v \to xv \to yu \to u$ (see Figure 9, where solid curves

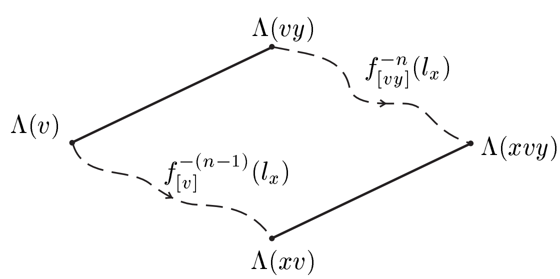


Fig. 8. Isomorphism Λ between T and X^*

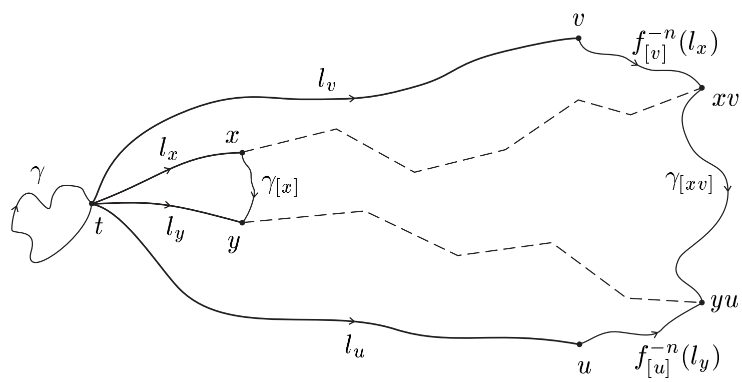


Fig. 9. Self-similar action of iterated monodromy group

represent paths in M and dashed lines represent paths in the tree X^*). We have

$$f^n(p) = l_x \gamma_{[x]} l_y^{-1}.$$

Thus the loop $\ell = l_x \gamma_{[x]} l_y^{-1}$ based at t represents the element of $\mathrm{IMG}(f)$ which moves v to u. The loop ℓ is independent of v (and u). Thus we have $\gamma|_x = l_x \gamma_{[x]} l_y^{-1}$. $\square$

CHAPTER III

SUSHCHANSKY GROUPS

The results of this chapter are published in paper [BS07] written jointly with I. Bondarenko. The structure of this chapter is as follows. In Section 1 we recall the original definition of Sushchansky groups. In Section 2 we describe the corresponding automata. The associated action on a rooted tree is not level-transitive and in Section 3 we describe its orbit tree and show that there exists a faithful level-transitive action given by finite initial automata. In Section 4 we show that every automorphism of the tree of infinite order has an infinite orbit on the boundary of the tree. The self-similar closure is studied in Section 5. The main results are presented in Section 6. It was pointed out in [Gri85a] that all Sushchansky p-groups have intermediate growth, but only the main idea of the proof was given. Here we provide a complete proof of this fact together with new estimates on the growth function, thus contributing to the Milnor question [Mil68], which was solved in [Gri83] by R.I. Grigorchuk. Also we give an upper bound on the period growth function. The main idea is to use G groups of intermediate growth introduced in [BŠ01] (see also [BGŠ03]). For each Sushchansky p-group we construct a G group of intermediate growth and prove that their growth functions are equivalent.

1 Original definition via tableaux

In this chapter let $X = \{0, 1, \ldots, p-1\}$ be a finite alphabet for some prime p. We identify X with the finite field $\mathbb{F}_p$ consisting of p elements. The set X^* of all finite words over X has a natural structure of a rooted p-ary tree.

The Sylow p-subgroup P_∞ of the profinite group $\operatorname{Aut} X^*$ is equal to the infinite wreath product of cyclic groups of order p, i.e. $P_\infty = \wr_{i\geq 1} C_p^{(i)}$. Using this description one can construct special "tableau" representation of P_∞. The "tableau" representation was initially introduced by L. Kaloujnine for Sylow p-subgroups of symmetric groups of order p^m in [Kal48].

The group P_∞ is isomorphic to the group of triangular tableaux of the form:

$$u = [a_1, a_2(x_1), a_3(x_1, x_2), \ldots],$$

where $a_1 \in \mathbb{F}_p$, $a_{i+1}(x_1, \ldots, x_i) \in \mathbb{F}_p[x_1, \ldots, x_i]/\langle x_1^p - x_1, \ldots, x_i^p - x_i\rangle$. The multiplication of tableaux is given by the formula:

$$[a_1, a_2(x_1), a_3(x_1, x_2), \ldots] \cdot [b_1, b_2(x_1), b_3(x_1, x_2), \ldots] =$$

$$= [a_1 + b_1, a_2(x_1) + b_2(x_1 + a_1), a_3(x_1, x_2) + b_3(x_1 + a_1, x_2 + a_2(x_1)), \ldots].$$

The action of the tableau u on the tree X^* is given by:

$$u(x_1 x_2 \ldots x_n) = y_1 y_2 \ldots y_n, \tag{3.1}$$

where $y_1 = x_1 + a_1$, $y_2 = x_2 + a_2(x_1), \ldots, y_n = x_n + a_n(x_1, \ldots, x_{n-1})$, where all calculations are made by identifying X with the field $\mathbb{F}_p$.

For the duration of the rest of the chapter we fix a prime $p > 2$.

Fix some order $\lambda = \{(\alpha_i, \beta_i), i = 1, \ldots, p^2\}$ on the set of pairs $\{(\alpha, \beta) | \alpha, \beta \in \mathbb{F}_p\}$. For $j > p^2$ we define $(\alpha_j, \beta_j) = (\alpha_i, \beta_i)$ where $i \equiv j \mod p^2$. Define two tableaux

$$A = [1, x_1, 0, 0, \ldots], \quad B_\lambda = [0, 0, b_3(x_1, x_2), b_4(x_1, x_2, x_3), \ldots],$$

where the coordinates of B_λ are defined by its values in the following way:

a) $b_3(2, 1) = 1$;

b) $b_i(0, 0, \ldots, 0, 1) = 1$ if $\beta_i \neq 0$;

c) $b_i(1,0,\dots,0,1) = -\frac{\alpha_i}{\beta_i}$ if $\beta_i \neq 0$ and $b_i(1,0,\dots,0,1) = 1$ if $\beta_i = 0$;

d) all the other values are zeroes.

The group $G_\lambda = \langle A, B_\lambda \rangle$ is called *the Sushchansky group of type* λ. The following theorem is proven in [Sus79].

Theorem III.1. *G_λ is infinite periodic p-group for any type λ.*

2 Automata approach

In this section we explicitly construct automata associated to Sushchansky groups. Let $\sigma = (0, 1, \dots, p-1)$ be a cyclic permutation of X. With a slight abuse of notation, depending on the context, σ will also denote the automorphism of X^* of the form $(1, 1, \dots, 1)\sigma$.

Given the order $\lambda = \{(\alpha_i, \beta_i)\}$ define words $u, v \in X^{p^2}$ in the following way:

$$u_i = \begin{cases} 0, & \text{if } \beta_i = 0; \\ 1, & \text{if } \beta_i \neq 0. \end{cases} \qquad v_i = \begin{cases} 1, & \text{if } \beta_i = 0; \\ -\frac{\alpha_i}{\beta_i}, & \text{if } \beta_i \neq 0. \end{cases}$$

The words u and v encode the actions of B_λ on the words $00\dots01*$ and $10\dots01*$, respectively. Using the words u and v we can construct automorphisms $q_1, \dots, q_{p^2}, r_1, \dots, r_{p^2}$ of the tree X^* by the following recurrent formulas:

$$q_i = (q_{i+1}, \sigma^{u_i}, 1, \dots, 1), \qquad r_i = (r_{i+1}, \sigma^{v_i}, 1, \dots, 1), \tag{3.2}$$

for $i = 1, \dots, p^2$, where the indices are considered modulo p^2, i.e. $i = i + np^2$ for any n.

Formula (3.1) implies that q_i and r_i are precisely the sections of B_λ at the words $00(0)^{i-1+np^2}$ and $10(0)^{i-1+np^2}$, respectively, for any $n \geq 0$.

The action of the tableau A is given by:

$$A = (1, \sigma, \sigma^2, \ldots, \sigma^{p-1})\sigma;$$

while B_λ acts trivially on the second level and the action on the rest is given by the sections:

$$B_\lambda|_{00} = q_1, \quad B_\lambda|_{10} = r_1, \quad B_\lambda|_{21} = \sigma$$

and all the other sections are trivial. In particular, the automorphisms A and B_λ are finite-state and Sushchansky group G_λ is generated by two finite initial automata. Denote the union of these two automata by $\mathcal{A}_{u,v}$. Its structure is shown in Figure 10. The particular automaton for $p = 3$ and the lexicographic order on $\{(\alpha, \beta) | \alpha, \beta \in \mathbb{F}_p\}$ is given in Figure 11 (all the arrows not shown in the figures go to the trivial state 1).

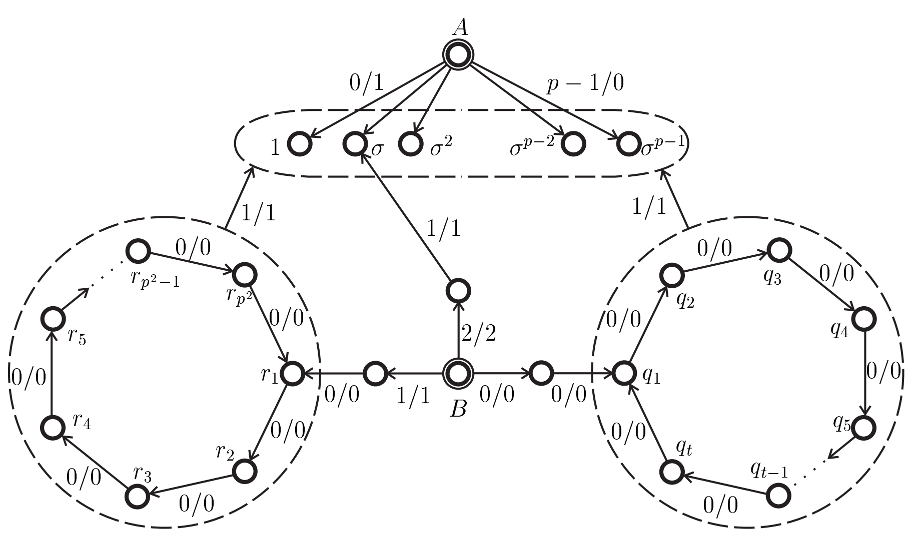


Fig. 10. The structure of Sushchansky automata

Notice that the word v cannot be periodic since it contains exactly $p - 1$ zeros and $p - 1 \nmid p^2$. On the contrary, u may be periodic with period p. In this case we have $q_i = q_{i+p}$ and the minimization of $\mathcal{A}_{u,v}$ contains $p^2 + 2p + 5$ states. If u is not periodic then $\mathcal{A}_{u,v}$ contains $2p^2 + p + 5$ states. Let t be the length of the minimal period in u (thus either $t = p$ or $t = p^2$).

Lemma III.2. *The group $\langle q_1, \ldots, q_t, r_1, \ldots, r_{p^2} \rangle$ is elementary abelian p-group.*

Proof. All q_i, r_j have order p because

$$q_i^p = (q_{i+1}^p, 1, 1, \ldots, 1), \qquad r_i^p = (r_{i+1}^p, 1, 1, \ldots, 1),$$

and therefore q_i^p and r_i^p act trivially on the tree.

All q_i, r_j commute with each other, because

$$\begin{aligned}
q_i q_j &= (q_{i+1} q_{j+1}, \sigma^{u_i + u_j}, 1, \ldots, 1), \quad q_j q_i = (q_{j+1} q_{i+1}, \sigma^{u_i + u_j}, 1, \ldots, 1);\\
r_i r_j &= (r_{i+1} r_{j+1}, \sigma^{v_i + v_j}, 1, \ldots, 1), \quad r_j r_i = (r_{j+1} r_{i+1}, \sigma^{v_i + v_j}, 1, \ldots, 1);\\
q_i r_j &= (q_{i+1} r_{j+1}, \sigma^{u_i + v_j}, 1, \ldots, 1), \quad r_j q_i = (r_{j+1} q_{i+1}, \sigma^{u_i + v_j}, 1, \ldots, 1),
\end{aligned}$$

so the corresponding pairs act equally on the tree. □

The last lemma implies that the order of B_λ is p. Since

$$A^p = (\sigma^{\frac{p(p-1)}{2}}, \sigma^{\frac{p(p-1)}{2}}, \ldots, \sigma^{\frac{p(p-1)}{2}})$$

and p is odd, the order of A is also p.

3 Actions on rooted trees

Here we describe the structure of the action of G_λ on a p-ary tree by means of the orbit tree. This notion is defined in [Ser03] and used in [GNS01] to establish a criterion determining when two automorphisms of a rooted tree are conjugate. Here we use it to simplify the definition of Sushchansky groups and show that they admit a faithful level-transitive action on a regular rooted tree.

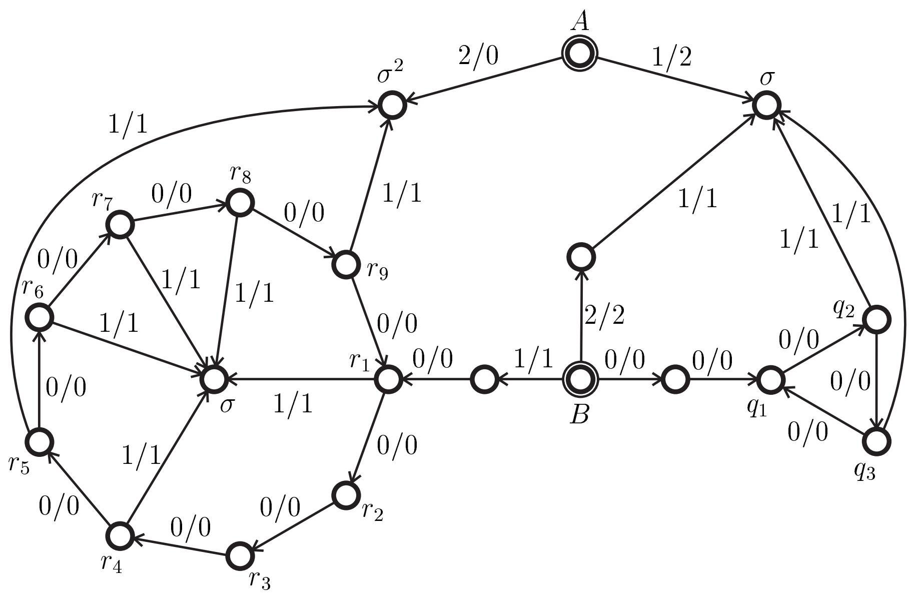


Fig. 11. Sushchansky automaton for $p = 3$ corresponding to the lexicographic order

Definition 22. Let G be a group acting on a regular p-ary tree X^*. The *orbit tree* of G is a graph whose vertices are the orbits of G on the levels of X^* and two orbits are adjacent if and only if they contain vertices that are adjacent in X^*.

Proposition III.3. *The structure of the orbit tree of* G_λ *does not depend on the type* λ *and is shown in Figure* 12*.*

Proof. Let T_O be the orbit tree of G_λ. Denote by $\operatorname{Orb}(w)$ the orbit of the word $w \in X^*$ under the action of G_λ. Define the set

$$V = \{xyw \in X^* | xy \in \operatorname{Orb}(00) \text{ and } w \in X^*\} \cup \{\emptyset\}, \tag{3.3}$$

where $\emptyset$ is the root of the tree.

The generator B_λ stabilizes the second level of the tree and hence the orbit $\operatorname{Orb}(00)$ coincides with the orbit of 00 under the action of the group generated by A.

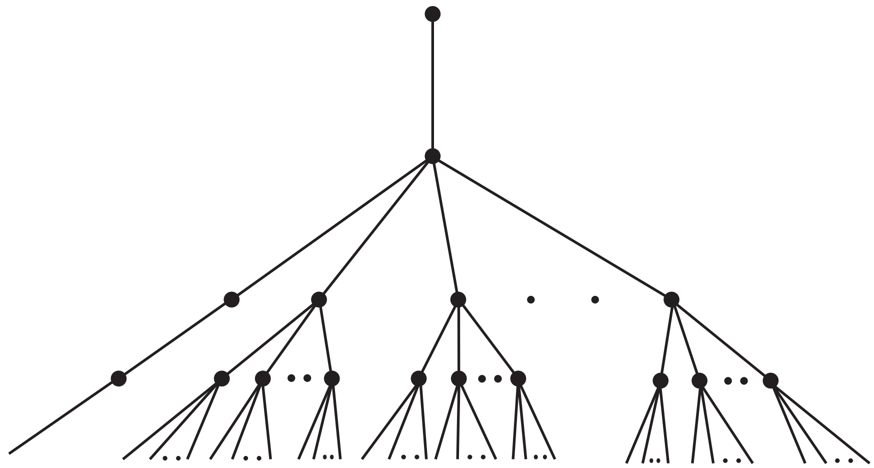


Fig. 12. The orbit tree of Sushchansky group

The set V and its compliment $W = X^* \setminus V$ are invariant under the action of G_λ.

Notice that $\{00, 10, 21\} \subset \mathrm{Orb}(00)$ and the generator B_λ acts trivially on all words that lie in the set W. Since the sections of A on all words of length ≥ 2 are trivial, every element $g \in G_\lambda$ that acts trivially on the second level of the tree must stabilize all the vertices of the set W. Hence, the orbits of G_λ on W coincide with the ones of A. Automorphism A acts transitively on the first level and has order p. Therefore the orbit of any word $w \in W$ consists of p vertices, namely the images of w under the action of the cyclic group of order p generated by A. Therefore the first two levels of T_O are exactly as shown in Figure 12 and $p-1$ vertices on the second level of T_O are the roots of regular p-ary trees.

Let us prove that G_λ acts transitively on the levels of the set V, i.e. for every $n \geq 1$ the group G_λ acts transitively on the set

$$V_n = \{xyw \in X^{n+1} | xy \in \mathrm{Orb}(00) \text{ and } w \in X^{n-1}\}.$$

We use induction on n. For $n = 1$ there is nothing to prove. Assume G_λ acts transitively on V_n and consider the $(n+1)$-th level. Since by construction either $u_{n-1}=1$ or $v_{n-1}=1$, the section of B_λ at either $00\ldots01$ or $10\ldots01$ is equal to σ. Denote this word as s (here $s\in V_n$) and note that B stabilizes s. To prove the induction step it suffices for an arbitrary $s'z'\in V_{n+1}$, where $s'\in V_n$ and $z'\in X$, to construct an element $g\in G_\lambda$ such that $g(s0)=s'z'$. By the inductive assumption there is an element $h\in G_\lambda$ such that $h(s)=s'$. Suppose $h^{-1}(s'z')=sz$ for some letter $z\in X$. Then for $g=(B_\lambda)^z h$ (here we consider z as an integer) we have

$$\begin{aligned} g(s0) &= h((B_\lambda)^z(s0)) = h(s(B_\lambda)^z|_s(0)) = h(s(B_\lambda|_s)^z(0)) = \\ &= h(s\sigma^z(0)) = h(sz) = s'z' \end{aligned}$$

as required. $\square$

The set V has a natural structure of a rooted p-ary tree T, where the root $\emptyset$ is connected by an edge with every vertex in $\mathrm{Orb}(00)$ and there is an edge between w and wx for all $w\in V$ and $x\in X$. In other words, there is a natural 1-to-1 correspondence between V and vertices of T given by $xyw\mapsto xw$ for $xy\in\mathrm{Orb}(00)$ and $w\in X^*$. Since the set V is invariant under the action of G_λ, the group G_λ acts by automorphisms on the tree T. This action has simpler structure and the following proposition holds.

Proposition III.4. *The action of Sushchansky group G_λ on the tree T is faithful, level transitive and has the following form*

$$\begin{aligned} A &= \sigma, \\ B_\lambda &= (q_1, r_1, \sigma, 1, \ldots, 1), \\ q_i &= (q_{i+1}, \sigma^{u_i}, 1, \ldots, 1), \\ r_i &= (r_{i+1}, \sigma^{v_i}, 1, \ldots, 1). \end{aligned} \tag{3.4}$$

Proof. The expressions (3.4) follow directly from the definition of Sushchansky groups.

Let us prove that this action is faithful. Take an arbitrary nontrivial element $g \in G_\lambda$. If g acts non-trivially on the second level of X^*, then the exponent of A in g is not divisible by p. But then g acts non-trivially on the first level of T as well because it is fixed under B_λ and A acts there as σ. If g acts trivially on the second level of X^* then it acts trivially on the complement of V in X^* according to Proposition III.3. Therefore to be nontrivial it must act nontrivially on T.

We proved in Proposition III.3 that G_λ acts transitively on every set V_n, which is precisely the n-th level of the tree T. $\square$

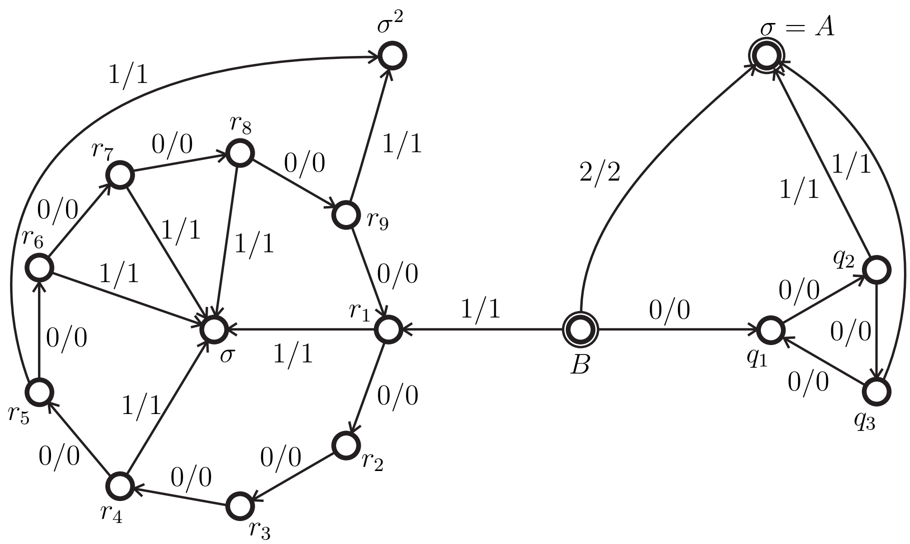


Fig. 13. Simplified Sushchansky automaton for $p = 3$ corresponding to the lexicographic order

Figure 13 shows the impact of Proposition III.4 on the original Sushchansky automaton for $p = 3$ and lexicographic order, shown in Figure 11.

4 Orbits of automorphisms of infinite order

Here we include a useful result obtained jointly with Yaroslav Vorobets. Obviously, if an automorphism g of the tree has infinite orbit on the boundary of the tree, it must have infinite order. The converse statement is not that obvious, but still holds. We will not need this result in subsequent sections, but it is interesting on its own.

Theorem III.5. *Every finite-state automorphism of a d-ary tree of infinite order has infinite orbit on the boundary of the tree.*

We give a proof for a binary tree to avoid technicalities and make the idea behind it more clear. The same proof with slight modifications (see remark after the proof) works also for d-ary tree.

Lemma III.6. *Let g be a finite-state automorphism of a binary tree, whose set of sections has cardinality s. If g has infinite order, then there is a fixed by g vertex v of the tree with $|v| \leq s$, such that $g|_v$ acts nontrivially on the first level and also has infinite order.*

Proof. Define the following subset of the set of sections of g

$$D = \{g|_v \bigm| v \in X^*,\ g(v) = v,\ g|_v \text{ acts nontrivially on the first level}\}.$$

If each element in D has finite order, then the order of g is equal to the least common multiple of these orders (note that D is finite since g has finitely many sections). Therefore, in order for g to have infinite order, at least one element from D must have infinite order as well. Suppose that this is a section of g at vertex w fixed by g. If length of w is greater than s then there will be two identical sections $g|_u = g|_v$ along the path joining the root of the tree to w. One can remove the part of this path between u and v to obtain another vertex w' fixed by g such that $g|_{w'} = g|_w$ and the length of w' is strictly shorter than the length of w. This process can be repeated until we reach a word whose length is greater than s. $\square$

Proof of Theorem III.5. Let g be a finite-state automorphism of a binary tree of infinite order, whose set of sections has cardinality s. By Lemma III.6 there is a fixed by g vertex v_1 with $|v_1| \leq s$, such that $g|_{v_1}$ acts nontrivially on the first level and has infinite order. Then the orbit of any vertex below v_1 under the action of g has length at least 2.

The automorphism $g|_{v_1}$ also has no more than s sections since all its section are the sections of g as well. Therefore, $g|_{v_1}^2 = g^2|_{v_1}$ is an automorphism of infinite order, whose set of sections has cardinality no more than s^2. Thus, by Lemma III.6 there is a fixed by $g^2|_{v_1}$ vertex v_2 with $|v_2| \leq s^2$, such that $\big(g^2|_{v_1}\big)|_{v_2} = g^2|_{v_1v_2}$ acts nontrivially on the first level and has infinite order. The orbit of any vertex below v_1v_2 under the action of g^2 has length at least 2. Hence, the orbit of any such vertex under g has length at least 4.

Continuing this way we get an infinite path $P = v_1v_2v_2\ldots$, such that the length of v_i is no more than s^{2^i} and the orbit of $v_1v_2\ldots v_k$ under the action of g has length at least 2^{k-1}. This shows that the orbit of P is infinite. $\square$

Remark. In the case of d-ary tree for $d > 2$ the proof goes similarly, except that in Lemma III.6 one proves the existence of a vertex v fixed by g for which there is a vertex x in the first level of the tree such that the orbit of x under $g|_v$ has size $k > 2$ and $g^k|_{vx}$ has infinite order.

5 Self-similar closure

The Sushchansky group G_λ is not generated by all the states of $\mathcal{A}_{u,v}$ and is not self-similar (see definition below). However, we can embed it into a larger self-similar group where we can use some known techniques to derive some important results about G_λ itself. In particular that G_λ is amenable (Corollary III.11) and that the word problem is solvable in polynomial time (Corollary III.12). For the definitions not given here and more information on self-similar groups we refer to [Nek05] and [BGŠ03].

Definition 23. The *self-similar* closure of $G < \operatorname{Aut} X^*$ is the group generated by all the sections of all the elements of G at words in X^*.

Let $\tilde{G}_\lambda$ be the self-similar closure of G_λ, i.e. $\tilde{G}_\lambda$ is generated by all the states of the automaton $\mathcal{A}_{u,v}$. Consider also the self-similar subgroup $K = \langle q_1, \ldots, q_t, r_1, \ldots, r_{p^2}, \sigma\rangle$ of $\tilde{G}_\lambda$.

Lemma III.7. *The group K is not periodic.*

Proof. First, consider the case $t = p$. Then all u_i's are equal to 1 except one equal to 0. In particular, $\sum_{i=1}^{p} u_i = p - 1$. Then the element $g = q_1 q_2 \cdots q_t \sigma^{p-1}$ has representation

$$g = (q_1 q_2 \cdots q_t, \sigma^{p-1}, 1, \ldots, 1)\sigma^{p-1}.$$

Therefore

$$g^p = (q_1 q_2 \cdots q_t \sigma^{p-1}, *, \ldots, *) = (g, *, \ldots, *).$$

Since g is nontrivial it must have infinite order.

In case $t = p^2$, exactly p of u_i's are zeros. We mark the vertices of the cycle of q_i's in the automaton by the corresponding u_i's. There are at most $\binom{p}{2}$ different distances between the zeros in the cycle. But the length of the cycle is p^2 so there are

$$\frac{p^2-1}{2} > \frac{p^2-p}{2} = \binom{p}{2}$$

possible distances in the cycle, so let d be a distance that is not attained as a distance between two zeros.

Now consider the element $g = q_1 q_{d+1} \sigma^{u_{p^2}+u_d}$. It can be written as

$$g = (q_2 q_{d+2}, \sigma^{u_1+u_{d+1}}, 1, \dots, 1)\sigma^{u_{p^2}+u_d}.$$

Since the distance between states q_{p^2} and q_d in the cycle is exactly d at least one of u_{p^2} and u_d is nonzero so $\sigma^{u_{p^2}+u_d}$ is a cycle of length p. Hence

$$g^p = (q_2 q_{d+2} \sigma^{u_1+u_{d+1}}, *, \dots, *).$$

Therefore if the order $|g|$ of g is finite, then it is not smaller than $p \cdot |q_2 q_{d+2} \sigma^{u_1+u_{d+1}}|$.

Now we repeat this procedure p^2 times and on the i-th iteration we get

$$q_i q_{d+i} \sigma^{u_{i-1}+u_{d+i-1}} = (q_{i+1} q_{d+i+1}, \sigma^{u_i+u_{d+i}}, 1, \dots, 1)\sigma^{u_{i-1}+u_{d+i-1}}.$$

Again, the distance between q_{i-1} and q_{d+i-1} is exactly d so $\sigma^{u_{i-1}+u_{d+i-1}}$ is a cycle of length p and

$$(q_i q_{d+i} \sigma^{u_{i-1}+u_{d+i-1}})^p = (q_{i+1} q_{d+i+1} \sigma^{u_i+u_{d+i}}, *, \dots, *).$$

Therefore

$$|q_i q_{d+i} \sigma^{u_{i-1}+u_{d+i-1}}| \geq p \cdot |q_{i+1} q_{d+i+1} \sigma^{u_i+u_{d+i}}|.$$

But after p^2 steps we will meet g again. So its order cannot be finite. □

Lemma III.8. *A self-similar group of binary tree automorphisms is level transitive if and only if it is infinite.*

Proof. The proof of this lemma is similar to the proof of transitivity in Proposition III.3. Let G be a self-similar group acting on a binary tree.

If G acts level transitively then G must be infinite (since the size of the levels is not bounded).

Assume now that the group G is infinite.

We first prove that all level stabilizers $\mathrm{Stab}_G(n)$ are different. Note that, since all level stabilizers have finite index in G and G is infinite, all level stabilizers are infinite. In particular, each contains a nontrivial element.

Let $n > 0$ and $g \in \mathrm{Stab}_G(n-1)$ be an arbitrary nontrivial element. Let $v = x_1 \dots x_k$ be a word of the shortest length such that $g(v) \neq v$. Since $g \in \mathrm{Stab}_G(n-1)$, we must have $k \geq n$. The section $h = g_{x_1 x_2 \dots x_{k-n}}$ is an element of G by the self-similarity of G. The minimality of the word v implies that $g \in \mathrm{Stab}_G(k-1)$, and therefore $h \in \mathrm{Stab}_G(n-1)$. On the other hand h acts nontrivially on $x_{k-n+1} \dots x_k$ and we conclude that $h \in \mathrm{Stab}_G(n-1) \setminus \mathrm{Stab}_G(n)$. Thus all level stabilizers are different.

We now prove level transitivity by induction on the level.

The existence of elements in $\mathrm{Stab}_G(0) \setminus \mathrm{Stab}_G(1)$ shows that G acts transitively on level 1.

Assume that G acts transitively on level n. Select an arbitrary element $h \in \mathrm{Stab}_G(n) \setminus \mathrm{Stab}_G(n+1)$ and let $w =\in X^n$ be a word of length n such that $h(w1) = w0$.

Let u be an arbitrary word of length n and let x be a letter in $X = \{0, 1\}$. We will prove that ux is mapped to $w0$ by some element of G, proving the transitivity of the action at level $n+1$. By the inductive assumption there exists $f \in G$ such that $f(u) = w$. If $f(ux) = w0$ we are done. Otherwise, $hf(ux) = h(w1) = w0$ and we are done again. □

Next, we prove the following proposition.

Proposition III.9. *$\tilde{G}_\lambda$ is a weakly regular branch group over K^p.*

Proof. First of all note that Lemma III.7 guarantees that K^p is nontrivial. At least one (in fact more) of the u_i's is non zero, say u_1. Then the relations (3.2) and

$$\sigma q_1 \sigma^{p-1} = (\sigma^{u_1}, 1, \dots, 1, q_2)$$

show that the set of sections of the elements of K, that stabilize the first level X of the tree, at letter 0 includes the generators of K and hence the whole group K (therefore conjugating by $\sigma \in K$ yields that K is self-replicating, i.e. for any $x \in X$ the projection of $\mathrm{St}_x(K)$ onto the vertex x coincides with K). Thus for any $v \in K$ there is $w \in K$ of the form

$$w = (v, \sigma^i, 1, \dots, 1, q_2^j)$$

for some i and j. But then by Lemma III.2

$$w^p = (v^p, \sigma^{ip}, 1, \dots, 1, q_2^{jp}) = (v^p, 1, \dots, 1).$$

Therefore $K^p \succ K^p \times 1 \times \cdots \times 1$. Since σ acts transitively on the first level and belongs to the normalizer of K^p in K (because $\sigma^{-1} v^p \sigma = (\sigma^{-1} v \sigma)^p$) by conjugation we get

$$K^p \succ K^p \times K^p \times \cdots \times K^p,$$

as geometric embedding.

The transitivity of $\tilde{G}_\lambda$ on levels follows from the fact that its subgroup K acts nontrivially on the first level and is self-replicating, and hence, level transitive. Another proof follows from Proposition III.8. $\square$

We summarize some general properties of $\tilde{G}_\lambda$ in the following proposition:

Proposition III.10. *The self-similar closure of G_λ is neither torsion, nor torsion free, level-transitive group of tree automorphisms. Moreover, it is generated by a bounded automaton, hence it is contracting and amenable.*

Proof. The first three assertions are already proved above. The automaton $\mathcal{A}_{u,v}$ is bounded by Corollary 14 in [Sid00a] (see the definition in Section 4). As a corollary $\tilde{G}_\lambda$ is contracting (see [BN03]) and amenable (see [BKN08]). □

Corollary III.11. *G_λ is amenable.*

Note also that the last corollary follows from Theorem III.19.

Corollary III.12. *The word problem in G_λ is solvable in polynomial time.*

Proof. See Proposition 2.13.10 in [Nek05]. □

6 Intermediate growth

Let G be a group finitely generated by a set S. *The growth function of* G is defined by

$$\gamma_G(n) = \left|\{g \in G | g = s_1 s_2 \ldots s_k \text{ for some } s_i \in S \cup S^{-1}, k \leq n\}\right|.$$

Two functions γ_1 and γ_2 are called *equivalent* if there exists a constant $C > 0$ such that $\gamma_1(\frac{1}{C}n) \leq \gamma_2(n) \leq \gamma_1(Cn)$ for all n. The growth function γ_G depends both on G and on S, but the equivalence class of γ_G does not depend on S.

In 1968 John Milnor asked about the existence of finitely generated groups with growth that is intermediate between polynomial and exponential. The first examples of such groups were provided by R.I. Grigorchuk in [Gri83], where he constructed uncountable family of such groups. In particular, it was shown, that there are groups of intermediate growth generated by automata with 5 states, namely, G_ω for $\omega = (012)^\infty$ (not to be confused with Sushchansky groups G_λ). These examples were generalized to the notion of G groups [BGŠ03]. Under some finiteness restriction all G groups have intermediate growth.

Recently it was proved [BP06] (see also [Nek07c]) that there is a 4-state automaton over a 2-letter alphabet generating a group of intermediate growth. This group itself is isomorphic to the iterated monodromy group of the map $f(z) = z^2 + i$. But it is still an open question whether there is a group of intermediate growth generated by a 3-state automaton over a 2-letter alphabet.

In view of the examples above it is not very surprising that the two of the pioneering examples of infinite finitely generated periodic groups introduced by S.V. Aleshin in [Ale72] and V.I. Sushchansky in [Sus79] also have intermediate growth. For Aleshin group it follows from the intermediate growth of Grigorchuk group and the result of Y.I. Merzlyakov [Mer83], who proved that Aleshin group contains a subgroup of finite index isomorphic to the subdirect product of four copies of Grigorchuk group. Also the relation between these two groups was studied in [Gri85b].

As was mentioned above in [Gri85a] R.I. Grigorchuk pointed out that all Sushchansky groups have intermediate growth, but only the idea of proof was given. In this chapter we give a complete proof of this fact based on the results from [BŠ01] and [BGŠ03].

At the present moment the main method of obtaining the upper bounds for growth functions of groups was originated by R.I. Grigorchuk in [Gri84]. Different modifications of this method in [Bar98, MP01, BŠ01] allowed to improve existing estimates and to prove the estimates for new groups.

As for the lower bounds for growth functions, there are several techniques. In [Gri84] R.I. Grigorchuk uses self-similarity to obtain the lower bound of the form $e^{\sqrt{n}}$ for most of his groups. Moreover, he shows that any group G that is abstractly commensurable with its own power G^k for some $k \geq 2$ has a growth function not smaller that e^{n^α} for some $0 < \alpha \leq 1$.

In [Gri89] R.I. Grigorchuk used bounds on the coefficients of Hilbert-Poincaré series of graded algebras associated with groups to bound their growth functions. Namely, it was obtained that any residually p-group whose growth function is not bounded above by polynomial, must grow at least as $e^{\sqrt{n}}$.

Y.G. Leonov [Leo01], L. Bartholdi and Z. Šunić [Bar98, BŠ01] used more advanced techniques (common in spirit to the ones used in [Gri84]) also based on certain self-similarity of the groups acting on trees. In obtaining the lower bounds for the growth functions of these groups the important role was played by the property, which is in some sense opposite to contraction. The main idea is that the sections of elements cannot be much shorter than the elements themselves.

A. Erschler used random walks and Poisson boundary to approach to this question. In particular, in [Ers04] it was shown that the growth function of Grigorchuk group G_ω for $\omega = (01)^\infty$, which is generated by 5-state automaton, grows faster than e^{n^α} for any $\alpha < 1$. The upper estimate of the same sort was obtained for this group in spirit of [Gri84], which shows that groups G_ω for $\omega = (012)^\infty$ and $\omega = (01)^\infty$ have essentially different growth functions.

Recall the definition of a G group.

Definition 24. Let R be a subgroup of $Sym(X)$, D be any group with a sequence of homomorphisms $w_i : D \to \mathrm{Sym}(X)$, $i \geq 1$. Then R acts on the first level of X^* and D acts on X^* in the following way. Each $d \in D$ defines the automorphism $\hat{d}$ that acts trivially on the first level and is given by its sections

$$\hat{d}\big|_{0^i1} = w_i(d), i \geq 1$$

and all the other sections act trivially on X. Denote $\hat{D} = \{\hat{d} \mid d \in D\}$.

The group $G = \langle R, \hat{D}\rangle$ is called a G *group* if the following conditions are satisfied:

(i) The groups R and $w_i(D), i \geq 1$, act transitively on X.

(ii) For each $d \in D$ the permutation $w_i(d)$ is trivial for infinitely many indices.

(iii) For each nontrivial $d \in D$ the permutation $w_i(d)$ is nontrivial for infinitely many indices.

The groups R and D are called *the root part* and *the directed part* of G correspondingly.

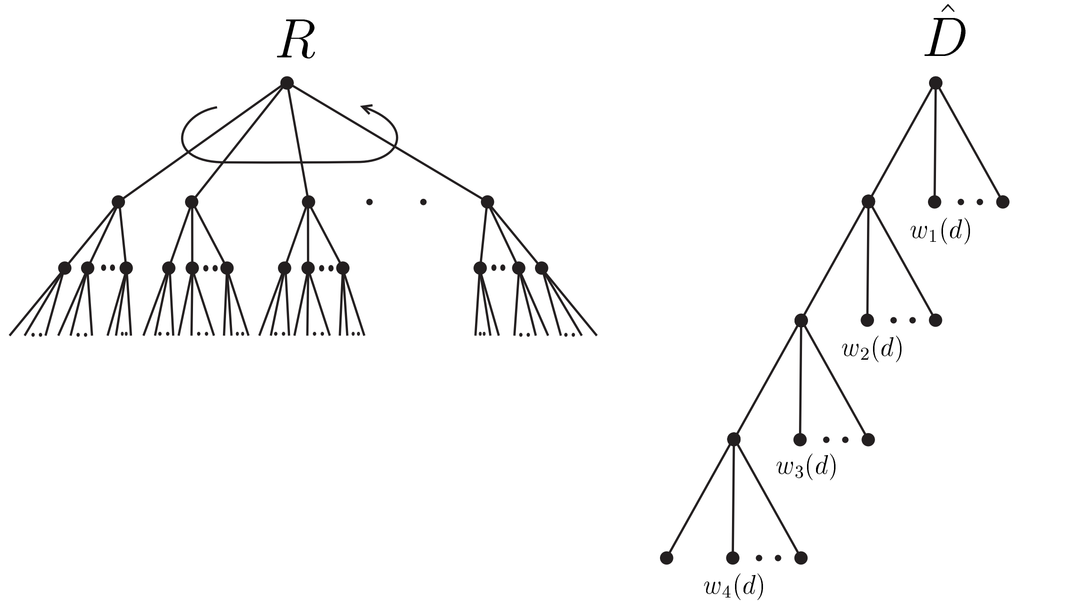


Fig. 14. The action of G group on the tree

The actions of R and $\hat{D}$ on the tree are schematically depicted in Figure 14. Note that in [BGŠ03] the definition of a G group is given in slightly more general settings. The results in [BŠ01] and [BGŠ03] imply the following theorem.

Theorem III.13. *All G groups with finite directed part have intermediate growth.*

There is a lower bound for the growth of such groups given in [BGŠ03]:

$$\gamma_G(n) \succeq e^{n^\alpha}, \tag{3.5}$$

where $\alpha = \frac{\log(|X|)}{\log(|X|)+\log(2)}$.

The sequence of homomorphisms w_i in the definition of a G group is called r-homogeneous, if for every finite subsequence of r consecutive homomorphisms

$w_{i+1}, w_{i+2}, \ldots, w_{i+r}$ every element of D is sent to the identity by at least one of the homomorphisms from this finite subsequence. In particular, if the sequence of homomorphisms $\{w_i, i \geq 1\}$ defining a G group is periodic with period r, it is also r-homogeneous.

It is proved in [BŠ01] that in case of r-homogeneous sequence of defining homomorphisms there is an estimate of the upper bound on the growth function. Moreover, in this case if the directed part has finite exponent there is an upper bound on the torsion growth function $\pi(n)$ (the maximal order of an element of length at most n).

Theorem III.14 (η-estimate)**.** *Let G be a G group defined by an r-homogeneous sequence of homomorphisms. Then the growth function of the group G satisfies*

$$\gamma_G(n) \preceq e^{n^\beta}, \tag{3.6}$$

where $\beta = \frac{\log(|X|)}{\log(|X|)-\log(\eta_r)} < 1$ and η_r is the positive root of the polynomial $x^r + x^{r-1} + x^{r-2} - 2$.

If the directed part D of G has finite exponent q, then the group G is torsion and there exists a constant $C > 0$, such that the torsion growth function satisfies

$$\pi(n) \leq Cn^{\log_{1/\eta_r}(q)}. \tag{3.7}$$

Sushchansky groups G_λ are not G groups, because the automorphism B_λ cannot be expressed as $\hat{d}$ for some homomorphisms w_i. On the other hand, the automorphisms q_i and r_i can, and the following proposition shows that the self-similar closure of G_λ contains a subgroup which is a G group. Since the simplified definition of G_λ from Proposition III.4 does not simplify considerably the proofs in this section, we will use the original definition in order to make this section independent from

Section 3.

Proposition III.15. *The group $H = \langle q_1, r_1, \sigma\rangle$ is a G group with finite directed part defined by a periodic sequence of homomorphisms with period p^2.*

Proof. We prove that the subgroups $\langle q_1, r_1\rangle$ and $\langle \sigma\rangle$ are the directed and the root parts of H.

First observe that $\langle q_1, r_1\rangle \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$. Indeed, the group $\langle q_1, r_1\rangle$ is elementary abelian p-group by Lemma III.2. Suppose that $r_1 \in \langle q_1\rangle$, $r_1 = q_1^k$. Comparing sections at words $0\dots01$ we get $v_i = ku_i$. Contradiction, since $u_i = 0$ and $v_i = 1$ for i with $\beta_i = 0$.

Consider the periodic sequence of homomorphisms $w_i : \langle q_1, r_1\rangle \to \mathrm{Sym}(X)$ with period p^2 given by $w_i(q_1) = \sigma^{u_i}$ and $w_i(r_1) = \sigma^{v_i}$. Then for any $d \in \langle q_1, r_1\rangle$ the associated $\hat{d}$ from the definition of a G group coincides with the automorphism d. To complete the proof we need to check the conditions (i)–(iii) from the definition of a G group.

(i) The root part generated by σ acts transitively on X. Furthermore, for any $i \geq 1$

$$\begin{aligned} w_i(q_1) &= \sigma, \quad \text{if } \beta_i \neq 0;\\ w_i(r_1) &= \sigma, \quad \text{if } \beta_i = 0. \end{aligned}$$

In any case $w_i(\langle q_1, r_1\rangle)$ contains σ and thus acts transitively on X.

(ii),(iii) Let $d = q_1^k r_1^l$, $k, l \in \mathbb{Z}_p$, be an arbitrary nontrivial element of $\langle q_1, r_1\rangle$. Since the sequence w_i is periodic it suffices to show at least one occurrence of trivial and one occurrence of nontrivial $w_i(d)$.

Find i such that

$$(\alpha_i, \beta_i) = (1, 0), \quad \text{if } l = 0;$$

$$(\alpha_i, \beta_i) = (k, l), \quad \text{if } l \neq 0.$$

Then

$$w_i(d) = \begin{cases} w_i(q_1^k) = \sigma^{ku_i} = 1, & \text{if } l = 0; \\ w_i(q_1^k r_1^l) = \sigma^{ku_i + lv_i} = \sigma^{k + l(-k/l)} = 1, & \text{if } l \neq 0. \end{cases}$$

For a nontrivial occurrence find i such that

$$(\alpha_i, \beta_i) = (0, 1), \quad \text{if } l = 0;$$

$$(\alpha_i, \beta_i) = (1, 0), \quad \text{if } l \neq 0.$$

Then

$$w_i(d) = \begin{cases} w_i(q_1^k) = \sigma^{ku_i} = \sigma^k, & \text{if } l = 0; \\ w_i(q_1^k r_1^l) = \sigma^{ku_i + lv_i} = \sigma^l, & \text{if } l \neq 0. \end{cases}$$

□

The last proposition shows that the growth function of H satisfies inequalities (3.5) and (3.6), for $r = p^2$. Also note that it is proved in [BGŠ03] that a G group is torsion if and only if its directed part D is torsion. Therefore, the group H is torsion. The next proposition exhibits another regular branch structure inside $\tilde{G}_\lambda$.

Proposition III.16. *The group $H = \langle q_1, r_1, \sigma \rangle$ is regular branch over its commutator subgroup H'.*

Proof. Let $H_k = \langle q_k, r_k, \sigma \rangle$, $k = 1, \dots, p^2$ be the subgroups of $\tilde{G}_\lambda$. First we show that

$$H'_k \succeq H'_{k+1} \times H'_{k+1} \times \cdots \times H'_{k+1} \tag{3.8}$$

for all k. Indeed, at least one of u_k and v_k is nonzero. Suppose $u_k \neq 0$. Then relations $q_k = (q_{k+1}, \sigma^{u_k}, 1, \ldots, 1)$ and $r_k = (r_{k+1}, \sigma^{v_k}, 1, \ldots, 1)$ imply

$$\begin{aligned}
[q_k, r_k] &= ([q_{k+1}, r_{k+1}], 1, \ldots, 1), \\
[q_k, (q_k^{\sigma^{-1}})^{1/u_k}] &= ([q_{k+1}, \sigma], 1, \ldots, 1), \\
[r_k, (q_k^{\sigma^{-1}})^{1/u_k}] &= ([r_{k+1}, \sigma], 1, \ldots, 1).
\end{aligned}$$

Since the projection of the stabilizer of the first level in H_k on the leftmost vertex coincides with H_{k+1} we get $H_k' \succeq H_{k+1}' \times 1 \times \cdots \times 1$. Conjugation by $\sigma \in H_k$ implies inclusion (3.8). Since $H_1 = H_{p^2+1} = H$, we obtain $H' \succeq H' \times H' \times \cdots \times H'$ as geometric embedding induced by the restriction on X^{p^2}.

The transitivity of H on the levels is proved by the method used in Proposition III.3.

Now H is a torsion p-group, hence, so is H/H', which is abelian. But each torsion finitely generated abelian group is finite. Thus, H' is a subgroup of finite index in H. □

When we deal with a group G of automorphisms of X^*, it is sometimes difficult to say something about the whole group, but we know something about the group P generated by all the sections of the elements in G at some level k of the tree. In case G is self-similar, P is a subgroup of G and if G is self-replicating, P coincides with G. Some properties of P are inherited by G itself. In particular, if P is finite or torsion then so is G (the converse is not true). But what we are interested in here is that the growth of G can be estimated in terms of the growth of P.

Let S be a finite generating set of G. Then P is generated by the set $\tilde{S}$ of the sections of all elements of S at all vertices of k-th level X^k of the tree. The following lemma holds.

Lemma III.17. *The growth function $\gamma_G(n)$ of the group G with respect to S is bounded from above by*

$$\gamma_G(n) \preceq \big(\gamma_P(n)\big)^{|X|^k}, \tag{3.9}$$

where $\gamma_P(n)$ is the growth function of the group P with respect to $\tilde{S}$. In particular, the growth type of G (finite, polynomial, intermediate or exponential) cannot exceed the one of P.

Proof. Let $g \in G$ be an element of length n with respect to the generating set S. This element induces a permutation π_k of the k-th level of the tree and $|X|^k$ sections $g|_v, v \in X^k$, at words of length k. Moreover, different automorphisms correspond to different tuples $(\pi_k, \{g|_v, v \in X^k\})$ of sections and permutations. Each such a section is a word of length not greater than n with respect to the generating set $\tilde{S}$ of P. So for each vertex $v \in X^k$ the number of possible sections at v is bounded from above by $\gamma_P(n)$. □

The following corollary shows an easy way to construct new examples of groups with intermediate (finite, polynomial, exponential) growth.

Corollary III.18. *Let F be a finite set of automorphisms from $\operatorname{Aut} X^*$, whose sections at some level k belong to G (in particular, F could be a set of finitary automorphisms). Then*

$$\gamma_G(n) \precsim \gamma_{\langle G,F\rangle}(n) \precsim \big(\gamma_G(n)\big)^{|X|^k}.$$

where $\gamma_{\langle G,F\rangle}(n)$ is the growth function of the group $\langle G, F\rangle$ with respect to the generating set $S \cup F$.

In particular the previous corollary shows that if a group G is generated by a finite automaton, then the growth type of this group depends only on the nucleus (see definition in [Nek05]) of this automaton.

An interesting question is whether it is true that if G grows faster than polynomially then $\gamma_G(n) \sim \gamma_{\langle G,F\rangle}(n)$.

We are ready to prove the main results.

Theorem III.19. *All Sushchansky p-groups have intermediate growth. The growth function of each Sushchansky p-group G_λ satisfies*

$$e^{n^\alpha} \preceq \gamma_{G_\lambda}(n) \preceq e^{n^\beta},$$

where $\alpha = \frac{\log(p)}{\log(p)+\log(2)}$, $\beta = \frac{\log(p)}{\log(p)-\log(\eta_r)}$ *and η_r is the positive root of the polynomial* $x^r + x^{r-1} + x^{r-2} - 2$, *where* $r = p^2$.

Proof. The group generated by all the sections of elements of G_λ at the second level is $H = \langle q_1, r_1, \sigma\rangle$, which is a G group of intermediate growth by Proposition III.15 and Theorems III.13 and III.14, whose growth function satisfies inequalities (3.5) and (3.6). Therefore by Lemma III.17 the Sushchansky group G_λ has subexponential growth function, which satisfies inequality

$$\gamma_G(n) \precsim (\gamma_H(n))^{p^2} \precsim \gamma_H(n). \tag{3.10}$$

The last part of this inequality follows from Proposition III.16, where it is proved that H is regular branch over H'.

Now consider the subgroup $L = \langle B_\lambda, AB_\lambda A^{p-1}, A^2B_\lambda A^{p-2}\rangle$ of G_λ. This subgroup stabilizes the second level of the tree and the sections of the generators at the second level look like:

$$\begin{aligned} B_\lambda &= (q_1, *, \ldots, *), \\ AB_\lambda A^{p-1} &= (r_1, *, \ldots, *), \\ A^2B_\lambda A^{p-2} &= (\sigma, *, \ldots, *). \end{aligned}$$

Each word of length n in L will be projected on the corresponding word of length n in H. Therefore $\gamma_L(n) \geq \gamma_H(n)$ for all $n \geq 1$. But L is a finitely generated subgroup of G_λ. Thus

$$\gamma_H(n) \precsim \gamma_L(n) \precsim \gamma_G(n). \tag{3.11}$$

Inequalities (3.10) and (3.11) imply

$$\gamma_G(n) \sim \gamma_H(n). \tag{3.12}$$

□

Finally, it was mentioned above that the group H is torsion as a G group with torsion directed part. But periodicity of H implies that G_λ is periodic as well. This gives a different proof of Theorem III.1 proved by V.I. Sushchansky. The theory of G groups allows to sharpen this result.

Theorem III.20. *There is a constant $C > 0$, such that the torsion growth function of each Sushchansky p-group G_λ satisfies inequality*

$$\pi_{G_\lambda}(n) \leq Cn^{\log_{1/\eta_r}(p)},$$

where η_r is the same as in the previous theorem.

Proof. By Proposition III.15 the group H is a G group defined by a p^2-homogenous sequence of homomorphisms, whose directed part $\langle q_1, r_1 \rangle$ is an elementary abelian p-group (see Lemma III.2). Therefore by Theorem III.14 the torsion growth function $\pi_H(n)$ satisfies inequality

$$\pi_H(n) \leq C_1 n^{\log_{1/\eta_r}(p)}$$

for some constant C_1.

For any element g of length n in G_λ, g^p stabilizes the second level of the tree and the sections of g^p at the vertices of the second level are the elements of H, whose length is not bigger than pn. Hence, the order of g^p cannot be bigger than the least common multiple of the orders of $g|_v$, $v \in X^2$. Since the orders of these sections are the powers of p, the least common multiple coincides with the maximal order among the sections. This implies

$$\mathrm{Order}(g) = p \cdot \mathrm{Order}(g^p) \leq p\pi_H(pn) \leq pC_1(pn)^{\log_{1/\eta_r}(p)} \leq Cn^{\log_{1/\eta_r}(p)}$$

for $C = C_1 p^{\log_{1/\eta_r}(p)+1}$.

□

CHAPTER IV

DUAL AUTOMATA AND FREE PRODUCTS OF GROUPS OF ORDER 2

The results of this chapter are presented in paper [SV08] written jointly with Y. Vorobets. The structure of this chapter is as follows. All necessary definitions are given in Section 1. The automaton generating the free product of 4 cyclic groups of order 2 is studied in Section 2. In Section 3 the family of automata generating the free products of groups of order 2 is considered.

1 Preliminaries

In this chapter let X be a finite alphabet of cardinality d. We start from introducing the notion of dual automaton. For any finite automaton one can construct a *dual automaton* defined by switching the states and the alphabet as well as switching the transition and the output functions.

Definition 25. Given a finite automaton $\mathcal{A}=(Q,X,\pi,\lambda)$ its *dual automaton* $\hat{\mathcal{A}}$ is a finite automaton $(X,Q,\hat{\lambda},\hat{\pi})$, where

$$\hat{\lambda}(x,q)=\lambda(q,x),$$
$$\hat{\pi}(x,q)=\pi(q,x)$$

for any $x\in X$ and $q\in Q$.

Note that the dual of the dual of an automaton $\mathcal{A}$ coincides with $\mathcal{A}$. The semigroup $\mathbb{S}(\hat{\mathcal{A}})$ generated by dual automaton $\hat{\mathcal{A}}$ of automaton $\mathcal{A}$ acts on the free monoid Q^*. This action induces the action on $\mathbb{S}(\mathcal{A})$. Similarly, $\mathbb{S}(\mathcal{A})$ acts on $\mathbb{S}(\hat{\mathcal{A}})$.

Definition 26. For an automaton semigroup G generated by automaton $\mathcal{A}$ the *dual semigroup* $\hat{G}$ to G is a semigroup generated by a dual automaton $\hat{\mathcal{A}}$.

A particularly important class of automata is the class of bireversible automata.

Definition 27. An automaton $\mathcal{A}$ is called *bireversible* if it is invertible, its dual is invertible, and the dual to $\mathcal{A}^{-1}$ are invertible.

In particular, for any group generated by a bireversible automaton $\mathcal{A}$ one can consider a dual group generated by the dual automaton $\hat{\mathcal{A}}$.

The following proposition is proved in [VV07] by induction on level. With a slight abuse of notations we will denote by the same symbol the element of a free monoid and its image under canonical epimorphism onto corresponding semigroup.

Proposition IV.1. *Let G be an automaton semigroup acting on X^* and generated by the finite set S. And let $\hat{G}$ be a dual semigroup to G acting on S^*. Then for any $g \in G$ and $v \in X^*$ we have $g|_v = v(g)$ in G. Similarly, for any $g \in S^*$ and $v \in \hat{G}$, $v|_g = g(v)$ in $\hat{G}$.*

2 Automaton generating $C_2 * C_2 * C_2 * C_2$

Consider the group generated by the 4-state automaton $\mathcal{B}_4$, whose

transition and output functions are given by wreath recursion (its Moore diagram is shown in the right half of Figure 1)

$$\begin{aligned} a &= (c, b), \\ b &= (b, c), \\ c &= (d, d)\sigma, \\ d &= (a, a)\sigma. \end{aligned} \tag{4.1}$$

In this chapter we will denote this group by $\mathcal{G}$. The next Theorem is the main result of this section.

Theorem IV.2. *Group $\mathcal{G}$ is isomorphic to $C_2 * C_2 * C_2 * C_2$.*

The proof of this theorem is split into a number of lemmas below.

First, we note that the automaton $\mathcal{B}_4$ is bireversible. The dual group Γ to $\mathcal{G}$ is generated by the following automaton (shown in Figure 15)

$$\begin{aligned} \mathbb{0} &= (\mathbb{0}, \mathbb{0}, \mathbb{1}, \mathbb{1})(a\,c\,d), \\ \mathbb{1} &= (\mathbb{1}, \mathbb{1}, \mathbb{0}, \mathbb{0})(a\,b\,c\,d). \end{aligned} \tag{4.2}$$

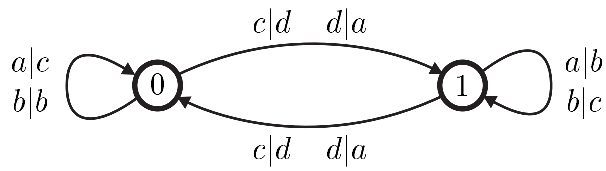


Fig. 15. Automaton dual to Bellaterra automaton $\mathcal{B}_4$

This group acts on a rooted 4-ary tree T whose vertices are labelled by the words over $\{a, b, c, d\}$. Since $a^2 = (c^2, b^2)$, $b^2 = (b^2, c^2)$, $c^2 = (d^2, d^2)$ and $d^2 = (a^2, a^2)$ we get that $a^2 = b^2 = c^2 = d^2 = 1$ in Γ and the image of any word containing any of a^2, b^2, c^2 or d^2 under any element of Γ will also contain one of these subwords. Therefore there is an invariant under Γ subtree $\hat{T}$ of T consisting of all words over $\{a, b, c, d\}$ that do not have a^2, b^2, c^2 and d^2 as subwords. The root of $\hat{T}$ has 4 descendants and all the other vertices in $\hat{T}$ have three (see Figure 16, where subtree $\hat{T}$ is drawn with bold edges).

The following simple proposition was obtained independently by Z. Šunić (private communication) and the proof is implicitly contained in the book of V. Nekrashevych [Nek05].

Proposition IV.3. *Let G be any semigroup generated by a finite automaton and $\hat{G}$ be its dual semigroup. Then G is finite if and only if $\hat{G}$ is finite.*

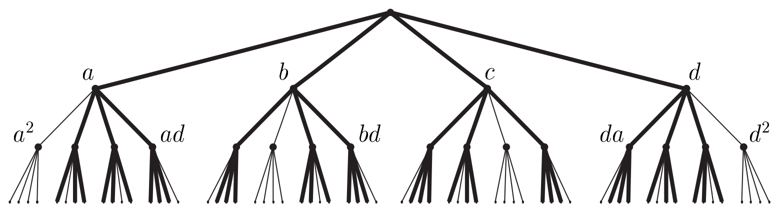


Fig. 16. Trees T and $\hat{T}$

Proof. Since dual of the dual of the automaton generating G coincides with this automaton, it is enough to show the implication in one direction.

Suppose G is finite. For any element $v \in \hat{G}$ and any vertex g of tree the semigroup $\hat{G}$ acts on, we have $v|_g = g(v)$ in $\hat{G}$ by Proposition IV.1. Therefore the number of different sections of v is bounded by the size of G. But there are only finitely many different automata with a fixed number of states. Thus $\hat{G}$ is finite. $\square$

Lemma IV.4. *The group $\mathcal{G}$ is infinite.*

Proof. The lemma follows from the fact that the group acts transitively on each level of the tree. To prove this we first observe that the group $G/\operatorname{Stab}_{\mathcal{G}}(2)$ is cyclic of order 4 and the portrait of depth 2 of every element of G (rooted binary tree of depth 2, where each vertex is labelled by the permutation induced by this element at this vertex) must coincide with one of the listed in Figure 17.

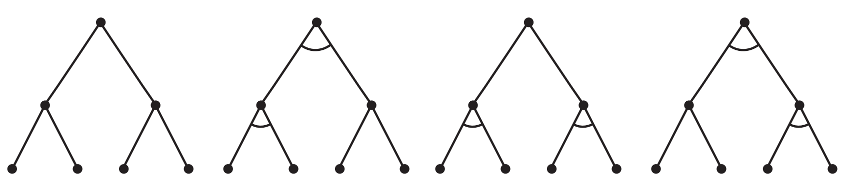


Fig. 17. Possible portraits of elements of G of depth 2

It is proved in [GNS01] that an automorphism g of the rooted binary tree acts level transitively if and only if on each level the number of sections of g at the vertices of this level acting nontrivially on the first level, is odd.

By induction on level it follows that each element g of G acting nontrivially on the first level acts spherically transitively. Indeed, if the number of sections of g at the vertices of the k-th level acting nontrivially on the first level (the number of "switches" on the k-th level) is odd, then each of these sections will produce exactly one switch on the $(k+1)$-st level, while the sections acting trivially on the first level will produce either none or two switches on the $(k+1)$-st level. Thus, the total number of switches on the $(k+1)$-st level will remain odd. $\square$

The direct corollary of Proposition IV.3 and Lemma IV.4 is

Corollary IV.5. *The group Γ is infinite.*

Corollary IV.6. *The stabilizers of levels of T in Γ are pairwise different.*

Proof. Since Γ is infinite by Corollary IV.5 and all stabilizers of levels are finite index subgroups in Γ, they are all infinite. Let $g \in \mathrm{Stab}_\Gamma(n)$ be arbitrary and nontrivial and let $m \geq n+1$ be the smallest level on which g acts nontrivially. Then there exists a vertex $v = x_1x_2\dots x_{m-1}$ of the tree, such that $g|_v$ acts nontrivially on the first level. Then $g|_{x_1x_2\dots x_{m-n-1}} \in \mathrm{Stab}_\Gamma(n) \setminus \mathrm{Stab}_\Gamma(n+1)$. $\square$

Lemma IV.7. *Let $\hat{T}_n$ be the subtree of $\hat{T}$ consisting of the first n levels. Then* $\mathrm{Stab}_\Gamma(n) = \mathrm{Stab}_\Gamma(\hat{T}_n)$.

Proof. Since the leaves of $\hat{T}_n$ are vertices of the n-th level of T we have $\mathrm{Stab}_\Gamma(n) \subset \mathrm{Stab}_\Gamma(T_n)$.

Suppose $v \in \mathrm{Stab}_\Gamma(\hat{T}_n) \setminus \mathrm{Stab}_\Gamma(n)$. Then there is a vertex g of the n-th level which is not in $\hat{T}$ and is not fixed under v. Since v fixes T_n it follows that $g = ftth$

and $v(g) = ftth'$ for some $f, h, h' \in G$ and $t \in \{a, b, c, d\}$. Then

$$v(fh) = v(f)v|_f(h) = f \cdot (v|_f)|_{tt}(h) = fv|_{ftt}(h) = fh'.$$

The second equality above holds since for any $t \in \{a, b, c, d\}$ we have $t^2 = 1$ and thus for any $w \in \Gamma$ by Proposition IV.1, $w|_{tt} = (tt)(w) = w$ in Γ and for any word $h \in T$ we have $w|_{tt}(h) = w(h)$.

We can repeat this procedure until we get an element of $\hat{T}_n$ not fixed under the action of v, obtaining contradiction. Thus $v \in \mathrm{Stab}_\Gamma(n) \setminus \mathrm{Stab}_\Gamma(T_n)$. □

The next statement follows directly from Corollary IV.6 and Lemma IV.7.

Corollary IV.8. *For any $n \geq 1$ there is an element in Γ fixing $\hat{T}_n$ but moving some vertex in $\hat{T}_{n+1}$.*

Lemma IV.9. *The sections of any element of $\mathrm{Stab}_\Gamma(n)$ at the vertices of the n-th level act on the first level by even permutations.*

Proof. By self-similarity it is enough to prove the Lemma for $n = 1$. The claim follows from the fact that $|\mathrm{Stab}_\Gamma(1)/\mathrm{Stab}_\Gamma(2)| = 3^3$ (this was computed using [MS08]). Therefore the sections of any element of $\mathrm{Stab}_\Gamma(1)$ at the vertices of the first level act on the first level by permutations, whose order is a power of 3, which are either cycles of length 3 or the trivial permutation. All these are even permutations.

Below we provide a proof that does not rely on computer computations. This proof is also important because it introduces certain notation that will be used later in Section 3.

First we show that if $v \in \Gamma$ fixes vertex d, then the parities of the actions of v and $v|_d$ on the first level coincide. For this purpose we introduce a new generating set in Γ. For any $x \in \mathrm{Sym}(\{a,b,c,d\})$ denote by $\bar{x}$ the automorphism of T defined by

$$\bar{x} = (\bar{x},\bar{x},\bar{x},\bar{x})x.$$

The portrait of $\bar{x}$ has x at each vertex of the tree. Since $(\mathbb{0}\mathbb{1}^{-1})^2 = (\mathbb{0}^{-1}\mathbb{1})^2 = 1$ we obtain

$$\mathbb{0}\mathbb{1}^{-1} = (\mathbb{0}\mathbb{1}^{-1},\mathbb{0}\mathbb{1}^{-1},\mathbb{0}\mathbb{1}^{-1},\mathbb{0}\mathbb{1}^{-1})(a\,b) = \overline{(a\,b)} \tag{4.3}$$

and

$$\mathbb{0}^{-1}\mathbb{1} = (\mathbb{0}^{-1}\mathbb{1},\mathbb{0}^{-1}\mathbb{1},\mathbb{0}^{-1}\mathbb{1},\mathbb{0}^{-1}\mathbb{1})(b\,c) = \overline{(b\,c)}. \tag{4.4}$$

This shows that $\overline{(a\,c)} \in \Gamma$. If we denote

$$\begin{array}{rcll} \alpha = \mathbb{1}\cdot\overline{(a\,c)} & = & (\alpha,\alpha,\beta,\beta) & (a\,b)(c\,d), \\ \beta = \mathbb{0}\cdot\overline{(a\,c)} & = & (\beta,\beta,\alpha,\alpha) & (c\,d), \end{array} \tag{4.5}$$

then $\alpha^2 = \beta^2 = 1$ and, taking into account that $\beta\alpha^{-1} = \overline{(a\,b)}$,

$$\Gamma = \langle \alpha,\beta,\overline{(b\,c)}\rangle.$$

Suppose now $v \in \Gamma$ is an arbitrary element fixing vertex d. Represent v as a word over $\{\alpha,\beta,\overline{(b\,c)}\}$

$$v = v_1 v_2 \cdots v_k,$$

then by (2.3)

$$v|_d = v_1|_d \cdot v_2|_{v_1(d)} \cdots v_k|_{v_1v_2\cdots v_{k-1}(d)}.$$

The parity of the action of v_i on the first level is different from the one of $v_i|_{v_1v_2\cdots v_{i-1}(d)}$ only in case v_i is α or β and $v_1v_2\cdots v_{i-1}(d) = c$ or $v_1v_2\cdots v_{i-1}(d) = d$.

Note that in this situation if $v_1v_2\cdots v_{i-1}(d) = c$ then $v_1v_2\cdots v_i(d) = d$, and if $v_1v_2\cdots v_{i-1}(d) = d$ then $v_1v_2\cdots v_i(d) = c$. The converse is also true in the following sense: if $v_1v_2\cdots v_{i-1}(d)\neq d$ and $v_1v_2\cdots v_i(d)=d$ then $v_1v_2\cdots v_{i-1}(d)=c$ and v_i is either α or β, and if $v_1v_2\cdots v_{i-1}(d)=d$ and $v_1v_2\cdots v_i(d)\neq d$, then $v_1v_2\cdots v_i(d)=c$ and v_i is either α or β. In other words, the parity of the action of v_i on the first level is different from the one of $v_i|_{v_1v_2\cdots v_{i-1}(d)}$ exactly when there is a change from d to anything else or from something to d in the sequence $\{d, v_1(d), \dots, v_1v_2\cdots v_k(d)\}$. But since $v_1v_2\cdots v_k(d)=v(d)=d$, there must be an even number of such changes. Hence, the parity is different in even number of places and the parities of the actions of v and $v|_d$ on the first level coincide.

By the above for any $g=(g|_a, g|_b, g|_c, g|_d)\in \mathrm{Stab}_\Gamma(1)$ the parity of the action of $g|_d$ on the first level is even. Furthermore, the conjugate $g^\beta = \beta^{-1}g\beta$ has decomposition

$$g^\beta = (*,*,*,(g|_c)^\alpha)\in \mathrm{Stab}_\Gamma(1),$$

which implies that $(g|_c)^\alpha$ and, hence, $g|_c$ acts on the first level by an even permutation. Finally,

$$g^{\overline{(a\,c)}} = (*,*,(g|_a)^{\overline{(a\,c)}},*)\in \mathrm{Stab}_\Gamma(1),$$
$$g^{\overline{(b\,c)}} = (*,*,(g|_b)^{\overline{(b\,c)}},*)\in \mathrm{Stab}_\Gamma(1).$$

This shows that all sections of g at the vertices of the first level act on the first level by even permutations. □

Lemma IV.10. *The group* Γ *acts transitively on the levels of* $\hat{T}$.

Proof. We proceed by induction on levels. The transitivity on the first level is clear. Assume Γ acts transitively on the n-th level of $\hat{T}$. By Corollary IV.8 there is an element $v\in\Gamma$ that fixes $\hat{T}_n$ and acts nontrivially on $\hat{T}_{n+1}$. This means that there is a vertex $g\in\hat{T}_n$ such that $v(g)=g$ and $v|_g$ acts nontrivially on the first level. By Lemma IV.9 the permutation induced by $v|_g$ on the first level is even, which implies that it is a cycle of length 3. Thus $v|_g$ acts transitively on the first level of the tree.

Without loss of generality assume that g ends with d. By induction assumption, for any vertex $h_1h_2\dots h_{n+1}$ of $\hat{T}_{n+1}$ there is an element $w\in\Gamma$ that moves g to $h_1h_2\dots h_n$. Then v^kw, where k is 0, 1 or 2 will move ga to $h_1h_2\dots h_{n+1}$. Thus, Γ acts transitively on $\hat{T}_{n+1}$. □

Finally, we have all ingredients for the proof of Theorem IV.2.

Proof of Theorem IV.2. For every $n\geq 1$ there is a nontrivial element $h\in\mathcal{G}$ that belongs to the n-th level of $\hat{T}$ ($h=(ab)^{\frac{n-1}{2}}c\neq 1$ for an odd n and $h=(ab)^{\frac{n}{2}-1}ac\neq 1$ for an even n). By Lemma IV.10 the group Γ acts transitively on each level of $\hat{T}$. Therefore for any word g from the n-th level of $\hat{T}$ (which is a word of length n without double letters) there exists $v\in\Gamma$ such that

$$g|_v=v(g)=h\neq 1.$$

Thus there are no relations in $\mathcal{G}$ except $a^2=b^2=c^2=d^2=1$. □

3 Family of automata generating the free products of C_2

Let us define a family of automata obtained from the automaton $\mathcal{B}_4$ by inserting new states on the path from c to d. Namely, for every integer $n>4$ and any permutations $\sigma_{n,i}\in\mathrm{Sym}(\{1,2\})$, $i=1,\dots,n-4$ consider an automaton with n states $a_n,b_n,c_n,q_{n1},q_{n2},\dots,q_{n,n-4},d_n$ whose transition and output functions are given via

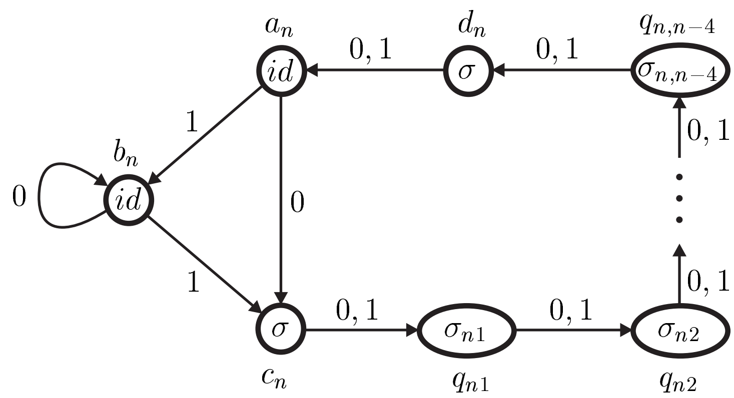


Fig. 18. Bellaterra automaton $\mathcal{B}^{(n)}$

the wreath recursion

$$\begin{aligned} a_n &= (c_n, b_n), \\ b_n &= (b_n, c_n), \\ c_n &= (q_{n1}, q_{n1})\sigma, \\ q_{n,i} &= (q_{n,i+1}, q_{n,i+1})\sigma_{n,i}, i = 1, \dots, n-5, \\ q_{n,n-4} &= (d_n, d_n)\sigma_{n,n-4}, \\ d_n &= (a_n, a_n)\sigma. \end{aligned} \tag{4.6}$$

With a slight abuse of notation we denote this automaton by $\mathcal{B}^{(n)}$ regardless of the choice of permutations $\sigma_{n,i}$. The Moore diagram of $\mathcal{B}^{(n)}$ is shown in Figure 18.

From the wreath recursion it is easy to observe that the generators of $\mathcal{B}^{(n)}$ are involutions.

This section is devoted to the proof of the following theorem.

Theorem IV.11. *The group $\mathcal{G}^{(n)}$ generated by automaton $\mathcal{B}^{(n)}$ is isomorphic to the free product of n copies of the cyclic group of order* 2*.*

The proof relies on the results of Section 2. The approach is similar. We prove that the dual automaton acts transitively on the invariant subtree consisting of words without double letters. This yields the structure of the free product in the group $\mathcal{G}^{(n)}$.

Note that the automaton $\mathcal{B}^{(n)}$ is bireversible so that the dual group $\Gamma^{(n)}$ to $\mathcal{G}^{(n)}$ is well defined. The group $\Gamma^{(n)}$ is generated by automaton acting on the rooted n-ary tree $T^{(n)}$ as follows

$$\begin{aligned} \mathbb{0}_n &= (\mathbb{0}_n, \mathbb{0}_n, \mathbb{1}_n, \mathbb{K}_{n1}, \dots, \mathbb{K}_{n,n-4}, \mathbb{1}_n)(a_n\, c_n\, q_{n1} \dots q_{n,n-4}\, d_n), \\ \mathbb{1}_n &= (\mathbb{1}_n, \mathbb{1}_n, \mathbb{0}_n, \mathbb{L}_{n1}, \dots, \mathbb{L}_{n,n-4}, \mathbb{0}_n)(a_n\, b_n\, c_n\, q_{n1} \dots q_{n,n-4}\, d_n), \end{aligned} \tag{4.7}$$

where $\mathbb{K}_{n,i} = \mathbb{0}_n$ and $\mathbb{L}_{n,i} = \mathbb{1}_n$ if $\sigma_{n,i}$ is a trivial permutation, and $\mathbb{K}_{n,i} = \mathbb{1}_n$ and $\mathbb{L}_{n,i} = \mathbb{0}_n$ otherwise.

Consider a subtree $\hat{T}^{(n)}$ of $T^{(n)}$ consisting of all words over the alphabet $Y^{(n)} = \{a_n, b_n, c_n, q_{n1}, q_{n2}, \dots, q_{n,n-4}, d_n\}$ without double letters. The root of $\hat{T}^{(n)}$ has n descendants and all other vertices have $n-1$. This subtree is invariant under the action of $\Gamma^{(n)}$.

Similarly to (4.3) and (4.4) we get that $\mathbb{0}_n \mathbb{1}_n^{-1} = \overline{(a_n\, b_n)}$ and $\mathbb{0}_n^{-1}\mathbb{1}_n = \overline{(b_n\, c_n)}$. Similarly to (4.5) we define transformations $\alpha_n = \mathbb{1}_n \cdot \overline{(a_n\, c_n)}$ and $\beta_n = \mathbb{0}_n \cdot \overline{(a_n\, c_n)}$ for which we have

$$\begin{aligned} \alpha_n &= (\alpha_n, \alpha_n, \beta_n, \gamma_{n1}, \dots, \gamma_{n,n-4}, \beta_n) \quad (a_n\, b_n)(c_n\, q_{n1} \dots q_{n,n-4}\, d_n), \\ \beta_n &= (\beta_n, \beta_n, \alpha_n, \delta_{n1}, \dots, \delta_{n,n-4}, \alpha_n) \qquad\qquad (c_n\, q_{n1} \dots q_{n,n-4}\, d_n), \end{aligned} \tag{4.8}$$

where $\gamma_{n,i} = \alpha_n$ and $\delta_{n,i} = \beta_n$ if $\sigma_{n,i}$ is a trivial permutation, and $\gamma_{n,i} = \beta_n$ and $\delta_{n,i} = \alpha_n$ otherwise.

Since $\alpha_n^{-1}\beta_n = \overline{(a_n\, b_n)}$ we get a new generating set for Γ_n,

$$\Gamma^{(n)} = \langle \alpha_n, \beta_n, \overline{(b_n\, c_n)} \rangle.$$

The following lemma establishes a relation between the actions of the groups

Γ and $\Gamma^{(n)}$. We consider the tree $\hat{T}$ naturally embedded in the tree $\hat{T}^{(n)}$ via a homomorphism of monoids induced by $a \mapsto a_n$, $b \mapsto b_n$, $c \mapsto c_n$, $c \mapsto c_n$, $d \mapsto d_n$. Then the group Γ acts also on $\hat{T}^{(n)}$ (the action on the letters not in the image of $\hat{T}$ is defined to be trivial).

Lemma IV.12. *For any $v \in \Gamma$ there exists $v' \in \Gamma^{(n)}$ with the following property. For any word g over $\{a_n, b_n, c_n\}$ such that $v(g)$ is also a word over $\{a_n, b_n, c_n\}$, we have $v(g) = v'(g)$.*

Proof. Let $x_1x_2\ldots x_k$ be the word over $\{\alpha, \beta, \overline{(b_n\, c_n)}\}$ representing v. Define $y_i \in \{\alpha_n, \beta_n, \overline{(b_n\, c_n)}\}$ by the following rule. If $x_i = \overline{(b_n\, c_n)}$, then put $y_i = x_i$. In the case $x_i = \alpha$ (resp. $x_i = \beta$) compute the total number of α and β among $x_1, x_2, \ldots, x_{i-1}$. If this number is even, then define $y_i = \alpha_n$ (resp. $y_i = \beta_n$). Otherwise, put $y_i = \alpha_n^{-1}$ (resp. $y_i = \beta_n^{-1}$).

Now let g be any word over $\{a_n, b_n, c_n\}$. We will show by induction on i that $y_1y_2\ldots y_i(g)$ is obtained from $x_1x_2\ldots x_i(g)$ by replacing all occurrences of d_n by q_{n1} when the total number of α and β among $x_1, x_2, \ldots, x_{i-1}$ is odd, and coincides with $x_1x_2\ldots x_i(g)$ otherwise.

The claim holds trivially for $i = 0$. Let us prove the induction step. First of all, if $x_{i+1} = y_{i+1} = \overline{(b_n\, c_n)}$ then the relation between $y_1y_2\ldots y_{i+1}(g)$ and $x_1x_2\ldots x_{i+1}(g)$ is the same as between $y_1y_2\ldots y_i(g)$ and $x_1x_2\ldots x_i(g)$. This is because $\overline{(b_n\, c_n)}$ fixes letters d_n and $q_{n,1}$. Hence we can assume that $x_{i+1} = \alpha$ or $x_{i+1} = \beta$.

Suppose first that there is an odd number of α and β among $x_1, x_2, \ldots, x_i$. By induction assumption $y_1y_2\ldots y_i(g)$ is obtained from $x_1x_2\ldots x_i(g)$ by replacing all occurrences of d_n by q_{n1} and, in particular, is a word over $\{a_n, b_n, c_n, q_{n1}\}$. If $x_{i+1} = \alpha$ ($x_{i+1} = \beta$), then by construction $y_{i+1} = \alpha_n^{-1}$ (respectively $y_{i+1} = \beta_n^{-1}$), for which we have

$$\begin{aligned}\alpha_n^{-1} &= (\alpha_n^{-1}, \alpha_n^{-1}, \beta_n^{-1}, \beta_n^{-1}, \gamma_{n1}^{-1}, \dots, \gamma_{n,n-4}^{-1})(a_n\, b_n)(c_n\, d_n \dots q_{n1}),\\ \beta_n^{-1} &= (\alpha_n^{-1}, \alpha_n^{-1}, \beta_n^{-1}, \beta_n^{-1}, \delta_{n1}^{-1}, \dots, \delta_{n,n-4}^{-1}) \qquad\qquad (c_n\, d_n \dots q_{n1}).\end{aligned} \tag{4.9}$$

Therefore the images of $y_1 y_2 \dots y_i(g)$ under the actions of α_n^{-1} and β_n^{-1} coincide with the images of $x_1 x_2 \dots x_i(g)$ under the actions of α and β correspondingly. Thus, $y_1 y_2 \dots y_{i+1}(g) = x_1 x_2 \dots x_{i+1}(g)$, which is exactly what we need since the number of α and β among $x_1, x_2, \dots, x_{i+1}$ is even.

In case of even number of occurrences of α and β among $x_1, x_2, \dots, x_i$ by induction assumption $y_1 y_2 \dots y_i(g)$ coincides with $x_1 x_2 \dots x_i(g)$. In particular, it is a word over $\{a_n, b_n, c_n, d_n\}$. Also by construction $y_{n+1} = \alpha_n$ or $y_{n+1} = \beta_n$.

It follows from (4.8) that y_{i+1} acts on the letters of $y_1 y_2 \dots y_i(g)$ exactly as x_{i+1}, except it everywhere moves c_n to q_{n1}, instead of moving it to d_n. Therefore, the resulting word $y_1 y_2 \dots y_{i+1}(g)$ can be obtained from $x_1 x_2 \dots x_{i+1}(g)$ by changing all occurrences of d_n by q_{n1}. This agrees with the fact that the total number of α and β among $x_1, x_2, \dots, x_{i+1}$ is odd.

Finally, to finish the proof of the lemma, it is enough to put $v' = y_1 y_2 \dots y_k$ and note that if $v(g)$ is a word over $\{a_n, b_n, c_n\}$, then $v'(g)$ must coincide with $v(g)$ regardless of the number of α and β in the word representing v. □

Lemma IV.13. *The group $\Gamma^{(n)}$ acts transitively on the levels of $\hat{T}^{(n)}$.*

Proof. We proceed by induction on levels. Obviously, $\Gamma^{(n)}$ acts transitively on the first level. Suppose it acts transitively on level m. We will show that any vertex of the $(m+1)$-st level can be moved to the vertex $a_n b_n a_n b_n \dots b_n a_n$ or $a_n b_n a_n b_n \dots a_n b_n$ (depending on the parity of m).

Let g be the vertex of the $(m+1)$-st level of $\hat{T}^{(n)}$. Then $g = ht$, where h is the vertex of the m-th level and $t \in Y^{(n)}$. For definiteness let us assume that m is even.

By induction assumption there is $v \in \Gamma^{(n)}$ that moves h to $a_n b_n a_n b_n \dots b_n$. Then

$$v(g) = a_n b_n a_n b_n \dots a_n b_n t'$$

for some $t' \in Y^{(n)}$. Since β_n fixes $a_n b_n a_n b_n \dots a_n b_n$ and $\beta_n|_{a_n b_n a_n b_n \dots a_n b_n} = \beta_n$, after applying, if necessary, a power of β_n we can assume that $t' \in \{a_n, b_n, c_n\}$. Now we invoke the transitivity of the group Γ on $\hat{T}$. By Lemma IV.10 there is $w \in \Gamma$ such that $w(a_n b_n a_n b_n \dots a_n b_n t') = a_n b_n a_n b_n \dots a_n b_n a_n$. Then by Lemma IV.12 there is $w' \in \Gamma^{(n)}$ such that $w'(a_n b_n a_n b_n \dots a_n b_n t') = w(a_n b_n a_n b_n \dots a_n b_n t') = a_n b_n a_n b_n \dots a_n b_n a_n$. This proves transitivity of $\Gamma^{(n)}$ on the levels of $\hat{T}^{(n)}$. $\square$

Finally, Theorem IV.11 is derived from Lemma IV.13 exactly in the same way as Theorem IV.2 is obtained from Lemma IV.10.

CHAPTER V

ON ITERATED MONODROMY GROUP $IMG(Z^2+I)$

The results of this chapter are published in paper [GSŠ07] written jointly with R. Grigorchuk and Z. Šunić. The structure of this chapter is as follows. We give very detailed calculation of the action of IMG(z^2+i) (denoted by $\mathcal{G}$ in the rest of this chapter) on the binary rooted tree in 1. Then in Section 2 we show that the group $\mathcal{G}$ is a regular branch group, thus presenting an example of a branch group which naturally appears in holomorphic dynamics. The main body of this chapter is devoted to the calculation of an L-presentation for $\mathcal{G}$. Section 4 deals with finding a self-similar measure on $\mathcal{G}$. Finally, in Section 5 we construct a rational map on $\mathbb{R}^3$ whose proper invariant set (shaped as a "strange attractor") gives the spectrum of the Markov operator acting on the boundary of the tree after intersection by a corresponding line.

1 Computing the action of IMG(z^2+i) on the tree

All necessary for this chapter definitions and notations were described in Section II.7. The only critical point of the map $z \mapsto z^2+i$ is $z=0$, which is preperiodic:

$$0 \xrightarrow{f} i \xrightarrow{f} (i-1) \overset{f}{\rightleftarrows} -i.$$

and, hence, the postcritical set of f is $\{i, i-1, -i\}$. Therefore the restriction of f on $M_1 = \mathbb{C} \setminus \{i, i-1, -i, 0\}$ is a 2-fold covering of $M = \mathbb{C} \setminus \{i, i-1, -i\}$.

Set $t = 0 \in \mathbb{C}$ as the base point. It has two preimages $e^{i3\pi/4}$ and $e^{i7\pi/4}$ which are identified with the letters 0 and 1, respectively (more precisely, we set $\Lambda(0) = e^{i3\pi/4}$ and $\Lambda(1) = e^{i7\pi/4}$). For the paths l_0 and l_1 connecting t to its preimages we choose

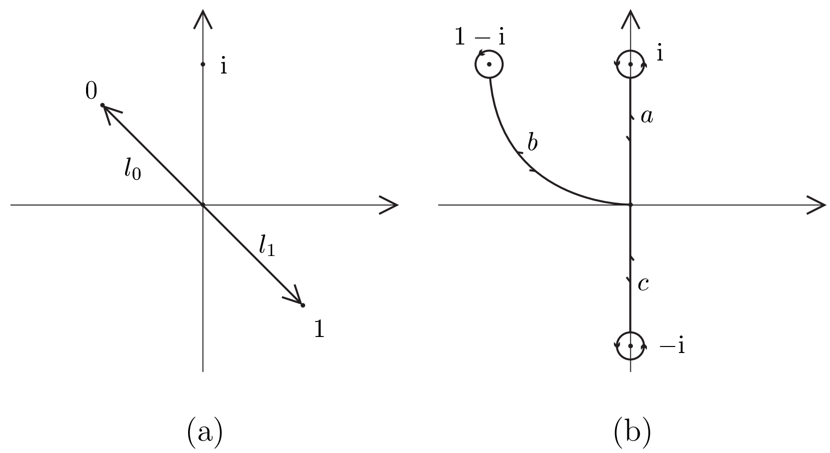


Fig. 19. Paths connecting $t = 0$ to its preimages and generators of the fundamental group $\pi(M, t)$

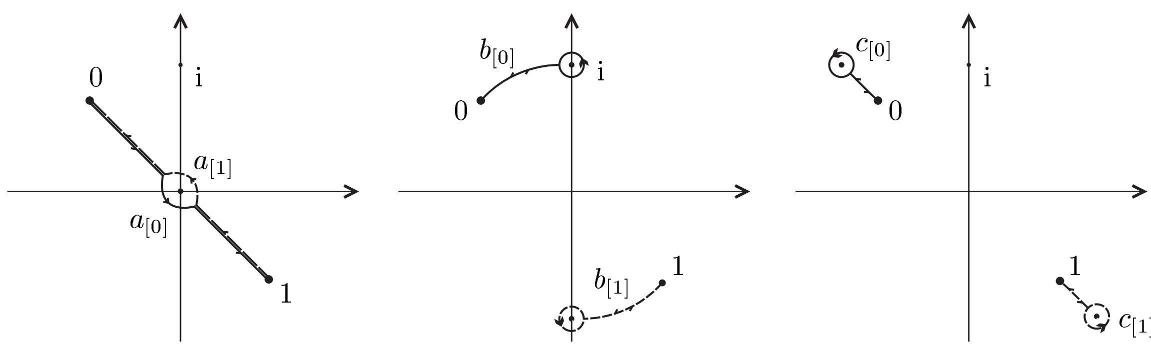


Fig. 20. Preimages of the generating loops

the straight segments shown in Figure 19(a).

The fundamental group $\pi_1(M, t)$ is generated by the 3 loops a, b, c shown in Figure 19(b) going around i, $-i$ and $i - 1$ respectively. Each of these loops has two preimages $a_{[i]}$, $b_{[i]}$ and $c_{[i]}$, $i = 0, 1$, shown in Figure 20.

According to formula (2.5) in Chapter II and Figures 19 and 20 the sections of

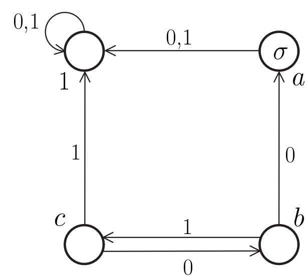


Fig. 21. Automaton generating group $\mathrm{IMG}(z^2+i)$

the generators a, b, c at 0 and 1 satisfy:

$$a|_0 = l_0 a_{[0]} l_1^{-1} = 1, \qquad a|_1 = l_1 a_{[1]} l_0^{-1} = 1,$$

$$b|_0 = l_0 b_{[0]} l_0^{-1} = a, \qquad b|_1 = l_1 b_{[1]} l_1^{-1} = c,$$

$$c|_0 = l_0 c_{[0]} l_0^{-1} = b, \qquad c|_1 = l_1 c_{[1]} l_1^{-1} = 1,$$

where 1 denotes the trivial loop at t, which represents the identity element of $\mathrm{IMG}(z^2+i)$.

Since a permutes the elements of $f^{-1}(t)$, while b and c do not, we obtain the following wreath recursion for the generators of $\mathrm{IMG}(z^2+i)$

$$a = (1,1)\sigma, \quad b = (a,c), \quad c = (b,1), \tag{5.1}$$

where σ is the nontrivial transposition in $\mathrm{Sym}(2)$.

These relations show that the set of all sections of the generators a, b, c of $\mathcal{G}$ is $\{1, a, b, c\}$ and that the group $\mathcal{G}$ is generated by the states of the finite automaton shown in Figure 21.

We now say a few words about the relation between the dynamics of the map $z \mapsto z^2 + i$ and the combinatorial properties of the action of $\mathcal{G}$ on the tree T.

Recall that if a group G acts on a set Y then the *Schreier graph* of this action with respect to the generating set S of G is an oriented graph, whose set of vertices is

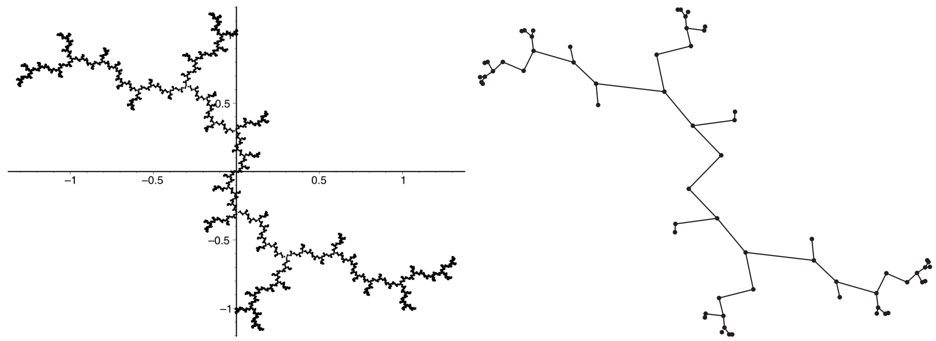


Fig. 22. Schreier graph of $\mathcal{G}$ and Julia set of $z^2 + i$

Y and there is an edge from $y \in Y$ to $z \in Y$ labeled by $s \in S$ if and only if $s(y) = z$. It is convenient sometimes to forget about the labels and/or the orientation of the edges.

Every group acting on a rooted tree acts on each level of the tree. The Schreier graphs of such actions are of particular interest, since in many situations (such as the one we are in) they can be used to find the spectrum of the Markov operator on the boundary of the tree (see Section 5).

Recent results of Nekrashevych [Nek05] show that the Schreier graphs of $\mathrm{IMG}(f)$ on the levels of the tree converge to the Julia set of the map f. Therefore the structure of the Julia set of f provides understanding of the structure of the Schreier graphs of $\mathrm{IMG}(f)$ (and vice versa). In our case the Julia set of $z^2 + i$ is the dendrite shown in the left half of Figure 22. The right half of this figure displays the Schreier graph of $\mathcal{G}$ on level 8. The set of vertices of this graph is just $f^{-8}(0)$ and the vertices are connected according to the action of $\mathcal{G}$ (no loops are drawn though to emphasize the relation between left and right halves of the figure).

2 Branch structure in $\mathrm{IMG}(z^2+i)$

In this section we prove that $\mathcal{G}$ is a self-replicating branch group (see Section II.6 for definitions). The fact that it is self-replicating is clear from the equalities

$$b = (a, c), \quad c = (b, 1), \quad aba = (c, a), \quad aca = (1, b).$$

Consider the normal subgroup N of $\mathcal{G}$ defined by

$$N = \langle [a, b], [b, c] \rangle^{\mathcal{G}}.$$

By definition, $[g, h] = g^{-1}h^{-1}gh$ and $\langle \cdot \rangle^{\mathcal{G}}$ denotes normal closure in $\mathcal{G}$.

Theorem V.1. *The group $\mathcal{G}$ is a regular branch group over N.*

Proof. First we observe that N has finite index in $\mathcal{G}$. Direct computation shows that $a^2 = b^2 = c^2 = (ac)^4 = (ab)^8 = (bc)^8 = 1$, so $\mathcal{G}/N$ is a homomorphic image of

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^4 = [a, b] = [b, c] = 1 \rangle \cong \mathrm{C}_2 \times \mathrm{D}_4,$$

where C_2 is the cyclic group of order 2 and D_4 is the dihedral group of order 8.

Further, we have

$$[b, c] = ([a, b], 1) \quad [c, b^a] = ([b, c], 1)$$

Since $[c, b^a] = cb[b, a]cb[b, a] = [b, a]^{bc}[c, b][b, a] \in N$ we have that $([a, b], 1)$ and $([b, c], 1)$ are elements in N. The fractalness of $\mathcal{G}$ enables us to conjugate the sections in $([a, b], 1)$ and $([b, c], 1)$ by arbitrary elements in $\mathcal{G}$ without leaving N. Thus we get the inclusion $N \times 1 \preceq N$. The transitivity of $\mathcal{G}$ on the first level then implies

$$N \times N \preceq N.$$

The level transitivity of $\mathcal{G}$ can be obtained almost for free. The fact that N is nontrivial along with the fact that $N \times N \preceq N$ implies that N is infinite, and hence so is $\mathcal{G}$. Now Lemma III.8 tells that a self-similar group acting on a binary rooted tree is infinite if and only if it acts transitively on all levels. Another way to show transitivity would be to use Proposition II.10. $\square$

3 *L*-presentation

The goal of this section is to prove the following result.

Theorem V.2. *The group $\mathcal{G}$ has the following L-presentation*

$$\begin{aligned}\mathcal{G} \cong \Big\langle a, b, c \;\Big|\; & \phi^n(a^2), \phi^n\big((ac)^4\big), \phi^n([c, ab]^2), \phi^n([c, bab]^2), \\ & \phi^n([c, ababa]^2), \phi^n([c, ababab]^2), \phi^n([c, bababab]^2), n \geq 0\Big\rangle, \end{aligned} \tag{5.2}$$

where ϕ is the substitution defined on words in the free monoid over the alphabet $\{a, b, c\}$ by

$$\phi\colon \begin{cases} a \to b, \\ b \to c, \\ c \to aba. \end{cases}$$

In order to prove Theorem V.2 we introduce some notation and prove a few intermediate results.

The group

$$\Gamma = \langle a, b, c \mid a^2, b^2, c^2, (ac)^4 \rangle$$

covers $\mathcal{G}$ (the relators of Γ are relators of $\mathcal{G}$). The action of $\mathcal{G}$ on the binary tree induces an action of the covering group Γ on the same tree, which is not faithful. Let Ω be the kernel of this action. Then, obviously, a set of generators of Ω as a normal subgroup in Γ, together with the relators in Γ constitutes a presentation for $\mathcal{G}$.

The embedding $\mathcal{G} \hookrightarrow \mathcal{G} \wr \mathrm{Sym}(2)$ induces a homomorphism

$$\Psi\colon \Gamma \to \Gamma \wr \mathrm{Sym}(2)$$

defined by

$$\Psi\colon \begin{cases} a \mapsto (1,1)\sigma, \\ b \mapsto (a,c), \\ c \mapsto (b,1). \end{cases}$$

Indeed, the relators of Γ are mapped to the trivial element $(1,1)$ of $\Gamma \wr \mathrm{Sym}(2)$:

$$\Psi(a^2) = (1,1)\sigma(1,1)\sigma = (1,1), \qquad \Psi(b^2) = (a,c)^2 = (a^2,c^2) = (1,1),$$
$$\Psi(c^2) = (b,1)^2 = (b^2,1) = (1,1), \qquad \Psi((ac)^4) = ((1,b)\sigma)^4 = (b^2,b^2) = (1,1).$$

The homomorphism Ψ induces homomorphisms $\Psi_n\colon \Gamma \to \Gamma \wr \left(\wr_{i=1}^n \mathrm{Sym}(2)\right)$ (here $\wr_{i=1}^n \mathrm{Sym}(2)$ denotes the iterated permutational wreath product) defined recursively by $\Psi_1 = \Psi$ and

$$\Psi_n\colon \Gamma \xrightarrow{\Psi_{n-1}} \Gamma \wr \left(\mathop{\wr}_{i=1}^{n-1} \mathrm{Sym}(2)\right) \xrightarrow{\Psi} \big(\Gamma \wr \mathrm{Sym}(2)\big) \wr \left(\mathop{\wr}_{i=1}^{n-1} \mathrm{Sym}(2)\right) = \Gamma \wr \left(\mathop{\wr}_{i=1}^{n} \mathrm{Sym}(2)\right).$$

If, for $g \in \Gamma$, we have $\Psi_n(g) = (g|_u, u \in X^n)\sigma_n$, with $g|_u \in \Gamma$ and $\sigma_n \in \wr_{i=1}^n \mathrm{Sym}(2)$ we call $g|_u$ the *section* of g at u.

For every $g \in \Gamma$ denote by $l(g)$ the length of the shortest word in a, b, c representing g in Γ. The following lemma shows that Γ possesses the so called *contraction* property.

Lemma V.3. *For every $g \in \Gamma$ and $u \in X^2$*

$$l(g|_u) \leq \frac{l(g)+1}{2}. \tag{5.3}$$

Proof. Observe that, because of the self-similarity, all generators satisfy inequality (5.3). Indeed,

$$\Psi_2(a) = (1,1,1,1)(02)(13), \quad \Psi_2(b) = (1,1,b,1)(01), \quad \Psi_2(c) = (a,c,1,1),$$

where, for ease of notation, the vertices on the second level are renamed by using the identifications $00 \leftrightarrow 0$, $01 \leftrightarrow 1$, $10 \leftrightarrow 2$, and $11 \leftrightarrow 3$.

All pairwise products of generators also satisfy inequality (5.3):

$$\begin{aligned}
\Psi_2(a^2) = 1, \quad \Psi_2(b^2) &= 1, \quad \Psi_2(c^2) = 1, \\
\Psi_2(ab) &= \big(\Psi_1(c), \Psi_1(a)\big)\sigma = (b,1,1,1)(0213), \\
\Psi_2(ba) &= \big(\Psi_1(a), \Psi_1(c)\big)\sigma = (1,1,b,1)(0312), \\
\Psi_2(ac) &= \big(\Psi_1(1), \Psi_1(b)\big)\sigma = (1,1,a,c)(02)(13), \\
\Psi_2(ca) &= \big(\Psi_1(b), \Psi_1(1)\big)\sigma = (a,c,1,1)(02)(13), \\
\Psi_2(bc) &= \big(\Psi_1(ab), \Psi_1(c)\big) = (c,a,b,1)(01), \\
\Psi_2(cb) &= \big(\Psi_1(ba), \Psi_1(c)\big) = (a,c,b,1)(01).
\end{aligned} \tag{5.4}$$

Any word w in a, b, c of length n can be split into a product of at most $(n+1)/2$ products of pairs of generators (if the length of w is odd one can pair the last letter in w with 1). Therefore the sections of w on the vertices of the second level are products of at most $(n+1)/2$ letters. Thus the inequality (5.3) holds for w as well. $\square$

Define an increasing sequence of subgroups of Γ by

$$\Omega_n = \ker \Psi_n.$$

Lemma V.4. *The kernel Ω of the canonical epimorphism $\Gamma \to \mathcal{G}$ satisfies*

$$\Omega = \bigcup_{n \geq 1} \Omega_n$$

Proof. Let h be a word in a, b, c of length at most $2^n + 1$ representing the trivial element in $\mathcal{G}$. Then, since for any words $u, v \in X^*$

$$h|_{uv} = h|_u|_v, \tag{5.5}$$

by Lemma V.3 we obtain that all sections of h have length at most 1 on the $2(n+1)$th level. Therefore they must be trivial, because h acts trivially on the tree. Hence, $h \in \Omega_{2(n+1)}$. □

The above lemma reduces the problem of finding generators for Ω to finding generators for Ω_n. We start from $\Omega_1 = \ker \Psi$ and, based on it, derive generators for Ω_n.

Let $H = \mathrm{St}_\Gamma(1)$ be the stabilizer of the first level of the tree in Γ.

Lemma V.5. *The group H has the following presentation*

$$H = \langle \beta, \delta, \gamma, \rho \mid \beta^2 = \delta^2 = \gamma^2 = \rho^2 = (\rho\delta)^2 = 1 \rangle,$$

where $\beta = b$, $\delta = c$, $\gamma = aba$, $\rho = aca$.

Proof. The index of H in Γ is 2 and the coset representatives are $\{1, a\}$. The Reidemeister-Schreier procedure gives the above presentation. □

Obviously, each Ω_n is a subgroup of H. Therefore one can restrict Ψ to H. Since H stabilizes the first level one can think of Ψ as a homomorphism $H \to \Gamma \times \Gamma$. This map is given by

$$\Psi\colon \begin{cases} \beta = b \to (a, c), \\ \gamma = aba \to (c, a), \\ \delta = c \to (b, 1), \\ \rho = aca \to (1, b), \end{cases}$$

which mimics the corresponding embedding of the generators b, aba, c, aca of $\mathrm{St}_G(1)$ into $G \times G$.

Define the following words in Γ:

$$U_1 = (ba)^8, \qquad U_2 = [c, ab]^2, \qquad U_3 = [c, bab]^2,$$
$$U_4 = [c, ababa]^2, \qquad U_5 = [c, ababab]^2, \qquad U_6 = [c, bababab]^2.$$

Lemma V.6. $\Omega_1 = \langle U_1, U_2, U_3, U_4, U_5, U_6 \rangle^\Gamma$.

Proof. In order to find a generating set for $\Omega_1 = \ker \Psi$ we first describe $\Psi(H)$. Since $\Psi(\delta) = (b, 1)$ and $\Psi(\rho) = (1, b)$, we get

$$B \times B \trianglelefteq \Psi(H),$$

where $B = \langle b \rangle^\Gamma$. Furthermore, $\Psi(H)/(B \times B) \cong \langle (a, c), (c, a) \rangle \cong \mathrm{D}_4$. Therefore

$$\Psi(H) \cong (B \times B) \rtimes \mathrm{D}_4.$$

Now we provide a presentation for B.

Define

$$\xi_1 = b, \qquad \xi_2 = b^a, \qquad \xi_3 = b^c, \qquad \xi_4 = b^{ca},$$
$$\xi_5 = b^{ac}, \qquad \xi_6 = b^{aca}, \qquad \xi_7 = b^{cac}, \qquad \xi_8 = b^{acac}.$$

Since $\Gamma = C_2 * D_4$ where the cyclic group C_2 of order 2 is generated by b and the dihedral group D_4 of order 8 is generated by a and c, it is clear that B is generated by all conjugates of b by the elements in $D_4 = \langle a, c \rangle$. Thus $\{ \xi_i \mid i = 1, \dots, 8 \}$ is a generating set for B. Moreover, it is clear that

$$B = \langle \xi_i, i = 1, \dots, 8 \mid \xi_i^2 = 1, i = 1, \dots, 8 \rangle,$$

i.e., B is a free product of 8 copies of the cyclic group of order 2 (indeed, none of the

b's in an expression of the form $\xi_{i_1}\xi_{i_2}\dots\xi_{i_m}$ can be canceled in Γ when $i_j \neq i_{j+1}$ for $j = 1, \dots, m-1$).

Therefore $B \times B$ is generated by 16 elements, namely, $\tilde{\xi}_i = (\xi_i, 1)$ and $\hat{\xi}_i = (1, \xi_i)$ and is presented by

$$B \times B = \langle \tilde{\xi}_i, \hat{\xi}_j, i, j = 1, \dots, 8 \mid \tilde{\xi}_i^2, \hat{\xi}_j^2, [\tilde{\xi}_i, \hat{\xi}_j], i, j = 1, \dots, 8 \rangle.$$

Now we compute the action of D_4 generated by $x = (a, c)$ and $y = (c, a)$ on $B \times B$.

$$\begin{aligned}
\tilde{\xi}_1^x &= (b, 1)^{(a,c)} = (aba, 1) = \tilde{\xi}_2, \\
\tilde{\xi}_2^x &= (aba, 1)^{(a,c)} = (b, 1) = \tilde{\xi}_1, \\
\tilde{\xi}_3^x &= (cbc, 1)^{(a,c)} = (acbca, 1) = \tilde{\xi}_4, \\
\tilde{\xi}_4^x &= (acbca, 1)^{(a,c)} = (cbc, 1) = \tilde{\xi}_3, \\
\tilde{\xi}_5^x &= (cabac, 1)^{(a,c)} = (acabaca, 1) = \tilde{\xi}_7, \\
\tilde{\xi}_6^x &= (cacbcac, 1)^{(a,c)} = (acacbcaca, 1) = \tilde{\xi}_8, \\
\tilde{\xi}_7^x &= (acabaca, 1)^{(a,c)} = (cabac, 1) = \tilde{\xi}_5, \\
\tilde{\xi}_8^x &= (acacbcaca, 1)^{(a,c)} = (cacbcac, 1) = \tilde{\xi}_6,
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\tilde{\xi}_1^y &= (b,1)^{(c,a)} = (cbc,1) = \tilde{\xi}_3,\\
\tilde{\xi}_2^y &= (aba,1)^{(c,a)} = (cabac,1) = \tilde{\xi}_5,\\
\tilde{\xi}_3^y &= (cbc,1)^{(c,a)} = (b,1) = \tilde{\xi}_1,\\
\tilde{\xi}_4^y &= (acbca,1)^{(c,a)} = (cacbcac,1) = \tilde{\xi}_6,\\
\tilde{\xi}_5^y &= (cabac,1)^{(c,a)} = (aba,1) = \tilde{\xi}_2,\\
\tilde{\xi}_6^y &= (cacbcac,1)^{(c,a)} = (acbca,1) = \tilde{\xi}_4,\\
\tilde{\xi}_7^y &= (acabaca,1)^{(c,a)} = (cacabacac,1) = (acacbcaca,1) = \tilde{\xi}_8,\\
\tilde{\xi}_8^y &= (acacbcaca,1)^{(c,a)} = (cacacbcacac,1) = (acabaca,1) = \tilde{\xi}_7.
\end{aligned} \tag{5.7}$$

The action on $\hat{\xi}_i$'s can be determined from the action on $\tilde{\xi}_i$. Namely, if $\tilde{\xi}_i^x = \tilde{\xi}_p$ and $\tilde{\xi}_i^y = \tilde{\xi}_q$, then

$$\hat{\xi}_i^x = \hat{\xi}_q \quad \text{and} \quad \hat{\xi}_i^y = \hat{\xi}_p. \tag{5.8}$$

Now we can write down a presentation for $\Psi(H)$.

$$\Psi(H) = \langle \tilde{\xi}_i, \hat{\xi}_j, i, j = 1, \ldots, 8, x, y \mid \tilde{\xi}_i^2 = \hat{\xi}_j^2 = [\tilde{\xi}_i, \hat{\xi}_j] = 1, i, j = 1, \ldots, 8,$$
$$x^2 = y^2 = (xy)^4 = 1, \text{relations (5.6),(5.7) and (5.8)}\rangle.$$

Note that relations (5.6), (5.7) and (5.8) show that $\Psi(H) = \langle \tilde{\xi}_1, \hat{\xi}_1, x, y\rangle$. The kernel of Ψ is generated by the preimages of the relators of $\Psi(H)$. We have

$$\Psi(c) = \tilde{\xi}_1, \quad \Psi(aca) = \hat{\xi}_1, \quad \Psi(b) = x, \quad \Psi(aba) = y.$$

We can now start writing down the generators of $\ker\Psi$ as a normal subgroup in H.

$$x^2 \to b^2 = 1 \quad y^2 \to (aba)^2 = 1 \quad (xy)^4 \to (baba)^4 = (ba)^8 = U_1$$

We use the fact that $(ba)^8 \in \ker\Psi$ to simplify the further calculations.

Each of $\tilde{\xi}_i$'s and $\hat{\xi}_i$'s has the form $\tilde{\xi}_1^{z_i}$ and $\hat{\xi}_1^{w_i}$, respectively, for some $z_i, w_i \in \langle x, y\rangle$. The lifts of all elements from $\langle x, y\rangle$ constitute the normal closure $\langle b\rangle^{\mathrm{D}_8}$ of b in $\mathrm{D}_8 = \langle a, b\rangle$. Therefore the lifts of $\tilde{\xi}_i$'s will have form

$$\tilde{\xi}_i \to c^z,$$

where z runs over all elements of $\langle b\rangle^{\mathrm{D}_8}$. In the same way the lifts of $\hat{\xi}_i$'s look like

$$\hat{\xi}_i \to (aca)^z = c^{az} = c^w,$$

where w runs over the complement of $\langle b\rangle^{\mathrm{D}_8}$ in D_8, which is just the coset $a\langle b\rangle^{\mathrm{D}_8}$.

Hence we get the remaining lifts of relators

$$\tilde{\xi}_i^2 \to (c^z)^2 = z^{-1}czz^{-1}cz = 1,$$

$$\hat{\xi}_i^2 \to (c^w)^2 = w^{-1}cww^{-1}cw = 1,$$

$$\big[\tilde{\xi}_i, \hat{\xi}_j\big], i, j = 1, \ldots, 8 \to [c^z, c^w], z \in \langle b\rangle^{\mathrm{D}_8}, w \in a\langle b\rangle^{\mathrm{D}_8}$$

Now we would like to simplify the generators we obtained. First of all, since $c^2 = 1$ we immediately get

$$[c^z, c^w] = [c, c^{wz^{-1}}]^z$$

so we can get rid of z (because $wz^{-1} \in a\langle b\rangle^{\mathrm{D}_8}$). Furthermore,

$$[c, c^w] = cw^{-1}cw \cdot cw^{-1}cw = [c, w]^2$$

We can discard 3 more generators since

$$[c, ba]^2 = \big([c, ab]^{-2}\big)^c, \quad [c, bababa]^2 = \big([c, ababab]^{-2}\big)^c, \quad [c, a]^2 = (ca)^4 = 1.$$

Thus we get 5 more generators for $\ker\Psi$:

$$[c,ab]^2=U_2,\quad [c,bab]^2=U_3,\quad [c,ababa]^2=U_4,$$
$$[c,ababab]^2=U_5,\quad [c,bababab]^2=U_6.$$

These generators, together with $U_1=(ba)^8$ generate $\ker\Psi$ as a normal subgroup in Γ. $\square$

Lemma V.7. $\Omega_n=\langle\phi^i(U_j), i=0,\dots,n-1, j=1,\dots,6\rangle^\Gamma$

Proof. We will use induction on n. For $n=1$ the statement holds by Lemma V.6. Assume it is true for some fixed n.

By the definition of Ω_{n+1} we have $\Psi(\Omega_{n+1})\leq\Omega_n\times\Omega_n$. We will show that $\Psi(\Omega_{n+1})\geq\Omega_n\times\Omega_n$. Observe that

$$\varphi_1\Big(\Psi\big(\phi^i(U_j)\big)\Big)=1 \tag{5.9}$$

in Γ (recall that, for an element $h=(h|_0,h|_1)$ in H, $\varphi_1(h)=h|_1$. Indeed, since $\varphi_1\left(\Psi\big(\phi(\Gamma)\big)\right)\leq\langle a,c\rangle=\mathrm{D}_4$ it's sufficient to check only that all U_i's are trivial in D_4. But this is true since all these words are squares of commutators and $[\mathrm{D}_4,\mathrm{D}_4]\cong\mathbb{Z}/2\mathbb{Z}$.

Equation (5.9) for $i=n+1$, together with the inductive assumption yields $\Omega_n\times 1\leq\Psi(\Omega_{n+1})$. Since Ω_{n+1} is normal in Γ conjugation by a yields $1\times\Omega_n\leq\Psi(\Omega_{n+1})$. Therefore

$$\Psi(\Omega_{n+1})=\Omega_n\times\Omega_n.$$

Equation (5.9) also implies that

$$\Psi(\phi^{n+1}(U_j))=(\phi^n(U_j),1),\quad \Psi(\phi^{n+1}(U_j)^a)=(1,\phi^n(U_j)),$$

i.e.,

$$\Psi(\langle\phi^i(U_j), i=1,\dots,n, j=1,\dots,6\rangle^\Gamma)=\Omega_n\times\Omega_n.$$

Therefore

$$\begin{aligned}\Omega_{n+1} &= \ker\Psi \cdot \langle \phi^i(U_j), i=1,\ldots,n, j=1,\ldots,6\rangle^\Gamma \\ &= \langle \phi^i(U_j), i=0,\ldots,n, j=1,\ldots,6\rangle^\Gamma. \end{aligned}$$ □

Lemmas V.7 and V.4 prove Theorem V.2. Since $\phi\big((ac)^4\big) = (ba)^8 = U_1$, $\phi(a^2) = b^2$, $\phi(b^2) = c^2$ and $\phi(c^2) = (aba)^2 = 1$ the presentation in (5.2) is slightly simplified.

Corollary V.8. *The group $\mathcal{G}$ embeds into an amenable finitely presented group of exponential growth*

$$\tilde{\mathcal{G}} = \langle a,b,c,s \mid a^2, (ac)^4, [c,ab]^2, [c,bab]^2, [c,ababa]^2, [c,ababab]^2, [c,bababab]^2, \\ a^s = b, b^s = c, c^s = aba\rangle,$$

which is an ascending HNN-extension of $\mathcal{G}$.

Proof. By Theorem V.2 and the fact that $\varphi_0(\Psi(\phi(u))) = u$ the substitution ϕ induces an injective endomorphism of $\mathcal{G}$. Thus the HNN-extension construction can be applied. Since $\tilde{\mathcal{G}}$ is an extension of the amenable group $\mathcal{G}$ (see Section 4) by the amenable group $\mathbb{Z}$ generated by s, $\tilde{\mathcal{G}}$ is amenable. The growth of $\tilde{\mathcal{G}}$ is exponential because it contains a free semigroup of rank 2 (it follows from the HNN-extension construction that, for example, s and sa generate such a semigroup). □

4 Self-affine measures and amenability

Although the amenability of $\mathcal{G}$ follows from the intermediate growth of this group, which was established in [BP06], we present here a different approach based on the tools developed in [BV05, Kai05]. More precisely, we construct a particular self-affine measure on $\mathcal{G}$, which proves the vanishing of the asymptotic entropy and, hence, amenability.

Let G be a self-similar group acting spherically transitively on a d-ary tree. Consider a nondegenerate probability measure μ on G (the support of μ generates G). Then for any $x \in X$ one can define a new probability measure $\mu|_x$ on G, which is called the *restriction* of μ on x. The details of the definition and proofs of relevant statements are given in [Kai05] and here we only give the basic idea.

We consider a right random walk $g_n = h_1 h_2 \dots h_n$ on G determined by μ, i.e., $\{h_n\}$ is a sequence of independent variables identically distributed according to the measure μ. We consider $\mathcal{G}$ as embedded in $\mathcal{G} \wr \mathrm{Sym}(2)$ and keep track of the xth coordinate of the image of g_n in $G \wr \mathrm{Sym}(d)$. Recall that h_n is an automorphism of X^*. For $x \in X$, $h_n(x)$ denotes the action of h_n on x. Since

$$g_{n+1}|_x = g_n|_x \cdot h_{n+1}|_{g_n(x)}$$

the probability distribution of $g_{n+1}|_x$ is completely determined by $g_n|_x$ and $g_n(x)$. Therefore the induced random walk

$$(g_n|_x, g_n(x)) \tag{5.10}$$

on $G \times X$ is again a Markov chain. The last random walk is called a *random walk with internal degrees of freedom*. Since X is finite and G acts transitively on X, the subset $G \times \{x\} \subset G \times X$ is recurrent with respect to (5.10). Therefore one can consider the trace of the random walk (5.10) on $G \times \{x\}$, which is also a random walk. Finally, we define the measure $\mu|_x$ as the transition law for the last random walk on $G \times \{x\}$ considered as a copy of G.

There is a convenient way to compute $\mu|_x$ using the properties of the random walk (5.10). The random walk with internal degrees of freedom is governed by the matrix

$$M = (\mu_{xy})_{x,y \in X}$$

whose entries μ_{xy} are subprobability measures on G such that $\mu_{xy}(h)$ is a transition probability of getting to the state (gh, y) from the state (g, x).

With slight abuse of notation, we denote by g the δ-measure concentrated at g. Then the matrix M can be expressed as

$$M = \sum_{g \in \operatorname{supp} \mu} \mu(g) M^g,$$

where

$$M^g_{xy} = \begin{cases} g|_x, & y = g(x), \\ 0, & y \neq g(x), \end{cases}$$

The following theorem is proved in [Kai05].

Theorem V.9. *The measure $\mu|_x, x \in X$ can be expressed in terms of the matrix M as*

$$\mu|_x = \mu_{xx} + M_{x\bar{x}}(1 - M_{\bar{x}\bar{x}})^{-1} M_{\bar{x}x}, \tag{5.11}$$

where $M_{x\bar{x}}$ (resp., $M_{\bar{x}x}$) denotes the xth row (column) of M from which the entry μ_{xx} is removed, and $M_{\bar{x}\bar{x}}$ is the matrix obtained from M by removing the xth row and the xth column.

One can define $\mu|_w$ for any $w = x_1 x_2 \cdots x_n \in X^*$ by

$$\mu_w = (\cdots(\mu|_{x_1})|_{x_2} \cdots)|_{x_n}.$$

Definition 28. The nondegenerate probability measure μ on a self-similar group G is called *self-affine* (*self-similar* in [Kai05]) if there is a word $w \in X^*$ such that

$$\mu|_w = \alpha e + (1 - \alpha)\mu,$$

where $0 < \alpha < 1$ and e is the identity element in G.

For simplicity, we write α instead of αe.

Theorem V.10 ([Kai05]). *If a self-similar group G carries a self-affine nondegenerate measure μ with finite entropy, then it is amenable.*

In this section we construct such a measure on $\mathcal{G}$. Since this measure should be nondegenerate and have finite entropy the most natural place to look for it is the space Q of positive convex linear combinations of δ-measures concentrated on the generators a, b, and c, i.e., measure μ of the form

$$\mu = xa + yb + zc, \quad x + y + z = 1,\ x, y, z > 0.$$

Suppose we want this measure to be self-affine with respect to $x \in X$. By definition this means $\mu|_x = \alpha + (1-\alpha)\mu$ or, equivalently,

$$\mu = \frac{\mu|_x - \alpha}{1-\alpha}.$$

Since $\mu(e) = 0$ we get $\alpha = \mu|_x(e)$. Thus the measure μ is a fixed point of the transformation

$$\Phi\colon \mu \mapsto \frac{\mu|_x - \mu|_x(e)}{1 - \mu|_x(e)}, \tag{5.12}$$

which is defined in [Kai05] and used to prove amenability of a family of groups generalizing Basilica group $\mathrm{IMG}(z^2-1)$.

Let's compute $\mu|_0$ and the corresponding transformation Φ in the case of $\mathcal{G}$. The support of μ is $\{a, b, c\}$ and the corresponding matrices M^g are given by

$$M^a = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}, \quad M^b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \quad M^c = \begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix}. \tag{5.13}$$

Hence,

$$M = xM^a + yM^b + zM^c = \begin{pmatrix} ya + zb & x \\ x & yc + z \end{pmatrix}.$$

By Theorem V.9

$$\mu|_0 = (ya + zb) + x^2(1 - yc - z)^{-1}.$$

Since c has order 2 in $\mathcal{G}$ it is easy see that in the group algebra $\mathbb{R}\mathcal{G}$

$$(1 - yc - z)^{-1} = \frac{y}{z^2 - 2z + 1 - y^2} \cdot c - \frac{z - 1}{z^2 - 2z + 1 - y^2}.$$

Therefore

$$\mu|_0 = y \cdot a + z \cdot b + \frac{yx^2}{z^2 - 2z + 1 - y^2} \cdot c - \frac{(z - 1)x^2}{z^2 - 2z + 1 - y^2}$$

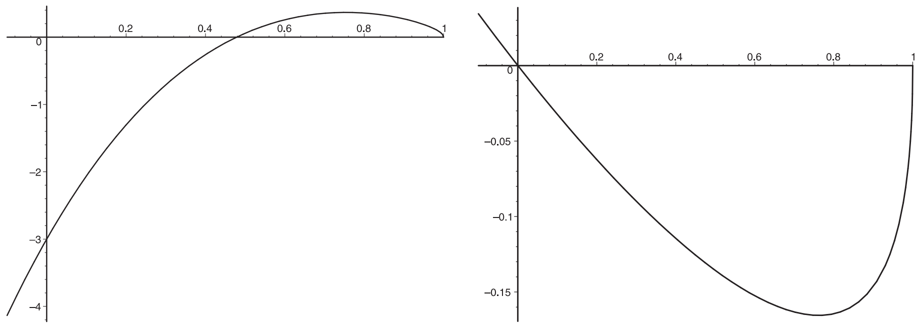
and the transformation Φ takes the form

$$\begin{aligned} \Phi(xa + yb + zc) &= \frac{y \cdot a + z \cdot b + yx^2/(z^2 - 2z + 1 - y^2) \cdot c}{1 + (z - 1)x^2/(z^2 - 2z + 1 - y^2)} \\ &= \frac{y(z^2 - 2z + 1 - y^2)}{z^2 - 2z + 1 - y^2 + x^2 z - x^2} \cdot a \\ &\quad + \frac{z(z^2 - 2z + 1 - y^2)}{z^2 - 2z + 1 - y^2 + x^2 z - x^2} \cdot b \\ &\quad + \frac{yx^2}{z^2 - 2z + 1 - y^2 + x^2 z - x^2} \cdot c. \end{aligned}$$

Now we are interested in a fixed point of the rational map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ induced by Φ, which maps (x, y, z) to the coefficients of $\Phi(xa + yb + zc)$. Moreover, we are searching for such a fixed point only in the invariant simplex $x+y+z = 1$, $x, y, z > 0$. Fortunately, there is such a fixed point. If $\zeta \approx 0.4786202932$ is the unique real root of the polynomial $Z^3 - 6Z^2 + 11Z - 4$, then the point

$$\bigl(\zeta,\ \zeta^2 - 4\zeta + 2,\ -1 + 3\zeta - \zeta^2\bigr)$$

is fixed under F, which produces a self-affine nondegenerate probabilistic measure on $\mathcal{G}$ with finite support, proving amenability of $\mathcal{G}$. This point is unique in the simplex of nondegenerate measures. Indeed, from the equation $F_1(x, y, 1 - x - y) = x$, where



(a) (b)

Fig. 23. Uniqueness of a self-affine measure

F_1 is the first coordinate of F, we get

$$y = \tfrac{1}{4}(-x^2 + x \pm x\sqrt{x^2 - 10x + 9}). \tag{5.14}$$

Substitution in $F_2(x, y, 1 - x - y) = y$ yields

$$(-x^4 + 12x^3 - 39x^2 + 40x - 12) \pm \sqrt{x^2 - 10x + 9}(x^3 - 7x^2 + 12x - 4) = 0. \tag{5.15}$$

Moving the second summand to the righthand side and squaring both sides produces the equation $x(x-1)(x^3 - 6x^2 + 11x - 4) = 0$, whose unique real root on the interval $(0, 1)$ is ζ. The graphs of the two functions in (5.15) are shown in Figure 23: (a) for "plus" and (b) for "minus." The solution comes from (a), so in (5.14) "plus" should be used. It is a routine to check that indeed $y = \zeta^2 - 4\zeta + 2$.

5 Spectral properties and Schur complement

Let H be a Hilbert space and M be an operator on H. Let $H = H_0 \oplus H_1$ and there are operators $A \in B(H_0)$, $D \in B(H_1)$, $B\colon H_1 \to H_0$ and $C\colon H_0 \to H_1$, such that the matrix of M in the basis consisting of the bases of H_0 and H_1 takes the form:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The following fact is of folklore type.

Proposition V.11. *Let D be invertible. The operator M is invertible if and only if $S_1(M) = A - BD^{-1}C$ is invertible.*

The matrix $S_1(M)$ is called *the first Schur complement* of M.

Proof. Indeed, the matrix

$$L = \begin{pmatrix} I_0 & 0 \\ -D^{-1}C & D^{-1} \end{pmatrix}$$

is invertible. Therefore M is invertible if and only if

$$ML = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & I_1 \end{pmatrix}$$

is invertible, which is equivalent to the nonsingularity of $S_1(M)$. □

In our case the action of $\mathcal{G}$ on the boundary X^ω (the set of infinite sequences over X) of the tree X^* induces a unitary representation $\pi_g(f)(x) = f(g^{-1}x)$ of $\mathcal{G}$ in $\mathcal{H} = B\big(L_2(X^\omega)\big)$. The Markov operator $M = \frac{1}{3}(\pi_a + \pi_b + \pi_c)$ corresponding to this unitary representation plays an important role (we do not include inverse elements because all generators are of order 2). The usual method to find the spectrum of M for a self-similar group G is to approximate M with finite dimensional operators arising from the action of $\mathcal{G}$ on the levels of the tree X^*. For more on this see [BG00a]. For simplicity we write g for π_g.

Let $\mathcal{H}_n$ be the subspace of $\mathcal{H}$ spanned by the 2^n characteristic functions f_v, $v \in X^n$, of the cylindrical sets, corresponding to the 2^n vertices of the nth level. Then

$\mathcal{H}_n$ is invariant under the action of $\mathcal{G}$ and $\mathcal{H}_n \subset \mathcal{H}_{n+1}$. Also $\mathcal{H}_n$ can be naturally identified with $L_2(X^n)$. By $\pi_g^{(n)}$ (or, with a slight abuse of notation, by g_n) we denote the restriction of π_g on $\mathcal{H}_n$. Then, for $n \geq 0$,

$$M_n = \tfrac{1}{3}(a_n + b_n + c_n)$$

are finite-dimensional operators whose spectra converge to the spectrum of M in the sense

$$\mathrm{sp}(M) = \overline{\bigcup_{n\geq 0} \mathrm{sp}(M_n)}.$$

Moreover, if P is the stabilizer of an infinite word from X^ω, then one can consider the Markov operator $M_{G/P}$ on the Schreier graph of G with respect to P. The following fact is observed in [BG00a] and can be applied in the case of $\mathcal{G}$.

Theorem V.12. *If G is amenable then*

$$\mathrm{sp}(M_{G/P}) = \mathrm{sp}(M).$$

Common practice for finding the spectrum of M, initiated in [BG00a], is to consider a pencil of operators

$$\widetilde{M}(y, z, \lambda) = a + yb + zc - \lambda$$

and find the set $\mathrm{sp}(y, z, \lambda)$ of points (y, z, λ) such that $\widetilde{M}(y, z, \lambda)$ is not invertible. Then the spectrum of M is just the intersection of $\mathrm{sp}(y, z, \lambda)$ with the line $y = z = 1$, shrunk by a factor of $\frac{1}{3}$. We take 1 as the coefficient at a to simplify the computation. Otherwise one can divide it out (we restrict our attention to the case when x, y, z are nonzero).

Let us consider the corresponding pencil $\tilde{M}_n(y, z, \lambda) = a_n + yb_n + zc_n - \lambda$ and find its matrix in the basis $\{f_v : v \in X^n\}$. The orthogonal subspaces $H_n^{(i)} =$

span$(f_{iv}, v \in X^{n-1})$, $i = 0, 1$ span H_n and are naturally isomorphic to H_{n-1}. The self-similar structure of $\mathcal{G}$ gives the following operator recursion (which coincides with the recursion (5.13))

$$a_n = \begin{pmatrix} 0 & I_{n-1} \\ I_{n-1} & 0 \end{pmatrix}, \quad b_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & c_{n-1} \end{pmatrix}, \quad c_n = \begin{pmatrix} b_{n-1} & 0 \\ 0 & I_{n-1} \end{pmatrix}, \tag{5.16}$$

for $n > 0$, where I_{n-1} denotes the identity matrix of size 2^{n-1}. The matrices a_0, b_0 and c_0 are equal to the 1×1 matrix [1]. For any constant r, we write r instead of rI_n. Thus we have,

$$\widetilde{M}_n(y, z, \lambda) = a_n + yb_n + zc_n - \lambda = \begin{pmatrix} ya_{n-1} + zb_{n-1} - \lambda & 1 \\ 1 & yc_{n-1} + z - \lambda \end{pmatrix}.$$

By Proposition V.11 in case $yc_{n-1} + z - \lambda$ is invertible the operator $\widetilde{M}_n(y, z, \lambda)$ is invertible if and only if $S_1(\widetilde{M}_n(y, z, \lambda))$ is invertible. The inverse of $yc_{n-1} + z - \lambda$ in $\mathbb{R}\mathcal{G}$ is

$$\frac{y}{-y^2 + z^2 - 2z\lambda + \lambda^2} \cdot c_{n-1} + \frac{-z + \lambda}{-y^2 + z^2 - 2z\lambda + \lambda^2}.$$

Hence, $yc_{n-1} + z - \lambda$ is not invertible if and only if $-y^2 + z^2 - 2z\lambda + \lambda^2 = (z - \lambda - y) \times (z - \lambda + y) = 0$. Denote the union of these 2 planes by Z_1. Note, that $\widetilde{M}_n(y, z, \lambda)$ is not necessary singular at each point of Z_1.

The first Schur complement of $\widetilde{M}$ is

$$\begin{aligned} &S_1(\widetilde{M}_n(y, z, \lambda)) \\ &\qquad = ya_{n-1} + zb_{n-1} - \lambda - (yc_{n-1} + z - \lambda)^{-1} \\ &\qquad = y \cdot a_{n-1} + z \cdot b_{n-1} + \frac{y}{-y^2 + z^2 - 2z\lambda + \lambda^2} \cdot c_{n-1} + \frac{-z + \lambda}{-y^2 + z^2 - 2z\lambda + \lambda^2} - \lambda. \end{aligned}$$

If $y = 0$ we get $S_1(\widetilde{M}_n(0, z, \lambda)) = zb_{n-1} + \frac{1}{\lambda - z} - \lambda$ is not invertible if and only if

$$\det \begin{pmatrix} 1/(\lambda - z) - \lambda & z \\ z & 1/(\lambda - z) - \lambda \end{pmatrix}$$
$$= \frac{1}{(\lambda - z)^2}(1 - (\lambda - z)(\lambda + z))(1 - \lambda + z)(1 + \lambda - z) = 0.$$

Denote corresponding union of a hyperbola and two lines in $\mathbb{R}^3$ by Z_2. Note that $Z_2 \cap Z_1 = \varnothing$.

If $y \neq 0$ then

$$\frac{1}{y} S_1(\widetilde{M}_n(y, z, \lambda)) = a_{n-1} + \frac{z}{y} \cdot b_{n-1} + \frac{1}{-y^2 + z^2 - 2z\lambda + \lambda^2} \cdot c_{n-1} - \frac{-\lambda y^2 + \lambda z^2 - 2z\lambda^2 + \lambda^3 + z - \lambda}{y(-y^2 + z^2 - 2z\lambda + \lambda^2)}$$
$$= \widetilde{M}_{n-1}(F(y, z, \lambda)),$$

where $F \colon \mathbb{R}^3 \to \mathbb{R}^3$ is the rational map defined by

$$F \colon (y, z, \lambda) \to \left(\frac{z}{y}, \frac{1}{-y^2 + z^2 - 2z\lambda + \lambda^2}, \frac{-\lambda y^2 + \lambda z^2 - 2z\lambda^2 + \lambda^3 + z - \lambda}{y(-y^2 + z^2 - 2z\lambda + \lambda^2)} \right).$$

Therefore the set $\mathrm{sp}_n(y, z, \lambda)$ of points (y, z, λ) where $\widetilde{M}_n(y, z, \lambda)$ is not invertible in this case ($y \neq 0$) is a preimage under F of the corresponding set $\mathrm{sp}_{n-1}(y, z, \lambda)$. To summarize,

$$Z_2 \cup F^{-1}(\mathrm{sp}_{n-1}(y, z, \lambda)) \subset \mathrm{sp}_n(y, z, \lambda) \subset Z_1 \cup Z_2 \cup F^{-1}(\mathrm{sp}_{n-1}(y, z, \lambda)). \tag{5.17}$$

Since $\widetilde{M}_0(y, z, \lambda) = (1 + y + z - \lambda)$ we have

$$\mathrm{sp}_0(y, z, \lambda) = \{(y, z, \lambda) : 1 + y + z - \lambda = 0\}. \tag{5.18}$$

Denote this plane by P.

Equations (5.17) and (5.18) show that

$$F^{-n}(P) \cup \bigcup_{i=0}^{n-1} F^{-i}(Z_2) \subset \mathrm{sp}_n(y,z,\lambda) \subset F^{-n}(P) \cup \bigcup_{i=0}^{n-1} F^{-i}(Z_1 \cup Z_2). \tag{5.19}$$

Since Z_2 consists of points with $y = 0$ every point (y, z, λ) from $F^{-1}(Z_2)$ must satisfy $z = 0$. But the preimages of all such points are empty. Hence, $F^{-2}(Z_2) = \varnothing$ and $\bigcup_{i=0}^{n-1} F^{-i}(Z_2) = Z_2 \cup F^{-1}(Z_2)$. Denote the last subset by Z_3.

One can easily check that $P \subset F^{-1}(P)$ and, hence, $F^{-n}(P) = \bigcup_{i=0}^{n} F^{-i}(P)$. Therefore equation (5.19) transforms to

$$Z_3 \cup \bigcup_{i=0}^{n} F^{-i}(P) \subset \mathrm{sp}_n(y,z,\lambda) \subset F^{-n}(P) \cup Z_3 \cup \bigcup_{i=0}^{n-1} F^{-i}(P \cup Z_1).$$

Thus, the spectrum of the operator $\widetilde{M}$ on $L_2(X^\omega)$ satisfies

$$\overline{Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P)} \subset \mathrm{sp}(\tilde{M}(y,z,\lambda)) = \overline{\bigcup_{i=0}^{\infty} \mathrm{sp}_n(y,z,\lambda)} \subset \overline{Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P \cup Z_1)}.$$

Note, that the sets $A = \overline{Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P)}$ and $B = \overline{Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P \cup Z_1)}$ are almost invariant with respect to F, in the sense that

$$F^{-1}(A) \cup Z_2 = A, \quad F^{-1}(B) \cup Z_1 \cup Z_2 = B,$$

which is an analog of [GŻ02b, Theorem 4.1] for the Basilica group.

The preimages of the plane P under F^4 and F^5 are shown in Figure 24.

Note that there are points in the spectrum of $\tilde{M}(y,z,\lambda)$ which do not belong to any preimage of the plane P. In particular, the point $(-\frac{1}{2}, 0, \frac{1}{2})$ belongs to Z_1 so it is not in the domain of F, but

$$\det \tilde{M}_n(-\tfrac{1}{2}, 0, \tfrac{1}{2}) = \det((a_{n-1}+1)(c_{n-1}+1) - 4) = 0$$

since 4 is an eigenvalue of $(a_{n-1}+1)(c_{n-1}+1)$. However, this point could be in the

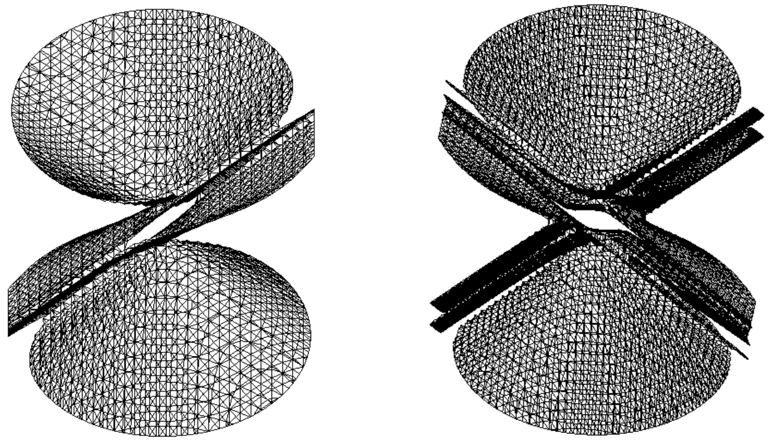


Fig. 24. Part of the spectrum of $\widetilde{M}(y, z, \lambda)$

closure of the union of all preimages of P.

On the other hand we can formulate a conjecture that the spectrum of $M = \frac{1}{3}(a + b + c)$ is the intersection of the line $y = z = 1$ with $A = \overline{Z_3 \cup \bigcup_{i=0}^{\infty} F^{-i}(P)}$, shrunk by a factor of $\frac{1}{3}$. This conjecture survives at least up to the 6th level.

Note also that the map F is conjugate to a simpler map

$$G\colon (y, z, \lambda) \to \left(\frac{z}{y}, \frac{\lambda}{y}(-2 + y\lambda), \frac{1}{\lambda}(-y + y\lambda^2 - \lambda)\right)$$

by the conjugator map

$$(y, z, \lambda) \to \left(\frac{1}{y}, \frac{1}{z}, y + z - \lambda\right).$$

The histogram for the spectral density of the operator M_n acting on 9th level is shown in Figure 25.

Further steps are required to identify the spectrum of the pencil $\widetilde{M}(y, z, \lambda)$ and of M more precisely. This is related to the problem of finding invariant subsets of the rational map F. Perhaps the spectrum of M is just the intersection of the "strange attractor" of F with the line $y = z = 1$, shrunk by factor of $\frac{1}{3}$. In any case here we

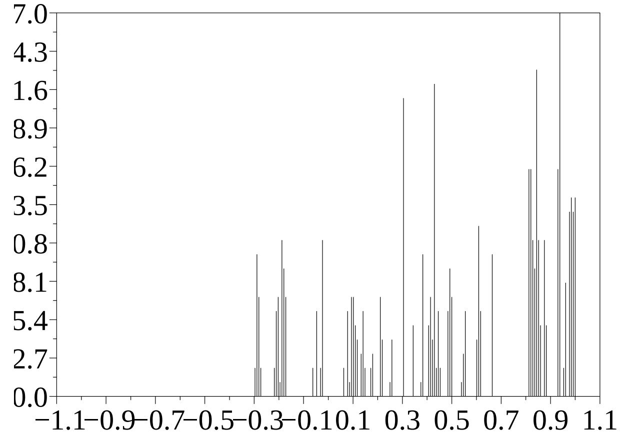


Fig. 25. Histogram of the spectrum on the 9th level

have one more example when the spectral problem is related to the dynamics of a multidimensional rational map. There is a hope that the methods developed for this type of transformations (see, for instance [Sib99]) could help to handle this case.

CHAPTER VI

PACKAGE `AutomGrp` FOR COMPUTATIONS IN SELF-SIMILAR GROUPS AND SEMIGROUPS

1 Functionality

The `AutomGrp` package provides methods for computations with groups and semigroups generated by finite automata or given by wreath recursions, as well as with their finitely generated subgroups, subsemigroups and elements.

The project originally started in 2000 mostly for personal use. It was gradually expanding during consequent years, including both addition of new algorithms and simplification of user interface. In this section we briefly outline main functionality which is currently available

- The package deals with
  - groups generated by finite-state automata
  - semigroups generated by finite-state automata
  - groups generated by recursively defined automata
  - semigroups generated by recursively defined automata
  - groups and semigroups acting on homogeneous rooted trees
  - contracting groups
  - initial and noninitial Mealy automata
- Properties for groups and semigroups:
  - testing whether the group or semigroup is abelian

  - testing self-replicating (fractalness) property
  - testing spherical transitivity
  - testing self-similarity for subgroups of self-similar groups
  - testing contracting property
  - testing amenability
  - working with automata generating the groups and semigroups

- Operations for groups and semigroups:
  - finding the size of the group
  - computing the action on the levels of the tree
  - computing stabilizers of vertices and levels
  - computing projections of the stabilizer of a vertex on this vertex
  - finding group or semigroup relations between generators of a group or semigroup up to a given length
  - iterating over the elements of groups and semigroups
  - computing first values of the growth function of a group
  - computing nucleus of a contracting group
  - computing the level of contraction to the nucleus
  - finding the type and the degree of growth of an automaton generating a group in the sense of Sidki (see Section II.4)
  - producing a random element of a group
  - computing the matrix of a finite dimensional Markov operator induced by the action of a group on the levels of the tree

- for groups and semigroups generated by recursively defined automata finding whether the generators are defined by finite initial automata and, if the answer is positive, computing these automata
- using rewriting systems to simplify the computations and the output, and also to introduce relators in groups or semigroups generated by recursively defined automata
- for finite groups finding isomorphic permutational group

• Properties and operations with group and semigroup elements

- testing transitivity on a given level
- testing spherical transitivity
- word problem (testing whether a given word represents the trivial element in the group or semigroup). The general algorithm has exponential complexity in the sense of the length of a given word, but for contracting groups the polynomial time complexity algorithm is implemented
- computing order of an element
- computing actions on levels of the tree
- computing sections of elements of groups and semigroups at the vertices of the tree
- finding decompositions of the elements on the levels of the tree, which are defined as images of the elements (acting on a d-ary tree) under the embedding

$$\operatorname{Aut} T \hookrightarrow \operatorname{Aut} T \wr \operatorname{Sym}(d)$$

- computing orbits of vertices of the tree under the action of the given element

  - for elements of groups and semigroups generated by recursively defined automata finding whether they are defined by finite initial automata
  - finding the contraction portraits of the elements of contracting groups

- Operations with noninitial automata

  - computing the minimization of an automaton
  - finding the type and the degree of growth of an automaton in the sense of Sidki (see Section II.4)
  - computing dual automaton
  - checking whether automaton is bireversible
  - computing the product of automata
  - computing minimal subautomaton containing given states
  - computing the nucleus of an automaton
  - generating a group (semigroup) by automaton

- The package also contains a library of predefined groups studied in the literature.

2 Example session

In this section we give several examples that exploit basic functionality of the package `AutomGrp` and explain how the user interface works. The package also contains a complete manual available on its webpage.

Here is how to define Grigorchuk group and Basilica group.

```
gap> GrigorchukGroup := AutomatonGroup("a=(1,1)(1,2),b=(a,c),c=(a,d),
d=(1,b)");
< a, b, c, d >
```

```
gap> Basilica := AutomatonGroup( "u=(v,1)(1,2), v=(u,1)" );
< u, v >
```

Similarly one can define a group (or semigroup) generated by a noninvertible automaton. As an example we consider the semigroup of intermediate growth generated by the two state automaton studied in [BRS06].

```
gap> SG := AutomatonSemigroup( "f0=(f0,f0)(1,2), f1=(f1,f0)[2,2]" );
< f0, f1 >
```

Another type of groups (semigroups) implemented in the package is the class of groups (semigroups) defined by wreath recursion (finitely generated self-similar groups).

```
gap> WRG:=SelfSimilarGroup("x=(1,y)(1,2),y=(z^-1,1)(1,2),z=(1,x*y)");
< x, y, z >
```

Now we can compute several properties of `GrigorchukGroup`, `Basilica` and `SG`

```
gap> IsFinite(GrigorchukGroup);
false
gap> IsSphericallyTransitive(GrigorchukGroup);
true
gap> IsFractal(GrigorchukGroup);
true
gap> IsAbelian(GrigorchukGroup);
false
gap> IsTransitiveOnLevel(GrigorchukGroup, 4);
true
```

We can also check that `Basilica` and `WRG` are contracting and compute their nuclei

```
gap> IsContracting(Basilica);
true
gap> GroupNucleus(Basilica);
[ 1, u, v, u^-1, v^-1, u^-1*v, v^-1*u ]
gap> IsContracting( WRG );
true
gap> GroupNucleus( WRG );
[ 1, y*z^-1*x*y, z^-1*y^-1*x^-1*y*z^-1, z^-1*y^-1*x^-1,
  y^-1*x^-1*z*y^-1,  z*y^-1*x*y*z, x*y*z ]
```

As was mentioned in Section 5 of Chapter II for contracting groups there is algorithm solving the word problem in a polynomial time. On practice, this algorithm indeed works much faster than the general exponential time algorithm for long words, but can actually work longer for short words if the nucleus of the group is sufficiently large. Below we provide an example comparing the two algorithms solving the word problem in the contracting group with nucleus consisting of 41 elements.

Note also then in order to use the polynomial time algorithm the, so-called, *contracting table* that stores the information on how the pairwise products of generators of the group contract to the nucleus, has to be computed first, which takes some time when the nucleus is big. This sometimes causes some delay.

```
gap> G := AutomatonGroup("a=(b,b)(1,2), b=(c,a), c=(a,a)");
< a, b, c >
gap> IsContracting(G);
true
```

```
gap> Size(GroupNucleus(G));
41
gap> ContractingLevel(G);
6
gap> ContractingTable(G);; time;
11336
gap> v := a*b*a*b^2*c*b*c*b^-1*a^-1*b^-1*a^-1;;
gap> w := b*c*a*b*a*b*c^-1*b^-2*a^-1*b^-1*a^-1;;
gap> UseContraction(G);;
gap> IsOne(Comm(v,w)); time;
true
251
gap> FindGroupRelations(G, 5);; time;
a^2
b^2
c^2
b*a*b*c*a*b*a*b*c*a
b*c*a*c*a*b*c*a*c*a
881
gap> DoNotUseContraction(G);;
gap> IsOne(Comm(v,w)); time;
true
3855
gap> FindGroupRelations(G, 5);; time;
a^2
b^2
```

```
c^2
b*a*b*c*a*b*a*b*c*a
b*c*a*c*a*b*c*a*c*a
451
```

Therefore for a word of length 48 the polynomial time algorithm outperformed the general one (251 vs. 3855). On the other hand, in the example below dealing with computing short relations of length up to 10 in the same group the standard algorithm was almost twice as fast as the polynomial one (451 vs. 881).

The group `GrigorchukGroup` is generated by a bounded automaton and, thus, is amenable (see [BKN08])

```
gap> IsGeneratedByBoundedAutomaton(GrigorchukGroup);
true
gap> IsAmenable(GrigorchukGroup);
true
```

We can compute the stabilizers of levels and vertices

```
gap> StabilizerOfLevel(GrigorchukGroup, 2);
< a*b*a*d*a^-1*b^-1*a^-1, d, b*a*d*a^-1*b^-1, a*b*c*a^-1, b*a*b*a*b^-1
*a^-1*b^-1*a^-1, a*b*a*b*a*b^-1*a^-1*b^-1 >
gap> StabilizerOfVertex(GrigorchukGroup, [2, 1]);
< a*b*a*d*a^-1*b^-1*a^-1, d, a*c*b^-1*a^-1, c, b, a*b*a*c*a^-1*b^-1*a^
-1, a*b*a*b*a^-1*b^-1*a^-1 >
```

In case of a finite group we can produce an isomorphism into a permutational group

```
gap> f := IsomorphismPermGroup(Group(a,b));
[ a, b ] ->
[ (1,2)(3,5)(4,6)(7,9)(8,10)(11,13)(12,14)(15,17)(16,18)(19,21)(20,
    22)(23,25)(24,26)(27,29)(28,30)(31,32), (1,3)(2,4)(5,7)(6,8)(9,
    11)(10,12)(13,15)(14,16)(17,19)(18,20)(21,23)(22,24)(25,27)(26,
    28)(29,31)(30,32) ]
gap> Size(Image(f));
32
```

Here is how to find relations in `Basilica` between elements of length not greater than 5.

```
gap> FindGroupRelations(Basilica, 6);
v*u*v*u^-1*v^-1*u*v^-1*u^-1
v*u^2*v^-1*u^2*v*u^-2*v^-1*u^-2
v^2*u*v^2*u^-1*v^-2*u*v^-2*u^-1
[ v*u*v*u^-1*v^-1*u*v^-1*u^-1, v*u^2*v^-1*u^2*v*u^-2*v^-1*u^-2,
  v^2*u*v^2*u^-1*v^-2*u*v^-2*u^-1 ]
```

Or relations in the subgroup $\langle p = uv^{-1}, q = vu\rangle$

```
gap> FindGroupRelations([u*v^-1,v*u], ["p", "q"], 5);
q*p^2*q*p^-1*q^-2*p^-1
[ q*p^2*q*p^-1*q^-2*p^-1 ]
```

Or relations in the semigroup `SG`

```
gap> FindSemigroupRelations(SG, 4);
f0^3 = f0
f0^2*f1 = f1
```

```
f1*f0^2 = f1
f1^3 = f1
[ [ f0^3, f0 ], [ f0^2*f1, f1 ], [ f1*f0^2, f1 ], [ f1^3, f1 ] ]
```

Some basic operations with elements are the following:

The function ‘IsOne’ computes whether an element represents the trivial automorphism of the tree

```
gap> IsOne( (a*b)^16 );
true
```

Here is how to compute the order (this function might not stop in some cases)

```
gap> Order(a*b);
16
gap> Order(u^22*v^-15*u^2*v*u^10);
infinity
```

One can check if a particular element acts spherically transitively on the tree (this function might not stop in some cases)

```
gap> IsSphericallyTransitive(a*b);
false
gap> IsSphericallyTransitive(u*v);
true
```

The sections of an element can be obtained as follows

```
gap> Section(u*v^2*u, 2);
u^2*v
gap> Decompose(u*v^2*u);
```

```
(v, u^2*v)
gap> Decompose(u*v^2*u, 3);
(v, 1, 1, 1, u*v, 1, u, 1)(1,2)(5,6)
```

One can try to compute whether the elements of group `WRG` defined by wreath recursion are finite-state and calculate corresponding automaton

```
gap> IsFiniteState(x*y^-1);
true
gap> AllSections(x*y^-1);
[ x*y^-1, z, 1, x*y, y*z^-1, z^-1*y^-1*x^-1, y^-1*x^-1*z*y^-1,
  z*y^-1*x*y*z, y*z^-1*x*y, z^-1*y^-1*x^-1*y*z^-1, x*y*z, y, z^-1,
  y^-1*x^-1, z*y^-1 ]
gap> A := MealyAutomaton(x*y^-1);
<automaton>
gap> NumberOfStates(A);
15
```

To get the action of an element on a vertex or on a particular level of the tree use the following commands

```
gap> [1,2,1,1]^(a*b);
[ 2, 2, 1, 1 ]
gap> PermOnLevel(u*v^2*v, 3);
(1,6,4,8,2,5,3,7)
```

The action of the whole group `GrigorchukGroup` on some level can be computed via ‘PermGroupOnLevel’ (see ”PermGroupOnLevel”).

```
gap> PermGroupOnLevel(GrigorchukGroup, 3);
Group([ (1,5)(2,6)(3,7)(4,8), (1,3)(2,4)(5,6), (1,3)(2,4), (5,6) ])
gap> Size(last);
128
```

The next example shows how to find all elements of Grigorchuk group of length at most 5, which have order 16.

```
gap> FindElements(GrigorchukGroup, Order, 16, 5);
[ a*b, b*a, c*a*d, d*a*c, a*b*a*d, a*c*a*d, a*d*a*b, a*d*a*c, b*a*d*a,
  c*a*d*a, d*a*b*a, d*a*c*a, a*c*a*d*a, a*d*a*c*a, b*a*b*a*c,
  b*a*c*a*c, c*a*b*a*b, c*a*c*a*b ]
```

3 Application to the classification of groups generated by 3-state automata over 2-letter alphabet

Among the major problems in many areas of mathematics are the classification problems. If the objects are given combinatorially then it is natural to try to classify them first by complexity and then within each complexity class.

A natural complexity parameter in our situation is the pair (m, n) where m is the number of states of the automaton generating the group and n is the cardinality of the alphabet.

There are 64 invertible 2-state automata acting on a 2-letter alphabet, but there are only six non-isomorphic $(2,2)$-automaton groups, namely, the trivial group, C_2, $C_2 \times C_2$, $\mathbb{Z}$, the infinite dihedral group D_∞, and the lamplighter group $\mathbb{Z} \wr C_2$ [GNS00, GŻ01]. A classification of semigroups generated by 2-state automata (not necessary invertible) over a 2-letter alphabet is provided by I. Reznikov and V. Sushchanskiĭ [RS02a]. Some examples from this class, including an automaton generating a semigroup of intermediate growth, were studied in the subsequent papers [RS02c, RS02b, BRS06].

It is not known how many pairwise non-isomorphic groups exist for any class (m, n) when either $m > 2$ or $n > 2$. Unfortunately, the number of automata that has to be treated grows super-exponentially with either of the two arguments (there are $m^{mn}(n!)^m$ invertible (m, n)-automata).

Nevertheless, a reasonable task is to consider the problem of classification for small values of m and n and try to classify the $(3, 2)$-automaton groups and $(2, 3)$-automaton groups.

The author of this dissertation is a part of a research group at Texas A&M University including also I. Bondarenko, R. Grigorchuk, R. Kravchenko, Y. Muntyan, V. Nekrashevych, and Z. Šunić which (with some contribution by Y. Vorobets and M. Vorobets) has been working on the problem of classification of $(3, 2)$-automaton groups for the last five yeas. In this section we outline main results of this project. The ongoing progress was published in [BGK⁺07a, BGK⁺09, BGK⁺07b] and the full report in [BGK⁺08] (see also [Mun09] for the latest progress). We start from introducing the numbering of automata in this class.

Every 3-state automaton $\mathcal{A}$ with set of states $S = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ acting on the 2-letter alphabet $X = \{0, 1\}$ is assigned a unique number as follows. Given the wreath recursion

$$\begin{cases} \mathbf{0} = (a_{12}, a_{13})\sigma^{a_{11}}, \\ \mathbf{1} = (a_{22}, a_{23})\sigma^{a_{21}}, \\ \mathbf{2} = (a_{32}, a_{33})\sigma^{a_{31}}, \end{cases}$$

defining the automaton $\mathcal{A}$, where $a_{ij} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ for $j \neq 1$ and $a_{i1} \in \{0, 1\}$, $i = 1, 2, 3$, assign the number

$$\begin{aligned}\text{Number}(\mathcal{A}) = \\ & a_{12} + 3a_{13} + 9a_{22} + 27a_{23} + 81a_{32} + \\ & 243a_{33} + 729(a_{11} + 2a_{21} + 4a_{31}) + 1\end{aligned}$$

to $\mathcal{A}$. With this agreement every $(3,2)$-automaton is assigned a unique number in the range from 1 to 5832. The numbering of the automata is induced by the lexicographic ordering of all automata in the class. Each of the automata numbered 1 through 729 generates the trivial group, since all vertex permutations are trivial in this case. Each of the automata numbered 5104 through 5832 generates the cyclic group C_2 of order 2, since both states represent the automorphism that acts by changing all letters in every word over X. Therefore the nontrivial part of the classification is concerned with the automata numbered by 730 through 5103.

Denote by $\mathcal{A}_n$ the automaton numbered by n and by G_n the corresponding group of tree automorphisms. Sometimes we may use just the number to refer to the corresponding automaton or group.

The following three operations on automata do not change the isomorphism class of the group generated by the corresponding automaton (and do not change the action on the tree in essential way):

(i) passing to inverses of all generators,

(ii) permuting the states of the automaton,

(iii) permuting the alphabet letters.

Definition 29. Two automata $\mathcal{A}$ and $\mathcal{B}$ that can be obtained from one another by using a composition of the operations (i)–(iii), are called *symmetric*.

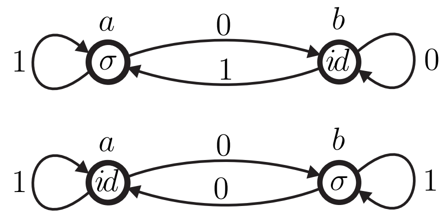


Fig. 26. Symmetric automata generating the lamplighter group

For instance, the two automata in Figure 26 are symmetric. The wreath recursion for the automaton obtained by permuting both the names of the states and the alphabet letters of the first of these two automata is

$$a = (b, a)$$

$$b = (b, a)\sigma$$

and this wreath recursion describes exactly the inverses of the tree automorphism defining the second of the two automata.

Additional identifications can be made after automata minimization is applied (for the algorithm of minimization see [Eil74]), since the automaton and its minimization always generate the same group.

Definition 30. If the minimization of an automaton $\mathcal{A}$ is symmetric to the minimization of an automaton $\mathcal{B}$, we say that the automata $\mathcal{A}$ and $\mathcal{B}$ are *minimally symmetric* and write $\mathcal{A} \sim \mathcal{B}$.

Our research goals moved in three main directions:

1. Search for new interesting groups and an attempt to use them to solve known problems. An example of such a group is the Basilica group (automaton [852]). It is the first example of an amenable group (shown in [BV05]) that is not sub-exponentially amenable group (shown in [GŻ02a]).

2. Recognition of already known groups as self-similar groups, and use of the self-

similar structure in finding new results and applications for such groups. As examples we can mention the free group of rank 3 (automaton [2240]), the free product of three copies of $\mathbb{Z}/2\mathbb{Z}$ (automaton [846]), Baumslag-Solitar groups $BS(1,\pm 3)$ (automata [870] and [2294]), the Klein bottle group (automaton [2212]), and the group of orientation preserving automorphisms of the 2-dimensional integer lattice (automaton [2229]).

3. Understanding of typical phenomena that occur for various classes of automaton groups, formulation and proofs of reasonable conjectures about the structure of self-similar groups.

The main general results on the class of groups generated by $(3,2)$-automata are as follows.

Theorem VI.1. *There are at most 115 non-isomorphic groups generated by* $(3,2)$-*automata.*

The numbers in brackets in the next two theorems are references to the numbers of the corresponding automata.

Theorem VI.2. *There are* 6 *finite groups in the class: the trivial group* $\{1\}$ [1], C_2 [1090], $C_2 \times C_2$ [730], D_4 [847], $C_2 \times C_2 \times C_2$ [802] *and* $D_4 \times C_2$ [748].

Theorem VI.3. *There are* 6 *abelian groups in the class: the trivial group* $\{1\}$ [1], C_2 [1090], $C_2 \times C_2$ [730], $C_2 \times C_2 \times C_2$ [802], $\mathbb{Z}$ [731] *and* $\mathbb{Z}^2$ [771].

Theorem VI.4. *The only nonabelian free group in the class is the free group of rank* 3 *generated by the Aleshin-Vorobets-Vorobets automaton* [2240].

Theorem VI.5. *There are no infinite torsion groups in the class.*

The short list of general results does not give full justice to the work that has been done. Namely, in most individual cases we have provided a lot of results and detailed information for the group in question. The variety is rather extreme and it is not surprising at all that one cannot formulate too many general results.

More work and, likely, some new invariants are required to further distinguish the 115 groups that are listed in this paper as potentially non-isomorphic. In some cases one could try to use the rigidity of actions on rooted trees (see [LN02]), since in many cases it is easier to distinguish actions than groups. In the contracting case one could use, for instance, the geometry of the Schreier graphs and limit spaces to distinguish the actions.

Next natural step would be to consider the case of $(2,3)$-automaton groups or 2-generated self-similar groups of binary tree automorphisms defined by recursions in which every section is either trivial, a generator, or an inverse of a generator. The cases $(4,2)$ and $(5,2)$ also seem to be attractive, as there are many remarkable groups in these classes.

Another possible direction is to study more carefully only certain classes of automata (such as bounded and polynomially growing automata in the sense of Sidki [Sid00b], etc.) and the properties of the corresponding automaton groups.

CHAPTER VII

CONCLUSIONS AND OPEN PROBLEMS

In this dissertation we study different questions and approaches emerging in the area of groups generated by automata, as well as some connections of this part of group theory to other branches of mathematics, such as holomorphic dynamics, spectral analysis and random walks. Below we outline major results and state main open problems and possible directions for further investigations.

The first main part of the dissertation is devoted to Sushchansky p-groups introduced by V. Sushchansky in [Sus79] in 1979 as one of the pioneering examples of Burnside groups. The results of this Chapter are published in [BS07]. Sushchansky used a different language, namely the language of tableaux, introduced by L. Kaloujnine to study properties of iterated wreath products [Kal48]. For each prime $p > 2$, V.I. Sushchansky constructed a finite family of infinite p-groups generated by two tableaux. Each such a tableau naturally defines an automorphism of a rooted tree and can be represented by a finite initial automaton. We describe these automata in Chapter III. The associated action on a rooted tree is not level-transitive and we describe its orbit tree and show that there exists a faithful level-transitive action given by finite initial automata.

Theorem VII.1. *The action of Sushchansky group G_λ on the tree T is faithful, level transitive and has the following form*

$$\begin{aligned} A &= \sigma, \\ B_\lambda &= (q_1, r_1, \sigma, 1, \ldots, 1), \\ q_i &= (q_{i+1}, \sigma^{u_i}, 1, \ldots, 1), \\ r_i &= (r_{i+1}, \sigma^{v_i}, 1, \ldots, 1). \end{aligned} \tag{7.1}$$

Unlike Grigorchuk group, Sushchansky groups are not self-similar. We consider a self-similar closure and prove the following theorem.

Theorem VII.2. *The self-similar closure of G_λ is neither torsion, nor torsion free, level-transitive group of tree automorphisms. Moreover, it is generated by a bounded automaton, hence it is contracting and amenable.*

The question if the self-similar closure of any of Sushchansky groups is branch (or regular branch) is still open.

Our main result about Sushchansky groups is the following theorem which gives bounds on the growth functions of Sushchansky groups.

Theorem VII.3. *All Sushchansky p-groups have intermediate growth. The growth function of each Sushchansky p-group G_λ satisfies*

$$e^{n^\alpha} \preceq \gamma_{G_\lambda}(n) \preceq e^{n^\beta},$$

where $\alpha = \frac{\log(p)}{\log(p)+\log(2)}$, $\beta = \frac{\log(p)}{\log(p)-\log(\eta_r)}$ and η_r is the positive root of the polynomial $x^r + x^{r-1} + x^{r-2} - 2$, where $r = p^2$.

This theorem is a contribution to the Milnor question on the existence of such groups, which was solved in [Gri83] by R.I. Grigorchuk. Also we give an upper bound on the period growth function. The main idea is to use, so called, G groups of intermediate growth introduced in [BŠ01] and [BGŠ03]. For each Sushchansky p-group we construct a G group of intermediate growth and prove that their growth functions are equivalent.

We were able to prove that the self-similar closures of Sushchansky groups are regular weakly branch. The questions of removing the word “weakly” from the above sentence is still open.

Problem 1. Are Sushchansky p-groups or their self-similar closures branch? regular branch?

Sushchansky p-groups are among the automata groups with extraordinary properties (infinite torsion in this case). But the class of automata groups is wide enough to contain previously known groups. The realization of a known group as a group generated by automata is also important because it may shed additional light on the structure of this group or revel some additional properties and create new links and applications for this group.

Chapter IV is devoted to the realizations of free products of groups of order 2 as self-similar groups acting on a binary rooted tree.

All transformations defined by states of finite invertible automata over a fixed alphabet form a group of automatic transformations over this alphabet. The structure of this large group is yet to be understood. An interesting question is the embedding of known groups into this group. For example, Brunner and Sidki proved in [BS98] that $GL_n(\mathbb{Z})$ can be generated by finite automata over the alphabet with 2^n letters. The first embedding of free products of groups of order 2 into the group of automatic transformations over the 2-letter alphabet was constructed by Olijnyk [Olī99].

The above construction lack the important property of self-similarity [Nek05]. In other words, the group is not generated by all states of a single automaton. The first self-similar example was provided by 3-state Bellaterra automaton $\mathcal{B}_3$ over 2-letter alphabet. It was proved (see, for example, [BGK$^+$08, Nek05]) that it generates the group isomorphic to the free product of 3 copies of groups of order 2.

The Bellaterra automaton gives rise to a family of bireversible automata in which all states define involutive transformations. The construction is very simple. Namely, we modify the automaton $\mathcal{B}_3$ by inserting new states on the path from c to a. More precisely, each automaton in the family is defined by wreath recursion

$$\begin{aligned}
a &= (c, b), \\
b &= (b, c), \\
c &= (q_1, q_1)\sigma_0, \\
q_i &= (q_{i+1}, q_{i+1})\sigma_i,\ \ i = 1, \dots, n-4, \\
q_{n-3} &= (a, a)\sigma_{n-3},
\end{aligned} \tag{7.2}$$

where $\sigma_i \in \mathrm{Sym}(\{0, 1\})$ is chosen arbitrarily.

Conjecture 2. Each automaton in the family (7.2) for which at least one of the σ_i is nontrivial, generates the free product of groups of order 2.

The first result supporting this conjecture was obtained by M. Vorobets and Y. Vorobets [VV06]. It was shown that if the number of states is odd and $\sigma_i = (12)$ for all i, then the conjecture holds. In the subsequent paper by the same authors and B. Steinberg [SVV06] the conjecture was proved for the automata with even number of states and additional condition that the number of nontrivial σ_i is odd.

In Chapter IV we prove that any n-state automaton from the family (1.1) with $n \geq 4$ satisfying $\sigma_0 = (12)$ and $\sigma_{n-3} = (12)$ generates the free product of groups of order 2. This result covers the series constructed in [VV06] except one, but the most important case $n = 3$, and partially overlaps with a family constructed in [SVV06]. More precisely, our main results are as follows.

For any $n \geq 4$ let $\mathcal{B}^{(n)}$ be the n-state automaton defined by the wreath recursion

$$\begin{array}{lcl} a_n & = & (c_n, b_n), \\ b_n & = & (b_n, c_n), \\ c_n & = & (q_{n1}, q_{n1})\sigma, \\ q_{n,i} & = & (q_{n,i+1}, q_{n,i+1})\sigma_{n,i}, i = 1, \ldots, n-5, \\ q_{n,n-4} & = & (d_n, d_n)\sigma_{n,n-4}, \\ d_n & = & (a_n, a_n)\sigma, \end{array}$$

where $\sigma_{n,i} \in \mathrm{Sym}(\{1,2\})$ are chosen arbitrarily.

Theorem VII.4. *The group generated by the automaton $\mathcal{B}^{(n)}$ is the free product of n copies of cyclic group of order 2.*

In papers [VV06] and [SVV06] other families of automata (so called, Aleshin families) generating free groups were considered. These families were obtained from the mentioned ones by postcomposing each state with automorphism $(1,1)\sigma$. The proof of the above theorem is simpler than the ones in [VV06] and [SVV06]. As a downside, the result cannot be automatically extended to the corresponding family of automata presumably generating free groups. More precisely, for any $n \geq 4$ let $\mathcal{A}^{(n)}$ be the n-state automaton defined by the wreath recursion

$$\begin{array}{lcl} a_n & = & (c_n, b_n)\sigma, \\ b_n & = & (b_n, c_n)\sigma, \\ c_n & = & (q_{n1}, q_{n1}), \\ q_{n,i} & = & (q_{n,i+1}, q_{n,i+1})\sigma_{n,i}, i = 1, \ldots, n-5, \\ q_{n,n-4} & = & (d_n, d_n)\sigma_{n,n-4}, \\ d_n & = & (a_n, a_n), \end{array}$$

where $\sigma_{n,i} \in \mathrm{Sym}(\{1,2\})$ are chosen arbitrarily.

Problem 2. Is it true that the automata $A^{(n)}$ generate free groups of rank n?

As was mentioned above all constructions producing self-similar groups with nonabelian free subgroups were based on bireversible automata. The structure of groups generated by this automata was not studied well yet but the studied examples naturally lead to the following conjecture.

Conjecture 3. Every group generated by bireversible automaton contains a nonabelian free group.

One of the most remarkable discoveries in the recent years is the observation, due to Nekrashevych, that the so-called iterated monodromy groups (IMG), which can be related to any self-covering map, belong to the class of self-similar groups and that, in the most natural situations, there is an explicit procedure representing them by finite automata [Nek05].

In Chapter V we study the connections of automata groups to holomorphic dynamics, spectral theory and random walks on the example of the iterated monodromy group $\mathrm{IMG}(z^2+i)$ of the complex self-covering mapping z to z^2+i. This group is generated by 3 nontrivial states of 4-state automaton over 2-letter alphabet. We show that the group $\mathrm{IMG}(z^2+i)$ is a regular branch group, thus presenting an example of a branch group which naturally appears in holomorphic dynamics. The main body of the chapter is devoted to the calculation of an L-presentation for $\mathrm{IMG}(z^2+i)$ (i.e., a presentation of a group by generators and relations which involves a finite set of relators and their iterations by substitution). More precisely, we prove the following theorem

Theorem VII.5. *The group* $\mathrm{IMG}(z^2+i)$ *has the following* L*-presentation*

$$\mathrm{IMG}(z^2+i) \cong \big\langle a,b,c \;\big|\; \phi^n(r), r \in \{a^2, (ac)^4, [c,ab]^2, [c,bab]^2, [c,ababa]^2, [c,ababab]^2, [c,bababab]^2\}, n \geq 0 \big\rangle, \quad (7.3)$$

where ϕ *is the substitution defined on words in the free monoid over the alphabet* $\{a,b,c\}$ *by*

$$\phi\colon \begin{cases} a \to b, \\ b \to c, \\ c \to aba. \end{cases}$$

Although it is known that L-presentations are quite common for groups of branch type the number of examples in which explicit computation is possible is rather small.

The presence of L-presentations is important from different points of view. Such presentations are at the first level of complexity after the finite presentations and quite often provide the simplest way to describe a group that is not finitely presented. Further, such presentations can be used to embed a group into a finitely presented group in a way that preserves many properties of the original group. We use the obtained L-presentation of $\mathrm{IMG}(z^2+i)$ to embed this group into a finitely presented group with 4 generators and 10 relators, which is amenable but not elementary amenable (this approach has been used for the first time in [Gri98]).

It was shown by K.-U. Bux and R. Pérez that the group $\mathrm{IMG}(z^2+i)$ has intermediate growth and, hence, is amenable. We find a self-similar measure on $\mathrm{IMG}(z^2+i)$ (first time introduced by Kaimanovich in [Kai05]) providing a different proof that the group is amenable. As was pointed out in Section 4 of Chapter II, similar methods were used in [BKN08] to prove amenability of groups generated by bounded automata. A more general question is still open.

Conjecture 4. Each group generated by polynomially growing automata (in the sense of Sidki) is amenable.

The self-similar measure is closely related to the problem of computation of the spectrum of a Hecke type operator that can be related to any group acting on a rooted tree and to the problem of computation of the spectrum of the discrete Laplace operator (or, what is almost the same, the Markov operator) on the boundary Schreier graph of a group (i.e., the graph of the action of the group on the orbit of a point of the boundary). Unfortunately, the spectral problem is not solved yet in our situation. What we are able to construct is a rational map on $\mathbb{R}^3$ whose proper invariant set (shaped as a "strange attractor") gives the spectrum of the Markov operator after intersection by a corresponding line. Here we have a situation analogous to the case of Basilica group [GŻ02b]. Further efforts in the description of the shape of the attractor (and hence of the spectrum) are needed.

Problem 3. Completely describe the spectrum of the Markov operator on the boundary Schreier graph of $\mathrm{IMG}(z^2+i)$.

Problem 4. Find an example of a group acting on the tree leading to counterexamples to Atiyah conjecture (asking to construct a Riemannian manifold with irrational L^2-Betti numbers).

Finally, Chapter VI presents the package `AutomGrp` for computations in groups and semigroups generated by automata. These groups are particularly nice from the computational point of view. Major algorithmic problems are unsolved so far in the general case but have solutions in certain special cases. The computations related to these groups are often cumbersome to be performed by hands and the computers may be of a great help here.

There was a strong need in the implementation of the algorithms related to automata groups and semigroups in some computer algebra system. The package `AutomGrp` [MS08] for `GAP` (Groups, Algorithms and Programming) system [GAP08] was developed jointly with Yevgen Muntyan to satisfy this need. The package was successfully used in the project of the classification of groups generated by 3-state automata over 2-letter alphabet [BGK+08], as well as by several other authors. Currently the status of the package is "deposited", but we are planning to submit it for refereeing in the nearest future.

The functionality of the package currently includes several methods for the word problem (including a polynomial time algorithm for contracting groups, whose complexity bounds were obtained in [Sav03] in terms of size of the nucleus of the group), finding relations, computing actions on the tree, computing various stabilizer subgroups, deciding whether group is contracting or not, finding the order of an element (two last problems currently have only partial solutions in certain cases), etc. The package is constantly developing with new releases published regularly.

There is a lot of room for extension of our package. In particular, we are planning to implement algorithms solving the conjugacy problem in certain classes of branch groups. We also have to add functionality related to branch groups and iterated monodromy groups.

The major application of the package `AutomGrp` by far is the project of classification of groups generated by 3-state automata over 2-letter alphabet. Similarly to other classes of groups the question of classification of groups generated by automata naturally arises. The first step in this direction was completed in [GNS00], where it was proven that there are 6 nonisomorphic groups generated by 2-state automata over 2-letter alphabet, namely the trivial group, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$, infinite dihedral group D_∞ and lamplighter group $\mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z})$.

During my studies at Texas A&M University I was involved in the project of classification of groups generated by 3-state automata over 2-letter alphabet. For simplicity, we will denote this class as (3,2)-groups. More generally, the class of (m, n)-groups consists of groups generated by m-state automata over n-letter alphabet. The results of our work were published in [BGK$^+$08, BGK$^+$07a, BGK$^+$09, BGK$^+$07b] (see also [Mun09]). The situation here is much more complicated than in the case of 2-state automata. The main results about the whole class include the following theorems.

Theorem VII.6. *There are no more than 115 nonisomorphic groups in the class of (3,2)-groups.*

Theorem VII.7. *There are* 6 *finite groups in the class of (3,2)-groups: the trivial group* $\{1\}$*,* C_2*,* $C_2 \times C_2$*,* D_4*,* $C_2 \times C_2 \times C_2$ *and* $D_4 \times C_2$*.*

Theorem VII.8. *There are* 6 *abelian groups in the class: the trivial group* $\{1\}$*,* C_2*,* $C_2 \times C_2$*,* $C_2 \times C_2 \times C_2$*,* $\mathbb{Z}$ *and* $\mathbb{Z}^2$*.*

Theorem VII.9. *The only nonabelian free group in the class is the free group of rank* 3 *generated by the Aleshin-Vorobets-Vorobets automaton.*

Theorem VII.10. *There are no infinite torsion groups in the class.*

A substantial information about these groups is obtained: all finite groups and all abelian groups are detected, it is proved that there are no infinite torsion groups, and there is only one noncommutative free group is in the class, namely F_3, etc.

Even though a lot of work has been done towards the classification of groups generated by 3-state automata over 2-letter alphabet, there are still much more open questions than the answered ones.

Problem 5. Classify all $(3, 2)$-groups up to isomorphism.

Problem 6. What is the minimal automaton over 2-letter alphabet generating infinite torsion group? group of intermediate growth?

The classification of $(2, 3)$-groups was started in the PhD dissertation of Yevgen Muntyan [Mun09].

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VITA

Dmytro Savchuk was born in Kyiv, Ukraine. He received his B.S. degree and M.S. degree in mathematics in June 2001 and June 2002, respectively, from National Taras Shevchenko University of Kyiv. He received his Ph.D. degree in mathematics from Texas A&M University in August 2009. His current research interests include groups generated by automata, iterated monodromy groups, Schreier graphs, Thompson's group F. His permanent address is: Department of Mathematics, MS3368, Texas A&M University, College Station, TX, 77843-3368, USA.

The typist for this dissertation was Dmytro Savchuk.