G-VARIETIES AND THE PRINCIPAL MINORS OF SYMMETRIC MATRICES

A Dissertation

by

LUKE AARON OEDING

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2009

Major Subject: Mathematics

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ABSTRACT

G-Varieties and the Principal Minors of Symmetric Matrices. (May 2009)

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The variety of principal minors of $n \times n$ symmetric matrices, denoted Z_n, can be described naturally as a projection from the Lagrangian Grassmannian. Moreover, Z_n is invariant under the action of a group $G \subset GL(2^n)$ isomorphic to $(SL(2)^{\times n}) \ltimes \mathfrak{S}_n$. One may use this symmetry to study the defining ideal of Z_n as a G-module via a coupling of classical representation theory and geometry. The need for the equations in the defining ideal comes from applications in matrix theory, probability theory, spectral graph theory and statistical physics.

I describe an irreducible G-module of degree 4 polynomials called the hyperdeterminantal module (which is constructed as the span of the G-orbit of Cayley's hyperdeterminant of format $2 \times 2 \times 2$) and show that it that cuts out Z_n set theoretically. This result solves the set-theoretic version of a conjecture of Holtz and Sturmfels and gives a collection of necessary and sufficient conditions for when it is possible for a given vector of length 2^n to be the principal minors of a symmetric $n \times n$ matrix.

In addition to solving the Holtz and Sturmfels conjecture, I study Z_n as a prototypical G-variety. As a result, I exhibit the use of and further develop techniques from classical representation theory and geometry for studying G-varieties.

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Shaowei Lin pointed out the reference [25]. Bernd Sturmfels suggested the addition of Corollary III.5.

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CHAPTER I

INTRODUCTION

The problem of finding the relations among principal minors of a matrix of indeterminants dates back (at least) 1897 when Nanson [25] found relations among the principal minors of an arbitrary 4×4 matrix. In 1928 Stouffer [29] found an expression for the determinant of a matrix in terms of a subset of its principal minors. Subsequently, interest in the subject seems to have diminished, however much more recently, there has been a renewed interest in the relations among principal minors and their application to matrix theory, probability, statistical physics and spectral graph theory. This renewed interest motivated Holtz and Sturmfels [14] to provide an algebraic framework for the relations among principal minors of symmetric matrices, namely, they introduced Z_n, the algebraic variety of principal minors of symmetric $n \times n$ matrices, and asked for generators of its ideal.

In the first nontrivial case, Holtz and Sturmfels showed that Z_3 is an irreducible hypersurface in $\mathbb{P}^7$ cut out by a special degree four polynomial, namely Cayley's hyperdeterminant of format $2 \times 2 \times 2$. In the next case they showed that the ideal of Z_4 is minimally generated by 20 degree four polynomials, but only 8 of these polynomials are copies of the hyperdeterminant found by natural substitutions. The geometric meaning of the remaining polynomials was somewhat mysterious. Because of the symmetry of the hyperdeterminant, Landsberg conjectured, and Holtz and Sturmfels showed, that Z_n is invariant under the action of $G \cong (SL(2)^n) \ltimes \mathfrak{S}_n \subset GL(2n)$ [14]. Holtz and Sturmfels named the span of the G-orbit of the hyperdeterminant *the hyperdeterminantal module*. It was then understood that the 20 degree four polynomials

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The journal model is *Representation Theory.*

are a basis of the hyperdeterminantal module when $n = 4$. This interpretation led to the following conjecture.

Conjecture I.1 (Conjecture 14 [14])**.** *The prime ideal of Z_n, the variety of principal minors of symmetric matrices, is generated in degree four by the hyperdeterminantal module for all $n \geq 3$.*

The dimension of the hyperdeterminantal module grows exponentially with n. The number of variables and the number of polynomials generating the hyperdeterminantal module renders computational methods ineffective already in the case $n = 5$. The symmetry of Z_n allows tools from representation theory to be used to study Z_n. In fact, Z_n is a prototypical (non-homogeneous) G-variety and we study it within the framework of G-varieties in spaces of tensors. By using a combination of representation theory and geometry, we prove Theorem III.3, which verifies the set theoretic version of the Holtz-Sturmfels Conjecture.

A unifying purpose of this work is to study Z_n as a prototypical (non-homogeneous) G-variety, and in so doing, we show the use of standard constructions in representation theory and geometry and further develop general tools for studying the symmetries and the ideals of such varieties. We anticipate these techniques will be applicable to other G-varieties in spaces of tensors such as those that arise naturally in computational complexity, probability, signal processing, and algebraic statistics for example. For references, see [3, 5, 16, 20, 26].

In Chapter II we recall basic definitions and concepts used in the study of G-varieties, establish notation, cover necessary background material and prove a couple of basic lemmas. We show how one can recover a symmetry group for a variety by finding projection of a homogeneous variety G/P and restricting the group G via the projection. This idea is used in Chapter III to give a geometric proof of the symmetry of Z_n.

Chapter III is a study of the algebraic and geometric properties of the variety of principal minors of symmetric matrices. Therein we prove the set theoretic version of the Holtz-Sturmfels conjecture. Additionally, we give a new proof of the symmetry of Z_n. We find a connection to the work of Landsberg and Weyman and their study of tangential varieties [21]. In particular, Theorem III.25 says that $\tau\left(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)\right)$, the tangential variety to the Segre product n copies of $\mathbb{P}^1$'s, is a subvariety of Z_n. Moreover, $\tau\left(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)\right)$, is the G-orbit of the image of the rank-1 symmetric matrices under the principal minor map. We define the hyperdeterminantal module in terms of Schur modules and study its properties. In particular, we generalize a property of the hyperdeterminantal module which we call augmentation since it constructs the hyperdeterminantal module in the $n+1$ case based on the hyperdeterminantal module in the n case. We use geometry to prove a crucial lemma (Lemma III.32) that characterizes the zero set of an augmented module in terms of the zero set of the original module. With this geometric characterization we show that the zero set of the hyperdeterminantal module is precisely Z_n, thereby proving the main theorem, Theorem III.3.

Chapter IV is an exposition of a known algorithm from representation theory that constructs polynomials in G-modules from representation theoretic data. Because we have not found any implementations of this algorithm in the literature, it was necessary to write our own. Our implementation in Appendix D, Section A computes an isotypic decomposition necessary for studying ideals of G-modules. We have included a Maple implementation for constructing highest weight vectors of Schur modules in Appendix D, Section B. The implementation in Appendix D, Section C is an example using lowering operators to construct a basis of a G-module when the highest weight vector is known. This implementation was used by S. Lin and B.

Sturmfels in their study of relations among the principal minors of (not necessarily symmetric) 4×4 matrices, [23].

Holtz and Sturmfels were working to answer questions posed by Holtz and Schneider [13], Wagner [30] and others. In Chapter V we briefly state these questions and show how Theorem III.3 answers these questions.

CHAPTER II

BACKGROUND

"I think those two modules are isomorphic." "Are you Schur?"

unknown

This chapter contains basic definitions and facts coming from representation theory and geometry. For more background, see [6, 7, 9, 11, 20].

A. G-varieties and representations: basic definitions

A representation of a group G on a finite dimensional complex vector space V is a group homomorphism $\rho : G \to GL(V)$. In this setting, V becomes a G-module, *i.e.* a vector space with a compatible G-action. It is common to call V a representation of G. These definitions, as well as a good introduction to representation theory can be found in [7]. Unless otherwise stated, we always consider reductive groups.

An algebraic variety $X \in \mathbb{P}V$ is said to be a *G-variety* if $g.x \in X$ for every $x \in X$ and $g \in G$. A variety X is said to be a *homogeneous variety* if for some $x \in X$, $X = G.x = G/P$, where P is the stabilizer of x. Homogenous varieties are often the first G-varieties that one encounters. They have rich geometric and algebraic properties and are well studied, see for instance [17] for a modern treatment.

If V and W are G-modules, a map $f : \mathbb{P}V \to \mathbb{P}W$ is said to be *G-equivariant* if $g.f(x) = f(g.x)$ for every $g \in G$ and every $x \in V$.

A rational mapping between projective spaces $f : \mathbb{P}V \dashrightarrow \mathbb{P}W$ can be defined in coordinates as follows. Let $\{x_0, \ldots, x_n\}$ and $\{y_0, \ldots, y_m\}$ be bases of V and W respectively. Let U be the open set $U = \{x_0 \neq 0\}$ and on U define coordinates

$z_i = \frac{x_i}{x_0}$. On $U \subset \mathbb{P}V$, there is a regular (polynomial) map f_U representing f,

$$f_U(z_1, \ldots, z_n) = [h_0(z_1, \ldots, z_n), \ldots, h_m(z_1, \ldots, z_n)],$$

where the h_i are polynomials.

The *image of the rational mapping* is understood to be $\overline{f_U(U)}$, the Zariski closure of $f_{|U}(U)$, where U is any open dense set. One can show that this definition is independent of the choice of open dense set U.

Note that if the action of G is transitive on V, *i.e.* for every $v, w \in V$ there is a $g \in G$ so that $g.w = v$, then any open set $U \subset \mathbb{P}V$ of the form $U_i = \{[x_0, \ldots, x_n] \mid v_i \neq 0\}$ is as good as any other.

The following lemma is useful for understanding maps between varieties and how various symmetries are preserved.

Lemma II.1. *Let T be a G-module and let $X \subset \mathbb{P}T$ be a G-variety. Let $H < G$ be a subgroup which splits T* - i.e. *$T = W \oplus W^c$ as an H-module. Let $\pi : \mathbb{P}(W \oplus W^c) \dashrightarrow \mathbb{P}((W \oplus W^c)/W^c) \simeq \mathbb{P}W$ be the projection map. The map π is obviously H-equivariant, so the image $\pi(X)$ is an H-invariant subvariety of $\mathbb{P}W$.*

Proof. We must consider the fact that π is only a rational map: certainly, $\pi(x) = 0$ if $x \in W^c$, so the map is not defined at all points. Let U be the open set defined by $U = \{[w_1 + w_2] \mid w_1 \neq 0, w_1 \in W, w_2 \in W^c\}$. Let $Y := \overline{\pi(U \cap X)}$. Then it is clear that our assumptions imply $H.(U \cap X) \subset U \cap X$.

Let $y \in \pi(U \cap X)$ and let $h \in H$. By definition, π is surjective onto its image, so let $x \in U \cap X$ be such that $\pi(x) = y$. Now we use the H-equivariance of π to conclude that $h.y = h.\pi(x) = \pi(h^{-1}.x) \in \pi(U \cap X)$.

Suppose $y \in \overline{\pi(U \cap X)}$. Then choose a sequence $y_i \to y \in Y$ such that $\exists x_i \in U \cap X$ and $\pi(x_i) = y_i$. If $h \in H$ then $h.y_i = h.\pi(x_i) = \pi(h^{-1}.x_i) \in Y$ for all i. If

$\{p_i\} \subset Y$ is a convergent sequence such that $p_i \to p$, and f is a polynomial which satisfies $f(p_i) = 0$, then by continuity, $f(p) = 0$ also. So Y must contain all of its limit points, and therefore $h.y_i \to h.y \in Y$, and we conclude that Y is an H-variety. $\square$

This lemma tells us that if we are presented with a variety that is the projection from a G-variety, then we should look for the symmetry group of our variety among subgroups of G. We carry out this procedure explicitly in Chapter III, Section A and arrive at a new proof of the symmetry of the variety of principal minors of symmetric matrices.

B. Spaces of tensors and G-varieties

Let $V_1, \ldots, V_d$ be complex vector spaces and let $V_1 \otimes \cdots \otimes V_d$ denote their tensor product. The group $GL(V_1) \times \cdots \times GL(V_d)$ acts by change of coordinates in each factor. If $V_i \simeq V$ for every i we can consider the induced action of $GL(V)$ on the tensor product,

$$\begin{array}{rcl} GL(V) \times V \otimes \cdots \otimes V & \longrightarrow & V \otimes \cdots \otimes V \\ (g, x_1 \otimes \cdots \otimes x_d) & \longmapsto & (g.x_1) \otimes (g.x_2) \otimes \cdots \otimes (g.x_d), \end{array}$$

where $g.x_i$ is the usual action of $GL(V)$ on V and we extend the action via linearity. There is also a natural action of the symmetric group $\mathfrak{S}_d$ on $V_1 \otimes \cdots \otimes V_d$ when $V_i = V$ for every i just by permuting the factors. More specifically, the left action is given (on a basis) by

$$\begin{array}{rcl} \mathfrak{S}_d \times V \otimes \cdots \otimes V & \longrightarrow & V \otimes \cdots \otimes V \\ (\sigma, (x_1 \otimes \cdots \otimes x_n)) & \longmapsto & x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(d)}. \end{array}$$

With this convention one may define a left action of the semi-direct product $GL(V) \ltimes \mathfrak{S}_n$ on $V^{\otimes n}$.

In fact there is a classical result of Weyl's, see also [9].

Theorem II.2 (The Double Commutant Theorem)**.** *Let* $V \simeq \mathbb{C}^n$*. Then* $\mathfrak{S}_d$ *and* $GL(V)$ *are commutants of each other,* i.e.

$$\mathfrak{S}_d = \{g \in GL(V^{\otimes d}) \mid g.A.(v_1 \otimes \cdots \otimes v_d) = A.g.(v_1 \otimes \cdots \otimes v_n)\ \forall A \in GL(V)\}$$

$$GL(V) = \{g \in GL(V^{\otimes d}) \mid g.\sigma.(v_1 \otimes \cdots \otimes v_d) = \sigma.g.(v_1 \otimes \cdots \otimes v_n)\ \forall \sigma \in \mathfrak{S}_d\}.$$

A natural subspace of the space of tensors are the symmetric tensors *i.e.* the space of $\mathfrak{S}_d$ invariants in $V^{\otimes d}$, $S^d(V) := (V \otimes \cdots \otimes V)^{\mathfrak{S}_d} \subset V^{\otimes d}$. The algebra of symmetric tensors is graded by degree,

$$Sym(V) = \bigoplus S^d V \tag{2.1}$$

and each graded piece is a $GL(V)$-module with the induced action,

$$\begin{array}{rcl} GL(V) \times S^d(V) & \longrightarrow & S^d(V) \\ (g, x^{\circ d}) & \longmapsto & (g.x)^{\circ d}. \end{array}$$

There is also a natural $GL(V)$-action on the dual, V^* - the vector space of linear maps $V \to \mathbb{C}$. If $\omega \in V^*$, is a linear map $\omega : V \to \mathbb{C}$, the dual action of $GL(V)$ is defined by $g.\omega(x) = \omega(g^{-1}.x)$ for every $x \in V$ and $g \in GL(V)$.

If $X \subset \mathbb{P}V$ is an algebraic variety, its *ideal* (also *vanishing ideal* or *defining ideal*), $\mathcal{I}(X) \subset \mathbb{P}(Sym(V^*))$ is the ideal of polynomials vanishing on X. Often algebraic varieties are given via an explicit parameterization by a rational map, but the vanishing ideal may be unknown. A basic question in algebraic geometry is to find generators for the ideal of a given variety. Though there are many known theoretical techniques, this remains a difficult practical problem.

1. Examples of classical G-varieties

The following are classic examples of G-varieties which happen to show up in the study of the variety of principal minors of symmetric matrices. These definitions can be found in many texts on algebraic geometry such as [11].

The space of all rank-one tensors (also called decomposable tensors) is the *Segre variety*, defined by the embedding,

$$\begin{array}{ccc} Seg : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n & \longrightarrow & \mathbb{P}\left(V_1 \otimes \cdots \otimes V_n\right) \\ ([v_1], \ldots, [v_n]) & \longmapsto & [v_1 \otimes \cdots \otimes v_n] \end{array} . \tag{2.2}$$

If $X_1 \subset \mathbb{P}V_1, \ldots, X_n \subset \mathbb{P}V_n$ are varieties, let $Seg\left(X_1 \times \cdots \times X_n\right)$ denote their Segre product. $Seg\left(V_1 \times \cdots \times V_n\right)$ is a G-variety for $G = GL(V_1) \times \cdots \times GL(V_n)$, moreover it is homogeneous since $Seg\left(V_1 \times \cdots \times V_n\right) = G.(x_1 \otimes \cdots \otimes x_n)$.

The space of all rank-one symmetric tensors is the *Veronese variety*, defined by the embedding,

$$\begin{array}{ccc} v_d : \mathbb{P}V & \longrightarrow & \mathbb{P}\left(S^d V\right) \\ [w] & \longmapsto & [(w)^d] \end{array} . \tag{2.3}$$

The Veronese variety is invariant under the action of $GL(V)$ and it is also homogeneous since $v_d(\mathbb{P}V) = GL(V).(w^d)$.

The *tangential variety* to a smooth variety $X \subset \mathbb{P}V$, denoted $\tau(X)$, is the Zariski closure of all embedded tangent $\mathbb{P}^1$'s to X, *i.e.*, if $\gamma(t) : [0,1] \to X$ is a smooth curve, $\mathbb{P}\overrightarrow{\gamma'(0)} \subset \mathbb{P}V$ is an embedded tangent line to X. Note: when the underlying variety X is not smooth, more care is needed in defining the tangential variety, see [21].

The r^{th} *secant variety* to a variety $X \subset \mathbb{P}V$, denoted $\sigma_r(X)$, is the Zariski closure of all embedded secant $\mathbb{P}^{r-1}$'s to X, *i.e.*,

$$\sigma_r(X) = \overline{\bigcup_{x_1,\ldots,x_r \in X} \mathbb{P}(span\{x_1,\ldots,x_r\})} \subset \mathbb{P}V. \tag{2.4}$$

Secant varieties and tangential varieties inherit the symmetry of the underlying variety. In particular, $\sigma_r\left(Seg\left(V_1 \times \cdots \times V_n\right)\right)$ and $\tau\left(Seg\left(V_1 \times \cdots \times V_n\right)\right)$ are G-varieties for $G = GL(V_1) \times \cdots \times GL(V_n)$. However, homogeneity is not preserved in general.

Remark II.3. Secant varieties and tangential varieties are classical, but were given a modern framework by Zak, in his work [31]. They come up in many applications such as computational complexity, signal processing and algebraic statistics.

The *Grassmannian* $Gr(k,V)$ is the space of k-planes in V. The *Plücker embedding* of $Gr(k,V)$ into projective space is the map

$$\begin{array}{rcl} Gr(k,V) & \hookrightarrow & \mathbb{P}\left(\bigwedge^k V\right) \\ \langle v_1,\ldots,v_k\rangle & \longmapsto & [v_1 \wedge \cdots \wedge v_k] \end{array} \tag{2.5}$$

which sends the k-plane spanned by the vectors $v_1,\ldots,v_k$ to their wedge product. One checks that this is well-defined independent of our choice of vectors $v_1,\ldots,v_k$ spanning a given k-plane. The Grassmannian is a homogeneous variety for $G = GL(V)$ and is a central object in the study of G-varieties.

2. Using representation theory to study the ideals of G-varieties

Let $\mathcal{I}(X) \subset Sym(\mathbb{P}V^*)$ denote the homogeneous ideal of polynomials vanishing on X. Often we want to have a greater understanding of $\mathcal{I}(X)$. The following proposition is a key observation because it allows us to use the representation theory of G-modules to study $\mathcal{I}(X)$.

Proposition II.4. *X is a G-variety if and only if $\mathcal{I}(X)$ is a G-module.*

Proof. By definition, $f(x) = 0$ for every $x \in X$ and every $f \in \mathcal{I}(X)$. But X is a G-variety, so, in particular, $g.x \in X$ $\forall g \in G$. We must show that G leaves $\mathcal{I}(X)$ invariant. Indeed $(g.f)(x) = f(g^{-1}.x) = 0$, because G is a group and $g^{-1}.x \in X$, so $\mathcal{I}(X)$ is a G-submodule of $\mathrm{Sym}(V^*)$. The proof in the other direction is similar. $\square$

The d^{th} graded piece of $\mathcal{I}(X)$ by $\mathcal{I}_d(X) = S^d(V^*) \cap \mathcal{I}(X)$. In particular, each $\mathcal{I}_d(X)$ is a G-module. If we want to study the ideal $\mathcal{I}(X)$, it makes sense to study the various graded pieces. Even more, we will restrict ourselves to studying reductive groups, *i.e.* those whose G-modules all split into a unique direct sum of irreducible G-modules.

A G-module said to be *irreducible* if it has no non-trivial G-invariant subspaces. Fact: If G is reductive, then M is an irreducible G module if and only if $M = \langle G.v \rangle$, *i.e.* it is the linear span of the orbit of a single vector.

Proposition II.5. *Let $z \in \mathbb{P}V$, and let $B \subset Sym(V^*)$ be a collection of polynomials (B is not necessarily a G-module). Then*

$$G.z \subset \mathcal{V}(B) \iff z \in \mathcal{V}(\langle G.B \rangle)$$

Proof. $G.z \subset \mathcal{V}(B)$ if and only if $f(g.z) = 0$ for all $g \in G$ and for all $f \in B$. But from the definition of the G-action on the dual space, $f(g.z) = (g^{-1}.f)(z)$, so $f(g.z) = 0$ for all $g \in G$ and for every $f \in B$. This happens if and only if $(g.f)(z) = 0$ for all $g \in G$ and for all $f \in B$, but, this is the condition that $z \in \mathcal{V}(\langle G.B \rangle)$. $\square$

Suppose $X \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$ is a variety in a space of tensors, and suppose X is invariant under the action of $G = GL(V_1) \times \cdots \times GL(V_n)$. To study $\mathcal{I}_d(X)$ as a G module, we need to understand how to decompose $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ into a direct sum of irreducible G modules. This is a standard computation in representation theory.

Lemma II.6 (Schur's Lemma [7]). *Let V and W be irreducible G-modules. If a homomorphism $f : V \to W$ is G-equivariant, then*

1. *Either f is an isomorphism or $f = 0$*
2. *If $V = W$, then $f = \lambda Id$ for some $\lambda \in \mathbb{C}$.*

Theorem II.7 (Proposition 15.47 [7]). *Let $S_\pi V = image\left(c_{\pi|_{V^{\otimes d}}}\right)$ where c_π is the Young symmetrizer associated to the partition π. Every irreducible representation of $GL(V)$ is isomorphic to one of the form $S_\pi V$.*

Young symmetrizers are the key objects used to construct polynomials in spaces of tensors. We study these maps in more detail in Chapter IV.

Proposition II.8 (Landsberg-Manivel [19] Proposition 4.1). *Let $V_1, \ldots, V_n$ be vector spaces and let $V = V_1 \otimes \cdots \otimes V_n$, and let $G = GL(A_1) \times \cdots \times GL(A_n)$. Then we have the following decomposition as a direct sum of irreducible G-modules:*

$$S^d(V_1 \otimes \cdots \otimes V_n) = \bigoplus_{|\pi_1| = \cdots = |\pi_k| = d} ([\pi_1] \otimes \cdots \otimes [\pi_k])^{\mathfrak{S}_d} \otimes S_{\pi_1} V_1 \otimes \cdots \otimes S_{\pi_n} V_n$$

where $([\pi_1] \otimes \cdots \otimes [\pi_k])^{\mathfrak{S}_d}$ denotes the space of $\mathfrak{S}_d$-invariants (i.e., instances of the trivial representation) in the tensor product.

Proof from [19]. Schur-Weyl duality is the assertion that the following map is an isomorphism of $GL(V)$-modules

$$\bigoplus_{|\pi| = d} [\pi] \otimes S_\pi V \longrightarrow V^{\otimes d}.$$

Apply Schur-Weyl duality separately to each of $V_1, \ldots, V_n$, take the tensor product of the corresponding isomorphisms, and compare with Schur duality for $V_1 \otimes \cdots \otimes V_n$. $\square$

The following is a special case of the previous theorem.

Theorem II.9. *We have the following decomposition into irreducible G-modules:*

$$S^d(V \otimes W) = \bigoplus_{|\pi|=d} S_\pi V \otimes S_\pi W$$

Proof. (sketch) In this case, we consider the fact that $[\pi_1] \otimes [\pi_2]$ is equivalent to a $\mathfrak{S}_d$-module homomorphism $\varphi : [\pi_1] \to [\pi_2]$ because $[\pi_1]$ and $[\pi_2]$ are self-dual. But $[\pi_1]$ and $[\pi_2]$ are irreducible $\mathfrak{S}_d$-modules, so Schur's lemma implies that $\varphi = \lambda Id$, and in particular, $[\pi_1] = [\pi_2]$. □

This is not such an unfamiliar concept since

$$S^2(V \otimes W) = (S^2V \otimes S^2W) \oplus (\bigwedge^2 V \otimes \bigwedge^2 W)$$

is just the statement from linear algebra that a square matrix can be decomposed into its skew symmetric and symmetric pieces.

When $V_i = V$, Proposition II.8 specializes to give the following decomposition formula found in [19]:

$$S^d(V \otimes \cdots \otimes V) = \bigoplus_{|\pi_1|=\cdots=|\pi_k|=d} (S_{\pi_1} V \otimes \cdots \otimes S_{\pi_n} V)^{\oplus N_{\pi_1,\ldots,\pi_k}}, \tag{2.6}$$

where the multiplicity $N_{\pi_1,\ldots,\pi_k}$ can be computed via characters. We give an implementation of this calculation in Appendix D, Section A.

3. Weight spaces

As mentioned before, the algebras $Sym(V)$ and $V^{\otimes}$ are graded by degree. We get a further decomposition by weights as follows. If we choose an ordered basis $e_1, \ldots, e_n$ of V, this induces a natural ordering on the decomposable tensors (*i.e.* the monomials) in $V^{\otimes d}$. A common way to assign weights to each e_i is via a weight function, *i.e.* an additive homomorphism

$$\begin{aligned} wt : V &\longrightarrow \mathbb{Z}^n \\ e_i &\longmapsto [0,\dots 0,1,0,\dots,0]. \end{aligned}$$

The requirement that wt is additive allows us to give a weight to each monomial in $V^{\otimes}$. For example $wt\left(e_1^{\otimes n_1} \otimes e_2^{\otimes n_2} \otimes \cdots \otimes e_k^{\otimes n_k}\right) = [n_1, n_2, \dots, n_k]$. This induces a grading by weights on $V^{\otimes}$ and $\mathrm{Sym}(V)$. This is also known as grading by multi-degree.

Each irreducible representation, $S_\pi V$ of $GL(V)$, has a *highest weight vector*, v_π and since $GL(V)$ is reductive, we have the nice property that $\langle GL(V).v_\pi\rangle$, *i.e. each irreducible representation of $GL(V)$ is the span of the orbit of a highest weight vector.*

We recall an algorithm of Landsberg and Manivel [19] for constructing highest weight vectors in Chapter IV and provide an implementations of this algorithm in Appendix D, Section B. We also provide an implementation of a standard algorithm for finding a weight basis of an irreducible module in Appendix D, Section C.

These facts along with the implementations provided allow us to carry out an ideal membership test (for small degree), which is present in [19]. The basic idea is that we can (in theory) write down a highest weight vector for each irreducible module of polynomials for a fixed small degree. Then we can test each highest weight vector on a general point of the variety X. If the highest weight vector vanishes, then the entire module is in the ideal $\mathcal{I}(X)$. There are complications that come up in practice and we give a full treatment of this in Chapter IV.

4. Kostant's theorem

An important theorem due to Kostant identifies the ideal of every homogeneous variety in the language of representation theory. Though this theorem does not appear to be published by Kostant, it can be found in [15] Corollary 10.1.11 pg 346. We quote a more recent formulation [20] Theorem 4.8.4.2.

Theorem II.10 (Kostant)**.** *Let V_λ be an irreducible G-module of highest weight λ, and let $X = G/P \subset \mathbb{P}V_\lambda$ be the orbit of a highest weight line. Then $\mathcal{I}(X)$ is generated in degree two by $V_{2\lambda}^{\perp} \subset S^2(V_\lambda)^*$.*

This theorem uses representation theory to treat all homogeneous varieties in the same way, and gives a uniform identification of their ideals. It serves as motivation for what can be done when representation theory is used to study questions in algebraic geometry.

CHAPTER III

THE VARIETY OF PRINCIPAL MINORS OF SYMMETRIC MATRICES

"Those smaller determinants
shouldn't drink - they're minors!"

unknown

In this chapter we will focus on the variety of principal minors of symmetric matrices. In order to give a precise definition of this variety, we need to introduce notation. Let $I = (i_1, \dots i_n)$ be a multi-index, with $i_k \in \{1, 2\}$ for $k = 1, \dots, n$, and let $|I|$ denote the number of 2's appearing. If A is an $n \times n$ matrix, then let $\Delta_I(A)$ denote the principal minor of A with row and column set indicated by the multi-index I, in the sense that the location of the 2's in I indicate which rows and columns are to be used in computing the minor determinant of A. If one includes the 0×0 minor, there are 2^n principal minors, therefore, a natural home for vectors of principal minors is $\mathbb{C}^{2^n}$. Because of the symmetry that will eventually become apparent, we will consider $\mathbb{C}^{2^n}$ as a space of tensors as follows: Let $V_1 \otimes V_2 \otimes \cdots \otimes V_n \simeq \mathbb{C}^{2^n}$, where each $V_i \simeq \mathbb{C}^2$. A choice of basis $\{x_i^1, x_i^2\}$ of V_i for each i determines a basis of $V_1 \otimes \cdots \otimes V_n$. We represent basis elements compactly by setting $X^I := x_1^{i_1} \otimes x_2^{i_2} \otimes \cdots \otimes x_n^{i_n}$. We use this basis to introduce coordinates on $\mathbb{P}\mathbb{C}^{2^n}$; if $P = [C_I X^I] \in \mathbb{P}\mathbb{C}^{2^n}$, the coefficients C_I are the coordinates of the point P.

The projective variety of principal minors of $n \times n$ symmetric matrices, Z_n, is defined by the following rational map,

$$\begin{aligned} \varphi : \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C}) \quad &\dashrightarrow \quad \mathbb{P}\mathbb{C}^{2^n} \\ [A, t] \quad &\longmapsto \quad \left[t^{n-|I|}\Delta_I(A)\, X^I\right]. \end{aligned}$$

The map φ is defined on the open set where $t \neq 0$. Moreover, φ is homogeneous of degree n, so it is well defined on projective space. In addition φ is generically finite-to-one and Z_n is a $\binom{n+1}{2}$-dimensional variety. The affine map (on the set $\{t = 1\}$) defines a closed subset of $\mathbb{C}^{2^n}$, [14].

Remark III.1. Griffin and Tsatsomeros [10] point out that the dimension of Z_n was essentially known to Stouffer in 1924, [28, 29]. In fact, Stouffer [29] claims that this result was already known to MacMahon in 1893 and later by Muir.

From this definition, the structure of this variety is not immediately apparent. In the first nontrivial case (the 3×3 case), the defining ideal is generated by Cayley's hyperdeterminant of format $2\times2\times2$, [14]. This polynomial, which was discovered over 150 years ago [4], is invariant under the action of $(SL(2) \times SL(2) \times SL(2)) \ltimes \mathfrak{S}_3$, (see also [8] p. 448). This implies that Z_3 is also invariant under the action of the same group. The symmetric group, $\mathfrak{S}_\mathfrak{n}$, was known to preserve Z_n, but it was not known that $SL(2) \times SL(2) \times SL(2)$ preserves Z_3. In fact, for the general case, Landsberg noticed the following theorem, which is proved in [14].

Theorem III.2. *The variety Z_n is invariant under the action of*

$$G = (SL(V_1) \times \cdots \times SL(V_n)) \ltimes \mathfrak{S}_n \subset GL(V_1 \otimes \cdots \otimes V_n),$$

where $V_i \cong \mathbb{C}^2$ for $1 \leq i \leq n$.

We often make the abbreviation, $(SL(V_1) \times \cdots \times SL(V_n)) \ltimes \mathfrak{S}_n \cong (SL(2)^{\times n}) \ltimes \mathfrak{S}_n$.

In Chapter III, Section A, we use geometric methods that exploit a connection to the well known Lagrangian Grassmannian to prove a stronger result (Theorem III.14), namely G is the largest subgroup of $GL(2n)$ that can leave Z_n invariant. Another consequence of this method of proof is that we find a subgroup of the symmetry group of the variety of principal minors of arbitrary square matrices (Proposition III.10). This theorem was originally proved by different methods in [2] and is also inherent in the work of [14], however neither of these proofs include the stronger result that this symmetry group is the largest possible among the subgroups of $GL(2n)$.

In fact, Theorem III.3 gives a G-module of set theoretic defining equations for Z_n. The form of this module implies the sharper result that G is the largest subgroup of $GL(V_1 \otimes \cdots \otimes V_n) \simeq GL(2^n)$ that can leave Z_n invariant.

This symmetry is the key for how we should study this variety and its defining ideal. Representation theory becomes an essential tool for this task in that it allows us to study the defining ideal as a G-module. One key use of representation theory comes from the work of Landsberg and Manivel, [19], where they use a classical test for ideal membership in the G-variety setting. A candidate irreducible G-module is either in the ideal or not, there is no in-between. To test whether a given G-module is in the ideal of a G-variety, it suffices to check whether the highest (or lowest) weight vector of that module vanishes at all points of the variety.

For what follows, $\mathcal{I}(X)$, (respectively $\mathcal{I}_d(X)$) denotes the ideal (respectively component of the ideal in degree d) of the variety X, and $\mathcal{V}(M)$ denotes the zero set of M.

The next idea comes from rephrasing the results of [14]; in the cases $n = 3$ and $n = 4$, a single irreducible module of degree 4 polynomials generates the defining ideals $\mathcal{I}(Z_3)$ and $\mathcal{I}(Z_4)$. Although the group gets larger from one case to the next, the module is generated by the span of the G-orbit of the same polynomial. (Since the polynomial is actually a hyperdeterminant, this module is called the hyperdeterminantal module in [14].) This idea led to Conjecture 14 of [14]: The prime ideal of the variety of principal minors of symmetric matrices is generated by the hyperdeterminantal module. Conjecture 14 was verified by computational methods for the cases of 3×3 and 4×4 matrices in [14]; however, due to the rapid growth of the number of variables and number of polynomials, the next case proved to be infeasible to verify on a computer.

The hyperdeterminantal module can be expressed precisely in the language of representation theory as follows. Let M denote the irreducible G-module

$$M = \bigoplus_{\sigma\in\Sigma} S_{(2,2)}V^*_{\sigma(1)} \otimes S_{(2,2)}V^*_{\sigma(2)} \otimes S_{(2,2)}V^*_{\sigma(3)} \otimes S_{(4)}V^*_{\sigma(4)} \otimes \cdots \otimes S_{(4)}V^*_{\sigma(n)},$$

where $\Sigma = \{\sigma \in \mathfrak{S}_n \mid \sigma(1) < \sigma(2) < \sigma(3), \text{ and } \sigma(4) < \sigma(5) < \cdots < \sigma(n)\}$. (We commit a standard abuse of notation in that we omit an implied permutation of the factors so that every module is still in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$, but since the V_i are all isomorphic, this is harmless.) We often abbreviate the notation to $M = S_{(2,2)}S_{(2,2)}S_{(2,2)}S_{(4)}\ldots S_{(4)}$.

In Chapter III, Section B, we investigate properties of M. As defined, we only know that $M \subset (V_1 \otimes \cdots \otimes V_n)^{\otimes 4}$. In Proposition III.17, we show that M is a module of degree 4 homogeneous polynomials. (We believe that this is the module that was intended to be the so-called hyperdeterminantal module; however, it has a different dimension than what is claimed for the hyperdeterminantal module in [14].)

Theorem III.3 (Main Theorem)**.** *The variety of principal minors of symmetric $n\times n$ matrices, Z_n, is cut out set theoretically by the irreducible $(SL(2)^{\times n}) \ltimes \mathfrak{S}_n$-module of degree 4 polynomials*

$$M = S_{(2,2)}S_{(2,2)}S_{(2,2)}S_{(4)}\ldots S_{(4)}.$$

Theorem III.3, verifies Conjecture 14 of [14] in the set theoretic version. As mentioned above, the ideal theoretic result is known to hold in the cases $n = 3$ and $n = 4$ [14]. A list of 250 polynomials which form a basis of M for the case $n = 5$ is available at

http://www.math.tamu.edu/~oeding/mypolys5.txt

or by request of the author.

For the benefit of readers not familiar with representation theory, we recall a standard algorithm (Remark III.16) to find a weight basis of a given irreducible module. We also include an example of Maple code that accomplishes this task in Appendix D, Section C. This allows one to compute a finite list of polynomials that cut out the variety. This list of polynomials gives necessary and sufficient conditions for a vector of length 2^n to actually be a vector of principal minors of a symmetric matrix; in other words, it is a complete algebraic solution to the principal minor assignment problem for symmetric matrices posed by [13].

Remark III.4. We note that Z_n is cut out by a single irreducible module of degree 4 polynomials. There are instances of varieties being cut out by a single irreducible module, for instance the 2 factor Segre variety, $Seg(\mathbb{P}^a \times \mathbb{P}^b)$, and its secant varieties exhibit this property. However in the case of the 4 factor Segre variety, the ideal is not generated by a single irreducible module. It may be interesting to know how often G-varieties are cut out by a single irreducible module, and what can be deduced from this property.

A practical membership test is the following:

Corollary III.5. *Suppose $v = v_I X^I \in \mathbb{C}^{2^n}$. Then v represents the principal minors of a symmetric $n \times n$ matrix if and only if for any element $g = (a_{i_1,j_1}) \times \cdots \times (a_{i_n,j_n}) \in SL(2)^{\times n}$ and any $\sigma \in \Sigma = \{\sigma \in \mathfrak{S}_n \mid \sigma(1) < \sigma(2) < \sigma(3)$, and $\sigma(4) < \sigma(5) < \cdots < \sigma(n)\}$, the transformed vector with coordinates w_I defined by $g.(\sigma.v) =$*

$a_{i_1,j_1}\dots a_{i_n,j_n}v_{\sigma(I)}X^J = w_I X^I$ *satisfies the* $2\times 2\times 2$ *hyperdeterminantal equation*

$$\begin{aligned}
(w_{I_{[1,1,1]}})^2(w_{I_{[2,2,2]}})^2 + (w_{I_{[2,1,1]}})^2(w_{I_{[1,2,2]}})^2\\
+(w_{I_{[1,2,1]}})^2(w_{I_{[2,1,2]}})^2 + (w_{I_{[1,1,2]}})^2(w_{I_{[2,2,1]}})^2\\
-2w_{I_{[1,1,1]}}w_{I_{[2,1,1]}}w_{I_{[1,2,2]}}w_{I_{[2,2,2]}} - 2w_{I_{[1,1,1]}}w_{I_{[1,2,1]}}w_{I_{[2,1,2]}}w_{I_{[2,2,2]}}\\
-2w_{I_{[1,1,1]}}w_{I_{[1,1,2]}}w_{I_{[2,2,1]}}w_{I_{[2,2,2]}} - 2w_{I_{[2,1,1]}}w_{I_{[1,2,1]}}w_{I_{[1,2,2]}}w_{I_{[2,1,2]}}\\
-2w_{I_{[2,1,1]}}w_{I_{[1,1,2]}}w_{I_{[1,2,2]}}w_{I_{[2,2,1]}} - 2w_{I_{[1,2,1]}}w_{I_{[1,1,2]}}w_{I_{[2,1,2]}}w_{I_{[2,2,1]}}\\
+4w_{I_{[1,1,1]}}w_{I_{[1,2,2]}}w_{I_{[2,1,2]}}w_{I_{[2,2,1]}} + 4w_{I_{[1,1,2]}}w_{I_{[1,2,1]}}w_{I_{[2,1,1]}}w_{I_{[2,2,2]}} = 0,
\end{aligned}$$

where $I_{[k,l,m]} = [k,l,m,1,\dots,1]$.

The main ideas that go into the proof of Theorem III.3 are as follows. In Proposition III.21 we show that the module M is in the ideal $\mathcal{I}(Z_n)$ using representation theory. We need a more geometric understanding of the zero set of the module. For this, we prove Lemmas, III.30 and III.32 about the zero sets of modules of the form $S_{\pi_1}V_1\otimes\cdots\otimes S_{\pi_n}V_n$ with at least one $\pi_i = (d)$. Finally in Chapter III, Section G, with the aid of Lemma III.43, we show that every point of the zero set has a matrix that maps to it via the principal minor map.

We anticipate that the same techniques used in this work will be applicable to other problems, especially to the case of principal minors of arbitrary matrices studied by Lin and Sturmfels, [23] and A. Borodin and E. Rains [2].

A. The symmetry of Z_n

Suppose V s a vector space and that $G\subset GL(V)$ is a group. A variety $X\subset\mathbb{P}V$ is said to be invariant under the action of G or a G-variety if $g.x\in X$ for every $x\in X$ and for every $g\in G$. In particular, $\mathcal{I}(X)$ is a G-module and we can study $\mathcal{I}(X)$ via the representation theory of G-modules.

In this section, we will give a geometric proof of Theorem III.2. We will consider the problem in a slightly more general context. In particular, in the case that a variety X is the linear projection from a G-variety, we give a method to identify a subgroup of G that leaves X invariant. (This will not be *a priori* the full symmetry group of X.)

As mentioned in the introduction to this chapter, we find a symmetry group for the variety of principal minors of arbitrary square matrices. We will then specialize this proof for the case of symmetric matrices. In both cases, we show that no larger subgroup of $GL(V)$ will preserve the variety.

1. A short introduction to the Grassmannian and the Lagrangian Grassmannian

Here we give a short exposition of the portion of the work of Landsberg and Manivel [17] that we will need for this paper. For more details as well as the generalization to all compact Hermitian symmetric spaces, see [17].

In the introduction to this chapter, we chose an ordered bases $\{x_i^1, x_i^2\}$ for each $V_i \simeq \mathbb{C}^2$. For this section, we will rename these elements by $x_i^1 = e_i$ and $x_i^2 = f_i$ and let $E = span\{e_1, \ldots, e_n\}$ and $F = span\{f_1, \ldots, f_n\}$. Finally, let $V = E \oplus F \simeq \mathbb{C}^{2n}$, and consider the Grassmannian of n-planes in V denoted $Gr(n, V)$.

The module $\bigwedge^n V = \bigwedge^n (E \oplus F)$ is irreducible as a $GL(V)$-module, but it decomposes into a sum of irreducible components as a $GL(E) \times GL(F)$-module:

$$\bigwedge^{n} (E \oplus F) = \bigoplus_{k=0}^{n} \left(\bigwedge^{n-k} E \otimes \bigwedge^{k} F \right). \tag{3.1}$$

Remark III.6. This decomposition will show the connection between points of the Grassmannian and vectors of minors. Later, we will actually want a finer decomposition as a $GL(V_1) \times \cdots \times GL(V_n)$-module. This will elucidate the connection to principal minors.

To fix notation, let $e^R = e^{r_1} \wedge e^{r_2} \wedge \cdots \wedge e^{r_k}$, with $R = \{1 \leq r_1 < r_2 < \cdots < r_k \leq n\}$ and define the size $|R| = k$. Similarly, let $f_S = f_{s_1} \wedge \cdots \wedge f_{s_k}$, with $S = \big\{1 \leq s_1 < s_2 < \cdots < s_k \leq n\big\}$ and $|S| = k$.

We can choose $e_1 \wedge \cdots \wedge e_n$ to be a volume form on E. Using this volume form, we can define an isomorphism $\bigwedge^{n-k} E \simeq \bigwedge^k E^*$, given on a basis by $e^R \mapsto e_{R^c}$, where R^c is the complement of R. We can use this isomorphism to write our decomposition as

$$\bigwedge^n (E \oplus F) = \bigoplus_{k=0}^{n} \left(\bigwedge^k E^* \otimes \bigwedge^k F \right). \tag{3.2}$$

The space $\bigwedge^k E^* \otimes \bigwedge^k F$ has the interpretation as the space generated by the $k \times k$ minors of $E^* \otimes F$, the space of $n \times n$ matrices, namely $e^R \otimes f_S(A)$ is the minor of a matrix $A \in (E^* \otimes F)^*$ with row set R and column set S. By convention, we take $e^\emptyset \otimes f_\emptyset = 1 \otimes 1$ - this is the 0×0 minor.

Now consider the rational map,

$$\begin{aligned} \psi : \mathbb{P}(E^* \otimes F \oplus \mathbb{C}) \quad & \dashrightarrow \quad \mathbb{P}\left(\bigwedge^n V \right) = \mathbb{P}\left(\bigoplus_{k=0}^{n} \left(\bigwedge^k E^* \otimes \bigwedge^k F \right) \right) \\ [(A), t] \quad & \longmapsto \quad \left[\sum_{|R|=|S|} t^{k-|R|} e^R \otimes f_S(A) \right]. \end{aligned}$$

The map ψ is a variant of the Plücker embedding of the Grassmannian, and it is compatible with the decomposition (3.2). In light of this mapping ψ and the decomposition (3.2), the Grassmannian $Gr(n, 2n)$ has the interpretation as the variety of minors of $n \times n$ matrices.

It is well known that $Gr(n, 2n)$ is a homogeneous variety for $GL(2n)$, and in particular, it is $GL(2n)$-invariant.

Now we consider the Lagrangian Grassmannian. Let $\omega \in \bigwedge^2 V^*$ be a non-degenerate symplectic form and let $Sp(2n) \subset GL(V)$ be the symplectic group preserving ω. Let $Gr_\omega(n, 2n)$ denote the Lagrangian Grassmannian – the variety of n planes in $V \simeq \mathbb{C}^{2n}$ that are isotropic for ω,

$$Gr_\omega(n, 2n) = \{E \in Gr(n, 2n) \mid \ \forall\, v, w \in E,\ \omega(v, w) = 0\}.$$

As a parallel to what we already did for $Gr(n, 2n)$, we can give a parameterization of $Gr_\omega(n, 2n)$, by modifying the map ψ as follows. Restrict the source to symmetric matrices and restrict the target to only the non-redundant minors. The vector space of all non-redundant minors of symmetric matrices is actually the $Sp(2n)$-module, $\Gamma_n := \bigwedge^n V / \left(\omega \wedge \bigwedge^{n-2} V\right)$. In order to see this fact, we would need to understand the decomposition of $\bigwedge^n(V)$ as an $Sp(2n)$- module just as we did in the classical Grassmannian case. For the sake of brevity, we do not include this here. (For more details, see [17].) Under these modifications, we have

$$\psi(S^2V \oplus \mathbb{C}) = Gr_\omega(n, 2n) \subset \mathbb{P}\Gamma_n.$$

As a parallel to the previous case, $Gr_\omega(n, 2n)$ has the interpretation as the variety of all (non-redundant) minors of $n \times n$ symmetric matrices.

$Gr_\omega(n, 2n)$ is a homogeneous variety for $Sp(2n)$ so, in particular, it is invariant under the action of $Sp(2n)$.

2. Finding the symmetry of the variety of principal minors of (arbitrary) matrices via a projection from the Grassmannian

Observation III.7. *The variety of principal minors of (arbitrary) $n \times n$ matrices, $\tilde{Z}_n \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$, is a linear projection from the Grassmannian, $Gr(n, 2n) \subset \mathbb{P}(\bigwedge^n \mathbb{C}^{2n})$.*

Proof. From the exposition of $Gr(n, 2n)$ above, it is clear that the projection is given by deleting the non-principal minors. $\square$

We will exploit this projection to find a subgroup of the symmetry group of $\tilde{Z}_n$ that lies in $GL(2n)$ by showing that this projection is equivariant for a subgroup $G' \subset GL(2n)$.

Recall from above that we have identified the space $\bigwedge^n(E \oplus F)$ as the vector space of all minors, $\bigoplus_{k=0}^n \left(\bigwedge^k E^* \otimes \bigwedge^k F\right)$. We have identified $e^R \wedge f_S$ with the minor with row set R and column set S. Therefore $\{e^R \wedge f_R\}$ are the minors which have the same row and column set - *i.e.* the principal minors. So, we may identify the space of principal minors, denoted W, in relation to the decomposition (3.2):

$$W = span\left\{e^R \wedge f_R \mid R \subset \{1, 2, \dots, n\}\right\} \subset \bigoplus_{k=0}^{n} \left(\bigwedge^k E^* \otimes \bigwedge^k F\right).$$

Often it is more convenient to use the isomorphism $\bigwedge^{n-k} E \simeq \bigwedge^k E^*$ and make the following identification related to the decomposition (3.1) above:

$$W \simeq span\left\{e_R \wedge f_S \mid R \cap S = \emptyset, |R| + |S| = n\right\} \subset \bigoplus_{k=0}^{n} \left(\bigwedge^{n-k} E \otimes \bigwedge^k F\right).$$

Notice that $\dim W = 2^n$, and in particular, W and $V_1 \otimes \cdots \otimes V_n$ are isomorphic. The isomorphism is given (on a basis and extended linearly) by the mapping $e_R \wedge f_S \mapsto x_1^{\epsilon_1} \otimes x_2^{\epsilon_2} \otimes \cdots \otimes x_n^{\epsilon_n}$ where $\epsilon_i = 1$ if $i \in R$ and $\epsilon_i = 2$ if $i \in S$. Indeed, $e_R \wedge f_S$ is the principal minor with row and column set equal to S. This isomorphism realizes W as a module for the group $G' = (GL(V_1) \times \cdots \times GL(V_n)) \ltimes \mathfrak{S}_n$ with the inherited action from $GL(V_1 \otimes \cdots \otimes V_n)$ - the natural group acting on $V_1 \otimes \cdots \otimes V_n$.

Notice that $\bigwedge^n \mathbb{C}^{2n}$ is naturally a $GL(2n)$-module. So it will also be a G-module for any subgroup $G \subset GL(2n)$. For our purposes, it is important to see $W \subset \bigwedge^n \mathbb{C}^{2n}$ as a G-submodule.

Lemma III.8. *There exists an embedding*

$$\left(GL(2)^{\times n}\right) \ltimes \mathfrak{S}_n \cong \left(GL(V_1) \times \cdots \times GL(V_n)\right) \ltimes \mathfrak{S}_n \subset GL(2n)$$

so that the vector space $W \simeq V_1 \otimes \cdots \otimes V_n$ *is a* $(GL(2)^{\times n}) \ltimes \mathfrak{S}_n$*-submodule of* $\bigwedge^n \mathbb{C}^{2n}$*. Moreover,* $(GL(2)^{\times n}) \ltimes \mathfrak{S}_n$ *is the largest subgroup of* $GL(2n)$ *preserving* W*.*

Proof. Let $\tilde{G} = GL(2)^{\times n} \simeq GL(V_1) \times \cdots \times GL(V_n) \subset GL(2n)$, and let $V \simeq \mathbb{C}^{2n}$. The group $\tilde{G} \ltimes \mathfrak{S}_n$ obviously acts on W. But we would like to see this action as the inherited action from $GL(V)$. Otherwise, there would be no reason to expect that $\tilde{G} \ltimes \mathfrak{S}_n$ acts on $\bigwedge^n V$.

We start with an arbitrary subgroup of $GL(V)$ acting on $\bigwedge^n(V)$ and consider only the conditions forced on any potential subgroup which could preserve W. This will show that indeed $\tilde{G}$ is the largest subgroup of $GL(V)$ which preserves W.

Step 1: $\mathfrak{S}_n$ invariance: The space $V = E \oplus F$ is left invariant under the action of the permutation group $\mathfrak{S}_{2n}$. Let $\mathfrak{S}_n^E$ be the subgroup of permutations preserving E, and similarly define $\mathfrak{S}_n^F$. Consider the diagonal action of $\mathfrak{S}_n$ inside of $\mathfrak{S}_n^E \times \mathfrak{S}_n^F \subset \mathfrak{S}_{2n}$ defined as follows. Let $e_R \wedge f_S$ be a basis element of $\bigwedge^n(E \oplus F)$. Then for $\sigma \in \mathfrak{S}_n$, the action is given by

$$\sigma.(e_R \wedge f_S) = e_{\sigma(R)} \wedge f_{\sigma(S)}.$$

Now suppose $e_R \wedge f_S \in W$. In this case $R \cap S = \emptyset$, so also $\sigma(R) \cap \sigma(S) = \emptyset$, and obviously $|R| = |\sigma(R)|$ (similarly for S), so $\sigma.P \in W\ \forall \sigma \in \mathfrak{S}_n,\ \forall P \in W$. Therefore the action of $\mathfrak{S}_n$ is defined on $\bigwedge^n(E \oplus F)$ and preserves the subspace W.

Step 2: $\tilde{G}$ invariance: Here, it is easier to work with the Lie algebra $\tilde{\mathfrak{g}}$ associated to the group $\tilde{G}$. It is sufficient to prove that W is invariant under the action of $\tilde{\mathfrak{g}}$. Also, the Lie algebra $\mathfrak{gl}(V) \simeq \mathfrak{gl}(2n)$ acts linearly on W, so it suffices to work on a basis of W and then extend by linearity.

Since we have chosen bases, we may express $\tilde{\mathfrak{g}}$ as a subgroup of

$$\mathfrak{gl}(2n) = \begin{pmatrix} E^* \otimes E & F^* \otimes E \\ E^* \otimes F & F^* \otimes F \end{pmatrix}.$$

Fix indices $1 \leq i, j \leq n$. We will consider an arbitrary element

$$\alpha = \begin{pmatrix} a^i_j & b^i_j \\ c^i_j & d^i_j \end{pmatrix} \in \mathfrak{gl}(2n),$$

and place restrictions on α so that it leaves W invariant. We calculate

$$\begin{aligned} \alpha.\left(e_1 \wedge \cdots \wedge e_n\right) &= \left(\sum_{i=1}^{n} a^i_i\right) e_1 \wedge \cdots \wedge e_n \\ &\quad + \sum_{j=1}^{n} e_1 \wedge \cdots \wedge e_{j-1} \wedge \left(\sum_i c^i_j f_i\right) \wedge e_{j+1} \wedge \cdots \wedge e_n. \end{aligned}$$

We see that $\alpha.\left(e_1 \wedge \cdots \wedge e_n\right) \notin W$ unless $c^i_j = 0$ for $i \neq j$, so for what follows, set $c^i_j = 0$ if $i \neq j$. Similarly, we compute $\alpha.\left(f_1 \wedge \cdots \wedge f_n\right)$ to find that $\alpha.\left(f_1 \wedge \cdots \wedge f_n\right) \notin W$ unless $b^i_j = 0$ for $i \neq j$, so for what follows, set $b^i_j = 0$ if $i \neq j$.

Next, let $E^k = e_1 \wedge \ldots e_{k-1} \wedge f_k \wedge e_{k+1} \cdots \wedge e_n$, and in general, let $E^{k_1,\ldots k_p}$ denote $E = e_1 \wedge \cdots \wedge e_n$ with e_{k_q} replaced with f_{k_q} for each $q \in \{1, \ldots, p\}$. In fact, the elements $E^{k_1,\ldots k_p}$ form a basis for W. We compute $\alpha.E^k$ and notice that,

$$\begin{aligned} \alpha.E^k \equiv & \sum_{j,\, j\neq k} e_1 \wedge \cdots \wedge e_{j-1} \wedge \left(a^k_j e_k\right) \wedge e_{j+1} \wedge \cdots \wedge e_{k-1} \wedge f_k \wedge e_{k+1} \wedge \cdots \wedge e_n \\ & + e_1 \wedge \cdots \wedge e_{k-1} \wedge \sum_l \left(d^l_k f_l\right) \wedge e_{k+1} \wedge \cdots \wedge e_n \quad mod(W). \end{aligned}$$

We see that the only way to have $\alpha.E^k \in W$ is if $a^i_j = d^i_j = 0$ for $i \neq j$. So set $a^i_j = d^i_j = 0$ if $i \neq j$. The restrictions that we have found are all necessary. We need to prove now, that these are, in fact, all of the restrictions that we get. For $\alpha \in \tilde{\mathfrak{g}}$, as restricted so far,

$$\alpha.E^{j,k} = \sum_{i\neq j, i\neq k} \left(a_i^i E^{j,k} + b_j^j E^k + b_k^k E^j + d_j^j E^{j,k} + d_k^k E^{j,k}\right) + \sum_{i\neq j, i\neq k} \left(c_i^i E^{i,j,k}\right) \in W.$$

A similar calculation for $\alpha.E^{k_1,\dots k_p}$ shows that in fact there are no more restrictions.

The restrictions we have found force

$$\tilde{\mathfrak{g}} = \left\{ \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \mid D_i \text{ diagonal matrices} \right\}.$$

Note that $\tilde{\mathfrak{g}} \cong (\mathfrak{gl}(2))^{\times n}$. Explicitly, each copy of $\mathfrak{gl}(2)$ is an $n \times n$ sub-matrix centered on the diagonal of $\left(\begin{smallmatrix} D_1 & D_2 \\ D_3 & D_4 \end{smallmatrix}\right)$. Notice that at every step, we have only considered the necessary restrictions. So $\tilde{G} \subset GL(V)$ cannot be any larger.

Step 3: We have shown that the diagonal subgroup $\mathfrak{S}_n \subset \mathfrak{S}_n^E \times \mathfrak{S}_n^F \subset \mathfrak{S}_{2n}$ is the largest possible subgroup that can preserve W, and we found an action of $GL(V_1) \times \cdots \times GL(V_n)$ as the largest possible subgroup of $GL(2n)$ preserving W. But there is a natural inclusion $\mathfrak{S}_{2n} \subset GL(2n)$, so $GL(2n) \ltimes \mathfrak{S}_{2n} = GL(2n)$. Therefore $(GL(V_1) \times \cdots \times GL(V_n)) \ltimes \mathfrak{S}_n \subset GL(2n) \ltimes \mathfrak{S}_{2n} = GL(2n)$ is the largest possible subgroup of $GL(2n)$ that can preserve W.

□

Remark III.9. Now we can consider W as a $\tilde{G}$ submodule of $\bigwedge^n(E \oplus F)$. But $\tilde{G}$ is a reductive group, so there must exist a complement W^c so that $\bigwedge^n(E \oplus F) = W \oplus W^c$. For a more detailed description of W^c as a sum of irreducible modules both in this case and in the case for symmetric principal minors, see Appendix C.

Theorem III.10. *The variety of principal minors of $n \times n$ matrices, $\tilde{Z}_n \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$, is invariant under the action of $(GL(2)^{\times n}) \ltimes \mathfrak{S}_n \subset GL(2n)$. Moreover, $(GL(2)^{\times n}) \ltimes \mathfrak{S}_n$ is the largest subgroup of $GL(2n)$ which preserves $\tilde{Z}$.*

Remark III.11. The first statement of this theorem was proved in [2] Theorem 4.2

and is implicit in [14] Theorem 12.

Proof. For the $\mathfrak{S}_n$ part, let $P \in \tilde{Z}_n$ be the vector of principal minors of an $n \times n$ matrix A. Notice that for $\sigma \in \mathfrak{S}_n$, the line through $P = \psi([A,t])$ is sent to the line through $\sigma.P = \psi([\sigma.A,t])$:

$$\begin{aligned}\sigma.P &= \sigma.\left[t^{|R_1|}\left(e_{R_1} \wedge f_{S_1}\right)(A), \dots, t^{|R_{2n}|}\left(e_{R_{2n}} \wedge f_{S_{2n}}\right)(A)\right] \\ &= \left[t^{|R_1|}\left(e_{R_1} \wedge f_{S_1}\right)(\sigma.A), \dots, t^{|R_{2n}|}\left(e_{R_{2n}} \wedge f_{S_{2n}}\right)(\sigma.A)\right],\end{aligned}$$

where $\sigma.A$ is the matrix constructed from A by permuting both its row set and column set by the same permutation σ. So $\sigma.P$ is a vector of principal minors of the matrix $\sigma.A$, and in particular, the cone over the variety of principal minors $\tilde{Z}_n$ is preserved by the action of $\mathfrak{S}_n$, and by passing to the projectivization, $\tilde{Z}_n$ is also preserved by the action of $\mathfrak{S}_n$.

For the continuous group we use the projection from the Grassmannian (Observation III.7), the invariance of the module $V_1 \otimes \cdots \otimes V_n$ under the action of $\tilde{G}$ (Lemma III.8), and the information about the preservation of symmetry under a linear projection (Lemma II.1) to conclude that $\tilde{Z}_n$ is a $\tilde{G}$-variety.

Finally, notice that we used the largest possible subgroup of $GL(2n)$ that could preserve $\mathbb{P}(V_1 \otimes \cdots \otimes V_n)$ (the ambient space containing $\tilde{Z}_n$) for our application of Lemma II.1, so this also implies that no larger subgroup of $GL(2n)$ will preserve $\tilde{Z}_n$ □

3. Finding the symmetry of Z_n via a projection from the Lagrangian Grassmannian

Now we will prove Theorem III.2 by specializing the proof for the case of principal minors of arbitrary square matrices.

Observation III.12. *The variety of principal minors of symmetric $n \times n$ matrices,*

$Z_n \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$, is a linear projection from the Lagrangian Grassmannian, $Gr_\omega(n, 2n) \subset \mathbb{P}\Gamma_n$.

Proof. From the exposition of $Gr_\omega(n, 2n)$ above, it is clear that the projection is given by deleting the non-principal minors. $\square$

As in the case of $\tilde{Z}_n$, we will exploit this projection to find a subgroup of the symmetry group of $\tilde{Z}_n$ that lies in $Sp(2n)$ by showing that, in fact, this projection is equivariant for a subgroup $G \subset Sp(2n)$. Similar to the $\tilde{Z}_n$ case, since Γ_n is a $Sp(2n)$-module, Γ_n is also G-module for any subgroup $G \subset Sp(2n)$. In the course of the proof we will find that the subgroup that does the job is $G = (SL(V_1) \times \cdots \times SL(V_n)) \ltimes \mathfrak{S}_n$. For our purposes, it is important to see $W \subset \Gamma_n$ as a G-submodule.

Lemma III.13. *There exists an embedding*

$$\left(SL(2)^{\times n}\right) \ltimes \mathfrak{S}_n \simeq (SL(V_1) \times \cdots \times SL(V_n)) \ltimes \mathfrak{S}_n \subset GL(2n)$$

so that the vector space $W \simeq V_1 \otimes \cdots \otimes V_n$ is a $(SL(2)^{\times n}) \ltimes \mathfrak{S}_n$-submodule of Γ_n. Moreover, $(SL(2)^{\times n}) \ltimes \mathfrak{S}_n$ is the largest subgroup of $Sp(2n)$ preserving W.

Proof. The same proof works as in the $GL(V)$ case, with the additional restriction that we start with elements in $\mathfrak{gl}(V)$ which preserve a symplectic form $\omega \in S^2V$, *i.e.* $\mathfrak{sp}(V) = \{g \in \mathfrak{gl}(V) \mid g.\omega + \omega.g^t = 0\}$. Since we have already done the work with $GL(2n)$, we hold off on applying the restriction coming from the symplectic form until the end. In particular, the proof concerning $\mathfrak{S}_n$ goes through without modification.

For the continuous group, we must consider the Lie algebra

$$\mathfrak{g} = \{\alpha \in \tilde{\mathfrak{g}} \subset \mathfrak{gl}(2n) \mid \omega\left(\alpha v, w\right) + \omega\left(v, \alpha w\right) = 0\}.$$

We show that $\mathfrak{g}$ preserves W. In matrices, the relation is that elements of $C \in \mathfrak{g}$ must satisfy the relation $C\Omega + \Omega({}^tC) = (0)$, where Ω is a matrix realization of ω. Recall that matrices in $\tilde{\mathfrak{g}}$ are of the form $C = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \in \tilde{\mathfrak{g}}$, with D_i diagonal matrices. With the particular choice

$$\Omega = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix},$$

the relation $C\Omega + \Omega({}^tC) = (0)$ implies that $D_1 + D_4 = (0)$, *i. e.*. $\mathfrak{g} \simeq \mathfrak{sl}_2^{\times n}$.

Again, $\mathfrak{S}_n$ and $SL(2)^{\times n}$ are commutants in $Sp(2n)$ and both actions leave W invariant. Since all of the restrictions on $\mathfrak{g}$ are necessary, we have also found that $G = (SL(2)^{\times n}) \ltimes \mathfrak{S}_n$ is the largest the subgroup of $Sp(2n) \subset GL(V)$ that could possibly act on W and leave it invariant. $\square$

A direct application of Observation III.12 and Lemmas III.13 and II.1 proves the following theorem, of which Theorem III.2 is a special case.

Theorem III.14. *The variety of principal minors of symmetric $n \times n$ matrices, $Z_n \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$, is invariant under the action of $(SL(2)^{\times n}) \ltimes \mathfrak{S}_n \subset Sp(2n)$. Moreover, $(SL(2)^{\times n}) \ltimes \mathfrak{S}_n$ is the largest subgroup of $Sp(2n)$ preserving Z_n.*

B. The hyperdeterminantal module

As a consequence of Theorem III.2, the defining ideal of Z_n, $\mathcal{I}(Z_n) \subset \mathrm{Sym}(V_1^* \otimes \cdots \otimes V_n^*)$, is a G-module for $G = (SL(V_1) \times \cdots \times SL(V_n)) \ltimes \mathfrak{S}_n$. As in the introduction to this chapter, we will consider the G-module $M = S_{(2,2)}S_{(2,2)}S_{(2,2)}S_{(4)} \dots S_{(4)}$ (called the hyperdeterminantal module in [14]).

Observation III.15. *The dimension of the module $M = S_{(2,2)}S_{(2,2)}S_{(2,2)}S_{(4)} \dots S_{(4)}$ is*

$$\binom{n}{n-3} 5^{n-3}.$$

Proof. The module $S_{(2,2)}\mathbb{C}^2$ is 1-dimensional and the module $S_{(4)}\mathbb{C}^2$ is 5-dimensional. $\square$

Remark III.16. For the sake of the reader not familiar with representation theory, we will recall a standard algorithm for finding an explicit basis of an irreducible G-module M. Start with a highest weight vector. In our case, the highest weight vector of M is the hyperdeterminant of format $2\times 2\times 2$ on the variables $X^{[i_1,i_2,i_3,1,\dots,1]}$. Consider the Lie algebra of lowering operators, $\mathfrak{g}_-$. In our case, the lowering operators are of the form $l_1+\dots+l_n$ where l_i are lower triangular. Successively apply lowering operators to the highest weight vector to get vectors of lower weight. Since M is a $\mathfrak{g}$-module, each new vector will be in M. Since M is finite dimensional, the process will stop at the lowest (nonzero) weight vector. The collection of weight vectors found in this way will be a weight basis of M, and will also be minimal set of generators of $\langle M\rangle$, the ideal generated by M.

Proposition III.17. *The module M occurs with multiplicity 1 in $S^4(V_1^*\otimes\dots\otimes V_n^*)$. Moreover, M is an irreducible G-module for $G=(SL(V_1)\times\dots\times SL(V_n))\ltimes\mathfrak{S}_n\simeq(SL(2)^{\times n})\ltimes\mathfrak{S}_n$.*

Remark III.18. The fact that M occurs with multiplicity 1 saves us a lot of work because we do not have to worry about which isomorphic copy of the module occurs in the ideal.

Proof. For the "moreover" part, notice that the module M is the span of the G-orbit of a single polynomial (namely the hyperdeterminant of format $2\times 2\times 2$ on the variables $X^{[i_1,i_2,i_3,1,\dots,1]}$) and therefore M is an irreducible module.

We need to examine the $SL(2)^{\times n}$-module decomposition of $S^4(V_1^*\otimes\dots\otimes V_n^*)$. It suffices to prove for any fixed permutation σ, that $S_{(2,2)}V_{\sigma(1)}^*\otimes S_{(2,2)}V_{\sigma(2)}^*\otimes S_{(2,2)}V_{\sigma(3)}^*\otimes S_{(4)}V_{\sigma(4)}^*\otimes\dots\otimes S_{(4)}V_{\sigma(n)}^*$ is an $SL(2)^{\times n}$-module which occurs with multiplicity 1 in the decomposition of $S^4(V_1^*\otimes\dots\otimes V_n^*)$.

We will follow the notation and calculations similar to [19]. Let χ_π denote the character of the representation $[\pi]$ in the group algebra $\mathbb{C}[\mathfrak{S}_d]$. The number of occurrences of $S_{\pi_1}V_1^* \otimes \cdots \otimes S_{\pi_n}V_n^*$ in the decomposition of $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ is computed by the dimension of the space of $\mathfrak{S}_d$ invariants, $\dim\left(([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}\right)$. This may be computed by the formula

$$\dim\left(([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}\right) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi_{\pi_1}(\sigma) \dots \chi_{\pi_n}(\sigma). \tag{3.3}$$

In our case, we need to compute

$$\begin{aligned} &\dim\left(([(2,2)] \otimes [(2,2)] \otimes [(2,2)] \otimes [(4)] \otimes \cdots \otimes [(4)])^{\mathfrak{S}_4}\right) \\ &\quad = \frac{1}{4!} \sum_{\sigma \in \mathfrak{S}_4} \chi_{(2,2)}(\sigma)\chi_{(2,2)}(\sigma)\chi_{(2,2)}(\sigma)\chi_{(4)}(\sigma) \dots \chi_4(\sigma). \end{aligned}$$

But, $\chi_{(4)}(\sigma) = 1 \ \forall \sigma \in \mathfrak{S}_4$. So, our computation reduces to the following

$$\begin{aligned} &\dim\left(([(2,2)] \otimes [(2,2)] \otimes [(2,2)] \otimes [(4)] \otimes \cdots \otimes [(4)])^{\mathfrak{S}_n}\right) \\ &\qquad\qquad = \frac{1}{4!} \sum_{\sigma \in \mathfrak{S}_4} \chi_{(2,2)}(\sigma)\chi_{(2,2)}(\sigma)\chi_{(2,2)}(\sigma) = 1, \end{aligned}$$

where the last equality is found by direct computation. The module $S_{(2,2)}V_1^* \otimes S_{(2,2)}V_2^* \otimes S_{(2,2)}V_3^*$ occurs with multiplicity 1 in $S^4(V_1^* \otimes V_2^* \otimes V_3^*)$. (The full decomposition of $S^4(V_1^* \otimes V_2^* \otimes V_3^*)$ was computed in (prop 4.3 [19]).) Therefore the module $S_{(2,2)}V_{\sigma(1)}^* \otimes S_{(2,2)}V_{\sigma(2)}^* \otimes S_{(2,2)}V_{\sigma(3)}^* \otimes S_{(4)}V_{\sigma(4)}^* \otimes \cdots \otimes S_{(4)}V_{\sigma(n)}^*$ occurs with multiplicity 1 in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$.

We have seen that each summand of M is an irreducible $SL(2)^{\times n}$-module which occurs with multiplicity 1 in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$. Therefore M is an irreducible G-module, and it occurs with multiplicity 1 in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$. $\square$

We remark that the above argument generalizes to:

Lemma III.19. *For every collection $\pi_1, \ldots, \pi_n$ of partitions of d,*

$$\dim\left(([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}\right) = \dim\left(([\pi_1] \otimes \cdots \otimes [\pi_n] \otimes [(d)])^{\mathfrak{S}_d}\right). \tag{3.4}$$

In particular, if M is any irreducible $SL(V_1) \times \cdots \times SL(V_n)$-module which occurs with multiplicity m in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$, then $M \otimes S^d V_{n+1}^*$ is an irreducible $SL(V_1) \times \cdots \times SL(V_n) \times SL(V_{n+1})$-module which occurs with multiplicity m in $S^d(V_1^* \otimes \cdots \otimes V_n^* \otimes V_{n+1}^*)$.

Proof. Use the formula

$$\dim\left(([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}\right) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi_{\pi_1}(\sigma) \ldots \chi_{\pi_n}(\sigma). \tag{3.5}$$

and note that $\chi_{(d)}(\sigma) = 1 \; \forall \sigma \in \mathfrak{S}_d$. $\square$

Let $\mathfrak{S}_n^{\hat{k}}$ denote the permutation group generated by the letters $\{1, \ldots, n\} \backslash \{k\}$ and let Let $\Sigma_n^k = \{\sigma \in \mathfrak{S}_n^{\hat{k}} \mid \sigma(1) < \sigma(2) < \sigma(3)$ and $\sigma(4) < \ldots \sigma(k-1) < \sigma(k+1) \cdots < \sigma(n)$. Then let M_k denote the following module

$$\begin{aligned} M_k &= \bigoplus_{\sigma \in \Sigma_n^k} S_{(2,2)} V_{\sigma(1)}^* \otimes S_{(2,2)} V_{\sigma(2)}^* \otimes S_{(2,2)} V_{\sigma(3)}^* \otimes S_{(4)} V_{\sigma(4)}^* \otimes \\ &\quad \cdots \otimes \widehat{S_{(4)} V_{\sigma(k)}^*} \otimes \otimes \cdots \otimes S_{(4)} V_{\sigma(n)}^* \\ &\subset S^4(V_1^* \otimes \cdots \otimes V_{k-1}^* \otimes V_{k+1}^* \otimes \cdots \otimes V_n^*). \end{aligned}$$

Remark III.20. There is a reduction we can make by realizing that $M = \sum_{i=1}^4 (M_i \otimes S_{(4)} V_i^*)$. This is because all of the modules that occur in $M_i \otimes S_{(4)} V_i^*$ for $5 \leq i \leq n$ have already occurred when $1 \leq i \leq 4$. More explicitly, $M_1 \otimes S_{(4)} V_1^*$ contains all the modules in M except for those that have an $S_{(2,2)} V_1^*$. The module $\left(M_1 \otimes S_{(4)} V_1^*\right) + \left(M_2 \otimes S_{(4)} V_2^*\right)$ contains all the modules in M except for those that have a factor which is $S_{(2,2)} V_1^* \otimes S_{(2,2)} V_2^*$. The sum $\left(M_1 \otimes S_{(4)} V_1^*\right) + \left(M_2 \otimes S_{(4)} V_2^*\right) + \left(M_3 \otimes S_{(4)} V_3^*\right)$

contains all the modules in M except for those that have $S_{(2,2)}V_1^* \otimes S_{(2,2)}V_2^* \otimes S_{(2,2)}V_3^*$, and these modules are contained in $M_4 \otimes S_{(4)}V_4^*$, so we have covered every possible module that occurs in the decomposition of M.

Proposition III.21. *Notation as above. We have the following inclusion*

$$M \subseteq \mathcal{I}(Z_n),$$

and in particular, $Z_n \subseteq \mathcal{V}(M)$.

Proof. Both M and $\mathcal{I}(Z_n)$ are G-modules and M is an irreducible G-module, so we only need to show that the highest weight vector of M vanishes on all points of Z_n. The highest weight vector of M is the hyperdeterminant of format $2 \times 2 \times 2$ on the variables $X^{[i_1,i_2,i_3,1,\dots,1]}$. The set $Z_n \cap span\{X^{[i_1,i_2,i_3,1,\dots,1]} \mid i_1, i_2, i_3 \in \{0,1\}\}$, is the set of principal minors of the upper 3×3 corner of $n \times n$ matrices. The hyperdeterminant vanishes on these principal minors because of the case $n = 3$, so there is nothing more to show. $\square$

Remark III.22. Our proof actually proves that if M is a module in $\mathcal{I}_d(Z_n)$, then $M \otimes S^dV_{n+1}$ is a module in $\mathcal{I}_d(Z_{n+1})$. The real utility of this is its contrapositive version. It gives a test for ideal membership for modules that have at least one $S_{(d)}V_i^*$ factor. Suppose we know $\mathcal{I}_d(Z_n)$ for some n. If we want to test whether $N = S_{\pi_1}V_1^* \otimes \cdots \otimes S_{\pi_{n+1}}V_{n+1}^*$ is in $\mathcal{I}_d(Z_{n+1})$ and we know that N has at least one $\pi_i = (d)$, then we can remove $S_{\pi_i}V_i^*$ and check whether the module we have left is in the ideal $\mathcal{I}_d(Z_n)$.

C. The tangential variety of the Segre product of $\mathbb{P}^1$'s

The tangential variety to the Segre, $\tau\left(Seg\left(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n\right)\right)$, is the set of points of the form

$$\left[a_1 \otimes \cdots \otimes a_n + \sum_i a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i' \otimes a_{i+1} \otimes \cdots \otimes a_n\right],$$

with $a_i, a_i' \in V_i$ and a_i nonzero. While the tangential variety is not homogeneous, it is invariant under the action of the group $SL(V_1) \times \cdots \times SL(V_n)$.

Remark III.23. $S_{(2,2)}S_{(2,2)}S_{(2,2)}$ is a 1-dimensional module. As mentioned in the introduction to this chapter, one can compute that this line is spanned by Cayley's hyperdeterminant of format $2 \times 2 \times 2$. The fact that $S_{(2,2)}S_{(2,2)}S_{(2,2)}$ gives a minimal generating set for the prime ideal of Z_3 was pointed out by [14]. $S_{(2,2)}S_{(2,2)}S_{(2,2)}$ also generates the prime ideal for two (identical) varieties. The first is the tangential variety to the Segre variety, $\tau(Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1))$, and the second is the dual variety to the Segre variety, $Seg^*(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$. This tells us that for the case $n = 3$, the tangential variety to the Segre, the dual variety to the Segre, and Z_3 are the same variety. When $n > 3$ the dual and tangential varieties of the Segre variety differ. While we were unable to exploit the dual variety, we found that the tangential variety is a proper subvariety of Z_n (cf. Proposition III.25).

Proposition III.24. *The G-orbit of the image of the zero matrix is the Segre variety,* i.e.

$$G.\varphi([(0), t]) = Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n).$$

Proof. Notice that $\varphi([(0), t]) = [t^n x_1^1 \otimes \cdots \otimes x_n^1] \in Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$. But the Segre variety is a homogeneous variety for the group G, so the result follows. $\square$

Consider the following variant of the Veronese embedding of $\mathbb{P}^{n-1}$ into the $n \times n$

matrices.

$$
\begin{array}{rcl}
v_2 : \mathbb{P}^n = \mathbb{P}\left(\mathbb{C}^n \oplus \mathbb{C}\right) & \longrightarrow & \mathbb{P}\left(S^2\mathbb{C}^n \oplus \mathbb{C}\right) \subset \mathbb{P}\left(\mathbb{C}^{n\times n} \oplus \mathbb{C}\right) \\
[w_1, w_2, \ldots, w_n, s] & \longmapsto & \left[\begin{pmatrix} w_1^2 & w_2w_1 & \ldots & w_nw_1 \\ w_1w_2 & w_2^2 & \ldots & w_nw_2 \\ \vdots & \vdots & \ddots & \vdots \\ w_1w_n & w_2w_n & \ldots & w_n^2 \end{pmatrix}, s^2\right] = [\mathbf{w}.{}^t\mathbf{w}, s^2].
\end{array}
$$

This parameterizes the projectivization of the rank 1 symmetric matrices.

Proposition III.25. *The closure of the G-orbit of the image (under φ) of the rank 1 symmetric matrices is the tangential variety to the n-factor Segre variety. In particular, $\tau(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)) \subset Z_n$.*

Proof. Let $Y = \varphi(v_2(\mathbb{P}^n))$. It remains to show that $\overline{G.Y} = \tau(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1))$.

Since $\mathbf{w}.{}^t\mathbf{w}$ is a rank 1 symmetric matrix, all $k \times k$ minors vanish for $k > 1$, and in particular, the $k \times k$ principal minors vanish for $k > 1$. Therefore a generic point in Y has the form

$$
P = \left[t\left(x_1^1 \otimes \cdots \otimes x_n^1\right) + \sum_i w_i^2\left(x_1^1 \otimes \cdots \otimes x_{i-1}^1 \otimes x_i^2 \otimes x_{i+1}^1 \cdots \otimes x_n^1\right)\right],
$$

where $w_i, t \in \mathbb{C}$. In particular, $Y \subset \tau\left(Seg\left(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n\right)\right)$.

Since $\tau\left(Seg\left(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n\right)\right)$ is a G-variety, $G.Y \subset \tau\left(Seg\left(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n\right)\right)$. The tangential variety is closed, so it contains $\overline{G.Y}$.

In the other direction, suppose we are given an arbitrary point $Q = \left[a_1 \otimes \cdots \otimes a_n + \sum_i a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i' \otimes a_{i+1} \otimes \cdots \otimes a_n\right] \in \tau\left(Seg\left(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n\right)\right)$.

Consider a generic element of $SL(2)^{\times n}$,

$$
g = \begin{pmatrix} a^1 & b^1 \\ c^1 & d^1 \end{pmatrix} \times \cdots \times \begin{pmatrix} a^n & b^n \\ c^n & d^n \end{pmatrix},
$$

with $a^i d^i - b^i c^i = 1$. The generic orbit of a generic point $P \in Y$ has the form

$$\begin{aligned} g.P = \quad & \left[t\left(a^1 x_1^1 + c^1 x_1^2\right) \otimes \cdots \otimes \left(a^n x_n^1 + c^n x_n^2\right)\right. \\ & + \textstyle\sum_i w_i^2 \left(a^1 x_1^1 + c^1 x_2^1\right) \otimes \cdots \otimes \left(a^{i-1} x_{i-1}^1 + c^{i-1} x_{i-1}^2\right) \otimes \\ & \left.\left(b^i x_i^1 + d^i x_i^2\right) \otimes \left(a^{i+1} x_{i+1}^1 + c^{i+1} x_{i+1}^2\right) \otimes \cdots \otimes \left(a^n x_n^1 + c^n x_n^2\right)\right]. \end{aligned}$$

We can choose $t, w_i, a^i, b^i, c^i, d^i$ so that the expressions $a_i = \left(a^i x_1^1 + c^i x_i^2\right)$ and $a_i' = w_i^2 \left(b^i x_i^1 + d^i x_i^2\right)$ hold for each i, and $Q = P$ is a point in the orbit. (The choice in w_i allows us to scale so that $a^i d^i - b^i c^i = 1$.) This implies that $\tau\left(Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)\right) \subset \overline{G.Y}$. Therefore $\overline{G.\varphi(v_2(\mathbb{P}^n))} = \tau\left(Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)\right)$. Finally, since $\varphi(v_2(\mathbb{P}^n)) \subset Z_n$, we know the closure of the G-orbit of $\varphi(v_2(\mathbb{P}^n))$ is a subvariety of Z_n, and we are done. $\square$

Landsberg and Weyman have studied tangential varieties to secant varieties and their defining ideals. We draw the following connections to their work [21]:

Theorem III.26 (Theorem 7.3 [21])**.** $\mathcal{I}\left(\tau\left(Seg\left(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n\right)\right)\right)$ *(when V_i are all 2-dimensional) is generated in degree less than or equal to 6.*

Conjecture III.27 (Conjecture 7.6 [21])**.** $\mathcal{I}\left(\tau\left(Seg\left(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n\right)\right)\right)$ *is generated by the quadrics in $S^2\left(V_1^* \otimes \cdots \otimes V_n^*\right)$ which have at least four $\bigwedge^2$ factors, the cubics with four $S_{2,1}$ factors and all other factors $S_{3,0}$, and the quartics with three $S_{2,2}$'s and all other factors $S_{4,0}$.*

D. Secant varieties and more geometry of Z_n

For an algebraic variety $X \subset \mathbb{P}^N$, let $\sigma_k(X) \subset \mathbb{P}^N$ denote the k^{th} *secant variety of* X, defined by

$$\sigma_k(X) = \overline{\bigcup_{p_1,\ldots,p_k \in X} \mathbb{P}_{p_1,\ldots,p_k}},$$

where the overline indicates Zariski closure.

Polarization of a polynomial is a tool used to study ideals of secant varieties in [18] and in [27]. For this section only, we will follow the notation of [27].

Lemma III.28 (Lemma 2.5(1) [27]). *If F is a homogeneous degree d polynomial, let $\overrightarrow{F}$ denote its polarization. Let $v = t_1x_1 + \cdots + t_kx_k$. Then the following expression holds:*

$$F(v) = \overrightarrow{F}(v,\ldots,v) = \sum_{\beta} \frac{1}{\beta!}\mathbf{t}^{\beta}\overrightarrow{F}\left(\mathbf{x}^{\beta}\right), \tag{3.6}$$

where $\beta = (\beta_1,\ldots,\beta_k)$, $|\beta| = d$, $\beta! = \beta_1!\ldots\beta_k!$, $\mathbf{t}^{\beta} = t_1^{\beta_1}\ldots t_k^{\beta_k}$, and $\overrightarrow{F}\left(\mathbf{x}^{\beta}\right) = \overrightarrow{F}\left(x_1^{\beta_1},\ldots,x_k^{\beta_k}\right)$.

In general, the polarization of the tensor product of two polynomials is not likely to be the product of the polarized polynomials; however, there is something we can say in the following special case:

Lemma III.29. *Let $F \in S^d(W^*)$ and let $\overrightarrow{F}$ denote its polarization. Then for $y \in V^*$ we have*

$$\overrightarrow{F \otimes (y)^d} = \overrightarrow{F} \otimes \overrightarrow{(y)^d} = \overrightarrow{F} \otimes (y)^d.$$

Proof. A standard fact about the polarization is that $\overrightarrow{F}$ is a symmetric multi-linear form. It is obvious that $\overrightarrow{(y)^d} = (y)^d$, because $(y)^d$ is already symmetric and multi-linear.

We will prove this by induction on the number of terms in F. Suppose F is a monomial, $F = \mathbf{w}^{\alpha} = w_1^{\alpha_1} \circ \cdots \circ w_n^{\alpha_n}$. Then use the isomorphism $W^{\otimes d} \otimes V^{\otimes d} \simeq$

$(W \otimes V)^{\otimes d}$, and write $\mathbf{w}^\alpha \otimes y^d = (w_1^{\alpha_1} \otimes y^{\alpha_1}) \circ \cdots \circ (w_n^{\alpha_n} \otimes y^{\alpha_n}) = (w_1 \otimes y)^{\alpha_1} \circ \cdots \circ (w_n \otimes y)^{\alpha_n} = (\mathbf{w} \otimes y)^\alpha$.

If F is not a monomial, suppose $F = F_1 + F_2$. It is clear that $\overrightarrow{F_1 + F_2} = \overrightarrow{F_1} + \overrightarrow{F_2}$. Also, the operation $\otimes y^d$ is distributive. So $\overrightarrow{F \otimes (y)^d} = \overrightarrow{F_1 \otimes y^d} + \overrightarrow{F_2 \otimes y^d}$. By induction, we know that $\overrightarrow{F_i \otimes y^d} = \overrightarrow{F_i} \otimes y^d$ for $i = 1, 2$. We conclude that $\overrightarrow{F_1 \otimes y^d} + \overrightarrow{F_2 \otimes y^d} = (\overrightarrow{F_1} + \overrightarrow{F_2}) \otimes y^d = \overrightarrow{F} \otimes y^d$. $\square$

We are studying a module M that is constructed by an augmentation procedure $M = \sum_i M_i \otimes S_{(4)} V_i$. This procedure is similar to prolongation, however augmentation does not change the degree. The following lemma was inspired by methods found in [18].

Lemma III.30 (Step Up Lemma)**.** *Let W and V be complex vector spaces. Let $X \subset \mathbb{P}W$ be a variety and suppose $\mathcal{I}_d(X) \subset S^d W^*$ is the ideal in degree d. Then*

$$\mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*) = Seg(\mathcal{V}(\mathcal{I}_d(X)) \times \mathbb{P}V) \sqcup \bigcup_{L \subset \mathcal{V}(\mathcal{I}_d(X))} \sigma_d(\mathbb{P}L \times \mathbb{P}V) \subset \mathbb{P}(W \otimes V), \tag{3.7}$$

where $L \subset \mathcal{V}(\mathcal{I}_d(X))$ are linear subspaces.

Remark III.31. Note that if X is generated in a single degree no larger than d, then one can replace $\mathcal{V}(\mathcal{I}_d(X))$ with X in the statement of the lemma. In particular, we will use the result of Lemma III.30 with $\mathcal{I}_4(X) = M$ and $X = \mathcal{V}(M)$.

Proof. Recall that we can choose a basis of $S^d V^*$ consisting of d^{th} powers of linear forms, $\{(y_1)^d, \ldots, (y_r)^d\}$, where $r = \binom{n+d-1}{d}$ and the y_i are in general linear position. So one can construct a basis of the module $\mathcal{I}_d(X) \otimes S^d V^*$, consisting of polynomials of the form $f \otimes y^d$.

First we prove $\supseteq$. Suppose $[v] = [tx \otimes a] \in Seg(\mathcal{V}(\mathcal{I}_d(X)) \times \mathbb{P}V)$ and evaluate $(f \otimes y^d)(tx \otimes a) = f(x) y^d(ta)$. But $x \in \mathcal{V}(\mathcal{I}_d(X))$, so $f(x) = 0\ \forall f \in \mathcal{I}_d(X)$, and in particular, $[v] \in \mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*)$.

Now suppose $[v] \in \sigma_d(\mathbb{P}L \times \mathbb{P}V)$ for some k-dimensional linear subspace $L \subset \mathcal{V}(\mathcal{I}_d(X))$ with $L = span\{x_1, \ldots, x_k\}$. We will show that v is in the zero set of $\mathcal{I}_d(X) \otimes S^dV^*$. Let $[v] = [t_1x_1 \otimes a_1 + \cdots + t_kx_k \otimes a_k]$ with $[x_i \otimes a_i] \in Seg(\mathbb{P}L \times \mathbb{P}V)$ and $t_i \in \mathbb{C}$. But $L \subset \mathcal{V}(\mathcal{I}_d(X)) \iff f(r_1x_1 + \cdots + r_kx_k) = 0$, for all scalars r_i and all $f \in \mathcal{I}_d(X)$. By allowing $r_1, \ldots, r_k$ to vary and using (3.6), we see that this condition is equivalent to $\overrightarrow{f}(\mathbf{x}^\beta) = 0$ for all β.

Then by Lemma III.29, $\overrightarrow{f \otimes y^d} = \overrightarrow{f} \otimes y^d$ and using the polarization formula (3.6), we write

$$\left(f \otimes y^d\right)(v) = \left(\overrightarrow{f} \otimes y^d\right)(v, \ldots, v) = \sum_\beta \frac{1}{\beta!}\mathbf{t}^\beta \overrightarrow{f}(\mathbf{x}^\beta)y^d(\mathbf{a}^\beta).$$

But every term of $(f \otimes y^d)(v)$ vanishes, so $(f \otimes y^d)(v) = 0$. Therefore, $[v] \in \mathcal{V}(\mathcal{I}_d(X) \otimes S^dV^*)$.

Now to prove $\subseteq$, we argue by cases depending on the rank of $[v] \in \mathbb{P}(W \otimes V)$. We will show for each k, if $[v] \in \mathcal{V}(\mathcal{I}_d(X) \otimes S^dV^*)$ has rank k, then $[v] \in Seg(\mathcal{V}(\mathcal{I}_d(X)) \times \mathbb{P}V) \sqcup \bigcup_{L \subset \mathcal{V}(\mathcal{I}_d(X))} \sigma_d(\mathbb{P}L \times \mathbb{P}V)$.

If $k = 1$, then consider $[v] = [x \otimes a] \in \mathcal{V}(\mathcal{I}_d(X) \otimes S^dV^*)$ for some $x \in \mathbb{P}W$ and $a \in \mathbb{P}V$. Let $f \otimes y^d$ be an element of $\mathcal{I}_d(X) \otimes S^dV^*$ such that $y(a) \neq 0$. Then the equation $(f \otimes y^d)(x \otimes a) = f(x)(y^d)(a) = 0$ must hold $\forall f \in \mathcal{I}_d(X)$. So $f(x) = 0$ for every $f \in \mathcal{I}_d(X)$ and $[v] \in Seg(\mathcal{V}(\mathcal{I}_d(X)) \times \mathbb{P}V)$.

If $k > 1$, let $[v] = [t_1x_1 \otimes a_1 + \cdots + t_kx_k \otimes a_k] \in \mathcal{V}(\mathcal{I}_d(X) \otimes S^dV^*)$, with $x_i \in W$, $a_i \in V$ and $t_i \in \mathbb{C}$. For any $f \otimes y^d \in \mathcal{I}_d(X) \otimes S^dV^*$ we can write

$$0 = (f \otimes y^d)(v) = \sum_\beta \frac{1}{\beta!}\mathbf{t}^\beta \overrightarrow{f}(\mathbf{x}^\beta)y^d(\mathbf{a}^\beta).$$

WLOG we can assume that $y(a_i) \neq 0 \ \forall i$. If not, re-choose $y \in V^*$. Now $y^d(\mathbf{a}^\beta)$ is a nonzero scalar, so by appropriate choices in $t_1, \ldots, t_k$ we can insist that $\frac{1}{\beta!}\mathbf{t}^\beta y^d(\mathbf{a}^\beta) =$

$1\ \forall\beta$. The expression above now reduces to

$$0 = \sum \overrightarrow{f}(\mathbf{x}^{\beta}).$$

But, of course, we could re-scale the x_i's so that we still define the same k-plane, and the previous expression is still true, so we must conclude that $0 = \overrightarrow{f}(\mathbf{x}^{\beta})$ for all β and for all $f \in \mathcal{I}_d(X)$. But this is the condition that the k-plane $L = span\{x_1, \dots, x_k\}$ must be contained in $\mathcal{V}(\mathcal{I}_d(X))$. Therefore $[v] \in \sigma_k(\mathbb{P}L \times \mathbb{P}V)$. $\square$

If A, B, C are vector spaces of polynomials such that $C = A + B$ then $\mathcal{V}(C) = \mathcal{V}(A) \cap \mathcal{V}(B)$. So a direct application of this fact and the Step Up Lemma III.30 yields the following

Lemma III.32 (Characterization Lemma)**.** *Assume that* $M = \sum_{i=1}^{n} M_i \otimes S^d V_i^* \subset S^d(V_1^* \otimes \cdots \otimes V_n^*)$, *then*

$$\mathcal{V}(M) = \bigcap_{i=1}^{n} \left(Seg(\mathcal{V}(M_i) \times \mathbb{P}V_i) \sqcup \bigcup_{L \subset V(M_i)} \sigma_d(\mathbb{P}L \times \mathbb{P}V_i) \right).$$

Additionally, we have the following inclusion of algebraic varieties:

$$\mathcal{V}(M) \subseteq \bigcap_{i=1}^{n} \sigma_d(\mathcal{V}(M_i) \times \mathbb{P}V_i).$$

Finally note that if $\dim(V) = s$, then $\sigma_d(\mathbb{P}W \times \mathbb{P}V) = \mathbb{P}(W \otimes V)\ \forall d \geq s$. Also, if $L \subset \mathcal{V}(\mathcal{I}_d(X))$ is a linear subspace, then $\sigma_d(\mathbb{P}L \times \mathbb{P}V) \subseteq \sigma_d(\mathcal{V}(\mathcal{I}_d(X)) \times \mathbb{P}V)$.

Remark III.33. A consequence of the characterization lemma is the following test. Suppose $[z] = [\zeta^1 \otimes x_i^1 + \zeta^2 \otimes x_i^2] \in \mathbb{P}^{2^n - 1}$. If either $[\zeta^1]$ or $[\zeta^2] \notin Z_{(n-1)\hat{i}}$, then $[z]$ is not a vector of principal minors of a symmetric matrix. This observation can be iterated, and each iteration cuts the size of the vector in question in half.

Remark III.34. We would like to have a better understanding of the algebraic implications of the procedure of augmentation. A natural guess for how the (ideal theoretic)

Holtz-Sturmfels Conjecture could be established is as follows. Attempt to define a ring homomorphism

$$f : \mathrm{Sym}(V_1^* \otimes \cdots \otimes V_{n+1}^*) \longrightarrow \mathrm{Sym}(V_1^* \otimes \cdots \otimes V_n^*)$$

so that $f^{-1}\left((\langle SL(2)^n) \ltimes \mathfrak{S}_n.hyp\rangle\right) = \langle (SL(2)^{n+1}) \ltimes \mathfrak{S}_{n+1}.hyp\rangle$. If such a homomorphism exists, then since prime ideals pull back under ring homomorphisms we could use this map to do induction on n to show that the hyperdeterminantal module always generates a prime ideal. This would be sufficient to prove the Holtz-Sturmfels Conjecture.

E. More structure of Z_n

Let $Z_{(n-1),\hat{i}}$ denote the isomorphic copy of Z_{n-1} inside of $\mathbb{P}(V_1 \otimes \cdots \otimes \widehat{V_i} \otimes \cdots \otimes V_n) \simeq \mathbb{P}(V_1 \otimes \cdots \otimes V_{n-1})$. Let $J_i = [j_1, \ldots, \hat{j}_i, \ldots, j_n]$ be a multi-index omitting the i^{th} entry. Then $Z_{(n-1),\hat{i}}$ can be described in the coordinates X^{J_i}.

The variety $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}\{x_i^1\})$ is an isomorphic copy of Z_{n-1} inside of $\mathbb{P}(V_1 \otimes \cdots \otimes V_{i-1} \otimes \{x_i^1\} \otimes V_{i+1} \otimes \cdots \otimes V_n) \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$, where $\{x_i^1\}$ indicates the span of x_i^1. Let $J_{i,1} = [j_1, \ldots, j_{i-1}, 1, j_{i+1}, \ldots, j_n]$, (respectively $J_{i,2} = [j_1, \ldots, j_{i-1}, 2, j_{i+1}, \ldots, j_n]$), be a multi-index that has a fixed 1 (respectively 2) in the i^{th} entry. Our convention is that $X^{J_{i,1}}$ are the principal minors which *do not* include the i^{th} row and column, and $X^{J_{i,2}}$ are the principal minors which *do* include the i^{th} row and column. Then $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}\{x_i^1\})$ can be described in the coordinates $X^{J_{i,1}}$, and we will see that points of $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i)$ have the interpretation as the principal minors of $n-1 \times n-1$ matrices.

It is obvious that we have isomorphisms, $Z_{(n-1),\hat{i}} \simeq Z_{n-1} \simeq Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}\{x_i^1\})$; however, $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}\{x_i^1\})$ is the only one that is actually a subvariety of Z_n.

Also, from Proposition III.21, we know $Z_n \subseteq \mathcal{V}(M)$, and in particular, we also get $Z_{(n-1),\hat{i}} \subset \mathcal{V}(M_i)$.

Proposition III.35. *The variety $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i)$ is a subvariety of Z_n. Moreover, any point of $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i)$ has an interpretation as the principal minors of a $(n-1) \times (n-1)$ matrix.*

Proof. Notice that $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}\{x_i^1\}) \subset Z_n$, and the $SL(2)^{\times n}$-orbit of $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}\{x_i^1\})$ is $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i)$. Since Z_n is a G-variety, it contains all of the G-orbits of points within Z_n, and in particular, it contains all of the $SL(2)^{\times n}$-orbits. For the "moreover" statement, notice that every point of $Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i)$ is in the G-orbit of a point which is the principal minors of an $n-1 \times n-1$ block of an $n \times n$ matrix, so after a change of basis, we have the result. □

With a little bit more work, one can show that a stronger result than Proposition III.35 holds:

Proposition III.36. *Let $Z_p \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_p)$ and $Z_q \subset \mathbb{P}(V_{p+1} \otimes \cdots \otimes V_n)$. Then $Seg(Z_p \times Z_q)$ is a subvariety of Z_{p+q}.*

Let $U_0 = \{[z] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \mid z = z_I X^I \in V_1 \otimes \cdots \otimes V_n, z_{[1,\dots,1]} \neq 0\}$. Then $\varphi([A,t]) \in Seg(Z_p \times Z_q) \cap U_0$, if and only if A is of the form

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where $P \in S^2\mathbb{C}^p$ and $Q \in S^2\mathbb{C}^q$.

Proof. Let φ^i denote the principal minor map on $i \times i$ matrices. Let $[x \otimes y] \in Seg(Z_p \times Z_q)$ be such that $[x] = \varphi^p([P,r])$ and $[y] = \varphi^q([Q,s])$. If $r = 0$ and $s = 0$, then $[x] = [0,\dots,0,det(P)]$ with $det(P) \neq 0$ and similarly $[y] = [0,\dots,0,det(Q)]$

with $det(Q) \neq 0$, so $[x \otimes y] = [0, \ldots, 0, det(P)det(Q)]$, But it is clear that

$$\varphi^{p+q}\left(\left[\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, 0\right]\right) = [0, \ldots, 0, det(P)det(Q)],$$

and therefore is in Z_{p+q}.

Now suppose that $s = 0$ but $r \neq 0$. Then we have $[x] = [r^p, \ldots, det(P)] = [r^n, \ldots, r^{n-p}det(P)]$ and $[y] = [0, \ldots, 0, det(Q)]$, therefore

$$[z] = [x \otimes y] = \left[det(Q)r^{n-|I|}\Delta_I(P)X^{2,\ldots,2,I}\right].$$

It suffices to assume $det(Q) = 1$. To make our computation easier, we use the group action to move to $g.[z] = [r^{n-|I|}\Delta_I(P)X^{1,\ldots,1,I}]$. Now we show that we can map to this point. Indeed,

$$\varphi^{p+q}\left(\left[\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, r\right]\right) = \left[r^{n-|I|}\Delta_I(P)X^{1,\ldots,1,I}\right].$$

Now, since $g.[z] \in Z_n$ we must also have $[z] \in Z_n$ so we are done with this case.

Now consider the case that $r \neq 0, s \neq 0$, and consider

$$[x \otimes y] = [(s^q x) \otimes (r^p y)] = \left[\left((s)^{n-|I|}\Delta_I(P)X^I\right) \otimes \left((r)^{n-|J|}\Delta_J(Q)X^J\right)\right].$$

Consider the following point:

$$[A, t] = \left[\begin{pmatrix} sP & 0 \\ 0 & rQ \end{pmatrix}, rs\right]. \tag{3.8}$$

We claim that $\varphi^{p+q}([A, t]) = [x \otimes y]$. Notice that we have the following:

$$\varphi^n([A, t]) = \left[(rs)^{n-|I|-|J|}\Delta_I(sP)\Delta_J(rQ)\ X^{I,J}\right].$$

where $[I,J]$ is an multi-index of the form $[I,J] = [i_1,\dots,i_p,j_1,\dots,j_q]$, $X^{[I,J]} = X^I \otimes X^J$. But now it is clear that

$$\left[(rs)^{n-|I|-|J|}\Delta_I(sP)\Delta_J(rQ)\ X^{I,J}\right] = \left[\left((r)^{n-|I|}\Delta_I(P)X^I\right) \otimes \left((s)^{n-|J|}\Delta_J(Q)X^J\right)\right].$$

For the second statement in the proposition, we need to show that every point in $Seg(Z_p \times Z_q) \cap U_0$ comes from a blocked matrix as in (3.8). Let $[z_I X^I] = [x \otimes y] \in Seg(Z_p \times Z_q)$ with $[x] = [x_{I_1} X^{I_1}]$ and $[y] = [y_{I_2} X^{I_2}]$. Since $[x \otimes y] \in Z_{p+q}$, let $C = (c_{i,j}) \in S^2\mathbb{C}^{p+q}$ be such that $\varphi^{p+q}([C,t]) = [x \otimes y]$.

By rescaling, we may assume that $z_{[1,\dots,1]} = x_{[1,\dots,1]} = y_{[1,\dots,1]} = 1$.

In coordinates, if $z_I X^I = x \otimes y$, then $z_{[I_1,I_2]} X^{[I_1,I_2]} = (x_{I_1} X^{I_1}) \otimes (y_{I_2} X^{I_2})$. From this, we conclude that $z_{[I_1[1,\dots,1]]} = x_{[I_1]}$ and $z_{[[1,\dots,1],I_2]} = y_{I_2}$, and therefore,

$$z_{[I_1,I_2]} = z_{I_1,[1,\dots,1]} z_{[1,\dots,1],I_2}. \tag{3.9}$$

Fix index ranges $1 \le \alpha \le p$ and $1 \le \gamma \le q$. It remains to show that $c_{\alpha,\gamma} = 0$ for each α, γ. By (3.9) with $|I_1| = |I_2| = 1$,

$$c_{\alpha,\alpha} c_{\alpha+\gamma,\alpha+\gamma} - c_{\alpha,\alpha+\gamma}^2 = c_{\alpha,\alpha} c_{\alpha+\gamma,\alpha+\gamma},$$

and therefore $c_{\alpha,\alpha+\gamma} = 0$. □

Remark III.37. By the same proof, this proposition still holds for the variety of principal minors of generic matrices.

Remark III.38. Proposition III.36 gives a useful tool in finding candidate modules for $I(Z_n)$: We are forced to consider $I(Z_n) \subset \bigcap_{p+q=n} I(Seg(Z_p \times Z_q))$.

Lemma III.39. *For Z_n and $\mathcal{V}(M)$ as above,*

$$Z_n \subset \sigma_2(Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i)),$$

and

$$\mathcal{V}(M) \subset \sigma_2(Seg(\mathcal{V}(M_i) \times \mathbb{P}V_i))$$

and moreover, in both cases, the containment is strict.

Proof. For the first part, if $\varphi([A,t]) = [z]$, then $z = z_{J_{i,1}} X^{J_{i,1}} + z_{J_{i,2}} X^{J_{i,2}}$ is such that $\varphi([A^{(i),t}]) = [z_{J_{i,1}} X^{J_{i,1}}]$ and $\varphi([Adj(A)^{(i)}, t]) = g.[z_{J_{i,2}} X^{J_{i,2}}]$, where $A^{(i)}$ is the principal submatrix of A formed by omitting the i^{th} row and column, $Adj(A)$ is the adjoint matrix, and $g \in SL(2)^{\times n}$ is the element that is the identity in the i^{th} factor and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the rest. So therefore $[z] \in \sigma_2(Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i))$.

For the second part, the Step Up Lemma III.30 applies directly and yields the result. The containment of varieties is strict since the point $X^{[1,\dots,1]} + X^{[2,\dots,2]}$ is in both secant varieties but is not on Z_n and does not vanish at the polynomial $hyp_{1,2,3} \otimes \left((x_4^1)^{(2)}(x_4^2)^{(2)}\right) \otimes \cdots \otimes \left((x_n^1)^{(2)}(x_n^2)^{(2)}\right) \in M$. □

We can restate the Characterization Lemma III.32 as follows

Lemma III.40 (Characterization Lemma again)**.** *Assume that $\mathcal{V}(M_i) = Z_{(n-1),\hat{i}}$. Then we have the following useful characterization of the zero set of M,*

$$\mathcal{V}(M) = \bigcap_{i=1}^{n} \left(Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i) \sqcup \bigcup_{L \subset Z_{(n-1),\hat{i}}} \mathbb{P}(L \otimes \mathbb{P}V_i) \right).$$

Additionally, we have the following inclusion of algebraic varieties:

$$\mathcal{V}(M) \subseteq \bigcap_{i=1}^{n} \sigma(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i).$$

Remark III.41*.* We can actually do better. Because of redundancies in the various M_i's, we made the reduction $M = \sum_{i=1}^{n} M_i \otimes S_{(d)}V_i = \sum_{i=1}^{4} M_i \otimes S_{(d)}V_i$, and on the variety side, we make the same reduction. In particular,

$$\mathcal{V}(M) \;=\; \bigcap_{i=1}^{4}\left(Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i) \sqcup \bigcup_{L\subset Z_{(n-1),\hat{i}}} \mathbb{P}(L\otimes \mathbb{P}V_i)\right).$$

For computational purposes, this will make things easier.

The following is another useful application of the Step Up Lemma (III.30).

Proposition III.42. *Suppose* $B^i \subset S^d(V_1^* \otimes \dots \widehat{V_i^*} \otimes \dots \otimes V_n^*)$ *is such that* $\mathcal{V}(B^i) = Seg(\mathbb{P}V_1 \times \dots \widehat{\mathbb{P}V_i} \times \dots \times \mathbb{P}V_n)$ *and that* $d \geq \dim(V_i)$ *for all* i*. Then* $\mathcal{V}(\bigoplus_i (B^i \otimes S^d V_i^*)) = Seg(\mathbb{P}V_1 \times \dots \times \mathbb{P}V_n)$*.*

Proof. Work by induction and use the Step Up Lemma (III.30). It is clear that $\mathcal{V}(\bigoplus_i (B^i \otimes S^d V_i^*)) \supset Seg(\mathbb{P}V_1 \times \dots \times \mathbb{P}V_n)$. All the linear spaces on $Seg(\mathbb{P}V_1 \times \dots \times \mathbb{P}V_n)$ are (up to permutation) of the form $V_1 \otimes \widehat{a_2} \otimes \dots \otimes \widehat{a_n}$, where $a_i \in V_i$ are nonzero and $\widehat{a_i}$ denotes the line through a_i. Then compute the intersection, $\bigcup_{L^i} \bigcap_{i=1}^n \mathbb{P}(L^i \otimes V_i)$, and notice that in the intersection of just 3 factors, all of the resulting linear spaces must live in $Seg(\mathbb{P}V_1 \times \dots \times \mathbb{P}V_n)$. $\square$

F. The Almost Lemma

Suppose we have a point $[z] \in \mathcal{V}(M)$ and a matrix A which satisfies $\varphi([A,t])_I = z_I$ for all $I \neq [2,\dots,2]$. In other words, we have determined that the matrix A almost maps to z in the sense that all of its principal minors except possibly for the determinant agree with the entries of z. What can we say about z?

Lemma III.43 (The Almost Lemma)**.** *Let* $n \geq 4$*. Suppose* $[z] = [z_I X^I] \in \mathcal{V}(M)$*, and* $[v_A] = [v_{A,I} X^I] = [\varphi([A,t])] \in Z_n$ *are such that* $z_I = v_{A,I}$ *for all* $I \neq [2,\dots,2]$*. If* $z_{[2,\dots,2]} \neq v_{A,[2,\dots,2]}$*, then*

$$[z] \in \bigcup_{\substack{|I_s| \leq 2 \\ 1 \leq s \leq m}} \left(Seg\left(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\right)\right) \subset Z_n.$$

Otherwise $[z] = [v_A] \in Z_n$.

Observation III.44. $Z_1 \simeq \mathbb{P}^1$ *and* $Z_2 \simeq \mathbb{P}^3$*, so Proposition III.36 implies that a point* $[A,t]$ *with* $t \neq 0$ *mapping to* $Seg\left(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\right)$ *with* $|I_s| \leq 2$ *for each* s *is permutation equivalent to a block diagonal matrix consisting of* 1×1 *and* 2×2 *blocks. Moreover, such a block diagonal matrix is a special case of a symmetric tri-diagonal matrix, and therefore none of its principal minors depend on the sign of the off diagonal terms.*

In what follows, we will show that if $v_{A,I} = z_I$ for all $I \neq [2,\dots,2]$ and $z_{[2,\dots,2]} \neq v_{A,[2,\dots,2]}$, then z is a zero of an auxiliary set of polynomials denoted B. We will then show that the zero set $\mathcal{V}(B)$ is contained in the union of Segre varieties. Finally, Proposition III.36 provides the inclusion into Z_n.

1. Reduction to one variable

Suppose $[v_A]$ and $[z]$ are as above. Both points are zeros of every polynomial in M, but the only coordinate in which they can differ is $[2,\dots,2]$. Now consider the coordinates z_I ($= v_{A,I}$) as fixed constants $\forall I \neq [2,\dots,2]$, and for $f \in M$ define f_z by the substitution $f(X^{[1,\dots,1]},\dots,X^{[2,\dots,2]}) \mapsto f(z_{[1,\dots,1]},\dots,z_{[1,2,\dots,2]},X^{[2,\dots,2]}) =: f_z(X^{[2,\dots 2]})$. Let $M_{[2,\dots,2]}(z) = \{f_z \mid f \in M\}$ denote the resulting set of univariate polynomials. Then $z_{[2,\dots,2]}$ and $v_{A,[2\dots,2]}$ are two (possibly different) roots of each univariate polynomial $f_z \in M_{[2,\dots,2]}(z)$.

Lemma III.45. *If* $f \in M$*, then the corresponding polynomial* $f_z \in M_{[2,\dots,2]}(z)$ *is either degree 0, 1, or 2 in* $X^{[2,\dots,2]}$.

Proof. It is sufficient to prove the lemma on a weight basis for M. In particular these polynomials have the property that all of their terms have the same weight. The weight of a monomial $X^I \dots X^J$ is a vector which is computed the following procedure (standard in representation theory). Take the index vectors $I, \dots, J$ and replace all the 1's with -1's and all the 2's with $+1$'s. The weight is the sum of the modified index vectors.

We recognize that $S_{(2,2)}S_{(2,2)}S_{(2,2)}$ is 1-dimensional, has weight zero, degree 4, and in particular, it is spanned by the hyperdeterminant, *hyp*. It is easy to see that, *hyp*, is a quadratic in $X^{[2,2,2]}$: $(X^{[2,2,2]})^2$ has weight $[2,2,2]$, and the only way to raise this to $[0,0,0]$ is to multiply by $(X^{[1,1,1]})^2$. We cannot have anything of lower weight because we will not be able to raise its weight back up to $[0,0,0]$ and still be degree 4.

Consider the module $S_{(2,2)}V_1^*\otimes S_{(2,2)}V_2^*\otimes S_{(2,2)}V_3^*\otimes S_{(4)}V_4^*\otimes\cdots\otimes S_{(4)}V_n^*$. A lowest weight vector in this module is constructed by taking the weight $[0,0,0]$ vector which spans $S_{(2,2)}V_1^*\otimes S_{(2,2)}V_2^*\otimes S_{(2,2)}V_3^*$, (*i.e.* $hyp_{1,2,3}$) and tensoring with $(x_4^2)^4\otimes\cdots\otimes(x_n^2)^4$ - the lowest weight vector for $S_{(4)}V_4^*\otimes\cdots\otimes S_{(4)}V_n^*$. The leading term is

$$\begin{aligned}(x_1^1\otimes x_2^1\otimes x_3^1)^2(x_1^2\otimes x_2^2\otimes x_3^2)^2\otimes(x_4^2)^4\otimes\cdots\otimes(x_n^2)^4\\ = (X^{[1,1,1,2,\dots,2]})^2(X^{[2,\dots,2]})^2.\end{aligned}$$

There cannot be any higher power of $X^{[2,\dots,2]}$ occurring in a polynomial in M, otherwise, the vector would have a weight that is lower than the lowest weight. $\square$

Now we know that $v_{A,[2,\dots,2]}$ and $z_{[2,\dots,2]}$ are both common zeros of univariate polynomials, all with degree 2 or less. A quadratic (not identically zero) in one variable has at most two solutions, and a linear polynomial (not identically zero) has at most one solution. The only way then for us to have $v_A \neq z$ and $v_A, z \in \mathcal{V}(M)$ is if *all* of the linear polynomials were identically zero and if *all* of the quadratics were scalar multiples of each other.

Therefore, we need to study the points $[z] \in \mathcal{V}(M)$ for which $f_z = \lambda_{f,g} g_z \ \forall f, g \in M_{[2,\ldots,2]}(z)$, for some $\lambda_{f,g} \in \mathbb{C}$. Define polynomials a_f,b_f, and c_f (which necessarily do not depend on $X^{[2,\ldots,2]}$) for each $f_z \in M_{[2,\ldots,2]}(z)$ by

$$f_z = a_f(z) \left(X^{[2,\ldots,2]}\right)^2 + b_f(z) \left(X^{[2,\ldots,2]}\right) + c_f(z).$$

The condition $f_z = \lambda_{f,g} g_z \ \forall f, g \in M_{[2,\ldots,2]}(z)$ is described (without reference to λ) by the polynomials $B' := span\{a_f(z)b_g(z) - a_g(z)b_f(z) \mid f, g \in M\}$.

Notice that B' is not G-invariant. Let $B = \langle G.B' \rangle$ denote the corresponding G-module.

The polynomials in B' have the property that if $h(z) \neq 0$ for some $h \in B'$, *i.e.* $[z] \notin \mathcal{V}(B')$, then the polynomials in $M_{[2,\ldots,2]}(z)$ must have a single common root, and therefore $v_{[2,\ldots,2]} = z_{[2,\ldots,2]}$. If, however $h(z) = 0 \ \forall h \in B'$ (*i.e.* $z \in \mathcal{V}(B')$), then it is possible that the polynomials in $M_{[2,\ldots,2]}(z)$ have 2 common roots.

Since $\mathcal{V}(M)$ and Z_n are G-varieties, $[z] \in \mathcal{V}(M)$ implies that $\overline{G.[z]} \subset \mathcal{V}(M)$, and similarly for Z_n. If $g.[z] \notin \mathcal{V}(B')$, then by our remarks above, $g.[z] \in Z_n$, and in particular, $[z] \in Z_n$.

So consider the case that $G.[z] \subset \mathcal{V}(B')$. This implies that $[z] \in \mathcal{V}(B)$. So, we need to look at the variety $\mathcal{V}(B)$. If $[z] \notin \mathcal{V}(B)$, then $[z] \in Z_n$. If $[z] \in \mathcal{V}(B)$ then we claim that (independent of v_A) $[z] \in Z_n$, and we will show this by the following:

Proposition III.46. *Let B be as above. Then*

$$\mathcal{V}(B) \subset \bigcup_{\substack{|I_s| \leq 2 \\ 1 \leq s \leq m}} Seg\left(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\right) \subset Z_n.$$

This proposition will be proved in the following sequence of lemmas.

1. $S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)}\dots S_{(5)} \subset B$.

2. $\mathcal{V}(S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)}\dots S_{(5)}) = \bigcup_{\substack{|I_s|\leq 2 \\ 1\leq s\leq m}} Seg\left(\mathbb{P}V_{I_1}\times\cdots\times\mathbb{P}V_{I_m}\right).$

3. $Seg\left(\mathbb{P}V_{I_1}\times\cdots\times\mathbb{P}V_{I_m}\right)\subset Z_n$ is by Proposition III.36.

Lemma III.47. *We have the following inclusion,* $S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)}\dots S_{(5)} \subset B$.

Suppose we can write down a polynomial h in the G-module B. Since B is a $\mathfrak{g}$-module, the following algorithm is a standard idea in representation theory and can be used to find submodules of B.

Input: $h \in B$.

Step 0**.** Choose an ordered basis of lowering operators $\mathfrak{g}_- = \{\alpha_1,\dots,\alpha_n\}$.

Step 1**.** Find the largest integer $k_1 \geq 0$ so that $\alpha_1^{k_1}.h \neq 0$, and let $h^{(1)} = \alpha_1^{k_1}.h$.

Step 2**.** Find the largest integer $k_2 \geq 0$ so that $\alpha_2^{k_2}.h^{(1)} \neq 0$, and let $h^{(2)} = \alpha_2^{k_2}.h^{(1)}$.

Step n**.** Find the largest integer $k_n \geq 0$ so that $\alpha_n^{k_n}.h^{(n-1)} \neq 0$.,and let $h^{(n)} = \alpha_n^{k_n}.h^{(n-1)}$.

Output: The vector $h^{(n)}$ is a lowest weight vector in B and $span\{G.h^{(n)}\}$ is a submodule of B.

Proof. We will carry out the steps in the algorithm given above. For this proof, we introduce some new notation. If i_1, i_2, i_3 are fixed, let $X^{I_{p,q,r}}$ denote the coordinate vector with $i_1 = p, i_2 = q, i_3 = r$ and $i_k = 2$ for $k \geq 4$.

Suppose $f_{[i_1,i_2,i_3]} \in S_{(2,2)}V_{i_1}^* \otimes S_{(2,2)}V_{i_2}^* \otimes S_{(2,2)}V_{i_3}^* \otimes S_{(4)}V_{i_4}^* \otimes\cdots\otimes S_{(4)}V_{i_n}^*$ is a lowest weight vector. Define $a_{[i_1,i_2,i_3]}$, $b_{[i_1,i_2,i_3]}$, $c_{[i_1,i_2,i_3]}$ by the equation $f_{[i_1,i_2,i_3]} = a_{[i_1,i_2,i_3]}(X^{[2,\dots,2]})^2 + b_{[i_1,i_2,i_3]}(X^{[2,\dots,2]}) + c_{[i_1,i_2,i_3]}$.

Since $f_{[i_1,i_2,i_3]}$ is a hyperdeterminant of format $2\times 2\times 2$, $a_I = (X^{I_{1,1,1}})^2$ and

$$b_{[i_1,i_2,i_3]} = X^{I_{1,1,1}}\left(X^{I_{2,1,1}}X^{I_{1,2,2}} + X^{I_{1,2,1}}X^{I_{2,1,2}} + X^{I_{1,1,2}}X^{I_{2,2,1}}\right) - 2X^{I_{2,1,1}}X^{I_{1,2,1}}X^{I_{1,1,2}}.$$

The weight of $a_{[i_1,i_2,i_3]}$ is (up to permutation) $[-2,-2,-2,2,\dots,2]$, where the -2's actually occur at $\{i_1,i_2,i_3\}$. The weight of $b_{[i_1,i_2,i_3]}$ is (up to permutation) $[-1,-1,-1,3,\dots,3]$, where the -1's actually occur at $\{i_1,i_2,i_3\}$. Now consider

$$h_{[i_1,i_2,i_3],[j_1,j_2,j_3]} = a_{[i_1,i_2,i_3]}b_{[j_1,j_2,j_3]} - a_{[j_1,j_2,j_3]}b_{[i_1,i_2,i_3]} \in B.$$

We notice that $h_{[i_1,i_2,i_3],[j_1,j_2,j_3]}$ can have 3 different (up to permutation) weights, depending on how $[i_1,i_2,i_3]$ and $[j_1,j_2,j_3]$ match up. The three possible weights of $h_{[i_1,i_2,i_3],[j_1,j_2,j_3]}$ are (up to permutation): $[-3,-3,1,1,5,\dots,5]$, $[-3,1,1,1,1,5,\dots,5]$, or $[1,1,1,1,1,1,5,\dots,5]$.

In each case, apply the algorithm above. The output in each case is a vector of weight (up to permutation) $[3,3,3,5,\dots,5]$. The fact that we are dealing with the tensor product of $SL(2)$ modules implies that the module with lowest weight $[3,3,3,5,\dots,5]$ is $S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)}\dots S_{(5)}$ and this must be a submodule of B. $\square$

Lemma III.48. *We have the following equality of sets*

$$\begin{aligned}&\mathcal{V}\left(S_{(4,1)}V_1^* \otimes S_{(4,1)}V_2^* \otimes S_{(4,1)}V_3^*\right)\\ &= Seg(\mathbb{P}(V_1\otimes V_2)\times \mathbb{P}V_3) \cup Seg(\mathbb{P}(V_1\otimes V_3)\times \mathbb{P}V_2) \cup Seg(\mathbb{P}(V_2\otimes V_3)\times \mathbb{P}V_1).\end{aligned}$$

Proof. The space $V_1^*\otimes V_2^*\otimes V_3^*$ has a finite number of orbits under the action of $SL(2)^{\times 3}$. We list these orbits and normal forms below.

- $Seg(\mathbb{P}V_1\times\mathbb{P}V_2\times\mathbb{P}V_3)$: Normal form $[x]=[a\otimes b\otimes c]$.
- $\tau(Seg(\mathbb{P}V_1\times\mathbb{P}V_2\times\mathbb{P}V_3))_{sing} = \mathfrak{S}_3.Seg(\mathbb{P}(V_1\otimes V_2)\times\mathbb{P}V_3)$: Normal form (up to permutation) $[x]=[a\otimes b\otimes c + a'\otimes b'\otimes c]$. This orbit is called the *singular orbit.*
- $\tau(Seg(\mathbb{P}V_1\times\mathbb{P}V_2\times\mathbb{P}V_3))$: Normal form $[x]=[a\otimes b\otimes c + a'\otimes b\otimes c + a\otimes b'\otimes c + a\otimes b\otimes c']$.
- $\sigma(Seg(\mathbb{P}V_1\times\mathbb{P}V_2\times\mathbb{P}V_3))$: Normal form $[x]=[a\otimes b\otimes c + a'\otimes b'\otimes c']$.

The lowest weight vector for $S_{(4,1)}S_{(4,1)}S_{(4,1)}$ is

$$f = (X^{[2,2,2]})^2\big(X^{[1,1,1]}(X^{[2,2,2]})^2 + 2X^{[2,1,2]}X^{[1,2,2]}X^{[2,2,1]} \\ - X^{[2,2,2]}(X^{[1,2,2]}X^{[2,1,1]} + X^{[2,1,2]}X^{[1,2,1]} + X^{[2,2,1]}X^{[1,1,2]})\big).$$

Since the orbits are nested, consider a generic point the singular obit. We find that $f(x) = 0$ for every $x \in \tau(Seg(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))_{sing}$. So therefore $\tau(Seg(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))_{sing} \subset \mathcal{V}(S_{(4,1)}S_{(4,1)}S_{(4,1)})$.

Next, we show that the other two orbits are not in $\mathcal{V}(S_{(4,1)}S_{(4,1)}S_{(4,1)})$. The orbits are nested, so consider the point $[x] = \big[X^{[2,2,2]} + X^{[1,2,2]} + X^{[2,1,2]} + X^{[2,2,1]}\big] \in \tau(Seg(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))$. But $f(x) = 2 \neq 0$, so the other two orbits are not in $\mathcal{V}(S_{(4,1)}S_{(4,1)}S_{(4,1)})$. Since we have considered all possible orbits, we are done. □

Notation III.49. *Let* $V_I = V_{i_1} \otimes \cdots \otimes V_{i_{|I|}}$ *and let* $\widehat{v_I} \in V_I$ *denote the line through* $v_{i_1} \otimes \cdots \otimes v_{i_{|I|}}$. *If* π *is a partition, let* $S_{[\pi]}V_I = S_\pi V_{i_1} \otimes \cdots \otimes S_\pi V_{i_{|I|}}$. *Note: This is* not *the same as* $S_\pi(V_{i_1} \otimes \cdots \otimes V_{i_{|I|}})$.

Observation III.50. *All the linear spaces on* $\bigcup_{|I|\leq 2,\, |J|\leq 2,\, |I|+|J|=3} Seg(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m})$, *are (up to permutation) of the form* $V_{I_1} \otimes \widehat{v_{I_2}} \otimes \cdots \otimes \widehat{v_{I_m}}$.

Let

$$\tilde{B} = \bigoplus_{|I|=n} \left(S_{[(4,1)]}V^*_{\{i_1,i_2,i_3\}} \otimes S_{[(5)]}V^*_{I\setminus\{i_1,i_2,i_3\}}\right),$$

and let

$$\tilde{B}_k = \bigoplus_{|I|=n-1,\, k\notin I} \left(S_{[(4,1)]}V^*_{\{i_1,i_2,i_3\}} \otimes S_{[(5)]}V^*_{I\setminus\{i_1,i_2,i_3\}}\right).$$

Notice that $\tilde{B} = \sum_{k=1}^n \tilde{B}_k \otimes S_{(5)}V^*_k$.

Lemma III.51. *Suppose*

$$\mathcal{V}\left(\tilde{B}_k\right) = \bigcup_{\substack{|I_s| \leq 2,\ k \notin I_s \\ \sum_s |I_s| = n-1}} Seg\left(\mathbb{P}V_{I_1} \times \mathbb{P}V_{I_2} \times \cdots \times \mathbb{P}V_{I_m}\right).$$

Then

$$\mathcal{V}\left(\tilde{B}_k \otimes S_{(5)}V_n^*\right) = \bigcup_{\substack{|I_s| \leq 2,\ k \notin I_s \\ \sum_s |I_s| = n-1}} Seg\left(\mathbb{P}V_{I_1\cup\{k\}} \times \mathbb{P}V_{I_2} \times \cdots \times \mathbb{P}V_{I_m}\right). \tag{3.10}$$

Proof. Apply the Step Up Lemma III.30 to the left hand side of (3.10). It remains to check that

$$\bigcup_{L\subset\mathcal{V}(\tilde{B}_k)} \mathbb{P}(L \otimes V_k) = \bigcup_{\substack{|I_s| \leq 2,\ k \notin I_s \\ \sum_s |I_s| = n-1}} Seg\left(\mathbb{P}V_{I_1\cup\{k\}} \times \mathbb{P}V_{I_2} \times \cdots \times \mathbb{P}V_{I_m}\right),$$

where $L \subset \mathcal{V}(\tilde{B}_k)$ are linear spaces. Because of symmetry, there is only one type of linear space to consider, $V_{I_1} \otimes \widehat{v_{I_2}} \otimes \cdots \otimes \widehat{v_{I_m}} \otimes V_k = V_{I_1\cup\{k\}} \otimes \widehat{v_{I_2}} \otimes \cdots \otimes \widehat{v_{I_m}}$. It is clear that each of these linear spaces is on one of the Segre varieties on the right hand side of (3.10), and moreover every point on the right hand side of (3.10) is on one of these linear spaces. $\square$

Proposition III.52.

$$\mathcal{V}\left(\tilde{B}\right) = \bigcup_{\substack{|I_s| \leq 2 \\ \sum_s |I_s| = n}} Seg\left(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\right).$$

Proof. Proof by induction. The base case is Lemma III.48. For the induction step, use Lemma III.51. We need to show that

$$\bigcap_{k=1}^{n} \bigcup_{\substack{|I_s| \leq 2,\ k \notin I_s \\ \sum_s |I_s| = n-1}} Seg\left(\mathbb{P}V_{I_1 \cup \{k\}} \times \mathbb{P}V_{I_2} \times \cdots \times \mathbb{P}V_{I_m}\right)$$
$$= \bigcup_{\substack{|I_s| \leq 2 \\ \sum_s |I_s| = n}} Seg\left(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\right).$$

It suffices to check that

$$Seg\left(\mathbb{P}V_{\{i_1,i_2,i_3\}} \times \mathbb{P}V_{i_3} \times \mathbb{P}V_{I_4} \times \cdots \times \mathbb{P}V_{I_m}\right)$$
$$\cap\, Seg\left(\mathbb{P}V_{\{i_1,i_2,i_4\}} \times \mathbb{P}V_{i_3} \times \mathbb{P}V_{I_5} \times \cdots \times \mathbb{P}V_{I_m}\right)$$
$$= Seg\left(\mathbb{P}V_{\{i_1,i_2\}} \times \mathbb{P}V_{i_3} \times \mathbb{P}V_{i_4} \times \mathbb{P}V_{I_5} \times \cdots \times \mathbb{P}V_{I_m}\right).$$

This can be done by writing a point $[p]$ in the first Segre variety in coordinates and then requiring the 2×2 minors in the ideal of the second Segre variety to vanish on $[p]$. $\square$

G. Proof of Theorem III.3

The outline of proof is as follows. Assume for induction that $\mathcal{V}(M_i) = Z_{(n-1),\hat{i}}$. In the cases of $n = 3, 4$, the ideal theoretic version of the theorem was proved with the aid of a computer in [14]. Recall that the Characterization Lemma (III.32) says that

$$\mathcal{V}(M) = \bigcap_{i=1}^{n} \left(Seg(Z_{(n-1),i} \times \mathbb{P}V_i) \bigcup \left(\bigcup_{L \subset Z_{(n-1),\hat{i}}} \mathbb{P}(L \otimes \mathbb{P}V_i) \right) \right).$$

Also, $Seg(Z_{(n-1),i} \times \mathbb{P}V_i) \subset \bigcup_{L \subset Z_{(n-1),\hat{i}}} \mathbb{P}(L \otimes \mathbb{P}V_i)$, therefore

$$\mathcal{V}(M) = \bigcap_{i=1}^{n} \bigcup_{L \subset Z_{(n-1),\hat{i}}} \mathbb{P}(L \otimes \mathbb{P}V_i). \tag{3.11}$$

Lemma III.21 says that $Z_n \subseteq \mathcal{V}(M)$. To show containment in the other direction, we will take any point $z \in \mathcal{V}(M)$ and use the characterization (3.11) to show that the restrictions placed on $[z]$ force it to live in Z_n. We will do this by constructing a matrix A so that $[A, t] \mapsto [z]$ or by using the Almost Lemma III.43 to conclude that $[z] \in Z_n$.

Suppose we take a point in the zero set

$$[z] \in \bigcap_{i=1}^{n} \bigcup_{L^i \subset Z_{(n-1),i}} \mathbb{P}(L^i \otimes V_i) = \mathcal{V}(M).$$

Now since z is fixed, we can also fix a single L^i for each i. So, now we have $[z] \in \bigcap_{i=1}^{n} \mathbb{P}(L^i \otimes V_i)$. For each i, we know that $[z] \in \mathbb{P}(L^i \otimes V_i)$, so after choosing a basis work in the cone over projective space write

$$\begin{aligned} z = z_J X^J &= \left(z_{J_{i,1}} X^{J_{i,1}}\right) + \left(z_{J_{i,2}} X^{J_{i,2}}\right) \\ &= \left(z_{J_{i,1}} X^{J_i}\right) \otimes x_i^1 + \left(z_{J_{i,2}} X^{J_i}\right) \otimes x_i^2 \\ &= \eta^i \otimes x_i^1 + \nu^i \otimes x_i^2, \end{aligned}$$

where $\eta^i := \eta^i_{J_i} X^{J_i} = z_{J_{i,1}} X^{J_i}$, and $\nu^i := \nu^i_{J_i} X^{J_i} = z_{J_{i,2}} X^{J_i}$, and $\eta^i, \nu^i \in L^i$. If either of η^i, ν^i is zero, then z is in a Segre variety, and this is in Z_n. We consider the case that neither η^i, ν^i is zero for each i. We now have n different expressions for our point z:

$$\begin{aligned} z &= \eta^1 \otimes x_1^1 + \nu^1 \otimes x_1^2 \\ &\vdots \\ z &= \eta^n \otimes x_n^1 + \nu^n \otimes x_n^2. \end{aligned}$$

Next, we compare the n different expressions for z in coordinates, one coordinate at a time. This gives $n2^n$ relations on the entries of z in terms of the entries of η^i and

ν^i. Our induction hypothesis says that $Z_{(n-1),i} = \mathcal{V}(M_i)$ for $1 \leq i \leq n$. In particular, η^i and ν^i have $(n-1) \times (n-1)$ matrices that map to them. Let $[A^{(i)}, t^{(i)}]$ be such that $\varphi([A^{(i)}, t^{(i)}]) = \eta^i$, where the parenthetical superscript on $t^{(i)}$ or $A^{(i)}$ indicates an index, not a power.

1. Building a matrix

Now we attempt to build a matrix A so that it agrees with all of the information we have. We consider the restrictions forced on us by the η's.

Consider the $[1, \dots, 1]$ coordinate of the η^i's. Our restrictions imply that $z_{[1,\dots,1]} = \eta^i_{[1,\dots,1]} = (t^{(i)})^n$ for each i. So, $t^{(i)}$ all agree up to a factor of a root of unity. But, with out loss of generality, we can just assume that they are all equal. This is because if not, then we can just re-scale each individual η^i. (We are allowed to do this because φ is a well-defined homogeneous degree n map on projective space, so $\varphi([\lambda A, \lambda t]) = [\lambda^n z_I X^I] = [z_I X^I] = \varphi([A, t])$.) So, we take $t^{(i)} = t$ for all i. We might need to consider two cases, depending on whether $t = 0$; however, we notice that without loss of generality, we may assume that $z_{[1,\dots,1]} \neq 0$. This is because of the following lemma:

Lemma III.53. *Let* $U_0 = \{[z] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \mid z = z_I X^I \in V_1 \otimes \cdots \otimes V_n, z_{[1,\dots,1]} \neq 0\}$. *Then* $\mathcal{V}(M) \cap U_0 \subset Z_n$ *implies that* $\mathcal{V}(M) \subset Z_n$.

Proof. Since Z_n and $\mathcal{V}(M)$ are G-varieties, and $G.U_0 = \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$ the result follows. $\square$

Therefore, by replacing $[z]$ with $g.[z]$ if necessary, we may assume that $z_{[1,\dots,1]} \neq 0$. The assumption $z_{[1,\dots,1]} \neq 0$ implies that $t \neq 0$ and if A, and A' are $n \times n$ matrices, $t^{n-|I|}\Delta_I(A) = t^{n-|I|}\Delta_I(A')$ then $\Delta_I(A) = \Delta_I(A')$. So it is no loss to set $t = 1$.

Next, we are working to build a matrix A so that $\varphi([A,1]) = z$. We want A to have the property that its principal submatrices are actually the matrices $A^{(1)}, \ldots, A^{(n)}$ from above. If we can do this, we will have determined that all of the principal minors of A, except possibly the determinant, match z, and by the Almost Lemma III.43, we will be done. However, we do not yet know if our choices of $A^{(i)}$ are consistent with each other. Let $(A^{(i)})^{(j)}$ be the submatrix of $A^{(i)}$ obtained by deleting the j^{th} row and column. The question of consistency comes down to the following question. If we have already chosen a matrix $A^{(1)}$, is it possible to choose a matrix $A^{(2)}$ so that it satisfies the two properties; that it maps to η^2, and $(A^{(2)})^{(1)} = (A^{(1)})^{(1)}$?

Define a candidate matrix,

$$A(x) = \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} \\ x_{1,2} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & a_{2,n} & \ldots & a_{n,n}, \end{pmatrix},$$

where the $x_{i,j}$ are indeterminants and the principal submatrix formed by omitting the first row and column of $A(x)$ is $A^{(1)}$. $A(x)$ is now a candidate for a matrix that will satisfy $\varphi([A,1]) = [z]$. We must have $z_{[2,1,\ldots,1]} = \Delta_{[2,1\ldots,1]}(A(x)) = x_{1,1}$, therefore we fix $x_{1,1} = a_{1,1}$. Also, the equations on the 2×2 minors, $x_{1,i}^2 - a_{1,1}a_{i,i} = z_{[2,1\ldots1,2,1,\ldots,1]}$, determine $x_{1,i}$ up to sign.

Throughout what follows, the term "works" will mean that the matrix in question has all of its principal minors matching the appropriate entries of z. So, of the 2^{n-1} choices of combinations of signs, we want to know if there is one choice that will work for all of the principal minors. Also, in light of the Almost Lemma III.43, it suffices to prove that all of the principal minors smaller than the determinant work. Our question then becomes the following:

Proposition III.54. *Suppose* $[z] \in \mathcal{V}(M)$ *and* $z_{[1,\dots,1]} = 1$. *Suppose* $A(x)$ *as above such that* $\Delta_{[1,i_2,i_3,\dots,i_n]}(A(x)) = z_{[1,i_2,i_3,\dots,i_n]}$. *Then either there exists a choice of* $y = (x_{1,2},\dots,x_{1,n})$ *so that* $\Delta_I(A(x)) = z_I$ *for all* I *and hence,* $z \in Z_n$, *or* $[z] \in Seg(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}) \subset Z_n$ *with* $|I_s| \leq 2$ *and* $\sum_s |I_s| = n$.

We work by induction. Suppose $A(x)^{(2)}$ is such that

$$\Delta_{[1,i_3,\dots,i_n]}(A(x)^{(2)}) = z_{[1,1,i_3,\dots,i_n]}.$$

There are two cases to consider. Case 1, there is a choice in $y_2 = (x_{1,2},\dots,x_{1,n})$ so that $\Delta_{I_{2,1}}(A(x))^{(2)} = z_{I_{2,1}}$ for all $I_{2,1}$. Case 2, $[z_{I_{2,1}} X^{I_2}] \in Seg(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m})$.

In the second case, we apply Proposition III.36 to conclude that there is a matrix that is permutation equivalent to a block diagonal matrix (with only 1×1 and 2×2 blocks) that maps to $[z_{I_{2,1}} X^{I_2}]$. If this happens, we start over by insisting that $A^{(2)}(x)$ is of this form and because of this, none of its principal minors depend on the sign of the off diagonal entries (see Observation III.44).

In either case, we are still left to choose $x_{1,2}$. We do not know if there is single choice of $x_{1,2}$ that works for all principal minors. Proposition III.54 will follow from the following proposition.

Proposition III.55. *Suppose* $[z] \in \mathcal{V}(M)$. *Work on the set where* $z_{[1,\dots,1]} = 1$. *Let*

$$A(x_{1,2}) = \begin{pmatrix} a_{1,1} & x_{1,2} & a_{1,3} & \dots & a_{1,n} \\ x_{1,2} & a_{2,2} & \dots & \dots & a_{2,n} \\ a_{1,3} & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \dots & & a_{n,n} \end{pmatrix},$$

with $a_{i,j}$ *fixed and* $x_{1,2}$ *variable. Assume that* $\Delta_I(A(x_{1,2})) = z_I$ *for* $I = [1, i_2, \dots, i_n]$ *or* $I = [i_1, 1, i_3, \dots, i_n]$ - i.e. *all principal minors not involving* $x_{1,2}$. *Then either there exists a choice of* $x_{1,2}$ *so that* $\Delta_I(A(x)) = z_I$ *for all* I *and hence,* $[z] \in Z_n$, *or*

$[z] \in Seg(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}) \subset Z_n$ *with* $|I_s| \leq 2$ *and* $\sum_s |I_s| = n$.

We will use the following observation often in the course of the proof:

Observation III.56. *Suppose A is an $n \times n$ matrix. The set of all of the principal minors of all principal submatrices of size $n-1 \times n-1$ is equal to the set of all of the principal minors of A except the determinant.*

Proof of Proposition III.55. We will see how to step up from 3×3 to 4×4, and then we will do the general case. Our hypotheses are that $n = 4$, $[z] \in \mathcal{V}(M)$ and that $A(x_{1,2})$ is a 4×4 matrix depending on $x_{1,2}$ with the property that the principal minors of $A(x_{1,2})$ which do not involve $x_{1,2}$ agree with z, *i.e.* $\varphi([A^{(1)}, 1]) = z_{I_1} X^{I_1}$ and $\varphi([A^{(2)}, 1]) = z_{I_2} X^{I_2}$. Now consider $A^{(3)}(x_{1,2})$ and $A^{(4)}(x_{1,2})$. We know, by induction, that one choice of $\pm p = x_{1,2}$ will work so that $\varphi([A^{(3)}, 1] \otimes (x_{1,2})) = z_{I_3} X^{I_3}$. Similarly, one choice of $\pm p = x_{1,2}$ will work for $A^{(4)}(x_{1,2})$. If the same choice works for both, then we will have determined that $A(p)$ works for the principal submatrices $A^{(1)}, \ldots, A^{(4)}$. By Observation (III.56), we know that all the principal minors of $A(x)$ work except possibly the determinant. So, by the Almost Lemma III.43 we are done.

Now suppose that $p = x_{1,2}$ works for $A^{(3)}(x_{1,2})$ and $-p = x_{1,2}$ works for $A^{(4)}(x_{1,2})$. Then by our construction, the principal minors $A^{(1)}, A^{(2)}$, and $A^{(3)}(+p)$ actually work for z. The submatrices $A^{(1,4)}, A^{(2,4)}, A^{(3,4)}$, are submatrices $A^{(1)}, A^{(2)}$, and $A^{(3)}(+p)$, so they work for $+p$, but they are also submatrices of $A^{(4)}$, so they must also work for $-p$. Therefore we have determined that all of the principal minors of $A^{(4)}$ except for $det(A^{(4)})$ actually work for $+p$ (by Observation III.56).

Now if $det(A^{(4)}(p)) \neq z_{[2,2,2,1]}$, then by the Almost Lemma III.43,

$$\eta^4 = z_{[i_1,i_2,i_3,1]} X^{[i_1,i_2,i_3,1]} \in Seg\left(\mathbb{P}(V_{j_1} \otimes V_{j_2}) \times \mathbb{P}V_{j_3} \times \mathbb{P}\{x_4^1\}\right),$$

for some $\{j_1, j_2, j_3\} = \{1, 2, 3\}$. But Proposition III.36 says that any matrix mapping to η^4 is permutation equivalent to a block diagonal matrix, and hence by Observation III.44 the principal minors of $A^{(4)}(p)$ do not depend on the sign of the off diagonal entries. In particular, we must have $det(A^{(4)}(p)) = det(A^{(4)}(-p))$, but we assumed that $det(A^{(4)}(p)) \neq z_{[2,2,2,1]}$ and $det(A^{(4)}(-p)) = z_{[2,2,2,1]}$, a contradiction. So $A^{(4)}(x_{1,2})$ must work for both signs and we are back to the previous case and we are done with the case $n = 4$.

This motivates the following lemmas.

Lemma III.57. *Suppose $[z] \in \mathcal{V}(M)$ and $A(x_{1,2})$ are as Proposition III.55. Suppose that we have found that the matrices $A^{(i)}(x_{1,2})$ all work for the same choice in sign of $x_{1,2}$. Then either $\varphi([A(x_{1,2}), 1]) = [z] \in Z_n$, or there is a matrix A such that $\varphi([A, 1]) = [z] \in Z_n$ and every such matrix A has the property that none of the principal minors of A depend on the sign of its off diagonal entries.*

Proof. In light of Observation III.56 we see that the hypotheses of the lemma imply that $\Delta_I(A(x_{1,2})) = z_I$ for all $I \neq [2, \dots, 2]$, so we may apply the Almost Lemma III.43 to conclude that either $\varphi([A(x_{1,2}), 1]) = [z]$ or $[z] \in Seg(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}) \subset Z_n$ with $|I_s| \leq 2$ and $\sum_s |I_s| = n$. In the latter case, we use Proposition III.36 and Observation III.44 to conclude. $\square$

Lemma III.58. *Suppose $[z] \in \mathcal{V}(M)$ and $A(x_{1,2})$ are as Proposition III.55 and that $\varphi([A(x_{1,2}), 1]) = [z]$. If all of the principal submatrices $A^{(i)}(x_{1,2})$ work for both signs $x_{1,2} = \pm p$, then $A(x_{1,2})$ also works for both signs.*

Proof. Suppose the determinant of $A(x_{1,2})$ matches $[z]$ for $x_{1,2} = +p$ but it is unknown whether $x_{1,2} = -p$ also works. Then since all of the principal submatrices $A^{(i)}(x_{1,2})$ work for $x_{1,2} = -p$, apply the Lemma III.57 to conclude that every matrix mapping to $[z]$ (and in particular $A(+p)$) must have the property that none of the principal minors of A depend on the sign of its off diagonal entries, so $x_{1,2} = -p$ must also work. $\square$

For the general case, suppose we know the proposition for $n-1$. More specifically, this will say that each of $A^{(i)}(x_{1,2})$ work for at least one choice of $x_{1,2} = \pm p$. In our construction, we will have matrices $A^{(1)}$ and $A^{(2)}$ independent of $x_{1,2} = \pm p$, and (possibly after a permutation), choose matrices $A^{(3)}, \ldots, A^{(k)}$ which work for $x_{1,2} = +p$ and matrices $A^{(k+1)}, \ldots, A^{(n)}$ which do not work for $x_{1,2} = +p$. We could do the same construction with $-p$ replaced with $+p$. If either construction ends up with all of the matrices $A^{(i)}$ working for the same sign, then apply Lemma III.57 to conclude.

For the sake of contradiction, suppose $3 \leq k < n$, that $A^{(3)}, \ldots, A^{(k)}$ work for $x_{1,2} = +p$ and that none of the matrices $A^{(k+1)}, \ldots, A^{(n)}$ work for $x_{1,2} = +p$. Consider $A^{(k+1,\ldots,n)}$ - the matrix formed from $A(x_{1,2})$ by omitting the rows and columns labeled $k+1, \ldots, n$. Each of the matrices $A^{(i,k+1,\ldots,n)}$ for $1 \leq i \leq k$, must work for both signs. This is because $A^{(i,k+1,\ldots,n)}$ is a submatrix of $A^{(n)}$ which is assumed to work for $-p$, and it is a submatrix of $A^{(i)}$ which is assumed to work for $+p$. By allowing i to vary, this determines that all of the principal minors of $A^{(k+1,\ldots,n)}$ work for both signs except possibly for the determinant. Now apply Lemma III.58 to conclude that the determinant of $A^{(k+1,\ldots,n)}$ also works for both signs.

Next, consider $A^{(k+1,\ldots,\hat{s},\ldots,n)}$, the submatrix of $A(x_{1,2})$ obtained by omitting the rows and columns $k+1, \ldots, n$, but not the row and column s, with $k+1 \geq s \geq n$. Again, we know that for each i, $A^{(i,k+1,\ldots,\hat{s},\ldots,n)}$ must work for both signs, and we have assumed that $A^{(k+1,\ldots,n)}$ works for both signs. So can apply Lemma III.58 to conclude that $A^{(i,k+1,\ldots,\hat{s},\ldots,n)}$ also works for both signs.

We continue to repeat this argument with $A^{(k+1,\ldots,\hat{r},\ldots,\hat{s},\ldots,n)}$ to conclude that this larger matrix works for both signs. This process will continue until we are forced to conclude that the matrices $A^{(s)}$ work for both signs, but we assumed for contradiction that these matrices do not work for $x_{1,2} = +p$. So this finishes the proof of the proposition. $\square$

CHAPTER IV

CONSTRUCTION OF HIGHEST WEIGHT VECTORS

> "Is this in bad character?"
> "I'll have to think about it a Weyl."
>
> unknown

The description of the various pieces of an ideal as irreducible G-modules is useful because it allows one to look at various pieces of an ideal as isotypic G-modules. Moreover, each irreducible G-module has a highest weight vector, and to check whether an irreducible G-module is in a G-invariant ideal, it suffices to check whether the highest weight vector is in the ideal. This is a significant dimensional reduction. In particular, this description allows one to avoid looking at the individual monomials in an ideal that may have a Groebner basis that is too large for many computations. In some applications, however, one might actually want to know how to write out a basis of polynomials in the ideal. Perhaps less ambitious, one might want to just write down a highest weight vector for each module in a given degree. Landsberg and Manivel [19] gave an algorithm (based on standard facts in representation theory) to accomplish this goal, and though this is a standard algorithm in representation theory, an implementation of this algorithm was not readily available.

We wrote two implementations of the Landsberg-Manivel algorithm. The first was an attempt to directly emulate the algorithm suggested in [19]. The second is a significant practical improvement. We describe the main algorithm and the direct implementation of it. We point out some undesirable aspects of this implementation and describe the second implementation of the algorithm.

A. Schur modules, Young symmetrizers and polynomials

It is necessary to recall some basic representation theory from Fulton and Harris [7]. To a partition π of d, there is an associated Young diagram Y_π of d boxes arranged in the shape π. When a Young diagram is filled with numbers, it is called a Young tableau T_π. A Young tableau is called standard when it is strictly increasing in its rows and columns. For example, the partition $(3,1)$ of 3 has the corresponding Young diagram $Y_{(3,1)} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$. $Y_{(3,1)}$ has three standard fillings, $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$.

The irreducible representations of $GL(V)$ in $V^{\otimes d}$ are indexed by standard Young tableau as $S_{T_\pi} V \subset V^{\otimes|\pi|}$ with $|\pi| = d$. Two irreducible representations are isomorphic if they are indexed by Young tableau of the same shape, therefore irreducible representations can also be indexed by the partitions π along with a multiplicity M_π. In this case, S_π is usually called a *Schur functor* and $S_\pi V$ is known as a *Schur module.*

A Young tableau provides a combinatorial recipe for constructing a highest weight vector. Suppose π is a partition of d such that $\pi = (p_1, \ldots, p_{l(\pi)})$, with conjugate partition $\pi' = (q_1, \ldots, q_{l(\pi')})$. Then for each Young tableau T_π one can associate a skew-symmetrization map $b_\pi : V^{\otimes d} \to \bigwedge^{q_1} V \otimes \cdots \otimes \bigwedge^{q_{l(\pi')}} V \subset V^{\otimes d}$, (*i.e.* skew-symmetrization of the vector spaces in positions marked by the indices in the columns of T_π) and a symmetrization map $a_\pi : V^{\otimes d} \to S^{p_1} V \otimes \cdots \otimes S^{p_{l(\pi)}} V \subset V^{\otimes d}$, (*i.e.* symmetrization of the vector spaces marked by the indices in the rows of T_π). From the maps a_π and b_π one constructs a Young symmetrizer $c_\pi = a_\pi b_\pi$. These maps depend on the filling T_π, and wiring diagrams are a convenient way to keep track of this dependence on filling.

The Landsberg-Manivel algorithm from [19] is described in more detail in the forthcoming book [20]. Here we just present an outline in the multiplicity 1 case.

1. Compute the decomposition of $S^d(V_1 \otimes \cdots \otimes V_n)$ and select a module $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n \subset S^d(V_1 \otimes \cdots \otimes V_n)$ which occurs with multiplicity 1.

2. Construct the pre-highest weight monomial for $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n$.

   If $\{e_1, \ldots, e_N\}$ is an ordered basis of V and $\pi = (p_1, \ldots, p_{l(\pi)})$ then $e^\pi = e_1^{\otimes p_1} \otimes e_2^{\otimes p_2} \cdots \otimes e_{l(\pi)}^{\otimes p_{l(\pi)}}$ is the pre-highest weight monomial for $S_\pi V$.

3. For each π_i, choose a (random) column standard filling T_{π_i} of the Young Tableau Y_{π_i} associated to π_i. Let $c_{\pi_i} = a_{\pi_i} b_{\pi_i}$ be the associated Young symmetrizer to T_{π_i}.

4. Construct a wiring diagram of skew-symmetrizations b_{π_i} and symmetrizations a_{π_i} based on the choice of filling T_{π_i}.

5. Braid the output strands of the n diagrams by selecting for each strand one wire from each diagram.

6. Symmetrize the output of the braiding (by appropriately replacing $\otimes$ with $\circ$) so that the output lives in $S^d(V_1 \otimes \cdots \otimes V_n)$.

7. If the result is non-zero, stop. If not, return to step 3.

This makes a black box which does the following: Given an input vector, $e^{\pi_1} \otimes \cdots \otimes e^{\pi_n} \in (V_1 \otimes \cdots \otimes V_n)^{\otimes d}$, produces an element in $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n \subset S^d(V_1 \otimes \cdots \otimes V_n)$ which has the same weight as the input. The final step is to make sure that the result is non-zero. If you start with a vector which has the highest possible weight that could live in $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n$, then the result will be a highest weight vector of the module, however this vector could actually be a complicated expression for the zero vector, and this can be a big annoyance.

When the multiplicity M_π of $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n \subset S^d(V_1 \otimes \cdots \otimes V_n)$ is greater than 1, the only modification we need is to repeat the multiplicity 1 algorithm as many times as it takes to find M_π (non-zero) linearly independent vectors. The number of times we have to repeat the algorithm may be much larger than M_π if good fillings are hard to find.

This algorithm has been used effectively to produce polynomials in low degree. An undesirable aspect of the algorithm is the random choice of filling. This is done because we do not have a sufficient understanding of the correct combinations of fillings that will construct a diagram that gives a non-zero highest weight vector. Since we know that such a combination exists, we just randomly search for one in the space of possibilities. Eventually we will find a set of acceptable fillings, so the algorithm will terminate, however it may take many iterations. The understanding we would need to fix this problem relies on some unsolved problems in combinatorics, however there may be a way to get enough information to narrow our search. A suggested route to follow is outlined in section D.

A second undesirable aspect of this algorithm is the complexity of the symmetrization and skew-symmetrization maps. *A priori* we need to do roughly $s!$ computations for each symmetrization or skew symmetrization of each monomial in a polynomial of degree s in the intermediate stages. The final symmetrization map is a potential nightmare of computation. This causes the computation time and the memory requirements of this algorithm to grow very quickly. The algorithm we present has two aspects that attempt to alleviate some of this burden.

B. Implementing the algorithm

Note, the algorithm we present works for more than 3 factors, but all the essential aspects are present in the 3-factor case. We have included Maple code for this algorithm in Appendix D, Section B.

Our goal is to compute a non-zero highest weight vector inside a given isotypic component in the following isotypic decomposition of $S^d(A \otimes B \otimes C)$ as $GL(A) \times GL(B) \times GL(C)$-modules:

$$S^d(A \otimes B \otimes C) = \bigoplus_{|\pi|=d} S_{\pi_1}A \otimes S_{\pi_2}B \otimes S_{\pi_3}C^{\oplus M_{\pi_1,\pi_2,\pi_3}}, \tag{4.1}$$

where the multiplicity M_{π_1,π_2,π_3} can be computed via characters. We have included Maple code for computing these decompositions in Appendix D, Section A.

When the multiplicity is greater than 1, we would like to compute a (natural) basis of the highest weight space inside $S_{\pi_1}A \otimes S_{\pi_2}B \otimes S_{\pi_3}C^{\oplus M_{\pi_1,\pi_2,\pi_3}}$. To avoid too many notational headaches, we give the description with an example. We would like to construct a highest weight vector of $S_{(2,2,2)}A \otimes S_{(2,2,2)}B \otimes S_{(3,1,1,1)}C$.

In general, if $e^i_1, \ldots, e^i_{N_i}$ is an ordered basis of V_i, Suppose $\pi_i = (p^1_i, \ldots, p^{l(\pi)}_i)$ is a partition, and let $\pi' = (q^1_i, \ldots, q^{l(\pi')}_i)$ denote the conjugate partition. Then $S_{\pi_i}V_i \subset \bigwedge^{q^1_i} V_i \otimes \bigwedge^{q^2_i} V_i \otimes \cdots \otimes \bigwedge^{q^{l(\pi')}_i} V_i \subset V_i^{\otimes d}$. Using this inclusion, a pre-highest weight vector for $S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n$ is

$$e^{\pi_1} \otimes e^{\pi_2} \otimes \cdots \otimes e^{\pi_n},$$

where $e^{\pi} = e_1^{\otimes p_1} \otimes e_2^{\otimes p_2} \cdots \otimes e_{l(\pi)}^{\otimes p_{l(\pi)}}$. The result of the skew-symmetrization stage has a notationally compact expression when we use the wedge product

$$\bigotimes_i \left(e^1_1 \wedge \cdots \wedge e^1_{q^1_i}\right) \otimes \cdots \otimes \left(e^n_1 \wedge \cdots \wedge e^n_{q_i^{l(\pi'_i)}}\right). \tag{4.2}$$

Here we specialize to the three factor case. Suppose $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$ and $\{c_1, c_2, c_3, c_4\}$ are ordered bases of A, B and C respectively. In our example, expression (4.2) becomes

$$\begin{aligned}((a_1 \wedge a_2 \wedge a_3) \otimes (a_1 \wedge a_2 \wedge a_3)) \otimes ((b_1 \wedge b_2 \wedge b_3) \otimes (b_1 \wedge b_2 \wedge b_3)) \\ \otimes ((c_1 \wedge c_2 \wedge c_3 \wedge c_4) \otimes c_1 \otimes c_1)\,.\end{aligned} \tag{4.3}$$

At this stage we notice our first improvement. The expression (4.3) looks like a product of determinants. A fact from linear algebra implies that the determinant of an $n \times n$ matrix can be computed in roughly n^3 operations rather than the $n!$ operations suggested by the naive formula. This speed up is built in to most mathematical programming languages. By programming in Maple, we let the Maple Kernel handle this speed up by telling it to compute a determinant rather than telling it to compute the naive definition of the skew-symmetrization maps.

The next step is to partially symmetrize the expression (4.3). But we haven't said which vectors we want to symmetrize. We are going to accomplish the symmetrization stage and the final braiding / symmetrization stage in one step by keeping track of labels on the various terms.

The expression (4.3) is a product of three degree 6 polynomials, $f_a f_b f_c$, each only depending on a's, or b's or c's respectively. The final result will be a polynomial of degree 6, on the variables $a_i \otimes b_j \otimes c_r =: Z_{i,j,r}$, where $1 \leq i, j, \leq 3$, $1 \leq r, \leq 4$. To construct this polynomial, consider a monomial in f_a, f_b, f_c. It will have 6 a's, 6 b's, and 6 c's. We just need a consistent rule for selecting six triples of an a, b and a c. The key to this is to decide on this rule *before skew-symmetrizing*. We do this by adding a label to each of the symbols and carry this label throughout the computation. For example a_1 will become $a_{1,l}$, and the l indicates that $a_{1,l}$ will eventually go towards building the l^{th} factor in the monomial $Z_{i_1,j_1,r_1} \dots Z_{i_6,j_6,r_6}$.

Fix indices $1 \leq i, j, \mu \leq 3$, $1 \leq r, \phi \leq 4$, $4 \leq \nu \leq 6$ and permutations α, β, γ of $\{1, \ldots, 6\}$ and construct a product of determinants from expression (4.3)

$$\left|a_{i,\alpha(\mu)}\right| \left|a_{i,\alpha(\nu)}\right| \left|b_{i,\beta(\mu)}\right| \left|b_{i,\beta(\nu)}\right| \left|c_{r,\gamma(\phi)}\right| \left|c_{1,\gamma(5)}\right| \left|c_{1,\gamma(6)}\right|, \tag{4.4}$$

where, for example if $\alpha = (6, 5, 4, 3, 2, 1)$,

$$\left|a_{i,\alpha(\mu)}\right| = \begin{vmatrix} a_{1,6} & a_{1,5} & a_{1,4} \\ a_{2,6} & a_{2,5} & a_{2,4} \\ a_{3,6} & a_{3,5} & a_{3,4} \end{vmatrix}.$$

Next, we do a replacement as follows: Iteratively extract the coefficient of the partial monomial $a_{i,1}b_{j,1}c_{r,1}$ in (4.4) and multiply this expression by the single variable $Z_{i,j,r}$. The new expression will be a polynomial on a's, b's, c's and Z's. Repeat this process, replacing $a_{i,2}b_{j,2}c_{r,2}$ with $Z_{i,j,r}$, and so on, until the final expression only involves the variables $Z_{i,j,r}$.

In this stage we have accomplished two symmetrizations at once. Since the product of determinants (4.4) was constructed with regular multiplication and not tensor product, we have already accomplished the first symmetrization stage. We have not lost the information of the tensor product because we have kept track of where each term should be in the expression of the tensor by the second index on, for example, $a_{i,\mu}$. Second, we made the grouping of one each of an a, b and a c and multiplied the results by standard multiplication.

The output is a polynomial in $S_{(2,2,2)}A \otimes S_{(2,2,2)}B \otimes S_{(3,1,1,1)}C$, however this polynomial may simplify to 0, so this process may have to be repeated with a new choice in permutations α, β, γ. In fact, in an example where the multiplicity is greater than one, we would just need to repeat the above procedure with random permutations α, β, γ as many times as it takes to get a basis of the highest weight space.

1. The keys to speed

This algorithm is fast because of the following features: The procedure "coeff " for finding the coefficient of a given monomial is built in to the kernel of Maple and is already optimized. The command "coeff" works well when every symbol occurs in degree at most 1 - this is the case for our application. (For other applications when this is not the case and a given symbol occurs in an expression in higher degree, more care needs to be taken.)

A key to the built in "coeff" procedure is that numerical methods can be used without sacrificing accuracy. A simple example of this is as follows. Suppose it is known that all the monomials in a polynomial $f(x_{1,\ldots,x_N})$ are square free. Then the coefficient of x_1 is found by evaluating $f(1, x_2, \ldots, x_N) - f(0, x_2, \ldots, x_N)$.

Second, this algorithm (specifically the command "coeff") does not require the expression 4.4 to be expanded. Though this expression of the polynomial is not the densest expression, it is better computationally because on a fundamental level it allows for quicker evaluation. Also, we have allowed any cancellations that might happen to happen as early as possible in the symmetrization stages, rather than all at once in the final symmetrization. Finally, the most significant savings is that we never had to implement a procedure that involved a sum with factorial-many terms. This symbolic symmetrization is preferable to the naive summation over all permutations.

The limitations of this algorithm lie in the memory requirements for the intermediate stages. For example, Maple quickly runs out of memory when trying to compute the degree 9 polynomial $S_{(3,3,3)}A \otimes S_{(3,3,3)} \otimes S_{(3,3,3)}$.

C. A theoretical advantage

Eventually we want to use the polynomials we constructed and evaluate them on a point of a given algebraic variety. It may be possible to get more out of the algorithm we presented above. For instance notice that in expression (4.3) we have a product of determinants. We know precisely when a product of polynomials is zero - if and only if one (or more) of the factors is zero. Therefore if we know that we will be evaluating our polynomial on a point that does not have "enough independent vectors" one of the determinants will automatically be zero.

To be more specific, let $Sub_{a,b,c}(A\otimes B\otimes C) = \{T \in \mathbb{P}(A\otimes B\otimes C) \mid \exists A' \subset A,\ \ B' \subset B,\ \ C' \subset C,\ \ \dim(A') = a,\ \ \dim(B') = b,\ \ \dim(C') = c,\ \ T \in \mathbb{P}(A' \otimes B' \otimes C')\}$. Let $n(\pi)$ be the length (the number of parts) of the partition π. We have shown

Proposition IV.1. *Let $T \in Sub_{a,b,c}(A \otimes B \otimes C)$, and let f_π be a non-zero highest weight vector of $S_{\pi_1}A^* \otimes S_{\pi_2}B^* \otimes S_{\pi_3}C^*$. Then $f(T) = 0$ if $a < l(\pi_1)$ or $b < l(\pi_2)$ or $c < l(\pi_3)$.*

Obviously this proposition holds for more than 3 factors. Therefore we recover a weaker version of a result of Landsberg and Weyman (cf. Theorem 3.1 [22]).

Remark IV.2. We would like to have a better understanding of the process of pairing a point $T \in \mathbb{P}(A \otimes B \otimes C)$ with the product of determinants (4.3) in the intermediate stage of our algorithm. In particular, if we could have a complete description of the kernel of this pairing, then we would be able to push our understanding of the highest weight vectors in Schur modules much further than a computer could ever take us as this would allow us to evaluate polynomials without actually constructing them explicitly.

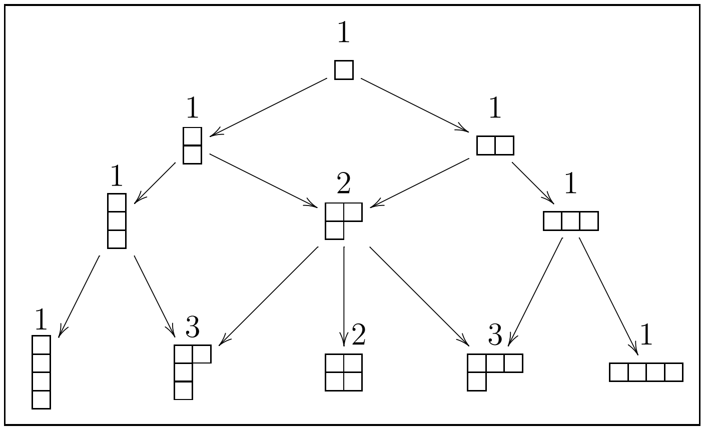

Fig. 1. An Example Young Lattice

D. Littlemann paths and good fillings

Figure 1 is an example of a Young lattice. In general, the Young lattice is a systematic way of enumerating all partitions. But the diagram does much more. For example, we have put labels above each the Young diagram Y_π to count the number of paths in the Young lattice (also called *Littlemann paths*) that end at the node for Y_π.

Proposition IV.3 (Young)**.** *There is a* $1-1$ *correspondence between the number of paths to* Y_π *in the Young lattice and the number of standard fillings of the Young tableau* T_π.

Proof. Proof by picture. See Figure 2. $\square$

But the standard Young tableau T_π index the irreducible representations $S_{T_\pi}V$ in $V^{\otimes|\pi|}$ [11]. Therefore there is a $1-1$ correspondence of paths in the Young lattice and representations $S_{T_\pi}V$ in $V^{\otimes|\pi|}$.

Now we come back to our question: How do we find good fillings $T_{\pi_1}, \ldots, T_{\pi_n}$

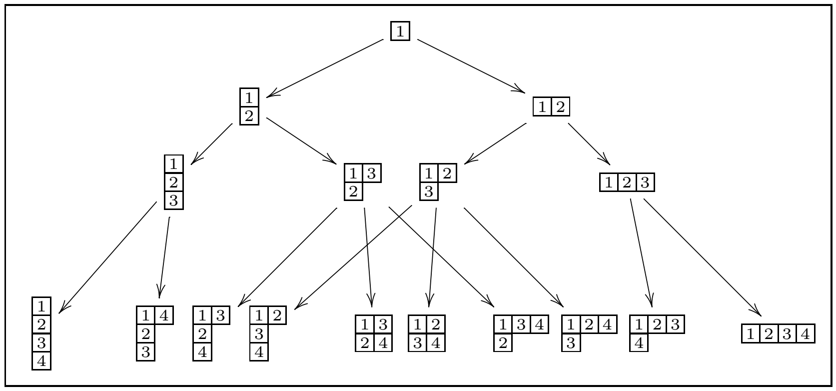

Fig. 2. Paths in the Young Lattice and Standard Young Tableau

so that our algorithm produces a non-zero vector in $S_{\pi_1}V_1 \otimes S_{\pi_n}V_n$? Since we are concerned with more than one filling, consider the paths in many overlaid Young lattices, henceforth called the *Young multi-lattice.* In this setting, we are now asking for a rule that tells us which combinations of Littlemann multi-paths in the Young multi-lattice are allowable. This idea deserves further investigation.

Goal IV.4. *Describe the allowable Littlemann multi-paths via graph theoretic properties of the Young multi-lattice.*

CHAPTER V

COROLLARIES AND RESTATEMENTS

"All algebras are associative."

PBW

The results of Theorem III.3 can be used to answer many different questions. Most of these questions can be found in the literature [1, 10, 13, 14] and the references therein. In this chapter we address a few of the applications of Theorem III.3.

A. GKK-τ matrices

In one fell swoop, O. Holtz [12] gave a counterexample to four conjectures, all of them involving the requirement of positivity of principal minors. The abstract of the paper states,

> Hermitian positive definite, totally positive, and nonsingular M-matrices enjoy many common properties, in particular
>
> (A) positivity of all principal minors
>
> (B) weak sign symmetry
>
> (C) eigenvalue monotonicity
>
> (D) positive stability
>
> The class of GKK matrices is defined by properties (A) and (B), whereas the class of nonsingular τ-matrices by (A) and (C).

Holtz proves that no combination of (A) with any of (B) or (C) imply (D).

In their list of open problems related to $GKK - \tau$ matrices, O. Holtz and H. Schneider, [13] asked the principal minor assignment problem (PMAP): Given a vector of length 2^n does there exist a matrix which has its vector of principal minors equal to the given vector? Their motivation for this problem comes from the following theorem.

Theorem V.1 (Gantmacher-Krein-Carlson)**.** *A P-matrix (all principal minors positive) is GKK if and only if its principal minors satisfy the generalized Hadamard-Fisher (HF) inequality*

$$A[\alpha]A[\beta] \geq A[\alpha \cup \beta]A[\alpha \cap \beta] \quad \forall \alpha, \beta \in \langle n \rangle$$

This can be used as follows. An answer to PMAP would tell whether or not there exist at least one matrix with a prescribed set of potential principal minors. If there is no such matrix, then stop. If there is such a matrix, then there exist GKK matrices with the prescribed principal minors if and only if the following two conditions are satisfied, (1) the vector satisfies the HF inequality and (2) has all positive entries.

We also know that the spectrum of a matrix is determined by its principal minors, so the outline above allows one to find (in principle) all possible spectra of GKK matrices.

Holtz and Schneider also point out that PMAP is also equivalent to the following inverse eigenvalue problem. Given a vector $v \in \mathbb{C}^{2^n}$, is there a matrix with its eigenvalues and the eigenvalues of all of its principal submatrices given by v? The equivalence of these problems comes from the fact that specifying all principal minors implies specifying all characteristic polynomials and all eigenvalues of the principal submatrices.

We should mention that Griffin and Tsatsomeros gave a partial numerical answer to PMAP [10]. Their main result is a MatLab program that can take a reasonably sized input vector of potential principal minors and either return a matrix with those principal minors or say that a matrix probably does not exist. However, their program is restricted to a subclass of all matrices, and it loses accuracy and reliability when the entries are close to 0.

In the case of symmetric matrices, Theorem III.3 answers the symmetric principal minor assignment problem and the equivalent inverse eigenvalue problem. Therefore due to the remarks in [13], the polynomials in the hyperdeterminantal module can be used to give a complete description of possible symmetric P-matrices, possible symmetric GKK-matrices, possible symmetric non-singular τ-matrices, etc.

B. Negatively correlated random variables

Consider a real symmetric $n \times n$ matrix A. The principal minors of A can be interpreted as values of a function $\omega : \mathcal{P}(\{1, \ldots, n\}) \to [0, \infty)$, where $\mathcal{P}$ is the power set. This function ω, under various restrictions, is of interest to statisticians. In particular, in D. Wagner's study of the covariance of random variables [30] he is interested in the following example.

Question V.2. *When is it possible to prescribe the principal minors of the matrix A as well as the off-diagonal entries of A^{-1}?*

When $A = (a_{i,j})$ is symmetric,

$$a_{i,j}^2 = \Delta_i(A)\Delta_j(A) - \Delta_{i,j}(A)\Delta_\emptyset(A).$$

So to prescribe the off-diagonal entries in a symmetric matrix A is equivalent to prescribing the 2×2 principal minors and a sign for each off-diagonal term.

Another useful fact is if A is invertible then

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)},$$

where $\operatorname{adj}(A)_{i,j} = ((-1)^{i+j} det(A_i^j))$ is the adjugate matrix.

This formula implies that up to scale, the vector of principal minors of A^{-1} is the vector of principal minors of A in reverse order. So Wagner's question specialized to symmetric matrices is equivalent to the following question:

Question V.3. *When is it possible to prescribe the principal minors and the signs of the off-diagonal terms of a symmetric matrix A*

Our main result immediately provides an answer to the first part of the question:

Corollary V.4. *It is possible to prescribe the principal minors of a symmetric matrix if and only if the candidate principal minors satisfy all the relations given the hyperdeterminantal module.*

C. Determinantal point processes

An important notion in statistical physics is that of a determinantal point process. Of particular interest to this study is the work of Borodin and Rains. In [2] they considered the space of all *determinantal points.* A non zero point $p_S \in \mathbb{C}^{2^n}$ is called determinantal if there is an integer m and an $(n+m) \times (n+m)$ matrix K such that for $S \subset \{1, 2, \ldots, n\}$

$$p_S = \det_{S \cup \{n+1, \ldots, n+m\}}(K).$$

Borodin and Rains were able to completely classify all such points for the case $n = 4$ (Theorem 4.6 [2]) by giving a nice geometric characterization. Lin and Sturmfels [23] studied the geometric and algebraic properties of the algebraic variety of determinantal points and independently arrived at the same result as Borodin and Rains, moreover Lin and Sturmfels gave a complete proof of the claim of [2] that the ideal of the variety is generated in degree 12 by 718 polynomials.

Consider the case where we impose the restrictions that the matrix K to be symmetric and the integer $m = 0$, and call these restricted determinantal points *symmetric determinantal points.*

Corollary V.5. *The variety of all symmetric determinantal points is cut out set theoretically by the hyperdeterminantal module.*

Corollary V.5 is useful because it provides a complete list of necessary and sufficient conditions for determining which symmetric determinantal points can possibly exist.

D. Spectral graph theory

Let Γ be a finite graph with vertex set $Q_0 = \{v_1, \dots, v_n\}$ and edge set $Q_1 = \{e_{i,j} \mid \overrightarrow{v_i v_j} \in \Gamma\}$. A weight $wt : Q_0 \times Q_0 \to \mathbb{C}$ on a graph is an assignment of a complex number to every edge and 0 if no edge exists between a pair of vertices.

The weighted Laplacian of a graph is the matrix $\Delta_{wt}(\Gamma)_{i,j} = wt(v_i, v_j)$. When no weight is indicated, the weighted Laplacian is the usual graph Laplacian of an undirected graph, $\Delta(\Gamma)$, *i.e.* the weighted Laplacian with

$$wt(v_i, v_j) = \begin{cases} -1 & \text{if } i \neq j \text{ and } e_{i,j} \in Q_1 \\ 0 & \text{if } i \neq j \text{ and } e_{i,j} \notin Q_1 \\ deg(v_i) & \text{if } i = j \end{cases}$$

The eigenvalues of $\Delta_{wt}(\Gamma)$ are invariants of the graph. The first example is with the standard graph Laplacian. The well known Kirchoff's Matrix-Tree theorem states that any $(n-1) \times (n-1)$ principal minor of $\Delta(\Gamma)$ counts the number of spanning trees of Γ.

There are many generalizations of the Matrix-Tree Theorem, such as the Matrix-Forest Theorem which states that $\Delta(\Gamma)_S^S$, the principal minor of the graph Laplacian formed by omitting rows and columns indexed by the set $S \subset \{1, \dots, n\}$, computes the number of spanning forests of Γ rooted at vertices indexed by S.

The principal minors of the graph Laplacian are graph invariants. The relations among principal minors are then also relations among graph invariants. Relations among graph invariants are central in the study of the theory of unlabeled graphs. In fact, Mikkonen holds that "the most important problem in graph theory of unlabeled graphs is the problem of determining graphic values of arbitrary sets of graph invariants," (p. 1 [24]).

Theorem III.3 gives relations among the graph invariants that come from principal minors, and in particular, since a graph can be reconstructed from a symmetric matrix, Theorem III.3 implies the following Corollary.

Corollary V.6. *There exists an undirected weighted graph Γ with invariants $[v] \in \mathbb{P}^{2^n-1}$ specified by the principal minors of a symmetric matrix $\Delta_{wt}(\Gamma)$ if and only if $[v]$ is a zero of all the polynomials in the hyperdeterminantal module.*

CHAPTER VI

SUMMARY

The variety of principal minors of symmetric matrices is a prototypical G-variety in a space of tensors. We have studied it in the setting of G-varieties using representation theory and geometry with a secondary goal that the techniques used and presented here will be useful in the study of other G-varieties in spaces of tensors. Groebner basis techniques were used successfully by Holtz and Sturmfels to prove that the hyperdeterminantal module gives a set of minimal generators of the prime ideal $\mathcal{I}(Z_n)$ only for the first two non-trivial cases [14]. Now the set theoretic result is established for all n, but there is still more work to be done for the full Holtz-Sturmfels Conjecture - *i.e.* the ideal theoretic case.

The set theoretic result is good enough for many applications related to principal minors of symmetric matrices. In particular, set theoretic defining equations of Z_n are necessary and sufficient conditions for a given vector of length 2^n to be expressed as the principal minors of a symmetric matrix. In Chapter V we pointed out several applications of this result.

The study of Z_n has raised several natural questions.

Question VI.1. *A single irreducible module cuts out the Z_n set theoretically. Under what conditions does the G-orbit of an irreducible polynomial generate a prime ideal?*

An answer to this question could help to resolve the Holtz-Sturmfels Conjecture by allowing us to decide whether the hyperdeterminantal module generates a prime ideal.

As far as applications are concerned, $\tilde{Z}_n$, the variety of principal minors of arbitrary square matrices is also interesting. Borodin-Rains [2] and Lin-Sturmfels [23] independently found that the ideal of $\tilde{Z}_4$ is generated in degree 12. It would be compelling to see how many of the techniques in this study could be used in the absence of the assumption that the matrix be symmetric. We reiterate a question of Lin and Sturmfels.

Question VI.2. *Does the* $(GL(2)^n) \ltimes \mathfrak{S}_n$ *orbit of* $\mathcal{I}_{12}(\tilde{Z}_4)$ *cut out* $\tilde{Z}_n$*?*

An affirmative answer to this question would completely resolve the principal minor assignment problem [13]. A negative answer might help to shed light on the subtleties of Question VI.1.

In the course of this study, we showed that $\tau\left(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)\right)$ is a natural subvariety of Z_n. Because of this we ask the following:

Question VI.3. *Can the inclusion* $\tau\left(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)\right) \subset Z_n$ *be used to verify the conjecture of Landsberg and Weyman [21] on the defining ideal of the tangential variety?*

Our hope is that these questions about G-varieties and their ideals can be answered using techniques similar to those which were used to study Z_n.

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APPENDIX A

THE DIMENSION OF THE ZERO SET OF THE HYPERDETERMINANTAL MODULE

Proposition A.1. *The hyperdeterminantal module M has $dim(\mathcal{V}(M)) = \binom{n+1}{2}$.*

Proof. To compute $dim(\mathcal{V}(M))$, first notice that $dim(\mathcal{V}(M)) \geq \binom{n+1}{2}$. This is because $\dim(Z_n) = \binom{n+1}{2}$ and $Z_n \subseteq \mathcal{V}(M)$. So we need to show that $dim(\mathcal{V}(M)) \geq \binom{n+1}{2}$, i.e. we only need to find at least $2^n - \binom{n+1}{2} - 1$ polynomials f_i in M so that their differential has maximal rank at a smooth point. We'll accomplish this by successively selecting polynomials that are involve new variables not used in the previous polynomials. This will construct an upper triangular matrix. It will have full rank as long as all of the diagonal entries are nonzero.

The selection will go as follows: Let $hyp_{i,j,k}$ denote the hyperdeterminant on factors $\{i, j, k\}$. Choose

$$f_0 := hyp_{1,2,3} \otimes (x_4^1)^4 \otimes \cdots \otimes (x_n^1)^4$$

- depends on $X^{[1,\dots,1]}$. Notice that f_0 is a hyperdeterminant on the variables $X^{[i_1,i_2,i_3,1,\dots,1]}$. Next, choose

$$f_j := hyp_{i_1,i_2,i_3} \otimes (x_{i_4}^1)^4 \otimes \cdots \otimes (x_{i_{j-1}}^1)^4 \otimes (x_{i_j}^2)^4 \otimes (x_{i_{j+1}}^1)^4 \otimes (x_{i_n}^1)^4$$

- depends on $X^{I_{j,1}}$ for $|I| = 1$, but not $X^{[1,\dots,1]}$. This selects $\binom{n}{1}$ independent polynomials, f_j, and each f_j is a hyperdeterminant on the variables X^I where I is such that the entries i_1, i_2, i_3 can be either 1 or 2, the entry j must be a 2 and the rest must be 1's.

Continue until finally choosing

$$f_{j_1,\ldots,j_{n-3}} := hyp_{i_1,i_2,i_3} \otimes (x_{i_4}^2)^4 \otimes \cdots \otimes (x_{i_n}^2)^4$$

- depends on $X^{I_{j_1,\ldots,j_{n-3},2,2,2}}$ for $|I| = n-3$, but not X^I when $|I| < n-3$. This selects $\binom{n}{n-3}$ independent polynomials, $f_{j_1,\ldots,j_{n-3}}$ which are each a hyperdeterminant on the variables X^I where, I is such that the entries i_1, i_2, i_3 can be either 1or 2, and each of the entries j_k must be a 2.

Recall that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Also recall, $\binom{n}{2} + \binom{n}{1} = \binom{n+1}{2}$.

We have constructed a set of polynomials which has size $1 + \binom{n}{1} + \cdots + \binom{n}{n-3} = 2^n - 1 - \binom{n+1}{2}$. Call this set of polynomials E and compute $rank\left(d(E)_{|p}\right)$. Notice that by our construction, each of our polynomials is actually the same polynomial, only the variables have different names. Order the variables consistently with the ordering of the polynomials so that $d(E)$ is upper triangular.

Now it remains to show that we can choose a point p so that $d(E)$ will have maximal rank. First, notice the diagonal entries are

$$\frac{df_{j_1,\ldots,j_k}}{dX^{I_{j_1,\ldots,j_k,2,\ldots,2}}},$$

and modulo change of names of variables in each case, this is actually

$$\frac{d(hyp_{123})}{dX^{[2,2,2]}}$$

In the next proposition, we will show that this quantity is always nonzero for a nice choice in form of matrix, and then we'll be done. □

Now we just need to select a point that behaves similarly no matter which set of minors we take.

Consider the following matrix:

$$C := \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \vdots & 2 & 2 & \dots & 2 \\ \vdots & \vdots & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$$

Because of the structure of Z_n, we may consider any set of 8 variables of the form $X^{I_{i_1,i_2,i_3}}$ (where all of the entries of I are fixed except for the entries in the i_1, i_2, i_3 positions) to be the principal minors of a certain 3×3 matrix. Every matrix that arises in this way when constructed from C, still has a nice enough form so that we can compute $\dfrac{d(hyp_{123})}{dX^{[2,2,2]}}$.

Proposition A.2. *Consider the projection*

$$\begin{aligned} \pi : Z_n &\rightarrow Z_3 \\ t^{|I|}\Delta_I(A)X^I &\mapsto t^{|I_{i_1,i_2,i_3}|}\Delta_{I_{i_1,i_2,i_3}}(A)X^{i_1,i_2,i_3} \end{aligned}$$

Then $\exists C_3$ *such that* $\varphi([C_3, t]) = t^{|I_{i_1,i_2,i_3}|}\Delta_{I_{i_1,i_2,i_3}}(C)X^{i_1,i_2,i_3}$ *and* $\dfrac{d(hyp_{123})}{dX^{[2,2,2]}}(\varphi(C_3)) \neq 0.$

Proof. $\varphi([C, t]) = [t^n, t^{n-1}(i_1), t^{n-2}(i_1(i_2 - i_1)), t^{n-3}(i_1(i_2 - i_1)(i_3 - i_2), \dots, i_1(i_2 - i_1) \dots (i_n - i_n - 1)]$ After reordering the indices, we may assume

$I_{i_1,i_2,i_3} = [i_1, i_2, i_3, i_4, \dots, i_k, \dots, i_n]$, where $i_4 = \dots = i_k = 2$, $i_{k+1} = \dots = i_n = 1$.

So,

$$\begin{aligned}\pi(\varphi([C,t])) = [&t^{n-k}c_4(c_5-c_4)(\dots)(c_k-c_{k-1}),\\ &t^{n-k-1}c_1(c_4-c_1)(c_5-c_4)(\dots)(c_k-c_{k-1}),\\ &t^{n-k-1}c_2(c_4-c_2)(c_5-c_4)(\dots)(c_k-c_{k-1}),\\ &t^{n-k-1}c_3(c_4-c_3)(c_5-c_4)(\dots)(c_k-c_{k-1}),\\ &t^{n-k-2}c_1(c_2-c_1)(c_4-c_2)(c_5-c_4)(\dots)(c_k-c_{k-1}),\\ &t^{n-k-2}c_1(c_3-c_1)(c_4-c_3)(c_5-c_4)(\dots)(c_k-c_{k-1}),\\ &t^{n-k-2}c_2(c_3-c_2)(c_4-c_3)(c_5-c_4)(\dots)(c_k-c_{k-1}),\\ &t^{n-k-3}c_1(c_2-c_1)(c_3-c_2)(c_4-c_3)(c_5-c_4)(\dots)(c_k-c_{k-1})]\end{aligned}$$

$$\begin{aligned}= [&t^3(c_4), t^2c_1(c_4-c_1), t^2c_2(c_4-c_2), t^2c_3(c_4-c_3),\\ &tc_1(c_2-c_1)(c_4-c_2), tc_1(c_3-c_1)(c_4-c_3),\\ &tc_2(c_3-c_2)(c_4-c_3), c_1(c_2-c_1)(c_3-c_2)(c_4-c_3)]\end{aligned}$$

So we notice that we can define the following matrix

$$[C_3,t] := \left[\frac{1}{c_4}\begin{pmatrix} c_1(c_4-c_1) & c_1(c_4-c_1) & c_1(c_4-c_1)\\ c_1(c_4-c_1) & c_2(c_4-c_2) & c_2(c_4-c_2)\\ c_1(c_4-c_1) & c_2(c_4-c_2) & c_3(c_4-c_3)\end{pmatrix}, t\right]$$

so that $\pi(\varphi([C,t])) = \varphi([C_3,t])$.

Remark A.3. We could have found this matrix using Schur complement and would have gotten the same answer - perhaps this gives a more streamlined approach to this problem.

We compute the differential:

$$\frac{d(hyp_{123})}{dX^{[2,2,2]}} = -2X^{[1,1,1]}X^{[1,2,1]}X^{[2,1,2]} - 2X^{[1,1,1]}X^{[1,2,2]}X^{[2,1,1]}$$
$$-2X^{[1,1,1]}X^{[1,1,2]}X^{[2,2,1]} + 2(X^{[1,1,1]})^2X^{[2,2,2]} + 4X^{[1,1,2]}X^{[1,2,1]}X^{[2,1,1]}$$

Then we evaluate on the point:

$$\varphi(C_3) = \left[\begin{array}{c} X^{[1,1,1]} = t^3c_4, X^{[2,1,1]} = t^2c_1(c_4 - c_1), \\ X^{[1,2,1]} = t^2c_2(c_4 - c_2), X^{[2,2,1]} = tc_1(c_4 - c_2)(c_2 - c_1), \\ X^{[1,1,2]} = t^2c_3(c_4 - c_3), X^{[2,1,2]} = tc_1(c_4 - c_3)(c_3 - c_1), \\ X^{[1,2,2]} = tc_2(c_4 - c_3)(c_3 - c_2), \\ X^{[2,2,2]} = c_1(c_4 - c_3)(c_3 - c_2)(c_2 - c_1) \end{array}\right]$$

Finally, we find that

$$\frac{d(hyp_{123})}{dX^{[2,2,2]}}(\varphi([C_3, 1])) = 4c_2c_1^2(c_4 - c_3)^2(c_4 - c_2)$$

So as long as we choose c_1, c_2 both nonzero, and $c_4 \neq c_3$ and $c_4 \neq c_2$, then we have a nonzero differential. This is what we wanted to show. □

Remark A.4. In the course of this proof, we have shown that the P-matrices with the same form as C above are in the smooth locus of $\mathcal{V}(M)$.

APPENDIX B

A GENERALIZATION OF TWO LEMMAS

This appendix generalizes and provides an alternate proof of the Step Up Lemma and the Characterization Lemma.

Suppose $X_n \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$ is a sequence of linearly non-degenerate varieties. Consider the linear projections $\pi : X_{i+1} \to \mathbb{P}(V_1 \otimes \cdots \otimes V_i \otimes \{x\})$. If for each i and each $x \in V_i$ we have $\pi(X_{i+1}) \subseteq Seg(X_i \times \mathbb{P}\{x\})$, then say the sequence (X_n) satisfies the cutting property (CP).

Lemma B.1. *Suppose $X_n \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$ is a sequence of linearly non-degenerate varieties which satisfy the cutting property (CP). Then*

$$\mathcal{I}_d(X_n) \otimes S^d(V_{n+1}^*) \subseteq \mathcal{I}_d(X_{n+1}).$$

Moreover,

$$X_{n+1} \subset \sigma_{m_{n+1}}(Seg(X_n \times \mathbb{P}V_{n+1})),$$

where $m_{n+1} = \dim(V_{n+1})$.

Proof. The space $\mathcal{I}_d(X_n) \otimes S^d V_{n+1}^*$ has a basis of the form $f \otimes (y^d)$ where $f \in \mathcal{I}_d(X_n)$ and $y \in V_{n+1}^*$. We need to show that $f \otimes (y^d)$ vanishes at all points of X_{n+1}.

Given such a polynomial $f \otimes (y^d)$, we know that $f \otimes (y^d) \subset S^d(V_1^* \otimes \cdots \otimes V_n^* \otimes \{y\})$, so in particular, $f \otimes (y^d)$ vanishes on all points in $V_1 \otimes \cdots \otimes V_n \otimes \{y\}^\perp$. So we only need to consider the image of the projection $\pi : X_{n+1} \to \mathbb{P}(V_1 \otimes \cdots \otimes V_n \otimes \{y\})$ which is contained in $Seg(X_i \times \mathbb{P}\{y\})$ by hypothesis.

But now if $[x \otimes a] \in Seg(X_i \times \mathbb{P}\{y\})$ it is clear that $f \otimes y^d(x \otimes a) = f(x)(y^d(a)) = 0$. So $f \otimes y^d$ vanishes at all points of X_{n+1} and we are done with the first part.

For the "moreover" statement, Let $v_1, \dots v_{m_{n+1}}$ be a basis of V_{n+1}. Notice that by the cutting property and the fact that X_{n+1} is linearly non-degenerate, every point $[z] \in X_{n+1}$ can be written in the form $[z] = [x_1 \otimes v_1 + \cdots + x_{n+1} \otimes v_{m_{n+1}}]$ with $x_j \in X_n$ for all j. $\square$

Remark B.2. This lemma could be used to replace the Characterization Lemma as follows. Show that $\mathcal{V}(M)$ satisfies the hypothesis, so the lemma implies that $\mathcal{V}(M) \subset \sigma_2(\mathcal{V}(M_i) \times \mathbb{P}V_i))$. Then proceed with the proof of the main theorem with a point in the intersection $\mathcal{V}(M) \cap \sigma_2(\mathcal{V}(M_i) \times \mathbb{P}V_i))$. This point will satisfy the property that it is the sum of two points and moreover, every point of $\mathcal{V}(M)$ has that property.

Lemma B.3. *An immediate corollary of the previous lemma is the following.*

$$\mathcal{I}_d(Z_{(n-1),\hat{i}}) \otimes S^d V_i \subset \mathcal{I}_d(Z_n)$$

and

$$Z_n \subset \sigma_2(Z_{(n-1),\hat{i}} \times \mathbb{P}V_i),$$

and moreover, the second containment is strict.

Proof. The varieties Z_n are linearly non-degenerate since they contain $Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$. Z_n satisfies the cutting property because of its symmetry. More specificially, the action of $SL(V_i)$ on V_i is transitive, so with out loss of generality, we only need to show the cutting property for $V_1 \otimes \cdots \otimes V_{n-1} \otimes \{x_n^1\}$. But this is no problem. Consider $[z] = \varphi([A,t])$. In our preferred basis, we may write $z = z_{J_{i,1}} X^{J_{i,1}} + z_{J_{i,2}} X^{J_{i,2}}$. We must show that $[z_{J_{i,1}} X^{J_{i,1}}] \in Seg(Z_{(n-1),\hat{i}} \times \mathbb{P}\{x_i^1\})$. Let $A^{(i)}$ denote the submatrix of A obtained by omitting the i^{th} row and column. It is clear that $\varphi([A^i,t]) = [z_{J_{i,1}} X^{J_i}] \in Z_{(n-1),\hat{i}}$, and this is what we needed to show.

The containment of varieties is strict since the point $X^{[1,\dots,1]} + X^{[2,\dots,2]}$ is in the secant variety, but not in Z_n. $\square$

Remark B.4. While the first part of this lemma proves Proposition III.21 , it is a bit of overkill. The real utility comes its contrapositive version. It gives a test for ideal membership for modules that have at least one $S_{(d)}$ factor. Suppose we know $\mathcal{I}_d(Z_n)$ for some n. If we want to test whether $N = S_{\pi_1}V_1^* \otimes \cdots \otimes S_{\pi_{n+1}}V_{n+1}^*$ is in $\mathcal{I}_d(Z_{n+1})$ and we know that N has at least one $\pi_i = (d)$ then we can just remove $S_{\pi_i}V_i^*$ and check whether the module we have left actually lives in the ideal $\mathcal{I}_d(Z_n)$. If not, then we know that N can't be in the ideal $\mathcal{I}_d(Z_{n+1})$.

APPENDIX C

MODULE DECOMPOSITIONS VIA REPRESENTATION RINGS

In this appendix, we show how to decompose a fundamental representation Γ_n of $Sp(2n)$ as $\mathfrak{g}$-module for $\mathfrak{g} = sl_2 \times \cdots \times sl_2$. Before we dive into this task, we recall some basic facts from representation theory found in Fulton and Harris, [7].

A. Representation rings

Here we follow Fulton and Harris [7] and introduce representation rings. The set of isomorphism classes $[V]$ of representations of a semi-simple (rank n) Lie algebra $\mathfrak{g}$ is a ring, denoted $R(\mathfrak{g})$ under the operations $[V] + [W] = [V \oplus W]$ and $[V] * [W] = [V \otimes W]$. Let $\Lambda = \Lambda_W$ be the weight lattice for $\mathfrak{g}$. If λ is a weight, write $e(\lambda)$ as the corresponding basis element of $\mathbb{Z}[\Lambda]$ of weight λ.

Proposition C.1 (Fact)**.** *The Character map*

$$Char: \ R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]$$

is a well defined injective ring homomorphism.

Let Γ_i be the fundamental representations of $Sp(2n)$. In particular,

$$\Gamma_n = \frac{\bigwedge^n \mathbb{C}^{2n}}{\left(\omega \wedge \bigwedge^{n-2} \mathbb{C}^{2n}\right)}.$$

Let $\mathbb{Z}\left[\Lambda^{\mathcal{W}}\right]$ denote the ring generated by the invariants of the Weyl group, $\mathcal{W}$, and let $P_i = Char(\Gamma_i)$.

Proposition C.2 (Fact)**.** *The following map is a ring isomorphism.*

$$Char: \ R(\mathfrak{g}) \to \mathbb{Z}\left[\Lambda^{\mathcal{W}}\right] \simeq \mathbb{Z}[P_1, \ldots, P_n].$$

Proposition C.3 (Fact)**.** *If* $\mathfrak{g}' \subset \mathfrak{g}$ *and* $\mathfrak{h}' \subset \mathfrak{h}$ *then the restriction map* $Res: \ R(\mathfrak{g}) \to R(\mathfrak{g}')$ *is a surjective ring homomorphism.*

So, all we need to do to describe the representation ring of a sub-algebra is to determine what happens to the polynomial generators under the restriction map. The generators are the characters of the fundamental representations.

1. The representation ring of $sl_{n+1}\mathbb{C}$

For $sl_{n+1}\mathbb{C}$ let $L_i \in \Lambda$ denote the weights and let $x_i = e(L_i) \in \mathbb{Z}[\Lambda]$ denote their corresponding basis element. Using WCF (or simpler formulas) one can determine that the representation ring of $sl_{n+1}\mathbb{C}$ is $\mathbb{Z}[A_1, \ldots, A_n]$, where A_i is the i^{th}elementary symmetric polynomial on the variables $x_1, \ldots, x_{n+1}$ with the additional requirement that $x_1 x_2 \ldots x_{n+1} = 1$.

2. The representation ring of $sp_{2n}\mathbb{C}$

For $sp_{2n}\mathbb{C}$, it can be determined that $R(sp_{2n}\mathbb{C}) \simeq \mathbb{Z}[C_1, \ldots, C_n]$ where C_i is the i^{th} elementary symmetric polynomial on the variables $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$. Also, the characters of the fundamental representations are given by $Char(\Gamma_1) = C_1$, $Char(\Gamma_2) = C_2 - C_1$, $Char(\Gamma_3) = C_3 - C_1$, $\ldots$, $Char(\Gamma_n) = C_n - C_{n-2}$.

Proposition C.4. $R(sp_2\mathbb{C}) \simeq R(sl_2\mathbb{C})$*.*

Proof. Notice that $A_1 = x_1 + x_2$,$A_2 = x_1 x_2 = 1$, $C_1 = (x_1) + \left(x_1^{-1}\right)$ but $C_2 = x_1 x_1^{-1} = 1$. So the identifications $x_2 = x_1^{-1}$, $A_1 = C_1$ and $A_2 = C_2$ make sense for $n = 2$, and we will use them in what follows. $\square$

3. Eigenvectors

As a subgroup of sp_{2n}, $sl_2 \times \cdots \times sl_2 = \mathfrak{g}$ has the same Cartan subalgebra (they both have the same rank). Therefore, the eigenvectors for the action of $\mathfrak{h}^*$ on $\mathfrak{g}$ are a subset of (in general a subspace of the span of) the eigenvectors for the action on sp_{2n}, which are

$$\{E_{i,j} - E_{n+i,n+j}, E_{i,n+j} - E_{j,n+i}, E_{n+i,j} - E_{n+j,i}, E_{i,n+i}, E_{n+i,i}\},$$

with corresponding roots $\{L_i - L_j, L_i + L_j, -L_i - L_j, 2L_i, -2L_i\}$. The eigenvectors for $\mathfrak{g}$ are $\{E_{i,n+i}, E_{n+i,i}\}$, and the corresponding roots are $\{2L_i, -2L_i\}$. We rescale so that the roots for $\mathfrak{g}$ are $\pm L_i$.

B. Decomposing the fundamental $sp(2n)$ modules as $sl(2)^{\times n}$-modules

Our goal is to determine a formula for the decomposition of the restriction corresponding to the inclusion $sl_2 \times \cdots \times sl_2 \subset sp_{2n}$, where we identify $sp_2 \simeq sl_2$.

The fundamental representations Γ_k of sp_{2n} are the kernels of the maps

$$\bigwedge^k \mathbb{C}^{2n} \to \bigwedge^{k-2} \mathbb{C}^{2n}.$$

So, we need to understand how to decompose the module

$$\bigwedge^k \mathbb{C}^{2n} = \bigwedge^k (V_1 \oplus \cdots \oplus V_n),$$

where $V_i \simeq \mathbb{C}^2$ as a $sl(2)^{\times n}$-module.

Theorem C.5. *The following decomposition holds as a $\mathfrak{g} \subset sp_{2n}$-module.*

$$\bigwedge^k (V_1 \oplus \cdots \oplus V_n) = \bigoplus_{(k_1,\ldots,k_n) \in \mathcal{P}_{n,2}(k)} \bigotimes_{i=1}^n \bigwedge^{k_i} V_i,$$

where $\mathcal{P}_{n,2}(k) = \{(k_1, \ldots, k_n) \mid k_1 + \cdots + k_n = k, k_i \in \{0, 1, 2\}\}$, the set of distinct *partitions of k with n parts, each part has size at most* 2.

Proof. Use induction on pairs (n, k). For $(n, k) = (1, 1)$ this is obvious. Assume the theorem is true for all pairs (i, j) such that $i < n$ and $j < k$.

We will use the following standard fact

$$\bigwedge^k (A \oplus B) = \bigoplus_{a+b=k} \bigwedge^a A \otimes \bigwedge^b B. \tag{C.1}$$

Apply formula (C.1) with $A = V_1$ and $B = V_2 \oplus \cdots \oplus V_n$ as follows:

$$\begin{aligned} \bigwedge^k (V_1 \oplus \cdots \oplus V_n) &= \bigoplus_{a+b=k} \bigwedge^a V_1 \otimes \bigwedge^b (V_2 \oplus \cdots \oplus V_n) \\ &= \left(\bigwedge^0 V_1 \otimes \bigwedge^k (V_2 \oplus \cdots \oplus V_n) \right) \\ &\quad \oplus \left(\bigwedge^1 V_1 \otimes \bigwedge^{k-1} (V_2 \oplus \cdots \oplus V_n) \right) \\ &\quad \oplus \left(\bigwedge^2 V_1 \otimes \bigwedge^{k-2} (V_2 \oplus \cdots \oplus V_n) \right). \end{aligned}$$

The summation ends after 3 steps because $dim(V_1) = 2$.

The induction hypothesis says that

$$\begin{aligned} \bigwedge^1 V_1 \otimes \bigwedge^{k-1} (V_2 \oplus \cdots \oplus V_n) &= \bigwedge^1 V_1 \bigoplus_{(k_2,\dots,k_n) \in \mathcal{P}_{n-1,2}(k-1)} \bigotimes_{i=2}^n \bigwedge^{k_i} V_i \\ &= \bigoplus_{(1,k_2,\dots,k_n) \in \mathcal{P}_{n,2}(k)} \bigotimes_{i=1}^n \bigwedge^{k_i} V_i, \end{aligned}$$

and

$$\bigwedge^2 V_1 \otimes \bigwedge^{k-2} (V_2 \oplus \cdots \oplus V_n) = \bigoplus_{(2,k_2,\dots,k_n) \in \mathcal{P}_{n,2}(k)} \bigotimes_{i=1}^n \bigwedge^{k_i} V_i.$$

But now, we need to see what to do with $\bigwedge^k (V_2 \oplus \cdots \oplus V_n)$. Using the formula

(C.1) again, we have

$$\begin{aligned}
\bigwedge^{k}(V_2 \oplus \cdots \oplus V_n) &= \left(\bigwedge^{0} V_2 \otimes \bigwedge^{k}(V_3 \oplus \cdots \oplus V_n)\right) \\
&\oplus \left(\bigwedge^{1} V_2 \otimes \bigwedge^{k-1}(V_3 \oplus \cdots \oplus V_n)\right) \\
&\oplus \left(\bigwedge^{2} V_2 \otimes \bigwedge^{k-2}(V_3 \oplus \cdots \oplus V_n)\right).
\end{aligned}$$

We can use the induction hypothesis for the factors involving $\bigwedge^{k-2}(V_3 \oplus \cdots \oplus V_n)$ and $\bigwedge^{k-1}(V_3 \oplus \cdots \oplus V_n)$. For the factor involving $\bigwedge^{k}(V_3 \oplus \cdots \oplus V_n)$, we continue to apply (C.1) to cut down the number of summands until the module $\bigwedge^{k}(V_i \oplus \cdots \oplus V_n)$ is zero just by dimension count. This completes the proof. □

Remark C.6. The statement and proof of the previous theorem works for the case of $\mathfrak{g} = gl(2)^{\times n} \subset gl(2n)$, and in particular, it allows us to conclude two things. First, we recognize that $V_1 \otimes \cdots \otimes V_n$ is in fact a $G = (Gl(2)^{\times n}) \ltimes \mathfrak{S}_n \subset Gl(2n)$-module where the G action is the induced action from $Gl(2n)$. And second, we can now identify the complementary G-module complement to $V_1 \otimes \cdots \otimes V_n$ in $\bigwedge^n (V_1 \oplus \cdots \oplus V_n) \simeq \bigwedge^n \mathbb{C}^{2n}$.

Using the decomposition from the previous theorem, we can now understand how the restriction map behaves: The terms on the right hand side are combinations of fundamental representations for $sp_2 \simeq sl_2$. We have

$$C_k\left(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\right) \mapsto \sum_{(k_1,\ldots,k_n)\in\mathcal{P}_{n,2}(k)} \bigotimes_{i=1}^{n} C^i_{k_i},$$

where $C^i_{k_i}$ are the elementary symmetric polynomials on the variables $\left\{x_i, x_i^{-1}\right\}$, i.e. $C^i_1 = x_i + x_i^{-1}$ and $C^i_0 = C^i_2 = 1$.

Proposition C.7. *The decomposition of the fundamental representation* Γ_n *of* sp_{2n}

as an $sl_2 \times \cdots \times sl_2$-module is given by the following character:

$$C_n - C_{n-2} \mapsto \sum_{(k_1,\ldots,k_n)\in\mathcal{P}_{n,2}(k)} \bigotimes_{i=1}^{n} C^i_{k_i} - \sum_{(k_1,\ldots,k_n)\in\mathcal{P}_{n,2}(k-2)} \bigotimes_{i=1}^{n} C^i_{k_i}.$$

Proposition C.8. *We can give a refinement:*

$$C_n - C_{n-2} \mapsto \sum_{l=0}^{n} \sum_{K_l} \left(\frac{1}{m+1}\right) \binom{2m}{m} C_1^{i_1} \otimes C_1^{i_2} \otimes \cdots \otimes C_1^{i_l} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-l},$$

where K_l is a partition with precisely l-1's. More specifically, $K_l = (k_1,\ldots,k_n) \in \mathcal{P}_{n,2}(k)$ with $k_{i_1} = \cdots = k_{i_l} = 1$ and $k_j \in \{0,2\}$ whenever $j \notin \{i_1,\ldots,i_l\}$ and we define m by, $2m = n-l$. Implicitly we are including the requirement that if n is even (odd) then l must be even (odd) also.

Proof. This proposition comes from counting the number of isomorphic modules in the image. There are $\binom{n}{l}$ such choices for the location of the ones in K_l.

Once the locations for the ones in K_l have been chosen, there are $\binom{n-l}{\frac{n-l}{2}} = \binom{2m}{m}$ choices for the remaining zeros and twos. Each of these modules is isomorphic. We can see that under the restriction map, C_k hits all $\binom{2m}{m}$ possibilities, whereas C_{k-2} only hits $\binom{n-l}{\frac{n-l-2}{2}} = \binom{2m}{m-1}$ of them. Using basic facts about binomial coefficients, we see that there are $\binom{2m}{m} - \binom{2m}{m-1} = \left(\frac{1}{m+1}\right)\binom{2m}{m}$ isomorphic copies of the module with l ones in the prescribed locations given by K_l. This is what we wanted to show. $\square$

Since all the modules with l ones are isomorphic (no matter their location) we see that the multiplicity for such modules in the image of $C_k - C_{k-2}$ is $\binom{n}{l}\binom{2m}{m}\left(\frac{1}{m+1}\right)$.

Finally, we arrive at our goal:

Theorem C.9. *The decomposition of Γ_n as an $sl_2 \times \cdots \times sl_2$ module into irreducibles is given by*

$$\Gamma_n \simeq \bigoplus_l \left[\bigotimes_{i=1}^{l} V_i \otimes \bigotimes_{k=1}^{n-l} \mathbb{C} \right]^{\oplus N_l}, \tag{C.2}$$

where $N_l = \binom{n}{l}\binom{2m}{m}\left(\frac{1}{m+1}\right)$ *and* $2m = n - l$*. (Implicitly, if* n *is even (odd) then* l *is even (odd)).*

Remark C.10. This theorem allows us to conclude two things. First, we recognize that $V_1 \otimes \cdots \otimes V_n$ is in fact a G-module where the G action is the induced action from $Sp(2n)$. And second, we can now identify the complementary G-module complement to $V_1 \otimes \cdots \otimes V_n$ in Γ_n.

APPENDIX D

MAPLE CODE

A. Using characters to compute an isotypic decomposition

```
with(combinat):

reverse := proc(L::list)
   [seq(L[nops(L)-i+1], i = 1 .. nops(L))]
end proc:

mypartition := proc(d ::integer)
   local X;
   X := reverse(partition(d)):
   return [seq(reverse(X[i]),i=1..numbpart(d))];
end proc:

myconjpart := proc (L::list)
   local preK, K;
   description "I needed to have a conjugate partition
      function that is in decreasing order.";
   return reverse(conjpart(reverse(L)));
end proc;

smash := proc (u, v)
   description "smash two vectors together";
   [seq(u[i], i = 1 .. nops(u)), seq(v[i], i = 1 .. nops(v))]
end proc;

dimModule := proc (LL::list, k)
   local L, m;
   description "this just applies a formula from
      Fulton and Harris pg 77";
   L := smash(LL, [seq(0, i = 1 .. k)]);
   m := `*`(seq(seq((L[i]-L[j]+j-i)/(j-i), j = i+1 .. k), i = 1 .. k-1));
   if k < nops(LL) then m := 0 end if;
   return m
end proc;

numbclass:= proc (L::list) local m, f, g, r, top, i, j, c, LL;
   description "see exercise in
   http://www.math.unibas.ch/~kraft/Papers/KP-Primer.pdf
   for the formula for computing the number of elements
   in a given conjugacy class.  This procedure computes
   the number of elements in the conjugacy class
   corresponding to a given partition as imput";
   top := max(seq(L[k], k = 1 .. nops(L)));
   for i to top do c := 0;
      for j to nops(L) do
         if L[j] = i then c := c+1 end if
      end do;
      r[i] := c
   end do;
   LL := [seq(r[i], i = 1 .. top)];
   m := `+`(seq(i*LL[i], i = 1 .. nops(LL)));
```

```
   f := `*`(seq(i^LL[i], i = 1 .. nops(LL)));
   g := `*`(seq(factorial(LL[i]), i = 1 .. nops(LL)));
   return factorial(m)/(f*g)
end proc;


mults := proc (degree::integer, numfacts::integer, maxdim::integer)
   local M, Par, i, p, X, IP2, N, NN, k, numpar, temp, stopper, d, n,
      count, myindicator, myoutput, breaker, cc, K;
   description "This procedure computes the multiplicites of the
      irreducible modules in the decomposition of
      S^d(A_1\otimes \dots \otimes A_n).  It expects three integers,
      the degree - d, the number of factors -n, and the maximum
      dimension of the A_i - maxdim.";
   d := degree;
   n := numfacts;
   if d = 1 then return start*over end if;
   K := partition(d);
   N := [seq(numbclass(K[i]), i = 1 .. nops(K))];
   M := character(d);
   Par := mypartition(d);
   numpar := nops(Par);
   count := 1;
   for i to numpar do
      if evalb(nops(Par[i]) <= maxdim) then
         myindicator[count] := i; count := count+1
      end if
   end do;
   stopper := count-1;
   count := 'count';
   for i to stopper do
      X[myindicator[i]] := S[Par[myindicator[i]]]
   end do;
   i := 'i'; cc := 1;
   if n <= 10 then
   breaker := [seq(1, i = 1 .. n), seq(0, i = n+1 .. 11)];
      for i[1] to stopper do
         for i[2] from i[1] to i[1]+breaker[2]*(stopper-i[1]) do
         for i[3] from i[2] to i[2]+breaker[3]*(stopper-i[2]) do
         for i[4] from i[3] to i[3]+breaker[4]*(stopper-i[3]) do
         for i[5] from i[4] to i[4]+breaker[5]*(stopper-i[4]) do
         for i[6] from i[5] to i[5]+breaker[6]*(stopper-i[5]) do
         for i[7] from i[6] to i[6]+breaker[7]*(stopper-i[6]) do
         for i[8] from i[7] to i[7]+breaker[8]*(stopper-i[7]) do
         for i[9] from i[8] to i[8]+breaker[9]*(stopper-i[8]) do
         for i[10] from i[9] to i[9]+breaker[10]*(stopper-i[9]) do
   temp := simplify((`+`(seq(N[q]*(`*`(
            seq(M[myindicator[i[p]], q], p = 1 .. n))),
            q = 1 .. nops(Par))))/factorial(d));
            if 0 < temp then
               myoutput[cc] :=
                  [temp, [seq(X[myindicator[i[p]]], p = 1 .. n)]];
               cc := cc+1;
         end if
         end do
         end do
         end do
         end do
         end do
         end do
```

```
          end do
          end do
          end do
        end do
     end if;
   return [seq(myoutput[i], i = 1 .. cc-1)]
end proc;

##examples ##

mults(3,6,3);

dimModule([3,2,1],4);
```

B. Making polynomials via Young symmetrizers

This Maple file contains procedures that construct highest weight vectors. Many of the examples we have included at the end of this file are found in [19]. In the final example we find that no submodule of $\left(S_{(3,2,1)}V_1 \otimes S_{(3,2,1)}V_1 \otimes S_{(3,1,1,1)}V_1\right)^{\oplus 4}$ occurs in the ideal of $\sigma_4\left(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)\right)$.

```
with(LinearAlgebra): with(combinat):

### some necessary procedures ###
reverse := proc(L::list)
   return [seq(L[nops(L)-i+1],i = 1 .. nops(L))]
end proc:

mypartition := proc(d ::integer)
   local X;
   X := reverse(partition(d)):
   return [seq(reverse(X[i]),i=1..numbpart(d))];
end proc:

myconjpart := proc (L::list) local preK, K;
   description "I needed to have a conjugate partition function that is in
   decreasing order.";
   return reverse(conjpart(reverse(L)));
end proc:

### procedures specific to this task ###

makeDets:=proc(a,LL::list,mu::list)
   local L:
   description "this procedure makes a product of
determinants of sizes determined by a partition LL. The
first index of each column is twisted by a permutation mu.";
   L:=myconjpart(LL):
   return `*`(seq(Determinant(Matrix([seq([seq(a[op(j+( `+`
(seq(op(p,L),p=1..k-1))),mu),i],j=1 .. op(k,L))], i = 1 .. op
(k,L))])) ,k=1..nops(L)))
end proc:

makeUnsymmetric:=proc(J::list,K::list)
```

```
description "this procedure takes in a list of partitions J and
a list of permutations K and produces the unsymmetrized
(and factored!) tensor";
local alpha;
   if(nops(J)<> nops(K)) then
      return "uneven";
   else
      alpha:= [a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z]:
      return `*`(seq(makeDets(alpha[i],J[i],K[i]),i=1..nops(J)))
   fi:
end proc:

unfactor:= proc(X,degree,L::list)
description "X is the tensor, d is the degree, L is the list of
dimensions of the vector spaces":
local temp,temp2,p,i;
if nops(L) >8 then
   return "too many factors";
fi:

if nops(L) = 1 then
   temp2 := X;
   for p to degree do
      temp := 0;
      for i[1] from 0 to op(1,L)-1 do
temp := coeff(temp2, a[p, i[1]+1])*Z[[i[1]]]+temp;
      end do;
      temp2 := temp; #print(nops(temp2))
   end do;
   return temp;
fi:

if nops(L) = 2 then
   temp2 := X;
   for p to degree do
      temp := 0;
         for i[1] from 0 to op(1,L)-1 do
            for i[2] from 0 to op(2,L)-1 do
temp := coeff(coeff(temp2, a[p, i[1]+1]), b[p, i[2]+1])
*Z[[ seq(i[p],p=1..nops(L) )]]+temp
            end do
         end do;
      temp2 := temp; #print(nops(temp2))
   end do;
   return temp;
fi:

if nops(L) = 3 then
   temp2 := X;
   for p to degree do
      temp := 0;
      for i[1] from 0 to op(1,L) do
         for i[2] from 0 to op(2,L) do
            for i[3] from 0 to op(3,L) do
temp := coeff(coeff(coeff(temp2, a[p, i[1]+1]), b[p, i
[2]+1]), c[p, i[3]+1])*Z[[ seq(i[p],p=1..nops(L) )]]+temp
         end do
      end do
   end do;
   temp2 := temp; #print(nops(temp2))
end do;
```

```
return temp;
fi:

if nops(L) = 4 then
   temp2 := X;
   for p to degree do
      temp := 0;
      for i[1] from 0 to op(1,L) do
         for i[2] from 0 to op(2,L) do
            for i[3] from 0 to op(3,L) do
               for i[4] from 0 to op(4,L) do
   temp := coeff(coeff(coeff(coeff(temp2, a[p, i[1]
+1]), b[p, i[2]+1]), c[p, i[3]+1]),d[p, i[4]+1])*Z[[ seq(i
[p],p=1..nops(L) )]]+temp ;
               end do
            end do
         end do
      end do;
      temp2 := temp; #print(nops(temp2))
   end do;
   return temp;
fi:

if nops(L) = 5 then
   temp2 := X;
   for p to degree do
      temp := 0;
      for i[1] from 0 to op(1,L) do
         for i[2] from 0 to op(2,L) do
            for i[3] from 0 to op(3,L) do
               for i[4] from 0 to op(4,L) do
                  for i[5] from 0 to op(5,L) do
      temp := coeff(coeff(coeff(coeff(coeff(temp2, a[p,
i[1]+1]), b[p, i[2]+1]), c[p, i[3]+1]),d[p, i[4]+1]),e[p,i[5]+1])*Z[[
seq(i[p],p=1..nops(L) )]]+temp
                  end do
               end do
            end do
         end do
      end do;
      temp2 := temp; #print(nops(temp2))
   end do;
   return temp;
fi:

if nops(L) = 6 then
   temp2 := X;
   for p to degree do
      temp := 0;
      for i[1] from 0 to op(1,L) do
         for i[2] from 0 to op(2,L) do
            for i[3] from 0 to op(3,L) do
               for i[4] from 0 to op(4,L) do
                  for i[5] from 0 to op(5,L) do
                     for i[6] from 0 to op(6,L) do
temp := coeff(coeff(coeff(coeff(coeff(coeff
(temp2, a[p, i[1]+1]), b[p, i[2]+1]), c[p, i[3]+1]),d[p, i[4]+1]),e
[p,i[5]+1]),f[i[6]+1])*Z[[ seq(i[p],p=1..nops(L) )]]+temp
                     end do
                  end do
```

```
                end do
             end do
          end do
       end do;
       temp2 := temp; #print(nops(temp2))
    end do;
    return temp;
fi:

if nops(L) = 7 then
    temp2 := X;
    for p to d do
       temp := 0;
       for i[1] from 0 to op(1,L) do
          for i[2] from 0 to op(2,L) do
             for i[3] from 0 to op(3,L) do
                for i[4] from 0 to op(4,L) do
                   for i[5] from 0 to op(5,L) do
                      for i[6] from 0 to op(6,L) do
                         for i[7] from 0 to op(7,L) do
    temp := coeff(coeff(coeff(coeff(coeff(coeff
(coeff(temp2, a[p, i[1]+1]), b[p, i[2]+1]), c[p, i[3]+1]),d[p, i[4]
+1]),e[p,i[5]+1]),f[i[6]+1]),g[i[7]+1])*Z[[ seq(i[p],p=1..nops
(L) )]]+temp
                         end do
                      end do
                   end do
                end do
             end do
          end do
    end do;
    temp2 := temp; #print(nops(temp2))
    end do;
    return temp;
fi:
end proc:

## examples ##

T:=makeUnsymmetric([[2,1,1,1],[3,1,1],[2,1,1,1]],[[1,2,3,4,5],
[1,5,3,4,2],[1,4,5,2,3]]):
unfactor(T,5,[3,2,3]): nops(expand(%));

T:=makeUnsymmetric([[2,2,2],[2,2,2],[3,1,1,1]],[[1,2,3,4,5,6],
[1,5,3,4,2,6],[1,4,5,2,3,6]]):
unfactor(T,6,[2,2,3]): nops(expand(%));

T:=makeUnsymmetric([[3,1,1],[3,1,1],[2,2,1]],[[1,2,3,4,5],
[1,4,5,2,3],[1,2,3,4,5]]):
unfactor(T,5,[2,2,2]): nops(expand(%));

T:=makeUnsymmetric([[3,1,1],[3,1,1],[2,2,1]],[[1,2,3,4,5],
[1,2,4,3,5],[1,3,5,2,4]]):
unfactor(T,5,[2,2,2]): nops(expand(%));

## Here is an example where there is multiplicity greater
than 1 and Landberg and Manivel have already guessed
the correct permutations to give linearly independent
elements of the highest weight space ##

sigma:=[1,2,3,5,6,4]:
tau := [3, 4, 5, 1, 2, 6]; mu := [1, 4, 5, 6, 2, 3];
```

```
T1:=makeUnsymmetric([[3,2,1],[3,2,1],[3,1,1,1]],
[sigma,tau,mu]):
P1:=unfactor(T1,6,[2,2,3]): nops(expand(%));

tau := [3, 4, 5, 1, 2, 6]; mu := [2, 3, 5, 6, 1, 4];
T2:=makeUnsymmetric([[3,2,1],[3,2,1],[3,1,1,1]],
[sigma,tau,mu]):
P2:=unfactor(T2,6,[2,2,3]): nops(expand(%));

tau := [3, 4, 5, 1, 2, 6]; mu := [2, 3, 4, 5, 1, 6];
T3:=makeUnsymmetric([[3,2,1],[3,2,1],[3,1,1,1]],
[sigma,tau,mu]):
P3:=unfactor(T3,6,[2,2,3]): nops(expand(%));

tau := [3, 4, 6, 1, 2, 5]; mu := [2, 3, 4, 5, 1, 6];
T4:=makeUnsymmetric([[3,2,1],[3,2,1],[3,1,1,1]],
[sigma,tau,mu]):
P4:=unfactor(T4,6,[2,2,3]): nops(expand(%));

PP:= ss*P1+tt*P2+uu*P3+vv*P4:

for count from 1 to 4 do
 for j to 6 do
mysegrepoint[j] := expand(unfactor(`+`(seq(`+`(seq(a[1,k]
*U[[k-1,i]],k=1..3))*`+`(seq(b[1,k]*V[[k-1,i]],k=1..3))*`+`(seq(c
[1,k]*W[[k-1,i]],k=1..4)),i=1..j)), 1, [2,2,3] ));
end do:
 for j to 6 do mysegrerandomizer := {}:
  for i to j do
  x:= RandomVector(3); y := RandomVector(3); z :=
RandomVector(4);
  mysegrerandomizer := `union`(mysegrerandomizer, {seq
(U[[k-1,i]] = x[k], k = 1 .. 3), seq(V[[k-1,i]] = y[k], k = 1 .. 3),
seq(W[[k-1,i]] = z[k], k = 1 .. 4)})
end do:
x := 'x'; y := 'y'; z := 'z':
end do:
for j to 6 do
 myrandomsegrepoint[j] := subs(mysegrerandomizer,
mysegrepoint[j])
end do:
for p to 6 do
mysegreevaluator || p := {seq(seq(seq(Z[[i, j, k]] = coeff
(mysegrepoint[p], Z[[i, j, k]]), i = 0 .. 2), j = 0 .. 2), k = 0 .. 3)}:

myrandomsegreevaluator || p := {seq(seq(seq(Z[[i, j, k]] =
coeff(myrandomsegrepoint[p], Z[[ i,j,k]]), i = 0 .. 2), j = 0 .. 2),
k = 0 .. 3)}:

end do:

val||count:=subs(myrandomsegreevaluator || 4, expand
(PP));
od:

solve({val1,val2,val3,val4});
```

C. Construction of a weight basis of a Schur module

This Maple file reads a file called "initialvectors" in "yourpath." It expects that the file contain highest weight polynomials F, G and H in the variables $xijkl$ with $0 \le i, j, k, l \le 1$. The output is put into a file "outputbasis". There is also a check that the polynomials generated by this program actually vanish on the variety of principal minors. Many aspects of this program have been tailored to the polynomials F, G, H given to us by Lin and Sturmfels, however this file can be easily adapted to many other applications.

```
restart:with(combinat):with(LinearAlgebra):

monoweight := proc (X)
   local K, Y;
   description "This procedure calculates the weight of a monomial.";
   if type(op(1, X), list) then K := op(1, X); return [seq(K[i], i =
      1 .. nops(K))]
   elif type(op(1, X), integer) then return X
   else return "bad imput"
   end if
end proc;

weight := proc (X)
   local i, S, Y, WP, count;
   description "This procedure calculates the weight of an expression.";
   if op(0, X) = `+` then
      Y := op(1, expand(X)) else Y := X end
   if;
   count := 1;
   if op(0, Y) = `*` then
      for i to nops(Y) do
         if not type(op(i, Y), integer) then
            if type(op(i, Y), atomic) then S[count] := monoweight(op(i,
               Y)); count := count+1
            end if;
            if op(0, op(i, Y)) = `^` then
               S[count] := monoweight(op(1, op(i, Y)))*op(2, op(i, Y));
               count := count+1
            end if
         end if
      end do;
      if 2 <= count then
         WP := S[1];
         for i from 2 to count-1 do
            WP := WP+S[i]
         end do
      end if;
   return WP
   elif type(X, atomic) then
      return monoweight(X)
```

```
   elif op(0, X) = `^` then
      return monoweight(op(1, X))*op(2, X)
   end if
end proc;

#Example:
weight(Z[[1,3,1,1]]*Z[[1,3,1,1]]);

raise := proc (x, f)
   local temp, i, j, k;
   description "this procedure raises the vector x in the f th factor
   (limited to 4 factors right now) and raises the p th coordinate.";
   temp := 0;
   if f = 1 then
   for i from 0 to 1 do
      for j from 0 to 1 do
         for k from 0 to 1 do
temp := temp+(diff(expand(x), Z[[0, i, j, k]]))*Z[[1, i, j, k]]
         end do
      end do
   end do;
   elif f = 2 then
       for i from 0 to 1 do
          for j from 0 to 1 do
             for k from 0 to 1 do
temp := temp+(diff(expand(x), Z[[i, 0, j, k]]))*Z[[i, 1, j, k]]
             end do
          end do
       end do;
   elif f = 3 then
      for i from 0 to 1 do
         for j from 0 to 1 do
            for k from 0 to 1 do
temp := temp+(diff(expand(x), Z[[i, j, 0, k]]))*Z[[i, j, 1, k]]
            end do
         end do
      end do;
   elif f = 4 then
      for i from 0 to 1 do
         for j from 0 to 1 do
            for k from 0 to 1 do
temp := temp+(diff(expand(x), Z[[i, j, k, 0]]))*Z[[i, j, k, 1]]
            end do
         end do
      end do;
   end if;
   return expand(temp)
end proc:

lower := proc (x, f)
   local temp, i, j, k;
   description "this procedure raises the vector x in the f th factor
   (limited to 4 factors right now) and lowers the p th coordinate.";
   temp := 0;
   if f = 1 then
      for i from 0 to 1 do for j from 0 to 1 do for k from 0 to 1 do
temp := temp+(diff(expand(x), Z[[1, i, j, k]]))*Z[[0, i, j, k]]
      end do end do end do;
   elif f = 2 then
      for i from 0 to 1 do for j from 0 to 1 do for k from 0 to 1 do
temp := temp+(diff(expand(x), Z[[i, 1, j, k]]))*Z[[i, 0, j, k]]
      end do end do end do;
   elif f = 3 then
      for i from 0 to 1 do for j from 0 to 1 do for k from 0 to 1 do
```

```
temp := temp+(diff(expand(x), Z[[i, j, 1, k]]))*Z[[i, j, 0, k]]
      end do end do end do;
   elif f = 4 then
      for i from 0 to 1 do for j from 0 to 1 do for k from 0 to 1 do
temp := temp+(diff(expand(x), Z[[i, j, k, 1]]))*Z[[i, j, k, 0]]
      end do end do end do;
   end if;
   return expand(temp)
end proc:

varschange:=seq(seq(seq(seq( cat(cat(cat(cat(x,i),j),k),l)
= Z[[i,j,k,l]],i=0..1),j=0..1),k=0..1),l=0..1);

varschangeback:=seq(seq(seq(seq( Z[[i,j,k,l]] = cat(cat(cat
(cat(x,i),j),k),l) ,i=0..1),j=0..1),k=0..1),l=0..1);

with(linalg);
sqtest := proc (f) local A, para, paraZ;
   A := randmatrix(4, 4);
   para := {x1111 = det(submatrix(A, [1, 2, 3, 4], [1, 2, 3, 4])),
   x1110 = det(submatrix(A, [1, 2, 3], [1, 2, 3])),
   x1101 = det(submatrix(A, [1, 2, 4], [1, 2, 4])),
   x1011 = det(submatrix(A, [1, 3, 4], [1, 3, 4])),
   x0111 = det(submatrix(A, [2, 3, 4], [2, 3, 4])),
   x1100 = det(submatrix(A, [1, 2], [1, 2])),
   x1010 = det(submatrix(A, [1, 3], [1, 3])),
   x1001 = det(submatrix(A, [1, 4], [1, 4])),
   x0110 = det(submatrix(A, [2, 3], [2, 3])),
   x0101 = det(submatrix(A, [2, 4], [2, 4])),
   x0011 = det(submatrix(A, [3, 4], [3, 4])),
   x0001 = det(submatrix(A, [4], [4])),
   x0010 = det(submatrix(A, [3], [3])),
   x0100 = det(submatrix(A, [2], [2])),
   x1000 = det(submatrix(A, [1], [1])),
   x0000 = 1};
   paraZ := subs(varschange, para);
   return subs(paraZ, f)
end proc;

read "/yourpath/initialvectors";


FZ:=subs(varschange,F):nops(%);sqtest(FZ);
GZ:=subs(varschange,G):nops(%);sqtest(GZ);
HZ:=subs(varschange,H):nops(%);sqtest(HZ);
weight(FZ);
lower(FZ,4):nops(%);
Hmodule[0]:=HZ:
for i from 1 to 6 do
   raise(Hmodule[i-1],1):
   if %<>0 then
      Hmodule[i]:=%:
      print(weight(%));
   fi:
od:

seq(nops(Hmodule[i]),i=0..6);
seq(sqtest(Hmodule[i]),i=0..6);

c:=1:
Gmodule[0,0]:=GZ:
for i from 1 to 4 do
```

```
   raise(Gmodule[i-1,0],1):
   if %<>0 then
      Gmodule[i,0]:=%:
      print(weight(%),[c]);
      c:=c+1:
   fi:
od:

for i from 0 to 4 do
   for j from 1 to 4 do
      raise(Gmodule[i,j-1],2):
         if %<>0 then
            Gmodule[i,j]:=%:
            print(weight(%),[c]);
            c:=c+1:
         fi:
   od:
od:
c;
seq(seq(nops(Gmodule[i,j]),i=0..4),j=0..4);
seq(seq(sqtest(Gmodule[i,j]),i=0..4),j=0..4);

c:=0:
for i from 0 to 4 do
   for j from 0 to 2 do
      for k from 0 to 2 do
         for l from 0 to 2 do
            Fmodule[i,j,k,l]:=0:c:=c+1:
         od:
      od:
   od:
od:

c:=1:
Fmodule[0,0,0,0]:=FZ:
for i from 1 to 4 do
   raise(Fmodule[i-1,0,0,0],1):
   if %<>0 then
      Fmodule[i,0,0,0]:=%:
      print(weight(%),[c]);
      c:=c+1:
   fi:
od:

for i from 0 to 4 do
   for j from 1 to 2 do
      raise(Fmodule[i,j-1,0,0],2):
      if %<>0 then
         Fmodule[i,j,0,0]:=%:
         print(weight(%),[c]);
         c:=c+1:
      fi:
   od:
od:

for i from 0 to 4 do
   for j from 0 to 2 do
      for k from 1 to 2 do
         raise(Fmodule[i,j,k-1,0],3):
         if %<>0 then
            Fmodule[i,j,k,0]:=%:
            print(weight(%),[c]);
            c:=c+1:
         fi:
      od:
```

```
   od:
od:

for i from 0 to 4 do
   for j from 0 to 2 do
      for k from 0 to 2 do
         for l from 1 to 2 do
            raise(Fmodule[i,j,k,l-1],4):
            if %<>0 then
               Fmodule[i,j,k,l]:=%:
               print(weight(%),[c]);
               c:=c+1:
            fi:
         od:
      od:
   od:
od:

seq(seq(seq(seq(nops(Fmodule[i,j,k,l]),i=0..4),j=0..2),k=0..2),l=0..2);

seq(seq(seq(seq(sqtest(Fmodule[i,j,k,l]),i=0..4),j=0..2),k=0..2),l=0..2);

subs(varschangeback,Fmodule[1,1,1,1]):op(1,%);

for i from 0 to 4 do
   for j from 0 to 2 do
      for k from 0 to 2 do
         for l from 0 to 2 do
            xFmodule[i,j,k,l]:=subs(varschangeback,Fmodule[i,j,k,l]):
         od:
      od:
   od:
od:

for i from 0 to 4 do
   for j from 0 to 4 do
      xGmodule[i,j]:=subs(varschangeback,Gmodule[i,j]):
   od:
od:

for i from 0 to 6 do
   xHmodule[i]:=subs(varschangeback,Hmodule[i]):
od:

fd := fopen("outputbasis", APPEND):
c:=1:
for i from 0 to 4 do for j from 0 to 2 do for k from 0 to 2 do
         for l from 0 to 2 do
            fprintf(fd, "F[%a] =  %a : \n\n", c, xFmodule[i,j,k,l]):
            c:=c+1:
         od:
od:od:od:
c:=1:
for i from 0 to 4 do for j from 0 to 4 do
      fprintf(fd, "G[%a] =  %a : \n\n", c, xGmodule[i,j]):
      c:=c+1:
od:od:
c:=1:
for i from 0 to 6 do
   fprintf(fd, "H[%a] =  %a : \n\n", c, xHmodule[i]):
c:=c+1:
od:
fclose(fd):
```

VITA

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