CONTRIBUTIONS TO THE COMPACTNESS THEORY OF THE $\overline{\partial}$-NEUMANN OPERATOR

A Dissertation

by

MEHMET ÇELİK

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2008

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Emil J. Straube
Committee Members,	Harold P. Boas
	Paulo Lima-Filho
	Michael Longnecker
Head of Department,	Al Boggess

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ABSTRACT

Contributions to the Compactness Theory

of the $\overline{\partial}$-Neumann Operator. (May 2008)

Mehmet Çelik, B.S., Marmara University, Turkey

Chair of Advisory Committee: Dr. Emil J. Straube

This dissertation consists of three parts. In the first part we explore compactness of the $\overline{\partial}$-Neumann operator via the ideal of compactness multipliers. We study the zero set of the compactness multipliers as the obstruction to compactness of the $\overline{\partial}$-Neumann problem. That is, the ideal of compactness multipliers on a given domain is used with the purpose to get algebraic information about the boundary of the domain.

The second part of the dissertation is about independence from the metric of some estimates for the $\overline{\partial}$-Neumann operator. It is a theorem by W. J. Sweeney in a very general manner which implies that coercive estimates are independent of the metric on the cotangent bundle, and it seems to be folklore that this is also true for subelliptic operators. The metric considered is smooth positive definite hermitian on the whole closure of the domain. Here we give a simple proof specifically for subellipticity of the $\overline{\partial}$-Neumann operator. We also show that compactness of the $\overline{\partial}$-Neumann operator is independent of the metric.

The third part of the dissertation is about compactness of the $\overline{\partial}$-Neumann operator on a transversal intersection of two smooth domains, both of which have a compact $\overline{\partial}$-Neumann operator. In order to understand the properties of compactness of the $\overline{\partial}$-Neumann problem, this question is of fundamental importance. In particular, this problem serves as a test to see whether there might be a reasonable notion of obstruction to compactness that lives in the boundary.

The results of our research show that on the domains where compactness is understood, one can identify the common zero set of the multiplier ideal, the obstruction for compactness of the $\overline{\partial}$-Neumann problem. Moreover, although the Sobolev estimates for the $\overline{\partial}$-Neumann operator are not independent of the metric our results also show that compactness and subelliptic estimates of the $\overline{\partial}$-Neumann operator are. Furthermore, on the intersection of two domains compactness of the $\overline{\partial}$-Neumann operator holds when one of the domains satisfies property (P).

To my wife Yuliya

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CHAPTER I

INTRODUCTION

A. Cauchy-Riemann Equations

Many of the differences between the function theories of one and several complex variables can be accounted for by the nature of holomorphic functions. A holomorphic function is a function which is locally a power series in the variables z_j where $j = 1,\ 2,\ ...,\ n$. Some of the differences are based on the following properties of holomorphic functions: isolated singularities, integral representation, the Riemann Mapping theorem, continuation of holomorphic functions on open sets. Besides these there is one difference which is considered as a principal tool in constructing holomorphic functions with some specific properties on a given domain in multidimensional complex case, inhomogeneous Cauchy-Riemann equations. In one complex variable theory there exist so many tools for constructing holomorphic functions: Weierstrass products, Runge's, Mergelyan's, Mittag-Leffler theorems and so on, none of which are useful on a large class of domains in multidimensional complex space. Existence of such powerful theorems suppresses the role of the inhomogeneous Cauchy-Riemann equations in one complex variable case. However in the multidimensional complex case these equations play a central role in the development of the theory.

In $\mathbb{C}$, $f = u + iv$, $z = x + iy$, the Cauchy-Riemann operator is $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. A way to express the operator $\frac{\partial}{\partial \overline{z}}$ acting on a function f is to use differential form $d\overline{z}$, dual to the vector field $\frac{\partial}{\partial \overline{z}}$. Define $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z}$. Therefore, for $g d\overline{z} = (Re(g) + iIm(g))d\overline{z}$

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a given 1-form, the inhomogeneous Cauchy - Riemann equations, $\overline{\partial} f = g$, that is

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = Re(g) \ \ \& \ \ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = Im(g)$$

form a system of two equations in two unknown real functions, u & v. However, in $\mathbb{C}^n$ for $n > 1$, the Cauchy-Riemann equations form a system of $2n$ equations in two unknown real functions; the system becomes overdetermined.

Today a large part of the study of several complex variables theory is about the domains of holomorphy. Topics related to this study include the space of square integrable holomorphic functions (Bergman space), classification of domains of holomorphy (pseudoconvex, strongly pseudoconvex, Reinhardt, Hartogs domains), $\overline{\partial}$-operator and the solution to the inhomogeneous Cauchy - Riemann equations

$$\overline{\partial} f = g \ \text{ with a necessary condition } \ \overline{\partial} g = 0. \tag{1.1}$$

That is, if $(z_1, ..., z_n) = (x_1 + iy_1, ..., x_n + iy_n)$ denotes the standard coordinates on $\mathbb{C}^n$ the equation (1.1) is $\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right) = g_j$, $\forall j \in \{1, ..., n\}$ with a necessary condition $\frac{\partial g_j}{\partial \bar{z}_k} = \frac{\partial g_k}{\partial \bar{z}_j} \ \ \forall j, k \in \{1, ..., n\}$.

The proof of one of the fundamental problems in several complex variables theory, the equivalence between domains of holomorphy and pseudoconvex domains, uses the solution of the inhomogeneous Cauchy-Riemann equations on that type of domains. For a given smooth data g on a domain finding a smooth solution f for the equation (1.1) on that domain yields the equivalence of the above two types domains. On the other hand, if one tries to understand the boundary behavior of holomorphic functions, for example, whether they can be extended smoothly to the boundary of a domain or not, one should study the boundary behavior of the Cauchy-Riemann equations. For a given smooth datum g on the closure of the domain one should look for a solution of the equation (1.1) which is smooth on the closure of that domain.

Suppose you found such a solution, say f. Now, since we are on a pseudoconvex domain which is equivalent to a domain of holomorphy [19], it is always possible to find a holomorphic function that can not be smoothly extended to a given boundary point, say that it is h. It is clear that $f + h$ is also a solution of (1.1), but the solution $f + h$ is not smooth on the closure of the domain. It is obvious that we should get rid of the holomorphic part of the solution, so one has to look for particular solutions. In the L^2-theory, such a particular solution is provided by the unique solution that is orthogonal to the holomorphic functions. Generally, the equation (1.1) is considered as an equation for a $(0, q)$-form f, given a $(0, q + 1)$-form g. Thus, in the above discussions holomorphic functions can be replaced by closed forms, elements of the kernel of $\overline{\partial}$. Moreover, on the last discussion the solution to (1.1) which is of interest to be orthogonal to holomorphic functions can be interpreted as a $(0, q)$-form orthogonal to the closed $(0, q)$-forms, that is orthogonal to the $\ker(\overline{\partial})$. On the other hand, a form g is said to be closed if $\overline{\partial} g = 0$ and exact if $g = \overline{\partial} f$ for some form f.

1. The relation between the $\overline{\partial}$-problem and the $\overline{\partial}$-Neumann problem

The Hodge decomposition states that any form g can be split into three L^2 components: $g = \overline{\partial} m_1 + \overline{\partial}^\star m_2 + \gamma$ where γ is harmonic. This follows by noting that exact and co-exact forms are orthogonal; the orthogonal complement then consists of forms that are both closed and co-closed. Orthogonality is defined with respect to the L^2 inner product on the given domain. Recall that we are looking for a solution orthogonal to the closed forms. The Hodge decomposition is providing us a useful machinery to define an explicit solution to the inhomogeneous $\overline{\partial}$-problem. This is exactly the part where the $\overline{\partial}$-Neumann problem connects with the $\overline{\partial}$-problem.

The $\overline{\partial}$-Neumann problem is a boundary value problem for the second order operator

$$\Box = \overline{\partial}\ \overline{\partial}^{\star} + \overline{\partial}^{\star}\ \overline{\partial}. \tag{1.2}$$

$\Box$ is closed, densely defined, selfadjoint operator. The $\overline{\partial}$-Neumann problem consists of inverting the $\Box$ operator. While the operator $\Box$ is elliptic, the boundary conditions are not coercive, so that from the partial differential equations view the classical elliptic theory does not apply. The solution to this boundary value problem by J. J. Kohn [21] on a special class of pseudoconvex domains thus marks an important milestone in the theory of linear PDE as well as a significant advance in the theory of functions of several complex variables. The $\overline{\partial}$-Neumann problem is a non-elliptic problem for which regularity theory was established. It is first of its kind. In an elliptic problem of order m the solution is m degrees smoother than the data, the $\overline{\partial}$-Neumann problem does not exhibit the maximal degree of smoothing; the solution gains a predictable number of derivatives, but that number is less than the degree of the operator with which one is working.

Let us amplify the connection between the $\overline{\partial}$-problem and $\overline{\partial}$-Neumann problem more explicitly. Assume the existence of the solution for the $\overline{\partial}$-Neumann problem in the space of certain level forms with coefficients in $L^2(\Omega)$. The notation for the solution operator of the $\overline{\partial}$-Neumann problem is N, $\Box^{-1} = N$. As we mentioned above, by the Hodge decomposition we have the orthogonal decomposition

$$g = \overline{\partial\partial}^{\star} Ng + \overline{\partial}^{\star}\overline{\partial} Ng + \gamma(g), \tag{1.3}$$

where the coefficients of the form g are in $L^2(\Omega)$ and $\gamma(g) \in Ker(\Box) = \{0\}$. Now, consider the necessary condition in the $\overline{\partial}$-problem at (1.1), $\overline{\partial} g = 0$. Then, $\overline{\partial}^{\star}\overline{\partial} Ng \in \ker(\overline{\partial}^{\star}) \cap \ker(\overline{\partial})$, from which follows that the second term on the right, in the decomposition (1.3), is vanishing. Consequently, $g = \overline{\partial}\left(\overline{\partial}^{\star} Ng\right)$ with $\|\overline{\partial}^{\star} Ng\|^2 \leq C\|g\|^2$.

As a result, the operator $\overline{\partial}^{\star} N g$ provides a solution for the $\overline{\partial}$-problem, which is orthogonal to closed forms. This solution is called canonical solution operator for the $\overline{\partial}$-problem, solving the inhomogeneous Cauchy-Riemann equations. The canonical solution operator, $\overline{\partial}^{\star} N$, involves the $\overline{\partial}$-Neumann operator N. Thus, in order to get a smooth solution f on the closure of Ω for the inhomogeneous Cauchy-Riemann equations, (1.1), we need to have the $\overline{\partial}$-Neumann operator, N mapping forms with smooth coefficients to forms with smooth coefficients, with smoothness on the closure of the domain. If N has this property it is said that it is globally regular.

B. Results in the Dissertation

This dissertation concerns compactness of the $\overline{\partial}$-Neumann problem [8, 14, 15, 24, 28, 38].

If the $\overline{\partial}$-Neumann operator is a compact operator, that is, the image under N of any bounded subset of the space of forms with L^2 coefficients is a relatively compact subset of that space, then it is globally regular [24].

In the first part of the dissertation we explore compactness of the $\overline{\partial}$-Neumann operator via the ideal of compactness multipliers. In 1979, J. J. Kohn [23] developed a theory of subelliptic multipliers. He invented an interesting algorithmic procedure for computing certain ideals. He was able to use these ideals at least in domains with real analytic boundary to see whether there is a complex analytic variety in the boundary and whether there is a subelliptic estimate of the $\overline{\partial}$-Neumann operator. Influenced by subelliptic multipliers we define a compactness multiplier notion associated to the compactness estimate of the $\overline{\partial}$-Neumann operator. It is possible to use a similar algorithmic procedure for compactness estimate as it is done for subelliptic estimates in [11, 23]. The algorithm works in the real analytic case for subelliptic estimates. A smooth analog of this procedure for the compactness estimate requires an extra generator involved in the algorithm, otherwise it gives no new information about the regularity of the operator other than what is known from subelliptic estimates. We chose as a direction to study the zero set of the compactness multipliers as the obstruction to compactness of the $\overline{\partial}$-Neumann problem. That is, the ideal of compactness multipliers on a given domain is used with the purpose to get algebraic information about the boundary of the domain. The compactness multiplier notion is of functional analytic flavor but interacts heavily with potential theoretic sufficient conditions, such as, Property (P). On the domains where compactness is understood, one can identify this obstruction (the common zero set of the multiplier ideal), and we do this for bounded convex domains in $\mathbb{C}^n$ and for complete pseudoconvex Hartogs domains with smooth boundary in $\mathbb{C}^2$.

Let A be the common zero set of the compactness multiplier ideal,

Theorem 1. *Let Ω be a bounded convex domain in $\mathbb{C}^n$ then*

$$A = \overline{\left(\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})\right)}$$

Here, the family $\{f_\alpha(\mathbb{D})\}_{\alpha\in\Lambda}$ denotes the family of nontrivial analytic discs on the boundary of the domain Ω.

A complete Hartogs domain Ω in $\mathbb{C}^2$ with base Ω_1 on open set in $\mathbb{C}^1$ is defined by $|w| < e^{-\phi(z)}$ for $z \in \Omega_1$, where $\phi(z)$ is an upper semi-continuous real valued function.

Theorem 2. *Let Ω be a smooth bounded pseudoconvex complete Hartogs domain in $\mathbb{C}^2$ and $K \subset \Omega_1$ be below the portion of $b\Omega$ with weakly pseudoconvex points. Assume that boundary points of the form $(z, 0)$ are strictly pseudoconvex. Then the common zero set of the multiplier ideal is equal to the portion of $b\Omega$ above $\overline{Int_f(K)}^E$, where the latter is the Euclidean closure of the fine interior points of the set K.*

Studying the obstructions for the existence of compactness estimate of the $\overline{\partial}$-Neumann operator will help to understand the necessary and sufficient conditions for the compactness of the $\overline{\partial}$-Neumann problem. This is a necessary step towards understanding global regularity of the $\overline{\partial}$-Neumann operator.

The second part of the dissertation is about independence from the metric of some estimates for the $\overline{\partial}$-Neumann operator. A paper by W. J. Sweeney [39] in a very general manner shows that coercive estimates are independent of the metric on the cotangent bundle, and folklore says that subelliptic estimates are likewise independent of the metric. The metric which is considered is smooth positive definite hermitian on the whole closure of the domain: we let $G = \{G_q\}_{q=1}^n$ be these metrics. In particular, the metric on higher level forms is not assumed to be induced by the metric on $(0,1)$-forms. The theorem we give here is specifically for subellipticity of the $\overline{\partial}$-Neumann operator.

Theorem 3. *The subellipticity of the $\overline{\partial}$-Neumann operator associated with the general metric G is equivalent to the subellipticity of the $\overline{\partial}$-Neumann operator associated with the Euclidean metric.*

Besides, a natural question to ask is whether the same is true for the compactness of the $\overline{\partial}$-Neumann operator. It is also shown that the answer is yes; compactness of the $\overline{\partial}$-Neumann operator is independent of the metric.

Theorem 4. *The $\overline{\partial}$-Neumann operator associated with the general metric G is compact from $L^2_{(0,q)}(\Omega, G)$ to itself if and only if the $\overline{\partial}$-Neumann operator associated with the Euclidean metric is compact from $L^2_{(0,q)}(\Omega)$ to itself.*

It is interesting to note that the Sobolev estimates of the $\overline{\partial}$-Neumann operator are not independent from such a metric. The Sobolev estimates are very important in the study of the global regularity for the $\overline{\partial}$-Neumann operator. Global regularity of the $\overline{\partial}$-Neumann operator has always been studied through Sobolev estimates, see [7, 38].

The third part of the dissertation is about the compactness of the $\overline{\partial}$-Neumann operator on a transversal intersection of two smooth domains. If the $\overline{\partial}$-Neumann operator is compact on two bounded smooth pseudoconvex domains then can we say that the $\overline{\partial}$-Neumann operator is compact on the transversal intersection of these two domains? In order to understand the properties of the compactness of the $\overline{\partial}$-Neumann operator this question is of fundamental importance. In particular, this problem serves as a test to see whether there might be a reasonable notion of obstruction to compactness that lives in the boundary. If it is absent from both boundaries, it should be absent from the boundary of the intersection. The difficulty is on the non-smooth part of the intersection. We are only able to give some partial answers to the question. For example, when one of the domains satisfies property (P), the answer is affirmative.

CHAPTER II

BACKGROUND FOR THE $\overline{\partial}$-NEUMANN OPERATOR AND COMPACTNESS OF THE $\overline{\partial}$-NEUMANN OPERATOR

Let $\Omega \subset \mathbb{C}^n$. We say that Ω has C^k-boundary, $1 \leq k \leq \infty$, if there exists a neighborhood, U, of the boundary of Ω, and a real valued C^k function ρ defined on U satisfying

1. $\Omega \cap U = \{z \in U : \rho(z) < 0\}$, $b\Omega \cap U = \{z \in U : \rho(z) = 0\}$,
   $U \setminus \overline{\Omega} = \{z \in U : \rho(z) > 0\}$, and
2. $d\rho \neq 0$ on the boundary of Ω.

In this case we call ρ a defining function for Ω.

Let $\Omega \subset \mathbb{C}^n$ be a domain with C^k-boundary and defining function ρ. The real tangent space, $T_p(b\Omega)$, to a point p in the boundary of Ω consists of the vectors $\xi = (\xi_1, ..., \xi_n) = (u_1 + iv_1, ..., u_n + iv_n)$ satisfying $\sum_{j=1}^n \left(\frac{\partial \rho(p)}{\partial x_j} u_j + \frac{\partial \rho(p)}{\partial y_j} v_j \right) = 0$.

The complex tangent space to $b\Omega$ at p consists of those vectors in $T_p(b\Omega)$ which remain in $T_p(b\Omega)$ after scalar multiplication by i. Specifically, the complex tangent space, $H_p(b\Omega)$, to a point in $b\Omega$ is given by the vectors $\xi = (\xi_1, ..., \xi_n)$ satisfying $\sum_{j=1}^n \frac{\partial \rho(p)}{\partial z_j} \xi_j = 0$.

If ρ is at least a C^2 defining function we define the complex Hessian of ρ at a boundary point $z \in b\Omega$ as follows:

$$L_\rho(z; \xi) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k} \xi_j \overline{\xi}_k,$$

where ξ is a vector of type $(1,0)$ in $\mathbb{C}^n$ with $\xi = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j}$.

An upper semicontinuous function, $\rho : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$, is plurisubharmonic on Ω if the restriction of ρ to every complex line that passes through Ω is subharmonic.

One can check that a C^2 function f is plurisubharmonic on Ω if and only if $L_f(z;\xi) \geq 0$, for $z \in \Omega$ and any vector ξ of type $(1,0)$.

Definition 1. *A domain $\Omega \subset \mathbb{C}^n$ is said to be pseudoconvex if there exists a continuous plurisubharmonic function ρ on Ω such that $\{z \in \Omega : \rho(z) < c\}$ is a precompact subset of Ω for any real number $c \geq 0$.*

Notice that in this definition no smoothness is assumed. For a proof of the following theorem and other equivalent definitions of pseudoconvexity we refer the reader to [25, 31].

Theorem 5. *A domain Ω with C^2-boundary in $\mathbb{C}^n$, $n \geq 2$, is pseudoconvex if and only if it has a defining function ρ such that $L_\rho(z;\xi) \geq 0$ for $z \in b\Omega$ and $\xi \in H_p(b\Omega)$.*

The restriction of the complex Hessian $L_\rho(z;\cdot)$ on the space of complex tangential vectors $H_p(b\Omega)$ is called the Levi form. It is easy to see that, in the above theorem, the Levi form of any defining function is non-negative on the boundary. Therefore, pseudoconvexity is independent of the defining function; it is a well defined notion.

A. The $\overline{\partial}$-Neumann Problem

With purpose to prove existence theorems for holomorphic functions on complex manifolds in fifties Kohn and Spencer introduced the $\overline{\partial}$-Neumann problem. Since then the main applications to complex analysis have centered around the solution of the inhomogeneous Cauchy-Riemann equations $\overline{\partial} f = g$ which arises from the solution of the $\overline{\partial}$-Neumann problem. This problem was first solved by Kohn [20, 21], who proved existence and regularity properties of the $\overline{\partial}$-Neumann operator N.

In this section we sketch the setup of the $\overline{\partial}$-Neumann problem. We refer the reader to the books [9, 13] and a survey [7] for a more detailed treatment of the topic.

Let Ω be a bounded domain in $\mathbb{C}^n, n \geq 2$, and $0 \leq q \leq n$. We restrict our attention to $(0,q)$-forms, modifications for (p,q)-forms are simple (because $\overline{\partial}$ operator does not see dz differentials). We denote the space of square integrable and smooth $(0,q)$-forms by $L^2_{(0,q)}(\Omega)$ and $C^\infty_{(0,q)}(\Omega)$, respectively. Let $z = (z_1, \ldots, z_n)$ denote the complex coordinates for $\mathbb{C}^n$. Any square integrable $(0,q)$-form u can be written uniquely as a sum

$$u = \sum_J{}' u_J d\bar{z}_J$$

where $J = (j_1, \ldots, j_q)$ is a multiindex set with $j_1 < j_2 < j_3 < \cdots < j_q$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$, and $\sum'$ denotes the summation over strictly increasing multiindices.

To simplify the notation sometimes we will suppress the indices from $\overline{\partial}_q$ and just write $\overline{\partial}$. $L^2_{(0,q)}(\Omega)$ is a Hilbert space with the inner product coming from the following norm:

$$\|u\|^2 = \sum_J{}' \int_\Omega |u_J|^2 dV$$

where dV is the volume element on $\mathbb{C}^n$. When u is a smooth $(0,q)$-form we define the action of $\overline{\partial}$ as follows:

$$\overline{\partial} u = \sum_J{}' \sum_k \frac{\partial u_J}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_J.$$

Then we extend $\overline{\partial}$ to weak closure of smooth $(0,q)$-forms and still denote it by $\overline{\partial}$. Hence $u \in Dom(\overline{\partial})$ if $u \in L^2_{(0,q)}(\Omega)$ and $\overline{\partial} u \in L^2_{(0,q+1)}(\Omega)$ where $\overline{\partial} u$ is defined in the distribution sense. One can check that $\overline{\partial}$ is a linear, closed, and densely defined operator. Then the Hilbert space adjoint $\overline{\partial}^* : L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega)$ is linear, closed and densely defined. A square integrable $(0,q)$-form u belongs to $Dom(\overline{\partial}^*)$ if there exists $v \in L^2_{(0,q)}(\Omega)$ such that

$$\langle u, \overline{\partial} \varphi \rangle = \langle v, \varphi \rangle \quad \text{for} \quad \varphi \in Dom(\overline{\partial}) \cap L^2_{(0,q)}(\Omega)$$

where $\langle \cdot\ ,\ \cdot \rangle$ is the inner product on the corresponding Hilbert spaces. When Ω is a bounded domain one can easily see that $C^\infty_{(0,q)}(\overline{\Omega}) \subset Dom(\overline{\partial})$. But for a $(0,q)$-form u to be in $Dom(\overline{\partial}^*)$ it must satisfy a boundary condition in the weak sense. In case Ω has C^1 boundary, using integration by parts, one can show that a C^1-smooth $(0,q)$-form u is in the domain of $\overline{\partial}^*$ if and only if it satisfies the following:

$$\sum_{k=1}^{n} u_{kK} \frac{\partial \rho}{\partial z_k} = 0 \quad \text{on} \quad b\Omega \tag{2.1}$$

for all strictly increasing multiindices K such that $|K| = q-1$.

Now we will define the complex Laplacian $\square_{(0,q)}$.

Definition 2. $\square_q = \overline{\partial}_{q-1}\overline{\partial}^*_q + \overline{\partial}^*_{q+1}\overline{\partial}_q$ *is a linear operator defined on* $L^2_{(0,q)}(\Omega)$ *such that a square integrable* $(0,q)$*-form* f *is in* $Dom(\square_q)$ *if and only if* $f \in Dom(\overline{\partial}_q) \cap Dom(\overline{\partial}^*_q)$ *and* $\overline{\partial} f \in Dom(\overline{\partial}^*_{q+1})$, $\overline{\partial}^* f \in Dom(\overline{\partial}_{q-1})$.

One can check that $\square_q$ is a densely defined, closed (unbounded) linear, self-adjoint operator on $L^2_{(0,q)}(\Omega)$. The $\overline{\partial}$-Neumann problem is defined as finding a solution to $\square_q f = g$ on Ω for $f \in Dom(\square_q)$. Existence of a solution for the $\overline{\partial}$-Neumann problem on pseudoconvex domains is guaranteed by the following theorem. We refer the reader to [9] for a proof.

Theorem 6 (Hörmander)**.** *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n, n \geq 2$, *and* e *be the base of the natural logarithm. For each* $1 \leq q \leq n$, *there exists a bounded operator, called the* $\overline{\partial}$*-Neumann operator,* $N_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$ *such that*

(1) *Range*$(N_q) \subset Dom(\square_q)$, *and*

$N_q \square_q = \square_q N_q = I$ *on* $Dom(\square_q)$.

(2) *For any* $f \in L^2_{(0,q)}(\Omega)$, $f = \overline{\partial}\overline{\partial}^* N_q f + \overline{\partial}^* \overline{\partial} N_q f$.

(3) $\overline{\partial} N_q = N_{q+1} \overline{\partial}$ *on* $Dom(\overline{\partial}), 1 \leq q \leq n-1$.

(4) $\overline{\partial}^* N_q = N_{q-1} \overline{\partial}^*$ *on* $Dom(\overline{\partial}^*), 2 \leq q \leq n$.

(5) Let δ be the diameter of Ω. The following estimates hold for any $f \in L^2_{(0,q)}(\Omega)$:

$$\|N_q f\| \leq \frac{e\delta^2}{q}\|f\|,$$

$$\|\overline{\partial} N_q f\| \leq \sqrt{\frac{e\delta^2}{q}}\|f\|,$$

$$\|\overline{\partial}^* N_q f\| \leq \sqrt{\frac{e\delta^2}{q}}\|f\|.$$

We note that N_0 has a similar existence theorem. The main difference between N_0 and N_q for $q \geq 1$ is that $\Box_0$ is not onto. We refer the reader to [9] for more information on this matter. Using the above theorem one can show that when Ω is bounded and pseudoconvex, an L^2 solution to the $\overline{\partial}$-problem exists. In fact, the solution operator with minimal norm in the L^2 sense is $\overline{\partial}^* N_q$, as the following corollary shows.

Corollary 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$. Assume that $1 \leq q \leq n$, $g \in L^2_{(0,q)}(\Omega)$, and $\overline{\partial} g = 0$. Then $f = \overline{\partial}^* N_q g$ satisfies $\overline{\partial} f = g$ and*

$$\|f\| \leq \sqrt{\frac{e\delta^2}{q}}\|g\|. \tag{2.2}$$

f is the unique solution to $\overline{\partial} u = g$ that is orthogonal to $Ker(\overline{\partial})$.

$\overline{\partial}^* N_q$ is called the canonical solution operator for the $\overline{\partial}$-problem.

B. Compactness of the $\overline{\partial}$-Neumann Operator and Its Applications

In this section we introduce compactness of the $\overline{\partial}$-Neumann problem. We refer the reader to [10, 14, 15, 16, 28, 38] for more information.

We will use the notation $W^s_{(p,q)}(\Omega)$ for $(0,q)$-forms with coefficient functions from the Sobolev space $W^s(\Omega)$. The norm on $W^s(\Omega)$ is denoted by $\|\cdot\|_s$. Compactness of the $\overline{\partial}$-Neumann problem can be formulated in several useful ways:

Lemma 1. *Let Ω be a bounded pseudoconvex domain, $1 \leq q \leq n$. Then the following are equivalent:*

i) The $\overline{\partial}$-Neumann operator, N_q, is compact from $L^2_{(0,q)}(\Omega)$ to itself.

ii) The embedding of the space $Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*)$, provided with the graph norm $u \to \|\overline{\partial}u\| + \|\overline{\partial}^* u\|$, into $L^2_{(0,q)}(\Omega)$ is compact.

iii) For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\|u\| \leq \varepsilon(\|\overline{\partial}u\| + \|\overline{\partial}^* u\|) + C_\varepsilon \|u\|_{-1}, \;\; for \;\; u \in Dom(\overline{\partial}^*) \cap Dom(\overline{\partial}).$$

iv) The canonical solution operators $\overline{\partial}^* N_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q-1)}(\Omega)$ and $\overline{\partial}^* N_{q+1} : L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega)$ are compact.

The statement in (iii) is called a compactness estimate. The equivalence of (ii) and (iii) is a result of Lemma 1.1 in [24]. The general L^2-theory and the fact that $L^2_{(0,q)}(\Omega)$ embeds compactly into $W^{-1}_{(0,q)}(\Omega)$ shows that (i) is equivalent to (ii) and (iii). Finally, the equivalence of (i) and (iv) follows from the formula

$$N_q = (\overline{\partial}^* N_q)^* \overline{\partial}^* N_q + \overline{\partial}^* N_{q+1} (\overline{\partial}^* N_{q+1})^*$$

(see [13], p.55, [31]). We refer the reader to [28] for similar calculations.

Remark 1. Compactness of the $\overline{\partial}$-Neumann operator is invariant under rotation and biholomorphism, see [15], or [33] Lemma 2. The key reason for the invariance is that $\overline{\partial}$ operator commutes with the pullback (precomposition with a differential).

The next property makes compactness a good tool to investigate global regularity of the $\overline{\partial}$-Neumann operator: compactness can be localized.

Lemma 2. *[15] Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$, and N_q be the $\overline{\partial}$-Neumann operator on $L^2_{(0,q)}(\Omega)$ where $1 \leq q \leq n$.*

1) *If for every boundary point p there exists a pseudoconvex domain U that contains p such that the $\overline{\partial}$-Neumann operator on (the domain) $U \cap \Omega$ is compact, then N_q is compact.*

2) *If U is smooth bounded and strictly pseudoconvex and $U \cap \Omega$ is a domain, then if N_q is compact, so is the corresponding $\overline{\partial}$-Neumann operator on $U \cap \Omega$.*

We say that the $\overline{\partial}$-Neumann operator is globally regular if it maps the space of $(0,q)$-forms with coefficients in $C^\infty(\overline{\Omega})$ into itself. We also say that the $\overline{\partial}$-Neumann operator is exactly regular when it maps $(0,q)$-forms with coefficient in the L^2-Sobolev spaces $W^s(\Omega)$ to themselves (for $s \geq 0$). One can see that exact regularity implies global regularity. The next theorem is by Kohn and Nirenberg, it is about compactness implies exact regularity. In fact, the theorem says more than that:

Theorem 7. *[24] Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. If N_q is compact on $L^2_{(0,q)}(\Omega)$, then N_q is compact (in particular, continuous) as an operator from $W^s_{(0,q)}(\Omega)$ to itself, for all $s \geq 0$.*

It is also known that, implication in the other direction is valid as well [24]; if N_q is a compact operator on $W^s_{(0,q)}(\Omega)$ for some $s \geq 0$, then N_q is compact in $L^2_{(0,q)}(\Omega)$.

In case of convex domains, compactness of the $\overline{\partial}$-Neumann problem is very well studied:

Theorem 8. *[14] Let Ω be a bounded convex domain in $\mathbb{C}^n$. Let $1 \leq q \leq n$. The following are equivalent:*

1) *There exists a compact solution operator for $\overline{\partial}$ on $(0,q)$-forms.*

2) The boundary of Ω does not contain any affine variety of dimension greater than or equal to q.

3) The boundary of Ω does not contain any analytic variety of dimension greater than or equal to q.

4) The $\overline{\partial}$-Neumann operator N_q is compact.

1. Sufficient conditions for compactness of the $\overline{\partial}$-Neumann operator

Kohn and Nirenberg proved in [24] that compactness estimate for the $\overline{\partial}$-Neumann operator implies global regularity. Later, Catlin in [8] started the program of classifying domains satisfying global regularity of the $\overline{\partial}$-Neumann operator through the existence of compactness estimate. Along the way, he provided a very elementary (to state) condition, Property (P), sufficient for the existence of compactness estimate. However, Catlin treated Property (P) as a necessary condition in his study of types of domains satisfying global regularity. As a sufficient condition for global regularity, property (P) was investigated by Sibony [34, 35]. He studied the condition by the name B-regularity. Sibony's work has shown that domains known to satisfy global regularity by Catlin's work actually consist of a much bigger class. Since then, investigating sufficient and necessary conditions for the existence of compactness estimate has became a classical approach to study global regularity of the $\overline{\partial}$-Neumann operator.

Definition 3. *[8] For a bounded pseudoconvex domain Ω, we say that $b\Omega$ satisfies property (P) if for every positive number M, there exists a neighborhood $U = U_M$ of $b\Omega$ and a C^2 smooth function $\lambda = \lambda_M$ on $U \cap \Omega$, such that*

(1.) $0 \leq \lambda_M \leq 1$, *and*

(2.) *the complex Hessian of* λ *has all its eigenvalues bounded below by* M *on* $U \cap \Omega$*:*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial z_j \partial \overline{z}_k}(z)\xi_j\overline{\xi}_k \geq M\left|\xi\right|^2, \quad \forall z \in U \cap \Omega, \quad \xi \in \mathbb{C}^n.$$

The following is a relaxed version of Catlin's condition. It is based on having the gradients bounded by the complex Hessian of the functions.

Definition 4. Ω *satisfies property* $(\widetilde{P})$ *if, for every* $M > 0$, $\exists \phi = \phi_M \in C^2(U_M \cap \Omega)$ *where* U_M *is an open neighborhood of* $b\Omega$ *such that*

1. $\exists C > 0$*, for* $z \in U_M \cap \Omega$*,* $\xi \in \mathbb{C}^n$

$$\left|\sum_{k=1}^{n} \frac{\partial \phi(z)}{\partial z_k}\xi_k\right|^2 \leq C \sum_{j,k=1}^{n} \frac{\partial^2 \phi(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k,$$

2. *for* $z \in U_M \cap \Omega$*,* $\xi \in \mathbb{C}^n$

$$\sum_{j,k=1}^{n} \frac{\partial^2 \phi(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k \geq M|\xi|^2.$$

The definition of Property $(\widetilde{P})$ comes from McNeal's paper [28] where he proved that on a bounded pseudoconvex domain with smooth boundary, property $(\widetilde{P_q})$ suffices the compactness of the $\overline{\partial}$-Neumann operator on the $(0,q)$-form level. One can easily see that (P) implies $(\widetilde{P})$ but the other direction is an open question. The equivalence of these two properties is known only on some special cases; locally convexifiable bounded domains in $\mathbb{C}^n$, $(n \geq 1)$ [14], bounded pseudoconvex complete Hartogs domains in $\mathbb{C}^2$ [10]. Proving or disproving the equivalence between (P) and $(\widetilde{P})$ on a general domain in $\mathbb{C}^n$ would shed new light on the compactness theory of the $\overline{\partial}$-Neumann operator.

The next proposition shows that property $(\widetilde{P})$ can be restricted on the weakly pseudoconvex directions on the complex tangent space on the boundary of a smooth bounded domain. This does not seem to have been observed before, and so we include a proof. Consider $H^{(1,0)}(b\Omega)$ as the holomorphic tangent bundle on $b\Omega$. For $p \in b\Omega$ set $\mathcal{N}_p := \left\{\xi \in H_p^{(1,0)}(b\Omega) \mid \sum_{j,k=1}^n \frac{\partial^2\rho(p)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k = 0\right\}$, the null space of the Levi form.

Proposition 1. *Let Ω be a bounded pseudoconvex domain with smooth boundary. Property $(\widetilde{P})$ restricted to $\mathcal{N}_p$ is equivalent to Property $(\widetilde{P})$.*

Proof. It suffices to verify conditions in the definition 4. As a first step, we lift the $(\widetilde{P})$ conditions from weakly pseudoconvex directions to complex tangential directions.

For every positive real number M define the following function $\widetilde{\lambda}_M(z) := \lambda_M(z) + A_M\rho(z)$. Then,

$$\sum_{j,k=1}^n \frac{\partial^2\widetilde{\lambda}_M(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k = \sum_{j,k=1}^n \frac{\partial^2\lambda_M(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k + A_M \sum_{j,k=1}^n \frac{\partial^2\rho(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k,$$

$$\left|\sum_{j=1}^n \frac{\partial\widetilde{\lambda}_M(z)}{\partial z_j}\xi_j\right|^2 = \left|\sum_{j=1}^n \frac{\partial\lambda_M(z)}{\partial z_j}\xi_j + A_M \sum_{j=1}^n \frac{\partial\rho(z)}{\partial z_j}\xi_j\right|^2.$$

By hypothesis, we have $\lambda_M(z)$ satisfying $(\widetilde{P})$ restricted to weakly pseudoconvex directions, $\rho(z)$ is a defining function for the domain Ω, and A_M is a constant depending on M, to be chosen below. Thus, for $z \in b\Omega$ and $\xi \in \mathcal{N}_z$ we have

$$(1^\circ.) \qquad \left|\sum_{k=1}^n \frac{\partial\lambda_M(z)}{\partial z_k}\xi_k\right|^2 \leq C \sum_{j,k=1}^n \frac{\partial^2\lambda_M(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k,$$

$$(2^\circ.) \qquad \sum_{j,k=1}^n \frac{\partial^2\lambda_M(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k \geq M|\xi|^2.$$

For $z \in b\Omega$ and $\xi \in \mathcal{N}_z$ we also have

$$(1^{\circ\circ}.) \qquad \left|\sum_{k=1}^n \frac{\partial\lambda_M(z)}{\partial z_k}\xi_k\right|^2 < 2C \sum_{j,k=1}^n \frac{\partial^2\lambda_M(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k,$$

$$\text{(since } \sum_{j,k=1}^n \frac{\partial^2\lambda_M(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k \geq M, \quad \text{if} \quad |\xi| = 1.\text{);}$$

and

$$(2^{\circ\circ}.)\qquad \sum_{j,k=1}^{n}\frac{\partial^2\lambda_M(z)}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k > \frac{M}{2}|\xi|^2.$$

The point is that ($1^{\circ\circ}$.) and ($2^{\circ\circ}$.) are inequalities, i.e. open conditions.

Let $H^{(1,0)}(b\Omega)$ be the holomorphic tangent bundle on $b\Omega$ and $\mathbb{S}H^{(1,0)}(b\Omega)$ the unit sphere bundle. The fiber over a point $p \in b\Omega$ is the set of all unit $(1,0)$-vectors in $H^{(1,0)}_p(b\Omega)$. Define the following set $K := \left\{(p,\xi) \in \mathbb{S}H^{(1,0)}(b\Omega) \mid \xi \in \mathcal{N}_p\right\}$. K is a compact set, so ($1^{\circ\circ}$.) and ($2^{\circ\circ}$.) hold in a neighborhood U of K in $\mathbb{S}H^{(1,0)}(b\Omega)$. Let $K \subset\subset U_1 \subset\subset U$, and set

$$a := \min\left\{\sum_{j,k=1}^{n}\frac{\partial^2\lambda_M}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k \mid (z,\xi) \in \mathbb{S}H^{(1,0)}(b\Omega)\backslash U_1\right\},$$

$$b := \min\left\{\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k \mid (z,\xi) \in \mathbb{S}H^{(1,0)}(b\Omega)\backslash U_1\right\}.$$

Choose A_M big enough so that $a + A_M b \geq M$, (note that $b > 0$). Then,

$$\sum_{j,k=1}^{n}\frac{\partial^2\widetilde{\lambda}_M}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k \geq \sum_{j,k=1}^{n}\frac{\partial^2\lambda_M}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k > \frac{M}{2}\left|\xi\right|^2, \text{ for } (z,\xi) \in U. \tag{2.3}$$

We have used that $\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k \geq 0$ for $(z,\xi) \in U$ (because Ω is pseudoconvex).

Moreover,

$$\sum_{j,k=1}^{n}\frac{\partial^2\widetilde{\lambda}_M}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k \geq (a + A_M b)|\xi|^2 \geq M|\xi|^2, \text{ for } (z,\xi) \in \mathbb{S}H^{(1,0)}(b\Omega)\backslash U_1. \tag{2.4}$$

(2.3) and (2.4) give ($2^{\circ\circ}$.).

We can also satisfy ($1^{\circ\circ}$.). Chosen A_M so big that also $\widetilde{a} \leq a + A_M b$, where $\widetilde{a} := \max\{|\sum_{j=1}^{n}\frac{\partial\lambda_M}{\partial z_j}\xi_j|^2 \ : \ (z,\xi) \in \mathbb{S}H^{(1,0)}(b\Omega)\}$. Then, for $(z,\xi) \in U \subset \mathbb{S}H^{(1,0)}(b\Omega)$

$$\left|\sum_{j=1}^{n}\frac{\partial\widetilde{\lambda}_M}{\partial z_j}\xi_j\right|^2 = \left|\sum_{j=1}^{n}\frac{\partial\lambda_M}{\partial z_j}\xi_j\right|^2 \ < \ 2C\sum_{j,k=1}^{n}\frac{\partial^2\lambda_M}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k$$

$$\leq \quad 2C\sum_{j,k=1}^{n}\frac{\partial^2\widetilde{\lambda}_M}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k. \tag{2.5}$$

We have used again that Ω is pseudoconvex, and so $\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k \geq 0$ for $(z,\xi)\in U$. Moreover, for $(z,\xi)\in \mathbb{S}H^{(1,0)}(b\Omega)\backslash U_1$

$$\begin{aligned}\left|\sum_{j=1}^{n}\frac{\partial\widetilde{\lambda}_M}{\partial z_j}\xi_j\right|^2 = \left|\sum_{j=1}^{n}\frac{\partial\lambda_M}{\partial z_j}\xi_j\right|^2 &\leq \widetilde{a}\\ &< 2(a+A_Mb)\\ &\leq 2\sum_{j,k=1}^{n}\frac{\partial^2\widetilde{\lambda}_M}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k. \end{aligned}\tag{2.6}$$

(2.5) and (2.6) give ($1^{\circ\circ}$.).

As a second step, we lift the $(\widetilde{P})$ conditions from the complex tangent space to the whole $\mathbb{C}^n$. For every positive real number M define the following function $\phi_M(z) := \widetilde{\lambda}_M(z) + B_M\rho^2(z)$. Then,

$$\begin{aligned}\sum_{j,k=1}^{n}\frac{\partial^2\phi_M(z)}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k &= \sum_{j,k=1}^{n}\frac{\partial^2\widetilde{\lambda}_M(z)}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k + 2B_M\rho(z)\sum_{j,k=1}^{n}\frac{\partial^2\rho(z)}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k\\ &+ 2B_M\left|\sum_{j=1}^{n}\frac{\partial\rho(z)}{\partial z_j}\xi_j\right|^2,\end{aligned}$$

$$\left|\sum_{j=1}^{n}\frac{\partial\phi_M(z)}{\partial z_j}\xi_j\right|^2 = \left|\sum_{j=1}^{n}\frac{\partial\widetilde{\lambda}_M(z)}{\partial z_j}\xi_j + 2B_M\rho(z)\sum_{j=1}^{n}\frac{\partial\rho(z)}{\partial z_j}\xi_j\right|^2.$$

By the first step of the proof $\widetilde{\lambda}_M(z)$ satisfies the $(\widetilde{P})$ conditions restricted to complex tangential directions; B_M is a constant depending on M, to be chosen below.

We use an argument similar to one at the first step. Let $Y := b\Omega\times\{\xi\in\mathbb{C}^n : |\xi| = 1\}$, then $\mathbb{S}H^{(1,0)}(b\Omega)$ embeds into Y and so it is compact subset of Y. Thus, by continuity (as above) we have the analog of ($1^{\circ\circ}$.) and ($2^{\circ\circ}$.) in a neighborhood $V(\subset Y)$ of $\mathbb{S}H^{(1,0)}(b\Omega)$. Let $V_1 \subset\subset V$. Set $c := \min\{\sum_{j=1}^{n}\frac{\partial^2\widetilde{\lambda}_M}{\partial z_j\partial\overline{z}_k}\xi_j\xi_k : (z,\xi)\in Y\backslash V_1\}$,

$d := \min\{|\sum_{j=1}^n \frac{\partial\rho}{\partial z_j}\xi_j|^2 : (z,\xi) \in Y\backslash V_1\}$, and $l := \max\{|\sum_{j=1}^n \frac{\partial\widetilde{\lambda}_M}{\partial z_j}\xi_j|^2 : (z,\xi) \in Y\}$. Choose B_M big enough so that $l \leq c + B_M d$ when $(z,\xi) \in Y$ (note that $d > 0$).

For $(z,\xi) \in Y\backslash V_1$

$$\begin{aligned} \left|\sum_{j=1}^n \frac{\partial\phi_M}{\partial z_j}\xi_j\right|^2 = \left|\sum_{j=1}^n \frac{\partial\widetilde{\lambda}_M}{\partial z_j}\xi_j\right|^2 \leq l &\leq c + 2B_M d \\ &\leq \sum_{j,k=1}^n \frac{\partial^2\phi_M}{\partial z_j \partial\overline{z}_k}\xi_j\overline{\xi}_k. \end{aligned} \tag{2.7}$$

(2.7) and the above discusion before (2.7) give ($1^{\circ\circ}$.) on Y.

Choose B_M so big that also $(c + 2B_M d) \geq M$, (note that $d > 0$). Then, for $(z,\xi) \in Y\backslash V_1$

$$\sum_{j,k=1}^n \frac{\partial^2\phi_M}{\partial z_j\partial\overline{z}_k}\xi_j\overline{\xi}_k \geq (c + 2B_M d)|\xi|^2 \geq M|\xi|^2, \text{ for } (z,\xi) \in Y\backslash V_1. \tag{2.8}$$

(2.8) and the above discusion before (2.7) give ($2^{\circ\circ}$.) on Y. □

C. Notions and Theorems Used in the Next Chapters

Let's first give the basic identity:

Proposition 2 (Twisted Kohn-Morrey Formula [7]). *Let Ω be a bounded domain in $\mathbb{C}^n$ with class C^2 boundary; let u be a $(0,q)$ form (where $1 \leq q \leq n$) that is in the domain of $\overline{\partial}^\star$ and that is continuously differentiable on the closure $\overline{\Omega}$; and let a and ϕ be real functions that are twice continuously differentiable on $\overline{\Omega}$, with $a \geq 0$. Then*

$$\begin{aligned} \|\sqrt{a}\ \overline{\partial}u\|_\phi^2 + \|\sqrt{a}\ \overline{\partial}_\phi^\star u\|_\phi^2 &= {\sum_K}' \sum_{j,k=1}^n \int_{b\Omega} a\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}u_{jK}\overline{u}_{kK}e^{-\phi}d\sigma \\ &+ {\sum_J}' \sum_{j=1}^n \int_\Omega a\left|\frac{\partial u_J}{\partial\overline{z}_j}\right|^2 e^{-\phi}dV + 2\mathcal{R}e({\sum_K}' \sum_{j=1}^n u_{jK}\frac{\partial a}{\partial z_j}d\overline{z}_K, \overline{\partial}_\phi^\star u)_\phi \\ &+ {\sum_K}' \sum_{j,k=1}^n \int_\Omega (a\frac{\partial^2\phi}{\partial z_j\partial\overline{z}_k} - \frac{\partial^2 a}{\partial z_j\partial\overline{z}_k})u_{jK}\overline{u}_{kK}e^{-\phi}dV \end{aligned} \tag{2.9}$$

Theorem 9 (Basic estimate [18]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, e the base of the natural logarithm, and D the diameter of Ω. Then, the following estimate is valid*

$$\|u\|^2 \leq \frac{eD^2}{q} \left(\|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2 \right) \tag{2.10}$$

$\forall u \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^\star) \subset L^2_{(0,q)}(\Omega)$, where $1 \leq q \leq n$.

Theorem 10 (Sobolev Interpolation). *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. For any $\varepsilon > 0$, $u \in W^{s_1}(\Omega)$, $s_1 > s > s_2$, we have the following inequality:*

$$\|u\|^2_s \leq \varepsilon \|u\|^2_{s_1} + C_\varepsilon \|u\|^2_{s_2}, \tag{2.11}$$

where C_ε is independent of u.

Usually in the theory of the $\overline{\partial}$-Neumann problem to be able to pass from an a priori estimate like Proposition 2 to a real estimate, the following lemma plays crucial importance, see [9], or section 1.2 in [18].

Theorem 11 (Density Lemma). *Let Ω be a bounded domain with at least C^2 boundary, and $\phi \in C^2(\overline{\Omega})$. Then $Dom(\overline{\partial}^\star) \cap C^2(\overline{\Omega})$ is dense in $Dom(\overline{\partial}^\star) \cap Dom(\overline{\partial})$ under the graph norm*

$$Q(u, u) = \|u\|^2_\phi + \|\overline{\partial} u\|^2_\phi + \|\overline{\partial}^\star u\|^2_\phi$$

Let Ω be a pseudoconvex domain with C^∞ boundary in $\mathbb{C}^n$ and defining function ρ. Ω is of finite type if one-dimensional complex varieties have bounded order of contact at boundary points. The precise definition is as follows. Fix $p \in b\Omega$. If f is a smooth vector-valued function defined near 0 in $\mathbb{C}$, let $\nu(f)$ denote its order of vanishing at 0. For a given non-constant germ of a holomorphic map $\gamma : \mathbb{C} \longrightarrow \mathbb{C}^n$

with $\gamma(0) = p$ define $\tau(\gamma) = \nu(r \circ \gamma)/\nu(\gamma - p)$. We say p is of finite type in $b\Omega$ if there exists a finite constant τ so that $\tau(\gamma) \leq \tau$ for all germs γ. We say Ω is of finite type if every point of its boundary is of finite type, see [11].

CHAPTER III

THE IDEAL OF COMPACTNESS MULTIPLIERS

A. Compactness Multipliers

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Recall that a *compactness estimate* of the $\overline{\partial}$-Neumann operator is said to hold on Ω if for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that the following estimate

$$\|u\|^2 \leq \varepsilon \left(\|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2 \right) + C_\varepsilon \|u\|_{-1}^2$$

is valid $\forall u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star) \subset L^2_{(0,q)}(\Omega)$. ($\| \cdot \|_{-1}$ is the L^2-Sobolev (-1)-norm.)

Definition 5. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. A function $f \in C(\overline{\Omega})$ is called a compactness multiplier on Ω if for every $\varepsilon > 0$ there is a constant $C_{\varepsilon,f} > 0$ such that the following estimate*

$$\|fu\|^2 \leq \varepsilon(\|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2) + C_{\varepsilon,f} \|u\|_{-1}^2 \tag{3.1}$$

is valid $\forall u \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^\star) \subset L^2_{(0,q)}(\Omega)$.

Remark 2. The definition of the compactness multiplier is equivalent to having multiplication operator M_f from $\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star)$ to $L^2_{(0,q)}(\Omega)$ as a compact operator. That is, $M_f(u) = fu$ is compact on $\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star)$: assume we have the above estimate (3.1) and a sequence $\{u_n\}_{n=1}^\infty$ in $\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star)$ bounded under the graph norm $Q(u,u) = \|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2$. By the basic estimate, which says $\forall u \in L^2_{(0,q)}(\Omega)$ $\|u\|^2 \leq C(\|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2)$ is valid, the sequence $\{u_n\}_{n=1}^\infty$ is also bounded in the $L^2_{(0,q)}(\Omega)$-norm. Since $L^2_{(0,q)}(\Omega)$ embeds compactly into $W^{-1}_{(0,q)}(\Omega)$ there is a subsequence $\{u_n\}_{n=1}^\infty$, which converges in $W^{-1}_{(0,q)}(\Omega)$. Assume that we have passed to a such subsequence then this gives that the subsequence is Cauchy in $L^2_{(0,q)}(\Omega)$ and hence converges: $\|f(u_n-u_m)\|^2 \leq \varepsilon \underbrace{Q(u_n-u_m, u_n-u_m)}_{<C} + C_{\varepsilon,f} \underbrace{\|u_n-u_m\|^2_{-1}}_{<\varepsilon} \lesssim \varepsilon$. Thus, we have M_f as a compact operator from $\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star)$ to $L^2_{(0,1)}(\Omega)$.

Let J^q_Ω be the set of the compactness multipliers defined as above, associated with $(0,q)$ forms, $1 \leq q \leq n$. We will develop the theory for $(0,1)$-forms. For the general case computations are the same.

Proposition 3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. The set of compactness multipliers J^1_Ω is an ideal, closed under the sup-norm.*

Proof. Note that $\|fu\|^2 \leq C_f\|u\|^2$. Replacing f by hg, it is easy to see that hg is a compactness multiplier whenever g is; $\|(hg)u\|^2 \leq C_h\|gu\|^2$. Thus, J^1_Ω is closed under multiplication by elements of $C(\overline{\Omega})$. The sum of two compactness multipliers is a compactness multiplier; $\|(g+f)u\|^2 \leq \|gu\|^2 + \|fu\|^2$. Thus, J^1_Ω is an ideal of $C(\overline{\Omega})$. J^1_Ω is closed under the *sup*-norm.

Let $\{f_j\}_{j=1}^\infty \in J^1_\Omega$ such that f_j converges to f in the *sup*-norm. Now, fix $\varepsilon > 0$ such that there exists $j_\circ(\varepsilon) = j_\circ \in \mathbf{N}$ for which $|f_{j_\circ} - f| < \varepsilon$. Thus,

$$\begin{aligned} \|fu\|^2 &= \|(f-f_{j_\circ})u + f_{j_\circ}u\|^2 \\ &\leq \|(f-f_{j_\circ})u\|^2 + \|f_{j_\circ}u\|^2 \\ &\leq \varepsilon\|u\|^2 + \|f_{j_\circ}u\|^2 \end{aligned} \tag{3.2}$$

Now, for the second term on the right hand side of (3.2) we use $f_{j_\circ} \in J^1_\Omega$ and for the first term we use the basic estimate on Ω, see Theorem 9; $\|u\|^2 \leq C(\|\overline{\partial}u\|^2 + \|\overline{\partial}^\star u\|)$. Therefore,

$$\|f_{j_\circ}u\|^2 \leq \varepsilon(\|\overline{\partial}u\|^2 + \|\overline{\partial}^\star u\|^2) + C_{\varepsilon,f_{j_\circ}}\|u\|^2_{-1}$$

we get

$$\|fu\|^2 \leq \varepsilon(\|\overline{\partial}u\|^2 + \|\overline{\partial}^\star u\|^2) + C_{\varepsilon,f}\|u\|_{-1}^2.$$

As a result, $f \in J_\Omega^1$ and so J_Ω^1 is closed ideal under the supremum norm. $\square$

Proposition 4. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and r be a continuous function on Ω with $r_{|b\Omega} \equiv 0$. Then the function r is a compactness multiplier.*

Proof. Fix $\varepsilon > 0$. Let $\Omega_\varepsilon := \{z \in \Omega \mid r(z) < -\varepsilon,\ 0 < \varepsilon \text{ arbitrarily small}\}$. Choose $\psi_\varepsilon \in C_\circ^\infty(\Omega)$ such that $\psi_\varepsilon \equiv 1$ on $\overline{\Omega}_\varepsilon$, $0 \leq \psi_\varepsilon \leq 1$. Then $\forall u \in C_{(0,1)}^\infty(\overline{\Omega}) \cap \mathrm{Dom}(\overline{\partial}^\star)$,

$$\begin{aligned} \|ru\|^2 &\leq \|r(1-\psi_\varepsilon)u\|_{\Omega\setminus\Omega_\varepsilon}^2 + \|r\psi_\varepsilon u\|_{\Omega_\varepsilon}^2 \\ &\leq \varepsilon\|u\|^2 + C\|\psi_\varepsilon u\|_{\Omega_\varepsilon}^2 \\ &\leq \varepsilon\left(\|\overline{\partial}u\|^2 + \|\overline{\partial}^\star u\|^2\right) + C\|\psi_\varepsilon u\|_{\Omega_\varepsilon}^2. \end{aligned}$$

Note that in the second inequality we have used the boundness of r on $\overline{\Omega}$ and on the third one the basic estimate, Theorem 9.

Because of the interior elliptic regularity of $\overline{\partial} \oplus \overline{\partial}^\star$ we get the following estimate

$$\|\psi_\varepsilon u\|_1^2 \leq C(\|\overline{\partial}u\|^2 + \|\overline{\partial}^\star u\|^2) \tag{3.3}$$

for $\forall u \in C_{(0,1)}^\infty(\overline{\Omega}) \cap \mathrm{Dom}(\overline{\partial}^\star)$ on Ω. Note that $\psi_\varepsilon u$ is still in $C_{(0,1)}^\infty(\overline{\Omega}) \cap \mathrm{Dom}(\overline{\partial}^\star)$. To be complete let's show (3.3); by the definition of the Sobolev 1-norm we have

$$\|\psi_\varepsilon u\|_1^2 = C\sum_{j,k=1}^n \int_\Omega \left\{\left|\frac{\partial(\psi_\varepsilon u_j)}{\partial z_k}\right|^2 + \left|\frac{\partial(\psi_\varepsilon u_j)}{\partial \overline{z}_k}\right|^2\right\} dV + \|\psi_\varepsilon u\|^2$$

It follows immediately from two integration by part that

$$\int_\Omega \left|\frac{\partial(\psi_\varepsilon u_j)}{\partial z_k}\right|^2 dV = \int_\Omega \left|\frac{\partial(\psi_\varepsilon u_j)}{\partial \overline{z}_k}\right|^2 dV$$

of course the boundary integrals are zero because of forms $(\psi_\varepsilon u)$ being supported in the interior of Ω. Then

$$\begin{aligned}
\|\psi_\varepsilon u\|_1^2 &= C\sum_{j,k=1}^{n}\int_\Omega \left\{\left|\frac{\partial(\psi_\varepsilon u_j)}{\partial z_k}\right|^2+\left|\frac{\partial(\psi_\varepsilon u_j)}{\partial \overline{z}_k}\right|^2\right\} dV+\|\psi_\varepsilon u\|^2\\
&\le C\left(\sum_{j,k=1}^{n}\|\frac{\partial(\psi_\varepsilon u_j)}{\partial \overline{z}_k}\|^2+\|u\|^2\right)\\
&\le C(\|\overline{\partial}u\|^2+\|\overline{\partial}^\star u\|^2)
\end{aligned}$$

the last inequality is coming from the Kohn-Morrey formula, the identity (2.9) in Proposition 2 (for $a\equiv 1$ and $\phi=0$, one obtains all the $\overline{\partial}$-derivatives of a form become bounded by the above estimate). The second term on the second row is estimated again by the basic estimate. Moreover, the constant C may change on every row. Then, by using the Sobolev interpolation, Theorem 10,

$$\|\psi_\varepsilon u\|_{\Omega_\varepsilon}^2 \le \|\psi_\varepsilon u\|_{\Omega}^2 \le \varepsilon\|\psi_\varepsilon u\|_1^2 + C_\varepsilon\|\psi_\varepsilon u\|_{-1}^2$$

by considering $|\psi|\le 1$ as a continuous multipier in $W_\circ^1(\Omega)$, $\|\psi_\varepsilon u\|_{-1}^2 \lesssim \|u\|_{-1}^2$ we have

$$\|\psi_\varepsilon u\|_{\Omega_\varepsilon}^2 \le \varepsilon\left(\|\overline{\partial}u\|^2+\|\overline{\partial}^\star u\|^2\right)+C_\varepsilon\|u\|_{-1}^2.$$

Therefore, by the density lemma, Theorem 11. $\forall u\in \mathrm{Dom}(\overline{\partial})\cap\mathrm{Dom}(\overline{\partial}^\star)$ on Ω

$$\|ru\|^2 \le \varepsilon\left(\|\overline{\partial}u\|^2+\|\overline{\partial}^\star u\|^2\right)+C_\varepsilon\|u\|_{-1}^2.$$

Thus, r is a compactness multiplier, $r\in J_\Omega^1$. $\square$

Remark 3. The above proposition says that the ideal of compactness multipliers, J_Ω is always nontrivial. In fact, in addition to r as above, every compactly supported continuous function in Ω is a compactness multiplier. The proof of this is the same as in Proposition 4.

Let's define $A_q := \{z \in \overline{\Omega} \mid g(z) = 0 \;\; \forall g \in J^q_\Omega\}$. A_q is a compact subset of $\overline{\Omega}$. Note that along the chapter usually we will suppress q or Ω on J^q_Ω and on A^q_Ω unless we need emphasis on these notions.

Theorem 12. *The ideal J_Ω of compactness multipliers on Ω, is equal to the set of functions continuous on the closure of Ω which are vanishing on $A \subset b\Omega$. That is, $J_\Omega = \{g \in C(\overline{\Omega}) \mid g_{|_A} \equiv 0\}$.*

Proof. If $f \in J$ then by the definition of A, f must vanish on A.
The other direction is by elementary functional analysis, with more general settings. The following proof can be found in [26] with a negligible modification.
A is subset of $b\Omega$ because r, defining function for Ω is an element of J and r does not vanish in the interior of Ω.
Let $f \in C(\overline{\Omega})$ such that $f_{|_A} \equiv 0$. We are trying to show that $f \in J$. Let $U := \{|f(z)| < \varepsilon\}$. U is open then $\overline{\Omega} \backslash U =: K$ is closed and so compact. Now for every $z \in K$ there is $g_z \in J$ such that $g_z \neq 0$ in some open set V_z. Then by compactness cover K by finitely many open sets $\{V_{z_j}\}_{j=1}^m$ (corresponding to $\{g_{z_j}\}_{j=1}^m$). Define $g := |g_{z_1}|^2 + |g_{z_2}|^2 + ... + |g_{z_m}|^2$. Clearly $g \in J$ and $g \geq c > 0$ on K. Now define $f_n := f\frac{ng}{1+ng} \in J$ and approximate f within ε on K. Since $0 \leq \frac{ng}{1+ng} \leq 1$ it follows that on U we have the estimate $0 \leq |f_n| < \varepsilon$ thus $\sup|f_n - f| < 2\varepsilon$. As a result, f is in the closure of J. $\square$

Corollary 2. *A compactness estimate of the $\overline{\partial}$-Neumann operator holds if and only if $A = \emptyset$. That is, a nonempty set A is the obstruction to compactness of the $\overline{\partial}$-Neumann operator.*

Remark 4. The set A_Ω is a subset of the set of infinite type points on $b\Omega$, $(b\Omega)_\infty$. That is, if $p \in A_\Omega$ and $p \notin (b\Omega)_\infty$ then we have compactness estimate in a neighborhood U of p. This gives the existence of a compactly supported continuous function ϕ with support in U and element of J_Ω. However, ϕ does not vanish on the point $p \in A_\Omega$. Therefore, $p \in (b\Omega)_\infty$.

Before we continue further let's introduce some notions. We say that a compact subset $K \subset \mathbb{C}^n$ is B-regular if any continuous functions on K can be approximated uniformly on K by plurisubharmonic functions continuous in a neighborhood of K. This definition of B-regularity is equivalent to the following one, see [34]; for any $M > 0$ there is a $\lambda \in C^\infty$ plurisubharmonic function in the neighborhood of K such that $0 < \lambda < 1$ and $\sum_{j,k=1}^n \frac{\partial^2\lambda}{\partial z_j \partial \overline{z}_k}(z)\xi_j\overline{\xi}_k \geq M\,|\xi|^2$. When $K = b\Omega$, where Ω is a bounded domain, then the second definition of B-regularity is essentially property (P) for $b\Omega$. In fact, property (P) requires the function λ to exist only in a neighborhood of $b\Omega$ intersected with Ω, rather than in a full neighborhood of $b\Omega$, see Definition 3. However, it is easy to see that on domains with relatively minimally regular boundary (for instance, when the boundary is locally a graph), the two notions coincide.

If A is not an empty set, then it can not be "too small", in the following sense.

Lemma 3. *If the set A is not empty then it can not satisfy property (P).*

Proof. Assume that $A \neq \emptyset$ satisfies property (P). Fix $\varepsilon > 0$. Then $\exists\lambda_\varepsilon$ such that on a neighborhood U_ε of the set A we have

(1.) $0 \leq \lambda_\varepsilon(z) \leq 1$

(2.) $\sum_{j,k=1}^n \frac{\partial^2\lambda_\varepsilon(z)}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k \geq \frac{1}{\varepsilon}\,|\xi|^2, \quad \forall\xi \in \mathbb{C}^n, \ \forall z \in U_\varepsilon.$

Now, choose $\chi_\varepsilon \in C^\infty_\circ(U_\varepsilon)$, $\chi_\varepsilon \equiv 1$ on A and $0 \leq \chi_\varepsilon \leq 1$ in U_ε. Then, for $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star)$ we have $\|u\|^2 \lesssim \|\chi_\varepsilon u\|^2 + \|(1-\chi_\varepsilon)u\|^2$. The second term on the right hand side of the inequality can be estimated by considering $(1-\chi_\varepsilon)$ as a compactness multiplier, element in J_Ω. As for the first term on the right hand side, we use the function λ_ε. By keeping in mind that $e^{-1} \leq e^{\lambda_\varepsilon - 1} \leq 1$, in the

Kohn-Morrey-Hörmander formula (2.9) consider $a = 1 - e^{\lambda_\varepsilon - 1}$ and $\phi = 1$ to get the following estimate:

$$\sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 \lambda_\varepsilon}{\partial z_j \overline{z}_k} (\chi_\varepsilon u_j)(\chi_\varepsilon \overline{u}_k) \leq C(\|\overline{\partial}(\chi_\varepsilon u)\|^2 + \|\overline{\partial}^\star(\chi_\varepsilon u)\|^2); \tag{3.4}$$

see [7] for details. Then, by using condition (2.) of property (P) we get

$$\begin{aligned} \|\chi_\varepsilon u\|^2 &\lesssim \varepsilon(\|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2 + \|\left(\nabla \chi_\varepsilon\right) u\|^2) \\ &\lesssim \varepsilon(\|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2) + C_{\varepsilon, \widetilde{\chi}} \|\widetilde{\chi}_\varepsilon u\|^2 \end{aligned} \tag{3.5}$$

where $\widetilde{\chi}$ is a smooth cut off function identically 1 on the support of $\nabla \chi$ whose support does not intersect with A_Ω, and $C_\chi := max|\nabla \chi|$. The last term on the right hand side of the inequality can be estimated by considering $\widetilde{\chi}_\varepsilon$ as compactness multiplier, element in J_Ω. Therefore, we have

$$\|\chi_\varepsilon u\|^2 \leq \varepsilon(\|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2) + C_\varepsilon \|u\|_{-1}^2. \tag{3.6}$$

As a result, $\chi_\varepsilon + (1 - \chi_\varepsilon) \in J_\Omega$; we have compactness estimate on Ω which contradicts with the Corollary 2. $\square$

Example 1. The set A can not have two dimensional Hausdorff measure zero. A set of infinite type points with two dimensional Hausdorff measure zero is benign for property (P), see [5, 34]. Now, combine this with Remark 4.

Example 2. The set A can not be contained in a subvariety Σ of holomorphic dimension 0. The reason is that every compact subset K in Σ has property (P), see [34] Proposition 12. Compare this example with Example 3 in Chapter V. A smooth submanifold Σ of $b\Omega$ with constant complex dimension of $H_p(\Sigma)$ $\forall p \in \Sigma$ has holomorphic dimension zero if $\forall p \in \Sigma$ and $\forall \xi \in H_p(\Sigma)$ we have $L_r(z, \xi) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k > 0$.

1. Characterization of A on a convex domain Ω in $\mathbb{C}^n$

In order to separate characterizations with different forms, as a notation A_q will be used for characterization with $(0,q)$-forms.

Theorem 13. *Let Ω be a bounded convex domain in $\mathbb{C}^n$ then*

$$A_1 = \overline{\left(\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})\right)}.$$

Here, the family $\{f_\alpha(\mathbb{D})\}_{\alpha\in\Lambda}$ denotes the family of nontrivial analytic discs on the boundary of Ω.

Proof. Choose $p \in b\Omega \backslash \left(\overline{\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})}\right)$. Then there exists $r > 0$ such that $B(p,r) \cap \left(\overline{\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})}\right) = \emptyset$. $B(p,r) =: B_p$ is a ball with center p and radius r.
Choose $\phi \in C^\infty_\circ(B_p)$. If $u \in \mathrm{Dom}(\overline{\partial}^\star) \cap \mathrm{Dom}(\overline{\partial}) \subset L^2_{(0,1)}(\Omega)$ then

$$(\phi u) \in \mathrm{Dom}(\overline{\partial}^\star) \cap \mathrm{Dom}(\overline{\partial}) \subset L^2_{(0,1)}(B_p \cap \Omega).$$

Now, since $B_p \cap \Omega$ is a convex domain and by assumption there is no disc in the boundary of the domain then by Theorem 11 in [15] the compactness estimate of the $\overline{\partial}$-Neumann operator on $B_p \cap \Omega$ exists, see [14] and [15]. Thus, $\forall \varepsilon > 0\ \exists C_\varepsilon > 0$ such that

$$\|\phi u\|^2_{B_p\cap\Omega} \leq \varepsilon \left(\|\overline{\partial}(\phi u)\|^2_{B_p\cap\Omega} + \|\overline{\partial}^\star(\phi u)\|^2_{B_p\cap\Omega}\right) + \|\phi u\|^2_{-1,B_p\cap\Omega}.$$

Therefore, we have

$$\begin{aligned} \|\phi u\|^2_\Omega &\leq \varepsilon \left(\|\overline{\partial}(\phi u)\|^2_\Omega + \|\overline{\partial}^\star(\phi u)\|^2_\Omega\right) + C_\varepsilon \|\phi u\|^2_{-1,\Omega} \\ &\leq \varepsilon \left(C_1\|\overline{\partial} u\|^2_\Omega + C_2\|\overline{\partial}^\star u\|^2_\Omega\right) + C_{\varepsilon,\phi}\|u\|^2_{-1,\Omega} \end{aligned}$$

$\forall u \in \mathrm{Dom}(\overline{\partial}^\star) \cap \mathrm{Dom}(\overline{\partial}) \subset L^2_{(0,1)}(\Omega)$. Here, we have used that $\|u\|^2_\Omega \lesssim \|\overline{\partial} u\|^2_\Omega + \|\overline{\partial}^\star u\|^2_\Omega$.

The part with the (-1) Sobolev norms in the above inequalities is as follows: There is an isometric embedding $W^1_\circ(\Omega \cap B_p) \longrightarrow W^1_\circ(\Omega)$. Now take the dual of this isometric embedding and get $W^{-1}(\Omega) \longrightarrow W^{-1}(\Omega \cap B_p)$, that is, $\|\phi u\|_{-1,\Omega\cap B_p} \lesssim \|\phi u\|_{-1,\Omega}$. The part with $\|\phi u\|_{-1,\Omega} \lesssim \|u\|_{-1,\Omega}$ uses $|\phi| \leq 1$ that ϕ is a continuous multiplier on $W^1_\circ(\Omega)$.

As a result, ϕ is a compactness multiplier on $\overline{\Omega}$ and does not vanish on $B_p \cap b\Omega$. Thus, for an arbitrarily chosen $p \in b\Omega \backslash \left(\overline{\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})}\right)$ there exists $\phi_p \in J^1_\Omega$ and $r' < r$ such that $\phi_p \neq 0$ on $B(p, r') \cup \Omega \subset B(p, r) \cap \Omega$, so $p \notin A_1$. Therefore, if $p \in A_1$ then $p \in \overline{\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})}$. That is, the set A_1 is a subset of $\left(\overline{\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})}\right)$.

Now, let's show the other way around, that $\left(\overline{\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})}\right)$ is contained in A_1. Let's choose $p \in b\Omega\backslash A_1$. Then, we have a ball with radius $r > 0$ and center p such that $B(p, r) \cap A_1 = \emptyset$. Moreover, we may take $r > 0$ small enough so that there is a compactness multiplier f on Ω that does not vanish on $\overline{\Omega} \cap \overline{B(p, r)}$.

In order to conclude that there is no disc in the boundary of $B(p, r) \cap \Omega$ by Fu and Straube's result in [15], Theorem 11, it suffices to derive a compactness estimate for the $(0, 1)$-forms on $B_p \cap \Omega$, a convex bounded domain. Then for an arbitrarily chosen p in $b\Omega\backslash A_1$ it can be seen that $p \notin \overline{\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})}$ then it can be concluded that $\overline{\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})}$ is contained in A_1.

We claim that there is a compactness estimate on $B(p, r) \cap \Omega$. Since $B(p, r)$ is a ball it is a strictly pseudoconvex domain, so we have property (P) on $bB(p, r)$; for fixed $\varepsilon > 0$ $\exists V_\varepsilon := \{z \in \mathbb{C}^n \mid \operatorname{dist}(z, bB(p, r)) < \varepsilon\}$ and $\exists \lambda_\varepsilon \in C^2(V_\varepsilon)$ in a neighborhood of $bB(p, r)$ such that $0 \leq \lambda_\varepsilon \leq 1$ and $\sum_{i,k=1}^n \frac{\partial^2 \lambda_\varepsilon}{\partial z_j \partial \overline{z}_k} \xi_j \overline{\xi}_k \geq \frac{1}{\varepsilon} |\xi|^2$ on V_ε for $\xi \in \mathbb{C}^n$.

Now, choose ϕ_ε as a smooth cut off function $(0 \leq \phi_\varepsilon \leq 1)$, $\phi_\varepsilon \equiv 1$ near $bB(p, r)$ and supported in V_ε. Let's take $u \in \operatorname{Dom}(\overline{\partial}) \cap \operatorname{Dom}(\overline{\partial}^\star)$ on $\Omega \cap B_p := \Omega \cap B(p, r)$ and estimate the following

$$\|u\|^2_{\Omega\cap B_p} \lesssim \|\phi_\varepsilon u\|^2_{\Omega\cap B_p} + \|(1-\phi_\varepsilon)u\|^2_{\Omega\cap B_p} \tag{3.7}$$

Note that $(1-\phi_\varepsilon)u$ can be considered as a form in $\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star) \subset L^2_{(0,q)}(\Omega)$

To find an estimate for $\|\phi_\varepsilon u\|^2_{\Omega\cap B_p}$ we use the inequality extracted from the Kohn-Morrey formula (2.9). We set $a := 1 - e^{\lambda_\varepsilon - 1}$ and apply Cauchy-Schwarz inequality to the real valued term which shows up and then absorb $\|e^{\frac{\lambda_\varepsilon-1}{2}}\overline{\partial}^\star u\|$ to the other side of the inequality. (For completeness only for the regularization part we consider $(0,q)$-forms.) By considering $a + e^{\lambda_\varepsilon-1} = 1$ we get

$$\begin{aligned}
&\sideset{}{'}\sum_{|K|=q-1} \int_{\Omega\cap B_p} e^{\lambda_\varepsilon - 1} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_\varepsilon}{\partial z_j \partial \overline{z}_k} (\phi_\varepsilon u_{jK}) \overline{(\phi_\varepsilon u_{kK})}\, dV(z) \\
\leq \quad & \|\overline{\partial}\,(\phi_\varepsilon u)\,\|^2_{\Omega\cap B_p} + \|\overline{\partial}^\star\,(\phi_\varepsilon u)\,\|^2_{\Omega\cap B_p}.
\end{aligned} \tag{3.8}$$

However, boundary of $\Omega \cap B_p$ is not a C^2 boundary so the Density Lemma (Theorem 11.) may not work (we need at least C^1 regular boundary, see [18]). We will use the regularization procedure, see [37], in order to make the inequality (3.8) work.

Let's exhaust $\Omega \cap B_p$ by $\{\Omega_\nu\}_{\nu=1}^\infty$ such that $\Omega_\nu \subset\subset \Omega_{\nu+1}$, Ω_ν has pseudoconvex C^2 boundary and $\Omega \cap B_p = \bigcup_{\nu=1}^\infty \Omega_\nu$.

On Ω_ν the inequality (3.8) will work and by having property (P) we will have an estimate for $\|\phi_\varepsilon u\|^2_{\Omega_\nu}$ but $(\phi_\varepsilon u)|_{b\Omega_\nu}$ may not be in the domain of $\overline{\partial}^\star$ on Ω_ν. However, if we define the following form on Ω_ν as follows:

$$(\phi_\varepsilon u)_\nu := \overline{\partial} N_{q-1,\nu} \vartheta(\phi_\varepsilon u) + \overline{\partial}^\star_\nu N_{q+1,\nu} \overline{\partial}(\phi_\varepsilon u) \tag{3.9}$$

for $(\phi_\varepsilon u) \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star)$ on $\Omega \cap B_p$, then $(\phi_\varepsilon u)_\nu \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star_\nu)$ on Ω_ν.

In (3.9), ϑ is the formal adjoint of $\overline{\partial}$ under the usual L^2 norm. When $q = 1$, the $\overline{\partial}$-Neumann operator N_{q-1} acts on functions, see [9] Theorem 4.4.3.

The idea is that when $\nu \longrightarrow \infty$ to get

$$\overline{\partial} N_{q-1} \overline{\partial}^{\star} (\phi_{\varepsilon} u) + \overline{\partial}^{\star} N_{q+1} \overline{\partial} (\phi_{\varepsilon} u) = (\phi_{\varepsilon} u).$$

Now, we can use the inequality (3.8) on Ω_ν's;

$$\begin{aligned} &\sum_{|K|=q-1}^{\prime} \int_{\Omega_\nu} e^{\lambda_\varepsilon - 1} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_\varepsilon}{\partial z_j \partial \overline{z}_k} \left(\phi_\varepsilon u_{jK}\right)_\nu \overline{\left(\phi_\varepsilon u_{kK}\right)_\nu} \, dV(z) \\ &\leq \|\overline{\partial} \left(\phi_\varepsilon u\right)_\nu \|^2_{\Omega_\nu} + \|\overline{\partial}^{\star}_\nu \left(\phi_\varepsilon u\right)_\nu \|^2_{\Omega_\nu}. \end{aligned} \tag{3.10}$$

Since we have property (P) on V_ε we derive the following from (3.10);

$$\frac{1}{\varepsilon} \|(\phi_\varepsilon u)_\nu\|^2_{\Omega_\nu} \leq \|\overline{\partial} \left(\phi_\varepsilon u\right)_\nu \|^2_{\Omega_\nu} + \|\overline{\partial}^{\star}_\nu \left(\phi_\varepsilon u\right)_\nu \|^2_{\Omega_\nu} \tag{3.11}$$

Now, by using the definition of $(\phi_\varepsilon u)_\nu$ in (3.9)

$$\begin{aligned} \overline{\partial}(\phi_\varepsilon u)_\nu &= \overline{\partial} \left(\overline{\partial} N_{q-1,\nu} \vartheta(\phi_\varepsilon u) + \overline{\partial}^{\star}_\nu N_{q+1,\nu} \overline{\partial}(\phi_\varepsilon u) \right) = \overline{\partial\partial^{\star}_\nu} N_{q+1,\nu} \overline{\partial}(\phi_\varepsilon u), \\ \overline{\partial}^{\star}_\nu(\phi_\varepsilon u)_\nu &= \overline{\partial}^{\star}_\nu \left(\overline{\partial} N_{q-1,\nu} \vartheta(\phi_\varepsilon u) + \overline{\partial}^{\star}_\nu N_{q+1,\nu} \overline{\partial}(\phi_\varepsilon u) \right) = \overline{\partial}^{\star}_\nu \overline{\partial} N_{q-1,\nu} \vartheta(\phi_\varepsilon u). \end{aligned}$$

Take into account that $\overline{\partial\partial^{\star}_\nu} N_{q+1,\nu}$ and $\overline{\partial}^{\star}_\nu \overline{\partial} N_{q-1,\nu}$ are orthogonal projections. They are continuous operators on $L^2_*(\Omega_\nu)$ hence considering their norms less than 1 in the respective L^2 spaces (3.11) becomes

$$\begin{aligned} \frac{1}{\varepsilon} \|(\phi_\varepsilon u)_\nu\|^2_{\Omega_\nu} &\leq \|\overline{\partial} \left(\phi_\varepsilon u\right) \|^2_{\Omega_\nu} + \|\overline{\partial}^{\star} \left(\phi_\varepsilon u\right) \|^2_{\Omega_\nu} \\ &\leq \|\overline{\partial} \left(\phi_\varepsilon u\right) \|^2_{\Omega \cap B_p} + \|\overline{\partial}^{\star} \left(\phi_\varepsilon u\right) \|^2_{\Omega \cap B_p}. \end{aligned} \tag{3.12}$$

Now, one can check that $\left(\phi_\varepsilon u\right)_\nu$ goes to $(\phi_\varepsilon u)$ weakly in $L^2_{(0,q)}(\Omega \cap B_p)$ as $\nu \longrightarrow \infty$, where $\left(\phi_\varepsilon u\right)_\nu$ is continued by zero on $(\Omega \cap B_p) \setminus \Omega_\nu$; Since $\overline{\partial} N_{q-1,\nu}$ and $\overline{\partial}^{\star}_\nu N_{q+1,\nu}$ are bounded in their norms in $L^2_{(0,q-1)}(\Omega_\nu)$ and $L^2_{(0,q+1)}(\Omega_\nu)$ respectively, $\left(\phi_\varepsilon u\right)_\nu$ are bounded in $L^2_{(0,q)}(\Omega \cap B_p)$ independently of ν. Thus, a suitable subsequence will converge weakly in $L^2_{(0,q)}(\Omega \cap B_p)$. By observing $(\phi_\varepsilon u)$ is in $\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^{\star}) \subset$

$L^2_{(0,q)}(\Omega \cap B_p)$, it can be seen that the limit equals $(\phi_\varepsilon u)$, see [29], proof of Theorem 1. As a result of combining the above argument with (3.12) we get

$$\frac{1}{\varepsilon}\|\phi_\varepsilon u\|^2_{\Omega\cap B_p} \leq \|\overline{\partial}\,(\phi_\varepsilon u)\,\|^2_{\Omega\cap B_p} + \|\overline{\partial}^\star\,(\phi_\varepsilon u)\,\|^2_{\Omega\cap B_p}. \tag{3.13}$$

Thus, from (3.7) and (3.13) we get

$$\begin{aligned}\|u\|^2_{\Omega\cap B_p} &\lesssim \varepsilon\left(\|\overline{\partial}\,(\phi_\varepsilon u)\,\|^2_{\Omega\cap B_p} + \|\overline{\partial}^\star\,(\phi_\varepsilon u)\,\|^2_{\Omega\cap B_p}\right) + \|(1-\phi_\varepsilon)u\|^2_{\Omega\cap B_p}\\ &\lesssim \varepsilon\left(\|\overline{\partial}u\|^2_{\Omega\cap B_p} + \|\overline{\partial}^\star u\|^2_{\Omega\cap B_p} + \|(\nabla\phi_\varepsilon)u\|^2_{\Omega\cap B_p}\right) + \|(1-\phi_\varepsilon)u\|^2_{\Omega\cap B_p}.\end{aligned}$$

$(1-\phi_\varepsilon)u$ and $\nabla\phi_\varepsilon u$ can be viewed as forms on Ω in $\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star)$. Let $\chi_\varepsilon \in C^\infty_\circ(B_p)$ such that $\chi_\varepsilon \equiv 1$ on $\{\{\mathrm{supp}(\nabla\phi_\varepsilon)\} \cup \{\mathrm{supp}(1-\phi_\varepsilon)\}\} \subset B_p$.

$$\|u\|^2_{\Omega\cap B_p} \lesssim \varepsilon\left(\|\overline{\partial}u\|^2_{\Omega\cap B_p} + \|\overline{\partial}^\star u\|^2_{\Omega\cap B_p}\right) + C_{\phi_\varepsilon}\|\chi_\varepsilon u\|^2_{\Omega\cap B_p} \tag{3.14}$$

$\forall u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star) \subset L^2_{(0,q)}(\Omega \cap B_p)$.

Now, we will try to estimate the last term in (3.14). $\chi_\varepsilon u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star) \subset L^2_{(0,q)}(\Omega)$ for any $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star) \subset L^2_{(0,q)}(\Omega \cap B_p)$, so we have the following estimate on Ω; $\forall \varepsilon' > 0$ there exists $C_{\varepsilon'} > 0$ such that for $f \in J_\Omega$, we have f-compactness multiplier estimate

$$\begin{aligned}C_{\phi_\varepsilon}\|f\chi_\varepsilon u\|^2_\Omega &\leq C_{\phi_\varepsilon}\left[\varepsilon'\left(\|\overline{\partial}(\chi_\varepsilon u)\|^2_\Omega + \|\overline{\partial}^\star(\chi_\varepsilon u)\|^2_\Omega\right) + C_{\varepsilon',f}\|\chi_\varepsilon u\|^2_{-1,\Omega}\right]\\ &\leq C_{\phi_\varepsilon}[\varepsilon'(\underbrace{\|\overline{\partial}(\chi_\varepsilon u)\|^2_\Omega}_{\lesssim \|\overline{\partial}u\|^2_{\Omega\cap B_p}} + \underbrace{\|\overline{\partial}^\star(\chi_\varepsilon u)\|^2_\Omega}_{\lesssim \|\overline{\partial}^\star u\|^2_{\Omega\cap B_p}} + \|\nabla\chi_\varepsilon u\|^2_\Omega)\\ &+ C_{\varepsilon',f}C_\varepsilon\|u\|^2_{-1,\Omega\cap B_p}].\end{aligned} \tag{3.15}$$

Let's point out that the idea is to use the following, $C_{\phi_\varepsilon}\|\chi_\varepsilon u\|^2_{\Omega\cap B_p} \leq C_{\phi_\varepsilon,f}\|f\chi_\varepsilon u\|^2_\Omega$ combined with the estimate in (3.15) and the existence of $f \neq 0 (\in J_\Omega)$ on $\overline{\Omega} \cap B_p$.

Let's see how we got the inequalities between the (-1) Sobolev norms, $\|\cdot\|_{-1,\Omega}$

and $\|\cdot\|_{-1,\Omega\cap B_p}$ with associated domains.

$$\begin{aligned}
\|\chi_\varepsilon u\|^2_{-1,\Omega} &= \sup_{0\neq v\in\left(W^1_{(0,q)}(\Omega)\right)_\circ:\ \|v\|_{1,\Omega}=1} |(\chi_\varepsilon u, v)_\Omega| \\
&= \sup_{0\neq v\in\left(W^1_{(0,q)}(\Omega)\right)_\circ:\ \|v\|_{1,\Omega}=1} |(u, \chi_\varepsilon v)_\Omega| \\
&\leq \|\chi_\varepsilon v\|_{1,\Omega\cap B_p}\|u\|_{-1,\Omega\cap B_p} \\
&\leq C_\varepsilon\|v\|_{1,\Omega}\|u\|_{-1,\Omega\cap B_p}.
\end{aligned}$$

Now, lets estimate $\|\nabla\chi_\varepsilon u\|^2_\Omega$ in (3.15)

$$\begin{aligned}
\|\nabla\chi_\varepsilon u\|^2_\Omega &\lesssim \|\nabla\chi_\varepsilon u\|^2_{\Omega\cap B_p} \\
&\lesssim C_\varepsilon\|u\|^2_{\Omega\cap B_p} \\
&\lesssim C_\varepsilon\left(\|\overline{\partial}u\|^2_{\Omega\cap B_p} + \|\overline{\partial}^\star u\|^2_{\Omega\cap B_p}\right).
\end{aligned}$$

On the last step we used the basic estimate on $\Omega\cap B_p$, $\|u\|^2_{\Omega\cap B_p}\leq C(\|\overline{\partial}u\|^2_{\Omega\cap B_p} + \|\overline{\partial}^\star u\|_{\Omega\cap B_p})$.

By the choice of p and its neighborhood B_p there is $f\in J_\Omega$ such that $f\neq 0$ on $\overline{\Omega}\cap B_p$. Thus, combining (3.15) and the discussion after it within (3.14) we have a compactness estimate on $\Omega\cap B_p$. That is, $\forall\varepsilon>0\ \exists C_\varepsilon>0$ such that

$$\|u\|^2_{B_p\cap\Omega} \lesssim \varepsilon\left(\|\overline{\partial}u\|^2_{B_p\cap\Omega} + \|\overline{\partial}^\star u\|^2_{B_p\cap\Omega}\right) + C_\varepsilon\|u\|^2_{-1,B_p\cap\Omega} \tag{3.16}$$

$\forall u\in\mathrm{Dom}(\overline{\partial})\cap\mathrm{Dom}(\overline{\partial}^\star)\subset L^2_{(0,q)}(\Omega\cap B_p)$.

As we explained before, by Fu and Straube's result [15], on a convex domain the above result implies that there is no any discs in the boundary of $\Omega\cap B_p$, so $p\notin\overline{\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})}$. Thus, since we have chosen $p\in b\Omega\backslash A$ we can conclude that $\overline{\left(\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})\right)}$ is contained in A and as a result

$$\overline{\left(\bigcup_{\alpha\in\Lambda} f_\alpha(\mathbb{D})\right)} = A.$$

□

Corollary 3. *Let Ω be a bounded convex domain in $\mathbb{C}^n$ and $1 \leq q \leq n$ then*

$$A_q = \overline{\left(\bigcup_{\alpha\in\Lambda} K_\alpha\right)}.$$

Here, the family $\{K_\alpha\}_{\alpha\in\Lambda}$ denotes the family of q-dimensional analytic varieties on the boundary of Ω.

Proof. The $\overline{\partial}$-Neumann operator N_q fails to be compact on $L^2_{(0,q)}(\Omega)$ iff the boundary of the convex domain Ω contains an analytic variety of dimension $\geq q$, see [15], Theorem 11. The proof of the corollary goes exactly along the lines of the proof of Theorem 13. □

2. Characterization of $A(\neq \emptyset)$ on a smooth bounded pseudoconvex complete Hartogs domain Ω in $\mathbb{C}^2$

Definition 6. *A complete Hartogs domain Ω in $\mathbb{C}^2$ with base Ω_1 on an open set in $\mathbb{C}$ is defined by $|w| < e^{-\phi(z)}$ for $z \in \Omega_1$, where $\phi(z)$ is an upper semi-continuous function.*

A complete Hartogs domain Ω is pseudoconvex if and only if $\phi(z)$ is a subharmonic function on the base Ω_1, see [30], section 1.3.7.

Let Ω be a smooth bounded complete Hartogs domain and $\pi : b\Omega \longrightarrow \overline{\Omega}_1$ be the continuous projection map $(z, w) \longmapsto (z, 0)$. It follows from the computation of the Levi form that the weakly pseudoconvex boundary points with $w \neq 0$ are

$$\{(z, w) \in b\Omega \mid |w| = e^{-\phi(z)} \text{ and } \Delta\phi(z) = 0\}.$$

On a smooth bounded pseudoconvex (not necessarily complete) Hartogs domain compactness of the $\overline{\partial}$-Neumann operator is equivalent to the existence of property (P)

is a result by Christ and Fu in [10]. Additionally, if the domain is also complete i.e. in the form $\Omega := \{(z,w) \in \mathbb{C}^2 \mid z \in \Omega_1 \subset \mathbb{C}^1,\ |w| < e^{-\phi(z)}\}$ and the boundary points with $w = 0$ are strictly pseudoconvex, then both of the above conditions (compactness and property (P)) are equivalent to the set of projected (onto Ω_1) infinite type points of $b\Omega$ having nonempty fine interior, a result by Sibony in [34], see also [16].

We will use the following notation: Let D be a subset of Ω_1. Define Ω in the following way:

$$\Omega := \{(z,w) \in \mathbb{C}^2 \mid z \in \Omega_1 \subset \mathbb{C}^1,\ |w| < e^{-\phi(z)}\}.$$

Since $\pi : b\Omega \longrightarrow \overline{\Omega}_1$ is a continuous projection map,

$$\pi^{-1}(D) := \{(z,w) \in b\Omega \mid z \in D,\ |w| = e^{-\phi(z)}\} \subset b\Omega.$$

The fine topology is the smallest topology that makes all subharmonic functions continuous; see, e.g. [17] for properties of this toplogy. It is strictly larger than the Euclidean topology, and there exist compact sets with empty Euclidean interior, but nonempty fine interior, see [2] example 7.9.3.

The following notations will be used: $Int_f(K)$ for the fine interior points (interior points under the Fine topology) of the set K, $Int_e(K)$ for the interior points under the Euclidean topology of the set K, and $\overline{Int_f(K)}^E$ for the Euclidean closure of the fine interior points of the set K.

The next three propositions are from Sibony's paper [34]. We will give them with the original notion of B-regularity, but when we use them in the proof of the theorem we will refer to it as property (P).

Proposition 5. *[34] Let $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$ be two compact subsets. Let π be a continuous map from X to Y. Suppose that the components of π belong to the algebra*

$H(X)$, algebra of the functions that can be uniformly approximated by holomorphic functions in a neighborhood of X. Suppose also that Y is B-regular and that for all $y \in Y$ compact $\pi^{-1}(y)$ is B-regular. Then X is B-regular.

Proposition 6. *[34] Let $X \subset \mathbb{C}$ be compact. X is B-regular if and only if it is of empty fine interior.*

Proposition 7. *[34] Let X be compact in $\mathbb{C}^n$. Suppose that $X = \cup_{k=1}^{\infty} X_k$ where every X_k is a B-regular compact set. Then X is B-regular.*

The following result is a characterization of the set A on a bounded pseudoconvex complete Hartogs domain with smooth boundary. Denote by K the set $\{z \in \Omega_1 \mid \Delta\phi(z) = 0\}$, the set of projected (onto Ω_1) infinite type points of $b\Omega$.

Theorem 14. *Let Ω be a smooth bounded pseudoconvex complete Hartogs domain in $\mathbb{C}^2$. Assume that the boundary points of the form $(z, 0)$ are strictly pseudoconvex. Then*

$$A = \pi^{-1}\left(\overline{Int_f(K)}^E\right).$$

Remark 5. The assumption about the boundary points of the form $(z, 0)$ to be strictly pseudoconvex forces the projected infinite type points to stay away from the boundary of the base set. In particular, K is a compact subset of Ω_1. The case without the assumption will be investigated at another time.

Proof. According to the Proposition 5 and Proposition 6 we can argue as follows: Let $p \in b\Omega \backslash \pi^{-1}\left(\overline{Int_f(K)}^E\right)$, then $p = (p_z, p_w)$ where $\pi(p) = p_z \in \Omega_1 \backslash \overline{Int_f(K)}^E$. Then $\exists D(p_z, r) \subset \Omega_1$ such that $D(p_z, r) \cap \overline{Int_f(K)}^E = \emptyset$.

Consider the portion of $b\Omega$ over $D(p_z, r)$, $\pi^{-1}(D(p_z, r)) = \{(z, w) \in \mathbb{C}^2 \mid z \in D(p_z, r),\ |w| = e^{-\phi(z)}\} \subset b\Omega$ and choose an open ball with center $p = \pi^{-1}(p_z)$, $B((p_z, p_w), r')$ such that $B(p, r') \cap b\Omega \subset \pi^{-1}(D(p_z, r))$.

Let $K := \{z \in \Omega_1 \mid \Delta\phi(z) = 0\}$, the projected (onto Ω_1) infinite type points of $b\Omega$, be written as the union of two disjoint compact sets $K = \overline{Int_f(K)}^E \cup K_\circ$ where $Int_f(K_\circ) = \emptyset$. Now, by Proposition 6 the compact set $K_\circ$ has property (P), and then by Proposition 5 it follows that $\pi^{-1}(K_\circ) \subset b\Omega$ has property (P), π is a continuous projection map from $b\Omega$ to K (sending (z, w) to $(z, 0)$). Thus, we have property (P) on every compact subset of $B(p, r') \cap b\Omega$ which implies the existence of compactness estimate on $B(p, r') \cap \Omega$, see [8]. From here on the argument proceeds as in the first part of the proof of Theorem 13.

Let's show the other way around, that $\pi^{-1}\left(\overline{Int_f(K)}^E\right)$ is a subset of A.
Let $p_z \in \Omega_1 \backslash \pi(A)$ then $\exists r > 0$ such that $D(p_z, r) \cap \pi(A) = \emptyset$ ($\pi(A)$ is closed!). Then, construct a new Hartogs domain Ω' with the base $D(p_z, r') \supset D(p_z, r)$. Choose $\varphi(z) \in C_\circ^\infty(D(p_z, r'))$ such that $\varphi(z) \equiv 1$ on $\overline{D(p_z, r)}$. Now, set $\psi(z) := \phi(z)$ on $D(p_z, r)$ and $\psi(z) := \phi(z)\varphi(z) + h(z)$ on $D(p_z, r')$, where $h(z)$ is a smooth radially symmetric strictly subharmonic function on $D(p_z, r') \backslash D(p_z, r)$, $h(z) = 0$ on $\overline{D(p_z, r)}$, and equals $\frac{1}{2}\log(r'^{\,2} - |z|^2)$ when $|z|$ is close to r'. (Choose $h(z) := v(|z|)\log(r'^{\,2} - |z|^2)^{1/2}$, where $v \in C^\infty(\mathbb{R})$, $\mathrm{supp}(v(z)) \subset [r, \infty)$ with v' and $v'' > 0$, and $v(r') = 1$.) Such a function can be chosen to have its second (radial) derivative as big as we wish on a given compact subset of $D(p_z, r') \backslash \overline{D(p_z, r)}$, in particular on $\{\Delta\psi(z) \leq 0\}$. Thus,

$$\Omega' := \{(z, w) \in \mathbb{C}^2 \mid z \in D(p_z, r'),\ |w| < e^{-\psi(z)}\}$$

is smooth bounded pseudoconvex complete Hartogs domain.

We claim that there exists a compactness estimate on Ω'. To prove the claim consider the boundary of the new domain Ω' as follows;

$$b\Omega' = \pi^{-1}\left(D(p_z, r)\right) \cup \pi^{-1}\left(D(p_z, r') \backslash \overline{D(p_z, r)}\right) \cup bD(p_z, r') \cup \pi^{-1}\left(bD(p_z, r)\right).$$

It will be sufficient to see that each part of the boundary of Ω' satisfies property

(P) or there exists a compactness estimate.

$\pi^{-1}(bD(p_z, r))$ has property (P) by Proposition 5; $bD(p_z, r)$ is a circle and so a totally real smooth submanifold in which case it has property (P).

$\pi^{-1}(D(p_z, r')\backslash\overline{D(p_z, r)})$ is strictly pseudoconvex part of the boundary of Ω. It can be seen as a countable union of compact sets each one having property (P); every compact subsets on a strictly pseudoconvex boundary is having Property (P).

$bD(p_z, r')$ by construction is having property (P); $bD(p_z, r')$ is also a part of $b\Omega'$ which we set it to be strongly pseudoconvex, so it has property (P).

Thus, the portion of $b\Omega'$ over $\overline{D(p_z, r')}\backslash D(p_z, r)$ satisfies property (P); it can be written as a countable union of compact sets each one satisfying property (P) then so does $\pi^{-1}\left(D(p_z, r')\backslash\overline{D(p_z, r)}\right) \cup bD(p_z, r') \cup \pi^{-1}\left(bD(p_z, r)\right)$, by Theorem 7.

As for the rest of the boundary of Ω' we do the following: For every $q \in \pi^{-1}(D(p_z, r))$ we can find an open ball $B(q, R)$ such that $B(q, R) \cap \Omega'$ is having compactness estimate because we have at least one compactness multiplier f on Ω that does not vanish on $B(q, R) \cap \overline{\Omega'} = B(q, R) \cap \overline{\Omega}$. Existence of non-vanishing compactness multiplier in this part of the boundary is because we have chosen A away from this portion of $b\Omega$. The existence of the compactness estimate can be obtained as in the second part of the proof of Theorem 13. For completeness of the argument we give a sketch of the proof:

Choose ϕ_ε as a smooth cut off function such that $(0 \leq \phi_\varepsilon \leq 1)$, $\phi_\varepsilon \equiv 1$ near $bB(q, R)$ and supported in V_ε.

Take $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star)$ on $\Omega \cap B(q, R) =: \Omega \cap B_q$ and

$$\|u\|^2_{\Omega\cap B_q} \lesssim \|\phi_\varepsilon u\|^2_{\Omega\cap B_q} + \|(1-\phi_\varepsilon)u\|^2_{\Omega\cap B_q}. \tag{3.17}$$

The estimate we get (through the regularization procedure, see the second part of the proof of Theorem 13.) for the first term on the right (3.17) is

$$\|\phi_\varepsilon u\|^2_{\Omega\cap B_q} \le \varepsilon \left(\|\overline{\partial} u\|^2_{\Omega\cap B_q} + \|\overline{\partial}^\star u\|^2_{\Omega\cap B_q} + \|(\nabla\phi_\varepsilon)u\|^2_{\Omega\cap B_q} \right). \tag{3.18}$$

Then, considering (3.17) in (3.18) we get

$$\|u\|^2_{\Omega\cap B_q} \lesssim \varepsilon \left(\|\overline{\partial} u\|^2_{\Omega\cap B_q} + \|\overline{\partial}^\star u\|^2_{\Omega\cap B_q} + \|(\nabla\phi_\varepsilon)u\|^2_{\Omega\cap B_q} \right) + \|(1-\phi_\varepsilon)u\|^2_{\Omega\cap B_q}. \tag{3.19}$$

View $(1-\phi_\varepsilon)u$ and $(\nabla\phi_\varepsilon)u$ as forms on Ω in $\mathrm{Dom}(\overline{\partial})\cap\mathrm{Dom}(\overline{\partial}^\star)$. Use the existence of compactness multiplier f on Ω such that $f \neq 0$ on $b\Omega \cap B_q$ to estimate third and fourth term in (3.19). Combining all gives compactness estimate on $B(q,R)\cap b\Omega' = B(q,R)\cap b\Omega$.

The smooth bounded pseudoconvex complete Hartogs domain Ω' with the base $D(p_z, r')$ has compactness estimate. Moreover, by a result of Christ and Fu, see [10] Theorem 1.1, existence of Property (P) and compactness estimate are equivalent on Ω'. Furthermore, $Int_f(K)\cap D(p_z,r) = \emptyset$ by Proposition 6. Recall that $p_z \in \Omega_1\backslash\pi(A)$ was chosen arbitrarily then $Int_f(K)\cap(\Omega_1\backslash\pi(A)) = \emptyset$. Therefore, $Int_f(K)\subset\pi(A)$.

Now, let us point out that the set A is invariant under rotation. If it was not, then on a neighborhood U of $\pi^{-1}(\pi(A))\backslash A$, which would not be an empty set, the existence of compactness estimate would fail. This would contradict invariance of compactness of the $\overline{\partial}$-Neumann operator under rotation. Thus, we have $\pi^{-1}(\pi(A)) = A$, and so

$$\pi^{-1}\left(\overline{Int_f(K)}^E\right) \subset A.$$

□

Remark 6. Assume boundary of a bounded (complete) pseudoconvex Reinhardt domain Ω to be Lipschitz. Consider the following two theorems. The first one attributed to Catlin (see Proposition 9 in [15]) is that the existence of the compactness estimate on a bounded pseudoconvex domain Ω in $\mathbb{C}^2$ with Lipschitz boundary implies the absence of an analytic disc from the boundary of the domain. The second one, which is in fact more general than we need, is Theorem 12 on the survey paper [15]; on a bounded pseudoconvex Reinhardt domain Ω absence of the disc from the boundary $b\Omega$ provides property (P) which implies the existence of the compactness estimate of the $\overline{\partial}$-Neumann operator on Ω. Thus, combining these two theorems we can say that on a bounded pseudoconvex Reinhardt domain Ω in $\mathbb{C}^2$ with Lipschitz boundary the absence of the disc from the boundary is equivalent to the existence of the compactness estimate of the $\overline{\partial}$-Neumann operator on Ω. The assumption about the boundary of such a domain to be Lipschitz regular makes the boundary stay free from the coordinate hyperplanes. The reason we want this is if a coordinate hyperplanes becomes part of the boundary of the domain it creates an analytic disc on the boundary but does not fail compactness of the $\overline{\partial}$-Neumann operator. Moreover, this regularity conditions on the boundary also gives flexibility to the boundary of a Reinhardt domain to be treated as locally convexifiable, see [15] section 5. Therefore, in a bounded pseudoconvex Reinhardt domains Ω in $\mathbb{C}^2$ with Lipschitz regularity on the boundary the characterization of the common zero set of the ideal of the multipliers is as in convex domains, see Theorem 13.

CHAPTER IV

INDEPENDENCE FROM THE METRIC OF SOME PROPERTIES OF THE $\overline{\partial}$-NEUMANN OPERATOR

A theorem by W. J. Sweeney in [39] shows that coercive estimates are independent of the metric on the tangent bundle, and it appears to be folklore that the same is true for subelliptic estimates. The metric considered is smooth positive definite hermitian on the whole closure of the domain. Here, we give a simple proof of this result, but specifically for subellipticity of the $\overline{\partial}$-Neumann operator. Besides, a natural question to ask is whether the same is true for compactness of the $\overline{\partial}$-Neumann operator. The answer is yes, the compactness of the $\overline{\partial}$-Neumann operator is also independent of the metric. First, we will present a study related with compactness of the $\overline{\partial}$-Neumann operator and the general metric. On the second part, we will give a proof for the subellipticity of the $\overline{\partial}$-Neumann operator independent of the general metric.

A. Notations

Unless otherwise specified Ω will be a pseudoconvex domain with smooth boundary in $\mathbb{C}^n$. $u = \sum'_{|J|=q} u_J d\overline{z}_J$ will be a $(0,q)$-form on Ω. The coefficients u_J are functions, belonging to various function classes on Ω. Consider A and B as multiindex sets;

$$A = (a_1,\ a_2,\ a_3,\ \ldots\ ,\ a_q),\quad 1 \leq a_1 < a_2 < \cdots < a_q \leq n.$$

Let $G_q = G_q^{AB} = G_q^{(a_1,\ldots,a_q)(b_1,\ldots,b_q)}$ for the $(0,q)$-form level be a metric tensor where G_q^{AB} is smooth and positive definite Hermitian in the sense of differential geometry at the closure of the domain Ω. The metric G_q^{AB} is on the q-fold exterior product of the cotangent bundle where the forms on Ω take their values. $L^2_{(0,q)}(\Omega, G_q)$ is a set of

$(0,q)$-forms u such that

$$J=(j_1,\ j_2,\ j_3,\ \dots\ ,\ j_q),\quad 1\le j_1<j_2<\cdots<j_q\le n$$

$$d\overline{z}_J=d\overline{z}_{j_1}\wedge d\overline{z}_{j_2}\wedge\cdots\wedge d\overline{z}_{j_n}$$

$$\begin{aligned}\langle u,v\rangle_{G_q}: &= \left\langle \sideset{}{'}\sum_{|J|=q} u_J d\overline{z}_J,\ \sideset{}{'}\sum_{|K|=q} v_K d\overline{z}_K\right\rangle_G\\ &= \sideset{}{'}\sum_{|J|=|K|=q} G_q^{JK}u_J(z)\overline{v_K(z)},\end{aligned}$$

$$(u,v)_{G_q} := \int_\Omega \langle u,v\rangle_{G_q} = \sideset{}{'}\sum_{|J|=|K|=q}\int_\Omega G_q^{JK}u_J(z)\overline{v_K(z)}dV,$$

and

$$\|u\|^2_{G_q} := \int_\Omega \langle u,u\rangle_{G_q} = \sideset{}{'}\sum_{|J|=|K|=q}\int_\Omega G_q^{JK}u_J(z)\overline{u_K(z)}dV.$$

$$\begin{aligned}\overline{\partial}u &= \overline{\partial}\left(\sideset{}{'}\sum_{|J|=q} u_J d\overline{z}_J\right) = \sideset{}{'}\sum_{|J|=q}\overline{\partial}u_J\wedge d\overline{z}_J\\ &= \sum_{j=1}^n \sideset{}{'}\sum_{|J|=q}\frac{\partial u_J}{\partial\overline{z}_j}d\overline{z}_j\wedge d\overline{z}_J.\end{aligned}$$

Remark 7. The general metric G_q does not have to be related to $G_{q'}$ for $q'\neq q$. In particular, it is not assumed that G_q is induced by G_1.

B. Compactness of the $\overline{\partial}$-Neumann Operator and the General Metric

To see that compactness of the $\overline{\partial}$-Neumann operator does not depend on the metric, we will express both N_q and N_q^G (the $\overline{\partial}$-Neumann operator induced with the metric G), in terms of the (G)-canonical (Kohn) solution operators. The key point is the fact that if f is a $\overline{\partial}$-closed form, then the canonical solution induced with a metric G, $\overline{\partial}^{\star}_{G}N^{G}$ may be composed with a suitable projection to obtain the unweighted canonical solution $\overline{\partial}^{\star}N$, and vice versa. This composition preserves compactness.

First we will develop a theory of the $\overline{\partial}$-Neumann operator induced with the metric G. It is analogous to the theory of the $\overline{\partial}$-Neumann operator with the usual (Euclidean) metric, δ.

$L^2_{(0,q)}(\Omega, G_q)$ is a Hilbert space via the above mentioned inner product; $L^2_{(0,q)}(\Omega)$ is independent of G_q as a set. With such a metric G_q all the induced norms are equivalent to the Euclidean norm, which is the one induced with the usual metric δ (Kronecker δ-delta). Thus, the $\overline{\partial}$ operator has the same domain and range as it has when it is associated with the Euclidean metric. In particular:

$$\overline{\partial}_q : L^2_{(0,q)}(\Omega, G_q) \twoheadrightarrow \ker(\overline{\partial}_{q+1}) \subset L^2_{(0,q+1)}(\Omega, G_{q+1})$$

is onto and the range of $\overline{\partial}_q$ is closed. $\overline{\partial}_q$ has an adjoint $\left(\overline{\partial}_q\right)^{\star}_{G}$.

In order to demonstrate that the domain of $\overline{\partial}^{\star}_{G}$ depends on the metric G we will calculate $\left(\overline{\partial}_{\circ}\right)^{\star}_{G_1} u$ and the (boundary) conditions for a smooth $(0,1)$-form to be in the domain of $\left(\overline{\partial}_{\circ}\right)^{\star}_{G_1}$. Assume $|\nabla\rho| = 1$. Let u be a $(0,1)$ form $u = \sum_{j=1}^{n} u_j \; d\overline{z}_j, \quad u_j \in C^1(\overline{\Omega})$ and $\alpha(z) \in C^{\infty}(\overline{\Omega})$ then

$$\begin{aligned} (u, \overline{\partial}\alpha)_{G_1} &= \left(\sum_{j=1}^{n} u_j d\overline{z}_j , \; \sum_{k=1}^{n} \frac{\partial \alpha}{\partial \overline{z}_k} d\overline{z}_k \right)_{G_1} \\ &= \sum_{j,k=1}^{n} \int_{\Omega} G_1^{jk} u_j \overline{\frac{\partial \alpha}{\partial \overline{z}_k}} dV. \end{aligned} \tag{4.1}$$

Note that $G_{\circ}$ is just a function, and we use $(G_{\circ})^{-1} = G^{\circ}$. Then, using integration by parts (4.1) becomes

$$
\begin{aligned}
&= -\sum_{j,k=1}^{n} \int_{\Omega} \frac{\partial\left(G_1^{jk} u_j(z)\right)}{\partial z_k}\, \overline{\alpha}\, dV + \sum_{j,k=1}^{n} \int_{b\Omega} G_1^{jk}\ u_j\ \overline{\alpha} \frac{\partial \rho}{\partial z_k} d\sigma \\
&= -\sum_{j,k,l_1=1}^{n} \int_{\Omega} G_{\circ} G^{\circ} \frac{\partial\left(G_1^{jk} u_j(z)\right)}{\partial z_k}\, \overline{\alpha}\, dV + \sum_{j,k=1}^{n} \int_{b\Omega} G_1^{jk}\ u_j\ \overline{\alpha} \frac{\partial \rho}{\partial z_k} d\sigma \\
&= \left(-\sum_{j,k=1}^{n} G^{\circ} \frac{\partial\left(G_1^{jk} u_j(z)\right)}{\partial z_k}\ ,\ \alpha \right)_{G_{\circ}} + \sum_{j,k=1}^{n} \int_{b\Omega} G_1^{jk}\ u_j\ \overline{\alpha} \frac{\partial \rho}{\partial z_k} d\sigma.
\end{aligned}
$$

The same computation with compactly supported smooth $(0,q)$-forms α annihilates the boundary term in the previous computation. Thus,

$$
(u, \overline{\partial}\alpha)_{G_1} = \left(-\sum_{j,k=1}^{n} G^{\circ} \frac{\partial\left(G_1^{jk} u_j(z)\right)}{\partial z_k}\ ,\ \alpha \right)_{G_{\circ}} . \tag{4.2}
$$

From (4.2) we get

$$
\left(\overline{\partial}_{\circ}\right)^{\star}_{G_1} u = -\left\{ -\sum_{j,k=1}^{n} G^{\circ} \frac{\partial\left(G_1^{jk} u_j(z)\right)}{\partial z_k} \right\},\ u \in \operatorname{Dom}\left(\left(\overline{\partial}_{\circ}\right)^{\star}_{G_1}\right). \tag{4.3}
$$

The condition for u to be in $\operatorname{Dom}(\overline{\partial}^{\star}_{G_1})$ is

$$
\sum_{j,k=1}^{n} \int_{b\Omega} G_1^{jk}\ u_j\ \overline{\alpha} \frac{\partial \rho}{\partial z_k} d\sigma = \sum_{j,k=1}^{n} \int_{b\Omega} \left(G_1^{jk}\ u_j \frac{\partial \rho}{\partial z_k} \right) \overline{\alpha} d\sigma = \ 0.
$$

Consequently, the boundary condition is

$$
\sum_{j,k=1}^{n} G_1^{jk}\ u_j \frac{\partial \rho}{\partial z_k} = \ 0 \ \text{ for } \ z \in b\Omega. \tag{4.4}
$$

Note that if G_1 is the usual Euclidean metric δ, that is, $G_1^{jk} = \delta_k^j$ (where δ_k^j is Kronecker-δ), we get the boundary conditions in (2.1).

The range of $\left(\overline{\partial}_q\right)^{\star}_{G_{q+1}}$ is closed because the range of $\overline{\partial}_q$ is, see for example, Lemma 4.1.1 in [9]. However, the range of $\left(\overline{\partial}_q\right)^\star_{G_{q+1}}$ is also dense in $\ker(\overline{\partial}_q)^{\perp_{G_q}}$, and so $\operatorname{Im}\left(\left(\overline{\partial}_q\right)^\star_{G_{q+1}}\right) = \ker(\overline{\partial}_q)^{\perp_{G_q}}$. It follows that

$$L^2_{(0,q)}(\Omega, G_q) = \underbrace{\ker(\overline{\partial}_q)}_{\operatorname{Im}(\overline{\partial}_{q-1})} \oplus \underbrace{\operatorname{Im}\left(\left(\overline{\partial}_q\right)^\star_{G_{q+1}}\right)}_{\ker\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right)}. \tag{4.5}$$

For $u, v \in \operatorname{Dom}(\overline{\partial}_q) \cap \operatorname{Dom}\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right)$, the Dirichlet form $Q_G(u,v)$ is defined as

$$Q_{G_q}(u,v) := \left(\overline{\partial} u, \overline{\partial} v\right)_{G_{q+1}} + \left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q} u, \left(\overline{\partial}_{q-1}\right)^\star_{G_q} v\right)_{G_{q-1}}. \tag{4.6}$$

Note that an operator T is closed iff its graph is closed which is the same as saying that the domain of the operator T, $\operatorname{Dom}(T)$ is complete under the norm

$$\|\psi\|_T = \|T\psi\| + \|\psi\|.$$

Similarly, $\operatorname{Dom}(\overline{\partial}_q) \cap \operatorname{Dom}\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right)$ is complete with respect to

$$|||\ u\ |||^2_G := Q_G(u,u) + \|u\|^2_G.$$

Then, as an application of theorem $VIII.15$ in Reed & Simon [32] (if Q is a closed symmetric quadratic form then Q is the quadratic form of a unique self adjoint operator) there is a unique non-negative selfadjoint operator $\Box^G_q$ associated to Q_G via

$$Q_{G_q}(u,v) = (\Box^G_q u, v), \quad u \in \operatorname{Dom}(\Box^G_q). \tag{4.7}$$

By (4.7), if $u \in \ker\left(\Box^G_q\right)$, then $Q_G(u,u) = 0$. Therefore, $\ker(\Box^G_q) = \ker(\overline{\partial}_q) \cap \ker\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right) = \{0\}$ by (4.5). Then, Hörmander's functional analysis Theorem 1.1.2 in [18] implies

$$\|u\|^2_{G_q} \lesssim \|\overline{\partial}_q u\|^2_{G_{q+1}} + \|\left(\overline{\partial}_{q-1}\right)^\star_{G_q} u\|^2_{G_{q-1}}, \quad \forall u \in \operatorname{Dom}(\overline{\partial}) \cap \operatorname{Dom}\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right). \tag{4.8}$$

From (4.8), it follows (as in Euclidean case) from general Hilbert space arguments that $\Box_q^G$ has a bounded inverse. In fact: $u, v \in \text{Dom}(\overline{\partial}_q) \cap \text{Dom}\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right)$ then

$$\begin{aligned} \left|(u,v)_{G_q}\right| &\leq \|u\|_{G_q}\|v\|_{G_q} \\ &\lesssim \|u\|_{G_q}\left(\|\overline{\partial}_q v\|^2_{G_{q+1}} + \|\left(\overline{\partial}_{q-1}\right)^\star_{G_q} v\|^2_{G_{q-1}}\right)^{\frac{1}{2}}. \end{aligned}$$

That is, this functional is continuous in the norm induced by Q_{G_q} on $\text{Dom}(\overline{\partial}_q) \cap \text{Dom}\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right)$. Thus, it is given by an inner product

$$(u,v)_{G_q} = Q_{G_q}(N_q^G u, v).$$

Both N_q^G and N_q can be expressed in terms of the canonical solution operators

$$N_q = \left(\overline{\partial}^\star N_q\right)^\star \left(\overline{\partial}^\star N_q\right) + \left(\overline{\partial}^\star N_{q+1}\right)\left(\overline{\partial}^\star N_{q+1}\right)^\star,$$

$$N_q^G = \left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q} N_q^G\right)^\star_{G_q}\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q} N_q^G\right) + \left(\left(\overline{\partial}_q\right)^\star_{G_{q+1}} N_{q+1}^G\right)\left(\left(\overline{\partial}_q\right)^\star_{G_{q+1}} N_{q+1}^G\right)^\star_{G_{q+1}}.$$

Denote by P_q^G the orthogonal projection from $L^2_{0,q}(\Omega, G)$ onto $\ker(\overline{\partial}_q)$. Since $(\overline{\partial}_{q-1})^\star_{G_q} N_q^G$ annihilates $\ker(\overline{\partial}_q)^{\perp_G}$, we have

$$\left(\overline{\partial}_{q-1}\right)^\star_{G_q} N_q^G = \left(\overline{\partial}_{q-1}\right)^\star_{G_q} N_q^G P_q^G.$$

Now, if f is a $\overline{\partial}$-closed $(0,q)$-form, then $\overline{\partial}^\star N_q f$ and $(\overline{\partial}_{q-1})^\star_{G_q} N_q^G f$ are both solutions of the equation $\overline{\partial} u = f$; orthogonal in the respective inner products to $\ker(\overline{\partial}_{q-1})$. Therefore the previous formula implies

$$\left(\overline{\partial}_{q-1}\right)^\star_{G_q} N_q^G = (I - P_{q-1}^G)\overline{\partial}^\star N_q P_q^G.$$

Similarly,

$$\left(\overline{\partial}_q\right)^\star_{G_{q+1}} N_{q+1}^G = (I - P_q^G)\overline{\partial}^\star N_{q+1} P_{q+1}^G,$$

and analogously,

$$\overline{\partial}^\star N_q = (I - P_{q-1}) \left(\overline{\partial}_{q-1}\right)^\star_{G_q} N_q^G P_q,$$

$$\overline{\partial}^\star N_{q+1} = (I - P_q) \left(\overline{\partial}_q\right)^\star_{G_{q+1}} N_{q+1}^G P_{q+1}.$$

Theorem 15. *N_q^G is compact if and only if N_q is compact.*

Proof. By using the fact that $A^\star A + BB^\star$ is compact if and only if A and B are compact, the above identities, and the fact that composition with bounded operators (projections in our case) preserves compactness, we get the result. $\square$

C. Subellipticity of the $\overline{\partial}$-Neumann Operator and the General Metric

Here, we only treat global subellipticity. A subelliptic estimate of order $\varepsilon > 0$ is said to hold, if

$$\|u\|_\varepsilon^2 \leq C \left(\|\overline{\partial} u\|^2 + \|\overline{\partial}^\star u\|^2 \right) \quad \text{for all} \quad u \in C^\infty_{(0,q)}(\overline{\Omega}) \cap \mathrm{Dom}\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q} \right) \tag{4.9}$$

where the norm on the left hand side is the L^2-Sobolev norm of order ε. This holds if and only if N_q^G maps $L^2_{(0,q)}(\Omega)$ continuously to $W^{2\varepsilon}_{(0,q)}(\Omega)$. The proof is the same as in the Euclidean case. Let δ represent the usual (Euclidean) metric and G the general metric.

Theorem 16. *N_q^G is subelliptic if and only if N_q^δ is subelliptic.*

Before we give the proof of the theorem let's first set up some notions. Let's take u and v from $L^2_{(0,q)}(\Omega)$. Then, define the following linear functional $u \longmapsto (u, v)_{L^2_{(0,q)}(\Omega)}$. Since

$$\left| (u, v)_{L^2_{(0,q)}(\Omega)} \right| \leq \|u\|_{L^2_{(0,q)}(\Omega)} \|v\|_{L^2_{(0,q)}(\Omega)} \lesssim \|u\|_{L^2_{(0,q)}(\Omega)} \|v\|_{L^2_{(0,q)}(\Omega, G)}$$

it is a bounded linear functional. Then, by the Riesz representation theorem, there is a unique element $T_q^G u$ in $L^2_{(0,q)}(\Omega)$ such that the functional is given by pairing with

$T_q^G u$

$$(u,v)_{L^2_{(0,q)}(\Omega)} = (T_q^G u, v)_{L^2_{(0,q)}(\Omega,G)}.$$

Thus we have a bounded linear operator $T_q^G : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q)}(\Omega, G)$, which is an isomorphism. T_q^G can be computed explicitly as in (4.1)-(4.4).

Lemma 4. *$u \in Dom(\overline{\partial}^\star_{q-1})$ if and only if $T_q^G u \in Dom\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right)$. Moreover,*

$$\left(\overline{\partial}_{q-1}\right)^\star_{G_q} T_q^G u = T_{q-1}^G \left(\overline{\partial}_{q-1}\right)^\star u. \tag{4.10}$$

Proof. Let $u \in \mathrm{Dom}(\overline{\partial}^\star)$ and $\alpha \in C^\infty_{(0,q-1)}(\overline{\Omega})$ then

$$\begin{aligned} (T_q^G u, \overline{\partial}\alpha)_{G_q} &= (u, \overline{\partial}\alpha) \\ &= (\overline{\partial}^\star u, \alpha) \\ &= (T_{q-1}^G \overline{\partial}^\star u, \alpha)_{G_{q-1}}. \end{aligned}$$

Thus, $T_q^G u \in \mathrm{Dom}\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right)$, and $\left(\overline{\partial}_{q-1}\right)^\star_{G_q} T_q^G u = T_{q-1}^G \left(\overline{\partial}_{q-1}\right)^\star u$.

Let $T_q^G u \in \mathrm{Dom}\left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q}\right)$ and $\alpha \in C^\infty_{(0,q-1)}(\overline{\Omega})$. Then,

$$\begin{aligned} (u, \overline{\partial}_q\alpha) &= (T_q^G u, \overline{\partial}\alpha)_{G_q} \\ &= \left(\left(\overline{\partial}_{q-1}\right)^\star_{G_q} T_q^G u, \alpha\right)_{G_{q-1}} \\ &= ((T_{q-1}^G)^{-1} \overline{\partial}^\star_G T_q^G u, \alpha). \end{aligned}$$

So $u \in \mathrm{Dom}(\overline{\partial}^\star_q)$, and $\overline{\partial}^\star_q u = \left(T_{q-1}^G\right)^{-1} \left(\overline{\partial}_{q-1}\right)^\star_{G_q} T_q^G u.$ □

Proof of the theorem: For simplicity we only do the case $q = 1$. The arguments for $q > 1$ are analogous. First assume that there is a subelliptic estimate of order $\varepsilon > 0$ in the norm induced with the G-metric:

$$\|u\|^2_{\varepsilon,G_1} \leq C\left(\|\overline{\partial}u\|^2_{G_2} + \|\overline{\partial}^\star u\|^2_{G_\circ}\right) \quad \forall u \in \mathrm{Dom}(\overline{\partial}_1) \cap \mathrm{Dom}\left(\left(\overline{\partial}_\circ\right)^\star_{G_1}\right) \quad \text{on} \ \ \Omega.$$

Now, let $u \in \text{Dom}(\overline{\partial}_1) \cap \text{Dom}(\overline{\partial}_1^\star)$. Then $T_1^G u \in \text{Dom}\left(\left(\overline{\partial}_\circ\right)_{G_1}^\star\right)$. Also,

$$\|\overline{\partial}\left(T_1^G u\right)\|_{G_2}^2 \lesssim \sum_{j,k=1}^{n} \|\frac{\partial u_k}{\partial \overline{z}_j}\|^2 + \|u\|^2, \tag{4.11}$$

in view of (4.1)-(4.4). (4.11) implies $T_1^G u \in \text{Dom}(\overline{\partial}_1)$.

We have

$$\begin{aligned}
\|u\|_\varepsilon^2 &\lesssim \|T_1^G u\|_{\varepsilon,G_1}^2 \\
&\lesssim \|\overline{\partial} T_1^G u\|_{G_1}^2 + \|\left(\overline{\partial}_\circ\right)_{G_1}^\star T_1^G u\|_{G_\circ}^2 \\
&\lesssim \sum_{j,k=1}^{n} \|\frac{\partial u_k}{\partial \overline{z}_j}\|^2 + \|u\|^2 + \|T_\circ^G \overline{\partial}^\star u\|^2 \\
&\lesssim \sum_{j,k=1}^{n} \|\frac{\partial u_k}{\partial \overline{z}_j}\|^2 + \|u\|^2 + \|\overline{\partial}_\circ^\star u\|^2 \\
&\lesssim \|\overline{\partial}_1 u\|^2 + \|\overline{\partial}_\circ^\star u\|^2.
\end{aligned} \tag{4.12}$$

The first inequality comes from the set up of T_1^G, the second inequality is from the assumption of having a subelliptic estimate associated with the metric G, the third inequality is coming from (4.11) and is from (4.10) in Lemma 4. The last inequality is the basic estimate together with the fact that the sum of bar derivatives of a form is bounded by the graph norm; both estimates are consequence of the basic identity (Kohn-Morrey formula (2.9) in Proposition 2.). Thus, we get a subelliptic estimate in the usual (Euclidean) metric.

Now, let's show how the subellipticity of N_1^δ (associated with the usual metric) implies the subellipticity of N_1^G (associated with the metric G). Note that the $\overline{\partial}$-Neumann operator $N_\circ^G$ acts on functions and the existence theory of $N_\circ^G$ goes along the lines of $N_\circ^\delta$, see [9] Theorem 4.4.3.

Let $u \in \mathrm{Dom}(\overline{\partial}_1) \cap \mathrm{Dom}\left(\left(\overline{\partial}_\circ\right)^\star_{G_1}\right) \subset L^2_{(0,1)}(\Omega, G_1)$ then we write

$$u = \left(\overline{\partial}_1\right)^\star_{G_2} N_2^G(\overline{\partial}_1 u) + \overline{\partial}_1 N_\circ^G \left(\left(\overline{\partial}_\circ\right)^\star_{G_1} u\right). \tag{4.13}$$

However, since $(I - P_1^G)$ projects onto the range of $\left(\overline{\partial}_1\right)^\star_{G_2}$, we write

$$\left(\overline{\partial}_1\right)^\star_{G_2} N_2^G(\overline{\partial} u) = \left(I - P_1^G\right) \overline{\partial}_1^\star N_2(\overline{\partial}_1 u). \tag{4.14}$$

As for the second term on the right of (4.13), by considering the commutativity property $\overline{\partial}_\circ N_\circ^G = N_1^G \overline{\partial}_\circ$ we rewrite it as follows:

$$\begin{aligned} N_1^G \overline{\partial}_\circ = \left(\left(\overline{\partial}_\circ\right)^\star_{G_1} N_1^G\right)^\star_G &= \left(\left(I - P_\circ^G\right) \overline{\partial}_\circ^\star N_1 P_1^G\right)^\star_G \\ &= P_1^G \left(\overline{\partial}_\circ^\star N_1\right)^\star_G \left(I - P_\circ^G\right). \end{aligned} \tag{4.15}$$

Thus, by using (4.14) and (4.15) we write (4.13) as

$$u = \left(I - P_1^G\right) \overline{\partial}^\star N_2(\overline{\partial}_1 u) + P_1^G \left(\overline{\partial}_\circ^\star N_1\right)^\star_G \left(\left(\overline{\partial}_\circ\right)^\star_{G_1} u\right); \tag{4.16}$$

note that $(I - P_\circ^G)\left(\overline{\partial}_\circ\right)^\star_{G_1} u = \left(\overline{\partial}_\circ\right)^\star_{G_1} u$.

Now, a subelliptic estimate associated with the metric G will follow if we can show the following two maps are continuous;

(i.) $P_1^G : W^\varepsilon_{(0,1)}(\Omega, G) \longrightarrow W^\varepsilon_{(0,1)}(\Omega, G)$

(ii.) $\left(\overline{\partial}^\star N_q\right)^\star_G : L^2_{(0,q)}(\Omega, G) \longrightarrow W^\varepsilon_{(0,q-1)}(\Omega, G),\ \ q = 1, 2$

Let's first show **(i.)**; let's write $P_1^G = \overline{\partial}\overline{\partial}^\star_G N_1^G$. Since N_1 is subelliptic it is compact and hence by Theorem 15, N_1^G is compact. Since N_1 is compact, therefore, N_2 is also compact see e.g. Proposition 3.5 in [36]. Then, $\overline{\partial}\overline{\partial}^\star_G N_1^G$, preserves Sobolev spaces. The proof is exactly the same as for $\overline{\partial}\overline{\partial}^\star N_1$, see for example Theorem 6.2.2 in [9].

As for **(ii.)**, note that $W^\varepsilon_{(0,q)}(\Omega, G) = W^\varepsilon_{(0,q)}(\Omega)$ and $\left(W^\varepsilon_{(0,q)}(\Omega)\right)^\star = W^{-\varepsilon}_{(0,q)}(\Omega)$, since $0 \leq \varepsilon \leq 1/2$, see [27]. Thus, the statement in **(ii.)** is equivalent to having continuous map

$$\overline{\partial}^{\star} N_q : W^{-\varepsilon}_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q-1)}(\Omega). \tag{4.17}$$

$L^2_{(0,q-1)}(\Omega)$ is dense in $W^{-\varepsilon}_{(0,q)}(\Omega)$, so to prove (4.17) we estimate

$$\begin{aligned}
\|\overline{\partial}^{\star} N_q u\|^2 + \|\overline{\partial} N_q u\|^2 &= \left(\overline{\partial}^{\star} N_q u, \overline{\partial}^{\star} N_q u\right) + \left(\overline{\partial} N_q u, \overline{\partial} N_q u\right) \\
&= (u, N_q u) \\
&\lesssim \|u\|_{-\varepsilon} \|N_q u\|_{\varepsilon} \\
&\leq (\text{s.c.}) \|N_q u\|^2_{\varepsilon} + (\text{l.c.}) \|u\|^2_{-\varepsilon} \\
&\lesssim (\text{s.c.}) \left(\|\overline{\partial}^{\star} N_q u\|^2 + \|\overline{\partial} N_q u\|^2\right) + (\text{l.c.}) \|u\|^2_{-\varepsilon}. \quad (4.18)
\end{aligned}$$

The last inequality comes from the subelliptic estimate associated with the usual metric. The first term on the right at the last inequality can be absorbed the left side to obtain

$$\|\overline{\partial}^{\star} N_q u\|^2 + \|\overline{\partial} N_q u\|^2 \lesssim \|u\|^2_{-\varepsilon}. \tag{4.19}$$

This was for $u \in L^2_{(0,q)}(\Omega)$. By density both $\overline{\partial}^{\star} N_q$ and $\overline{\partial} N_q$ extend to $W^{-\varepsilon}_{(0,q)}(\Omega)$, and (4.17) holds. Thus, N_1^G is subelliptic. $\square$

Remark 8. Barrett in [4] shows that on the Diederich-Fornæss worm domains [12] the Bergman projection does not map the Sobolev space W^k into itself when $k \geq 0$ is big. Then, a result by Boas and Straube in [6] implies that the $\overline{\partial}$-Neumann operator fails to map the space of $(0,1)$-forms with coefficients in W^k into itself for $k > 0$ big. However, by using a suitable metric on the $(0,1)$-forms one can make the $\overline{\partial}$-Neumann operator map the space of $(0,1)$-forms with coefficients in W^k into itself for any fixed $k \geq 1$. For example, Kohn in [22] uses weighted the L^2 space, $L^2(\Omega, e^{-\phi_t})$, where

$\phi_t(z) = t\,|z|^2$), see Chapter 6 in [9]. Thus, the Sobolev estimates are not independent of the metric. That is, for a given $k \geq 0$ one can choose $G^{jl} = \phi_t \delta_l^j$ with t big enough, such that $\|N_1^G u\|_{k,G} \lesssim \|u\|_{k,G}$ holds. On the other hand, for k big enough the following estimate (induced with Euclidean norm) $\|N_1^\delta u\|_k \lesssim \|u\|_k$ does not hold. Note that the difference does not lie in the norms (they are equivalent), but in the operators N_1^G and N_1^δ, respectively.

CHAPTER V

COMPACTNESS OF THE $\overline{\partial}$-NEUMANN OPERATOR ON THE INTERSECTION OF TWO DOMAINS

Let

$$\Omega_1 := \{z \in \mathbb{C}^n \mid \rho_1(z) < 0\}$$

and

$$\Omega_2 := \{z \in \mathbb{C}^n \mid \rho_2(z) < 0\}$$

be bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundaries, and $\nabla\rho_1$ and $\nabla\rho_2$ be $\neq 0$ on $b\Omega_1$ and $b\Omega_2$ respectively. Assume that the compactness estimates for the $\overline{\partial}$-Neumann operator exist on both domains, Ω_1 and Ω_2. Then, the question is whether there is a compactness estimate for the $\overline{\partial}$-Neumann operator on the transversal intersection of Ω_1 and Ω_2.

By the local property of the compactness of the $\overline{\partial}$-Neumann operator, [15], some parts of the intersection, $\Omega_1 \cap \{z \in \mathbb{C}^n \mid \rho_2(z) = 0\}$ and $\Omega_2 \cap \{z \in \mathbb{C}^n \mid \rho_1(z) = 0\}$ are having compact $\overline{\partial}$-Neumann operator locally. The part that needs to be checked is non-smooth part of the intersection, the set

$$S := \{z \in \mathbb{C}^n \mid \rho_1(z) = 0 = \rho_2(z)\}.$$

In order to understand the properties of the compactness of the $\overline{\partial}$-Neumann operator this question is of fundamental importance. In particular, this problem serves as a test to see whether there might be a reasonable notion of obstruction to compactness that lives in the boundary. If it is absent from both boundaries it should be absent from the boundary of the intersection. The difficulty is on the non-smooth part of the resultant domain. For example, on a pseudoconvex domain with Lipchitz boundary the smooth forms need not be dense in $\text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^{\star})$ under the graph norm, see [9, 18]. But even for smooth forms, the estimate on the intersection is not clear at all. In this part we present an answer to the question on some special domains.

A. Transversal Intersections

From now on we will assume that on both domains Ω_1 and Ω_2 the $\overline{\partial}$-Neumann problem is compact and both are with smooth boundaries unless otherwise stated. We will also assume that the intersection of the two domains is transversal. That is, $b\Omega_1 \cap b\Omega_2 = S$ is a smooth manifold.

1. Results on special domains when S is a smooth manifold

Lemma 5. *The $\overline{\partial}$-Neumann operator is compact on $\Omega_1 \cap \Omega_2$ where at least one of the domains satisfies property (P).*

Proof. The proof goes along the lines of proof that compactness is a local property, see [15]. One can also compare with the second part of the proof of Theorem 13; the only difference is that instead of using compactness multipliers one uses the compactness estimate induced from one of the domains. □

Thus $\Omega_1 \cap \Omega_2$ has a compact $\overline{\partial}$-Neumann operator if at least one of the domains belongs to a class of domains where property (P) is known to actually be equivalent to compactness. These classes are the locally convexifiable domains [15] and Hartogs domains in $\mathbb{C}^2$ [10].

2. Special cases in $\mathbb{C}^2$ for more general domains

We will use $A \pitchfork B$ to mean that the sets A and B intersect each other transversally. Thus, the set $S = b\Omega_1 \pitchfork b\Omega_2$ is a two real dimensional smooth submanifold. If a point $p \in S$ has a non-trivial complex tangent space $H_p(S)$ we call it an exceptional point of S. Let K represent the set of exceptional points on S. The set K is a compact subset of the set S.

Lemma 6. *[1] (Chapter 17, Lemma 17.2) Let S be a totally real smooth submanifold of an open set in $\mathbb{C}^2$. Let $d_S(x) := dist(x, S) = \inf\{|x-y| \mid y \in S\}$. Then, there is a neighborhood U_S of S such that $d_S^2(x)$ is smooth and strictly plurisubharmonic in U_S.*

Example 3. Let $K \neq \emptyset$ such that $S \backslash K \neq \emptyset$. Then $S \backslash K$ is a smooth manifold with real dimension 2. By the above lemma we can say that on a neighborhood U_L of every compact subset L of $S \backslash K$, the part of S with non-exceptional points (i.e. the part which is totally real) can have a smooth strictly plurisubharmonic function, $d_L^2(x)$. Then it is easy to see that L satisfies Property (P), (see also Example 2); we have $\left(\frac{\partial^2 d_L^2}{\partial z_j \partial \bar{z}_k}(z)\right)_{j,k} \geq C > 0$ for $z \in U_L$. Thus, for a given $M > 0$ set $\lambda_M(z) := \frac{2M}{C} d_L^2(z)$ on $z \in U_L$, then $\left(\frac{\partial^2 \lambda_M}{\partial z_j \partial \bar{z}_k}(z)\right)_{j,k} \geq M$ on U_L. There is a neighborhood $U_M (\subset U_L)$ of L with $0 \leq \lambda_M(z) \leq 1$. It is possible to write $S \backslash K$ as a union of countably many compact subsets $\{L_j\}_{j=1}^{\infty}$, where each $\overline{L}_j = L_j \subset\subset S \backslash K$ and has property (P). Thus, if the set K also has property (P) then $S \left(= \overline{S} \subset\subset \mathbb{C}^2\right)$ can be written as $\left(\bigcup_{j=1}^{\infty} L_j\right) \cup K$. Since, each of these compact subsets has property (P) then, by theorem 7 in chapter III, the set S is also having property (P). See the next example for some examples of such set K.

Example 4. **(a.)** If K, the set of exceptional points on the smooth manifold S, is a discrete set.

(b.) If K is a smooth curve, say Γ, then the real dimension of Γ is 1. Therefore, Γ is actually a totally real 1-dimensional smooth manifold. By using again Lemma 6 we can see that the curve Γ has property (P).

(c.) Forthermore, if K is a set of 2-dimensional Hausdorff measure zero then by a result of Sibony in [34] (for better exposition of the matter see also Boas [5]) the set K has Property (P) and then so does the set S as in the previous example. The last case includes cases **(a.)** and **(b.)**.

For $p \in S$, let $H_p(S)$ be the complex tangent space and $T_p(S)$ be the real tangent space at the point $p \in S$. We have $\dim_{\mathbb{R}}(T_p(b\Omega_j)) = 3$ for $j = 1,\ 2$ and $\dim_{\mathbb{R}}(H_p(b\Omega_j)) = 2$ for $j = 1,\ 2$. Then since $S := b\Omega_1 \pitchfork b\Omega_2$ we have

$$\dim_{\mathbb{R}}(T_p(S)) = \dim_{\mathbb{R}}\left(T_p(b\Omega_1) \pitchfork T_p(b\Omega_2)\right) = 2.$$

Thus, if complex tangents exist at a point on S then $T_p(S) = H_p(S)$ at that point. In other words, if the complex normals are linearly dependent (over $\mathbb{C}$), then and only then, we have a complex tangent to S.

Lemma 7. *$p \in S$ is not an exceptional point if and only if*

$$\partial\rho_1(p) \wedge \partial\rho_2(p) \neq 0.$$

(That is, S is totally real at p if and only if det $\left(\frac{\partial\rho_j}{\partial z_k}(p)\right)_{1\leq j,k\leq 2} \neq 0$.)

Proof. By using the argument on the paragraph before the lemma, S being totally real at the point p means that the complex normals, $\partial\rho_1(p)$ and $\partial\rho_2(p)$, to $H_p(b\Omega_1)$ and $H_p(b\Omega_2)$ respectively are not parallel to each other. Then $\partial\rho_1(p) \wedge \partial\rho_2(p) \neq 0$. The converse is the same argument but backward. $\square$

Example 5. Assume the set of exceptional points K on the smooth manifold S is having an inner point (relative to the set S), that is, Interior$(K) =: K^\circ \neq \emptyset$. Now, K° as a subset in $\mathbb{C}^2$ is a real smooth submanifold S all of whose tangents are complex tangents. Such a submanifold is a Riemann surface see [3]. Thus, we would have an analytic disc on the boundaries of Ω_1 and Ω_2. Existence of the disc on the boundary contradicts the assumption of having the $\overline{\partial}$-Neumann operator compact on both domains. The contradiction comes from the result of Fu and Straube's paper [15], Theorem 4.9: on a bounded pseudoconvex Lipschitz domain in $\mathbb{C}^2$, the existence of an analytic disc in the boundary contradicts the compactness of the $\overline{\partial}$-Neumann operator. Thus, the set of exceptional points has empty interior according to the relative topology on S.

B. An Example of a Non-transversal Intersection

Let $\Omega_1 := \{z \in \mathbb{C}^n \mid \rho_1(z) < 0\}$ and $\Omega_2 := \{z \in \mathbb{C}^n \mid \rho_2(z) < 0\}$ be bounded pseudoconvex domains in $\mathbb{C}^n$, $n \geq 2$ with smooth boundaries, and $\nabla\rho_1$ and $\nabla\rho_2$ are $\neq 0$ on $b\Omega_1$ and $b\Omega_2$ respectively.

Proposition 8. *Let Ω_1 and Ω_2 intersect each other such that, if $S := \{\rho_1(z) = \rho_2(z) = 0\} \subset b(\Omega_1 \cap \Omega_2)$, then the boundary of S is union of two disjoint boundary components, $bS := S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Assume that the boundary of the resultant domain, $b(\Omega_1 \cap \Omega_2)$, is a piecewise smooth boundary and the non-smooth parts are S_1 and S_2.*

If there exists a compactness estimate for the $\overline{\partial}$-Neumann operator on Ω_1 and on $\Omega_2 \subset \mathbb{C}^n$, then there exists a compactness estimate for the $\overline{\partial}$-Neumann operator on $\Omega_1 \cap \Omega_2$.

Proof. Let $K_1 := \overline{\{\rho_2(z) = 0\} \cap \Omega_1}$ and $K_2 := \overline{\{\rho_1(z) = 0\} \cap \Omega_2}$. K_1 and K_2 are relatively disjointly closed in $\mathbb{C}^n$. Then, we can find a smooth function $\phi(z)$ on $\mathbb{C}^n$ such that $\phi \equiv 1$ on K_1 and $\phi \equiv 0$ on K_2.

For $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star) \subset L^2_{(0,1)}(\Omega_1 \cap \Omega_2)$ write $u = \phi u - (\phi - 1)u$. Let

$$\begin{aligned} v_1 &:= (\phi - 1)u \quad \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star) \subset L^2_{(0,1)}(\Omega_1), \\ &\quad\ \& \\ v_2 &:= \phi u \qquad\quad\ \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star) \subset L^2_{(0,1)}(\Omega_2). \end{aligned} \tag{5.1}$$

Now, from the hypothesis $\forall \varepsilon > 0$ $\exists C_\varepsilon > 0$ such that for $j = 1, 2$

$$\|v_j\|^2_{\Omega_j} \leq \varepsilon(\|\overline{\partial} v_j\|^2_{\Omega_j} + \|\overline{\partial}^\star v_j\|^2_{\Omega_j}) + C_\varepsilon \|v_j\|^2_{-1,\Omega_j}. \tag{5.2}$$

Thus, $\forall \varepsilon > 0$ $\exists C_\varepsilon > 0$ such that for $j = 1, 2$

$$\|v_j\|^2_{\Omega_1\cap\Omega_2} \leq \varepsilon(\|\overline{\partial} u\|^2_{\Omega_1\cap\Omega_2} + \|\overline{\partial}^\star u\|^2_{\Omega_1\cap\Omega_2} + \|(\nabla\phi)u\|^2_{\Omega_1\cap\Omega_2}) + C_\varepsilon \|u\|^2_{-1,\Omega_1\cap\Omega_2}. \tag{5.3}$$

Let's see how we got the inequalities between the (-1)-Sobolev norms, $\|\cdot\|_{-1,\Omega_j}$ and $\|\cdot\|_{-1,\Omega_1\cap\Omega_2}$ with associated domains.

$$\begin{aligned} \|v_2\|^2_{-1,\Omega_2} &= \sup_{0\neq\psi\in\left(W^1_{(0,q)}(\Omega_2)\right)_\circ:\ \|\psi\|_{1,\Omega_2}=1} \left|(v_2,\psi)_{\Omega_2}\right| \\ &= \sup_{0\neq\psi\in\left(W^1_{(0,q)}(\Omega_2)\right)_\circ:\ \|\psi\|_{1,\Omega_2}=1} \left|(u,\phi\psi)_{\Omega_2}\right| \\ &\leq \|\phi\psi\|_{1,\Omega_1\cap\Omega_2}\|u\|_{-1,\Omega_1\cap\Omega_2} \\ &\leq C_\phi\|\psi\|_{1,\Omega_2}\|u\|_{-1,\Omega_1\cap\Omega_2}. \end{aligned}$$

By the same way we can get

$$\|v_1\|^2_{-1,\Omega_1} \leq C_{(1-\phi)}\|\psi\|_{1,\Omega_1}\|u\|_{-1,\Omega_1\cap\Omega_2}. \tag{5.4}$$

Now, consider the basic estimate on $\Omega_1 \cap \Omega_2$,

$$\begin{aligned} \|u\|^2_{\Omega_1\cap\Omega_2} &\leq C(\|\overline{\partial} u\|^2_{\Omega_1\cap\Omega_2} + \|\overline{\partial}^\star u\|^2_{\Omega_1\cap\Omega_2}) \\ \forall u &\in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^\star) \subset L^2_{(0,q)}(\Omega_1 \cap \Omega_2). \end{aligned} \tag{5.5}$$

By using (5.5), we can estimate $\|\nabla\phi u\|^2_{\Omega_1\cap\Omega_2}$ in (5.3):

$$\begin{aligned} \|(\nabla\phi)u\|^2_{\Omega_1\cap\Omega_2} &\leq \max_{z\in\overline{\Omega_1\cap\Omega_2}} \{|\nabla\phi(z)|\} \|u\|^2_{\Omega_1\cap\Omega_2} \\ &\leq C(\|\overline{\partial}u\|^2_{\Omega_1\cap\Omega_2} + \|\overline{\partial}^\star u\|^2_{\Omega_1\cap\Omega_2}). \end{aligned} \tag{5.6}$$

Thus, combining estimates at (5.3) and (5.6) we have

$$\|\phi u\|^2_{\Omega_1\cap\Omega_2} \leq \varepsilon(\|\overline{\partial}u\|^2_{\Omega_1\cap\Omega_2} + \|\overline{\partial}^\star u\|^2_{\Omega_1\cap\Omega_2}) + C_\varepsilon\|u\|^2_{-1,\Omega_1\cap\Omega_2} \tag{5.7}$$

and

$$\|(\phi-1)u\|^2_{\Omega_1\cap\Omega_2} \leq \varepsilon(\|\overline{\partial}u\|^2_{\Omega_1\cap\Omega_2} + \|\overline{\partial}^\star u\|^2_{\Omega_1\cap\Omega_2}) + C_\varepsilon\|u\|^2_{-1,\Omega_1\cap\Omega_2}. \tag{5.8}$$

Therefore, ϕ and $(\phi-1)$ are compactness multipliers, that is $\phi,\ (\phi-1) \in J_{\Omega_1\cap\Omega_2}$. As a result, $\phi-(\phi-1) = 1 \in J_{\Omega_1\cap\Omega_2}$ which implies the existence of the compactness estimate on $\Omega_1\cap\Omega_2$, see Chapter III. $\square$

CHAPTER VI

SUMMARY

In chapter I we gave a glimpse of the theory of inhomogeneous Cauchy-Riemann equations and how they are related to the $\overline{\partial}$-Neumann problem. We also gave some motivation to study the compactness property of the $\overline{\partial}$-Neumann operator, related with the results in the dissertation.

In the first part of chapter II we gave the set-up of the $\overline{\partial}$-Neumann problem. Then, we defined compactness of the $\overline{\partial}$-Neumann operator and presented its basic properties. In addition, we gave a characterization of property $(\widetilde{P})$; one of the sufficient conditions for compactness of the $\overline{\partial}$-Neumann operator.

In chapter III, we introduced a compactness multiplier notion associated to the compactness estimate of the $\overline{\partial}$-Neumann operator. Then, we used the common zero set of the ideal of compactness multipliers as an obstruction to compactness of the $\overline{\partial}$-Neumann problem. We characterized this new obstruction on domains in $\mathbb{C}^n$ where compactness is understood; on convex domains and on complete pseudoconvex Hartogs domains with smooth boundary in $\mathbb{C}^2$. We believe studying the obstructions to compactness of the $\overline{\partial}$-Neumann operator will help understand necessary and sufficient conditions for the compactness.

According to a theorem by W. J. Sweeney in [39] coercive estimates are independent of the metric on the tangent bundle. It appears to be folklore that the same is true for subelliptic estimates. The metric considered is smooth positive definite hermitian on the whole closure of the domain. In chapter IV, we give a simple proof of this result, but specifically for subellipticity of the $\overline{\partial}$-Neumann operator. Moreover, we show that the compactness of the $\overline{\partial}$-Neumann operator is also independent of the metric. This is of interest because Sobolev estimates are not independent of the metric. The Sobolev estimates are very important in the study of the global regularity for the $\overline{\partial}$-Neumann operator.

In chapter V we studied the compactness of the $\overline{\partial}$-Neumann problem on a transversal intersection of two smooth domains. If the $\overline{\partial}$-Neumann operator is compact on two bounded smooth pseudoconvex domains, then can we say that the $\overline{\partial}$-Neumann problem is compact on the transversal intersection of these two domains? In order to understand the properties of compactness for the $\overline{\partial}$-Neumann problem, this question is of fundamental importance. In particular, this problem serves as a test to see whether there might be a reasonable notion of obstruction to compactness that lives in the boundary. The difficulty is on the non-smooth part of the intersection. We were only able to give some partial answers to the question.

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VITA

Mehmet Çelik received his B.S. in 1999 in mathematics education at Marmara University, Turkey, and attended Institute for Graduate Studies in Basic and Engineering Sciences in mathematics at the same university from fall 1999 until spring 2001. He started his Ph.D. in mathematics in Fall 2001 at Texas A&M University and graduated in May 2008. His research is in several complex variables.

Mr. Çelik can be reached at Texas A&M University, Mathematics Department, College Station, TX 77843-3368. His e-mail address is celik@math.tamu.edu.