THE ENDOMORPHISM NEAR RING ON THE QUATERNION GROUP

A Thesis

by

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ABSTRACT

The Endomorphism Near Ring on the Quaternion Group. (August 1969)

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The study of near rings is motivated by consideration of the system generated by the endomorphisms of a group. In this thesis, the near ring generated by the endomorphisms on the quaternion group of order eight is displayed.

In addition, certain subrings, right ideals, and the radical of the near ring are displayed.

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CHAPTER I

INTRODUCTION

Endomorphism near rings furnish the motivation for distributively generated near rings, as well as for near rings in general. Distributively generated near rings have been studied by Biedleman [2], Fröhlich [6] and Laxton [10]. Fröhlich [7] also studied the near ring generated by the inner automorphisms of a finite simple group. Chandy [4] gave a necessary and sufficient condition that the near ring generated by the inner automorphisms of a group be a ring. Gupta [8] presented a necessary and sufficient condition that the near ring generated by the mappings of the form $\rho_x : g \to -g - x + g + x$ and $\lambda_x : g \to -x - g + x + g$ of a group be a ring. Malone and Lyons [11] have investigated the endomorphism near ring on S_3. Guthrie [9] has investigated the endomorphism near ring on the dihedral group of order eight. However, there is at present no general theory of the structure of endomorphism near rings. This thesis will provide another example of such a near ring and thus, hopefully, contribute to the formulation of the general theory.

The citations on the following pages follow the style of the Proceedings of the American Mathematical Society.

Definition 1.1. A near ring is an ordered triple $(R, +, \cdot)$ such that

a) $(R, +)$ is a group,

b) $(R, \cdot)$ is a semigroup,

c) $\cdot$ is left distributive over $+$, i.e. $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$ for each $r_1, r_2, r_3 \in R$.

Definition 1.2. A near ring is distributively generated (d.g.) if there exists $S \subset R$ such that

a) $(S, \cdot)$ is a subsemigroup of $(R, \cdot)$,

b) each element of S is right distributive,

c) S is an additive generating set for $(R, +)$.

The near ring generated additively by the endomorphisms of a group $(G, +)$ is d.g. with S the set of endomorphisms. Such a near ring will be called an endomorphism near ring and will be denoted by $E(G)$.

Some basic theorems on the decomposition of near rings follow.

Theorem 1.3. [3] Let e be an idempotent element in the near ring R. Then each $r \in R$ has two unique decompositions $r = (r - er) + er = er + (- er + r)$. Thus $R = A_e + M_e = M_e + A_e$ where $A_e = \{r - er \mid r \in R\} = \{t \in R \mid et = 0\}$, $M_e = \{er \mid r \in R\}$, and $A_e \cap M_e = 0$.

Theorem 1.4. [11] Let R be a near ring such that (R, +) is generated by $\{r_\gamma \mid \gamma \in \Gamma$, an index set$\}$. Then A_e is the normal subgroup generated by $\{r_\gamma - er_\gamma \mid \gamma \in \Gamma\}$ and M_e is the subgroup generated by $\{er_\gamma \mid \gamma \in \Gamma\}$.

Corollary 1.5. [9] Let A' be the subgroup generated by $\{r_\gamma - er_\gamma\}$. Then A_e consists of the elements of A' and any conjugate of an element of A' by an element of M_e.

CHAPTER II

THE ELEMENTS OF E(Q)

Let (Q, +) designate the non-abelian group of order eight with addition as given in Table I. This group is called the quaternion group of order eight. In this chapter the elements of E(Q) will be displayed.

Since Q is finite each element of E(Q) can be expressed as a finite sum of endomorphisms of Q. It is obvious that each function in E(Q) maps 0 to 0. Each function in E(Q) can be represented as a seven-tuple, the first coordinate being the image of a, the second the image of 2a, etc. For example, the seven-tuple (a, 2a, 3a, b, a+b, 2a+b, 3a+b) represents the identity function. In E(Q) the addition of elements is pointwise and multiplication is composition of functions.

Table II, a list of the endomorphisms of Q, is taken from [5].

Maxson and Clay [12] have shown that no quaternion group has an idempotent endomorphism other than the identity and the zero. Consequently, the question of how to produce a non-trivial idempotent arises. The idempotent is necessary in order to use Theorem 1.3. For this particular example, 1 + 1 + 8 + 12 is an idempotent element of E(Q). In general there has been no investigation of necessary or sufficient conditions to produce

TABLE I

THE QUATERNION GROUP OF ORDER EIGHT

x	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	a	2a	3a	b	a+b	2a+b	3a+b
a	a	2a	3a	0	a+b	2a+b	3a+b	b
2a	2a	3a	0	a	2a+b	3a+b	b	a+b
3a	3a	0	a	2a	3a+b	b	a+b	2a+b
b	b	3a+b	2a+b	a+b	2a	a	0	3a
a+b	a+b	b	3a+b	2a+b	3a	2a	a	0
2a+b	2a+b	a+b	b	3a+b	0	3a	2a	a
3a+b	3a+b	2a+b	a+b	b	a	0	3a	2a

$G(a,b \mid 2a = 2b = [a,b],\ 4a = 1)$

TABLE II

THE ENDOMORPHISMS OF Q

	a	2a	3a	b	a+b	2a+b	3a+b
0.	0	0	0	0	0	0	0
1.	a	2a	3a	b	a+b	2a+b	3a+b
2.	a	2a	3a	2a+b	3a+b	b	a+b
3.	a	2a	3a	a+b	2a+b	3a+b	b
4.	a	2a	3a	3a+b	b	a+b	2a+b
5.	3a	2a	a	b	3a+b	2a+b	a+b
6.	3a	2a	a	2a+b	a+b	b	3a+b
7.	3a	2a	a	a+b	b	3a+b	2a+b
8.	3a	2a	a	3a+b	2a+b	a+b	b
9.	b	2a	2a+b	a	3a+b	3a	a+b
10.	b	2a	2a+b	3a	a+b	a	3a+b
11.	b	2a	2a+b	a+b	a	3a+b	3a
12.	b	2a	2a+b	3a+b	3a	a+b	a
13.	2a+b	2a	b	a	a+b	3a	3a+b
14.	2a+b	2a	b	3a	3a+b	a	a+b
15.	2a+b	2a	b	a+b	3a	3a+b	a
16.	2a+b	2a	b	3a+b	a	a+b	3a
17.	a+b	2a	3a+b	a	b	3a	2a+b
18.	a+b	2a	3a+b	3a	2a+b	a	b
19.	a+b	2a	3a+b	b	3a	2a+b	a
20.	a+b	2a	3a+b	2a+b	a	b	3a
21.	3a+b	2a	a+b	a	2a+b	3a	b
22.	3a+b	2a	a+b	3a	b	a	2a+b
23.	3a+b	2a	a+b	b	a	2a+b	3a
24.	3a+b	2a	a+b	2a+b	3a	b	a
25.	0	0	0	2a	2a	2a	2a
26.	2a	0	2a	0	2a	0	2a
27.	2a	0	2a	2a	0	2a	0

a non-trivial idempotent in E(G), for an arbitrary group G.

From Table II, 1 + 1 + 8 + 12 = (a+b, 0, a+b, 0, a+b, 0, a+b) is an idempotent of E(Q). Using e = 1 + 1 + 8 + 12, Theorem 1.3 is applied to determine the elements of E(Q). The decompositions of the endomorphisms (in the sense of Theorem 1.4) are given in Table III and Table IV.

From Table III and Theorem 1.4 it follows that $M_e = \{(y, 0, y, 0, y, 0, y) | y \in Q\}$. Also, A', the group generated by $\{r_\gamma - er_\gamma\}$, is

$$\{(x, 0, x, x, 0, x, 0) | x \in Q\} + ((0, 2a, 2a, 0, 0, 2a, 2a)).$$

Let $B = \{(x, 0, x, x, 0, x, 0) | x \in Q\}$ and let $C_1 = ((0, 2a, 2a, 0, 0, 2a, 2a))$. Then $A' = B + C_1 = C_1 + B$. Let $m \in M_e$, $b \in B$, and $c_1 = (0, 2a, 2a, 0, 0, 2a, 2a)$. Then $m + b - m = (y, 0, y, 0, y, 0, y) + (x, 0, x, x, 0, x, 0) - (y, 0, y, 0, y, 0, y)$. Since Z(Q), the center of Q, is {0, 2a} and the commutator of every pair of elements lies in the center, $-y + x + y = x$ or $-y + x + y = x + 2a$. Consequently $m + b - m = (x, 0, x, x, 0, x, 0)$ or $(x + 2a, 0, x + 2a, x, 0, x, 0)$. Thus $m + b - m \in B + C_2$ where $C_2 = ((2a, 0, 2a, 0, 0, 0, 0))$ and $m + b + c_1 - m = m + b - m + c_1$ since every entry of $C_1 \in Z(Q)$. Thus $m + b + c_1 - m \in B + C_2 + C_1$. From Corollary 1.5 it follows that $A_e = B + C_2 + C_1$ and the order of A_e is 32. Since the

TABLE III

$\{r_\gamma - er_\gamma\}$

	a	2a	3a	b	a+b	2a+b	3a+b
0.	0	0	0	0	0	0	0
1.	b	2a	2a+b	b	0	2a+b	2a
2.	2a+b	2a	b	2a+b	0	b	2a
3.	a+b	2a	3a+b	a+b	0	3a+b	2a
4.	3a+b	2a	a+b	3a+b	0	a+b	2a
5.	b	2a	2a+b	b	0	2a+b	2a
6.	2a+b	2a	b	2a+b	0	b	2a
7.	a+b	2a	3a+b	a+b	0	3a+b	2a
8.	3a+b	2a	a+b	3a+b	0	a+b	2a
9.	a	2a	3a	a	0	3a	2a
10.	3a	2a	a	3a	0	a	2a
11.	a+b	2a	3a+b	a+b	0	3a+b	2a
12.	3a+b	2a	a+b	3a+b	0	a+b	2a
13.	a	2a	3a	a	0	3a	2a
14.	3a	2a	a	3a	0	a	2a
15.	a+b	2a	3a+b	a+b	0	3a+b	2a
16.	3a+b	2a	a+b	3a+b	0	a+b	2a
17.	a	2a	3a	a	0	3a	2a
18.	3a	2a	a	3a	0	a	2a
19.	b	2a	2a+b	b	0	2a+b	2a
20.	2a+b	2a	b	2a+b	0	b	2a
21.	a	2a	3a	a	0	3a	2a
22.	3a	2a	a	3a	0	a	2a
23.	b	2a	2a+b	b	0	2a+b	2a
24.	2a+b	2a	b	2a+b	0	b	2a
25.	2a	0	2a	2a	0	2a	0
26.	0	0	0	0	0	0	0
27.	2a	0	2a	2a	0	2a	0

TABLE IV

$\{er_\gamma\}$

	a	2a	3a	b	a+b	2a+b	3a+b
0.	0	0	0	0	0	0	0
1.	a+b	0	a+b	0	a+b	0	a+b
2.	3a+b	0	3a+b	0	3a+b	0	3a+b
3.	2a+b	0	2a+b	0	2a+b	0	2a+b
4.	b	0	b	0	b	0	b
5.	3a+b	0	3a+b	0	3a+b	0	3a+b
6.	a+b	0	a+b	0	a+b	0	a+b
7.	b	0	b	0	b	0	b
8.	2a+b	0	2a+b	0	2a+b	0	2a+b
9.	3a+b	0	3a+b	0	3a+b	0	3a+b
10.	a+b	0	a+b	0	a+b	0	a+b
11.	a	0	a	0	a	0	a
12.	3a	0	3a	0	3a	0	3a
13.	a+b	0	a+b	0	a+b	0	a+b
14.	3a+b	0	3a+b	0	3a+b	0	3a+b
15.	3a	0	3a	0	3a	0	3a
16.	a	0	a	0	a	0	a
17.	b	0	b	0	b	0	b
18.	2a+b	0	2a+b	0	2a+b	0	2a+b
19.	3a	0	3a	0	3a	0	3a
20.	a	0	a	0	a	0	a
21.	2a+b	0	2a+b	0	2a+b	0	2a+b
22.	b	0	b	0	b	0	b
23.	a	0	a	0	a	0	a
24.	3a	0	3a	0	3a	0	3a
25.	2a	0	2a	0	2a	0	2a
26.	2a	0	2a	0	2a	0	2a
27.	0	0	0	0	0	0	0

order of M_e is 8, the order of E(Q) is 256. That is, every element of E(Q) is of the form $\{(x, 0, x, x, 0, x, 0) | x \varepsilon Q\}$ + ((0, 2a, 2a, 0, 0, 2a, 2a)) + ((2a, 0, 2a, 0, 0, 0, 0)) + $\{(y, 0, y, 0, y, 0, y) | y \varepsilon Q\}$.

Table V gives a complete listing of the idempotent elements of E(Q). These idempotents can be used to induce other decompositions of E(Q).

TABLE V

IDEMPOTENTS OF E(Q)

	a	2a	3a	b	a+b	2a+b	3a+b
1.	0	0	0	0	0	0	0
2.	0	0	0	b	b	b	b
3.	0	0	0	a+b	a+b	a+b	a+b
4.	0	0	0	2a+b	2a+b	2a+b	2a+b
5.	0	0	0	3a+b	3a+b	3a+b	3a+b
6.	0	2a	2a	0	2a	2a	0
7.	0	2a	2a	2a	0	0	2a
8.	0	2a	2a	2a	2a	0	0
9.	0	2a	2a	0	0	2a	2a
10.	a	0	a	0	a	0	a
11.	a	0	a	a	0	a	0
12.	2a	2a	0	0	0	2a	2a
13.	2a	2a	0	0	2a	2a	0
14.	2a	2a	0	2a	0	0	2a
15.	2a	2a	0	2a	2a	0	0
16.	3a	0	3a	0	3a	0	3a
17.	3a	0	3a	3a	0	3a	0
18.	b	0	b	b	0	b	0
19.	a+b	0	a+b	0	a+b	0	a+b
20.	2a+b	0	2a+b	2a+b	0	2a+b	0
21.	3a+b	0	3a+b	0	3a+b	0	3a+b
22.	0	2a	2a	b	b	2a+b	2a+b
23.	0	2a	2a	b	2a+b	2a+b	b
24.	0	2a	2a	a+b	a+b	3a+b	3a+b
25.	0	2a	2a	3a+b	a+b	a+b	3a+b
26.	a	0	a	b	a+b	b	a+b
27.	a	0	a	b	3a+b	b	3a+b

28.	a	0	a	2a+b	a+b	2a+b	a+b
29.	a	0	a	2a+b	3a+b	2a+b	3a+b
30.	3a	0	3a	b	a+b	b	a+b
31.	3a	0	3a	b	3a+b	b	3a+b
32.	3a	0	3a	2a+b	a+b	2a+b	a+b
33.	3a	0	3a	2a+b	3a+b	2a+b	3a+b
34.	2a	2a	0	b	b	2a+b	2a+b
35.	2a	2a	0	b	2a+b	2a+b	b
36.	2a	2a	0	a+b	a+b	3a+b	3a+b
37.	2a	2a	0	3a+b	a+b	a+b	3a+b
38.	a	2a	3a	0	a	2a	3a
39.	a	2a	3a	0	3a	2a	a
40.	a+b	2a	3a+b	0	a+b	2a	3a+b
41.	3a+b	2a	a+b	0	a+b	2a	3a+b
42.	a	2a	3a	a	0	3a	2a
43.	a	2a	3a	3a	0	a	2a
44.	b	2a	2a+b	b	0	2a+b	2a
45.	2a+b	2a	b	b	0	2a+b	2a
46.	a	2a	3a	2a	a	0	3a
47.	a	2a	3a	2a	3a	0	a
48.	a+b	2a	3a+b	2a	a+b	0	3a+b
49.	3a+b	2a	a+b	2a	a+b	0	3a+b
50.	a	2a	3a	a	2a	3a	0
51.	a	2a	3a	3a	2a	a	0
52.	b	2a	2a+b	b	2a	2a+b	0
53.	2a+b	2a	b	b	2a	2a+b	0
54.	a	2a	3a	b	a+b	2a+b	3a+b

CHAPTER III

CERTAIN SUBRINGS OF E(Q)

Definition 3.1. A subset K of a near ring R is a subnear ring of R *if (K, +, ·) is also a near ring with respect to the operations + and · of R.*

Definition 3.2. A group G is metabelian if G', the commutator subgroup, is abelian.

Definition 3.3. A group is n-metabelian if every n-generator subgroup is metabelian.

Definition 3.4. [8]. With each element $x \in G$ associate a mapping $\rho_x : g \to -g - x + g + x$ of G into G. Let R(G) denote the smallest near ring containing all sums and products of ρ's.

Theorem 3.5. [8]. R(G) is a ring if and only if G is 3-metabelian.

Theorem 3.6. $R(G) \subseteq I(G)$, the near ring generated by the inner automorphisms of G for any group G.

Proof: Let ω_x be the inner automorphism of G induced by x, i.e. $\omega_x : g \to -x + g + x$ and $\omega_o : g \to g$. Note that $g(-\omega_o) = -(g\omega_o) = -g$. Certainly $-\omega_o \in I(G)$ and $\rho_x = -\omega_o + \omega_x$. Thus each $\rho_x \in I(G)$ and consequently all finite sums of ρ's $\in I(G)$. Since $\rho_{x+y} = \rho_y + \rho_x + \rho_x \rho_y$ (see (8) of [8]), $-\rho_x - \rho_y + \rho_{x+y} = \rho_x \rho_y$. Thus products of ρ's are contained in I(G). Consequently $R(G) \subseteq I(G)$.

Definition 3.7. A group is nilpotent of class 2 if $G/Z(G)$ is abelian.

Theorem 3.8. $\rho_x g: \to -g - x + g + x$ is an endomorphism if and only if G is nilpotent of at most class 2.

Proof: Suppose G is nilpotent of at most class 2. This implies the commutator of every pair of elements of G is in Z(G). Thus $(g_1 + g_2)\rho_x = -g_2 - g_1 - x + g_1 + g_2 + x,$

$$= -g_2 - g_1 - x + g_1 + x - x + g_2 + x,$$

$$= -g_2 + [g_1, x] - x + g_2 + x,$$

$$= [g_1, x] - g_2 - x + g_2 + x,$$

$$= [g_1, x] + [g_2, x],$$

$$= g_1\rho_x + g_2\rho_x.$$

Now suppose ρ_x is an endomorphism for every g_1, g_2, $x \in G$. Then $g_1 \rho_x + g_2\rho_x = (g_1 + g_2)\rho_x$,

$$-g_1 - x + g_1 + x - g_2 - x + g_2 + x = -g_2 - g_1 - x + g_1 + g_2 + x,$$

$$-g_1 - x + g_1 + x - g_2 - x = -g_2 - g_1 - x + g_1,$$

$$-g_1 - x + g_1 + x - g_2 = -g_2 - g_1 - x + g_1 + x,$$

$$[g_1, x] - g_2 = -g_2 + [g_1, x].$$

Since every $x \in G$ determines a ρ_x and g_1 and g_2 are arbitrary, $G' \subseteq Z(G)$. Thus G is nilpotent of at most class 2.

Since Q is obviously 3-metabelian, R(Q) is a subring of E(Q) and since Q is nilpotent of class 2 each ρ_x is in fact an endomorphism. R(Q) = {0, 25, 26, 27} of Table II. $(R(Q), +) \cong (C_2, +) \oplus (C_2, +)$ where C_2 represents a cyclic group of order 2; the product of two elements of R(Q) is the zero map.

Definition 3.9. A group is an L-group if $[[x,y],y] = 0$ for all $x, y \in G$. $[x,y] = -x - y + x + y$.

Chandy [4] has shown that the near ring I(G) is a ring if and only if G is an L-group and that any group nilpotent of class at most 2 is an L-group. Consequently, I(Q) is a subring of E(Q) and I(Q) = ({1, 2, 5, 6}) where 1, etc. are from Table II. $(I(Q), +) \cong (C_4, +) \oplus (C_2, +) \oplus (C_2, +)$. The elements of I(Q) are displayed in Table VI.

Theorem 3.10. G is nilpotent of class 2 if and only if $\rho_x\rho_y = 0$, the map which takes every element of G to the identity of G.

Proof: Since
$$\begin{aligned} g(\rho_x\rho_y) &= (g\rho_x)\rho_y \\ &= (-g - x + g + x)\rho_y \\ &= -x - g + x + g - y - g - x + g + x + y \\ &= [x, g] - y + [g, x] + y \end{aligned}$$

TABLE VI

THE ELEMENTS OF I(Q)

	a	2a	3a	b	a+b	2a+b	3a+b
1.	0	0	0	0	0	0	0
2.	a	2a	3a	b	a+b	2a+b	3a+b
3.	a	2a	3a	2a+b	3a+b	b	a+b
4.	3a	2a	a	b	3a+b	2a+b	a+b
5.	3a	2a	a	2a+b	a+b	b	3a+b
6.	2a	0	2a	2a	2a	2a	2a
7.	2a	0	2a	0	0	0	0
8.	0	0	0	2a	0	2a	0
9.	0	0	0	0	2a	0	2a
10.	3a	2a	a	2a+b	3a+b	b	a+b
11.	3a	2a	a	b	a+b	2a+b	3a+b
12.	a	2a	3a	2a+b	a+b	b	3a+b
13.	a	2a	3a	b	3a+b	2a+b	a+b
14.	0	0	0	2a	2a	2a	2a
15.	2a	0	2a	0	2a	0	2a
16.	2a	0	2a	2a	0	2a	2a

$$= \; -y + [x, g] + [g, x] + y$$
$$= 0.$$

The converse follows in a similar manner.

Corollary 3.11. If G is nilpotent of class 2, $\rho_x + \rho_y = \rho_{x+y}$.

Theorem 3.12. If G is nilpotent of class 2, $R(G) \subseteq Z(E(G), +)$.

Proof: Let e represent any element of E(G). Since G is nilpotent of class 2 any element of R(G) is a ρ_x for some $x \in G$. Let $ge = g'$. Then $g(\rho_x + e) = [g, x] + g'$

$$= g' + [g, x]$$
$$= g(e + \rho_x).$$

Thus $R(G) \subseteq Z(E(G), +)$.

CHAPTER IV

THE RADICAL OF E(Q)

Since the radical of E(Q) depends on certain right ideals more definitions and theorems are necessary.

Definition 4.1. A subset K of a near ring R is a left ideal provided

a) (K, +) is a normal subgroup of R,

b) $rk \in K$ for each $r \in R$ and $k \in K$.

K is a right ideal provided

a) (K, +) is a normal subgroup of R,

b) $(r_1 + k)r_2 - r_1r_2 \in K$ for each r_1, $r_2 \in R$ and $k \in K$.

K is an ideal if it is a left ideal and a right ideal.

It is noted in [6] that in a d.g. near ring $(r_1 + k)r_2 - r_1r_2 \in K$ is equivalent to $kr \in K$. Since E(Q) is d.g., $kr \in K$ will be sufficient as a condition for a normal subgroup K to be a right ideal.

Theorem 4.2. [11]. Let T be a non-empty subset of a group G. Let $I_T = \{r \in E(G) | Tr = 0\}$, then I_T is a right ideal in E(G).

Definition 4.3. A subgroup K of the near ring R is an R-subgroup if $KR \subseteq K$.

Definition 4.4. If K is an R-subgroup such that $K^n = 0$

for some positive integer n, K is said to be a nilpotent R-subgroup.

Definition 4.5. The radical, J(R), of a d.g. near ring is the intersection of the right ideals of R which are maximal R-subgroups. If no such right ideals exist J(R) is defined to be R.

Recall that $C_1 = ((0, 2a, 2a, 0, 0, 2a, 2a))$, $C_2 = ((2a, 0, 2a, 0, 0, 0, 0))$, $B = \{(x, 0, x, x, 0, x, 0) | x \in Q\}$, and $M_e = \{(y, 0, y, 0, y, 0, y) | y \in Q\}$. Let $B_1 = ((2a, 0, 2a, 2a, 0, 2a, 0))$ and $M_1 = ((2a, 0, 2a, 0, 2a, 0, 2a))$. Some of the right ideals of E(Q) along with their respective subsets T (in the sense of Theorem 4.2) are given below:

T	I_T
$\{a, b, a + b\}$	C_1
$\{2a, b, a + b\}$	C_2
$\{2a\}$	$B + C_2 + M_e$
$\{b\}$	$C_1 + C_2 + M_e$
$\{a + b\}$	$B + C_1 + C_2$.

Also it is easily seen that B_1 and M_1 are right ideals. Since, in a d.g. near ring, the sum of right ideals is a right ideal, it follows that $B_1 + C_1 + C_2 + M_e$ and $B + C_1 + C_2 + M_1$ are right ideals.

Since $B + C_2 + M_e$ is obviously a maximal right ideal, it follows that $B + C_2 + M_e$ is a maximal E(Q)-subgroup. Now $B_1 + C_1 + C_2 + M_e$ and $B + C_1 + C_2 + M_1$ have been shown to be right ideals. If they are also maximal E(Q)-subgroups, it will follow that $J(E(Q)) \subset (B + C_2 + M_e) \cap (B_1 + C_1 + C_2 + M_e) \cap (B + C_1 + C_2 + M_1)$. If $B + C_1 + C_2 + M_1$ is not maximal, there exists a proper E(Q)-subgroup, R', such that $B + C_1 + C_2 + M_1 \subset R' \subset E(Q)$. If $B + C_1 + C_2 + M_1 \subset R'$, there is at least one $r \in E(Q)$ such that $r \in R'$ and $r \notin B + C_1 + C_2 + M_1$. Consequently the M_e-term of r must be in $M_e - \{(2a, 0, 2a, 0, 2a, 0, 2a),(0, 0, 0, 0, 0, 0, 0)\}$, and, in fact, this M_e-term is in R'. But, an E(Q)-subgroup containing an element of the form (x, 0, x, 0, x, 0, x), $x \neq 0$, 2a, must contain M_e since there exist functions in E(Q) which map such an x to any given element in Q. Hence $R' = E(Q)$ and $B + C_1 + C_2 + M_1$ is a maximal E(Q)-subgroup. Similarly, it can be shown $B_1 + C_1 + C_2 + M_e$ is a maximal E(Q)- subgroup. Thus

$$J(E(Q)) \subset (B + C_2 + M_e) \cap (B_1 + C_1 + C_2 + M_e) \cap (B + C_1 + C_2 + M_1) = B_1 + C_1 + M_1.$$

Theorem 4.6. [9]. The radical of a d.g. near ring R contains all the nilpotent R-subgroups of R.

Since B_1, C_2, and M_1 are each nilpotent E(Q)-subgroups it follows that $B_1 + C_2 + M_1 \subset J(E(Q))$.

Thus it has been shown that $J(E(Q)) = B_1 + C_2 + M_1$.

Because the radical is an ideal [1], we may consider E(Q)/J(E(Q)). Since E(Q) has a multiplicative identity, 1 of Table II, E(Q)/J(E(Q)) has the multiplicative identity $\overline{1}$, the coset containing 1. From 1 + 1 = (2a, 0, 2a, 2a, 2a, 2a) ε J(E(Q)) it follows that $\overline{1}$ has additive order 2 in E(Q)/J(E(Q)). Thus this quotient near ring has characteristic 2. But this implies that the additive group of E(Q)/J(E(Q)) is abelian. Since it was shown in [6] that a d.g. near ring whose additive group is abelian is a ring, E(Q)/J(E(Q)) is a ring of order 32.

REFERENCES

1. J.C. Beidleman, On near-rings and near-ring modules, Doctoral Dissertation, Penn. State University, 1964.

2. ____________, Strictly prime distributively generated near-rings, Math. Z. 100(1967), 97-105.

3. G. Berman and R.G. Silverman, Near-rings, Amer. Math. Monthly 66(1959), 23-34.

4. A.J. Chandy, Rings generated by inner automorphisms of non-abelian groups, Doctoral Dissertation, Boston University, 1965.

5. J.R. Clay, Computing near-rings on the non-abelian groups of order 8, unpublished.

6. A. Fröhlich, Distributively generated near-rings, Proc. London Math. Soc. (3) 8 (1958), 76-94.

7. __________, The near-ring generated by the inner automorphisms of a finite simple group, J. London Math. Soc. 33(1958), 95-107.

8. N.D. Gupta, Commutation near-rings of a group, J. Aust. Math. Soc. 7(1967), 135-140.

9. E.R. Guthrie, The endomorphism near-ring on D_8, Master's Thesis, Texas A&M University, 1969.

10. R.R. Laxton, A radical and its theory for distributively generated near-rings, J. London Math. Soc. 38(1963), 40-49.

11. J.J. Malone and C.G. Lyons, Endomorphism near-rings, unpublished.

12. C. Maxson and J.R. Clay, The near-rings with identities on the generalized quaternion groups, unpublished.

VITA

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