THE ENDOMORPHISM NEAR RING ON D_8

A Thesis

by

Edgar Raymond Guthrie

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Approved as to style and content by:

(Chairman of Committee)

(Head of Department)

(Member)

(Member)

(Member)

(Member)

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ABSTRACT

The Endomorphism Near Ring on D_8. (August 1969)

Edgar R. Guthrie, B.S., Sam Houston State;

Directed by: Dr. J. J. Malone, Jr.

The study of near rings is motivated by consideration of the system generated by the endomorphisms of a group. In this thesis, the near ring generated by the endomorphisms on the dihedral group of order eight is offered.

In addition, certain right ideals and the radical of the near ring are displayed.

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TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

INTRODUCTION

While the subject of endomorphism near rings has been explored [6] there is still a lack of specific examples of endomorphism near rings. It is the purpose of this thesis to display the near ring generated by the endomorphisms on the dihedral group of order eight, D_8. It is hoped that this example will contribute to the study of near rings.

Definition 1.1. A near ring is a triple (R, +, .) such that

1) (R, +) is a group,

2) (R, .) is a semigroup,

3) $r_1(r_2 + r_3) = r_1r_2 + r_1r_3$ for each $r_1, r_2, r_3 \in R$.

Definition 1.2. A near ring is distributively generated if there exists $S \subset R$ such that

1) (S, .) is a subsemigroup of (R, .),

2) each element of S is right distributive,

3) S is an additive generating set for (R, +).

The near ring generated additively by the endomorphisms of a group (G, +) is distributively generated, S being the set of

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endomorphisms. Such a near ring is called an endomorphism near ring and is denoted by E(G).

Definition 1.3. A subset K of a near ring R is a left ideal if

1) (K, +) is a normal subgroup of R,

2) $rk \in K$ for all $r \in R$ and $k \in K$.

K is a right ideal if

1) (K, +) is a normal subgroup of R,

2) $(r_1 + k)r_2 - r_1r_2 \in K$ for all r_1, $r_2 \in R$ and for all $k \in K$.

K is an ideal if it is a left ideal and a right ideal.

It is noted in [4] that in a distributively generated near ring the statement $(r_1 + k)r_2 - r_1r_2 \in K$ is equivalent to $kr \in K$. Since E(G) is a distributively generated near ring, $kr \in K$ will suffice as a condition for a normal subgroup K to be a right ideal.

Definition 1.4. A subgroup H of the near ring R is an R-subgroup if $HR \subset H$.

Definition 1.5. If H is an R-subgroup such that $H^n = 0$ for some positive integer n, H is said to be a nilpotent R-subgroup.

Definition 1.6. The radical, J(R), of the distributively generated near ring R is the intersection of the right ideals of R which are maximal R-subgroups. If no such right ideals exist, J(R) is defined to be R.

CHAPTER II

THE FORMULATION OF $E(D_8)$

The group D_8 is displayed in Table 1. It should be pointed out that the center of D_8 consists of the elements 0 and 2a. Also, the commutator of every pair of elements in D_8 lies in the center of D_8. It follows then, that for every x and y belonging to D_8, either $x + y - x = y$ or $x + y - x = 2a + y$.

The endomorphisms of D_8, from which the endomorphism near ring $E(D_8)$ will be formed, are displayed in Table 2. This table is taken from [3].

For purposes of computation, it will be desirable to represent each element of $E(D_8)$ as a seven-tuple. A seven-tuple is sufficient since each endomorphism and each sum of endomorphisms maps 0 to 0. In each seven-tuple: The first coordinate is the image of a, the second coordinate is the image of 2a, and so on. For example, the identity mapping M_1 is represented as (a, 2a, 3a, b, a+b, 2a+b, 3a+b).

Addition of elements in $E(D_8)$ is done by addition of coordinates and multiplication is composition of functions. The following theorems will be of some value in determining $E(D_8)$.

<u>Theorem 2.1</u>. [2] Let e be an idempotent element in the near ring R. Then each $r \in R$ has two unique decompositions $r = (r - er) + er = er + (-er + r)$. Thus $R = A_e + M_e = M_e + A_e$ where

TABLE 1

THE DIHEDRAL GROUP D_8

+	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	a	2a	3a	b	a+b	2a+b	3a+b
a	a	2a	3a	0	a+b	2a+b	3a+b	b
2a	2a	3a	0	a	2a+b	3a+b	b	a+b
3a	3a	0	a	2a	3a+b	b	a+b	2a+b
b	b	3a+b	2a+b	a+b	0	3a	2a	a
a+b	a+b	b	3a+b	2a+b	a	0	3a	2a
2a+b	2a+b	a+b	b	3a+b	2a	a	0	3a
3a+b	3a+b	2a+b	a+b	b	3a	2a	a	0

TABLE 2

THE ENDOMORPHISMS OF D_8

	0	a	2a	3a	b	a+b	2a+b	3a+b
M_1	0	a	2a	3a	b	a+b	2a+b	3a+b
M_2	0	a	2a	3a	a+b	2a+b	3a+b	b
M_3	0	a	2a	3a	2a+b	3a+b	b	a+b
M_4	0	a	2a	3a	3a+b	b	a+b	2a+b
M_5	0	3a	2a	a	b	3a+b	2a+b	a+b
M_6	0	3a	2a	a	a+b	b	3a+b	2a+b
M_7	0	3a	2a	a	2a+b	a+b	b	3a+b
M_8	0	3a	2a	a	3a+b	2a+b	a+b	b
M_9	0	0	0	0	2a	2a	2a	2a
M_{10}	0	0	0	0	b	b	b	b
M_{11}	0	0	0	0	a+b	a+b	a+b	a+b
M_{12}	0	0	0	0	2a+b	2a+b	2a+b	2a+b
M_{13}	0	0	0	0	3a+b	3a+b	3a+b	3a+b
M_{14}	0	2a	0	2a	2a	0	2a	0
M_{15}	0	b	0	b	b	0	b	0
M_{16}	0	a+b	0	a+b	a+b	0	a+b	0
M_{17}	0	2a+b	0	2a+b	2a+b	0	2a+b	0
M_{18}	0	3a+b	0	3a+b	3a+b	0	3a+b	0
M_{19}	0	2a	0	2a	0	2a	0	2a
M_{20}	0	b	0	b	0	b	0	b
M_{21}	0	a+b	0	a+b	0	a+b	0	a+b

M_{22}	0	2a+b	0	2a+b	0	2a+b	0	2a+b
M_{23}	0	3a+b	0	3a+b	0	3a+b	0	3a+b
M_{24}	0	2a	0	2a	b	2a+b	b	2a+b
M_{25}	0	2a	0	2a	a+b	3a+b	a+b	3a+b
M_{26}	0	2a	0	2a	2a+b	b	2a+b	b
M_{27}	0	2a	0	2a	3a+b	a+b	3a+b	a+b
M_{28}	0	a+b	0	a+b	2a	3a+b	2a	3a+b
M_{29}	0	a+b	0	a+b	3a+b	2a	3a+b	2a
M_{30}	0	3a+b	0	3a+b	2a	a+b	2a	a+b
M_{31}	0	3a+b	0	3a+b	a+b	2a	a+b	2a
M_{32}	0	b	0	b	2a	2a+b	2a	2a+b
M_{33}	0	b	0	b	2a+b	2a	2a+b	2a
M_{34}	0	2a+b	0	2a+b	2a	b	2a	b
M_{35}	0	2a+b	0	2a+b	b	2a	b	2a
M_{36}	0	0	0	0	0	0	0	0

$A_e = \{r - er : r \varepsilon R\} = \{t \varepsilon R : et = 0\}$, $M_e = \{er : r \varepsilon R\}$, and $A_e \cap M_e = 0$.

Theorem 2.2. [6] Let R be a near ring such that (R, +) is generated by $\{r_z : z \varepsilon Z, Z \text{ an index set}\}$.
Then A_e is the normal subgroup generated by $\{r_z - er_z : z \varepsilon Z\}$ and M_e is the subgroup generated by $\{er_z : z \varepsilon Z\}$.

Corollary 2.3. Let A' be the subgroup generated by $\{r_z - er_z\}$. Then A_e consists of the elements of A' and any conjugate of an element of A' by an element of M_e.

In the discussion of $E(D_8)$ which follows the endomorphism M_{13} will serve as the idempotent e. The sets $\{M_i - M_{13}.M_i\}$ and $\{M_{13}.M_i\}$ are displayed, respectively, in Table 3 and in Table 4. The group additively generated by $\{M_i - M_{13}.M_i\}$ is A' and the group additively generated by $\{M_{13}.M_i\}$ is M_e. Note that A' is $\{(x, 0, x, x, 0, x, 0), (x, 2a, 2a+x, x, 2a, 2a+x, 0) : x \varepsilon D_8\}$ and that M_e is $\{(0, 0, 0, y, y, y, y) : y \varepsilon D_8\}$.

Theorem 2.4. The order of $E(D_8)$ is 256.

Proof. Let $H = \{(x, 0, x, x, 0, x, 0) : x \varepsilon D_8\}$ and let $K_2 = ((0, 2a, 2a, 0, 2a, 2a, 0))$. Then $A' = H + K_2$. Let $m \varepsilon M_e$, $h \varepsilon H$, and $k_2 = (0, 2a, 2a, 0, 2a, 2a, 0) \varepsilon K_2$. Now $m + h - m =$ $(0, 0, 0, y, y, y, y) + (x, 0, x, x, 0, x, 0) - (0, 0, 0, y, y, y, y)$ $= (x, 0, x, y + x - y, 0, y + x - y, 0)$. Since $y + x - y = x$ or $y + x - y = 2a + x$, $m + h - m$ will be $(x, 0, x, x, 0, x, 0)$ or

TABLE 3

$\{M_i - M_{13} \cdot M_i\}$

M_1	(a, 2a, 3a, a, 2a, 3a, 0)
M_2	(a, 2a, 3a, a, 2a, 3a, 0)
M_3	(a, 2a, 3a, a, 2a, 3a, 0)
M_4	(a, 2a, 3a, a, 2a, 3a, 0)
M_5	(3a, 2a, a, 3a, 2a, a, 0)
M_6	(3a, 2a, a, 3a, 2a, a, 0)
M_7	(3a, 2a, a, 3a, 2a, a, 0)
M_8	(3a, 2a, a, 3a, 2a, a, 0)
M_9	(0, 0, 0, 0, 0, 0, 0)
M_{10}	(0, 0, 0, 0, 0, 0, 0)
M_{11}	(0, 0, 0, 0, 0, 0, 0)
M_{12}	(0, 0, 0, 0, 0, 0, 0)
M_{13}	(0, 0, 0, 0, 0, 0, 0)
M_{14}	(2a, 0, 2a, 2a, 0, 2a, 0)
M_{15}	(b, 0, b, b, 0, b, 0)
M_{16}	(a+b, 0, a+b, a+b, 0, a+b, 0)
M_{17}	(2a+b, 0, 2a+b, 2a+b, 0, 2a+b, 0)
M_{18}	(3a+b, 0, 3a+b, 3a+b, 0, 3a+b, 0)
M_{19}	(2a, 0, 2a, 2a, 0, 2a, 0)
M_{20}	(b, 0, b, b, 0, b, 0)
M_{21}	(a+b, 0, a+b, a+b, 0, a+b, 0)
M_{22}	(2a+b, 0, 2a+b, 2a+b, 0, 2a+b, 0)

M_{23}	(3a+b, 0, 3a+b, 3a+b, 0, 3a+b, 0)
M_{24}	(2a, 0, 2a, 2a, 0, 2a, 0)
M_{25}	(2a, 0, 2a, 2a, 0, 2a, 0)
M_{26}	(2a, 0, 2a, 2a, 0, 2a, 0)
M_{27}	(2a, 0, 2a, 2a, 0, 2a, 0)
M_{28}	(a+b, 0, a+b, a+b, 0, a+b, 0)
M_{29}	(a+b, 0, a+b, a+b, 0, a+b, 0)
M_{30}	(3a+b, 0, 3a+b, 3a+b, 0, 3a+b, 0)
M_{31}	(3a+b, 0, 3a+b, 3a+b, 0, 3a+b, 0)
M_{32}	(b, 0, b, b, 0, b, 0)
M_{33}	(b, 0, b, b, 0, b, 0)
M_{34}	(2a+b, 0, 2a+b, 2a+b, 0, 2a+b, 0)
M_{35}	(2a+b, 0, 2a+b, 2a+b, 0, 2a+b, 0)
M_{36}	(0, 0, 0, 0, 0, 0, 0)

TABLE 4

$\{M_{13} \cdot M_i\}$

M_1	(0, 0, 0, 3a+b, 3a+b, 3a+b, 3a+b)
M_2	(0, 0, 0, b, b, b, b)
M_3	(0, 0, 0, a+b, a+b, a+b, a+b)
M_4	(0, 0, 0, 2a+b, 2a+b, 2a+b, 2a+b)
M_5	(0, 0, 0, a+b, a+b, a+b, a+b)
M_6	(0, 0, 0, 2a+b, 2a+b, 2a+b, 2a+b)
M_7	(0, 0, 0, 3a+b, 3a+b, 3a+b, 3a+b)
M_8	(0, 0, 0, b, b, b, b)
M_9	(0, 0, 0, 2a, 2a, 2a, 2a)
M_{10}	(0, 0, 0, b, b, b, b)
M_{11}	(0, 0, 0, a+b, a+b, a+b, a+b)
M_{12}	(0, 0, 0, 2a+b, 2a+b, 2a+b, 2a+b)
M_{13}	(0, 0, 0, 3a+b, 3a+b, 3a+b, 3a+b)
M_{14}	(0, 0, 0, 0, 0, 0, 0)
M_{15}	(0, 0, 0, 0, 0, 0, 0)
M_{16}	(0, 0, 0, 0, 0, 0, 0)
M_{17}	(0, 0, 0, 0, 0, 0, 0)
M_{18}	(0, 0, 0, 0, 0, 0, 0)
M_{19}	(0, 0, 0, 2a, 2a, 2a, 2a)
M_{20}	(0, 0, 0, b, b, b, b)
M_{21}	(0, 0, 0, a+b, a+b, a+b, a+b)
M_{22}	(0, 0, 0, 2a+b, 2a+b, 2a+b, 2a+b)

M_{23}	(0, 0, 0, 3a+b, 3a+b, 3a+b, 3a+b)
M_{24}	(0, 0, 0, 2a+b, 2a+b, 2a+b, 2a+b)
M_{25}	(0, 0, 0, 3a+b, 3a+b, 3a+b, 3a+b)
M_{26}	(0, 0, 0, b, b, b, b)
M_{27}	(0, 0, 0, a+b, a+b, a+b, a+b)
M_{28}	(0, 0, 0, 3a+b, 3a+b, 3a+b, 3a+b)
M_{29}	(0, 0, 0, 2a, 2a, 2a, 2a)
M_{30}	(0, 0, 0, a+b, a+b, a+b, a+b)
M_{31}	(0, 0, 0, 2a, 2a, 2a, 2a)
M_{32}	(0, 0, 0, 2a+b, 2a+b, 2a+b, 2a+b)
M_{33}	(0, 0, 0, 2a, 2a, 2a, 2a)
M_{34}	(0, 0, 0, b, b, b, b)
M_{35}	(0, 0, 0, 2a, 2a, 2a, 2a)
M_{36}	(0, 0, 0, 0, 0, 0, 0)

$(x, 0, x, 2a + x, 0, 2a + x, 0)$. Hence $m + h - m \varepsilon H + K_1$ where $K_1 = ((0, 0, 0, 2a, 0, 2a, 0))$ and $m + h + k_2 - m = m + h - m + k_2 \varepsilon H + K_1 + K_2$. It follows that $A_e = H + K_1 + K_2$ and the order of A_e is 32. Since the order of M_e is 8, the order of $E(D_8)$ is 256.

CHAPTER III

THE RADICAL OF $E(D_8)$

Since the radical of $E(D_8)$, $J\{E(D_8)\}$, depends on the right ideals of $E(D_8)$ which are maximal $E(D_8)$-subgroups, an investigation of some of the right ideals is essential.

Theorem 3.1. [6] Let T be a non-empty subset of a group G. Let $I_T = \{r \in E(G) : Tr = 0\}$. If I_T is non-empty, then I_T is a right ideal in E(G).

Recall that $K_1 = ((0, 0, 0, 2a, 0, 2a, 0))$ and $K_2 = ((0, 2a, 2a, 0, 2a, 2a, 0))$. Let $N = ((0, 0, 0, 2a, 2a, 2a, 2a)) \subset M_e$ and $H_1 = ((2a, 0, 2a, 2a, 0, 2a, 0)) \subset H$ where M_e and H are defined as before. Some of the right ideals of $E(D_8)$ along with their respective subset T are

T	I_T
{0, a, 2a, 3a, a+b, 3a+b}	K_1
{0, a, b, 3a+b}	K_2
{0, 2a, a+b, 3a+b}	$H + K_1$
{0, a, 2a, 3a}	$K_1 + M_e$
{0, 2a}	$H + K_1 + M_e$.

N and H_1 are also right ideals. To show this, let $n \in N$ and $r \in E(D_8)$. Since 0 and 2a belong to the center of D_8, it follows that $r + n = n + r$ and N is a normal subgroup of $E(D_8)$. Now $(x)n = 0$ or $(x)n = 2a$ for all $x \in D_8$. Also $(0)r = 0$ and either $(2a)r = 2a$

or $(2a)r = 0$ for all $r \in E(D_8)$. Hence, $(x)nr = 0$ or $(x)nr = 2a$ and $nr \in N$ making N a right ideal.

Similarly, it can be shown that H_1 is a right ideal.

Since the sum of right ideals is a right ideal, it follows that $H + K_1 + K_2 + N$ and $H_1 + K_1 + K_2 + M_e$ are also right ideals.

Theorem 3.2. $H + K_1 + K_2 + N$, $H_1 + K_1 + K_2 + M_e$, and $H + K_1 + M_e$ are right ideals which are maximal $E(D_8)$-subgroups.

Proof. Since right ideals are R-subgroups it remains only to show that $H + K_1 + M_e$, $H + K_1 + K_2 + N$, and $H_1 + K_1 + K_2 + M_e$ are maximal $E(D_8)$- subgroups.

To prove that $H + K_1 + K_2 + N$ is a maximal $E(D_8)$-subgroup, we will suppose that it is not maximal and show that our supposition leads to a contradiction.

If $H + K_1 + K_2 + N$ is not maximal then there exists a proper $E(D_8)$-subgroup, call it R', such that $H + K_1 + K_2 + N \subset R' \subset E(D_8)$. If $H + K_1 + K_2 + N \subset R'$ there is at least one $r \in E(D_8)$ such that $r \in R'$ and $r \notin H + K_1 + K_2 + N$. Hence, the M_e-term of r must be one of the following:

$r_1 = (0, 0, 0, a, a, a, a)$,

$r_2 = (0, 0, 0, 3a, 3a, 3a, 3a)$,

$r_3 = (0, 0, 0, b, b, b, b)$,

$r_4 = (0, 0, 0, a+b, a+b, a+b, a+b)$,

$r_5 = (0, 0, 0, 2a+b, 2a+b, 2a+b, 2a+b)$, or

$r_6 = (0, 0, 0, 3a+b, 3a+b, 3a+b, 3a+b)$.

Then, in fact, at least one r_i is in R'.

Since R' is an $E(D_8)$-subgroup, then R' . $E(D_8) \subset R'$. The following equations based on R'. $E(D_8) \subset R'$ demonstrate that if any r_i is in R', then $M_e \subset R'$ and R' coincides with $E(D_8)$. This contradicts the hypothesis that R' is a proper $E(D_8)$-subgroup of $E(D_8)$. Consider

$r_1(3a, 0, 3a, 3a, 0, 3a, 0) = r_2$

$r_2(b, 0, b, b, 0, b, 0) = r_3$

$r_3(a, 0, a, a, 0, a, 0) = r_1$

$r_3r_4 = r_4 \quad r_4r_3 = r_3 \quad r_5r_3 = r_3 \quad r_6r_4 = r_4$

$r_3r_5 = r_5 \quad r_4r_5 = r_5 \quad r_5r_4 = r_4 \quad r_6r_5 = r_5$

$r_3r_6 = r_6 \quad r_4r_6 = r_6 \quad r_5r_6 = r_6 \quad r_6r_3 = r_3.$

Hence, $H + K_1 + K_2 + N$ is a maximal $E(D_8)$-subgroup.

Similarly, it can be shown that $H_1 + K_1 + K_2 + M_e$ is a maximal $E(D_8)$-subgroup. Since $H + K_1 + M_e$ contains 128 elements, it is immediate that it is a maximal $E(D_8)$-subgroup. The intersection of these three ideals is $H_1 + K_1 + N$.

According to Definition 1.6, $J\{E(D_8)\} \subset H_1 + K_1 + N$. For if there exists another right ideal which is a maximal $E(D_8)$-subgroup, call it L, it will either contain $H_1 + K_1 + N$ or it will not. If L does contain $H_1 + K_1 + N$, then the intersection is still $H_1 + K_1 + N$. If L does not contain $H_1 + K_1 + N$, then the intersection is contained in $H_1 + K_1 + N$. Hence, the radical of E(D) cannot be any larger than $H_1 + K_1 + N$ and $J\{E(D_8)\} \subset H_1 + K_1 + N$.

The definition of the radical of a distributively generated near ring given by Laxton [5] does not appear to be the same as the definition we have given. In fact, the only difference is in the terminology employed. The following theorem is a restatement in our terminology of Theorem 1.5 of [5].

Theorem 3.3. The radical of a distributively generated near ring R contains all the nilpotent R-subgroups of R.

Since H_1, K_1 and N are each nilpotent $E(D_8)$-subgroups it follows that $H_1 + K_1 + N \subset J\{E(D_8)\}$. Thus we have shown

Theorem 3.4. $J\{E(D_8)\} = H_1 + K_1 + N$.

Since the radical is an ideal [1], we may consider the quotient near ring $E(D_8)/J\{E(D_8)\}$.

Theorem 3.5. $E(D_8) / J\{E(D_8)\}$ has order 32 and is a ring of characteristic 2.

Since $E(D_8)$ has a multiplicative identity, the M_1 in Table 2, then $E(D_8)/J\{E(D_8)\}$ has the multiplicative identity $\bar{M}_1$, the coset containing M_1. From $M_1 + M_1 = (2a, 0, 2a, 0, 0, 0, 0) \in J\{E(D_8)\}$, it follows that $\bar{M}_1$ has (additive) order 2 in $E(D_8)/J\{E(D_8)\}$. Thus, this quotient near ring has characteristic 2. But this implies that the additive group of $E(D_8)/J\{E(D_8)\}$ is abelian. Since it was shown in [4] that a distributively generated near ring whose additive group is abelian is a ring, the theorem follows.

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VITA

Edgar Raymond Guthrie was born on March 6, 1934 in Salina, Oklahoma. His parents are William Spurgeon and Lucille Guthrie. A graduate of Sam Houston State Teachers College in 1958, he taught Mathematics and Physics for three years at A. and M. Consolidated High School. He has also taught Mathematics as a graduate assistant at Texas A. and M. University.

His permanent address is 1604 Armistead, College Station, Texas.

Typist for this thesis was Mrs. Fatema Islam.