HAUSDORF MEANS AND MULTIPLIERS

A Thesis

by

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CHAPTER I

INTRODUCTION

Let V be the set of sequences of real (or complex) numbers. Define addition and scalar multiplication in the usual manner; that is, if $\{a_n\}$ and $\{b_n\}$ are sequences and α is a real (or complex) number, then define $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ and $\alpha\{a_n\} = \{\alpha a_n\}$. Then V is a linear space with scalar field the real (or complex) numbers.

Let f be a function from a real (or complex) linear space B to its scalar field F, such that if a, b are in B and α is in F, then $f(a+b) = f(a) + f(b)$ and $f(\alpha a) = \alpha f(a)$. f is said to be a linear functional of B.

A summability method is a linear functional of any subspace of V; that is, a summability method is a function that maps each element of a subspace U of V to a real (or complex) number in such a manner that for every $\{a_n\}$ and $\{b_n\}$ in U and α real (or complex),

$$f(\{a_n\} + \{b_n\}) = f(\{a_n\}) + f(\{b_n\}) \text{ and}$$

$$f(\alpha\{a_n\}) = \alpha f(\{a_n\}) .$$

If $\{s_n\}$ is in the domain U of the functional f, then f is said to sum $\{s_n\}$ to $f(\{s_n\})$. We say f sums a series $\sum_{n=0}^{\infty} a_n$ if f sums the sequence of partial sums $\{s_n\}$ and we write $\sum_{n=0}^{\infty} a_n = s(f)$ when $f(\{s_n\}) = s$. As a matter of convenience we write Σa_n for $\sum_{n=0}^{\infty} a_n$ and the limits are 0 to ∞ unless otherwise indicated.

A summability method f is said to satisfy (C) if for every Σa_n that f sums, f sums $\sum_{n=0}^{\infty} a_n$ to $a_0 + s$ where f sums $\sum_{n=0}^{\infty} a_{n+1}$ to s. A summability method f is said to be regular if it sums every convergent series to its ordinary sum; in other words, if Σa_n converges then f sums Σa_n and $f(\{s_n\}) = \Sigma a_n$.

Our interest lies in summability methods generated by a $\omega \times \omega$ matrix $T = (c_{mn})$; given a sequence $\{s_n\}$, we define $t_m = \Sigma c_{mn} s_n$. If t_m exists for each m and $\lim_{m\to\infty} t_m = p$, then we say T sums $\{s_n\}$ to p. Unlike the abstract definition of a summability method, a matrix method has a natural domain, viz., the set of sequences $\{s_n\}$ for which $\lim t_m$ exists. It is in this context that we consider a matrix as a summability method. We will also be interested in the algebra of the linear sequence to sequence transformation defined by a matrix T, as $T\{s_n\} = \{t_m\}$. We see that, except for questions of convergence, this transformation is the same as in matrix multiplication:

$$t = \begin{bmatrix} t_0 \\ t_1 \\ t_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad T = \begin{bmatrix} c_{00} & c_{01} & c_{02} & \cdots \\ c_{10} & c_{11} & c_{12} & \cdots \\ c_{20} & c_{21} & c_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix},$$

then $t = Ts$ or

$$\begin{bmatrix} t_0 \\ t_1 \\ t_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} c_{00} & c_{01} & c_{02} & \cdots \\ c_{10} & c_{11} & c_{12} & \cdots \\ c_{20} & c_{21} & c_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

We call the numbers $\{t_m\}$ the T means of the sequence $\{s_n\}$.

In this thesis a particular type of matrix method, the general Hausdorf method, is studied.

Let δ be the $\omega x \omega$ matrix $\delta = (\delta_{ij})$ where $\delta_{ij} = (-1)^j \binom{i}{j}$, $i = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$ As a summability method, the δ means t_m are given by

$$t_m = \Delta^m s_0 = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \cdot s_n .$$

It is clear that δ is its own reciprocal: that is, if $t = \delta s$ then $s = \delta t$ or $\delta\delta = I$ where I is the identity transformation [4, p. 247].

A Hausdorf transformation is any transformation of the form $t = (\delta \mu \delta)s$, where $\mu = (\mu_n)$ is any diagonal transformation. The matrix of the transformation is called a Hausdorf matrix. Ordinary convergence and (C, 1) summability are Hausdorf transformations where $\mu_n = 1$ and $\mu_n = \frac{1}{n+1}$, respectively. It is interesting to note that any two Hausdorf transformations commute. For if

$$H_1 = \delta \mu \delta \text{ and } H_2 = \delta \mu' \delta \quad \text{then}$$

$$H_1 H_2 = (\delta \mu \delta)(\delta \mu' \delta) = \delta \mu \mu' \delta = \delta \mu' \mu \delta = (\delta \mu' \delta)(\delta \mu \delta) = H_2 H_1$$

since any two diagonal matrices commute and $\delta \delta = I$ [4, p. 249].

The class of Hausdorf transformations is the class of transformations that commute with the (C, 1) transformation [4, p. 249]. For suppose $B = \delta \mu \delta$ is a Hausdorf transformation; that $\mu_n \neq \mu_m$ for $m \neq n$; and that λ is a transformation that commutes with B. Now if $A = \delta \lambda \delta$ then we have $\lambda = \delta A \delta$ and since $B = \delta \mu \delta$ we have $\mu = \delta B \delta$. Therefore

$$A\mu = (\delta \lambda \delta)(\delta B \delta) = (\delta \lambda B \delta) = \delta B \lambda \delta = (\delta B \delta)(\delta \lambda \delta) = \mu A \quad .$$

If A has means t_m where $t_m = \Sigma c_{mn} s_n$ then the calculations above show $\Sigma c_{mn} \mu_n s_n = \mu_m \Sigma c_{mn} s_n = \Sigma c_{mn} \mu_m s_n$ for all s_n. Since $\mu_n \neq \mu_m$ for $m \neq n$, then $c_{mn} = 0$ for $m \neq n$ and A must be a diagonal transformation. In particular we can consider B as the (C, 1) transformation since $\frac{1}{n+1} \neq \frac{1}{m+1}$ for $n \neq m$.

We now state an important theorem concerning the means t_m for any Hausdorf transformation.

(1.1) THEOREM. [4, p. 250].

The general Hausdorf transformation is

$$t_m = \sum_{n=0}^{m} \binom{m}{n} \Delta^{m-n} \mu_n s_n .$$

Therefore the general Hausdorf matrix is $H = (c_{mn})$ where

$$c_{mn} = \binom{m}{n} \Delta^{m-n} \mu_n \quad \text{for} \quad n \leqq m \text{ and } c_{mn} = 0 \text{ for } m < n .$$

We will denote the Hausdorf transformation $(\delta \mu \delta)$ by (H, μ) .

A sequence $\{a_n\}$ is said to be totally monotone if $\Delta^p a_n \geqq 0$ for $n = 0, 1, 2, \ldots$ and $p = 0, 1, 2, \ldots$. The conditions of regularity for a Hausdorf transformation (H, μ) become (1.2) THEOREM [4, p. 256]. In order that the transformation (H, μ) should be regular, it is necessary and sufficient that $\{\mu_n\}$ should be the difference of two totally monotone sequences, that $\Delta^m \mu_0 \to 0$ as $m \to \infty$, and that $\mu_0 = 1$.

If $\{a_n\}$ is a sequence such that Σa_n converges, it is a well known fact that there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$ and $\Sigma a_n \lambda_n$ converges. The analogous theorem for (C, 1) summability holds as was suggested by Salem [6] and investigated in detail by Bryant [2, lemma 2.1]. We consider the problem for any Hausdorf transformation (H, μ): if Σa_n is summable (H, μ), does there exist a sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$ and $\Sigma a_n \lambda_n$ is summable

(H, μ)? A simple example will be presented to show the answer is no. The same example will also show that other "nice" properties of ordinary convergence do not hold for the general Hausdorf method.

CHAPTER II

RESULTS

Let (H, μ) be the Hausdorf transformation such that $\mu_n = n$. Since

$$\Delta^0 \mu_n = \mu_n = n ,$$

$$\Delta^1 \mu_n = \mu_n - \mu_{n+1} = -1 \quad \text{and} \quad \Delta^p \mu_n = 0 \quad \text{for} \quad p \geqq 2 ,$$

the means for (H, μ) are

$$t_m = \sum_{n=0}^{m} \binom{m}{n} \Delta^{m-n} \mu_n s_n = \sum_{n=m-1}^{m} \binom{m}{n} \Delta^{m-n} \mu_n s_n$$

$$= \binom{m}{m-1} \Delta \mu_{m-1} s_{m-1} + \binom{m}{m} \Delta^0 \mu_m s_m = -m s_{m-1} + m s_m$$

$$= m(s_m - s_{m-1}) = m a_m \quad \text{where} \quad s_n = \sum_{i=0}^{n} a_i .$$

Now consider the sequence $\{a_m\}$ where $a_0 = 1$ and $a_m = \frac{1}{m}$, $m \geqq 1$. (H, μ) transforms the series Σa_n into

$$t_m = m a_m = m(\frac{1}{m}) = 1, \quad \text{and} \quad t_m \to 1 \quad \text{as} \quad m \to \infty .$$

Let $\{\lambda_n\}$ be any sequence such that $\lambda_n \to \infty$. (H, μ) transforms the series $\Sigma a_n \lambda_n$ into

$$t_m = m(\lambda_m a_m) = m(\frac{1}{m}) \lambda_m = \lambda_m \to \infty .$$

Therefore no sequence $\{\lambda_n\}$ where $\lambda_n \to \infty$ can exist such that $\Sigma a_n \lambda_n$ is summable (H, μ).

Now (H, μ) is not regular, for notice that it is necessary that $\mu_0 = 1$ for (H, μ) to be regular. We could also show that (H, μ) is not regular by considering the following sequence $\{a_n\}$: if $a_0 = 2$ and $a_n = \frac{(-1)^n}{n}$ then Σa_n converges since $|a_n|$ is strictly decreasing to 0. (H, μ) does not sum Σa_n since

$$t_m = m a_m = m \frac{(-1)^m}{m} = \begin{cases} -1 & \text{if } m \text{ is odd} \\ 1 & \text{if } m \text{ is even} \end{cases}$$

and $\lim t_m$ does not exist.

This same Hausdorf transformation shows that other theorems from ordinary convergence do not hold. For example, it is a fact that if Σa_n is absolutely convergent that it is also convergent. For (H, μ) the theorem would be as follows: if $\Sigma |a_n|$ is summable (H, μ) then Σa_n is summable (H, μ).

As a counter example, consider the series Σa_m where $a_m = \frac{(-1)^m}{m}$ $m \geq 1$, $a_0 = 0$. Then for $\Sigma |a_n|$, $t_m = m \left| \frac{(-1)^m}{m} \right| = 1$ and (H, μ) sums $\Sigma |a_n|$ to 1. But for Σa_n we have means

$$t_m = m \frac{(-1)^m}{m} = \begin{cases} -1 & \text{if } m \text{ is odd} \\ 1 & \text{if } m \text{ is even or } 0 \end{cases}$$

and (H, μ) does not sum Σa_n.

Another theorem from ordinary convergence which does not hold for the general Hausdorf method is as follows: if Σa_n converges and $a_n \geqq 0$ then any rearrangement of $\{a_n\}$ will also converge to the same number.

For Hausdorf summability this becomes: suppose Σa_n is summable (H, μ) and $a_n \geqq 0$, then any rearrangement of $\{a_n\}$ is also summable (H, μ) to the same number. As a counter example consider the sequence $\{a_n\}$ where $a_0 = 0$ and $a_n = \frac{1}{n}$ for $n \geqq 1$. (H, μ) sums Σa_n to 1. Now if j is any odd integer, interchange a_j and a_{2j}. The rearranged sequence is

$$a_0, a_2, a_1, a_6, a_4, a_{10}, a_3, a_{14}, a_8, a_{18}, a_5, a_{22}, a_{12}, \ldots$$

Note that the only terms affected are those of the form a_{2k+1} and a_{4k+2}, where k is an integer; also note this interchange is a rearrangement. The (H, μ) means t_m for the series of rearranged terms is: if j is an odd integer then $t_j = j\,(\frac{1}{2j}) = \frac{1}{2}$ and $t_{2j} = 2j\,(\frac{1}{j}) = 2$. Therefore $\lim t_m$ does not exist and (H, μ) does not sum the rearranged series.

Perhaps even more importantly, this example demonstrates that the general Hausdorf method does not satisfy (C). For if $\{a_n\}$ is the sequence such that $a_0 = 1$, $a_n = \frac{1}{n}$, $n \geqq 1$ then (H, μ) sums Σa_n to 1. Let $b_n = a_{n+1}$, $n = 0, 1, 2, \ldots$; that is, $b_n = \frac{1}{n+1}$

for all positive integers n. Then (H, μ) sums

$$\Sigma b_n \text{ to } 1 \text{ since } t_m = m b_m = \frac{m}{m+1} \to 1 \text{ as } m \to \infty .$$

If (H, μ) sums a series Σx_n to x, denote x by $(H, \mu)[\Sigma x_i]$. Then we have, for this example, $(H, \mu)[\sum_{n=0}^{\infty} a_i] \neq a_0 + (H, \mu)[\sum_{n=1}^{\infty} a_i]$ and (H, μ) does not satisfy (C).

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