AQUIFER BEHAVIOR WITH REINJECTION

A Thesis

By

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TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

When fluid is injected into an aquifer, the asymmetric pressure and velocity distributions, as well as the injection rate and cumulative influx, are very useful parameters in planning production operations and related problems.

This thesis develops analytical expressions showing the cumulative influx, velocity and pressure distributions for several linear and radial injection systems with different boundary conditions.

Since interference between injection wells with each other and with the aquifer is readily solved by superimposition, it is only necessary to consider a single injection well of constant strength. The porous medium is assumed to be homogeneous isotropic and of constant thickness.

Several figures showing the velocity, rate and cumulative influx are presented for the linear closed aquifer, together with an example of pressure distribution inside the aquifer. Influx rate and cumulative influx are presented for a radial aquifer. The figures are presented in dimensionless parameters and may be used generally for aquifers of the same geometry and boundary conditions.

Two computer programs applying to linear and radial cases are attached.

INTRODUCTION

It is fairly common practice to reinject water into the aquifer near the oil-water interface in water-driven reservoirs. Many studies describing aquifer behavior without reinjection have been made[1, 2, 3, 4, 5] but apparently no analytical studies have been made to indicate the aquifer performance when injection wells are present in the aquifer.

The diffusivity equation may be used to describe the pressure behavior in an aquifer. When sources are present it can be written as:

$$\nabla^2 p = \frac{1}{\eta} \frac{\partial p}{\partial t} - \frac{\mu_w}{k_w} i_w \delta(\vec{r} - \vec{r}_o) \tag{1}$$

A general solution of Equation (1) may be obtained as the sum of two solutions. One is, a general solution of the homogeneous equation, obtained by dropping the source term in Equation (1) and still satisfying the required boundary conditions. Another is a general solution of Equation (1) satisfying homogeneous boundary conditions[10]. The first solution which corresponds to the aquifer without reinjection has been thoroughly reported in the literature[1, 2, 3, 4,5]. The latter will be treated here.

Solutions with variable injection rates are obtained by

1 References shown at end of thesis.

superimposing solutions with constant rates, so only the latter is considered here.

Six cases have been treated here. They are:

1) A linear aquifer of finite width and extent. Water is injected into the aquifer at point x_o, y_o at constant rate. The pressure along the line x = o is maintained at zero pressure. See Figure 1. Equations showing the potential distribution, rates, velocity and cumulative influx have been developed. Certain numerical results are shown.

2) A linear aquifer of finite width and extent. Water is injected into the aquifer at x_o, y_o at constant rate. The rate of production along the oil-water water contact is maintained constant. Equations showing the potential distribution inside the aquifer are presented.

3) A linear aquifer of finite width and extent. Constant pressures are maintained at opposite ends of the aquifer. Water is injected at constant rates at x_o, y_o. Equations showing the potential distribution are shown.

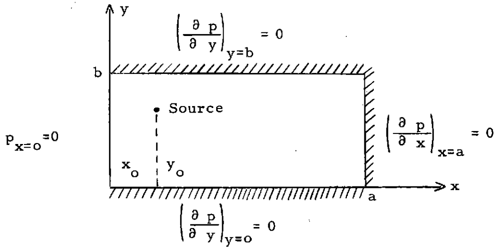
4) A linear aquifer of finite width and infinite extent. The pressure is maintained constant at the oil-water contact. Water is injected at a constant rate at $x_o y_o$.

5) A linear aquifer of finite width, and finite extent. The pressure at the oil-water is maintained constant. Water is injected at constant rate per unit length along a line x_o.

6) A radial aquifer of finite extent. The pressures are maintained constant at the external and internal boundary. Water is injected at a constant rate at point r_o, θ_o. Equations showing the potential distribution, velocity, rate of flux and cumulative flux along the oil-water contact are presented. Numerical solutions are presented in terms of dimensionless parameters.

THE LINEAR AQUIFER

The linear closed aquifer, performing at constant terminal pressure was selected as the first case to be analysed. The geometric parameters and the boundary values are depicted in Figure 1 below:



LINEAR AQUIFER - GEOMETRIC PARAMETERS

FIGURE 1

The pressure as derived in Appendix A for a unit thickness medium can be expressed by a double Fourier series as:

$$p(x, y, t, x_o, y_o) = \frac{4}{ab}\, i_w \frac{\mu_w}{k_w} \left\{ \frac{1}{2} \sum_{m=1}^{\infty} \frac{\sin(\frac{2m-1}{2a} \pi x_o)}{(\frac{2m-1}{2a} \pi)^2} \right.$$

$$\left[1-e^{-\eta\left(\frac{2m-1}{2a}\pi\right)^2 t}\right]\sin\frac{2m-1}{2a}\pi x+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2m-1}{2a}\pi x_o\right)\cos\left(\frac{n\pi}{b}y_o\right)}{\left[\left(\frac{n\pi}{b}\right)^2+\left(\frac{2m-1}{2a}\pi\right)^2\right]}$$

$$\left.\left[1-e^{-\eta\left[\left(\frac{n\pi}{b}\right)^2+\left(\frac{2m-1}{2a}\pi\right)^2\right]t}\right]\sin\frac{2m-1}{2a}\pi x\cos\frac{n\pi}{b}y\right\} \tag{2}$$

The velocity at the oil-water contact is given by:

$$(v_x)_{x=o}=\frac{4i_w\pi}{ab}\left\{\frac{1}{2}\sum_{m=1}^{\infty}\frac{2m-1}{2a}\cdot\frac{\sin\left(\frac{2m-1}{2a}\pi x_o\right)}{\left(\frac{2m-1}{2a}\pi\right)^2}\right.$$

$$\left[1-e^{-\eta\left(\frac{2m-1}{2a}\pi\right)^2 t}\right]+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{2m-1}{2a}\frac{\sin\left(\frac{2m-1}{2a}\pi x_o\right)\cos\left(\frac{n\pi}{b}y_o\right)}{\left[\left(\frac{n\pi}{b}\right)^2+\left(\frac{2m-1}{2a}\pi\right)^2\right]}$$

$$\left.\left[1-e^{-\eta\left[\left(\frac{n\pi}{b}\right)^2+\left(\frac{2m-1}{2a}\pi\right)^2\right]t}\right]\cos\frac{n\pi}{b}y\right\} \tag{3}$$

The influx rate is given by:

$$e_w=i_w\left[1-\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{2m-1}{2a}\pi x_o\right)}{2m-1}e^{-\eta\left(\frac{2m-1}{2a}\pi\right)^2 t}\right] \tag{4}$$

The cumulative influx by:

$$W_e=W_i\left[1-\frac{ax_o}{\eta t}\left(1-\frac{x_o}{2a}\right)+\frac{4}{\pi}\sum_{m=1}\frac{\sin\left(\frac{2m-1}{2a}\pi x_o\right)}{\eta\left(\frac{2m-1}{2a}\pi\right)^2 t(2m-1)}\right.$$

$$\left.e^{-\eta\left(\frac{2m-1}{2a}\pi\right)^2 t}\right] \tag{5}$$

As seen in Equations (4) and (5), the influx rate and cumulative influx are independent of the source's ordinate.

Equations (4) and (5) expressed in dimensionless variables were evaluated on the computer and the results are presented in Figures 3 and 4, respectively. The curves can be used for any linear aquifer satisfying the hypothesis for which the figures were constructed. As depicted in these figures, unless the source is located at a great distance from oil-water contact, the amount of injected water that does not reach the oil zone is very small even for small times, i. e., the transient period is of small duration.

Equations (2) and (3) converge slowly in our case, since normally $b << a$. For this reason, superposition of point sources was used to calculate the pressure inside the aquifer and velocity at the oil-water contact. The equations utilized, referring to one well, are: (See Figure 2).

$$p = p_e - \frac{i_w \mu_w}{4\pi k_w} \, Ei \left[- \frac{(y_o-y)^2+(x_o-x)^2}{4\eta t} \right] \tag{6}$$

$$v_x = \frac{i_w \mu_w (x_o-x)}{8\pi k_w \eta t} \left\{ \frac{4\eta t}{(y_o-y)^2+(x_o-x)^2} + \sum_{n=o}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} \left[\frac{(y_o-y)^2+(x_o-x)^2}{4\eta t} \right] \right\}^{6} \tag{7}$$

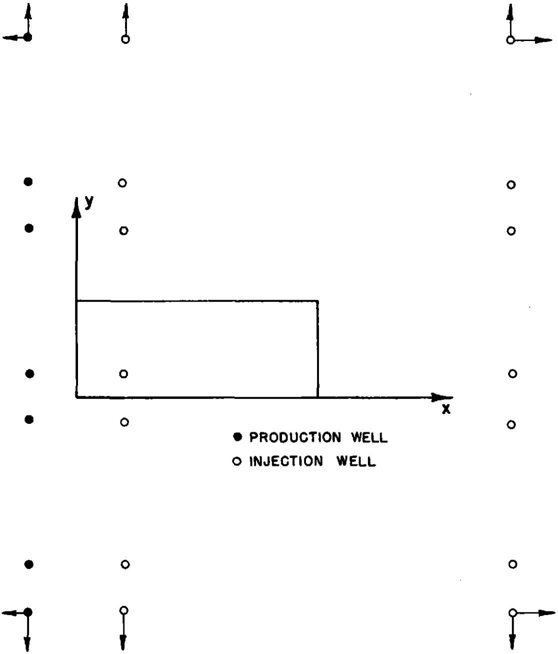


FIGURE 2. LAY-OUT OF PRODUCTION AND INJECTION WELLS FOR THE LINEAR CLOSED AQUIFER

Equations (6) and (7) were expressed in dimensionless quantities and then evaluated on the computer for various values of the parameters. Some of the results are shown in Figures 5a, 5b and 6. Comparison of superposition and the analytical solution for this problem can be seen in Figures 3 and 5b. The knowledge of this asymmetric velocity distribution can be very useful in planning production operations; the pressure distributions inside the aquifer can be very useful if it is desired to study interference between injection wells.

Since many parameters are involved and, consequently, many tables are necessary to cover all variations, only some values are presented. The computer program in Aggie language for calculation of Equation (6) is presented in Appendix C. With minor modifications this can be utilized for different boundary conditions and Equation (7).

Figure 6 shows that when the source is close to the water-oil contact, the asymmetric velocity distribution manifests almost immediately implying that the significant part of the transient time is of small duration. When the source is removed away from the line of contact and in case that the width of the aquifer is smaller than the source's ordinate, the velocity distribution is almost symmetric, as indicated by physical considerations.

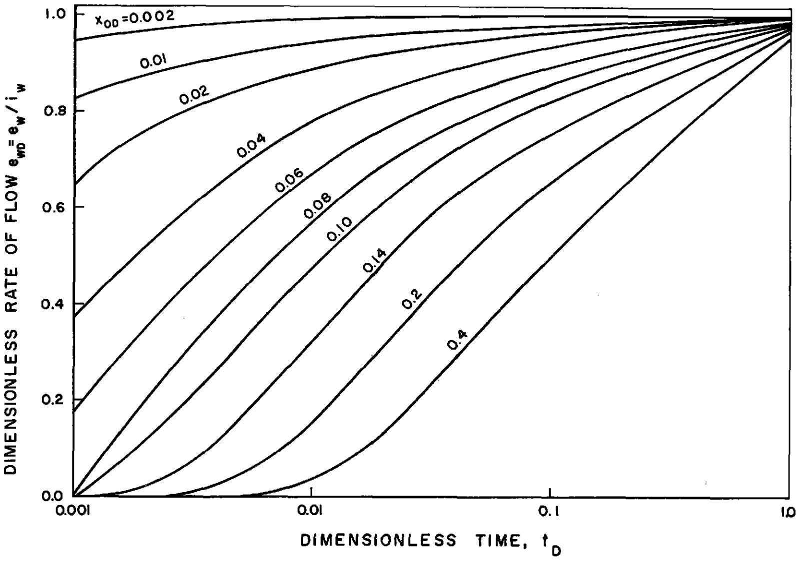


FIGURE 3. RATE OF FLOW FROM A LINEAR CLOSED AQUIFER

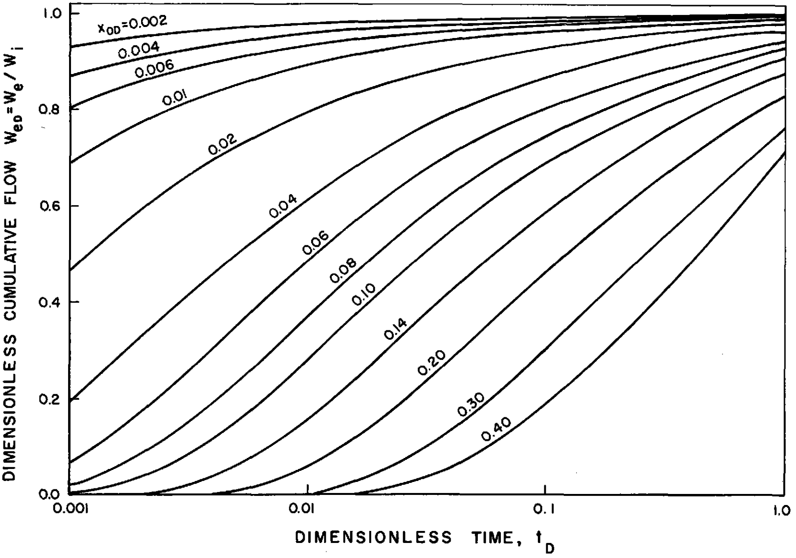


FIGURE 4. CUMULATIVE FLOW FROM LINEAR CLOSED AQUIFER

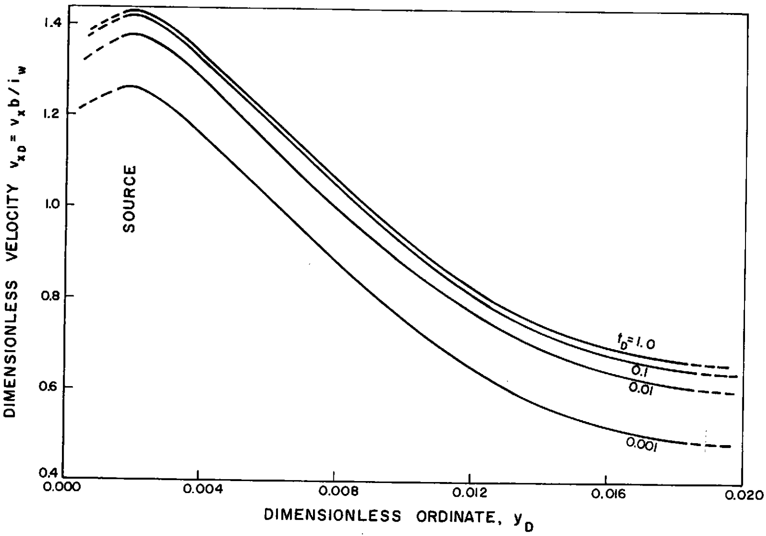


FIGURE 5a. VELOCITY IN LINEAR CLOSED AQUIFER FOR $b_D = \frac{b}{a} = 0.02; y_{oD} = \frac{y_o}{a} = 0.002; x_{oD} = \frac{x_o}{a} = 0.01$

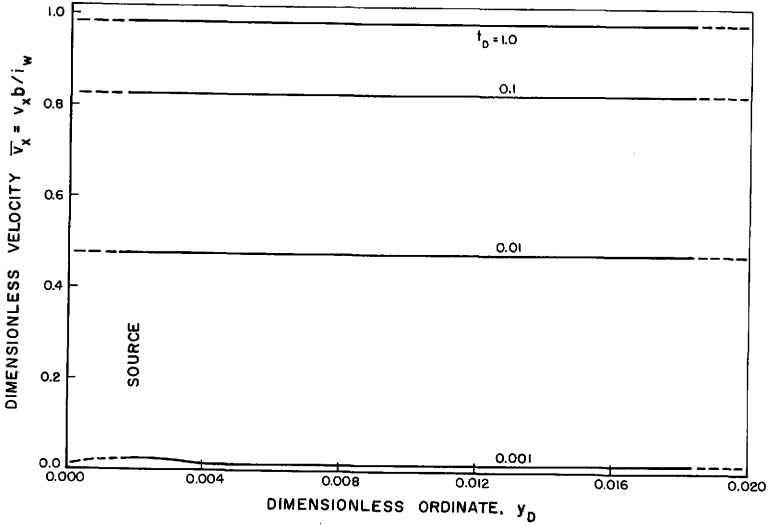


FIGURE 5b. VELOCITY IN LINEAR CLOSED AQUIFER FOR b_D=0.02; y_{OD}=0.002; x_{OD}=0.1

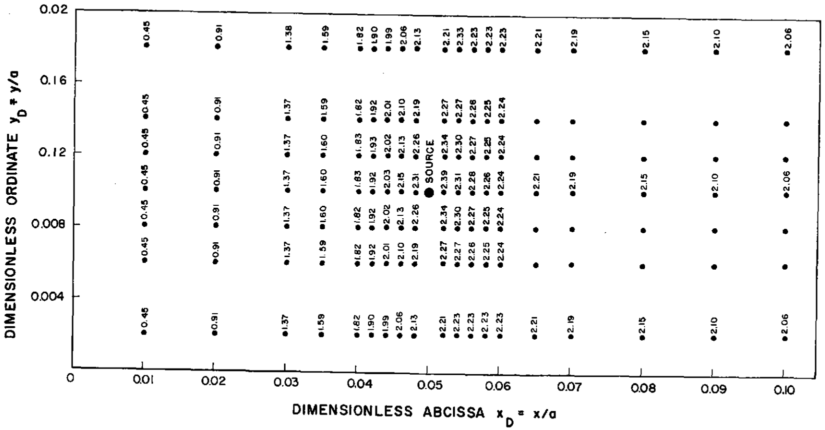
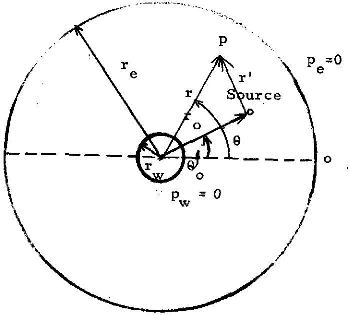


FIGURE 6. PRESURE IN LINEAR CLOSED AQUIFER FOR $t_D = 0.1$; $p_D = \frac{k_w}{\mu_w} i_w$

The equations for the linear system for Cases 2, 3, 4 and 5 are in Appendix A.

THE RADIAL AQUIFER

A radial system with constant pressures at the external and internal boundaries has also been studied. A source of constant strength is located at r_o, θ_o. Geometrical parameters are presented in Figure 7. The pressure at r_w is maintained at zero.



RADIAL AQUIFER - GEOMETRIC PARAMETERS

FIGURE 7

The pressure, for a medium of unit thickness, with a source of unit strength per radian, can be expressed as:*

*See a similar expression, but neglecting the presence of r_w, in Reference 5, page 661.

$$p(r, \theta, t, r_o, \theta_o) = -\frac{\mu_w}{k_w}\left\{\frac{1}{2}\ \ln\left[\left(\frac{r_o}{r_e}\right)^2 - 2\,\frac{r_o r}{r_e^2}\cos(\theta_o - \theta) + \frac{r^2}{r_e^2}\right]\right.$$

$$+\frac{1}{2}\ \ln\left[1 - 2\,\frac{r_o r}{r_e^2}\ \cos\ (\theta_o - \theta) + \frac{r_o^2 r^2}{r_e^4}\right] + \frac{\ln\frac{r_e}{r_o}\ \ln\frac{r_e}{r}}{\ln\frac{r_e}{r_w}}$$

$$+\sum_1^{\infty}\frac{1}{n}\ \frac{\left[\left(\frac{r_o r}{r_e^2}\right)^n - 2\left(\frac{r_o r}{r_w^2}\right)^n - \left(\frac{r_e^2}{r r_o}\right)^n + \left(\frac{r}{r_o}\right)^n + \left(\frac{r_o}{r}\right)^n\right]\cos n(\theta_o - \theta)}{1 - \left(\frac{r_e}{r_w}\right)^{2n}}$$

$$+\sum_1^{\infty} 2\ \frac{Z_{o,o}(mr_o)Z_{o,o}(mr)}{\left[r_e Z'_{o,o}(mr_e)\right]^2\ \left[r_w Z'_{o,o}(mr_w)\right]^2}\qquad e^{-\eta m^2 t}$$

$$\left.+\sum_1^{\infty}\sum_1^{\infty} 4\ \frac{Z_{n,n}(mr_o)Z_{n,n}(mr)}{\left[r_e Z'_{n,n}(mr_e)\right]^2 - \left[r_w Z'_{n,n}(mr_w)\right]^2}\ \cos\ m(\theta_o - \theta)\ e^{-\eta m^2 t}\right\} \tag{8}$$

where:

$$Z_{p,n}(mr) = J_p(mr_w)Y_n(mr) - Y_p(mr_w)J_n(mr),$$

J_n and Y_n are Bessel's functions of first and second kind and order n,

m is chosen such that $Z_{n,n}(mr_e) = 0$,

primes indicate derivatives with respect to r.

It is readily seen that the Green's function (8) is symmetric in r, r_o and θ, θ_o, i.e., the response in r and θ due to a source in r_o and θ_o is the same as the response in r_o and θ_o due to a source in r and θ.

The rate of influx at r_w can be written as:

$$e_w = 2\pi \left\{ \frac{\ln \frac{r_e}{r_o}}{\ln \frac{r_e}{r_w}} + 2 \sum_1^{\infty} \frac{Z_{o,o}(mr_o) r_w Z_{o}{}'_{,o}(mr_w)}{\left[r_e Z_o{}'_{,o}(mr_e)\right]^2 - \left[r_w Z_o{}'_{,o}(mr_w)\right]^2} e^{-\eta m^2 t} \right\} \tag{9}$$

The cumulative influx is expressed by:

$$W_e = 2\pi \left\{ t \frac{\ln \frac{r_e}{r_o}}{\ln \frac{r_e}{r_w}} + 2 \sum_1^{\infty} \frac{Z_{o,o}(mr_o) r_w Z_o{}'_{,o}(mr_w)}{m^2 t \left\{ \left[r_e Z_o{}'_{,o}(mr_e)\right]^2 - \left[r_w Z_o{}'_{,o}(mr_w)\right]^2 \right\}} \left[1 - e^{-\eta m^2 t}\right] \right\} \tag{10}$$

Equations (9) and (10) were calculated and some of the results obtained are shown in Figures 8a to e and 9a to e, respectively. Similar to the linear case, the figures are presented in dimensionless parameters and consequently, can be used for any radial system satisfying the hypothesis in which they are based.

The first five zeros of $Z_{o,o}(mr_e)=0$, which refers to Equations (9) and (10) where obtained from Reference 12, and the following ones by McMahon asymptotic expansion.[12] The Bessel's function was calculated using polynomial approximations.[12]

In evaluating equations (9) and (10) the values of e_w and W_e, for small values of t_D showed small errors due to numerical approximations. In Figure 8a some points do show these oscillations.

In this case, it can be very inefficient to reinject at a considerable distance from the oil-water contact in the case of small r_e, since this, being a line of constant pressure, the water will go in this direction.

For small values of r_o, as practiced in actual operations, the transient period is of insignificant duration and all injected water readily reaches the oil field.

The computer program used in calculating Equation (9) is attached, and with minor modifications can be used to calculate Equation (10).

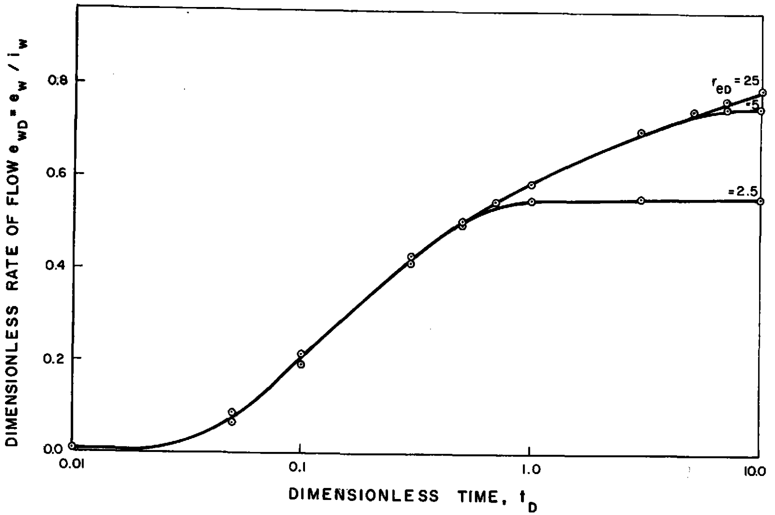


FIGURE 8a. RATE OF FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS $r_{oD} = 1.5$

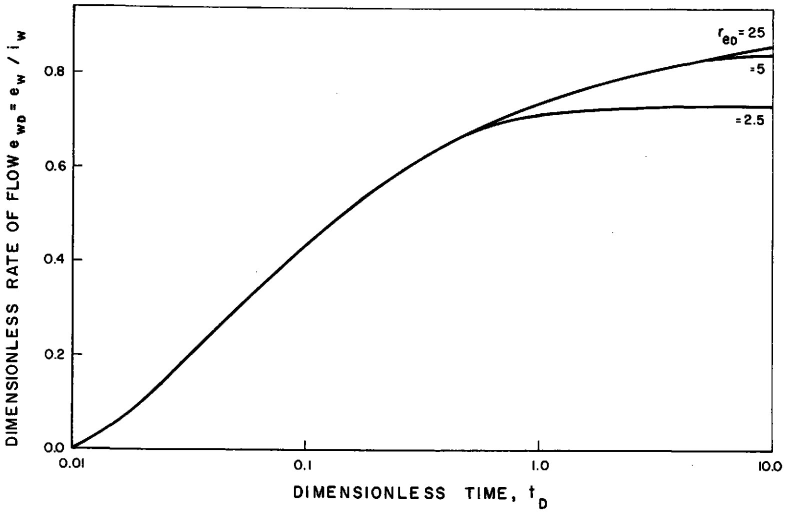


FIGURE 8b. RATE OF FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS r_{OD} = 1.3

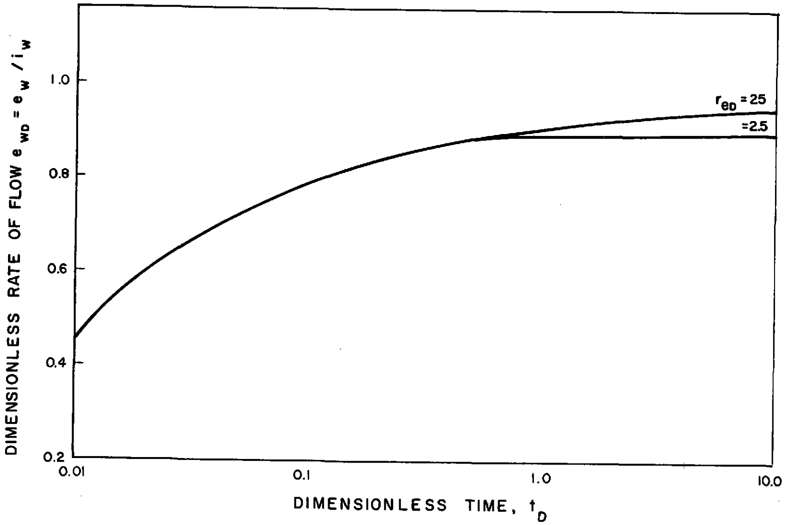


FIGURE 8c. RATE OF FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS $r_{OD} = 1.1$

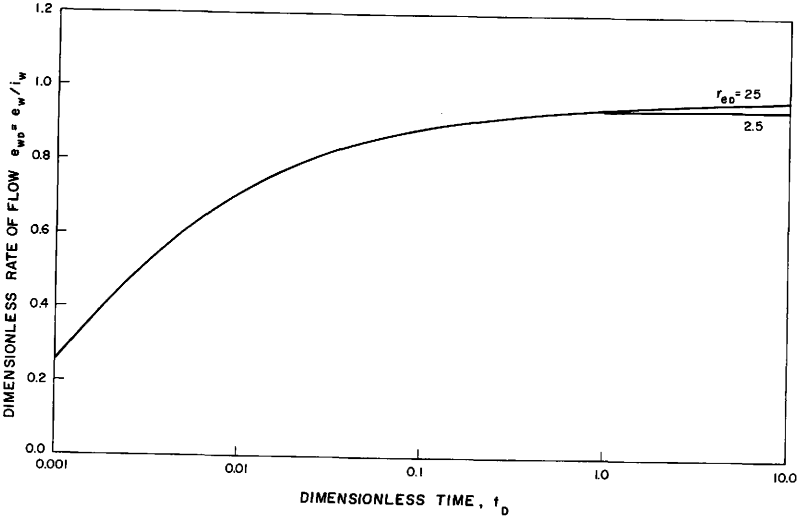


FIGURE 8d. RATE OF FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS $r_{OD} = 1.05$

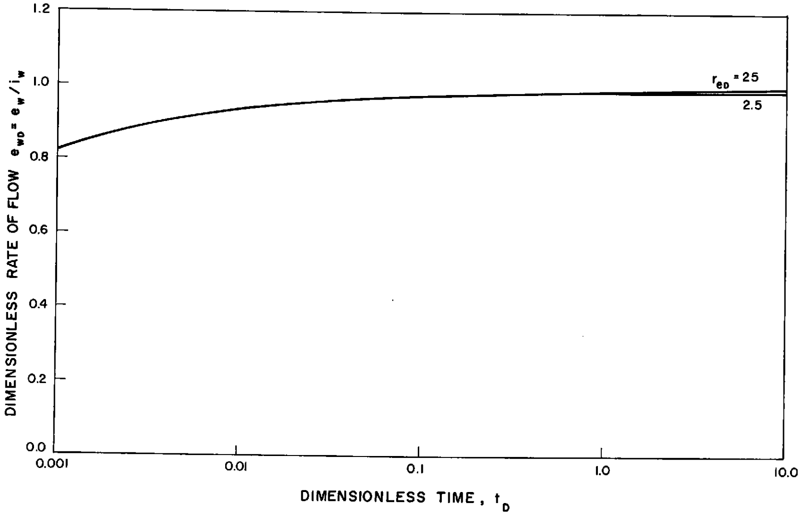


FIGURE 8e. RATE OF FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS $r_{OD} = 1.01$

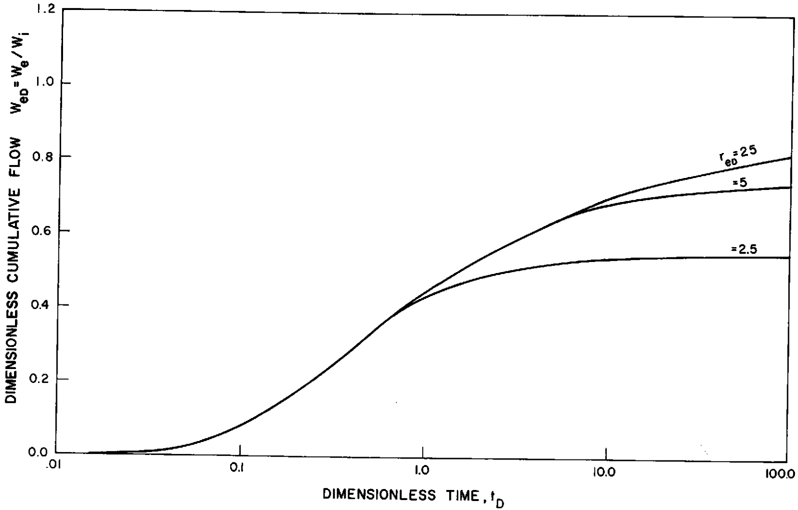


FIGURE 9a. CUMULATIVE FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS $r_{OD} = 1.5$

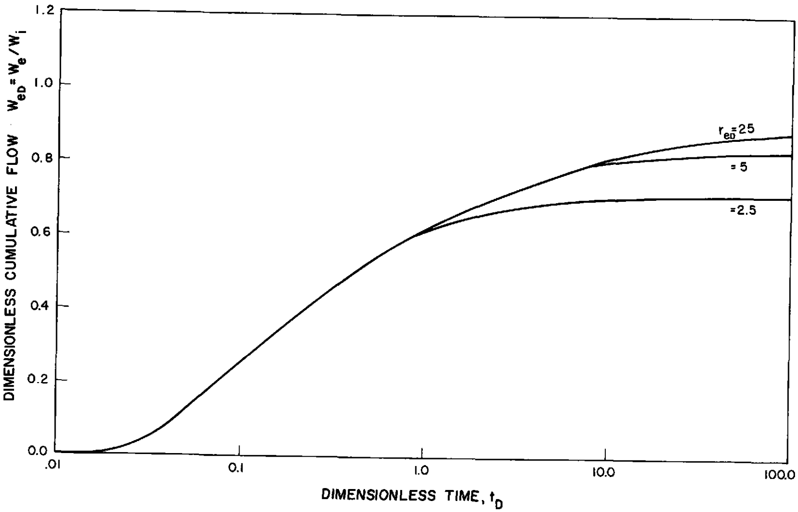


FIGURE 9b. CUMULATIVE FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS $r_{OD} = 1.3$

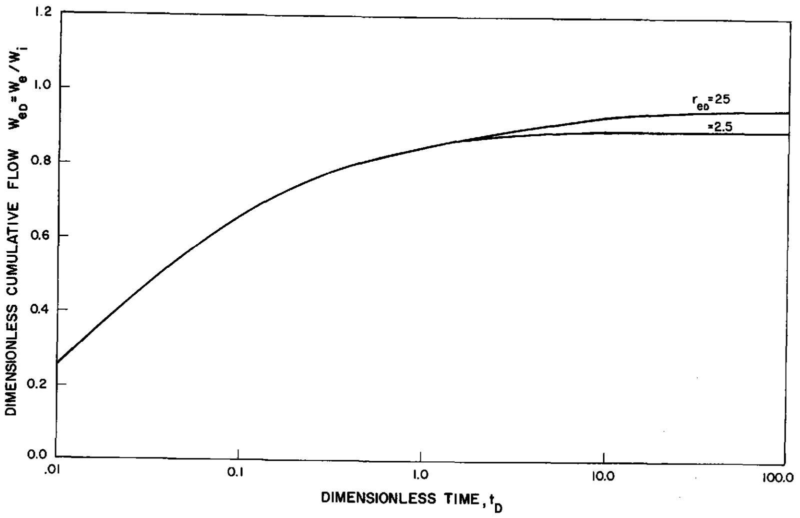


FIGURE 9c. CUMULATIVE FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS $r_{OD} = 1.1$

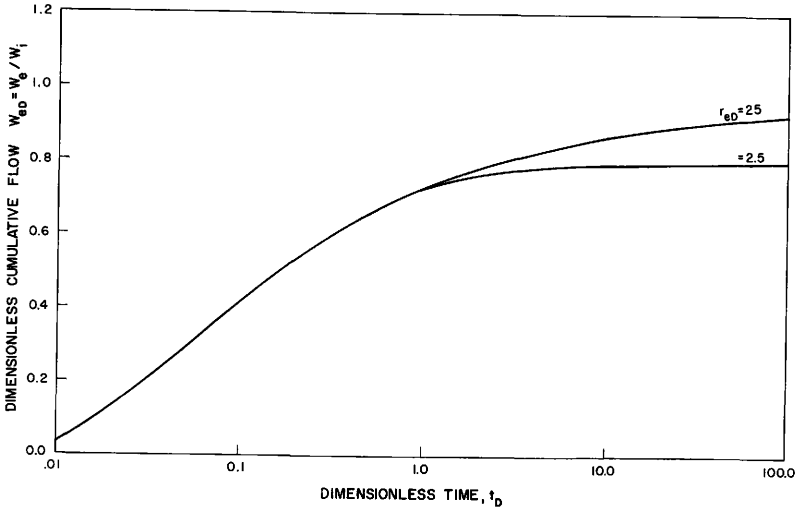


FIGURE 9d. CUMULATIVE FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS $r_{OD} = 1.2$

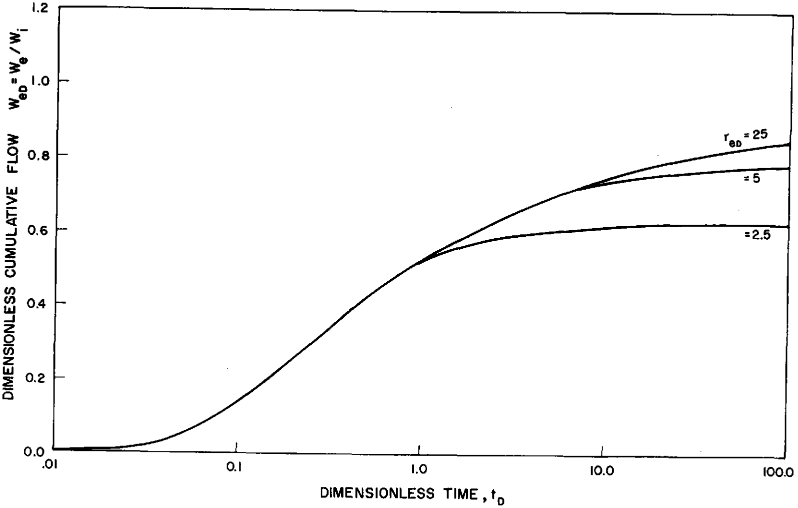


FIGURE 9e. CUMULATIVE FLOW FROM A RADIAL AQUIFER WITH CONSTANT EXTERNAL PRESSURE; SOURCE AT A DIMENSIONLESS RADIUS $r_{OD} = 1.4$

CONCLUSIONS

1. For an injection well located close to the oil-water contact, the transient period is of relatively small duration, i. e., in a short period of time all the parameters attain approximately a steady state value.
2. For closed systems, the water lost to the aquifer is relatively small, increasing with increasing distance between injection well and oil-water contact.
3. In aquifers with constant pressure at the external boundary the water lost can be very important when placing the injection well far from the water-oil contact for small values of r_e.
4. At the internal boundary the velocity rapidly approaches a steady asymmetric distribution when the source is close to the contact. When the source is far removed from the contact, the velocity is fairly uniform through all the contact, this being anticipated from physical considerations.

NOMENCLATURE

a = length of linear aquifer

b = width of linear aquifer

c_w = water compressibility

e_w = water influx rate

h = net pay thickness

i_w = water injection rate

k_w = effective permeability to water

p = pressure

p_d = dimensionless pressure

p_e = external boundary pressure

p_w = pressure at oil-water contact

r = radial distance

r_d = dimensionless radial distance

r_e = external boundary radius

r_w = internal radius of aquifer

t = time

t_d = dimensionless time

v = velocity

W_e = cumulative water influx

W_i = cumulative water injected

x = distance

y = distance

η = hydraulic diffusivity $(k_w / \phi \mu_w c_w)$

μ_w = water viscosity

ϕ = porosity

δ = Dirac delta function

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APPENDIX A

1. Linear Closed, Constant Terminal Pressure, One Point Source Aquifer.

The pressure in this system can be obtained by solving the following boundary value problem:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\eta}\frac{\partial p}{\partial t} - i_w \frac{\mu_w}{k_w}\,\delta(x-x_o)\,\delta(y-y_o) \tag{11}$$

$$\left(\frac{\partial p}{\partial y}\right)_{y=o} = \left(\frac{\partial p}{\partial y}\right)_{y=b} = \left(\frac{\partial p}{\partial x}\right)_{x=a} = \left(p\right)_{x=o} = \left(p\right)_{t=o} = 0 \tag{12}$$

Multiplication and integration* transforms (11) into:

$$+\int_o^a\int_o^b \frac{\partial^2 p}{\partial x^2}\sin\frac{2m-1}{2a}\pi\cos\frac{n\pi y}{b}\,dxdy + \int_o^a\int_o^b \frac{\partial^2 p}{\partial y^2}\sin\frac{2m-1}{2a}\pi x$$

$$\cos\frac{n\pi y}{b}\,dxdy = \frac{1}{\eta}\int_o^a\int_o^b \frac{\partial p}{\partial t}\sin\frac{2m-1}{2a}\pi x\cos\frac{n\pi y}{b}\,dxdy$$

$$-\int_o^a\int_o^b i_w\frac{\mu_w}{k_w}\,\delta(x-x_o)\,\delta(y-y_o)\sin\frac{2m-1}{2a}\pi x\cos\frac{n\pi y}{b}\,dxdy \tag{13}$$

---

*See Reference 11, page 236, for a similar steady state problem.

Integration by parts and using the boundary conditions it is readily obtained:

$$\frac{1}{\eta}\frac{d}{dt}\int_0^a\int_0^b p \sin\frac{2m-1}{2a}\pi\cos\frac{n\pi}{b}y\,dxdy + \left[\left(\frac{n\pi}{b}\right)^2+\left(\frac{2m-1}{2a}\pi\right)^2\right]$$

$$\int_0^a\int_0^b p \sin\frac{2m-1}{2a}\pi\, x\cos\frac{n\pi y}{b}\,dxdy = i_w\,\frac{\mu_w}{r_w}$$

$$\int_0^a\int_0^b \delta(x-x_o)\,\delta(y-y_o)\sin\frac{2m-1}{2a}\pi x\cos\frac{n\pi y}{b}\,dxdy \qquad (14)$$

Using the property of Dirac delta function, namely

$$\int_{-\infty}^{\infty} f(x)\,\delta(x_o) = f(x_o)$$

And recalling that the double integral can be interpreted as a successive Fourier finite, cosine, sine[11, 13] transform

$$\bar{\bar{p}}_{sc} = \int_0^a\int_0^b p \sin\frac{2m-1}{2a}\pi\, x\cos\frac{n\pi y}{b}\,dxdy$$

The Equation (14) transforms into Equation (15):

$$\frac{d\bar{\bar{p}}_{sc}}{dt} + \eta\left[\left(\frac{n\pi}{b}\right)^2+\left(\frac{2m-1}{2a}\pi\right)^2\right]\bar{\bar{p}}_{sc}$$

$$= \eta\, i_w\,\frac{\mu_w}{k_w}\sin\frac{2m-1}{2a}\pi\, x_o\cos\frac{n\pi}{b}y_o \qquad (15)$$

The general solution for the ordinary differential Equation (15) is:

$$\bar{\bar{p}}_{sc} = i_w \frac{\mu_w}{k_w} \quad \frac{\sin\left(\frac{2m-1}{2a}\ \pi x_o\right) \cos\left(\frac{n\pi}{b}\ y_o\right)}{\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{2m-1}{2a}\ \pi\right)^2\right]}$$

$$\left[1 - e^{-\eta\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{2m-1}{2a}\ \pi\right)^2\right] t}\right] \tag{16}$$

The inversion is readily written as:

$$p(x, y, t, x_o, y_o) = \frac{4}{ab} \quad i_w \quad \frac{\mu_w}{k_w} \left\{ \frac{1}{2} \sum_{m=1}^{\infty} \frac{\sin \frac{2m-1}{2a}\ \pi x_o}{\left(\frac{2m-1}{2a}\ \pi\right)^2}\right.$$

$$\left[1 - e^{-\eta\left(\frac{2m-1}{2a}\ \pi\right)^2 t}\right] \sin \frac{2m-1}{2a}\ \pi\, x$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin\left(\frac{2m-1}{2a}\ \pi x_o\right) \cos\left(\frac{n\pi}{b}\ y_o\right)}{\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{2m-1}{2a}\ \pi\right)^2\right]}$$

$$\left.\left[1 - e^{-\eta\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{2m-1}{2a}\ \pi\right)^2\right] t}\right] \sin \frac{2m-1}{2a}\ \pi\, x \cos \frac{n\pi}{b}\ y \right\} \tag{17}$$

Derivation with respect to x, readily gives formula (3) on page 5. Integration with respect to y, and noting that all terms with $n \neq 0$, drops, gives:

$$e_w = \int_o^b \frac{4i_w}{\pi b} \sum_{m=1} \frac{\sin \frac{2m-1}{2a} \pi x_o}{2m-1} \left[1 - e^{-\eta \left(\frac{2m-1}{2a} \pi \right)^2 t} \right] dy \tag{18}$$

Carrying out the integration and recalling that:

$$\sum_{m=1}^{\infty} \frac{\sin (2m-1)\, x}{2m-1} = \frac{\pi}{4}$$

We obtain:

$$e_w = i_w \left[1 - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin \left(\frac{2m-1}{2a} \pi x_o \right)}{2m-1} e^{-\eta \left(\frac{2m-1}{2a} \pi \right)^2 t} \right] \tag{19}$$

Considering that

$$W_e \quad \int_o^t e_w \, dt$$

And recalling that:

$$\sum_{m=1}^{\infty} \frac{\sin(2m-1)x}{(2m-1)^3} = \frac{\pi}{8} \; x(\pi - x)$$

We can write

$$W_e = W_i \left[1 - \frac{ax_o}{\eta\, t}\left(1 - \frac{x_o}{2a}\right) + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin \frac{2m-1}{2a} \pi x_o}{\eta \left(\frac{2m-1}{2a} \pi \right)^2 t(2m-1)} \right.$$

$$\left. e^{-\eta \left(\frac{2m-1}{2a} \pi \right)^2 t} \right] \qquad (20)$$

2. Linear, Closed, Constant Terminal Rate, One Point Source Aquifer.

Similar to Case 1, we can write:

$$p(x, y, t, x_o, y_o) = c\, \frac{4}{ab}\, i_w\, \frac{\mu_w}{k_w} \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} \frac{\cos \frac{m\pi x_o}{a} \cos \frac{n\pi y_o}{b}}{\left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]}$$

$$x \left[1 - e^{-\left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] t} \right] \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \qquad (21)$$

where

$$c = \begin{cases} 1 \text{ for } n \text{ and } m \neq 0 \\ 1/2 \text{ for } n \text{ or } m = o \end{cases}$$

3. Linear, Constant External Pressure, Constant Terminal Pressure, One Point Source Aquifer.

$$p(x, y, t, x_o, y_o) = c \frac{4}{ab} i_w \frac{\mu_w}{k_w} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{\sin \frac{m\pi x_o}{a} \cos \frac{n\pi y_o}{b}}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}$$

$$x \left[1 - e^{-\eta\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]t}\right] \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{22}$$

where

$$c = \begin{cases} 1 \text{ for } n \neq 0 \\ 1/2 \text{ for } n=o \end{cases}$$

4. Linear, Infinite, Constant Terminal Pressure, One Point Source Aquifer.

$$p(x, y, t, x_o, y_o) = c \frac{4}{b\pi} i_w \frac{\mu_w}{k_w} \sum_{n=o}^{\infty} \int_o^{\infty} \frac{\cos \frac{n\pi y_o}{b} \sin (px_o)}{\eta\left[\left(\frac{n\pi}{b}\right)^2 + p^2\right]}$$

$$\left[1 - e^{-\eta\left[\left(\frac{n\pi}{b}\right)^2 + p^2\right] t}\right] \cos \frac{n\pi y}{b} \sin (px)\, dp \tag{23}$$

where

$$c = \begin{cases} 1 \text{ for } n \neq 0 \\ 1/2 \text{ for } n = o \end{cases}$$

5. Linear, Closed, Constant Terminal Pressure, Line Source Aquifer.*

$$p(x, t, x_o) = \frac{2}{a}\, i_w \frac{\mu_w}{k_w} \sum_{n=1}^{\infty} \frac{\sin \frac{2n-1}{2a} \pi x_o}{\eta \left(\frac{2n-1}{2a} \pi\right)^2} \left[1 - e^{-\eta \left(\frac{2n-1}{2a}\pi\right)^2 t}\right] \sin \frac{2n-1}{2a} \pi x \qquad (24)$$

Systems of different boundary conditions as well as three dimensional problems based on similar hypothesis can be treated by this method.

*See Reference 10, page 300, for another approach for deriving the solution of a similar problem giving the same results.

APPENDIX B

The pressure distribution in a radial system with homogeneous Dirichlet boundary conditions at external and internal boundaries, with a point source, of unit strength per radian, is obtained by solving the following boundary value problem (See Figure 7):

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = \frac{1}{\eta}\frac{\partial p}{\partial t} - \frac{\mu_w}{k_w}\,\delta(r - r_o)\,\delta(\theta - \theta_o) \tag{25}$$

$$p(r_e, \theta, t) = p(r_w, \theta, t) = 0 \tag{26}$$

$$p(r, \theta, 0) = 0 \tag{27}$$

The source can be represented by:

$$-\frac{k_w}{\mu_w}\, p_s = \ln\frac{r'}{r_e} = \frac{1}{2}\ln\left[\frac{r_o^2}{r_e^2} - \frac{2 r_o r}{r_e^2}\cos(\theta_o - \theta) + \frac{r^2}{r_e^2}\right] \tag{28}$$

An expression containing Equation (28) and which is a solution of Equation (25) is:

$$-\frac{k_w}{\mu_w}\, p_a = \frac{1}{2}\left\{ \ln\left[\frac{r_o^2}{r_e^2} - \frac{2 r_o r}{r_e^2}\cos(\theta_o - \theta) + \frac{r^2}{r_e^2}\right]\right.$$

$$+ \ln \left[1 - \frac{2r_o r}{r_e^2} \cos(\theta_o - \theta) + \frac{r_o^2 r^2}{r_e^4} \right] \Bigg\} \tag{29}$$

Observing that Equation (28) can be expanded in Fourier series as follows:

$$-\frac{k_w}{\mu_w} p_a = \begin{cases} \ln \frac{r_e}{r_o} + \sum_1^{\infty} \frac{1}{n} \left[\left(\frac{r_o r}{r_e^2} \right)^n + \left(\frac{r}{r_o} \right)^n \right] \cos n(\theta_o - \theta) \text{ for } r < r_o \\ \\ \ln \frac{r_e}{r} + \sum_1^{\infty} \frac{1}{n} \left[\left(\frac{r_o r}{r_e^2} \right)^n + \left(\frac{r_o}{r} \right)^n \right] \cos n(\theta_o - \theta) \text{ for } r > r_o \end{cases}$$

Then it is easily found that a particular solution of Equation (25), containing Equation (28) and satisfying Equation (26) is:

$$-\frac{k_w}{\mu_w} p_p = \frac{k_w}{\mu_w} p_a + b_o + a_o \ln r + \sum_1^{\infty} (b_n r^n + c_n r^{-n}) \cos n\theta \tag{30}$$

where

$$a_o = - \frac{\ln \frac{r_e}{r_o}}{\ln \frac{r_e}{r_w}}$$

$$b_o = -a_o \ln r_e$$

$$b_n = \frac{1}{n} \frac{\left[\left(\frac{r_o r_w}{r_e^2}\right)^n + \left(\frac{r_w}{r_o}\right)^n\right] r_w^{\,n} - 2 r_o^{\,n}}{r_w^{\,2n} - r_e^{\,2n}}$$

$$c_n = \left[\frac{1}{n} \quad 2 \left(\frac{r_o}{r_e}\right)^n - b_n r_e^{\,n}\right] r_e^{\,n}$$

The general solution, consequently, will be:

$$-\frac{k_w}{\mu_w} p = \frac{k_w}{\mu_w} p_p + \sum_{n=o}^{\infty} \sum_{1}^{\infty} A_{mn} Z_{nn}(mr) \cos n\,(\theta_o - \theta)\, e^{-\ m^2 t} \tag{31}$$

Where we define:

$$Z_{p,n}(mr) = J_p(mr_w)\ Y_n(mr) - Y_p(mr_w) J_n(mr_w)$$

And m will be determined such that:

$$Z_{n,n}(mr_e) = 0 \tag{32}$$

According to the theory of Fourier-Bessel series the Equation (27) is satisfied by:

$$A_{mo} = \frac{I_{10}+I_{20}+I_{30}}{D_o} \; ; \quad A_{m,n} = \frac{I_{1n}+I_{2n}+I_{3n}}{D_n}$$

where

$$I_{10} = \int_{r_w}^{r_o} r \ln \frac{r_e}{r_o} \, Z_{o,o}(mr) \, dr$$

$$I_{20} = \int_{r_o}^{r_e} \ln \frac{r_e}{r} \, r \, Z_{o,o}(mr) \, dr$$

$$I_{30} = -\int_{r_w}^{r_e} (b_o + a_o \ln r) \, r Z_{o,o}(mr) \, dr$$

$$D_o = \int_{r_w}^{r_e} r Z_{o\ o}^{\ 2}(mr) \, dr$$

$$I_{1n} = \int_{r_w}^{r_o} \frac{1}{n} \left[\left(\frac{r_o r}{r_e^2} \right)^n + \left(\frac{r}{r_o} \right)^n \right] r Z_{n,n}(mr) \, dr$$

$$I_{2n} = \int_{r_o}^{r_e} \frac{1}{n} \left[\left(\frac{r_o r}{r_e^2} \right)^n + \left(\frac{r_o}{r} \right)^n \right] r Z_{n,n}(mr) dr$$

$$I_{3n} = - \int_{r_w}^{r_e} (b_n r^n + c_n r^{-n}) r Z_{n,n}(mr) dr$$

$$D_n = \int_{r_w}^{r_e} r Z^2_{n,n}(mr)\ dr$$

Using the recurrence relations for Bessel's functions, and the boundary conditions, it is readily verified that:

$$A_{mo} = 2 \frac{Z_{o,o}(mr_o)}{\left[r_e Z'_{o,o}(mr_e)\right]^2 - \left[r_w Z'_{o,o}(mr_w)\right]^2}$$

$$A_{mn} = 4 \frac{Z_{n,n}(mr_o)}{\left[r_e Z'_{n,n}(mr_e)\right]^2 - \left[r_w Z'_{n,n}(mr_w)\right]^2} \tag{33}$$

Consequently, the pressure is given by Equation (31), m determined by Equation (32) and A_{mn} by Equation (33), which, after some manipulation is identified with Equation (8).

The rate of influx is obtained by recalling that

$$\int_o^{2\pi} \left(\frac{\partial p_a}{\partial r} \right)_{r=r_w} d\theta = 0$$

and that all terms containing cos $(\theta_o - \theta)$ gives a null integral.

It remains only:

$$e_w = \frac{k_w}{\mu_w} \int_o^{2\pi} \frac{\partial}{\partial r} \left(b_o + a_o \ln r + \sum_1^{\infty} A_{o,o} Z_{o,o}(mr) e^{-\eta m^2 t}\right) r_w d\theta \quad r = r_w$$

From this Equation (9) is readily obtained.

The cumulative influx

$$W_e = \int_o^t e_w \, dt$$

Which gives Equation (10).

```
C                              APPENDIX C
C
$EXECUTE       AGGIE
C
C      *******   EUCLIDES JOSE BONET   **********
C      RATIO INFLUX RATE INJECTED WATER IN RADIAL AQUIFER WITH
C      CONSTANT PRESSURE IN RE AND RW
C
       DIMENSION EIV(10,20),TI(20),RE(10),RO(10)
C      THE EIGENVALUES ARE TAKEN FROM PG.415 HANDBOOK OF MATH.FUNCTIONS
       READ (5,2) (TI(J), J=1,20)
       READ (5,3) (RO(K),K=1,10)
       READ (5,4) ((EIV(N,I),I=1,5),RE(N),N=1,5)
       EXTERNAL BZJL3
       EXTERNAL BZJG3
       EXTERNAL BOJL3
       EXTERNAL BOJG3
       EXTERNAL BZYL3
       EXTERNAL BZYG3
       EXTERNAL BOYL3
       EXTERNAL BOYG3
       EXTERNAL EIGEN
       DO 115 K = 1,10
       DO 115 N = 1,5
       DO 115 J = 1,20
       I = 1
       WLOST = ALOG(RE(N)/RO(K))/ALOG(RE(N))
    10 IF (I - 5) 25,25,15
    15 EM = I
       EIG = EIGEN(EM,RE(N))
    16 CONTINUE
       GO TO 30
    25 EIG = EIV(N,I)
    30 IF (EIG - 3.) 35,35,40
    35 BZX = BZJL3(EIG)
```

```
36 CONTINUE
   YZX = BZYL3(EIG)
32 CONTINUE
   BOX = BOJL3(EIG)
37 CONTINUE
   YOX = BOYL3(EIG)
38 CONTINUE
   GO TO 45
40 BZX = BZJG3(EIG)
41 CONTINUE
   BOX = BOJG3(EIG)
42 CONTINUE
   YOX = BOYG3(EIG)
43 CONTINUE
   YZX = BZYG3(EIG)
44 CONTINUE
45 AF = EIG*RO(K)
   IF (AF-3.) 50,50,55
50 BZXRO = BZJL3(AF)
51 CONTINUE
   GO TO 60
55 BZXRO = BZJG3(AF)
56 CONTINUE
60 AF = EIG*RE(N)
   IF (AF-3.) 65,65,70
65 BOXRE = BOJL3(AF)
66 CONTINUE
   GO TO 75
70 BOXRE = BOJG3(AF)
71 CONTINUE
75 AS = EIG*RO(K)
   IF (AS - 3.) 80,80,85
80 YZXRO = BZYL3(AS)
81 CONTINUE
   GO TO 90
```

```
 85 YZXRO = BZYG3(AS)
 86 CONTINUE
 90 AS = EIG*RE(N)
    IF (AS - 3.) 95,95,100
 95 YOXRE = BOYL3(AS)
 96 CONTINUE
    GO TO 105
100 YOXRE = BOYG3(AS)
101 CONTINUE
105 ANUM=(BZX*YZXRO-YZX*BZXRO)*(BZX*YOX-YZX*BOX)*F/(EIG*TI(J))
    ADEN=(EIG*RE(N)*(BZX*YOXRE - YZX*BOXRE))**2 - (EIG*(BZX*YOX -
   1YZX*BOX))**2
    PLOST =  2.*ANUM/ADEN
    WLOST = WLOST + PLOST
    IF (ABS(PLOST) - 1.E-4) 108,110,110
110 I = I + 1
    GO TO 10
108 WRITE (6,9) RO(K),RE(N),TI(J),WLOST
115 CONTINUE
  2 FORMAT (1F10.0)
  3 FORMAT (1F10.0)
  4 FORMAT (6F10.0)
  9 FORMAT (3F12.4,1E20.6)
    END
C
    FUNCTION EIGEN(EM,RE)
    AL = EM*3.141593/(RE-1.)
    PE = -1./(8.*RE)
    QE = 4.*(-1.)*(-25.)*(RE**3-1.)/(3.*(8.*RE)**3*(RE-1.))
    RA = (32.*(-1.)*(+1073.)*(RE**5-1.))/(5.*(8.*RE)**5*(RE-1.))
    DA = PE/AL
    DB = (QE-PE**2)/(AL**3)
    DC = (RA-4.*PE*QE+2.*PE**3)/(AL**5)
    EIGEN = AL + DA + DB + DC
    RETURN
```

```
      END
C
      FUNCTION BZJL3(X)
C     POLYNOMIAL APPROXIMATION ACCORDING FORMULA 9.4.1 P.369 HANBOOK
      BZJL3 = 1.-2.25000*(X/3.)**2+ 1.26562*(X/3.)**4 - 0.31639*(X/3.)**
     16 + 0.04445*(X/3.)**8 - 0.00394*(X/3.)**10 + 0.00021*(X/3.)**12
      RETURN
      END
C
      FUNCTION BZJG3(U)
C     FORMULA 9.4.3 HANDBOOK
      THETO = U - 0.78540 - 0.04166*(3./U) - 0.00004*(3./U)**2 + 0.00263
     1*(3./U)**3 - 0.00054*(3./U)**4 - 0.00029*(3./U)**5 + 0.00014*
     2(3./U)**6
      FO = 0.79788 - 0.00553*(3./U)**2- 0.00010*(3./U)**3 + 0.00137*
     1(3./U)**4 - 0.00072*(3./U)**5 + 0.00014*(3./U)**6
      BZJG3 = FO*COS(THETO)/SQRT(U)
      RETURN
      END
C
      FUNCTION BZYG3(U)
      THETO = U - 0.78540 - 0.04166*(3./U) - 0.00004*(3./U)**2 + 0.00263
     1*(3./U)**3 - 0.00054*(3./U)**4 - 0.00029*(3./U)**5 + 0.00014*
     2(3./U)**6
      FO = 0.79788 - 0.00553*(3./U)**2- 0.00010*(3./U)**3 + 0.00137*
     1(3./U)**4 - 0.00072*(3./U)**5 + 0.00014*(3./U)**6
      BZYG3 = FO*SIN(THETO)/SQRT(U)
      RETURN
      END
C
      FUNCTION BZYL3(V)
      EXTERNAL BZJL3
      AUXZ3 = BZJL3(V)
  201 CONTINUE
      BZYL3 = (2./3.141593)*ALOG(V/2.)*AUXZ3    + 0.36747 + 0.60559*
```

```
     1(V/3.)**2 - 0.74350*(V/3.)**4 + 0.25300*(V/3.)**6- 0.04261*(V/3.)
     2**8 + 0.00427*(V/3.)**10 - 0.00025*(V/3.)**12
      RETURN
      END
C
      FUNCTION BOJL3(D)
      BOJL3 =D*(1./2. - 0.56250*(D/3.)**2 + 0.21094*(D/3.)**4 - 0.03954*
     1(D/3.)**6 + 0.00443*(D/3.)**8 - 0.00032*(D/3.)**10)
      RETURN
      END
C
      FUNCTION BOYL3(B)
      EXTERNAL BOJL3
      AUX03 = BOJL3(B)
  202 CONTINUE
      BOYL3 = ((2./3.141593)*B*ALOG(B/2.)*AUX03   - 0.63662 + 0.22121*
     1(B/3.)**2 + 2.16827*(B/3.)**4 - 1.31648*(B/3.)**6 + 0.31240*
     2(B/3.)**8 - 0.04010*(B/3.)**10 + 0.00279*(B/3.)**12)/B
      RETURN
      END
C
      FUNCTION BOJG3(C)
      THET1 = C - 2.35619 +  0.12500*(3./C) + 0.00006*(3./C)**3 +
     10.00074*(3./C)**4 + 0.00080*(3./C)**5 - 0.00029*(3./C)**6
      F1 = 0.79788 + 0.01659*(3./C)**2 + 0.00017*(3./C)**3 - 0.00250*
     1(3./C)**4 + 0.00113*(3./C)**5 - 0.00020*(3./C)**6
      BOJG3 = F1*COS(THET1)/SQRT(C)
      RETURN
      END
C
      FUNCTION BOYG3(C)
      THET1 = C - 2.35619 +  0.12500*(3./C) + 0.00006*(3./C)**3 +
     10.00074*(3./C)**4 + 0.00080*(3./C)**5 - 0.00029*(3./C)**6
      F1 = 0.79788 + 0.01659*(3./C)**2 + 0.00017*(3./C)**3 - 0.00250*
     1(3./C)**4 + 0.00113*(3./C)**5 - 0.00020*(3./C)**6
```

```
BOYG3 = F1*SIN(THET1)/SQRT(C)
RETURN
END
```

```
C     PRESSURE CALCULATION IN LINEAR CLOSED AQUIFER CONSTANT TERMINAL
C     PRESSURE BY SUPERPOSITION OF CONTINUOUS POINT SOURCE
      DIMENSION X(20),TI(10),Y(10),XOA(5),YOA(5)
      READ (5,4) (Y(I),I=1,7)
      READ (5,5) (YOA(L),L=1,2)
      READ (5,6) (X(J),J=1,19)
      READ (5,7) (TI(K),K=1,5)
      BA = 0.02
      XOA(L) = 0.05
      DO 100 MM = 1,2
      DO 100 K = 1,3
      DO 100 J = 1,19
      DO 100 I = 1,7
      XA = XOA(L)
      PD = 0.0
      N = 1
   10 DD = ((-1.)**(N+1)*XA  - X(J))**2
      M = 2
      MN = 2
   18 EL = 0.
      DS = ((2.*EL*BA)*(-1.)**MN+(-1.)**(M)*YOA(MM)-Y(I))**2
      DDS = DD + DS
      AR = DDS/(4.*TI(K))
      IF (AR-5.) 30,30,50
   30 C = 1.
      B = -1.
      DEN = 1.
      FEI = ALOG(AR) + 0.5772
   15 PEI = B*(AR**C)/DEN
      FEI = FEI + PEI
      IF (ABS(PEI)-1.E-4) 35,35,25
   25 C = C + 1.
      DEN = C*C*DEN/(C-1.)
      B = -B
      GO TO 15
```

```
35 PD = PD - ((-1.)**(N+1))*FEI/(4.*3.141593)
19 EL = EL + 1.
   DO 40 M = 1,2
   DS = ((2.*EL*BA)*(-1.)**MN+(-1.)**(M)*YOA(MM)-Y(I))**2
   DDS = DD + DS
   AR = DDS/(4.*TI(K))
   IF (AR-5.) 31,31,50
31 C = 1.
   B = -1.
   DEN = 1.
   FEI = ALOG(AR) + 0.5772
16 PEI = B*(AR**C)/DEN
   FEI = FEI + PEI
   IF (ABS(PEI)-1.E-4) 36,36,26
26 C = C + 1.
   DEN = C*C*DEN/(C-1.)
   B = -B
   GO TO 16
36 PD = PD - ((-1.)**(N+1))*FEI/(4.*3.141593)
40 CONTINUE
   GO TO 19
50 IF (MN-1)70,70,60
60 MN = 1
   M = 1
   GO TO 18
70 IF (N-2)80,80,85
80 IF (XA-XOA(L)) 81,81,82
81 XA = 2.-2.*XOA(L)
   GO TO 10
82 N = 4
   XA = XOA(L)
   GO TO 10
85 IF (XA-XOA(L))90,86,86
86 XA = -2.-2.*XOA(L)
   GO TO 10
```

```
 90 WRITE (6,9) PD,XOA(L),YOA(MM),BA,TI(K),X(J),Y(I)
100 CONTINUE
  4 FORMAT (1F10.0)
  5 FORMAT (1F10.0)
  6 FORMAT (1F10.0)
  7 FORMAT (1F10.0)
  9 FORMAT (1E20.6,6F10.5)
    END
```