ANTI-DE SITTER BLACK HOLES IN SUPERGRAVITY

A Dissertation

by

ZHIWEI CHONG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2006

Major Subject: Physics

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Approved by:

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ABSTRACT

Anti-de Sitter Black Holes in Supergravity. (August 2006)

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Chair of Advisory Committee: Dr. Christopher Pope

In this dissertation, we systematically construct non-extremal charged rotating anti-de Sitter black hole solutions in four, five and seven dimensions. In four dimensions, we first obtain the rotating Kerr-Taub-NUT metric with four independent charges, as solutions of $\mathcal{N} = 2$ supergravity coupled to three abelian vector multiplets by the solution generating technique. Then we generalise the four-dimensional rotating solutions to the solutions of gauged $\mathcal{N} = 4$ supergravity with charges set pairwise equal. In five dimensions, the most general charged rotating black hole solution has three charge and two rotation parameters. We obtain several special cases of the general solution. To be specific, we obtain the first example of a non-extremal rotating black hole solution with two independent rotation parameters, which has two charge parameters set equal and the third vanishing. In another example, we obtain the non-extremal charged rotating black hole solution with three charge parameters set equal and non-equal rotation parameters. We are also able to construct the single-charge solution with two independent rotation parameters. In seven dimensions, we obtain the solution for non-extremal charged rotating black holes in gauged supergravity, in the case where the three rotation parameters are set equal. There are two independent charges, corresponding to gauge fields in the $U(1) \times U(1)$ abelian subgroup of the $SO(5)$ gauge group.

To My Parents

ACKNOWLEDGMENTS

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CHAPTER I

INTRODUCTION

A.   Fundamental Problems in Black Hole Physics

A black hole is an object in the theory of general relativity, that represents the end-point of gravitational collapse. It is "black" because nothing, not even light, can escape from it. It is not merely a mathematical solution in the theory of general relativity, it is an object of study in experimental astrophysics as well. The black hole as a classical object has already been very fascinating; the "baldness" of black holes or "no-hair" theorem of four dimensional black holes, the separability of the Klein-Gordon and Dirac equations in the Kerr metric, etc. It became even more fascinating after Hawking discovered in 1974 that a black hole is not really black; it can radiate. Since then, the black hole has become a fundamental object of study in theoretical physics. There are several puzzles, which are so fundamental and important that it is generally believed that a proper understanding of these will shed new light on the understanding of both gravitational and quantum physics, leading to the hope that one day these two fundamental theories can be reconciled to produce a consistent unified theory of matter and quantum gravity.

Before the discovery of Hawking radiation, there was already a puzzle called the entropy problem. When matter is thrown into a black hole, the only effect is an increase in the mass and the area of the event horizon of the black hole, after it settles down to a stationary state. If one does not associate an entropy with a black hole, one will risk the violation of the second law of thermodynamics. Bekenstein

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˙The journal model is *Nuclear Physics B.*

solved this problem by attributing an entropy to a black hole, proportional to the area of the event horizon, so that the total entropy of the matter and the black hole does not decrease. This is not the end of the story. Now that there is an entropy associated with a black hole, it is natural to ask what the microscopic foundation of the black hole is, i.e. how to understand the entropy from the point of view of statistical mechanics. This is the core of the entropy problem. A four-dimensional black hole is characterised by just its mass, charge and angular momentum. This is the so called "no-hair theorem". Now the problem is what are the microscopic states associated with the black hole entropy? This problem was answered ten years ago in the framework of string theory, for certain extremal and near extremal black holes which will be discussed later.

Given a black hole with fixed entropy and energy, there is a temperature associated with it, assuming thermodynamics is valid. If a black hole absorbs matter with a given absorption cross section, it must also radiate with a rate proportional to that absorption cross section to achieve thermal equilibrium. This is not possible for a classical black hole. Hawking showed that black hole radiation is possible, by taking into account the vacuum fluctuation of the matter fields in the background of the black hole. One of the particles created by the fluctuation at the event horizon falls toward the inside of the horizon, while the other particle escapes to infinity, as is observed as radiation. This is the Hawking radiation.

The vacuum fluctuation of the matter fields in the black hole background nicely explains the temperature of the black hole and the expected radiation from it. However, this introduces another problem, i.e. the information paradox. A black hole continues evaporating through Hawking radiation until we are left only with the radiation. In this process, whatever initially makes the black hole is transformed into the radiation, which retains no information about where it comes from. This violates

the unitarity of quantum theory and is called the information paradox.

Black hole physics nicely knits together the fundamental theories, i.e. general relativity, quantum physics and thermodynamics, and raises intriguing paradoxes. Progress in solving these problems will yield new insights in the understanding of Nature at a fundamental level. String theory is a promising theory of quantum gravity and a candidate for the unification of all the known interactions, so in principle the problems related with black holes have a good chance to be explained in the framework of string theory. Indeed string theory partially meets this expectation.

B.   Counting Black Hole Microstates in String Theory

String theory has successfully answered the question of the origin of entropy for certain * black holes. It reproduces the black hole entropy by counting the number of the string states that are in one to one correspondence with the microstates of a black hole. [1, 2, 3] are excellent reviews †. Note that it is still not known what the precise nature of the microstates of a black hole is, but through the one to one correspondence, the number of black hole microstates is indirectly counted as the number of the string states. Basically, the string states and black hole involved in the discussion are two descriptions of the same object, in two distinct limits. On the one hand, we can count the number of string states, while on the other hand, we can calculate the entropy of the black hole. The outcome is that both calculations match precisely for certain class of black holes. To achieve this, three problems need to be answered. The first question is what string states are involved and how to count the number. The second is what kind of black hole is involved in the discussion. The

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*It will be shown below that the black hole involved is supersymmetric and multiply charged in supergravity theory.

†The author benefited greatly from these reviews in preparation for Chapter I.

third is what guarantees the one to one correspondence between the string states and the black hole microstates from one limit to the other.

To make the presentation concrete, it may be helpful to digress a little into the concept of string coupling g and two kinds of charges in string theory, i.e. NS-NS and R-R charges. When a string is quantized in flat spacetime, not merely is the dimension of the spacetime is determined, but an infinite tower of states corresponding to the spectrum of particles is created by the oscillations of the string. Within this spectrum, the massless particles are of special interest; the massless spin one particle is the gauge field, the massless spin two particle is the graviton and a massless scalar field called dilaton, whose asymptotic value at infinity determines the string coupling g. This string coupling g plays a central role in understanding the NS-NS and R-R charges (both electric and magnetic) in string theory. It is also important for understanding the correspondence between black holes and string states. The two kinds of charges in string theory are distinguished by the different ways they couple to the dilaton. This difference in the coupling results in different masses for the solitons which carry each kind of charge. Because string theory lives in higher spacetime dimensions than four, these solitons are higher dimensional extended objects surrounded by horizons, including strings, membranes and p-branes. To be precise, the mass of an extremal soliton with one unit of electric NS-NS charge is of order one, the mass of a soliton with one unit of magnetic NS-NS charge is of order $1/g^2$, while with one unit of R-R charge (either electric or magnetic) is of order $1/g$. So when the string coupling g is very small, the R-R charged solitons and magnetic NS-NS charged solitons are both very massive [2]. These basic facts are crucial in the argument of the correspondence between string states and black holes.

One of the two limits mentioned above is the weak coupling limit, i.e. $g \to 0$. Before we try to answer the first question, we first argue that there does exist a

description in this weak coupling limit. The gravitational field produced by the charged solitons is proportional to GM, where G is Newton's constant and M is the mass of the solitons discussed in the previous paragraph. Noting the fact that G is proportional to g^2, one sees that the spacetime becomes flat for both the NS-NS and R-R charged solitons in the weak coupling limit $g \to 0$. This shows that there does exit a nonsingular description of these solitons in the weak coupling limit. It turns out that the electrically charged NS-NS solitons are described by perturbative string states, due to the fact that a fundamental string couples with NS-NS charge and thence carries NS-NS charge. For the flat spacetime description of R-R charged solitons, it is a different matter. Strings do not carry R-R charges, because strings do not couple to the potential, but rather to the field strength of an R-R charge. One therefore cannot expect R-R charged solitons to be described by the usual perturbative string states. This is where D-branes enter the picture. It turns out that the appropriate flat spacetime description of the R-R charged solitons is via D-branes. A D-brane is an extended object in string theory, on which open strings can end. The ends of the open strings can move freely on the D-branes and the open string states describe the dynamics of the D-branes. The point is that in the D-brane picture the number of string states with R-R charge can be counted, just as the number of usual perturbative string states can be counted for NS-NS charged solitons.

The other end of the limit is at strong coupling. In this limit the charged solitons are described by the geometry of black holes. Because of Hawking radiation, non-extremal black holes or black branes are not stable, and so most attention has been focused on extreme charged black holes. Furthermore, since for a singly charged black hole the horizon becomes singular * when the extremal limit is taken, one is forced

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*This singularity prevents one from quantizing the solitons.

to consider multiply charged black holes. The reason for this singularity is that the dilaton diverges at the inner horizon and in the extremal limit the outer and inner horizon coalesce. For multiply charged black holes, more fields are turned on, and the dilaton is now regular at the horizon. This regular extremal black hole is what we are interested in, and it can be regarded as the strong coupling limit of bound states of D-branes.

To relate the string-state description at weak coupling with the black hole description at strong coupling, and to make possible the counting of string states to reproduce the black hole entropy, we still lack a mechanism to establish a one to one correspondence between string states and black hole microstates. This is made possible by supersymmetry. With supersymmetry the number of supersymmetric states is topologically invariant, and it does not vary when the string coupling goes from weak to strong. Each supersymmetric state with a given charge in the weak coupling limit is described by a black hole with the same charge at strong coupling. There are many string states carrying the same charge, and all of them are described by the same black hole at strong coupling, so we can identify the number of supersymmetric string states in weak coupling with the number of microstates for the black hole at strong coupling. In this manner string theory succeeds in reproducing the entropy calculated from the metric of the multiply charged black hole, and it accounts for the origin of the black hole microstates[*].

As a concrete example and also the first success in counting black hole microstates, we will sketch how this was achieved by Strominger and Vafa [4] nearly ten years ago. On the one hand, in the strong coupling limit the black hole involved

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[*]We still have no idea about what the nature of black hole microstates is at strong coupling, we only know that they evolve from supersymmetric string states at weak coupling.

is constructed in a five dimensional supergravity theory obtained by compactifying ten dimensional type IIB supergravity on a compact five dimensional manifold. It is charged under two separate $U(1)$ gauge fields such that it is supersymmetric and regular in the extremal limit. On the other hand, at weak coupling the black hole is described by intersecting D-branes wrapped on the compact five dimensional manifold. The resulting 1+1 dimensional gauge field theory describing the intersection is complicated. Nevertheless, the number of supersymmetric states can be counted. Because of supersymmetry we are assured that all these supersymmetric states are described by the same black hole in the strong coupling limit, and that the number of black hole microstates is the same as the number of supersymetric states at weak coupling. It turns out that for large charge the number of supersymmetric states is just the exponential of the black hole entropy. Progress in this vein has also been achieved for near-extremal black holes [5] and a correspondence principle [6] has been established for *any* black hole entropy counting. Although the exact numerical coefficient does not coincide for the calculations on each side, the mass and charge dependence are the same, and the difference in the numerical coefficient is close to unity. Further striking progress is in the reproduction [3] of the Hawking radiation rate and absorption cross-section of black holes in the weak coupling D-brane models. This success in deriving black hole absorption cross-section is not merely a valuable extension beyond the explanation of the black hole entropy, similar comparison made for black three branes plays a significant role in leading to the acclaimed AdS/CFT correspondence, which will be discussed later.

C.   From Black Branes to AdS/CFT Correspondence

Owing to the complexity of the 1+1 dimensional gauge theory for intersecting D-branes, Klebanov [2] investigated a parallel D-brane system, which is simpler than the intersecting D-brane system. There is a better understanding of the gauge theory living on the world volume of the D-brane, leading to the hope of obtaining more examples of counting the microstates for the corresponding black object. The D-brane system under investigation is N coincident D3-branes, on which a $U(N)$ gauge field theory lives. The strong coupling description of this D-brane system is an R-R charged black 3-brane. The entropy at strong coupling is obtained from the metric of the black 3-brane, which is regular everywhere[*]. This entropy is compared with that calculated from the $U(N)$ supersymmetric gauge field theory heated to the same temperature as that of the black brane. It turned out that both calculations agree up to a numerical factor of 4/3,[†] and remarkably they give the same temperature dependence. Deriving the absorption cross-sections for massless particles is a natural step beyond the derivation of the entropy, as was done for black holes. Klebanov et. al. also calculated the absorption cross-sections in the D-brane picture, using the Dirac-Born-Infield action for coincident D-branes coupled to massless bulk fields. Remarkably the two calculations agree exactly in the low energy limit. The absorption cross-section is related to the imaginary part of the two point correlation function in the supersymmetric Yang-Mills theory on the world volume of coincident D-branes. The effort to explain this agreement and other considerations, culminated in the conjecture of the AdS/CFT correspondence, which was formulated explicitly

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[*]This is due to the fact that dilaton is constant everywhere in the black 3-brane solution. All other black brane solutions do not have this nice property.

[†]This constant factor 4/3 was later explained as a prediction of strongly coupled gauge theory at finite temperature.

by Maldacena in [7]. This had led to a new era in the study of string theory.

According to the AdS/CFT correspondence, string theory in AdS space is dual to a conformal field theory on the conformal boundary of AdS space. The study of AdS black holes should therefore help understanding the non-perturbative structure of field theories by studying the supergravity solutions. To be specific, the Hawking-Page phase transition of AdS Schwarzschild black holes can be interpreted as a thermal phase transition from confinement to deconfinement in the dual $D = 4, N = 2$ super Yang-Mills theory, and the thermodynamics of large Schwarzschild black holes in AdS_5 space matches the thermodynamics of the gauge theory, up to a constant [8]. This work revived the interest in studying AdS black holes in the context of AdS/CFT correspondence. Significant progress in this direction was achieved with the construction of the five-dimensional rotating AdS black hole in Einstein gravity [9]. Shortly a generalisation to arbitrary dimensions is given by [10, 11]. These work provide the foundation to the construction of AdS black hole solution in five and seven dimensions, which will be discussed below.

This correspondence should in principle help to understand the problems in both black hole physics and the gauge field theory. However, there is only a qualitative, but not a quantitative, understanding of black hole physics, because one does not know how to compute in strongly coupled gauge theories. For example, one does not know how to reproduce the entropy of a Schwarzschild-AdS black hole from calculations on the gauge field theory side. One way to circumvent this problem is to look at supersymmetric black holes, because with supersymmetry one may go safely from the weakly coupled theory to the strongly coupled theory*. This is the direct motivation

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*To be more precise, black holes preserving some amount of supersymmetry correspond to an expansion around nonzero vacuum expectation values of certain CFT operators.

for seeking further supersymetric black hole solutions in gauged supergravity. Because the AdS_5/CFT_4 correspondence is better understood than AdS_4/CFT_3, much work was focused on five-dimensional supersymmetric black holes. In this line, [12] obtained BPS black holes in $U(1)$ gauged five-dimensional supergravity by solving the Killing spinor equations, but the solution has a naked singularity. To avoid this problem one can either look for *rotating* supersymmetric black holes, or for non-extremal solutions. The rotating supersymmetric solution found in [13] does have a regular horizon instead of the naked singularity of the static solution, but it has closed time-like curves(CTC's) outside the horizon. Significant progress in the construction of supersymmetric black holes was made by Gutowski and Reall [14, 15], who obtained a regular supersymmetric black hole without CTC in five dimensional gauged supergravity, using a method related to the so called algebraic Killing spinor*. This work triggered an interest in finding non-extremal black holes in five dimensional gauged supergravity.

D.   AdS Black Hole Solutions in Supergravity

After collecting higher dimensional black hole solutions in both asymptotically flat and anti de Sitter spacetimes in Chapter II, We study four dimensional non-extremal charged Kerr-Taub-NUT solution in ungauged and gauged supergravities [16]. In the ungauged case, we obtain Kerr-Taub-NUT solution with four independent charges, as solutions of $\mathcal{N} = 2$ supergravity coupled to three abelian vector multiplets. This is done by reducing the theory along the time direction to three dimensions, where

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*In this approach one assumes the existence of Killing spinor and then explore the constraints imposed on the geometry by supersymmetry. It is powerful, and exact solutions can be obtained. This approach also helps in finding supersymmetric black ring solutions, which have horizons of the topology $S^2 \times S^1$. For new black hole solutions in higher dimensions, see the recent review by Horowitz [17].

it has an $O(4,4)$ global symmetry. Applied to the reduction of the uncharged Kerr metric, $O(1,1)^4 \subset O(4,4)$ transformations generate new solutions that correspond, after lifting back to four dimensions, to the introduction of four independent electromagnetic charges. In the case where these charges are set pairwise equal, we then generalise the four-dimensional rotating Kerr-Taub-NUT to solutions of gauged $\mathcal{N} = 4$ supergravity, with mass, angular momentum and two independent electromagnetic charges. The dilaton and axion fields are non-constant. The solutions in gauged supergravity provide new gravitational backgrounds for a further study of the $\text{AdS}_4/\text{CFT}_3$ correspondence at non-zero temperature.

Generally black holes in five dimensional gauged supergravity can have two independent angular momenta and three independent charges. However, it is too difficult to find the most general solution, and in fact this still remains an open problem. There has, however, been much progress in the past year toward our final goal. The first progress in obtaining non-extremal AdS black holes with three equal charges and equal angular momenta was made by Cvetic, Lu and Pope in [18]. Shortly after, it was generalized to a solution with three independent charges [19]. It was a challenge to find solutions with two independent angular momenta, because a solution with two rotations depends intrinsically on two coordinate variables rather than one, and so the equations of motion are partial differential equations, which makes the problem much more involved. The first example with two independent angular momenta was discovered in [20], which has three charges with a certain constraint among them. Major progress was made in [21], where a solution with two rotations and three equal charges was obtained. This solution makes possible the comparison between the entropy of supersymmetric five-dimensional AdS black holes and that calculated from the counting of microstates in the D-brane models involving giant gravitons in the very recent paper [22]. In Chapter IV, we will present the details in constructing

these five-dimensional charged rotating AdS black hole solutions.

In Chapter V, we obtain the solution for non-extremal charged rotating black holes in seven-dimensional gauged supergravity [23], in the case where the three rotation parameters are set equal. There are two independent charges, corresponding to gauge fields in the $U(1) \times U(1)$ abelian subgroup of the $SO(5)$ gauge group. A new feature in these solutions, not seen previously in lower-dimensional examples, is that the first-order "odd-dimensional self-duality" equation for the 4-form field strength plays a non-trivial rôle. Our results are of significance for the AdS_7/CFT_6 correspondence in M-theory. We conclude this dissertation in Chapter VI.

CHAPTER II

SOME PRELIMINARIES

A.   Introduction

The construction of charged AdS black hole solutions in gauged supergravity in various dimensions is based on two recent developments. One is the construction of AdS black hole solution in all dimensions in pure Einstein gravity [9, 10, 11], and the other is the systematic construction of charged asymptotically flat black hole solutions in ungauged supergravity obtained by solution generating techniques [24]. Both of the two developments are based on the construction of higher dimensional asymptotically flat black hole solutions in pure Einstein gravity [25].

In the following, firstly I will present the asymptotically flat black hole solutions in higher dimensions [25]. For the anti-de Sitter black holes, I will present the five-dimensional solution [9], which is the first anti-de Sitter black hole solution in a dimension higher than four, and its generalisation to all dimensions [10]. As for the charged black hole solutions in ungauged supergravities, the four-dimensional Kerr-Taub-NUT metrics are constructed in Chapter III, the seven-dimensional charged rotating black solutions are constructed in Chapter V, and the general five -dimensional charged rotating black hole solutions [26] are presented in Appendix A.

B.   Black Hole Solutions in Higher Dimensions

There are two reasons to study higher dimensional black holes [1]. The first one comes from string theory. String theory, which is widely accepted as a promising candidate for the theory of quantum gravity, predicts that the spacetime dimension is higher than four. The second reason is for the understanding of black hole physics itself.

There are some nice properties such as "no hair theorem" for four-dimensional black holes. It is of interest to see if these properties survive in higher dimensions than four.

Originally the rotating Kerr black hole solution in four dimensions was found in the so called Kerr-Schild form, and so was its generalisation [25] to higher dimensions in asymptotically flat spacetimes. In the Kerr-Schild form the full metric can be expressed as a sum of a flat space-time metric, which is in a special coordinate system, and a vector squared.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{2M}{U}\left(k_\mu dx^\mu\right)^2 , \qquad (2.1)$$

where the vector k_μ is null and geodesic with respect to both the flat space metric $\eta_{\mu\nu}$ and the full metric $g_{\mu\nu}$.

Odd and even dimensions have separate solutions. For even dimensions $D = 2n \geq 4$, in flat spacetime coordinates x_i, y_i, z, t $(i = 1, \cdots n - 1)$, the null vector is

$$k = k_\mu dx^\mu = dt + \sum_{i=1}^{n-1} \frac{r(x_i dx_i + y_i dy_i) + a_i(x_i dy_i - y_i dx_i)}{r^2 + a_i^2} + \frac{zdz}{r} , \qquad (2.2)$$

with

$$U = \frac{1}{r}\left(1 - \sum_{i=1}^{n-1}\frac{a_i^2\left(x_i^2 + y_i^2\right)}{(r^2 + a_i^2)^2}\right)\prod_{j=1}^{n-1}(r^2 + a_j^2) , \qquad (2.3)$$

and

$$\sum_{i=1}^{n-1}\frac{x_i^2 + y_i^2}{r^2 + a_i^2} + \frac{z^2}{r^2} = 1 . \qquad (2.4)$$

where a_i are $(n - 1)$ independent rotation parameters in $(n - 1)$ orthogonal spatial 2-planes.

In odd spacetime dimensions $D = 2n + 1$, there is no z coordinate, and so the terms involving z are omitted. U is then $1/r$ times the right-hand side of equation (2.3).

Cvetic and Youm [26] used these solutions to obtain charged black hole solutions in ungauged supergravity theories in various dimensions by using the solution generating techniques [24]. Recently Gibbons et. al generalised these asymptotically flat black holes in pure Einstein gravity to asymptotically anti-de Sitter black holes in all dimensions [10].

C.   AdS Black Hole Solutions in Higher Dimensions

It is of interest to briefly retrace the history of four-dimensional black hole solution before I present the higher dimensional AdS black hole solutions. The Schwarzschild-like non-rotating black hole in asymptotically Anti de-Sitter space was found shortly after the discovery of Schwarzschild black hole. It took more than fifty years to discover the rotating, asymptotically flat Kerr black holes in four dimensions. The rotating AdS black hole solution in four dimensions was originally discovered as a "pure geometric curiosity" by Carter [27] in an attempt to rederive the Kerr solution based on some assumptions on separability.

Due to AdS/CFT correspondence in string theory, there is more and more interest in higher dimensional AdS black holes. The first example of this kind is the black hole solution with two independent rotation parameters in five dimensions obtained by Hawking et.al [9].

1.   AdS Black Holes in Five Dimensions

The metric for the two parameter five-dimensional rotating black hole is given by

$$
\begin{aligned}
ds^2 \;=\;& -\frac{\Delta}{\rho^2}(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi)^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}(adt - \frac{(r^2+a^2)}{\Xi_a}d\phi)^2 \\
& +\frac{\Delta_\theta \cos^2\theta}{\rho^2}(bdt - \frac{(r^2+b^2)}{\Xi_b}d\psi)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2
\end{aligned}
\tag{2.5}
$$

$$+\frac{(1+r^2g^2)}{r^2\rho^2}\left(abdt-\frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi-\frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2,$$

where

$$
\begin{aligned}
\Delta &= \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2g^2)-2M; \\
\Delta_\theta &= \left(1-a^2g^2\cos^2\theta-b^2g^2\sin^2\theta\right); \\
\rho^2 &= \left(r^2+a^2\cos^2\theta+b^2\sin^2\theta\right); \\
\Xi_a &= (1-a^2g^2); \quad \Xi_b=(1-b^2g^2).
\end{aligned}
\tag{2.6}
$$

After some coordinate transformations, this metric can be put into a very nice form [28]

$$
\begin{aligned}
ds^2 &= (x+y)\left(\frac{dx^2}{4X}+\frac{dy^2}{4Y}\right)-\frac{X}{x(x+y)}(dt+y\,d\phi)^2+\frac{Y}{y(x+y)}(dt-x\,d\phi)^2 \\
&\quad+\frac{a^2b^2}{xy}\left(dt-xyd\chi-(x-y)d\phi\right)^2,
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
X &= (1+g^2x)(x+a^2)(x+b^2)-2Mx \\
&= g^2x^3+(1+(a^2+b^2)g^2)x^2+(a^2+b^2+a^2b^2g^2-2M)x+a^2b^2, \\
Y &= -(1-g^2y)(a^2-y)(b^2-y)+2Ly \\
&= g^2y^3-(1+(a^2+b^2)g^2)y^2+(a^2+b^2+a^2b^2g^2+2L)y-a^2b^2.
\end{aligned}
\tag{2.8}
$$

2.   Anti-de Sitter Black Holes in All Dimensions

Higher dimensional Kerr-de Sitter metrics [10] are obtained by natural generalisations to the previously known $D=4$ and $D=5$ Kerr-de Sitter metrics casted in Kerr-Schild form. Note that odd and even dimensions have separate solutions. We copied the result from [10] in the following.

We begin by introducing $n = [D/2]$ coordinates μ_i, which are subject to the constraint

$$\sum_{i=1}^{n} \mu_i^2 = 1 \,, \tag{2.9}$$

together with $N = [(D-1)/2]$ azimuthal angular coordinates ϕ_i, the radial coordinate r, and the time coordinate t. When the total spacetime dimension D is odd, $D = 2n + 1 = 2N + 1$, there are n azimuthal coordinates ϕ_i, each with period 2π. If D is even, $D = 2n = 2N + 2$, there are only $N = (n-1)$ azimuthal coordinates ϕ_i, which we take to be $(\phi_1, \phi_2, \ldots, \phi_{n-1})$. When D is odd, all the μ_i lie in the interval $0 \le \mu_i \le 1$, whereas when D is even, the μ_i all lie in this interval except μ_n, for which $-1 \le \mu_n \le 1$.

The Kerr-de Sitter metrics ds^2 satisfy the Einstein equation

$$R_{\mu\nu} = (D-1)\,\lambda\, g_{\mu\nu} \,. \tag{2.10}$$

We first make the definitions

$$W \equiv \sum_{i=1}^{n} \frac{\mu_i^2}{1 + \lambda\, a_i^2} \,, \qquad F \equiv \frac{r^2}{1 - \lambda\, r^2} \sum_{i=1}^{n} \frac{\mu_i^2}{r^2 + a_i^2} \,. \tag{2.11}$$

In $D = 2n + 1$ dimensions the Kerr-de Sitter metrics are given by

$$ds^2 = d\bar{s}^2 + \frac{2M}{U}\, (k_\mu\, dx^\mu)^2 \,, \tag{2.12}$$

where the de Sitter metric $d\bar{s}^2$, the null one-form k_μ, and the function U are given by

$$\begin{aligned}
d\bar{s}^2 &= -W\,(1 - \lambda\, r^2)\, dt^2 + F\, dr^2 + \sum_{i=1}^{n} \frac{r^2 + a_i^2}{1 + \lambda\, a_i^2}\, (d\mu_i^2 + \mu_i^2\, d\phi_i^2) \\
&\quad + \frac{\lambda}{W\,(1 - \lambda\, r^2)} \left(\sum_{i=1}^{n} \frac{(r^2 + a_i^2)\, \mu_i\, d\mu_i}{1 + \lambda\, a_i^2} \right)^2 \,,
\end{aligned} \tag{2.13}$$

$$k_\mu\, dx^\mu = W\, dt + F\, dr - \sum_{i=1}^{n} \frac{a_i\, \mu_i^2}{1 + \lambda\, a_i^2}\, d\phi_i \,, \tag{2.14}$$

$$U \;=\; \sum_{i=1}^{n} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{n} (r^2 + a_j^2) \,. \tag{2.15}$$

Note that the null vector corresponding to the null one-form is

$$k^\mu \, \partial_\mu = -\frac{1}{1 - \lambda \, r^2} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} - \sum_{i=1}^{n} \frac{a_i}{r^2 + a_i^2} \frac{\partial}{\partial \phi_i} \,. \tag{2.16}$$

In $D = 2n$ dimensions, the Kerr-de Sitter metrics are given by

$$ds^2 = d\bar{s}^2 + \frac{2M}{U} \, (k_\mu \, dx^\mu)^2 \,, \tag{2.17}$$

where the de Sitter metric $d\bar{s}^2$, the null vector k_μ, and the function U are now given by

$$\begin{aligned}
d\bar{s}^2 \;=\;& -W \, (1 - \lambda \, r^2) \, dt^2 + F \, dr^2 + \sum_{i=1}^{n} \frac{r^2 + a_i^2}{1 + \lambda \, a_i^2} \, d\mu_i^2 + \sum_{i=1}^{n-1} \frac{r^2 + a_i^2}{1 + \lambda \, a_i^2} \, \mu_i^2 \, d\phi_i^2 \\
& + \frac{\lambda}{W \, (1 - \lambda \, r^2)} \Big(\sum_{i=1}^{n} \frac{(r^2 + a_i^2) \, \mu_i \, d\mu_i}{1 + \lambda \, a_i^2} \Big)^2 \,,
\end{aligned} \tag{2.18}$$

$$k_\mu \, dx^\mu \;=\; W \, dt + F \, dr - \sum_{i=1}^{n-1} \frac{a_i \, \mu_i^2}{1 + \lambda \, a_i^2} \, d\phi_i \,, \tag{2.19}$$

$$U \;=\; r \sum_{i=1}^{n} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{n-1} (r^2 + a_j^2) \,. \tag{2.20}$$

In this even-dimensional case, where there is no azimuthal coordinate ϕ_n, there is also no associated rotation parameter, and so $a_n = 0$. In this case k^μ is given by

$$k^\mu \, \partial_\mu = -\frac{1}{1 - \lambda \, r^2} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} - \sum_{i=1}^{n-1} \frac{a_i}{r^2 + a_i^2} \frac{\partial}{\partial \phi_i} \,. \tag{2.21}$$

The vector field k^μ is tangent to a null-geodesic congruence in both even and odd dimensions.

These higher dimensional anti-de Sitter black hole solutions, especially those in five and seven dimensions, together with the charged asymptotically flat black holes in ungauged supergravities, which will be presented in the related chapters, provide

the foundations for the search of charged rotating anti-de Sitter black hole solutions in gauged supergravities.

CHAPTER III

CHARGED KERR-TAUB-NUT SOLUTION IN FOUR DIMENSIONAL GAUGED
AND UNGAUGED SUPERGRAVITIES

A. Introduction *

Rotating charged black hole solutions of ungauged supergravity play an important role in the microscopic study of black hole entropy. It turns out that the microscopic properties can be addressed quantitatively not only for BPS black holes, but also for black holes that are close to extremality. (For a recent review see [3], and references therein.) The prerequisite for these studies is to obtain explicit black hole solutions on the supergravity side. These solutions are typically characterised by multiple electromagnetic charges, in addition to the mass and angular momenta. In four dimensions, such explicit non-extremal solutions, specified by their mass, angular momentum parameter and four charges, were found in [29]. The explicit metric and scalar fields for four-dimensional four-charge rotating black holes were obtained in [29]. Unfortunately, the explicit form of the four gauge potentials for this solution was not given explicitly in [29]. These solutions of ungauged supergravity all provide gravitational backgrounds for the microscopic study of black hole entropy within the string theory framework.

By contrast, black holes in gauged supergravity provide gravitational backgrounds that are relevant to the AdS/CFT correspondence. In particular, such non-extremal solutions play an important role in the study of the dual field theory at non-zero temperature. (An early study of the implications of static charged AdS black holes [30] in

the dual theory was carried out in [31, 32]. For recent related work see [33, 34, 35] and references therein.) However, the explicit form of charged AdS black hole solutions that are also rotating has remained elusive until recently. In four dimensions, there should exist rotating black hole solutions in gauged $\mathcal{N} = 8$ supergravity with four independent electromagnetic charges. Until now, the only known solutions of this type were the Kerr-Newman-AdS black holes [27, 36], which correspond to setting the four electromagnetic charges equal.

One of the purposes of this chapter is first to construct the complete and explicit form of the general rotating black holes of four-dimensional *ungauged* supergravity, with four independent electromagnetic charges. They can be viewed as solutions in ungauged $\mathcal{N} = 2$ supergravity coupled to three vector multiplets, which in turn can be embedded in $\mathcal{N} = 8$ maximal supergravity. We employ a solution-generating technique in which the $\mathcal{N} = 2$ theory is reduced to three dimensions on the time direction, where it has an $O(4, 4)$ global symmetry. By acting on the reduction of the uncharged Kerr solution with an $O(1, 1)^4 \subset O(4, 4)$ subgroup of the global symmetry, we obtain a new solution that lifts back to a solution of the four-dimensional theory with four independent electromagnetic charges. In this formulation, two of the $U(1)$ charges are electric and two are magnetic. By obtaining the explicit form of the four $U(1)$ gauge potentials, as well as the other fields, we therefore complete the results in [29], where the metric and scalar fields were found. We then apply the same generating technique to generalise these results by the inclusion also of the NUT parameter.

The second goal of this chapter is to obtain charged rotating black hole solutions in four-dimensional *gauged* supergravity. We have been able to do this in the case where the four charges of the ungauged theory are first set pairwise equal. With this restriction, we are able to conjecture, and then explicitly verify, the expression for the two-charge rotating black hole solutions of $\mathcal{N} = 4$ gauged supergravity. The solutions

have varying dilaton and axion fields.

This chapter is organized as follows. In Section B we describe the solution-generating technique for constructing charged rotating black holes of four-dimensional ungauged supergravity. In fact the procedure can be used to introduce four independent charges in any Ricci-flat four dimensional metric admitting a timelike Killing vector. In Section C we present the explicit form of the four-charge rotating black hole solution, generated from the Plebanski metric, in which in addition to the Kerr metric the NUT parameter is non-zero, and give its specialisation to the case where the charges are set pairwise equal. In Section D we present the generalisation of this latter case to a solution in $\mathcal{N} = 4$ gauged supergravity. We conclude this chapter in Section E.

B.   Charge-Generating Procedure

In this section, we set up the basic formalism for generating four-dimensional configurations carrying 4 independent charges, that are solutions of ungauged $\mathcal{N} = 2$ supergravity coupled to three vector multiplets. This theory, and hence also its solutions, can be consistently embedded in four-dimensional $\mathcal{N} = 8$ supergravity. The procedure involves starting with an uncharged four-dimensional solution that has a timelike Killing vector $\partial/\partial t$, and reducing it to three dimensions on the t direction. The reduction of the $\mathcal{N} = 2$ theory itself yields a three-dimensional theory with an $O(4, 4)$ global symmetry, after all the three-dimensional vector fields have been dualised to axions. By acting with an $O(1, 1)^4$ subgroup of $O(4, 4)$ on the dimensionally reduced solution, we generate new solutions involving four parameters δ_i characterising the $O(1, 1)^4$ transformation. Upon undualising the transformed dualised axions back to vectors again, and lifting back to $D = 4$, we thereby arrive at supergravity

solutions carrying 4 electromagnetic charges, parameterised by the δ_i.

In this section, we shall present the three-dimensional results for the reduction and $O(1,1)^4$ transformation of a general four-dimensional uncharged solution with a timelike Killing vector $\partial/\partial t$. One cannot abstractly "undualise" the three-dimensional scalars that originate from vectors in $D = 4$ (or from the Kaluza-Klein vector), since dualisation is intrinsically a non-local procedure. However, once one has an explicit solution, the process of undualisation can be implemented explicitly. Thus, in subsequent sections we shall apply these results to particular cases, and implement the complete and explicit construction of the charged four-dimensional solutions. (A solution-generating technique of the type we are using here was first employed in [37], to obtain electrically charged rotating black holes in ungauged supergravity.) As we shall see later, our solutions will carry two electric and two magnetic charges.

1. $O(4,4)$ Symmetry of the Reduced $D = 3$ Theory

The four-dimensional Lagrangian for the bosonic sector of the $\mathcal{N} = 2$ supergravity coupled to three vector multiplets can be written as[*]

$$
\begin{aligned}
\mathcal{L}_4 \;=\;& R*\mathbb{1} - \tfrac{1}{2}*d\varphi_i \wedge d\varphi_i - \tfrac{1}{2}e^{2\varphi_i}*d\chi_i \wedge d\chi_i - \tfrac{1}{2}e^{-\varphi_1}\Big(e^{\varphi_2-\varphi_3}*\hat{F}_{(2)1}\wedge\hat{F}_{(2)1} \\
&+e^{\varphi_2+\varphi_3}*\hat{F}_{(2)2}\wedge\hat{F}_{(2)2} + e^{-\varphi_2+\varphi_3}*\hat{\mathcal{F}}^1_{(2)}\wedge\hat{\mathcal{F}}^1_{(2)} + e^{-\varphi_2-\varphi_3}*\hat{\mathcal{F}}^2_{(2)}\wedge\hat{\mathcal{F}}^2_{(2)}\Big) \\
&-\chi_1\big(\hat{F}_{(2)1}\wedge\hat{\mathcal{F}}^1_{(2)} + \hat{F}_{(2)2}\wedge\hat{\mathcal{F}}^2_{(2)}\big),
\end{aligned}
\tag{3.1}
$$

---

[*]Our conventions for dualisation are that a p-form ω with components defined by $\omega = 1/p!\,\omega_{i_1\cdots i_p}\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ has dual $*\omega$ with components $(*\omega)_{i_i\cdots i_{D-p}} = 1/p!\,\epsilon_{i_1\cdots i_{D-p}j_i\cdots j_p}\,\omega^{j_1\cdots j_p}$.

where the index i labelling the dilatons φ_i and axions χ_i ranges over $1 \le i \le 3$. The four field strengths can be written in terms of potentials as

$$
\begin{aligned}
\hat{F}_{(2)1} &= d\hat{A}_{(1)1} - \chi_2 \, d\hat{\mathcal{A}}^2_{(1)} \,, \\
\hat{F}_{(2)2} &= d\hat{A}_{(1)2} + \chi_2 \, d\hat{\mathcal{A}}^1_{(1)} - \chi_3 \, d\hat{A}_{(1)1} + \chi_2 \, \chi_3 \, d\hat{\mathcal{A}}^2_{(1)} \,, \\
\hat{\mathcal{F}}^1_{(2)} &= d\hat{\mathcal{A}}^1_{(1)} + \chi_3 \, d\hat{\mathcal{A}}^2_{(1)} \,, \\
\hat{\mathcal{F}}^2_{(2)} &= d\hat{\mathcal{A}}^2_{(1)} \,.
\end{aligned}
\tag{3.2}
$$

Note that we are placing hats on the four-dimensional field strength and gauge potentials, to distinguish them from the three-dimensional fields. The four-dimensional theory can be obtained from six-dimensions, by reducing the bosonic string action

$$
\mathcal{L}_6 = R * \mathbb{1} - \tfrac{1}{2} e^{-\sqrt{2}\phi} * F_{(3)} \wedge F_{(3)}
\tag{3.3}
$$

on T^2. Thus the four-dimensional Lagrangian itself has an $O(2,2) \sim SL(2,R) \times SL(2,R)$ global symmetry, which enlarges at the level of the equations of motion to include a third $SL(2,R)$ factor when electric/magnetic S-duality transformations are included. We are going to reduce it one stage further, to $D = 3$. If left in its raw form, the three-dimensional Lagrangian would have an $O(3,3)$ global symmetry. However, if the 1-form potentials in $D = 3$ are dualised to axions (so that there are only dilatons and axions, plus the metric, in $D = 3$), then as is well known, the global symmetry will be enhanced to $O(4,4)$. The reduction from $D = 4$ to $D = 3$ will be performed on the time coordinate. This will imply that the coset parameterised by the dilatons and axions will not be $O(4,4)/(O(4) \times O(4))$, as would be the case for a spacelike reduction, but instead $O(4,4)/O(4,C)$.

To proceed, we first reduce the fields in the Lagrangian (3.1), according to the standard Kaluza-Klein reduction scheme adapted to the case of a timelike reduction.

Thus we write the following reduction ansätze for the metric and for 1-form potentials:

$$d\hat{s}_4^2 \;=\; -e^{\varphi_4}\left(dt + \mathcal{B}_{(1)}\right)^2 + e^{-\varphi_4}\, ds_3^2 \,, \tag{3.4}$$

$$\hat{A}_{(1)} \;=\; A_{(1)} + A_{(0)}\left(dt + \mathcal{B}_{(1)}\right). \tag{3.5}$$

The field strengths reduce according to the rule

$$\hat{F}_{(2)} = F_2 + F_1 \wedge \left(dt + \mathcal{B}_{(1)}\right). \tag{3.6}$$

In order to abbreviate the description, we shall directly present the fully-dualised form of the three-dimensional Lagrangian that results from reducing (3.1) according to this scheme, and then indicate afterwards how the three-dimensional fields are related to the four-dimensional ones. We find that the fully-dualised three-dimensional Lagrangian can be written as

$$
\begin{aligned}
e^{-1}\,\mathcal{L}_3 \;=\;& R - \tfrac{1}{2}(\partial\varphi_i)^2 - \tfrac{1}{2}e^{2\varphi_1}\left(\partial\chi_1\right)^2 - \tfrac{1}{2}e^{2\varphi_2}\left(\partial\chi_2\right)^2 - \tfrac{1}{2}e^{3\varphi_1}\left(\partial\chi_3\right)^2 \\
& -\tfrac{1}{2}e^{-2\varphi_4}\left(\partial\chi_4 + \sigma_1\,\partial\psi_1 + \sigma_2\,\partial\psi_2 + \sigma_3\,\partial\psi_3 + \sigma_4\,\partial\psi_4\right)^2 \\
& +\tfrac{1}{2}e^{-\varphi_1+\varphi_2-\varphi_3-\varphi_4}\left(\partial\sigma_1 - \chi_2\,\partial\sigma_4\right)^2 \\
& +\tfrac{1}{2}e^{-\varphi_1+\varphi_2+\varphi_3-\varphi_4}\left(\partial\sigma_2 + \chi_2\,\partial\sigma_3 - \chi_3\,\partial\sigma_1 + \chi_2\,\chi_3\,\partial\sigma_4\right)^2 \\
& +\tfrac{1}{2}e^{-\varphi_1-\varphi_2+\varphi_3-\varphi_4}\left(\partial\sigma_3 + \chi_3\,\partial\sigma_4\right)^2 + \tfrac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3-\varphi_4}\left(\partial\sigma_4\right)^2 \\
& +\tfrac{1}{2}e^{\varphi_1-\varphi_2+\varphi_3-\varphi_4}\left(\partial\psi_1 + \chi_3\,\partial\psi_2 - \chi_1\,\partial\sigma_3 - \chi_1\,\chi_3\,\partial\sigma_4\right)^2 \\
& +\tfrac{1}{2}e^{\varphi_1-\varphi_2-\varphi_3-\varphi_4}\left(\partial\psi_2 - \chi_1\,\partial\sigma_4\right)^2 \\
& +\tfrac{1}{2}e^{\varphi_1+\varphi_2-\varphi_3-\varphi_4}\left(\partial\psi_3 - \chi_2\,\partial\psi_2 - \chi_1\,\partial\sigma_1 + \chi_1\,\chi_2\,\partial\sigma_4\right)^2 \\
& +\tfrac{1}{2}e^{\varphi_1+\varphi_2+\varphi_3-\varphi_4}\bigl(\partial\psi_4 + \chi_2\,\partial\psi_1 - \chi_3\,\partial\psi_3 - \chi_1\,\partial\sigma_2 + \chi_2\,\chi_3\,\partial\psi_2 \\
& \qquad\qquad\qquad\quad -\chi_1\,\chi_2\,\partial\sigma_3 + \chi_1\,\chi_3\,\partial\sigma_1 - \chi_1\,\chi_2\,\chi_3\,\partial\sigma_4\bigr)^2 , \tag{3.7}
\end{aligned}
$$

where now the i index ranges over $1 \le i \le 4$. Note that the last eight terms have

the non-standard sign for their kinetic terms, in consequence of the timelike reduc-
tion, and the subsequent dualisations in a Euclidean-signature metric. The axion χ_4
corresponds to the dual of the Kaluza-Klein vector $\mathcal{B}_{(1)}$ in (3.4); the axions σ_i corre-
spond to the components $A_{(0)}$ (as in (3.5)) of the reductions of the 4 four-dimensional
potentials, in the order $(A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)})$; and the axions ψ_i correspond to the
dualisations of the three-dimensional 1-forms $A_{(1)}$ in (3.5), taken in the same order as
the σ_i.

In detail, the dualisations are performed as follows.[*] The field strength $G_{(2)} = dB_{(1)}$ for the Kaluza-Klein 1-form is replaced by

$$e^{2\varphi_4} *G_{(2)} = d\chi_4 + \sigma_1\, d\psi_1 + \sigma_2\, d\psi_2 + \sigma_3\, d\psi_3 + \sigma_4\, d\psi_4\,. \tag{3.8}$$

The four field strengths coming from the four field strengths in four dimensions are
replaced by

$$
\begin{aligned}
-e^{-\varphi_1+\varphi_2-\varphi_3+\varphi_4} *F_{(2)1} &= d\psi_1 + \chi_3\, d\psi_2 - \chi_1\, d\sigma_3 - \chi_1\,\chi_3\, d\sigma_4\,, \\
-e^{-\varphi_1+\varphi_2+\varphi_3+\varphi_4} *F_{(2)2} &= d\psi_2 - \chi_1\, d\sigma_4\,, \\
-e^{-\varphi_1-\varphi_2+\varphi_3+\varphi_4} *\mathcal{F}^1_{(2)} &= d\psi_3 - \chi_2\, d\psi_2 - \chi_1\, d\sigma_1 + \chi_1\,\chi_2\, d\sigma_4\,, \\
-e^{-\varphi_1-\varphi_2-\varphi_3+\varphi_4} *\mathcal{F}^2_{(2)} &= d\psi_4 + \chi_2\, d\psi_1 - \chi_3\, d\psi_3 - \chi_1\, d\sigma_2 + \chi_2\,\chi_3\, d\psi_2 \\
&\quad -\chi_1\,\chi_2\, d\sigma_3 + \chi_1\,\chi_3\, d\sigma_1 - \chi_1\,\chi_2\,\chi_3\, d\sigma_4\,. \tag{3.9}
\end{aligned}
$$

The Lagrangian (3.7) can be re-expressed as

$$\mathcal{L}_3 = R *1\!\!1 - \tfrac{1}{2}*d\varphi_i \wedge d\varphi_i - \tfrac{1}{2}\sum_{\alpha=1}^{12} \eta_\alpha\, e^{\vec{a}_\alpha\cdot\vec{\varphi}} *F^\alpha \wedge F^\alpha\,, \tag{3.10}$$

where each of the twelve 1-form field strengths F^α can be read off by comparison with

-----

[*]These results are obtained by applying the standard procedure of introducing the
dual potential as a Lagrange muliplier for the Bianchi identity of the original field
strength.

the twelve axion kinetic terms in (3.7). Likewise, the corresponding dilaton vector $\vec{a}_\alpha$ can be read off from the dilatonic prefactor of each axion kinetic term, and the coefficients $\eta_\alpha = \pm 1$ can be read off from the signs of the kinetic terms. One can then easily see that, as in the constructions in [38], we have

$$dF^\alpha = \tfrac{1}{2} f^\alpha{}_{\beta\gamma} \, F^\beta \wedge F^\gamma \,. \tag{3.11}$$

Introducing generators E_α and defining $\mathcal{F} \equiv F^\alpha \, E_\alpha$, one can express (3.11) as $d\mathcal{F} = \mathcal{F} \wedge \mathcal{F}$, where

$$[E_\alpha, E_\beta] = f^\gamma{}_{\alpha\beta} \, E_\gamma \,. \tag{3.12}$$

As in the discussions in [38], the E_α are easily seen to be the positive-root generators of the global symmetry group $O(4,4)$. We also introduce the four Cartan generators $\vec{H}$, which satisfy

$$[\vec{H}, E_\alpha] = \vec{a}_\alpha \, E_\alpha \,. \tag{3.13}$$

In an obvious notation, we may label the twelve positive-root generators by

$$E_\alpha = (E_{\chi_1}, E_{\chi_2}, \ldots, E_{\sigma_1}, E_{\sigma_2}, \ldots, E_{\psi_1}, E_{\psi_2}, \ldots) \,. \tag{3.14}$$

The simple root generators are $(E_{\chi_1}, E_{\chi_2}, E_{\chi_3}, E_{\sigma_4})$. The non-vanishing commutators are given by

$$[E_{\sigma_1}, E_{\psi_1}] = E_{\chi_4} \,, \; [E_{\sigma_2}, E_{\psi_2}] = E_{\chi_4} \,, \; [E_{\sigma_3}, E_{\psi_3}] = E_{\chi_4} \,, \; [E_{\sigma_4}, E_{\psi_4}] = E_{\chi_4} \,, \tag{3.15}$$

$$[E_{\chi_2}, E_{\sigma_4}] = -E_{\sigma_1} \,, \; [E_{\chi_2}, E_{\sigma_3}] = E_{\sigma_2} \,, \; [E_{\chi_3}, E_{\sigma_1}] = -E_{\sigma_2} \,, \; [E_{\chi_3}, E_{\sigma_4}] = E_{\sigma_3} \,,$$

$$[E_{\chi_3}, E_{\psi_2}] = E_{\psi_1} \,, \; [E_{\chi_1}, E_{\sigma_3}] = -E_{\psi_1} \,, \; [E_{\chi_1}, E_{\sigma_4}] = -E_{\psi_2} \,, \; [E_{\chi_2}, E_{\psi_2}] = -E_{\psi_3} \,,$$

$$[E_{\chi_1}, E_{\sigma_1}] = -E_{\psi_3} \,, \; [E_{\chi_2}, E_{\psi_1}] = E_{\psi_4} \,, \; [E_{\chi_3}, E_{\psi_3}] = -E_{\psi_4} \,, \; [E_{\chi_1}, E_{\sigma_2}] = -E_{\psi_4} \,.$$

Following [38] we now define a Borel-gauge coset representative $\mathcal{V}$ as follows:

$$\mathcal{V} = e^{\frac{1}{2}\vec{\varphi}\cdot\vec{H}}\,\mathcal{U}_\chi\,\mathcal{U}_\sigma\,\mathcal{U}_\psi\,, \tag{3.16}$$

where

$$
\begin{aligned}
\mathcal{U}_\chi &= e^{\chi_1\,E_{\chi_1}}\,e^{\chi_2\,E_{\chi_2}}\,e^{\chi_3\,E_{\chi_3}}\,e^{\chi_4\,E_{\chi_4}}\,, \\
\mathcal{U}_\sigma &= e^{\sigma_1\,E_{\sigma_1}}\,e^{\sigma_2\,E_{\sigma_2}}\,e^{\sigma_3\,E_{\sigma_3}}\,e^{\sigma_4\,E_{\sigma_4}}\,, \\
\mathcal{U}_\psi &= e^{\psi_1\,E_{\psi_1}}\,e^{\psi_2\,E_{\psi_2}}\,e^{\psi_3\,E_{\psi_3}}\,e^{\psi_4\,E_{\psi_4}}\,.
\end{aligned}
\tag{3.17}
$$

A straightforward calculation shows that if we define $\mathcal{U} \equiv \mathcal{U}_\chi\,\mathcal{U}_\sigma\,\mathcal{U}_\psi$ then

$$d\mathcal{U}\,\mathcal{U}^{-1} = \mathcal{F} = \sum_\alpha F^\alpha\,E_\alpha\,, \tag{3.18}$$

and we also have

$$d\mathcal{V}\,\mathcal{V}^{-1} = \tfrac{1}{2}d\vec{\varphi}\cdot\vec{H} + \sum_\alpha e^{\frac{1}{2}\vec{a}_\alpha\cdot\vec{\varphi}}\,F^\alpha\,E_\alpha\,. \tag{3.19}$$

Defining

$$\mathcal{M} \equiv \mathcal{V}^T\,\eta\,\mathcal{V}\,, \tag{3.20}$$

it can be seen that the fully-dualised three-dimensional Lagrangian (3.7) can be written as

$$e^{-1}\,\mathcal{L}_3 = R - \tfrac{1}{8}\mathrm{tr}(\partial\mathcal{M}^{-1}\,\partial\mathcal{M})\,. \tag{3.21}$$

This makes the $O(4,4)$ global symmetry manifest. Note that the constant matrix η in (3.20) is chosen so that the required distribution of positive and negative signs in the kinetic terms in (3.7) is obtained. Specifically, η is preserved under an $O(4,C)$ subgroup of $O(4,4)$ matrices K:

$$K^T\,\eta\,K = \eta\,. \tag{3.22}$$

(If we had instead performed a spacelike reduction to $D = 3$, so that all the kinetic

terms were of the standard sign, we would take $\eta = \mathbb{1}$, and the subgroup of $O(4,4)$ matrices satisfying $K^T K = \mathbb{1}$ would be $O(4) \times O(4)$.)

In order to generate the 4-charge solution, we shall act on the dimensional reduction of the uncharged Kerr black hole with an $O(1,1)^4$ subgroup of $O(4,4)$. Specifically, we shall take the $O(1,1)^4$ generators to be

$$\lambda_1 = E_{\psi_1} + E_{\psi_1}^T, \qquad \lambda_2 = E_{\sigma_2} + E_{\sigma_2}^T,$$

$$\lambda_3 = E_{\psi_3} + E_{\psi_3}^T, \qquad \lambda_4 = E_{\sigma_4} + E_{\sigma_4}^T, \tag{3.23}$$

and the $O(1,1)^4$ matrix

$$\Lambda \equiv e^{\delta_i \lambda_i} \tag{3.24}$$

will be used to act on $\mathcal{V}$ by right multiplication. In principle, we can calculate the resulting transformations of the fields from

$$\mathcal{V}' = \mathcal{O}\,\mathcal{V}\,\Lambda\,, \tag{3.25}$$

where $\mathcal{O}$ is an $O(4,C)$ compensating transformation that restores the coset representative to the Borel gauge as in (3.16). In practice, the drawback to this approach is that finding the required compensating transformation can be rather tricky. Instead, we can calculate the field transformations using

$$\mathcal{M}' = \Lambda^T \mathcal{M} \Lambda\,, \tag{3.26}$$

which avoids the need to find the compensating transformation. The price to be paid for this is that $\mathcal{M}$ is a much more complicated matrix than $\mathcal{V}$. However, by using an explicit realisation for the $O(4,4)$ matrices, the problem is easily tractable by computer.

2. $O(1,1)^4$ Transformation of a Reduced Uncharged Solution

When we implement the solution-generating procedure, our starting point in all cases will be an uncharged four-dimensional solution of the ungauged $\mathcal{N} = 2$ supergravity coupled to three vector multiplets, whose bosonic equations of motion are described by the Lagrangian (3.1). More specifically, in all our examples the starting point will be a solution of pure four-dimensional gravity, i.e. a Ricci-flat metric, admitting a timelike Killing vector $\partial/\partial t$. After reduction on the t direction, it follows that the only non-trivial three-dimensional fields will be the 3-metric ds_3^2, the Kaluza-Klein vector $\mathcal{B}_{(1)}$ and the Kaluza-Klein scalar φ_4. The Kaluza-Klein vector is then dualised to an axion, χ_4, using (3.8). In view of the fact that all the scalars σ_i and ψ_i (associated with the reduction of the 4 four-dimensional vector fields) are zero in the starting configuration, the dualisation of $\mathcal{B}_{(1)}$ at this stage is therefore simply given by

$$e^{2\varphi_4} *d\mathcal{B}_{(1)} = d\chi_4 \,. \tag{3.27}$$

We now implement the $O(1,1)^4$ transformations, as described in section B, taking as our starting point a three-dimensional configuration where only ds_3^2, φ_4 and χ_4 are non-trivial. For convenience, we shall denote these starting expressions for φ_4 and χ_4 by $\tilde{\varphi}_4$ and $\tilde{\chi}_4$, and then we denote the final expressions for all the $O(1,1)^4$-transformed fields by their symbols without tildes. (Since the three-dimensional metric is inert under $O(4,4)$ transformations, we don't need to introduce a tilde on the starting ds_3^2.)

The starting coset representative $\mathcal{V}$ is therefore given by

$$\mathcal{V} = e^{\frac{1}{2}\tilde{\varphi}_4 H_4} e^{\tilde{\chi}_4 E_{\chi_4}} \,. \tag{3.28}$$

Constructing $\mathcal{M} = \mathcal{V}^T \eta \mathcal{V}$, and acting with the $O(1,1)^4$ matrix Λ as in (3.26), we can obtain the transformed three-dimensional solution. Our results for the three-

dimensional fields after $O(1,1)^4$ transformation are as follows:

$$\sigma_1 = \frac{\tilde{\chi}_4\left(h_1(c_{234}s_1 - c_1 s_{234}\, e^{\tilde{\varphi}_4}) + c_1 s_1 s_{1234}\, \tilde{\chi}_4^2\right)}{W^2},$$

$$\sigma_2 = \frac{c_2}{s_2} - \frac{c_2 h_1 h_3 h_4 + (c_{134}s_2 - c_2 s_{134}\, e^{\tilde{\varphi}_4})\, s_{134}\, \tilde{\chi}_4^2}{s_2 W^2},$$

$$\sigma_3 = \frac{\tilde{\chi}_4\left(h_3(c_{124}s_3 - c_3 S_{124}\, e^{\tilde{\varphi}_4}) + c_3 s_3 s_{1234}\, \tilde{\chi}_4^2\right)}{W^2},$$

$$\sigma_4 = \frac{c_4}{s_3} - \frac{c_4 h_1 h_2 h_3 + (c_{123}s_4 - c_4 s_{123}\, e^{\tilde{\varphi}_4})\, s_{123}\, \tilde{\chi}_4^2}{s_4 W^2},$$

$$\psi_1 = \frac{c_1}{s_1} - \frac{c_1 h_2 h_3 h_4 + (c_{234}s_1 - c_1 s_{234}\, e^{\tilde{\varphi}_4})\, s_{234}\, \tilde{\chi}_4^2}{s_1 W^2},$$

$$\psi_2 = -\frac{\tilde{\chi}_4\left(h_2(c_{134}s_2 - c_2 s_{134}\, e^{\tilde{\varphi}_4}) + c_2 s_2 s_{1234}\, \tilde{\chi}_4^2\right)}{W^2},$$

$$\psi_3 = \frac{c_3}{s_3} - \frac{c_3 h_1 h_2 h_4 + (c_{124}s_3 - c_3 s_{124}\, e^{\tilde{\varphi}_4})\, s_{124}\, \tilde{\chi}_4^2}{s_3 W^2},$$

$$\psi_4 = -\frac{\tilde{\chi}_4\left(h_4(c_{123}s_4 - c_4 s_{123}\, e^{\tilde{\varphi}_4}) + c_4 s_4 s_{1234}\, \tilde{\chi}_4^2\right)}{W^2},$$

$$e^{\varphi_1} = \frac{h_1 h_3 + s_{13}^2 \tilde{\chi}_4^2}{W}, \qquad e^{\varphi_2} = \frac{h_2 h_3 + s_{23}^2 \tilde{\chi}_4^2}{W},$$

$$e^{\varphi_3} = \frac{h_1 h_2 + s_{12}^2 \tilde{\chi}_4^2}{W}, \qquad e^{\varphi_4} = \frac{e^{\tilde{\varphi}_4}}{W}, \qquad \chi_1 = \frac{(c_{13}s_{24} - c_{24}s_{13})\tilde{\chi}_4}{h_1 h_3 + s_{13}^2 \tilde{\chi}_4^2},$$

$$\chi_2 = \frac{(c_{14}s_{23} - c_{23}s_{14})\tilde{\chi}_4}{h_2 h_3 + s_{23}^2 \tilde{\chi}_4^2}, \qquad \chi_3 = \frac{(c_{12}s_{34} - c_{34}s_{12})\tilde{\chi}_4}{h_1 h_2 + s_{12}^2 \tilde{\chi}_4^2},$$

$$\begin{aligned}
\chi_4 = \frac{\tilde{\chi}_4}{W^2(h_1 h_2 + s_{12}^2 \tilde{\chi}_4^2)}\Big\{ &h_1 h_2 \Big[c_{1234}(1 + s_2^2 + s_4^2) + s_{1234}(1 + s_2^2 + s_4^2)e^{2\tilde{\varphi}_4} \\
&- \Big(c_{1234}(s_2^2 + s_4^2) + s_{1234}(2 + s_2^2 + s_4^2)\Big)e^{\tilde{\varphi}_4}\Big] + s_{1234}s_{12}^2(1 + s_2^2 + s_4^2)\tilde{\chi}_4^4 \\
&+ s_{12}\tilde{\chi}_4^2 \Big[c_{12}(c_{12}s_{34} + c_{34}s_{12})(1 + s_2^2 + s_4^2) - \Big((c_{1234}s_{12}(s_2^2 + s_4^2) \\
&+ s_{34}(s_1^2 + s_2^2 + 5s_{12}^2 + s_2^4 + 3s_1^2 s_2^4 + s_1^2 s_4^2 + s_{24}^2 + 3s_{124}^2)\Big)e^{\tilde{\varphi}_4} \\
&+ 2s_{1234}s_{12}(1 + s_2^2 + s_4^2)e^{2\tilde{\varphi}_4}\Big]\Big\},
\end{aligned} \tag{3.29}$$

where

$$h_i = c_i^2 - s_i^2\, e^{\tilde{\varphi}_4}, \qquad c_{i_1\cdots i_n} = \cosh\delta_i \cdots \cosh\delta_{i_n}, \qquad s_{i_1\cdots i_n} = \sinh\delta_i \cdots \sinh\delta_{i_n},$$

$$W^2 = h_1 h_2 h_3 h_4 + \tilde{\chi}_4^2 \Big(2c_{1234}s_{1234} - (s_{123}^2 + s_{124}^2 + s_{134}^2 + s_{234}^2 + 4s_{1234}^2)\, e^{\tilde{\varphi}_4}$$

$$+ 2s_{1234}^2 e^{2\tilde{\varphi}_4} \Big) + s_{1234}\tilde{\chi}_4^4. \tag{3.30}$$

C. 4-Charge Rotating NUT Solution in Ungauged Supergravity

In this section, we implement the procedure described in section B to generate the solution for a 4-charge rotating black hole in four dimensions. Our starting point, therefore, is simply the four-dimensional Plebanski [39, 40] solution, namely

$$ds_4^2 = -\frac{\widetilde{\Delta}_r}{a^2(r^2+u^2)} \left[adt + u^2\, d\phi\right]^2 + \frac{\widetilde{\Delta}_u}{a^2(r^2+u^2)} \left[adt - r^2\, d\phi\right]^2 + (r^2+u^2)\left(\frac{dr^2}{\widetilde{\Delta}_r} + \frac{du^2}{\widetilde{\Delta}_u}\right),$$

$$\tag{3.31}$$

where the functions $\widetilde{\Delta}_r$ and $\widetilde{\Delta}_u$ are given by

$$\widetilde{\Delta}_r = a^2 + r^2 - 2mr\,, \qquad \widetilde{\Delta}_u = a^2 - u^2 + 2\ell\, r\,. \tag{3.32}$$

Here m is the mass, ℓ is the NUT parameter, and a is the rotation parameter.

Recasting it in the form (3.4), we can read off the reduced three-dimensional metric, Kaluza-Klein vector and Kaluza-Klein scalar. After dualisation, using (3.27), the Kaluza-Klein 1-form $\mathcal{B}_{(1)}$ becomes the axion $\tilde{\chi}_4$.

All the other axions and dilatons in the three-dimensional theory described by (3.7) are zero. Note that, in line with the notation of section 1, we have placed tildes on the starting expressions for the fields $\tilde{\varphi}_4$ and $\tilde{\chi}_4$. The post-transformation fields are then written without tildes.

After the $O(1,1)^4$ transformation, the fields are given by (3.29). Before lifting the solution back to four dimensions, we must dualise the transformed axions ψ_i and χ_4 back to 1-form potentials, so that we can retrace the reduction steps. After performing the dualisations in three dimensions, we find that in three dimensions the

4-charge solution is given by

$$\chi_1 = \frac{2\left(c_{24}\,s_{13} - c_{13}\,s_{24}\right)\left(\ell\,r - m\,u\right)}{r_1\,r_3 + u_1\,u_3}, \qquad \chi_2 = \frac{2\left(c_{14}\,s_{23} - c_{23}\,s_{14}\right)\left(\ell\,r - m\,u\right)}{r_2\,r_3 + u_2\,u_3},$$

$$\chi_3 = \frac{2\left(c_{34}\,s_{12} - c_{12}\,s_{34}\right)\left(\ell\,r - m\,u\right)}{r_1\,r_2 + u_1\,u_2}, \qquad e^{\varphi_1} = \frac{r_1\,r_3 + u_1\,u_3}{W},$$

$$e^{\varphi_2} = \frac{r_2\,r_3 + u_2\,u_3}{W}, \qquad e^{\varphi_3} = \frac{r_1\,r_2 + u_1\,u_2}{W}, \qquad e^{\varphi_4} = \frac{r\,r_0 + u\,u_0}{W},$$

$$\sigma_1 = \frac{2}{W^2}\left(\ell\,r - m\,u\right)\left[c_1\,s_{234}\left(r_0\,r_1 + u_0\,u_1\right) - s_1\,c_{234}\left(r\,r_1 + u\,u_1\right)\right],$$

$$\sigma_3 = \frac{2}{W^2}\left(\ell\,r - m\,u\right)\left[c_3\,s_{124}\left(r_0\,r_3 + u_0\,u_3\right) - s_3\,c_{124}\left(r\,r_3 + u\,u_3\right)\right],$$

$$\sigma_2 = \frac{2}{W^2}\Big[c_2\,s_2\Big(m\,r_1\,r_3\,r_4 + \ell\,u_1\,u_3\,u_4 + r\,u\,(\ell\,r + m\,u) + 4\,\ell\,m\,r\,u\,(s_1^2 + s_3^2 + s_4^2)$$

$$+\,4\,\ell\,m\,(\ell\,r + m\,u)(s_{13}^2 + s_{14}^2 + s_{34}^2) + 16\,\ell^2\,m^2\,s_{134}^2\Big)$$

$$+\,2\,(\ell\,r - m\,u)^2\Big(c_{134}\,s_{134}\,(c_2^2 + s_2^2) - c_2\,s_2\,(s_{13}^2 + s_{14}^2 + s_{34}^2 + 2\,s_{134}^2)\Big)\Big]$$

$$\sigma_4 = \frac{2}{W^2}\Big[c_4\,s_4\Big(m\,r_1\,r_2\,r_3 + \ell\,u_1\,u_2\,u_3 + r\,u\,(\ell\,r + m\,u) + 4\,\ell\,m\,r\,u\,(s_1^2 + s_2^2 + s_3^2)$$

$$+\,4\,\ell\,m\,(\ell\,r + m\,u)(s_{12}^2 + s_{13}^2 + s_{23}^2) + 16\,\ell^2\,m^2\,s_{123}^2\Big)$$

$$+\,2\,(\ell\,r - m\,u)^2\Big(c_{123}\,s_{123}\,(c_4^2 + s_4^2) - c_4\,s_4\,(s_{12}^2 + s_{13}^2 + s_{23}^2 + 2\,s_{123}^2)\Big)\Big]$$

$$B_{(1)} = \frac{2}{a\,\bar\rho}\Big[c_{1234}\Big(\ell\,u\,(a^2 + r^2) + m\,r(a^2 - u^2)\Big)$$

$$-\,s_{1234}\Big(\ell\,u_0\,(a^2 + r^2) + m\,r_0\,(a^2 - u^2) + 4\,\ell\,m\,(\ell\,r - m\,u)\Big)\Big]d\phi,$$

$$A_{(1)1} = \frac{2\,c_1\,s_1}{a\,\bar\rho}\,[\,a^2\,(\ell\,r - m\,u) - r\,u\,(m\,r_0 + \ell\,u_0)\,]d\phi,$$

$$A_{(1)}^1 = \frac{2\,c_3\,s_3}{a\,\bar\rho}\,[\,a^2\,(\ell\,r - m\,u) - r\,u\,(m\,r_0 + \ell\,u_0)\,]d\phi,$$

$$A_{(1)2} = \frac{2}{a\,\bar\rho}\Big[s_2\,c_{134}\Big(r\,u\,(\ell\,r - m\,u) + a^2\,(m\,r + \ell\,u)\Big)$$

$$-\,c_2\,s_{134}\Big(r_0\,u_0\,(\ell\,r - m\,u) + a^2\,(m\,r_0 + \ell\,u_0)\Big)\Big]d\phi,$$

$$A_{(1)}^2 = \frac{2}{a\,\bar\rho}\Big[s_4\,c_{123}\Big(r\,u\,(\ell\,r - m\,u) + a^2\,(m\,r + \ell\,u)\Big)$$

$$-\,c_4\,s_{123}\Big(r_0\,u_0\,(\ell\,r - m\,u) + a^2\,(m\,r_0 + \ell\,u_0)\Big)\Big]d\phi. \tag{3.33}$$

with the definitions

$$r_0 = r - 2\,m\,, \qquad u_0 = r - 2\,\ell\,, \qquad \bar{\rho} = r\,r_0 + u\,u_0\,, \tag{3.34}$$

and

$$
\begin{aligned}
W^2 \;=\;& r_1\,r_2\,r_3\,r_4 + u_1\,u_2\,u_3\,u_4 + 2\,u^2\,r^2 + 2\,r\,u\,(\ell\,r + m\,u)(s_1^2 + s_2^2 + s_3^2 + s_4^2) \\
& - \; 4\,(\ell\,r - m\,u)^2(s_{123}^2 + s_{124}^2 + s_{134}^2 + s_{234}^2 + 2\,s_{1234}^2 - 2\,c_{1234}\,s_{1234}) \\
& + \; 8\,m\,\ell\,r\,u\,(s_{12}^2 + s_{13}^2 + s_{14}^2 + s_{23}^2 + s_{24}^2 + s_{34}^2) \\
& + \; 8\,m\,\ell(m\,u + \ell\,r)\,(s_{123}^2 + s_{124}^2 + s_{134}^2 + s_{234}^2) + 32\,m^2\,\ell^2\,s_{1234}^2 \tag{3.35}
\end{aligned}
$$

The final step is to lift this three-dimensional solution back to $D = 4$, using the Kaluza-Klein reduction rules (3.4) and (3.5). Thus the four-dimensional metric for the 4-charge rotating black hole solution is given by

$$ds_4^2 \;=\; -\frac{\bar{\rho}}{W}\,(dt + B_{(1)})^2 + W\left(\frac{dr^2}{\Delta_r} + \frac{du^2}{\Delta_u} + \frac{\Delta_r\,\Delta_u}{a^2\,\bar{\rho}}\,d\phi^2\right), \tag{3.36}$$

and the 4 four-dimensional gauge potentials are given in terms of the three-dimensional expressions in (3.33) by

$$
\begin{aligned}
\hat{A}_{(1)1} \;&=\; (A_{(1)1} + \sigma_1\,\mathcal{B}_{(1)}) + \sigma_1\,dt\,, \\
\hat{A}_{(1)2} \;&=\; (A_{(1)2} + \sigma_2\,\mathcal{B}_{(1)}) + \sigma_2\,dt\,, \\
\hat{\mathcal{A}}^1_{(1)} \;&=\; (\mathcal{A}^1_{(1)} + \sigma_3\,\mathcal{B}_{(1)}) + \sigma_3\,dt\,, \\
\hat{\mathcal{A}}^2_{(1)} \;&=\; (\mathcal{A}^2_{(1)} + \sigma_4\,\mathcal{B}_{(1)}) + \sigma_4\,dt\,. \tag{3.37}
\end{aligned}
$$

The dilatons $(\varphi_1, \varphi_2, \varphi_3)$ and axions (χ_1, χ_2, χ_3) are simply given by their three-dimensional expressions in (3.33).

D. Charged Rotating Black Holes in Gauged Supergravity

A simple special case arises if we set the two electric charges equal, by taking $\delta_4 = \delta_2$, and also set the two magnetic charges equal, by taking $\delta_3 = \delta_1$. In this section, we look for generalisations of the charged rotating black holes to the case of *gauged* four-dimensional supergravity. Since there is no longer a solution-generating technique for deriving the solutions in the gauged theories, we instead resort to a technique of "inspired guesswork," followed by brute-force verification that the equations of motion are satisfied. The verification is purely mechanical, but the process of guessing, or making an ansatz, for the form of the gauged solution is not so straightforward. In fact, so far we have succeeded in guessing the form of the gauged solution only in the case that the four charges are set pairwise equal. Thus in this section, we shall present our results for the gauged generalisation of the pairwise-equal ungauged solutions obtained in section C.

The easiest way to discuss the solution is by simply augmenting the bosonic Lagrangian (3.1) by the subtraction of the scalar potential that arises in the gauged supergravity. For the discussion of the solutions with pairwise-equal charges, which we are considering here, we can take the scalar potential to be that of the $\mathcal{N} = 4$ gauged $SO(4)$ theory, namely

$$V = -g^2 \sum_{i=1}^{3} \left(2 \cosh \varphi_i + \chi_i^2 \, e^{\varphi_i} \right). \tag{3.38}$$

with $\varphi_2 = \varphi_3 = \chi_2 = \chi_3 = 0$. The two electromagnetic charges in our pairwise-equal solution will then be carried by fields in $U(1)$ subgroups of the two $SU(2)$ factors in $SO(4) \sim SU(2) \times SU(2)$. Thus we may consider the bosonic Lagrangian

$$\mathcal{L}_4 \;=\; R *\mathbb{1} - \tfrac{1}{2} * d\varphi_1 \wedge d\varphi_1 - \tfrac{1}{2} e^{2\varphi_1} * d\chi_1 \wedge d\chi_1 - \tfrac{1}{2} e^{-\varphi_1} \left(*F_{(2)1} \wedge F_{(2)1} + *F_{(2)2} \wedge F_{(2)2} \right)$$

$$-\tfrac{1}{2}\chi_1\left(F_{(2)1}\wedge F_{(2)1}+F_{(2)2}\wedge F_{(2)2}\right)-g^2\left(4+2\cosh\varphi_1+e^{\varphi_1}\chi_1^2\right)*\mathbb{1}.\qquad(3.39)$$

It should be emphasised that the Lagrangian (3.39) is *not*, as it stands, the bosonic sector of any supergravity theory. First of all, we have included only the $U(1)\times U(1)$ subset of the $SU(2)\times SU(2)$ gauge fields of $SO(4)$-gauged $\mathcal{N}=4$ supergravity. This abelian truncation is consistent as a bosonic trunction, but not as a supersymmetric truncation. Secondly, even if we included all the $SU(2)\times SU(2)$ gauge fields in (3.39), the Lagrangian would still not be the bosonic sector of the $SO(4)$-gauged $\mathcal{N}=4$ supergravity. The gauge fields of one of the $SU(2)$ factors would have had to have been dualised prior to turning on the gauging, in order to get a supersymmetrisable theory. Since bare potentials appear in the expressions for the field strengths in the non-abelian gauged theory, dualisation can no longer be performed.

The upshot of the above discussion is that for the purposes of conjecturing, and then verifying, a solution of the gauged theory, it suffices to work with the generalisations of the ungauged solutions with two pair-wise equal charges, which is obtained from (3.36) by puting $\delta_4=\delta_2$ and $\delta_3=\delta_1$, and look at the equations of motion following from Lagrangian (3.39). Having successfully obtained charged rotating black-hole solutions, we can, if we wish, dualise one of the two field strengths. In this dualised form, the black hole can be directly viewed as a solution within $SO(4)$-gauged $\mathcal{N}=4$ supergravity, with the non-zero gauge fields of the solution residing within a $U(1)\times U(1)$ subgroup of $SU(2)\times SU(2)$. It is this dualised formulation that one would need to use if one wanted to test the supersymmetry of the solution.

By studying the form of the known Kerr-Newman-AdS black hole, as well that of the pairwise-equal charge solution, which is obtained from (3.36) by puting $\delta_4=\delta_2$ and $\delta_3=\delta_1$, we have been able to conjecture the form of the rotating black

hole solution of gauged supergravity with two pair-wise equal charges. Verifying the correctness of the conjecture is then a mechanical procedure, which we have performed using Mathematica. Our solution takes the form

$$ds_4^2 = -\frac{\Delta_r}{a^2\,W}\,[a\,dt + u_1\,u_2\,d\phi]^2 + \frac{\Delta_u}{a^2\,W}\,[a\,dt - r_1\,r_2\,d\phi]^2 + W\left(\frac{dr^2}{\Delta_r} + \frac{du^2}{\Delta_u}\right),\quad (3.40)$$

where

$$\Delta_r = \widetilde{\Delta}_r + g^2\,r_1\,r_2\,(r_1\,r_2 + a^2)\,,\qquad \Delta_u = \widetilde{\Delta}_u + g^2\,u_1\,u_2\,(u_1\,u_2 - a^2)\,,$$

$$W = r_1\,r_2 + u_1\,u_2\,,\qquad r_i = r + 2m\,s_i^2\,,\qquad u_i = u + 2\ell\,s_i^2\,,\qquad (3.41)$$

and $s_i = \sinh\delta_i$. The remaining fields are given by

$$e^{\varphi_1} = \frac{r_1^2 + u_1^2}{W}\,,\qquad \chi_1 = \frac{2(s_1^2 - s_2^2)\,(\ell\,r - m\,u)}{r_1^2 + u_1^2}\,,$$

$$A_{(1)1} = \frac{2\sqrt{2}s_1\,c_1}{a\,W}\left\{m\,u_1\,[a\,dt - r_1\,r_2\,d\phi] - \ell\,r_1\,[a\,dt + u_1\,u_2\,d\phi]\right\}\,,$$

$$A_{(1)2} = \frac{2\sqrt{2}s_2\,c_2}{a\,W}\left\{\ell\,u_1\,[a\,dt - r_1\,r_2\,d\phi] + m\,r_1\,[a\,dt + u_1\,u_2\,d\phi]\right\}\,.\qquad (3.42)$$

The ungauged case is, of course, obtained by setting $g = 0$. The verification of our conjectured result for $g \neq 0$ is straightforward; we used Mathematica to check that the equations of motion following from (3.39) are indeed satisfied.

If the two charge parameters are set equal, $\delta_1 = \delta_2$, then one has $\varphi_1 = \chi_1 = 0$ and the solution reduces to the charged AdS-Kerr-Taub-NUT solution of Einstein-Maxwell theory with a cosmological constant, as given in [39, 40].

E.   Conclusions

In this chapter we have presented new charged rotating solutions of four-dimensional ungauged and gauged supergravities. Our new ungauged solutions can be viewed as being embedded within $\mathcal{N} = 2$ supergravity coupled to three vector multiplets.

This is itself, of course, embedded within $\mathcal{N} = 8$ supergravity. We first constructed the general ungauged Kerr-Taub-NUT solution with four independent charges. We did this by employing a solution-generating technique, which involved reducing the four-dimensional theory on the time direction and then acting with global symmetry generators $O(1,1)^4 \subset O(4,4)$ to introduce the charges.

For the charged Kerr-Taub-NUT solution, we were able to conjecture the generalisations of the above ungauged solutions to the case of gauged supergravity, after making the specialisation that the four charges are set pairwise equal. We verified the correctness of our conjectured solutions by explicitly confirming that all the equations of motion are satisfied. These solutions are most appropriately viewed as being embedded in $SO(4)$-gauged $\mathcal{N} = 4$ supergravity.

The four-dimensional charged Kerr-Taub-NUT solution that we obtained in this chapter provide new gravitational backgrounds for four-dimensional vacua in compactified string theory. In particular, the non-extreme Kerr-Taub-NUT solution of gauged supergravity provide asymptotically AdS backgrounds that are characterised by their mass, angular momentum and two pair-wise equal charges (implying that they can be viewed as solutions in $\mathcal{N} = 4$ gauged supergravity). The gauged solutions should provide new information on the dual three-dimensional conformal field theory at non-zero temperature.

CHAPTER IV

CHARGED ROTATING BLACK HOLES IN FIVE DIMENSIONAL GAUGED SUPERGRAVITY

A.   Introduction

Charged black holes with non-zero cosmological constant provide important gravitational backgrounds for testing the AdS/CFT correspondence [1, 2]. In particular, for charged black holes in the anti-de Sitter background, the black hole charge plays the role of the R-charge [3] in dual field theory. In addition, the thermodynamic stability as well as analogs of the Hawking-Page phase transition for such configurations shed light [3, 4, 5] on the phase structure of the strongly coupled dual field theory.

Because the AdS_5/CFT_4 correspondence is better understood than AdS_4/CFT_3, much work was focused on five dimensional black holes. First examples of non-extremal charged black holes in five dimensions, as solutions of a gauged supergravity theory, were obtained in [6]. An important generalisation of static charged black holes is to allow for the rotation. General five-dimensional rotating charged black holes with two non-equal rotation parameters in the zero cosmological constant background were obtained in [9] by employing generating techniques associated with the underlying non-compact duality symmetries. Five-dimensional uncharged rotating black holes with non-zero cosmological constant, the five-dimensional Kerr-de Sitter metrics, were obtained a few years ago in [10]. In addition, certain five-dimensional extremal charged rotating solutions with non-zero cosmological constant have been found [11, 12, 13].

Constructing non-extremal charged rotating black hole solutions in gauged supergravity is quite a complicated problem. This is because, unlike the case of ungauged

supergravity, there are no known solution-generating techniques that could be used to add charges to the already-known neutral rotating black hole solutions. Due to this complication in the constructions of rotating black hole solutions in gauged supergravity, though the four-dimensional charged rotating black hole solutions with non-zero cosmological constant, the Kerr-Newman-de Sitter metrics, were found long ago [27], analogs of five-dimensional solutions have only been constructed in several special cases.

In the first special case [19], the problem was simplified greatly by taking the *a priori* independent rotation parameters of the orthogonal 2-planes in the transverse space to be equal. This reduces the problem to studying cohomogeneity-1 metrics, with non-trivial coordinate dependence on only the radial variable, rather than metrics of cohomogeneity 2. In section B, we will present these results obtained by constructing a general class of non-extremal charged rotating black hole solutions in the five-dimensional $U(1)^3$ gauged theory of $\mathcal{N} = 2$ supergravity coupled to two vector multiplets. They are the general non-extremal solutions of this dilatonic theory, with three independent electric charges, subject to the specialisation that the two angular momenta in the orthogonal 4-space are set equal. These 3-charge solutions are important for probing fully the microscopic degrees of freedom associated with the three R-charges in the dual $\mathcal{N} = 4$ CFT on the boundary, without the loss of information that would be inherent if the three charges were set equal.

In section C, we construct new non-extremal rotating black hole solutions in $SO(6)$ gauged five-dimensional supergravity. Our solutions are the first such examples in which the two rotation parameters are independently specifiable, rather than being set equal. The black holes carry charges for all three of the gauge fields in the $U(1)^3$ subgroup of $SO(6)$, albeit with only one independent charge parameter.

In section D, we construct the general solution for charged rotating black holes

in five-dimensional minimal gauged supergravity, with unequal angular momenta and three equal charge parameters, by a process involving a considerable amount of trial and error, followed by an explicit verification that the equations of motion are satisfied.

Finally, we obtain another independent class of new rotating non-extremal black hole solutions with just one non-vanishing charge.

B. Charged Black Holes with Equal Rotation Parameters in $U(1)^3$ Gauged $\mathcal{N} = 2$ Supergravity

The 3-charge solutions are generalisations to the gauged theory of the 3-charge spinning black hole solutions (with two rotation parameters set equal) of the corresponding five dimensional *ungauged* supergravity, obtained in [26]. They also, of course, specialise to the previously constructed results in [18] if one sets the three electric charges equal, under which circumstance the two dilatonic scalars decouple and become constant.

The bosonic sector of the five-dimensional $\mathcal{N} = 2$ gauged supergravity coupled to two vector multiplets is described by the Lagrangian

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}\partial\vec{\varphi}^2 - \tfrac{1}{4}\sum_{i=1}^{3} X_i^{-2}\left(F^i\right)^2 - \lambda \sum_{i=1}^{3} X_i^{-1} + \tfrac{1}{24}\epsilon_{ijk}\,\epsilon^{\mu\nu\rho\sigma\lambda}F^i_{\mu\nu}\,F^j_{\rho\sigma}\,A^k_\lambda, \quad (4.1)$$

where $\vec{\varphi} = (\varphi_1, \varphi_2)$, and

$$X_1 = e^{-\frac{1}{\sqrt{6}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}, \qquad X_2 = e^{-\frac{1}{\sqrt{6}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}, \qquad X_3 = e^{\frac{2}{\sqrt{6}}\varphi_1}. \qquad (4.2)$$

The gauge-coupling constant g is related to λ by $\lambda = -g^2$.

The solutions that we have obtained are as follows:

$$ds_5^2 = -\frac{Y - f_3}{R^2}\,dt^2 + \frac{r^2 R}{Y}\,dr^2 + R\,d\Omega_3^2 + \frac{f_1 - R^3}{R^2}\left(\sin^2\theta\,d\phi + \cos^2\theta\,d\psi\right)^2$$

$$-\frac{2f_2}{R^2}\,dt\,(\sin^2\theta d\phi + \cos^2\theta d\psi)\,, \tag{4.3}$$

$$A^i \;=\; \frac{\mu}{r^2\,H_i}\left(s_i\,c_i\,dt + \ell\,(c_i\,s_j\,s_k - s_i\,c_j\,c_k)\,(\sin^2\theta d\phi + \cos^2\theta d\psi)\right), \tag{4.4}$$

$$X_i \;=\; \frac{R}{r^2\,H_i}\,, \quad i=1,2,3 \tag{4.5}$$

where

$$R \;\equiv\; r^2\,(\prod_{i=1}^{3} H_i)^{\frac{1}{3}}\,, \qquad H_i \equiv 1 + \frac{\mu\,s_i^2}{r^2}\,,$$

$$d\Omega_3^2 \;=\; d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2\,, \tag{4.6}$$

and s_i and c_i are shorthand notations for

$$s_i \equiv \sinh\delta_i\,, \qquad c_i \equiv \cosh\delta_i\,, \qquad i=1,2,3\,. \tag{4.7}$$

Note that in the expressions (4.4) for the vector potentials A^i, the triplet indices (i,j,k) are all unequal: $(i \neq j \neq k \neq i)$. The functions (f_1, f_2, f_3, Y) are given by

$$
\begin{aligned}
f_1 \;&=\; R^3 + \mu\,\ell^2\,r^2 + \mu^2\,\ell^2\left[2\Big(\prod_i c_i - \prod_i s_i\Big)\prod_j s_j - \sum_{i<j} s_i^2\,s_j^2\right],\\
f_2 \;&=\; \gamma\,\ell\,\lambda\,R^3 + \mu\,\ell\left(\prod_i c_i - \prod_i s_i\right)r^2 + \mu^2\,\ell\prod_i s_i\,,\\
f_3 \;&=\; \gamma^2\,\ell^2\,\lambda^2\,R^3 + \mu\,\ell^2\,\lambda\left[2\gamma\left(\prod_i c_i - \prod_i s_i\right) - \Sigma\right]r^2\\
&\quad + \mu\,\ell^2 - \lambda\,\Sigma\,\mu^2\,\ell^2\left[2\Big(\prod_i c_i - \prod_i s_i\Big)\prod_j s_j - \sum_{i<j} s_i^2\,s_j^2\right] + 2\lambda\,\gamma\,\mu^2\,\ell^2\prod_i s_i\,,\\
Y \;&=\; f_3 - \lambda\,\Sigma\,R^3 + r^4 - \mu\,r^2\,,
\end{aligned}
\tag{4.8}
$$

where

$$\Sigma \equiv 1 + \gamma^2\,\ell^2\,\lambda\,. \tag{4.9}$$

It is helpful to note that $\sqrt{-g}$ takes a simple form, namely $\sqrt{-g} = r\,R\,\sin\theta\,\cos\theta$.

In order to make the global structure of the metrics more apparent, it is conve-

こ

nient to rewrite the metric (4.3) in terms of left-invariant 1-forms σ_i on S^3. Defining

$$
\begin{aligned}
\sigma_1 &= \cos\tilde{\psi}\,d\tilde{\theta} + \sin\tilde{\psi}\,\sin\tilde{\theta}\,d\tilde{\phi}\,, \\
\sigma_2 &= -\sin\tilde{\psi}\,d\tilde{\theta} + \cos\tilde{\psi}\,\sin\tilde{\theta}\,d\tilde{\phi}\,, \\
\sigma_3 &= d\tilde{\psi} + \cos\tilde{\theta}\,d\tilde{\phi}\,,
\end{aligned}
\tag{4.10}
$$

where

$$
\psi - \phi = \tilde{\phi}\,, \qquad \psi + \phi = \tilde{\psi}\,, \qquad \theta = \tfrac{1}{2}\tilde{\theta}\,,
\tag{4.11}
$$

we find that (4.3) can be rewritten as

$$
ds_5^2 = -\frac{R\,Y}{f_1}\,dt^2 + \frac{r^2\,R}{Y}\,dr^2 + \tfrac{1}{4}R\,(\sigma_1^2 + \sigma_2^2) + \frac{f_1}{4R^2}\,(\sigma_3 - \frac{2f_2}{f_1}\,dt)^2\,,
\tag{4.12}
$$

whilst the vector potentials in (4.4) become

$$
A^i = \frac{\mu}{r^2\,H_i}\left(s_i\,c_i\,dt + \tfrac{1}{2}\ell\,(c_i\,s_j\,s_k - s_i\,c_j\,c_k)\,\sigma_3\right).
\tag{4.13}
$$

C. Five-Dimensional Black Holes in Gauged Supergravity with Independent Rotation Parameters *

To avoid loosing information that would be inherent if the two rotation parameters are set equal, we construct non-extremal rotating black hole solutions in $SO(6)$ gauged five-dimensional supergravity in which the two rotation parameters are independently specifiable. Before we construct the solution with two independent rotation parameters, we first show how we obtain a solution with one rotation parameter. The reason that we present this solution is that we benefited from the construction of it in obtaining the solution with two independent rotation parameters.

---

1. Charged Solution with 1-Rotation

The Kerr-de Sitter solutions with one rotation in five dimensions in [9] is

$$
\begin{aligned}
ds^2 &= (-\frac{V_0 - 2\,m}{\rho^2}(dt - a\,\sin^2\theta\,d\phi)^2 \\
&\quad + \frac{\sin^2\theta}{\rho^2}(a\,dt - (a^2 + r^2 + 2\,m\,s^2)\,d\phi)^2 \\
&\quad + \frac{\rho^2}{V_0 - 2\,m}\,dr^2 + \rho^2\,d\theta^2 + r^2\,\cos^2\theta\,d\psi^2 \qquad (4.14)
\end{aligned}
$$

$$
\rho^2 = r^2 + a^2\cos^2\theta, \quad V_0 = a^2 + r^2. \qquad (4.15)
$$

The charged ungauged solution [26], with two of the three charge parameters set equal and the third vanishing, can be written in a similar form that is comparable with (4.15)

$$
\begin{aligned}
ds^2 &= H^{-\frac{4}{3}}\Big(-\frac{V_0 - 2\,m}{\rho^2}(dt - a\,\sin^2\theta\,d\phi)^2 \\
&\quad + \frac{\sin^2\theta}{\rho^2}(a\,dt - (a^2 + r^2 + 2\,m\,s^2)\,d\phi)^2\Big) \\
&\quad + H^{\frac{2}{3}}\Big(\frac{\rho^2}{V_0 - 2\,m}\,dr^2 + \rho^2\,d\theta^2 + r^2\,\cos^2\theta\,d\psi^2\Big) \qquad (4.16)
\end{aligned}
$$

$$
H = \frac{\tilde{\rho}^2}{\rho^2}, \quad \tilde{\rho}^2 = r^2 + a^2\cos^2\theta + 2\,m\,s^2,
$$

$$
\rho^2 = r^2 + a^2\cos^2\theta, \quad V_0 = a^2 + r^2. \qquad (4.17)
$$

$$
A^1 = A^2 = \frac{2\,m\,s\,c}{\tilde{\rho}^2}(dt - a\,\sin^2\theta\,d\phi), \qquad A^3 = \frac{2\,m\,a\,s^2}{\rho^2}\cos^2\theta\,d\psi. \,(4.18)
$$

Comparing with the solution in (4.15), we propose the following ansatz with only one function $V(r)$ to be determined

$$
\begin{aligned}
ds^2 &= H^{-\frac{4}{3}}\Big(-\frac{V(r) - 2m}{\rho^2}(dt - a\,\sin^2\theta d\phi)^2 \\
&\quad + \frac{\Delta_\theta\,\sin^2\theta}{\rho^2}(a\,dt - (a^2 + r^2 + 2ms^2)d\phi)^2\Big) \\
&\quad + H^{\frac{2}{3}}\Big(\frac{\rho^2}{V(r) - 2\,m}\,dr^2 + \frac{\rho^2}{\Delta_\theta}\,d\theta^2 + r^2\,\cos^2\theta\,d\psi^2\Big) \qquad (4.19)
\end{aligned}
$$

After plugging this ansatz into the equations of motion derived from the Larangian of minimal gauged supergravity, we were able to determine $V(r)$ to be

$$V(r) = a^2 + r^2 + g^2 (r^2 + 2ms^2)(r^2 + a^2 + 2ms^2) \tag{4.20}$$

with $\Delta_\theta = 1 - a^2 g^2 \cos^2 \theta$. It is helpful to observe that the difference between V_0 in (4.18) and $V(r)$ in (4.20) is the following replacement

$$r^2 \rightarrow r^2 + 2ms^2 \tag{4.21}$$

This observation helps a lot in obtaining the following solution with two rotation parameters, because we believe that we only need to make this replacement in proper places in finding gauged solutions with two rotations.

2. Charged Solutions with Two Rotation Parameters

We construct this solution by casting the two limiting cases, i.e. the five-dimensional Kerr-de Sitter solution and the charged solution in ungauged supergravity, into a specific form and then conjecture a solution that interpolates between them followed by a direct verification that the conjectured solution does solve the equations of motion. The observation in (4.21) plays a key role in making the conjecture.

The rotating black hole metrics in ungauged supergravity are charged under the $U(1)^3$ Cartan subgroup of $SO(6)$. The two rotation parameters can be specified independently. The relevant part of the supergravity Lagrangian that describes these solutions is given by

$$e^{-1} \mathcal{L} = R - \tfrac{1}{2} \partial \vec{\varphi}^2 - \tfrac{1}{4} \sum_{i=1}^{3} X_i^{-2} (F^i)^2 + \tfrac{1}{24} \epsilon_{ijk} \, \epsilon^{\mu\nu\rho\sigma\lambda} F^i_{\mu\nu} F^j_{\rho\sigma} A^k_\lambda \,, \tag{4.22}$$

where $\vec{\varphi} = (\varphi_1, \varphi_2)$, and

$$X_1 = e^{-\frac{1}{\sqrt{6}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}, \qquad X_2 = e^{-\frac{1}{\sqrt{6}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}, \qquad X_3 = e^{\frac{2}{\sqrt{6}}\varphi_1}. \qquad (4.23)$$

The complete metrics of charged black hole solution are given in [26]. In this section, unlike the general solution with three independent charge parameters s_1, s_2 and s_3, we consider a special case with $s_1 = s_2 \equiv s, s_3 = 0$. where $s_i \equiv \sinh \delta_i, c_i \equiv \cosh \delta_i$.

We then cast the charged solution with two independent rotation parameters in ungauged supergravity into the form that is comparable with the metrics found in [9]

$$
\begin{aligned}
ds^2 &= H^{-\frac{4}{3}} \Big[-\frac{X}{\rho^2}(dt - a \sin^2\theta \frac{d\phi}{\Xi_a} - b \cos^2\theta \frac{d\psi}{\Xi_b})^2 \\
&\quad + \frac{C}{\rho^2}(\frac{ab}{f_3}dt - \frac{b}{f_2}\sin^2\theta\frac{d\phi}{\Xi_a} - \frac{a}{f_1}\cos^2\theta\frac{d\psi}{\Xi_b})^2 \\
&\quad + \frac{Z\sin^2\theta}{\rho^2}(\frac{a}{f_3}dt - \frac{1}{f_2}\frac{d\phi}{\Xi_a})^2 + \frac{W\cos^2\theta}{\rho^2}(\frac{b}{f_3}dt - \frac{1}{f_1}\frac{d\psi}{\Xi_b})^2 \Big] + H^{\frac{2}{3}}(\frac{\rho^2}{X}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2),
\end{aligned}
$$
$$(4.24)$$

with ρ and $\tilde{\rho}$ defined by

$$\rho^2 = r^2 + a^2 \cos^2\theta + b^2 \sin^2\theta, \quad \tilde{\rho}^2 = \rho^2 + 2\,m\,s^2. \qquad (4.25)$$

The gauge potentials and scalar fields for the charged solution are given by

$$
\begin{aligned}
A^1 &= A^2 = \frac{2\,m\,s\,c}{\tilde{\rho}^2}(dt - a\sin^2\theta\frac{d\phi}{\Xi_a} - b\cos^2\theta\frac{d\psi}{\Xi_b}) \\
A^3 &= \frac{2\,m\,s^2}{\rho^2}(b\sin^2\theta\frac{d\phi}{\Xi_a} + a\cos^2\theta\frac{d\psi}{\Xi_b}) \\
X_1 &= X_2 = H^{-\frac{1}{3}}, \quad X_3 = H^{\frac{2}{3}}.
\end{aligned}
$$
$$(4.26)$$

All other functions in the metric are given in the Table I. The left column is for Kerr-de Sitter metrics, and the right one is for the charged ungauged solution. We were able to make a conjecture of the functions listed in the table so that they interpolate

Table I. Comparison between Kerr-de Sitter and charged black hole solutions in un-
gauged supergravity in five dimensions

Kerr de Sitter Black Hole Metrics	Charged Black Hole Metrics in Ungauged Supergravity
$H = 1$	$H = \tilde{\rho}^2/\rho^2$
$f_1 = a^2 + r^2$	$f_1 = a^2 + r^2$
$f_2 = b^2 + r^2$	$f_2 = b^2 + r^2$
$f_3 = (a^2 + r^2)(b^2 + r^2)$	$f_3 = (a^2 + r^2)(b^2 + r^2) + 2mr^2s^2$
$\Delta_\theta = 1 - a^2 g^2 \cos^2\theta - b^2 g^2 \sin^2\theta$	$\Delta_\theta = 1$
$X = \frac{1}{r^2}(a^2 + r^2)(b^2 + r^2) - 2m + g^2(a^2 + r^2)(b^2 + r^2)$	$X = \frac{1}{r^2}(a^2 + r^2)(b^2 + r^2) - 2m$
$C = f_1 f_2(X + 2m)$	$C = f_1 f_2(X + 2m - 4m^2s^4/\rho^2)$
$Z = -b^2 C + \frac{f_2 f_3}{r^2}[f_3 - g^2 r^2(a^2 - b^2)(a^2 + r^2)\cos^2\theta]$	$Z = -b^2 C + \frac{f_2 f_3}{r^2} f_3$
$W = -a^2 C + \frac{f_1 f_3}{r^2}[f_3 + g^2 r^2 (a^2 - b^2)(b^2 + r^2)\sin^2\theta]$	$W = -a^2 C + \frac{f_1 f_3}{r^2} f_3$
$\Xi_a = 1 - a^2 g^2$	$\Xi_a = 1$
$\Xi_b = 1 - b^2 g^2$	$\Xi_b = 1$

between the left and the right column. We verified that the following indeed solves the equations of motion derived from the Larangian of $SO(6)$ gauged five-dimensional supergravity

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}\partial\vec{\varphi}^2 - \tfrac{1}{4}\sum_{i=1}^{3}X_i^{-2}\left(F^i\right)^2 + 4g^2\sum_{i=1}^{3}X_i^{-1} + \tfrac{1}{24}\epsilon_{ijk}\,\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}^i\,F_{\rho\sigma}^j\,A_\lambda^k, \quad (4.27)$$

where $\vec{\varphi} = (\varphi_1, \varphi_2)$, and

$$X_1 = e^{-\frac{1}{\sqrt{6}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}, \qquad X_2 = e^{-\frac{1}{\sqrt{6}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}, \qquad X_3 = e^{\frac{2}{\sqrt{6}}\varphi_1}. \qquad (4.28)$$

The metric, gauge potentials and scalar fields have the same form as (4.24) and (4.26), but the functions listed in the table are replaced by

$$
\begin{aligned}
H &= \tilde{\rho}^2/\rho^2, \quad \rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \quad \tilde{\rho}^2 = \rho^2 + 2\,m\,s^2, \\
f_1 &= a^2 + r^2, \quad f_2 = b^2 + r^2, \quad f_3 = (a^2 + r^2)(b^2 + r^2) + 2\,m\,r^2\,s^2; \\
\Delta_\theta &= 1 - a^2\,g^2\cos^2\theta - b^2\,g^2\sin^2\theta, \\
X &= \frac{1}{r^2}(a^2 + r^2)(b^2 + r^2) - 2m + g^2(a^2 + r^2 + 2ms^2)(b^2 + r^2 + 2ms^2), \,(4.29) \\
C &= f_1\,f_2(X + 2m - 4\,m^2\,s^4/\rho^2), \\
Z &= -b^2\,C + \frac{f_2\,f_3}{r^2}[f_3 - g^2\,r^2\,(a^2 - b^2)(a^2 + r^2 + 2\,m\,s^2)\cos^2\theta], \\
W &= -a^2\,C + \frac{f_1\,f_3}{r^2}[f_3 + g^2\,r^2\,(a^2 - b^2)(b^2 + r^2 + 2\,m\,s^2)\sin^2\theta], \\
\Xi_a &= 1 - a^2 g^2, \quad \Xi_b = 1 - b^2 g^2,
\end{aligned}
$$

We noticed that the difference of the functions X, Z and W from those for Kerr-de Sitter metrics by [9] is the replacement (4.21)

In this section, we have constructed non-extremal black hole solutions in five-dimensional $SO(6)$ gauged supergravity. The solutions go beyond what has been found previously, by having unequal values for the angular momenta in the two orthogonal 2-planes in the transverse space. This means that the metrics are considerably

more complicated, since they are of cohomogeneity 2, rather than the cohomogeneity 1 of all the previously known examples.

The conserved angular momenta and charges are calculated in [21]. The energy is obtained by integrating the first law of thermodynamics. Using these results the BPS limits are taken to obtain supersymmetric backgrounds. In general, the BPS solutions have closed timelike curves outside a Killing horizon, and hence they describe "naked time machines". However, for a special choice of the relation between the mass and the rotation parameters, we obtain a completely regular black hole, with neither CTCs nor singularities outside the event horizon. Since the two rotation parameters still remain as free parameters, these black hole solutions provide a continuous supersymmetric interpolation between certain previously obtained equal-rotation solutions. The new solutions provide the first examples of supersymmetric black holes in gauged supergravity in which there are independent rotation parameters. It is also found that, in another special case, a solution describing a completely non-singular soliton.

D. Another Example with Independent Rotation Parameters *

Though our final goal is to find a general solution that has three independent charge parameters and two non-equal rotation parameters, we are still unable to achieve this. However, we succeeded in obtaining another special case with two independent rotation parameters, in which the three charge parameters are set equal. In this section, we will show the details of how this solution was obtained.

Our strategy is to employ the property of Kerr-Schild form of black hole metrics.

In Kerr-Schild form, the metric can be written as a sum of two parts, one is simply the metric for flat spacetimes, if the black hole is asymptotically flat, or the metric for Anti-de Sitter spacetimes, if the solution is asymptotically AdS. And the other is a null vector squared, which depends on mass and charge parameters of the black hole solution. We were not able to put the general charged solution into the Kerr-Schild form described above, however we managed to put the charged solution in ungauged supergravity with three equal charge parameters and two independent rotations into a *Kerr-Schild-like* form, then we may focus our attention on the part that depends on mass and charge parameters, which is supposed to be modified for the gauged supergravity solutions. In the generalisation to gauged supergravity solutions, we may simply replace the metric for the flat spacetimes by the metric for AdS spacetimes, and then figure out the necessary modification in the part that is dependent on mass and charge parameters so that the equations of motion derived from the Larangian of gauged supergravity can be solved.

In the following, we will first show that the solution with independent rotation parameters and charge parameters satisfying $s_1 = s_2 \equiv s, s_3 = 0$ in ungauged supergravity, which is the solution (4.29) with vanishing gauge coupling parameter g, can be put into a Kerr-Schild-like form after a series of coordinate transformations. Then we show the simple structure of inverse metric of solution (4.29) after similar coordinate transformations, though we were not able to put it into a Kerr-Schild-like form. We find that the structure of the inverse metric is quite simple. This encourages us to study the structure of the inverse metric in ungauged supergravity with three equal charge parameters, i.e. $s_1 = s_2 = s_3 \equiv s$ and two independent rotation parameters after a series of much more complicated coordinate transformations than those done before and some redefinition of coordinate. The structure is simple enough that it enables us to find the generalisation in gauged supergravity.

1. Kerr-Schild-Like Form of Charged Black Hole Metrics in Ungauged Supergravity

As an exercise, we show that the charged rotating black hole metrics with $s_1 = s_2 \equiv s, s_3 = 0, a \neq b$, which can be obtained from (4.29) with vanishing gauge coupling constant g, can be put in a Kerr-Schild-like from, which have extra warp factors. One can check the following vector is null in the solution

$$
\begin{aligned}
k_\mu dx^\mu &= dt + \frac{r^2 \tilde\rho^2}{\Delta} dr - a \, \sin^2\theta \, d\phi - b \, \cos^2\theta \, d\psi, \\
k^\mu \partial_\mu &= H^{\frac{1}{3}} (\partial_r - \frac{f_3}{\Delta}\partial_t - \frac{a \, r_b^2}{\Delta}\partial_\phi - \frac{b \, r_a^2}{\Delta}\partial_\psi). \\
r_a^2 &= r^2 + a^2, \quad r_b^2 = r^2 + b^2, \\
f_3 &= r_a^2 r_b^2 + 2\,M\,r^2\,s^2, \quad \Delta = r_a^2 r_b^2 - 2\,M\,r^2, \\
H &= \tilde\rho^2/\rho^2, \quad \rho^2 = r^2 + a^2 \cos^2\theta + b^2 \sin^2\theta, \quad \tilde\rho^2 = \rho^2 + 2\,M\,s^2 (4.30)
\end{aligned}
$$

Perform the following coordinate transformations that is related to the congruence of the null vector

$$
\begin{aligned}
dt &\rightarrow dt - \frac{2\,M\,r^2\,f_3}{r_a^2 r_b^2 \Delta} dr - \frac{2\,M\,s^2}{\tilde\rho^2}(a\,\sin^2\theta\,d\phi - b\,\cos^2\theta\,d\psi), \\
d\phi &\rightarrow H^{-1}\,d\phi - \frac{2\,a\,M\,r^2}{r_a^2 \Delta} dr, \quad d\psi \rightarrow H^{-1}\,d\psi - \frac{2\,b\,M\,r^2}{r_b^2 \Delta} dr. \tag{4.31}
\end{aligned}
$$

The solution then can be put into a Kerr-Schild-like form

$$
\begin{aligned}
ds^2 &= H^{-\frac{4}{3}}(-dt^2 + r_a^2\,\sin^2\theta\,d\phi^2 + r_b^2\,\cos^2\theta\,d\psi^2) + H^{\frac{2}{3}}(\frac{r^2 \rho^2}{r_a^2 r_b^2}dr^2 + \rho^2\,d\theta^2) \\
&+ H^{-\frac{4}{3}}(dt + \frac{r^2 \tilde\rho^2}{\Delta}dr - a\,\sin^2\theta\,d\phi - b\,\cos^2\theta\,d\psi)^2 \\
A^1 = A^2 &= \frac{2\,M\,s\,c}{\tilde\rho^2}(dt - a\,\sin^2\theta\,d\phi - b\,\cos^2\theta\,d\psi) - \frac{4\,a\,b\,M^2\,r^2}{r_a^2 r_b^2 \Delta}s\,c\,dr, \\
A^3 &= \frac{2\,M\,s^2}{\tilde\rho^2}(b\,\sin^2\theta\,d\phi + a\,\cos^2\theta\,d\psi) - \frac{2\,a\,b\,M\,r^2}{r_a^2 r_b^2 \Delta}dr) \tag{4.32}
\end{aligned}
$$

Note that if $H = 1$ the first line of the metric is just flat space, the second line is the null vector squared.

2. Simplicity of the Inverse Metric for Gauged Solution

Though the solution (4.29) can not be put into a Kerr-Schild-like form, we find that its inverse metric is surprisingly simple and the mass and charge dependent terms can be separated in a neat way. One can check that the following vector is null for the solution (4.29)

$$
\begin{aligned}
k_\mu dx^\mu &= dt + \frac{r^2\,\tilde{\rho}^2}{\Delta}dr - \frac{a\,\sin^2\theta}{\Xi_a}d\phi - \frac{b\,\cos^2\theta}{\Xi_b}d\psi, \\
k^\mu \partial_\mu &= H^{\frac{1}{3}}\left(\partial_r - \frac{f_3}{\Delta}\partial_t - \frac{a\,\Xi_a\,r_b^2}{\Delta}\partial_\phi - \frac{b\,\Xi_b\,r_a^2}{\Delta}\partial_\psi\right). \\
r_a^2 &= r^2 + a^2, \quad r_b^2 = r^2 + b^2, \\
\tilde{r}_a^2 &= r^2 + a^2 + q, \quad \tilde{r}_b^2 = r^2 + b^2 + q, \quad q = 2\,M\,s^2, \\
\Xi_a &= 1 - a^2\,g^2, \quad \Xi_b = 1 - b^2\,g^2, \\
\Delta_r &= 1 + g^2\,r^2, \quad \Delta_\theta = 1 - a^2\,g^2\,\cos^2\theta - b^2\,g^2\,\sin^2\theta, \\
f_3 &= r_a^2\,r_b^2 + q\,r^2, \quad \Delta = r_a^2\,r_b^2 - 2\,M\,r^2 + g^2\,r^2\,\tilde{r}_a^2\,\tilde{r}_b^2, \\
H &= \tilde{\rho}^2/\rho^2, \quad \rho^2 = r^2 + a^2\,\cos^2\theta + b^2\,\sin^2\theta, \quad \tilde{\rho}^2 = \rho^2 + q. \quad (4.33)
\end{aligned}
$$

The congruence of the null vector is defined by

$$
\begin{aligned}
dt &= -H^{\frac{1}{3}}\frac{f_3}{\Delta}d\lambda, \quad dr = H^{\frac{1}{3}}\,d\lambda, \quad d\theta = 0, \\
d\phi &= -\frac{a\,\Xi_a\,r_b^2}{\Delta}H^{\frac{1}{3}}\,d\lambda, \quad d\psi = -\frac{b\,\Xi_b\,r_a^2}{\Delta}H^{\frac{1}{3}}\,d\lambda,
\end{aligned}
\qquad (4.34)
$$

Then one perform the coordinate transformations

$$
dt \rightarrow dv - \frac{f_3}{\Delta}dr, \quad d\phi \rightarrow d\phi - \frac{a\,\Xi_a\,r_b^2}{\Delta}dr, \quad d\psi \rightarrow d\psi - \frac{b\,\Xi_b\,r_a^2}{\Delta}dr, \qquad (4.35)
$$

After making this coordinate transformation, we were not able to put it in a Kerr-Schild-like form. However, we find that that the inverse of the transformed metric is surprisingly simple

$$
\begin{aligned}
\partial s'^2 &= H^{-\frac{4}{3}}\left[\partial s_{AdS}^2 + \rho^{-2}\left[(-2\,M + q\,g^2\,(a^2 + b^2 + 2\,r^2 + q)\,)\partial_r^2 + 2\,q\,\partial_v\,\partial_r\right]\right], \\
A^1 &= A^2 = \frac{2\,M\,s\,c}{\tilde{\rho}^2}\,(dt - \frac{a\,\sin^2\theta}{\Xi_a}\,d\phi - \frac{b\,\cos^2\theta}{\Xi_b}\,d\psi) - 2\,M\,s\,c\,\frac{r^2}{\Delta}\,dr, \\
A^3 &= \frac{2\,M\,s^2}{\rho^2}(\frac{b\,\sin^2\theta}{\Xi_a}\,d\phi + \frac{a\,\cos^2\theta}{\Xi_b}) - 2\,M\,s^2\,\frac{a\,b}{\Delta}\,dr.
\end{aligned}
\tag{4.36}
$$

In this way, we successfully separate the involved part of the pure AdS space, and the mass and charge dependent terms are grouped in a relatively simple form. Encouraged by this, we will make proper coordinate transformations to the solution in ungauged supergravity with three equal charges and independent rotation parameters, hoping to separate the charge and mass dependent part from that of the flat spacetime.

3.   Inverse Metric for 3-Equal-Charge Solution and Its Gauged Generalisation

Though it is not easy to put the 3-equal-charge, unequal angular momenta charged black hole solution into Kerr-Schild form, it is encouraging to find that the inverse metric is quite simple and one can easily locate the place to be modified when going to the gauged solution. One finds that the following vector is null, which is the starting point of a series of coordinate transformations

$$
\begin{aligned}
k_\mu\,dx^\mu &= dt + \frac{r\,\tilde{r}\,\tilde{\rho}^2}{\Delta}\,dr - \alpha\,\sin^2\theta\,d\phi - \beta\,\cos^2\theta\,d\psi, \\
k^\mu\,\partial_\mu &= -\frac{f_3}{\Delta}\,\partial_t + \frac{\tilde{r}}{r}\,\partial_r - \alpha\,\frac{f_2}{\Delta}\,\partial_\phi - \beta\,\frac{f_1}{\Delta}\,\partial_\psi, \\
\alpha &= a\,c - b\,s, \quad \beta = b\,c - a\,s, \\
\rho^2 &= r^2 + a^2\,\cos^2\theta + b^2\,\sin^2\theta, \quad \tilde{\rho}^2 = \rho^2 + 2\,M\,s^2, \\
r_a^2 &= r^2 + a^2, \quad r_b^2 = r^2 + b^2,
\end{aligned}
$$

$$\tilde{r}_a^2 = r^2 + a^2 + 2Ms^2, \quad \tilde{r}_b^2 = r^2 + b^2 + 2Ms^2,$$

$$\tilde{r}^2 = \tilde{r}_a^2 - \alpha^2 = \tilde{r}_b^2 - \beta^2,$$

$$f_1 = \tilde{r}_a^2 + 2Msc\frac{\alpha}{\beta}, \quad f_2 = \tilde{r}_b^2 + 2Msc\frac{\beta}{\alpha},$$

$$f_3 = \tilde{r}_a^2\tilde{r}_b^2 + 2Msc\alpha\beta, \quad \Delta = r_a^2 r_b^2 - 2Mr^2, \tag{4.37}$$

This null vector defines the congruence

$$dt = -\frac{f_3}{\Delta}d\lambda, \quad dr = \frac{\tilde{r}}{r}d\lambda, \quad d\phi = -\alpha\frac{f_2}{\Delta}d\lambda, \quad d\psi = -\beta\frac{f_1}{\Delta}d\lambda. \tag{4.38}$$

Then performing the following coordinate transformation

$$dt \to dv - \frac{f_3}{\Delta}dR, \quad d\phi \to d\phi - \alpha\frac{f_2}{\Delta}dR, \quad d\psi \to d\psi - \beta\frac{f_1}{\Delta}dR \tag{4.39}$$

with $dR \equiv h\,dr, h = \frac{r}{\tilde{r}}$. This redefinition of radial coordinate plays a key role in eliminating c's and s'. The "good" coordinate is R. The null vector becomes very simple

$$k_\mu\,dx^\mu = dt - \alpha\sin^2\theta\,d\phi - \beta\cos^2\theta\,d\psi, \quad k^\mu\partial_\mu = \partial_R. \tag{4.40}$$

It is more helpful to look at the inverse metric instead of the metric itself

$$\partial s^2 = \partial s_{R^5}^2 - \frac{2m}{\hat{\rho}^2}\partial_R^2 + \frac{q}{R^2\,\hat{\rho}^2}[(q + 2\alpha\,\beta)\partial_R^2 + 2\partial_R(\alpha\,\beta\,\partial_v + \beta\,\partial_\phi + \alpha\,\partial_\psi)]$$

$$A^1 = A^2 = A^3 = \frac{q}{\hat{\rho}^2}(dv - \alpha\sin^2\theta\,d\phi - \beta\cos^2\theta\,d\psi),$$

$$\hat{\rho}^2 = R^2 + \alpha^2\cos^2\theta + \beta^2\sin^2\theta,$$

$$q = 2Msc, \quad m = \sqrt{M^2 + q^2}. \tag{4.41}$$

Here I have ignored the trivial r-component of the gauge potential.

To go to the gauged extension the result is surprisingly (disappointingly) simple, though the process to find it involved with ten unknown r-dependent functions,

solving only for three constants, among which one is just a redefinition of the mass parameter.

$$
\begin{aligned}
\partial s^2 &= \partial s^2_{AdS_5} - \frac{2\,m}{\hat{\rho}^2}\partial_R^2 + \frac{q}{R^2\,\hat{\rho}^2}[(q + 2\,\alpha\,\beta)\partial_R^2 + 2\,\partial_R\,(\alpha\,\beta\,\partial_v + \beta\,\Xi_a\partial_\phi + \alpha\,\Xi_b\partial_\psi)] \\
A^1 &= A^2 = A^3 = \frac{q}{\hat{\rho}^2}(dv - \alpha\,\sin^2\theta\,\frac{d\phi}{\Xi_a} - \beta\,\cos^2\theta\,\frac{d\psi)}{\Xi_b}, \\
\hat{\rho}^2 &= R^2 + \alpha^2\,\cos^2\theta + \beta^2\,\sin^2\theta, \\
q &= 2\,M\,s\,c, \quad m = \sqrt{M^2 + q^2}.
\end{aligned}
\tag{4.42}
$$

In one word it is as simple as

$$
d\phi \to \frac{d\phi}{\Xi_a}, \quad d\psi \to \frac{d\psi}{\Xi_b}, \quad \partial_\phi \to \Xi_a\,\partial_\phi, \quad \partial_\psi \to \Xi_b\,\partial_\psi.
\tag{4.43}
$$

and the flat R^5 is replaced by AdS_5.

4. The Solution in Boyer-Linquist Form

For the purpose to simplify the analysis of the event horizons and the causal structure of the metrics, it is useful to write the metric in the so called Boyer-Lindquist Coordinates, in which there are no cross-terms between dr and the other coordinate differentials. The Lagrangian of minimal gauged five-dimensional supergravity is

$$
\mathcal{L} = (R + 12g^2)*\mathbb{1} - \tfrac{1}{2}*F \wedge F + \frac{1}{3\sqrt{3}}F \wedge F \wedge A,
\tag{4.44}
$$

where $F = dA$, and g is assumed to be positive, without loss of generality.

In terms of Boyer-Lindquist type coordinates $x^\mu = (t, r, \theta, \phi, \psi)$ that are asymptotically static (i.e. the coordinate frame is non-rotating at infinity), we find that the metric and gauge potential for our new rotating solutions can be expressed as

$$
ds^2 = -\frac{\Delta_\theta\,[(1 + g^2r^2)\rho^2 dt + 2q\nu]\,dt}{\Xi_a\,\Xi_b\,\rho^2} + \frac{2q\,\nu\omega}{\rho^2} + \frac{f}{\rho^4}\left(\frac{\Delta_\theta\,dt}{\Xi_a\Xi_b} - \omega\right)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta}
$$

$$+\frac{r^2+a^2}{\Xi_a}\sin^2\theta d\phi^2 + \frac{r^2+b^2}{\Xi_b}\cos^2\theta d\psi^2 \,, \tag{4.45}$$

$$A = \frac{\sqrt{3}q}{\rho^2}\left(\frac{\Delta_\theta\,dt}{\Xi_a\,\Xi_b} - \omega\right) , \tag{4.46}$$

where

$$\begin{aligned}
\nu &= b\sin^2\theta d\phi + a\cos^2\theta d\psi \,,\\
\omega &= a\sin^2\theta\frac{d\phi}{\Xi_a} + b\cos^2\theta\frac{d\psi}{\Xi_b} \,,\\
\Delta_\theta &= 1 - a^2 g^2\cos^2\theta - b^2 g^2\sin^2\theta \,,\\
\Delta_r &= \frac{(r^2+a^2)(r^2+b^2)(1+g^2 r^2)+q^2+2abq}{r^2} - 2m \,,\\
\rho^2 &= r^2 + a^2\cos^2\theta + b^2\sin^2\theta \,,\\
\Xi_a &= 1 - a^2 g^2 \,, \quad \Xi_b = 1 - b^2 g^2 \,,\\
f &= 2m\rho^2 - q^2 + 2abqg^2\rho^2 \,.
\end{aligned} \tag{4.47}$$

A straightforward calculation shows that these configurations solve the equations of motion of minimal gauged five-dimensional supergravity.

Rotating black hole solutions in five-dimensional gauged supergravity provide backgrounds whose AdS/CFT duals describe four-dimensional field theories in the rotating Einstein universe on the boundary of anti-de Sitter spacetime. With the general solutions in minimal gauged supergravity that we have now found, this aspect of the AdS/CFT correspondence can be studied in a framework that also allows one to take a BPS or near-BPS limit, where the mapping from the bulk to the boundary is better controlled. In particular, it is of great interest to provide the microscopic interpretation from the boundary CFT for the entropy of the supersymmetric black holes with two general rotations [42, 43].

E.   Single-Charge Black Holes

1.   Single-Charge Black Holes with One Rotation

In this section we obtain another new solution describing a non-extremal rotating black hole in gauged five-dimensional supergravity. In this case, just one of the two rotation parameters is non-zero, and only one of the three gauge fields in the $U(1)^3$ subgroup of $SO(6)$ is turned on. This solution is therefore not a special case of any other previously-obtained solutions.

The ungauged solution with 1-charge and 1-rotation can be written as

$$
\begin{aligned}
ds^2 &= -H^{-\frac{2}{3}} \frac{Y_0(r)}{\hat{\rho}^2}(cdt - a \sin^2 \theta \, d\phi)^2 \\
&\quad + H^{\frac{1}{3}}\left(\frac{\sin^2 \theta}{\hat{\rho}^2}(adt - c(a^2 + r^2)d\phi)^2 \right. \\
&\quad \left. + \frac{\rho^2}{Y_0(r)}dr^2 + \rho^2 d\theta^2 + r^2 \cos^2 \theta d\psi^2 \right)
\end{aligned}
\tag{4.48}
$$

$$
H = \frac{\tilde{\rho}^2}{\rho^2}, \quad Y_0(r) = a^2 + r^2 - 2m, \quad \hat{\rho}^2 = r^2 + a^2 \cos^2 \theta + s^2(a^2 + r^2). \tag{4.49}
$$

$$
A^1 = \frac{2ms}{\tilde{\rho}^2}(cdt - a\sin^2 \theta). \tag{4.50}
$$

Then we try the following ansatz

$$
\begin{aligned}
ds^2 &= -H^{-\frac{2}{3}} \frac{Y(r)}{F(r,\theta)}(c\,dt - a\sin^2 \theta \, d\phi)^2 \\
&\quad + H^{\frac{1}{3}}\left(\frac{\Delta_\theta \sin^2 \theta}{F(r,\theta)}(f_1(r)dt - c f_2(r)d\phi)^2 + \frac{\rho^2}{Y(r)}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + r^2 \cos^2 \theta d\psi^2\right)
\end{aligned}
\tag{4.51}
$$

We notice the fact that the determinant of the metric for charged black holes in gauged supergravity remains the same as that for the charged solutions in ungauged supergravity. We use this fact to reduce the number of unknown functions in the metric ansatz (4.51) before we go to equations of motion derived from the Lagrangian for minimal gauged supergravity. To be specific, we find that the function $F(r,\theta)$ can

be expressed as

$$F(r,\theta) = c^2 f_2(r) - a \sin^2\theta f_1(r). \tag{4.52}$$

This helps to avoid solving partial differential equations. Instead, the task is reduced to finding the three functions $f_1(r)$, $f_2(r)$ and $Y(r)$, all of which depend only on one coordinate r. We were able to determine these three functions after we substitute the ansatz (4.51) into the equations of motion. They can be determined up to a constant w

$$
\begin{aligned}
f_1 &= \frac{a}{w}(1 - w\, s^2\, g^2\, (a^2 + r^2)), \quad f_2 = w\,(a^2 + r^2), \\
Y &= (a^2 + r^2)(1 + g^2\, r^2) - 2\, m\, f_1\, /\, a, \\
\Delta_\theta &= 1 - a^2\, g^2\, \cos^2\theta, \quad \Xi = 1 - a^2\, g^2. \tag{4.53}
\end{aligned}
$$

where again $c = \cosh\delta$, $s = \sinh\delta$, and the constant w, which satisfies $c^2 w^2 - s^2 w\Xi = 1$, is given by

$$w = \frac{\Xi s^2 + \sqrt{4(1+s^2) + \Xi^2 s^4}}{2(1+s^2)}. \tag{4.54}$$

The gauge potentials and scalar fields are given by

$$
\begin{aligned}
A^1 &= \frac{2\, m\, s\, \sqrt{w}}{\tilde\rho^2}(c\, dt - a\sin^2\theta\,\frac{d\phi}{w\,\Xi}), \quad A^2 = A^3 = 0, \\
X_1 &= H^{-\frac{2}{3}}, \quad X_2 = X_3 = H^{\frac{1}{3}}. \tag{4.55}
\end{aligned}
$$

The conserved angular momentum, charge and energy are calculated in [21]. The BPS limit is also studied. In this case, there are no regular supersymmetric black holes or solitons, but rather, the BPS solutions describe backgrounds with closed timelike curves outside a Killing horizon.

2. Single-Charge Black Holes with Independent Rotations

In this subsection, we extend this previous result by obtaining the general solution for a five-dimensional rotating black hole with arbitrary rotation parameters a and b, in the case that just one of the three $U(1)$ gauge fields carries a charge. In some sense this can be viewed as the most general "basic" solution. Our approach to constructing this solution involves first recasting the metrics into a form that leads eventually to a rather simple presentation of the result. We also find that the same type of transformation, applied to previously-known cases, leads to rather simple expressions in those cases too.

In the following, we first introduce our general ansatz for the types of metric we shall consider. Then, we give our specific results for the general single-charge rotating black holes. In an appendix, we show how all previously-known rotating black holes in five-dimensional gauged supergravity fit elegantly within the formulation that we have adopted. In addition, we show that the general 3-charge solution in ungauged five-dimensional supergravity, which was constructed in [26], also has a very simple expression when written in this formalism.

The bosonic sector of the relevant $\mathcal{N} = 2$ theory can be derived from the Lagrangian

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}\partial\vec{\varphi}^2 - \tfrac{1}{4}\sum_{i=1}^{3} X_i^{-2}\left(F^i\right)^2 + 4g^2\sum_{i=1}^{3} X_i^{-1} + \tfrac{1}{24}|\epsilon_{ijk}|\,\epsilon^{\mu\nu\rho\sigma\lambda}F^i_{\mu\nu}\,F^j_{\rho\sigma}\,A^k_\lambda\,, \quad (4.56)$$

where $\vec{\varphi} = (\varphi_1, \varphi_2)$, and

$$X_1 = e^{-\frac{1}{\sqrt{6}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}\,, \qquad X_2 = e^{-\frac{1}{\sqrt{6}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}\,, \qquad X_3 = e^{\frac{2}{\sqrt{6}}\varphi_1}\,. \quad (4.57)$$

All the solutions that we shall consider, comprising the new general single-charge rotating black holes, and also the previously-known solutions with two equal charges

[20] or three equal charges [21], as well as the general solutions in the ungauged theory with three unequal charges [26, 24], can all be cast in a simple manner within the following formalism. We write the metrics as

$$
\begin{aligned}
ds_5^2 &= (H_1 H_2 H_3)^{1/3} (x + y) d\hat{s}_5^2, \\
d\hat{s}_5^2 &= -\Phi (dt + \mathcal{A})^2 + ds_4^2,
\end{aligned}
\tag{4.58}
$$

with the scalars and gauge potentials given by

$$
\begin{aligned}
X_i &= H_i^{-1} (H_1 H_2 H_3)^{1/3}, \\
A^1 &= \frac{2m}{x + y} H_1^{-1} \{ s_1 c_1 dt + s_1 c_2 c_3 [abd\chi + (y - a^2 - b^2) d\sigma] \\
&\quad + c_1 s_2 s_3 (abd\sigma - y d\chi) \}, ,
\end{aligned}
\tag{4.59}
$$

with A^2 and A^3 given by cyclically permuting the subscripts on the right-hand side. The functions H_i are given by

$$
H_i = 1 + \frac{2m s_i^2}{x + y},
\tag{4.60}
$$

and we are using the shorthand notation

$$
s_i = \sinh \delta_i, \qquad c_i = \cosh \delta_i,
\tag{4.61}
$$

where δ_i are the charge parameters. The four-dimensional base metric in (4.58) takes the form

$$
ds_4^2 = \left(\frac{dx^2}{4X} + \frac{dy^2}{4Y} \right) + \frac{U}{G} \left(d\chi - \frac{Z}{U} d\sigma \right)^2 + \frac{XY}{U} d\sigma^2,
\tag{4.62}
$$

where X is a function of x, Y is a function of y, and G, U and Z are functions of both x and y. The "Kaluza-Klein" 1-form $\mathcal{A}$ appearing in (4.58) lives purely in the

four-dimensional base space, and takes the form

$$\mathcal{A} = f_1 \, d\sigma + f_2 \, d\chi \, . \tag{4.63}$$

The functions f_1 and f_2 depend only on x and y, as does Φ, which is given by

$$\Phi = \frac{G}{(x+y)^3 \, H_1 \, H_2 \, H_3} \, . \tag{4.64}$$

The inverse of the metric $d\hat{s}_5^2$ is given by

$$
\begin{aligned}
\left(\frac{\partial}{\partial \hat{s}_5}\right)^2 &= -\frac{1}{\Phi}\left(\frac{\partial}{\partial t}\right)^2 + 4X\left(\frac{\partial}{\partial x}\right)^2 + 4Y\left(\frac{\partial}{\partial y}\right)^2 + \frac{G}{U}\left(\frac{\partial}{\partial \chi} - f_2 \frac{\partial}{\partial t}\right)^2 \\
&\quad + \frac{1}{UXY}\left(U\frac{\partial}{\partial \sigma} + Z\frac{\partial}{\partial \chi} - (f_1 \, U + f_2 \, Z)\frac{\partial}{\partial t}\right)^2 .
\end{aligned}
\tag{4.65}
$$

Since there is no solution-generating technique for deriving charged black holes from neutral black holes in gauged supergravity (unlike the situation in ungauged supergravity), there is really no way other than a combination of guesswork, followed by explicit verification, for obtaining the charged solutions. We were led to write the ansatz for the metric, gauge potentials and scalar fields in the manner we have presented above by considering all the previously-obtained examples. The specific results for the new general single-charged rotating black holes, which we shall present below, were obtained by making a detailed comparison of various known cases, transformed into the format of the ansatz above, and then making a conjecture for the form of the solution. Finally, we substituted this into the equations of motion following from (4.56), to verify that it was indeed a solution. In doing this, we made extensive use of the Mathematica algebraic computing language.

Our new results for the general single-charge rotating black hole in five-dimensional gauged $\mathcal{N} = 2$ supergravity are as follows. Taking $\delta_2 = \delta_3 = 0$, and writing $\delta_1 = \delta$,

we find

$$
\begin{aligned}
X &= (x + a^2)(x + b^2) - 2mx + g^2(x + a^2)(x + b^2)[x + 2ms^2 - (a^2 + b^2)s^2 + 2absc]\,, \\
Y &= -(a^2 - y)(b^2 - y)[1 - g^2(y + (a^2 + b^2)s^2 - 2absc)]\,, \\
G &= (x + y)(x + y - 2m) + g^2(x + y)^2\,(x - y + a^2 + b^2)H\,, \\
U &= yX - xY + s^2W\,, \qquad Z = ab(X + Y) + scW\,, \\
W &= -2g^2m(a^2 - y)(b^2 - y)x + g^4(x + a^2)(x + b^2)(a^2 - y)(b^2 - y)(x + y + 2ms^2)\,, \\
\Phi &= \frac{G}{(x + y)^3\,H}\,, \\
\mathcal{A} &= s\,(xd\chi + abd\sigma) + c\,[abd\chi - (x + a^2 + b^2)d\sigma] \\
&\quad + \frac{1}{G}\Big[-s\,(x + y - 2m)(xd\chi + abd\sigma) - c\,(x + y)[abd\chi - (x + a^2 + b^2 - 2m)d\sigma] \\
&\qquad + g^2(x + a^2)(x + b^2)(x + y + 2ms^2)(cd\sigma - sd\chi)\Big]\,.
\end{aligned}
\tag{4.66}
$$

The gauge potentials in (4.59) reduce to $A^2 = A^3 = 0$ and

$$
A^1 = \frac{2ms}{x + y + 2ms^2}\,[cdt + abd\chi + (y - a^2 - b^2)d\sigma]\,,
\tag{4.67}
$$

and the H_i functions are given by $H_2 = H_3 = 1$ and

$$
H_1 \equiv H = 1 + \frac{2ms^2}{x + y}\,.
\tag{4.68}
$$

The solution we have presented here has four non-trivial parameters, namely m, δ, a and b (with $s = \sinh\delta$, $c = \cosh\delta$), which characterise the mass, charge and two angular momenta respectively.

F. Discussion

In this chapter, we present the details of how we obtained the new solutions with specific charge configurations and independent rotation parameters. Unlike in ungauged

supergravity, where there is a solution generating technique by employing the global symmetry of the theory, the difficulty in constructing new solutions in gauged supergravity lies in the fact that there is no systematic method of finding new solutions. Each case requires specific insights to find the answer. Though we succeed in finding solutions for several special cases, we are still trying to obtain the general solution with three independent charges and two rotation parameters.

CHAPTER V

CHARGED ROTATING BLACK HOLES IN SEVEN

DIMENSIONAL GAUGED SUPERGRAVITY

A. Introduction [*]

Charged black holes in gauged supergravities provide gravitational backgrounds of importance in the study of the AdS/CFT correspondence. Non-extremal black hole solutions are relevant for studying the dual field theory at non-zero temperature. This has been discussed extensively for static AdS black holes in, for example, [30, 31, 32]. See also [33, 34, 35], for recent related work. For non-extremal charged rotating black holes in gauged supergravities, little has been known until recently. In [18, 19] the first examples of non-extremal rotating charged AdS black holes in five-dimensional $\mathcal{N} = 4$ gauged supergravity were obtained, in the special case where the two angular momenta J_i are set equal. These solutions are characterised by their mass, three electromagnetic charges, and the angular momentum parameter $J = J_1 = J_2$. By taking appropriate limits, one obtains the various supersymmetric charged rotating $D = 5$ black holes obtained in [14, 15, 13]. If instead the charges are set to zero, the solutions reduce to the rotating AdS$_5$ black hole constructed in [9], with $J_1 = J_2$. In four dimensions, the charged Kerr-Newman-AdS black hole solution of the Einstein-Maxwell system with a cosmological constant has long been known [27, 36]. This can be viewed as a solution in gauged $\mathcal{N} = 8$ supergravity, in which the four electromagnetic fields in the $U(1)^4$ abelian subgroup of the $SO(8)$ gauge group are set equal. Recently, a more general class of non-extremal charged rotating solutions

in the four-dimensional gauged theory were constructed, in which the four electric charges are set pairwise equal [16].

Another case of interest from the AdS/CFT perspective is non-extremal charged rotating black holes in seven-dimensional gauged supergravity, and this forms the subject of the present chapter. The maximally-supersymmetric theory has $\mathcal{N} = 4$ supersymmetry, and the gauge group is $SO(5)$ [44]. It was shown in [45, 46] that this theory can be obtained as a consistent reduction of eleven-dimensional supergravity on S^4. A convenient presentation of the Lagrangian for the bosonic sector, in the conventions we shall be using, appears in [47]. The theory is capable of supporting black holes carrying two independent electric charges, carried by gauge fields in the $U(1) \times U(1)$ abelian subgroup of the full $SO(5)$ gauge group. For the purposes of discussing the solutions, it therefore suffices to perform a (consistent) truncation of the full supergravity theory to the relevant sector, in which all except the $U(1) \times U(1)$ subgroup of gauge fields are set to zero. The fields retained in the consistent truncation comprise the metric, two dilatons, the $U(1) \times U(1)$ gauge fields and a 4-form field strength that satisfies an odd-dimensional self-duality equation. The equations of motion can be derived from the Lagrangian

$$
\begin{aligned}
\mathcal{L}_7 \;=\;& R * \mathbb{1} - \tfrac{1}{2} * d\varphi_i \wedge d\varphi_i - \tfrac{1}{2} \sum_{i=1}^{2} X_i^{-2} * F_{(2)}^i \wedge F_{(2)}^i - \tfrac{1}{2}(X_1\,X_2)^2 * F_{(4)} \wedge F_{(4)} \\
&+ 2g^2 \left[(X_1\,X_2)^{-4} - 8X_1\,X_2 - 4X_1^{-1}X_2^{-2} - 4X_1^{-2}\,X_2^{-1} \right] \\
&- g\,F_{(4)} \wedge A_{(3)} + F_{(2)}^1 \wedge F_{(2)}^2 \wedge A_{(3)} \,,
\end{aligned}
\tag{5.1}
$$

where

$$
\begin{aligned}
F_{(2)}^i \;=\;& dA_{(1)}^i \,, \qquad F_{(4)} = dA_{(3)} \,, \\
X_1 \;=\;& e^{-\frac{1}{\sqrt{2}}\varphi_1 - \frac{1}{\sqrt{10}}\varphi_2} \,, \qquad X_2 = e^{\frac{1}{\sqrt{2}}\varphi_1 - \frac{1}{\sqrt{10}}\varphi_2} \,,
\end{aligned}
\tag{5.2}
$$

together with a first-order "odd-dimensional self-duality" equation to be imposed after the variation of the Lagrangian. This condition is most conveniently stated by introducing an additional 2-form potential $A_{(2)}$, which can be gauged away in the gauged theory, and defining

$$F_{(3)} = dA_{(2)} - \tfrac{1}{2} A^1_{(1)} \wedge dA^2_{(1)} - \tfrac{1}{2} A^2_{(1)} \wedge dA^1_{(1)} \,. \tag{5.3}$$

The odd-dimensional self-duality equation then reads[*]

$$(X_1 X_2)^2 * F_{(4)} = -2g A_{(3)} - F_{(3)} \,. \tag{5.4}$$

This is a first integral of the equation of motion for $A_{(3)}$ that follows directly from (5.1). (Note that one can alternatively write a Lagrangian that yields the equations of motion directly, with no need for an additional constraint. See, for example, [44, 47].)

The fact that the 3-form $A_{(3)}$ satisfies the odd-dimensional self-duality equation presents an interesting new challenge when constructing the charged rotating solutions in the gauged seven-dimensional supergravity. The trickiest part of finding charged rotating solutions in any of the gauged supergravities is that one has little *a priori* guidance as to how the dimensionless quantity $a\,g$ enters the solution, where a is the rotation parameter and g the gauge coupling constant. In the cases that have been constructed previously, in five dimensions [18, 19] and in four dimensions [16], the gauge coupling constant appeared always quadratically in the relevant equations of motion, and thus the dimensionless product entered the solutions in the combination $a^2 g^2$. In seven dimensions, by contrast, the gauge coupling constant g appears linearly in the odd-dimensional self-duality equation (5.4), and so in turn the solution

---

[*]Note that if $g \neq 0$, one can absorb A_2 by making a gauge transformation of $A_{(3)}$. If, on the other hand, $g = 0$, then (5.4) just becomes the defining equation for $F_{(3)}$ as the dual of $F_{(4)}$. When $g = 0$ one can equivalently work either with $A_{(3)}$, or with $A_{(2)}$ in a dual formulation of the theory.

involves linear powers of the product $a\,g$. This considerably complicates the task of parameterising possible forms for the solution, in the process of formulating a conjecture and then verifying that it works. It is intriguing that having found the charged rotating black hole, we have the rather uncommon situation of obtaining a solution of seven-dimensional gauged supergravity in which the odd-dimensional self-duality equation (5.4) is satisfied in a non-trivial way.

In the following sections, we shall construct non-extremal charged rotating black-hole solutions in the seven-dimensional gauged supergravity. In our approach, we begin with the previously-known charged rotating solutions in the ungauged theory. Then, we formulate a conjecture for the generalisation to the gauged theory, and verify it by an explicit checking of all the equations of motion. The charged solutions in the ungauged supergravity were constructed (with the full complement of three independent rotation parameters) in [48]. In order to give a uniform presentation of our results, and also to eliminate some typographical errors that arose in [48], we begin in section B by rederiving the charged rotating black holes in the ungauged seven-dimensional supergravity, in the special case we are addressing in this chapter where the three angular momenta are set equal. Then, in section C, we formulate our conjectured generalisation to the gauged supergravity theory, and verify that it does indeed satisfy the equations of motion. As well as obtaining the non-extremal solutions with two independent charges, we also present a somewhat simpler form of the metric in the special case where the two charges are set equal. In section D, we discuss the BPS limit, showing how supersymmetric rotating black hole solutions in seven-dimensional gauged supergravity arise for a suitable restriction of the parameters. The chapter ends with conclusions in section E.

B.   Charged Rotating Black Holes in the Ungauged Theory

Charged solutions in ungauged supergravity can be obtained from uncharged ones by making use of global symmetries of the theory, employed as solution-generating transformations. In the present case, one way of doing this is to recognise that in the ungauged ($g = 0$) limit, the seven-dimensional theory described by (5.1) can be obtained as the dimensional reduction of the eight-dimensional "bosonic string" theory described by

$$\mathcal{L}_8 = \hat{R} \,\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}d\varphi \wedge d\varphi - \tfrac{1}{2}e^{-\frac{2}{\sqrt{3}}\varphi}\,\hat{*}\hat{F}_{(3)} \wedge \hat{F}_{(3)}\,. \tag{5.5}$$

This yields the seven-dimensional theory in a formulation in which the 4-form $F_{(4)}$ has been dualised to the 3-form $F_{(3)}$ (see footnote 1). The strategy for introducing charges is then to begin with an uncharged, Ricci-flat solution in seven dimensions, take its product with a circle, and hence obtain a Ricci-flat solution of the eight-dimensional theory. Next, one performs a Lorentz transformation in the (t, z) plane, with Lorentz boost parameter δ_1, where

$$t \longrightarrow t\,\cosh\delta_1 + z\,\sinh\delta_1\,, \qquad z \longrightarrow z\,\cosh\delta_1 + t\,\sinh\delta_1\,, \tag{5.6}$$

where z is the circle coordinate of the eighth dimension. Upon reduction to $D = 7$ on the Lorentz-transformed circle coordinate z, one obtains a seven-dimensional solution in which the Kaluza-Klein vector carries an electric charge. The next step is to use the discrete Z_2 subgroup of the seven-dimensional global symmetry group that exchanges the Kaluza-Klein and winding vectors. This allows one to repeat the lifting, Lorentz boosting and reduction steps, with a second boost parameter δ_2, thereby ending up with a seven-dimensional solution where each of the Kaluza-Klein and winding vectors carries an electric charge.

In principle we can apply this charge-generating procedure starting from any Ricci-flat metric in seven dimensions. In our present case, we take as our starting point the generalisation of the rotating Kerr black hole to seven dimensions, obtained by Myers and Perry [25]. The most general such solution has three independent rotation parameters in the three orthogonal 2-planes of its six-dimensional transverse space. For reasons of simplicity, we restrict attention to the case where the three rotation parameters are set equal.

The uncharged seven-dimensional rotating black hole, with the three rotation parameters set equal, can be written as

$$ds_7^2 = -dt^2 + \frac{2m}{\rho^4}(dt - a\,\sigma)^2 + \frac{\rho^4\,dr^2}{V - 2m} + \rho^2\,(d\Sigma_2^2 + \sigma^2)\,, \tag{5.7}$$

where

$$\rho^2 \equiv (r^2 + a^2)\,, \qquad V \equiv \frac{1}{r^2}(r^2 + a^2)^3\,, \tag{5.8}$$

$d\Sigma_2^2$ is the standard Fubini-Study metric on CP^2, and σ is the connection on the $U(1)$ fibre over CP^2 whose total bundle is the unit 5-sphere. Thus we may write [49]

$$
\begin{aligned}
d\Sigma_2^2 &= d\xi^2 + \tfrac{1}{4}\sin^2\xi\,(\sigma_1^2 + \sigma_2^2) + \tfrac{1}{4}\sin^2\xi\,\cos^2\xi\,\sigma_3^2\,, \\
\sigma &= d\tau + \tfrac{1}{2}\sin^2\xi\,\sigma_3\,,
\end{aligned}
\tag{5.9}
$$

where σ_i denotes a set of left-invariant 1-forms on $SU(2)$, satisfying $d\sigma_i = -\tfrac{1}{2}\epsilon_{ijk}\,\sigma_j \wedge \sigma_k$. Note that we have

$$d\sigma = 2J\,, \tag{5.10}$$

where J is the Kähler form on CP^2.

After implementing the sequence of steps described above in order to introduce electric charges, we find that the charged rotating non-extremal seven-dimensional

black holes are given by

$$
\begin{aligned}
ds_7^2 &= (H_1\,H_2)^{1/5}\left[-\frac{\rho^4-2m}{\rho^4\,H_1\,H_2}\,dt^2 - \frac{4m\,a}{\rho^4\,H_1\,H_2}\,dt\,\sigma + \frac{2m\,a^2}{\rho^4\,H_1\,H_2}\left(1 - \frac{2m\,s_1^2\,s_2^2}{\rho^4}\right)\sigma^2 \right.\\
&\qquad\left. + \rho^2\,(d\Sigma_2^2 + \sigma^2) + \frac{\rho^4\,dr^2}{V - 2m}\right],\\
A_{(1)}^1 &= \frac{2m\,s_1}{\rho^4\,H_1}\,(c_1\,dt - a\,c_2\,\sigma), \qquad A_{(1)}^2 = \frac{2m\,s_2}{\rho^4\,H_2}\,(c_2\,dt - a\,c_1\,\sigma),\\
A_{(2)} &= \frac{m\,a\,s_1\,s_2}{\rho^4}\left(\frac{1}{H_1} + \frac{1}{H_2}\right)dt \wedge \sigma,\\
X_i &= (H_1\,H_2)^{2/5}\,H_i^{-1},
\end{aligned}
\tag{5.11}
$$

where

$$
H_i = 1 + \frac{2m\,s_i^2}{\rho^4}\,,
\tag{5.12}
$$

and we have defined

$$
s_i \equiv \sinh\delta_i\,, \qquad c_i \equiv \cosh\delta_i\,.
\tag{5.13}
$$

(A different solution-generating technique, making use of global symmetries of the three-dimensional theory obtained by dimensional reduction, was used in [48] to construct rotating charged black holes in D-dimensional supergravities for $4 \leq D \leq 9$, with 2 independent charges and $[(D-1)/2]$ independent angular momenta. When the three angular momenta in the $D = 7$ solution are set equal, the situation considered in [48] reduces to the one we have considered in this chapter.[*])

Note that the 3-form $F_{(3)} = dA_{(2)} - \frac{1}{2}A_{(1)}^1 \wedge dA_{(1)}^2 - \frac{1}{2}A_{(1)}^2 \wedge dA_{(1)}^1$ can be dualised to the 4-form $F_{(4)} = dA_{(3)}$, in which case one has

$$
A_{(3)} = \frac{2m\,a\,s_1\,s_2}{r^2 + a^2}\,\sigma \wedge J
\tag{5.14}
$$

in place of the expression for $A_{(2)}$ in (5.11).

---

[*]The general solution in [48] (eq. (12) of [48]) has a few typographical errors: a term $2N\,\ell_i^2\,\mu_i^2$ in the metric coefficient for $d\phi_i^2$ should be $2N\,\Delta\,\ell_i^2\,\mu_i^2$, the 2-form potential components $B_{\phi_i\,\phi_j}$ should be set to zero, and the quantity mr in $B_{t\,\phi_i}$ should be N.

C.   Charged Rotating Black Holes in the Gauged Theory

In this section, we construct non-extremal charged rotating solutions in the gauged seven-dimensional supergravity theory. Note that the global symmetries that allowed us to generate charged solutions from uncharged ones are broken in the gauged theory, and so there is no longer a procedure available that delivers the charged solutions by mechanical means. Instead, we have constructed the charged solutions by means of "educated guesswork," followed by an explicit verification that all the seven-dimensional equations of motion are indeed satisfied. In order to conjecture the form of the solution, we have made extensive use of previously-known limiting cases, including, especially, the charged solutions of the ungauged theory, which we described in the previous section.

We find that the charged and rotating non-extremal black hole solution of the seven-dimensional gauged supergravity is given by*

$$ds_7^2 = (H_1 H_2)^{1/5} \left[-\frac{Y\, dt^2}{f_1\, \Xi_-^2} + \frac{\rho^4\, d\rho^2}{Y} + \frac{f_1}{\rho^4\, H_1 H_2\, \Xi^2} \left(\sigma - \frac{2f_2}{f_1}\, dt \right)^2 + \frac{\rho^2}{\Xi}\, d\Sigma_2^2 \right],$$

$$A_{(1)}^i = \frac{2m\, s_i}{\rho^4\, \Xi\, H_i} \left(\alpha_i\, dt + \beta_i\, \sigma \right),$$

$$A_{(2)} = \frac{m\, a\, s_1\, s_2}{\rho^4\, \Xi_-^2} \left(\frac{1}{H_1} + \frac{1}{H_2} \right) dt \wedge \sigma, \qquad A_{(3)} = \frac{2m\, a\, s_1\, s_2}{\rho^2\, \Xi\, \Xi_-}\, \sigma \wedge J,$$

$$X_i = (H_1 H_2)^{2/5}\, H_i^{-1}, \qquad H_i = 1 + \frac{2m\, s_i^2}{\rho^4}, \qquad \rho^2 = r^2 + a^2,$$

$$\alpha_1 = c_1 - \tfrac{1}{2}(1 - \Xi_+^2)(c_1 - c_2), \qquad \alpha_2 = c_2 + \tfrac{1}{2}(1 - \Xi_+^2)(c_1 - c_2),$$

$$\beta_1 = -a\, \alpha_2, \qquad \beta_2 = -a\, \alpha_1,$$

---

*It should be emphasised that in the solution (5.15), $A_{(3)}$ is the potential for the fundamental field $F_{(4)} = dA_{(3)}$ in the gauged supergravity, while, as discussed in footnote 1, $A_{(2)}$ is a term that could, if one wished, be viewed as being absorbed into $A_{(3)}$ via a gauge transformation of $A_{(3)}$. It happens to be convenient to present it in the form we have done; we are *not* saying that $A_{(2)}$ is an independent fundamental field.

$$\Xi_\pm \;=\; 1 \pm a\,g\,, \qquad \Xi = 1 - a^2\,g^2 = \Xi_-\,\Xi_+\,, \tag{5.15}$$

where the functions f_1, f_2 and Y are given by

$$
\begin{aligned}
f_1 &= \Xi\,\rho^6\,H_1 H_2 - \frac{4\Xi_+^2\,m^2\,a^2\,s_1^2\,s_2^2}{\rho^4} + \tfrac{1}{2}m\,a^2\left[4\Xi_+^2 + 2c_1\,c_2\,(1-\Xi_+^4) + (1-\Xi_+^2)^2\,(c_1^2+c_2^2)\right], \\
f_2 &= -\tfrac{1}{2}g\,\Xi_+\,\rho^6\,H_1 H_2 + \tfrac{1}{4}m\,a\left[2(1+\Xi_+^4)\,c_1\,c_2 + (1-\Xi_+^4)\,(c_1^2+c_2^2)\right], \\
Y &= g^2\,\rho^8\,H_1 H_2 + \Xi\,\rho^6 + \tfrac{1}{2}m\,a^2\left[4\Xi_+^2 + 2(1-\Xi_+^4)\,c_1\,c_2 + (1-\Xi_+^2)^2(c_1^2+c_2^2)\right] \\
&\quad - \tfrac{1}{2}m\,\rho^2\left[4\Xi + 2a^2 g^2(6+8ag+3a^2 g^2)\,c_1\,c_2 - a^2 g^2(2+ag)(2+3ag)(c_1^2+c_2^2)\right]. \tag{5.16}
\end{aligned}
$$

It is a purely mechanical exercise, which we performed with the aid of Mathematica, to verify that this configuration indeed satisfies the equations of motion of seven-dimensional gauged supergravity, following from (5.1) together with the odd-dimensional self-duality equation (5.4). Note that as mentioned in the introduction, unlike the charged rotating AdS black holes in $D = 5$ and $D = 4$, the metric in $D = 7$ depends on odd powers of g as well as even powers, in consequence of the odd-dimensional self-duality equation.

If one specialises to the case where the two charges are set equal, the solution may be written in a somewhat simpler form, as

$$
\begin{aligned}
ds_7^2 &= H^{2/5}\left[-\frac{V-2m}{\rho^4\,H^2\,\Xi^2}\,(dt-a\,\sigma)^2 + \frac{1}{r^2\,H^2\,\Xi^2}\,(h_1\,dt - h_2\,\sigma)^2 + \frac{\rho^4\,dr^2}{V-2m} + \frac{\rho^2}{\Xi}\,d\Sigma_2^2\right], \\
A^1_{(1)} &= A^2_{(1)} = \frac{2m\,s\,c}{\rho^4\,H\,\Xi}\,(dt-a\,\sigma), \\
A_{(2)} &= \frac{2m\,s^2\,a}{\rho^4\,H\,\Xi_-^2}\,dt\wedge\sigma, \\
A_{(3)} &= \frac{2m\,a\,s^2}{\Xi\,\Xi_-\,(r^2+a^2)}\,\sigma\wedge J, \tag{5.17}
\end{aligned}
$$

where

$$
V = \frac{1}{r^2}\left((r^2+a^2)^3\,(1+g^2\,r^2) + 2g\,m\,(2g\,r^4 + 3a^2\,g\,r^2 - 2a^3)\,s^2 + 4g^2\,m^2\,s^4\right),
$$

$$h_1 = -a + g\,r^2\,H + \frac{2a^2\,g\,m\,s^2}{\rho^4}\,, \qquad h_2 = -a^2 - r^2\,H + \frac{2a^3\,g\,m\,s^2}{\rho^4}\,,$$

$$H = 1 + \frac{2ms^2}{\rho^4}\,, \qquad \rho^2 = r^2 + a^2\,, \qquad s \equiv \sinh\delta\,, \qquad c \equiv \cosh\delta\,, \tag{5.18}$$

D. The Supersymmetric Limit

The charged rotating black hole solutions in seven-dimensional gauged supergravity that we have derived in this chapter are in general non-extremal, with the mass and the electric charges freely specifiable. It is of interest also to study the extremal limit, in which one obtains supersymmetric BPS black hole solutions. For simplicity, we shall just present the results for the case where the two electric charges are set equal here.

The criterion for supersymmetry is that there should exist supersymmetry parameters ϵ such that the supersymmetry variations of the spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fields ψ_μ and λ_i in the gauged supergravity theory should vanish. This is most easily checked by looking at the integrability condition for the spin-$\frac{3}{2}$ field, and by looking directly at the transformation rule for the spin-$\frac{1}{2}$ field. This latter, in the case where the electric charges are set equal, takes the form

$$\delta\lambda_i = -\tfrac{1}{4}\Gamma^\mu\,\epsilon\,X^{-1}\,\partial_\mu X + \tfrac{i}{40}\,X^{-1}\,F_{\mu\nu}\,\Gamma^{\mu\nu}\,\epsilon - \tfrac{1}{480}\,X^2\,F_{\mu\nu\rho\sigma}\,\Gamma^{\mu\nu\rho\sigma}\,\epsilon + \tfrac{1}{5}g\,(X - X^{-4})\,\epsilon\,. \tag{5.19}$$

By studying the eigenvalues of the matrix that acts on ϵ, we find that there can exist Killing spinors if the parameter δ satisfies

$$\tanh\delta = \frac{\pm 1}{1 + a\,g}\,, \tag{5.20}$$

which implies that

$$s \equiv \sinh \delta = \frac{\pm 1}{\sqrt{\Xi_+^2 - 1}} \, , \qquad c \equiv \cosh \delta = \frac{\Xi_+}{\sqrt{\Xi_+^2 - 1}} \, . \qquad (5.21)$$

Specifically, we find that the 8×8 matrix has two zero eigenvalues if equation (5.20) holds.

Our results for supersymmetric black holes reduce to previously-known cases if we specialise to $g = 0$ or $a = 0$. In either of these cases, we find that the number of zero eigenvalues increases to four, with the BPS condition (5.20) reducing now to the familiar one that the "extremality parameter" is given by $\delta \to \pm\infty$ and $m \to 0$ with $m \sinh^2 \delta$ fixed in the BPS limit. Thus we see that, as is the case also in four dimensions, BPS rotating black holes in gauged supergravity have only one half of the supersymmetry that occurs if either the rotation or the gauge coupling is set to zero.

It is not uncommon, for certain ranges of the parameters, for a rotating black hole to have naked closed timelike curves (CTCs). In our solution, with the two charges set equal, it is easy to see that

$$
\begin{aligned}
H^{-2/5} g_{00} &= \left(\frac{4 f_2^2}{\rho^4 H^2 \Xi^2} - \frac{Y}{\Xi_-^2} \right) \frac{1}{f_1} \\
&= \frac{2m(1 - (\Xi_+^2 - 1)s^2) - \Xi_+^2 \rho^4}{\Xi^2 H^2 \rho^4} \, .
\end{aligned}
\qquad (5.22)
$$

The horizon is located at the outer root of $Y = 0$. The absence of CTCs requires that $f_1 > 0$, and so a necessary condition for no naked CTCs is that on the horizon, the expression on the second line be non-negative. This can be satisfied if $s^2 < s_0^2 \equiv 1/(\Xi_+^2 - 1)$, provided that m is sufficiently large. However, in the BPS limit, where $s = s_0$, the metric will necessarily have naked CTCs. (In fact recently an alternative supersymmetric limit of our seven-dimensional non-extremal black hole solution has

been found, which does include a regular black hole with no CTCs or singularities on or outside the event horizon [41].)

E.   Conclusions

In this chapter, we have constructed non-extremal charged rotating black hole solutions in seven-dimensional gauged supergravity. The solutions carry two independent charges, associated with gauge fields in the $U(1) \times U(1)$ abelian subgroup of the $SO(5)$ gauge group. In order to simplify the problem we set the three angular momenta of a generic rotating black hole equal. An interesting new feature that arises in seven dimensions is that the 4-form field $F_{(4)}$, which is also non-zero when the two electric charges are both non-vanishing, satisfies a first-order "odd-dimensional self-duality" equation. This implies that the structure of the solutions is considerably more complicated than in previous examples that were studied in four and in five dimensions. As well as obtaining the non-extremal black hole solutions, we also considered their BPS limits, showing that one can obtain supersymmetric rotating black hole solutions of seven-dimensional gauged supergravity.

   The results presented in this chapter are of significance for the $\mathrm{AdS}_7/\mathrm{CFT}_6$ correspondence in M-theory.

CHAPTER VI

CONCLUSIONS

In this dissertation, we systematically construct non-extremal charged rotating anti-de Sitter black hole solutions in four, five and seven dimensions. In four dimensions, after we obtain rotating Kerr-Taub-NUT metric with four independent charges, as solutions of $\mathcal{N} = 2$ supergravity coupled to three abelian vector multiplets by the solution generating technique, we then generalise the four-dimensional rotating solutions to the solutions of gauged $\mathcal{N} = 4$ supergravity with charges set pairwise equal. The four-dimensional charged Kerr-Taub-NUT solution that we obtained in chapter III provide new gravitational backgrounds for four-dimensional vacua in compactified string theory. In particular, the non-extreme Kerr-Taub-NUT solution of gauged supergravity provide asymptotically AdS backgrounds that are characterised by their mass, angular momentum and two pair-wise equal charges (implying that they can be viewed as solutions in $\mathcal{N} = 4$ gauged supergravity). The gauged solutions should provide new information on the dual three-dimensional conformal field theory at non-zero temperature.

In five dimensions, the most general charged rotating black hole solution has three charge and two rotation parameters. In chapter IV we obtain several special cases of the general solution. To be specific, we obtain the first example of non-extremal rotating black hole solution with two independent rotation parameters, which has two charge parameters set equal and the third vanishing. In another example, we obtain the non-extremal charged rotating black hole solution with three charge parameters set equal and non-equal rotation parameters. We are also able to construct the single-charge solution with two independent rotation parameters. Rotating black hole solutions in five-dimensional gauged supergravity provide backgrounds whose

AdS/CFT duals describe four-dimensional field theories in the rotating Einstein universe on the boundary of anti-de Sitter spacetime. With the general solutions in minimal gauged supergravity that we have now found, this aspect of the AdS/CFT correspondence can be studied in a framework that also allows one to take a BPS or near-BPS limit, where the mapping from the bulk to the boundary is better controlled. In particular, it is of great interest to provide the microscopic interpretation from the boundary CFT for the entropy of the supersymmetric black holes with two general rotations [42, 43].

In seven dimensions, We obtain the solution for non-extremal charged rotating black holes in gauged supergravity, in the case where the three rotation parameters are set equal. There are two independent charges, corresponding to gauge fields in the $U(1) \times U(1)$ abelian subgroup of the $SO(5)$ gauge group. The results presented in chapter V are of significance for the $\text{AdS}_7/\text{CFT}_6$ correspondence in M-theory.

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APPENDIX A

PREVIOUSLY-KNOWN ROTATING BLACK HOLES

In this appendix we present the previously-known rotating black hole solutions of five-dimensional supergravity, using the formalism that we have introduced in Chapter IV. These amount to three cases. The first is the case found in [20] with two charges set equal and the third related to this, in gauged $\mathcal{N} = 2$ supergravity. The second case, obtained in [21], is where all three charges are equal in $\mathcal{N} = 2$ gauged supergravity; this can be viewed also as the general solution in minimal gauged supergravity. The third case is the general solution in ungauged $\mathcal{N} = 2$ supergravity, with three unequal charges, which was obtained in [24, 26]. All three of these cases can be represented elegantly within the formulation of Chapter IV, and thus to present them we need only specify the various functions and gauge potentials.

CASE 1: Two equal charges in gauged supergravity

In this solution, obtained in [20], we have $\delta_1 = \delta_2 = \delta$, with $\delta_3 = 0$. In the ungauged theory, this choice of charge parameters would imply that two of the three physical conserved charges were equal and non-vanishing, whilst the third vanished. As was shown in [20], in the case of the solution in gauged supergravity the third physical charge is actually non-vanishing too, with a value related to those of the other two. We find that in the formalism of Chapter IV, this solution is given by

$$
\begin{aligned}
X &= (x+a^2)(x+b^2)-2mx+g^2(x+a^2+2ms^2)(x+b^2+2ms^2)x\,, \\
Y &= -(a^2-y)(b^2-y)(1-g^2\,y)\,, \\
G &= (x+y)(x+y-2m)+g^2(x+y)^2\,(x-y+a^2+b^2+2ms^2)H\,,
\end{aligned}
$$

$$U = yX - xY, \qquad Z = ab(X+Y),$$

$$\Phi = \frac{G}{(x+y)^3 \, H^2},$$

$$\mathcal{A} = abd\chi - (x+a^2+b^2+2ms^2)d\sigma$$

$$+\frac{1}{G}\left[-(x+y+2ms^2)[abd\chi-(x+a^2+b^2-2m)d\sigma]\right.$$

$$\left.+g^2(x+a^2+2ms^2)(x+b^2+2ms^2)(x+y+2ms^2)d\sigma\right]. \qquad (A.1)$$

That the gauge potentials in (4.59) reduce to

$$A^1 = A^2 = \frac{2msc}{x+y+2ms^2}\,[dt+abd\chi+(y-a^2-b^2)d\sigma],$$

$$A^3 = \frac{2ms^2}{x+y}\,(abd\sigma-yd\chi), \qquad (A.2)$$

and the functions H_i reduce to $H_3 = 1$, and

$$H_1 = H_2 = H = 1+\frac{2ms^2}{x+y}. \qquad (A.3)$$

CASE 2: Three equal charges in gauged supergravity

This solution, obtained in [21], which can also be viewed as the general rotating black hole solution in five-dimensional minimal gauged supegravity, corresponds in the formalism of Chapter IV to taking $\delta_1 = \delta_2 = \delta_3 = \delta$. We find that it then takes the form

$$X = (x+a^2)(x+b^2)-2mx$$

$$+g^2(x+a^2+2ms^2)(x+b^2+2ms^2)[x+2ms^2-(a^2+b^2)s^2+2absc],$$

$$Y = -(a^2-y)(b^2-y)[1-g^2(y+(a^2+b^2)s^2-2absc)],$$

$$G = (x+y)(x+y-2m)+g^2(x+y)^2\,(x-y+a^2+b^2+2ms^2)H^2,$$

$$U = yX-xY+s^2W, \qquad Z = ab(X+Y)+scW,$$

$$
\begin{aligned}
W &= -2g^2 m(a^2-y)(b^2-y)[x(c^2+s^2)+(a^2+b^2)s^2+2ms^4] \\
&\quad +g^4(x+a^2+2ms^2)(x+b^2+2ms^2)(a^2-y)(b^2-y)(x+y+2ms^2)\,, \\
\Phi &= \frac{G}{(x+y)^3\,H^3}\,, \\
\mathcal{A} &= s(xd\chi+abd\sigma)+c[abd\chi-(x+a^2+b^2+2ms^2)d\sigma]+2ms^3\,d\chi \\
&\quad +\frac{H}{G}\Big[-s(x+y-2m)(xd\chi+abd\sigma)-c(x+y)[abd\chi-(x+a^2+b^2-2m)d\sigma] \\
&\quad +g^2(x+a^2+2ms^2)(x+b^2+2ms^2)(x+y+2ms^2)(cd\sigma-sd\chi)\Big]\,.
\end{aligned}
\tag{A.4}
$$

The gauge potentials in (4.59) reduce to

$$
A^1 = A^2 = A^3 = \frac{2msc}{x+y+2ms^2}\left\{dt+c[abd\chi+(y-a^2-b^2)d\sigma]+s(abd\sigma-yd\chi)\right\}\,, \tag{A.5}
$$

and the functions H_i are given by

$$
H_1 = H_2 = H_3 = H \equiv 1+\frac{2ms^2}{x+y}\,. \tag{A.6}
$$

CASE 3: Three unequal charges in ungauged supergravity

This solution was first obtained in [26], by applying a solution-generating proce-
dure to add charges to the neutral five-dimensional rotating black hole of Myers and
Perry [25]. We find that in the formulation of Chapter IV, it takes the simple form

$$
\begin{aligned}
X &= (x+a^2)(x+b^2)-2mx\,, \qquad Y = -(a^2-y)(b^2-y)\,, \\
G &= (x+y)(x+y-2m)\,, \qquad U = yX-xY\,, \qquad Z = ab(X+Y)\,, \\
\Phi &= \frac{G}{(x+y)^3\,H_1 H_2 H_3}\,, \\
\mathcal{A} &= \frac{2mc_1c_2c_3}{G}[(a^2+b^2-y)d\sigma-abd\chi]-\frac{2ms_1s_2s_3}{x+y}(abd\sigma-yd\chi)\,.
\end{aligned}
\tag{A.7}
$$

VITA

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