

GEOMETRIC SUFFICIENT CONDITIONS FOR
COMPACTNESS OF THE $\bar{\partial}$ -NEUMANN OPERATOR

A Dissertation

by

SAMANGI MUNASINGHE

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2006

Major Subject: Mathematics

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ABSTRACT

Geometric Sufficient Conditions for
Compactness of the $\bar{\partial}$ -Neumann Operator. (August 2006)

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For smooth bounded pseudoconvex domains in \mathbb{C}^n , we provide geometric conditions on (the points of infinite type in) the boundary which imply compactness of the $\bar{\partial}$ -Neumann operator. This is an extension of a theorem of Straube for smooth bounded pseudoconvex domains in \mathbb{C}^2 .

To my parents.

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CHAPTER I

INTRODUCTION

The existence and regularity properties of the solution to the $\bar{\partial}$ -equation, $\bar{\partial}u = \alpha$, are important problems in several complex variables. On a bounded pseudoconvex domain Ω in \mathbb{C}^n , the existence of the solution was first given by Hörmander in ([13]). The inverse N of the complex laplacian $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is the $\bar{\partial}$ -Neumann operator. The operator N can be used to give the solution to the $\bar{\partial}$ -equation that is orthogonal to the space of holomorphic functions (or in general to the space of $\bar{\partial}$ -closed forms). This solution is given by $\bar{\partial}^* N\alpha$, and since it is the one orthogonal to the kernel of $\bar{\partial}$, it is the solution with the minimal norm.

Here we are interested in the compactness of the $\bar{\partial}$ -Neumann operator. Consequences of compactness of N include global regularity of N ([14]) and Fredholm theory of Toeplitz operators ([12], [22]).

A well known sufficient condition for compactness of N introduced by McNeal is condition (\tilde{P}) ([16]). Condition (\tilde{P}) is a generalization of the sufficient condition property (P) of Catlin ([3]). A domain satisfies property (P) if there exist sufficiently smooth functions with uniform bound and arbitrarily large Hessians. Condition (\tilde{P}) replaces the uniform bound, with functions whose gradients are uniformly bounded in the metric induced by the Hessians of the functions.

Examples of domains satisfying condition (\tilde{P}) are strictly pseudoconvex domains, or more generally domains of finite type ([3]). Condition (\tilde{P}) , however, is more general than finite type. Domains with finitely many infinite type points, or more generally domains with infinite type points having 2-dimensional Hausdorff measure zero, also satisfy property (P) ([1]).

This thesis follows the style of Transactions of the American Mathematical Society.

The equivalence of condition (\tilde{P}) and compactness of N is known only on some special classes of domains. This equivalence on smooth bounded pseudoconvex Hartogs domains in \mathbb{C}^2 was recently proved by Christ and Fu ([6]). On domains that are locally convexifiable, compactness of the $\bar{\partial}$ -Neumann problem, condition (\tilde{P}) , and absence of analytic discs in the boundary are all equivalent ([10]). If the boundary contains discs, then the condition (\tilde{P}) fails ([16]). However, the absence of discs from the boundary is not enough to imply condition (\tilde{P}) or compactness of N ([20]). In \mathbb{C}^2 , compactness excludes discs from the boundary ([11]). For domains in higher dimensions, however, it is not known whether discs in the boundary obstruct compactness in general. However, if the Levi form has at most one degenerate eigenvalue, discs in the boundary obstruct compactness even in higher dimensions ([18]).

Until recently, all known compactness results were proved via verifying property (P) / condition (\tilde{P}) . Given that it is not understood how much stronger than compactness property (P) / condition (\tilde{P}) are, it is of interest to have an approach to compactness that does not rely on verifying property (P) or condition (\tilde{P}) . This dissertation is a contribution to that circle of ideas.

For domains in \mathbb{C}^2 , Straube gave new geometric conditions that are sufficient for compactness, moreover, the compactness proof does not proceed via verifying condition (\tilde{P}) . In fact, whether these geometric conditions are sufficient for condition (\tilde{P}) to be satisfied is not yet known. Our result here is an extension of Straube's theorem to higher dimensions. The theorem in \mathbb{C}^2 is the following ([21]).

Theorem 1. *Let Ω be a C^∞ -smooth bounded pseudoconvex domain in \mathbb{C}^2 . Denote by K the set of boundary points of infinite type. Assume that there exist constants $C_1, C_2 > 0, C_3$ with $1 \leq C_3 < 3/2$, and a sequence $\epsilon_j > 0_{j=1}^\infty$ with $\lim_{j \rightarrow \infty} \epsilon_j = 0$ so that the following holds. For every $j \in \mathbb{N}$ and $p \in K$ there is a (real) complex tangential vector*

field $Z_{p,j}$ of unit length defined in some neighborhood of p in $b\Omega$ with $\max|\operatorname{div}Z_{p,j}| \leq C_1$ such that $\mathcal{F}_{Z_{p,j}}^{\epsilon_j}(B(p, C_2(\epsilon_j)^{C_3}) \cap K) \subseteq b\Omega \setminus K$. Then the $\bar{\partial}$ -Neumann operator on Ω is compact.

Here $\mathcal{F}_{Z_{p,j}}^{\epsilon_j}(B(p, C_2(\epsilon_j)^{C_3}) \cap K)$ is the flow of the vector field $Z_{p,j}$ at time t , which we assume exist for all initial points in $(B(p, C_2(\epsilon_j)^{C_3}) \cap K)$.

The idea of the proof of the theorem above is the following. To estimate the \mathcal{L}^2 norm of u near a point of infinite type, one expresses u there in terms of u in a patch which meets the boundary in a relatively compact subset of the set of finite type points, plus the integral of the derivative of u in the direction $Z_{P,j}$. While the first term is estimated using subelliptic estimates, the second is estimated by the length of the curve ϵ_j times the \mathcal{L}^2 -norm of $Z_{P,j}u$. In \mathbb{C}^2 , this \mathcal{L}^2 norm can be estimated by the \mathcal{L}^2 -norm of $\bar{\partial}u$ and $\bar{\partial}^*u$, because $Z_{P,j}$ is complex tangential, and domains in \mathbb{C}^2 satisfy so-called maximal estimates. Finally, the overlap and divergence issues coming from the integral of $Z_{P,j}u$ are taken care of by the uniformity built into the assumption in the theorem.

Straube's theorem fails in \mathbb{C}^n for $n \geq 3$ when transcribed verbatim (see [21]). For the extension of this theorem to \mathbb{C}^n for $n \geq 3$, we need more control over the vector fields given above. In the higher dimensional case, we impose control over the vector fields in the theorem by requiring the vector fields to be in complex tangential direction satisfying additional conditions. We denote by $H_\rho(X(\xi), \overline{X(\xi)})$ the Levi form of $\rho(\xi)$ applied to the vector $X(\xi)$.

Theorem 2. *Let Ω be a C^∞ -smooth bounded pseudoconvex domain in \mathbb{C}^n . Denote by K the set of boundary points of infinite type. For all points ξ in a neighborhood of K in $b\Omega$, denote by $\lambda_0(\xi)$ the smallest eigenvalue of the Levi form at ξ . Assume that there exist smooth complex tangential unit vector fields X_1, \dots, X_m , defined on*

$b\Omega$ near K so that $H_p(X_i(\xi), \overline{X_i(\xi)}) \leq C\lambda_0(\xi)$, for some constant C , a sequence $\{\epsilon_j\}_{j=1}^\infty$ with $\lim_{j \rightarrow \infty} \epsilon_j = 0$, and constants $C_1, C_2 > 0, C_3$ with $1 \leq C_3 < \frac{n+1}{n}$ so that the following holds. For every $j \in \mathbb{N}$ and $p \in K$ there is a vector field $Z_{p,j} \in \text{span}_{\mathbb{R}}(\text{Re}X_1, \text{Im}X_1, \dots, \text{Re}X_m, \text{Im}X_m)$ of unit length, defined in some neighborhood of p in $b\Omega$ with $\max_{p,j} |\text{div}Z_{p,j}| \leq C_1$, such that $\mathcal{F}_{Z_p}^{\epsilon_j}(B(p, C_2(\epsilon_j)^{C_3}) \cap K) \subseteq b\Omega \setminus K$. Then the $\bar{\partial}$ -Neumann operator on Ω is compact.

The proof of the above theorem differs from Straube's theorem in estimating $\|Z_p u\|^2$. The techniques used to obtain the necessary estimates are from a paper of Derridj ([8]), where he obtains the equivalence of "maximal estimates" (see Definition 27) with the condition that all the eigenvalues of the Levi form be comparable.

We use these methods on the vector fields $Z_{p,j}$. The terms which involve $\|\bar{Z}_{p,j} u\|^2$ can be easily estimated using the Kohn-Morrey formula. Using integration by parts, estimates on $\|Z_{p,j} u\|^2$, and finally the hypothesis on the vector fields, we can show estimates similar to maximal estimates hold for $\|Z_{p,j} u\|^2$.

CHAPTER II

BASICS OF $\bar{\partial}$ -NEUMANN PROBLEM

Let $\Omega \in \mathbb{C}^n$, $n \geq 2$ be a smooth bounded domain. A smooth (p, q) form on Ω can be expressed as

$$f = \sum'_{I,J} f_{I,J} dz^I \wedge d\bar{z}^J,$$

where $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are multiindices, $\sum'_{I,J}$ means summation over strictly increasing multiindices, $dz^I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$ and $d\bar{z}^J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$, and $f_{I,J} \in C^\infty(\Omega)$. Functions $f_{I,J}$ are defined for arbitrary I and J so that they are antisymmetric.

Denote the space of all smooth (p, q) forms on Ω by $C^\infty_{(p,q)}(\Omega)$. For $f \in C^\infty_{(p,q)}(\Omega)$, define

$$\bar{\partial}_q f = \sum_{I,J} \sum_{k=1}^n \frac{\partial f_{I,J}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz^I \wedge d\bar{z}^J.$$

Let $\mathcal{L}^2_{(p,q)}(\Omega)$ be the (p, q) forms whose coefficients are square integrable with respect to Lebesgue measure on \mathbb{C}^n , and let $dV = i^n dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n$ be the volume element. If $f = \sum'_{I,J} f_{I,J} dz^I \wedge d\bar{z}^J$ and $g = \sum'_{I,J} g_{I,J} dz^I \wedge d\bar{z}^J$ are two (p, q) forms in $\mathcal{L}^2_{(p,q)}(\Omega)$, we define the inner product on $\mathcal{L}^2_{(p,q)}(\Omega)$ by

$$\langle f, g \rangle = \sum'_{I,J} \langle f_{I,J}, g_{I,J} \rangle = \sum'_{I,J} \int_{\Omega} f_{I,J} \overline{g_{I,J}} dV$$

$$\|f\|^2 = \sum'_{I,J} \int_{\Omega} |f_{I,J}|^2 dV.$$

A (p, q) form $u \in \mathcal{L}^2_{(p,q)}(\Omega)$ is in $Dom(\bar{\partial}_q)$ if $\bar{\partial}_q u$ defined in the sense of distribution

belongs to $\mathcal{L}^2_{(p,q+1)}(\Omega)$. Since Ω is bounded we have $C^\infty_{(p,q)}(\overline{\Omega}) \subseteq \text{Dom}(\overline{\partial}_q)$.

Lemma 3. *The operator $\overline{\partial}_q : \mathcal{L}^2_{(p,q)}(\Omega) \longrightarrow \mathcal{L}^2_{(p,q+1)}(\Omega)$ is a closed, densely defined operator.*

By this lemma, the operator $\overline{\partial}_q$ has a Hilbert space adjoint. We denote this adjoint by $\overline{\partial}_q^*$. A $(p, q+1)$ form f belongs to the $\text{Dom}(\overline{\partial}_q^*)$, if there exist a (p, q) form $g \in \mathcal{L}^2_{p,q}(\Omega)$ such that for every $\phi \in \text{Dom}(\overline{\partial}_q) \cap \mathcal{L}^2_{(p,q)}(\Omega)$ we have

$$\langle f, \overline{\partial}\phi \rangle = \langle g, \phi \rangle.$$

Then $\overline{\partial}^* f = g$. For a $(p, q+1)$ form f to be in the $\text{Dom}(\overline{\partial}_q^*)$ it must satisfy the boundary condition given below.

Lemma 4. *Let Ω be a bounded domain with C^1 boundary $b\Omega$ and ρ be a C^1 defining function for Ω . For any $f \in \text{Dom}(\overline{\partial}_q^*) \cap C^1_{(p,q+1)}(\overline{\Omega})$, f must satisfy*

$$\sum_k f_{I,kJ} \frac{\partial \rho}{\partial z_k} = 0 \quad \text{on } b\Omega \quad \text{for all } I, J$$

with $|I| = p$ and $|J| = q$.

This adjoint $\overline{\partial}^* f$ can also be expressed explicitly as,

$$\overline{\partial}_q^* g = (-1)^{p-1} \sum'_{I,J} \sum_{k=1}^n \frac{\partial g_{I,kJ}}{\partial \bar{z}_k} dz^I \wedge d\bar{z}^J$$

for a $(p, q+1)$ form in the $\text{Dom}(\overline{\partial}_q^*)$, where the individual derivatives are taken as distributions (with the resulting forms in $\mathcal{L}^2_{(p,q)}(\Omega)$).

Proposition 5. *Let Ω be a bounded domain in \mathbb{C}^n having C^2 defining function ρ with $|\nabla \rho| \equiv 1$ on $b\Omega$. Let f be a $(0, q)$ -form ($1 \leq q \leq n$) that is in the domain of $(\overline{\partial}^*)$ and that is continuously differentiable on $\overline{\Omega}$, and let a be a real-valued function that*

is twice continuously differentiable on $\bar{\Omega}$, with $a \geq 0$. Then

$$\begin{aligned} \|\sqrt{a} \bar{\partial} f\|^2 + \|\sqrt{a} \bar{\partial}^* f\|^2 &= \sum'_K \sum_{j,k=1}^n \int_{b\Omega} a \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} f_{jK} \bar{f}_{kK} d\sigma \\ &+ \sum'_J \sum_{j=1}^n \int_{\Omega} a \left| \frac{\partial u_J}{\partial \bar{z}_j} \right|^2 dV \\ &+ 2\text{Re} \left\langle \sum'_K \sum_{j=1}^n u_{jK} \frac{\partial a}{\partial z_j} d\bar{z}_K, \bar{\partial}^* u \right\rangle \\ &- \sum'_K \sum_{j,k=1}^n \int_{\Omega} \frac{\partial^2 a}{\partial z_j \partial \bar{z}_k} u_{jK} \bar{u}_{kK} dV. \end{aligned}$$

The proposition given above can be used to show that compactly supported forms are not dense in $Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$ in the graph norm $f \rightarrow \|f\| + \|\bar{\partial} f\| + \|\bar{\partial}^* f\|$. The case $a = 1$ is the Kohn-Morrey formula, and will be used in the proof of Theorem 26. Therefore, we state it here for convenience.

$$\begin{aligned} \|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 &= \sum'_K \sum_{j,k=1}^n \int_{b\Omega} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} f_{jK} \bar{f}_{kK} d\sigma \\ &+ \sum'_J \sum_{j=1}^n \int_{\Omega} \left| \frac{\partial u_J}{\partial \bar{z}_j} \right|^2 dV \end{aligned} \tag{2.1}$$

With the operators $\bar{\partial}$ and $\bar{\partial}^*$ we now have for $0 \leq p \leq n$ and $0 \leq q \leq n-1$, the $\bar{\partial}$ -complex

$$\dots \mathcal{L}_{(p,q-1)}^2(\Omega) \begin{array}{c} \xrightarrow{\bar{\partial}_{(p,q-1)}} \\ \xleftarrow{(\bar{\partial}_{(p,q-1)})^*} \end{array} \mathcal{L}_{(p,q)}^2(\Omega) \begin{array}{c} \xrightarrow{\bar{\partial}_{(p,q)}} \\ \xleftarrow{(\bar{\partial}_{(p,q)})^*} \end{array} \mathcal{L}_{(p,q+1)}^2 \dots$$

Definition 6. Let $\square_{(p,q)} = \bar{\partial}_{(p,q-1)}(\bar{\partial}_{(p,q-1)})^* + (\bar{\partial}_{(p,q)})^*\bar{\partial}_{(p,q)}$ be the operator from $\mathcal{L}_{(p,q)}^2(\Omega)$ to $\mathcal{L}_{(p,q)}^2(\Omega)$ such that $Dom(\square_{(p,q)}) = \{f \in \mathcal{L}_{(p,q)}^2(\Omega) \mid f \in Dom(\bar{\partial}_{(p,q)}) \cap Dom((\bar{\partial}_{(p,q-1)})^*), \bar{\partial}_{(p,q)} f \in Dom((\bar{\partial}_{(p,q)})^*) \text{ and } (\bar{\partial}_{(p,q)})^* f \in Dom(\bar{\partial}_{q-1})\}$.

The operator $\square_{(p,q)}$ defined above is a linear, closed, densely defined, and self-adjoint

operator ([5], Proposition 4.2.3). This and the \mathcal{L}^2 existence theorem for $\bar{\partial}$ on pseudoconvex domains (Definition 7), can be used to show the existence of the inverse of $\square_{(p,q)}$ on pseudoconvex domains. The inverse of $\square_{(p,q)}$ is the $\bar{\partial}$ -Neumann operator N_q . Before we state the \mathcal{L}^2 existence theorem for the $\bar{\partial}$ -Neumann operator, we need the definition of a pseudoconvex domain. There are several equivalent definitions of pseudoconvexity. We first give a definition that applies to smooth domains ([5]).

Definition 7. Let $\Omega \in \mathbb{C}^n$ be a smooth bounded domain, and ρ a (smooth) defining function of Ω . We say that Ω is pseudoconvex at $p \in \text{b}\Omega$ if

$$\sum_{k,l=1}^n \frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_l}(p) \xi_k \bar{\xi}_l \geq 0, \quad \text{for all } \xi \in \mathbb{C}^n \text{ satisfying } \sum_{k=1}^n \frac{\partial \rho}{\partial z_k} \xi_k = 0. \quad (2.2)$$

We say that Ω is a pseudoconvex domain if every point p in the boundary of Ω is pseudoconvex. We call the domain strictly pseudoconvex if the inequality (2.2) is strict at each point of Ω .

Definition 8. A function $\phi : \Omega \rightarrow \mathbb{R}$ on an open subset Ω in \mathbb{R}^n , is called an exhaustion function for Ω if for every $c \in \mathbb{R}$ the set $\{x \in \Omega | \phi(x) < c\}$ is relatively compact in Ω .

Definition 9. A function ϕ defined on an open set $\Omega \subset \mathbb{C}^n$, $n \geq 2$, with values in $[-\infty, +\infty)$ is called plurisubharmonic if

1. ϕ is upper semicontinuous,
2. for any $z \in \Omega$ and $\omega \in \mathbb{C}^n$, $\phi(z + \tau\omega)$ is subharmonic in τ .

For non-smooth or unbounded domains Ω , we define pseudoconvexity by the following ([5]).

Definition 10. An open domain Ω in \mathbb{C}^n is called pseudoconvex if there exists a smooth strictly plurisubharmonic exhaustion function ϕ on Ω .

We now state the theorem which shows the \mathcal{L}^2 -existence of a solution to the $\bar{\partial}$ -equation, $\bar{\partial}u = f$, when the domain is pseudoconvex ([5]).

Theorem 11. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n . For every $f \in \mathcal{L}^2_{(p,q)}(\Omega)$, with $0 \leq p \leq n$, $1 \leq q \leq n$ and $\bar{\partial}f = 0$, one can find $u \in \mathcal{L}^2_{(p,q-1)}(\Omega)$ such that $\bar{\partial}u = f$ and*

$$q \int_{\Omega} |u|^2 dV \leq e\delta^2 \int_{\Omega} |f|^2 dV,$$

where $\delta = \sup_{z,z'} |z - z'|$ is the diameter of Ω .

The following theorem gives the existence of the $\bar{\partial}$ -Neumann operator on pseudoconvex domains and some basic properties of the operator ([5]).

Theorem 12. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n , $n \geq 2$. For each $0 \leq p \leq n$ and $1 \leq q \leq n$, there exist a bounded operator $N_{(p,q)} : \mathcal{L}^2_{(p,q)}(\Omega) \rightarrow \mathcal{L}^2_{(p,q)}(\Omega)$ such that*

1. $\mathcal{R}(N_{(p,q)}) \subset \text{Dom}(\square_{(p,q)})$,

$$N_{(p,q)}\square_{(p,q)} = \square_{(p,q)}N_{(p,q)} = I \text{ on } \text{Dom}(\square_{(p,q)}).$$

2. For any $f \in \mathcal{L}^2_{(p,q)}(\Omega)$, $f = \bar{\partial}\bar{\partial}^*N_{(p,q)}f + \bar{\partial}^*\bar{\partial}N_{(p,q)}f$.

3. $\bar{\partial}N_{(p,q)} = N_{(p,q+1)}\bar{\partial}$ on $\text{Dom}(\bar{\partial})$, $1 \leq q \leq n-1$.

4. $\bar{\partial}^*N_{(p,q)} = N_{(p,q-1)}\bar{\partial}^*$ on $\text{Dom}(\bar{\partial}^*)$, $2 \leq q \leq n$.

5. Let δ be the diameter of Ω . The following estimates hold for any $f \in \mathcal{L}^2_{(p,q)}(\Omega)$:

$$\|N_{(p,q)}f\| \leq \frac{e\delta^2}{q}\|f\|,$$

$$\|\bar{\partial}N_{(p,q)}f\| \leq \sqrt{\frac{e\delta^2}{q}}\|f\|,$$

$$\|\bar{\partial}^*N_{(p,q)}f\| \leq \sqrt{\frac{e\delta^2}{q}}\|f\|.$$

A. Compactness of the $\bar{\partial}$ -Neumann operator

Compactness of $N_{(p,q)}$ can be reformulated in several different ways. These equivalent conditions are given in the lemma below ([11]). In our proof we will use the third condition, which is called a compactness estimate. We drop the subscripts to $\bar{\partial}$ from now on when there is no confusion as to the form level where $\bar{\partial}$ acts.

Lemma 13. *Let Ω be a bounded pseudoconvex domain, $1 \leq q \leq n$. Then the following are equivalent:*

1. *The $\bar{\partial}$ -Neumann operator $N_{(p,q)}$ is compact from $\mathcal{L}_{(p,q)}^2(\Omega)$ to itself.*
2. *The embedding of the space $Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$, provided with the graph norm $u \rightarrow \|\bar{\partial}u\| + \|\bar{\partial}^*u\|$, into $\mathcal{L}_{(p,q)}^2(\Omega)$ is compact.*
3. *For every $\epsilon > 0$ there exist a constant $C_\epsilon > 0$ such that*

$$\|u\|^2 \leq \epsilon(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) + C_\epsilon\|u\|_{-1}^2 \quad (2.3)$$

when $u \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^)$.*

4. *The canonical solution operators $\bar{\partial}^*N_q : \mathcal{L}_{(0,q)}^2(\Omega) \rightarrow \mathcal{L}_{(0,q-1)}^2(\Omega)$ and $\bar{\partial}^*N_{q+1} : \mathcal{L}_{(0,q+1)}^2(\Omega) \rightarrow \mathcal{L}_{(0,q)}^2(\Omega)$ are compact.*

Note that compactness is a local property. The $\bar{\partial}$ -Neumann operator N_q on Ω (Ω sufficiently regular) is compact if and only if every boundary point has a neighborhood U such that the corresponding $\bar{\partial}$ -Neumann operator on $U \cap \Omega$ is compact (see for example [11] Lemma 1.2). Therefore, when verifying a compactness estimate (2.3) it is enough to consider forms in $Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$ supported in a small neighborhood of a boundary point.

On smooth bounded pseudoconvex domains a theorem of Kohn and Nirenberg ([14]) shows that compactness of N_q implies global regularity, i.e., N_q preserves

Sobolev spaces with positive indices. Therefore, compactness of the $\bar{\partial}$ -Neumann operator gives regularity of the solution to the $\bar{\partial}$ equation, $\bar{\partial}u = f$. In fact, the canonical solution operator $\bar{\partial}^* N_q$ also satisfies exact Sobolev estimates. Before stating the theorem, we need a few definitions.

The Sobolev space $W^s(\Omega)$, for any domain $\Omega \subseteq \mathbb{R}^n$ and $s \geq 0$, is defined as the space of restrictions of all functions $u \in W^s(\mathbb{R}^n)$ to Ω . The norm on $W^s(\Omega)$ is given by

$$\|u\|_s = \inf_{\substack{\mathcal{U} \in W^s \\ \mathcal{U}|_{\Omega} = u}} \|\mathcal{U}\|_{s(\mathbb{R}^n)}.$$

The theorem of Kohn and Nirenberg is the following.

Theorem 14. ([14]) *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n with smooth boundary. Let $1 \leq q \leq n$. If N_q is compact on $\mathcal{L}_{(0,q)}^2(\Omega)$, then N_q is compact (in particular, continuous) as an operator from $W_{(0,q)}^s(\Omega)$ to itself, for all $s \geq 0$.*

It is also true that if N_q is compact as an operator from $W_{(0,q)}^s(\Omega)$ to itself for some $s \geq 0$, then N_q is compact in $\mathcal{L}_{(0,q)}^2(\Omega)$. So, compactness of N_q on one $W_{(0,q)}^{s_0}(\Omega)$ gives compactness on all $W_{(0,q)}^s(\Omega)$, $s \geq 0$ ([11]).

Another consequence of compactness of the $\bar{\partial}$ -Neumann operator is compactness of the commutators between the Bergman projection (projection onto the space of holomorphic functions on $\mathcal{L}_{(0,q)}^2(\Omega)$) and multiplication operators. This property is important for the Fredholm theory of Toeplitz operators. ([22], [12])

Theorem 15. ([4], [11]) *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n . Assume that for some q , $0 \leq q \leq n-1$, the canonical solution operator $\bar{\partial}^* N_{q+1}$ is compact. Let M be a function that has bounded first order partial derivatives on Ω . Then the commutators $[P_q, M]$ between the Bergman projection P_q and the multiplication operator by M is compact on $\mathcal{L}_{(0,q)}^2(\Omega)$.*

Note that by Lemma (13) compactness of either N_q or N_{q+1} implies the compactness of $[P_q, M]$. It is also true that, the compactness of $[P_q, M]$ imply compactness of the canonical solution operator $\bar{\partial}^* N_{q+1}$ restricted to forms with holomorphic coefficients ([19]).

B. Sufficient conditions for the compactness of the $\bar{\partial}$ -Neumann operator.

We now consider some well known sufficient conditions for compactness. We will from now on only consider the case $q = 1$, i.e., the compactness of the operator N_1 . It is enough to show compactness of N_1 for the following reason. Compactness of N_q implies the compactness of N_{q+1} ([17]). Therefore, if the compactness of N_1 can be established we have the compactness of N_q for all $1 \leq q \leq n$. Catlin showed in ([3]), that a compactness estimate holds in any pseudoconvex domain which satisfies the condition property (P).

Definition 16. For a bounded pseudoconvex domain Ω we say that $b\Omega$ satisfies property (P) if for every positive number M there is a plurisubharmonic function $\lambda \in C_\infty(\bar{\Omega})$, with $0 \leq \lambda \leq 1$, such that for all $z \in b\Omega$

$$\sum_{k,j=1}^n \frac{\partial^2 \lambda}{\partial z_k \partial \bar{z}_j}(z) w_k \bar{w}_j \geq M |w|^2, \quad \text{for all } w \in \mathbb{C}^n.$$

We have given Catlin's original definition here, although there are weaker versions that suffice for compactness. For example, it is enough to have, for every $M > 0$, a C^2 -smooth function λ , with $0 \leq \lambda \leq 1$, defined in Ω only near $b\Omega$ such that the Hessian of λ is at least M .

McNeal ([16]) introduced a generalization of property (P), condition (\tilde{P}), that still implies compactness. Condition (\tilde{P}) replaces the boundedness condition in property (P) by that of a self-bounded complex gradient.

Definition 17. ([16]) We say that the domain Ω satisfies condition (\tilde{P}) if, for every $M > 0$, there exists $\phi = \phi_M \in C^2(\overline{\Omega})$ such that

1. ϕ has self-bounded complex gradient:

$$\left| \sum_{k=1}^n \frac{\partial \phi}{\partial z_k}(z) \zeta_k \right|^2 \leq \sum_{j,k=1}^n \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l}(z) \zeta_k \bar{\zeta}_l$$

for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ and $z \in \Omega$.

2. The sum of any q eigenvalues of the matrix $(\frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l})(z)$ is greater than or equal to M , for all $z \in b\Omega$.

To see that property (P) implies condition (\tilde{P}) , it suffices to consider the family of functions $\mu_M := e^{\lambda_M}$, where λ_M is the family of functions given by property (P) . McNeal's theorem mentioned above is the following.

Theorem 18. ([16]) *Let $\Omega \in \mathbb{C}^n$ be a smooth bounded pseudoconvex domain and $1 \leq q \leq n$. If Ω satisfies the condition (\tilde{P}) , then N_q is compact.*

The simplest examples of domains satisfying property (P) are strictly pseudoconvex domains. Another more general class of domains that satisfy property (P) are ones that satisfy a condition called finite 1-type. There are different definitions of type. While they are all the same when the domain is in \mathbb{C}^2 , they differ for domains in \mathbb{C}^n for $n > 2$. We use as our definition of 1-type D'Angelo's type ([7]).

A point $p \in b\Omega$ is called a finite 1-type point if the following holds:

Definition 19. $b\Omega$ is of finite 1-type at p in the sense of D'Angelo if the maximal order of contact with $b\Omega$ at p , of 1-dimensional analytic varieties is finite.

The finite type condition is an open condition ([7]). Therefore the set of infinite type points is a compact set in $b\Omega \subseteq \mathbb{C}^n$. A domain is of finite type if all points in the

boundary of the domain are of finite type. The two theorems by Catlin given below show, that if a domain is of finite type, then the domain satisfies property (P).

Theorem 20. ([3]) *Let Ω be a smoothly bounded pseudoconvex domain of finite type in \mathbb{C}^n . Then the boundary of Ω satisfies property (P).*

Since the $\bar{\partial}$ -Neumann operator is compact in domains that satisfy property (P), compactness holds in all domains of finite type. Catlin also proved that a subelliptic estimate holds on domains that satisfy finite 1-type. Subellipticity is a stronger condition than compactness.

Definition 21. Let p be a boundary point of the bounded pseudoconvex domain Ω . A subelliptic estimate of order ϵ is said to hold near p if there are a neighborhood U of p , $\epsilon > 0$, and a constant $C > 0$, such that

$$\|u\|_{\epsilon, U \cap \Omega} \leq C(\|\bar{\partial}u\|_0 + \|\bar{\partial}^*u\|_0), \quad (2.4)$$

for all $u \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$.

Theorem 22. ([2]) *Let z_0 be a point in the boundary of a smooth bounded pseudoconvex domain. Then there is a neighborhood V of z_0 such that 2.4 holds for some $\epsilon > 0$ if and only if the domain is of finite 1-type at p .*

The exact relationship between property (P) (or condition (\tilde{P})) and compactness of the $\bar{\partial}$ -Neumann operator is not known. There are, however, special classes of domains on which this relationship is completely understood. One such class of domains is the class of locally convexifiable domains. In fact, more is known when a domain is locally convexifiable.

Definition 23. A domain is called locally convexifiable if for every boundary point there is a neighborhood, and a biholomorphic map defined on this neighborhood, that takes the intersection of the domain with the neighborhood onto a convex domain.

For locally convexifiable domains the following is true.

Theorem 24. ([11], [10]) *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n which is locally convexifiable, and let $1 \leq q \leq n$. The following are equivalent:*

1. *The $\bar{\partial}$ -Neumann operator N_1 is compact.*
2. *The boundary of Ω does not contain any analytic variety of dimension greater than or equal to 1.*
3. *The boundary of Ω satisfies property (P_1) .*
4. *The boundary of Ω satisfies condition (\tilde{P}) .*

In fact, there are versions of property (P) and condition (\tilde{P}) for q -forms ([11], [16]). Then, theorem 24 remains true for $q \geq 1$ as well. Recall that we mentioned as a consequence of compactness of N_q , the commutator $[P_{q-1}, \bar{z}_j]$ is compact and also that the compactness of $[P_{q-1}, \bar{z}_j]$ gives the compactness of $\bar{\partial}^* N_q$ restricted to forms with holomorphic coefficients. When the domain is convex, compactness $\bar{\partial}^* N_q$ on forms with holomorphic coefficients gives compactness on all of $\mathcal{L}_{(0,q)}^2(\Omega)$ (see [10], Remark (2)), which by (2.3, part (4)) is equivalent to the compactness of the $\bar{\partial}$ -Neumann operator. Therefore, on convex domains the equivalent conditions above are also equivalent to compactness of $[P_{q-1}, \bar{z}_j]$, $1 \leq j \leq n$.

The equivalence of compactness of N_1 , property (P) , and condition (\tilde{P}) on smooth bounded pseudoconvex Hartogs domains in \mathbb{C}^2 was recently established by Christ and Fu ([6]). Pseudoconvex Reinhardt domains are “almost” locally convexifiable, and on such domains we have the following theorem (see [11], Theorem 5.2).

Theorem 25. *Let Ω be a bounded pseudoconvex Reinhardt domain in \mathbb{C}^n , $1 \leq q \leq n$. If the boundary of Ω does not contain an analytic variety of dimension greater than or equal to q , then the $\bar{\partial}$ -Neumann operator N_q on $(0, q)$ -forms is compact.*

Condition (\tilde{P}_1) always excludes discs from the boundary. However, unlike the convexifiable domains, in general domains absence of analytic discs from the boundary is not enough to guarantee condition (\tilde{P}_1) ([20]). Absence of discs in the boundary is also not sufficient for the compactness of the $\bar{\partial}$ -Neumann operator ([15]).

There are no known examples of domains with compact $\bar{\partial}$ -Neumann operator that do not satisfy property (P) (hence condition (\tilde{P})), but the exact relationship between property (P) and compactness of the $\bar{\partial}$ -Neumann operator is not known for general domains. Since compactness is usually proved by verifying property (P) , it is of considerable interest that the theorem in ([21]) and the main result here are not proved by verifying property (P) / condition (\tilde{P}) . However, whether our theorem will provide examples of domains with compact $\bar{\partial}$ -Neumann operator that do not satisfy property (P) / condition (\tilde{P}) is open.

CHAPTER III

A GEOMETRIC CONDITION THAT IMPLIES COMPACTNESS OF THE
 $\bar{\partial}$ -NEUMANN OPERATOR

Theorem 26. *Let Ω be a C^∞ -smooth bounded pseudoconvex domain in \mathbb{C}^n . Denote by K the set of boundary points of infinite type. For all points ξ in a neighborhood of K in $b\Omega$, denote by $\lambda_0(\xi)$ the smallest eigenvalue of the Levi form at ξ . Assume that there exist smooth complex tangential unit vector fields X_1, \dots, X_m , defined on $b\Omega$ near K so that $H_\rho(X_i(\xi), \overline{X_i(\xi)}) \leq C \lambda_0(\xi)$, for some constant C , a sequence $\{\epsilon_j\}_{j=1}^\infty$ with $\lim_{j \rightarrow \infty} \epsilon_j = 0$, and constants $C_1, C_2 > 0, C_3$ with $1 \leq C_3 < \frac{n+1}{n}$, so that the following holds. For every $j \in \mathbb{N}$ and $p \in K$ there is a real vector field $Z_{p,j} \in \text{span}_{\mathbb{R}}(\text{Re}X_1, \text{Im}X_1, \dots, \text{Re}X_m, \text{Im}X_m)$ of unit length, defined in some neighborhood of p in $b\Omega$ with $\max_{p,j} |\text{div}Z_{p,j}| \leq C_1$, such that $\mathcal{F}_{Z_p}^{\epsilon_j}(B(p, C_2(\epsilon_j)^{C_3}) \cap K) \subseteq b\Omega \setminus K$. Then the $\bar{\partial}$ -Neumann operator on Ω is compact.*

Remark: The assumption of the existence of the family of vector fields $X_1 \dots X_m$ satisfying

$$H_\rho(X_j(\xi), \overline{X_j(\xi)}) \leq C \lambda_0(\xi) \quad (3.1)$$

is an additional hypothesis needed to extend the theorem from \mathbb{C}^2 . In \mathbb{C}^2 , $X_1 = L$, where L is the unique complex tangential vector field of type $(1,0)$ on $b\Omega$. In particular, Theorem 26 generalizes the main result in ([21]). This main result does not generalize to \mathbb{C}^n without some further assumption, such as the one made here. To see this, consider a smooth bounded convex domain in \mathbb{C}^3 which is strictly convex, except for an analytic (affine) disc in the boundary. Then one can flow along complex tangential directions from points of the disc into the set of strictly (pseudo) convex boundary points as required in the second part of the assumption in the theorem.

Nonetheless, by a result of Fu and Straube ([11]), the $\bar{\partial}$ -Neumann operator on $(0, 1)$ -forms is not compact on such a domain.

The assumption (3.1) is also satisfied in the cases given below.

(1) When there exist a smooth complex tangential vector field in the direction of the smallest eigenvalue, this vector field X_1 satisfies the condition (3.1). Such a vector field always exists when the Levi form has at most one degenerate eigenvalue: the unit vector field in the direction of the smallest eigenvalue (well defined near K) is smooth.

(2) When the eigenvalues of the Levi form are all comparable, any finite collection of complex tangential vector fields X_1, \dots, X_m will satisfy the condition (3.1).

Definition 27. A domain Ω satisfies a maximal estimate at 0 for $(0, 1)$ forms, if there is a neighborhood V of 0 and a constant $C > 0$ so that

$$\sum_{j,k=1}^n \|\bar{L}_j u_k\|^2 + \sum_{\substack{j=1 \\ k=1, \dots, n}}^{n-1} \|L_j u_k\|^2 \leq C(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2 + \|u\|^2),$$

for all $u \in \Lambda_{(0,1)}(V)$.

It is shown in ([8]) that the eigenvalues of the Levi form are all comparable if and only if Ω satisfies "maximal estimates". In this case any vector field X_i will give

$$\begin{aligned} H_\rho(X_i(\xi), \overline{X_i(\xi)}) &\leq \lambda_{max}(\xi) |X_i(\xi)|^2 \\ &\leq C \lambda_0(\xi) |X_i(\xi)|^2, \end{aligned}$$

where C is independent of ξ . Consequently, the condition that $Z_{p,j} \in \text{span}_{\mathbb{R}}(\text{Re}X_1, \text{Im}X_1, \dots, \text{Re}X_m, \text{Im}X_m)$ is void.

(3) We return to case (1) above: assume the Levi form has at most one degenerate eigenvalue. Then, the Levi form can be diagonalized near each point $p \in K$

([18]). This gives a different perspective on (1), and also a constructive way to find fields X not necessarily in the direction of the smallest eigenvalue, but still satisfying (3.1). The construction is as follows. For every $p \in K$, there is a basis (not necessarily orthogonal) of the complex tangent vectors $X^p(\xi), Y_1^p(\xi), \dots, Y_{n-2}^p(\xi)$ for ξ in a neighborhood V_p of p , with eigenvalues $\mu_i^p(\xi)$ corresponding to $Y_i^p(\xi)$ all positive in this neighborhood and $X^p(\xi)$ the eigenvector associated to the smallest eigenvalue $\mu_0^p(\xi)$ of the Levi form, which in this new basis is given by a diagonal matrix. Note that the non-zero eigenvalues will change (from the eigenvalues of the Levi form in Euclidean basis) in general, when the basis is not orthogonal. Denote by $\lambda_0(\xi), \lambda_1(\xi), \dots, \lambda_{n-2}(\xi)$, the eigenvalues of the Levi form in Euclidean coordinates, with $\lambda_0(\xi)$ the smallest eigenvalue. Since we have assumed that the Levi form has at most one zero eigenvalue, $\lambda_1(p), \dots, \lambda_{n-2}(p)$ and $\mu_1^p(p), \dots, \mu_{n-2}^p(p)$ are all non-zero and $\lambda_0(p) = 0 = \mu_0^p(p)$, we may assume that on the neighborhood V_p of p

$$\lambda_0(\xi) \leq \eta \quad \text{and} \quad \mu_0^p(\xi) \leq \eta,$$

where $\eta = \frac{1}{2} \min_{\xi \in V_p} \{\mu_1^p(\xi), \dots, \mu_{n-2}^p(\xi), \lambda_1(\xi), \dots, \lambda_{n-2}(\xi)\}$. Also, note that $\lambda_0(\xi) = 0$ if and only if $\mu_0^p(\xi) = 0$ (although these two eigenvalues may not be the same when they are different from zero).

Denote by $T^{(1,0)}(b\Omega)$ the $(1, 0)$ -forms in the (complex) tangent space of the $b\Omega$. Any $Y \in T^{(1,0)}(b\Omega)$ can be expressed as

$$Y = b_0 X^p + b_1 Y_1^p + \dots + b_{n-2} Y_{n-2}^p.$$

Then,

$$\begin{aligned}
H_\rho(Y, \overline{Y}) &= \mu_0^p |b_0|^2 + \mu_1 |b_1|^2 + \cdots + \mu_{n-2} |b_{n-2}|^2 \\
&\geq \mu_0^p (|b_0|^2 + \cdots + |b_{n-2}|^2) \\
&\geq \mu_0^p C |Y|^2
\end{aligned}$$

Since $\lambda_0(\xi)$ is the smallest eigenvalue of the Levi form in Euclidean coordinates, we have

$$\lambda_0(\xi) \geq \mu_0^p(\xi) C \quad \text{or} \quad \mu_0^p(\xi) \leq \frac{1}{C} \lambda_0(\xi), \quad \text{for all } \xi \in V_p.$$

This gives us that

$$H_\rho(X^p(\xi), \overline{X^p(\xi)}) = \mu_0^p(\xi) \leq C \lambda_0(\xi),$$

for all $\xi \in V_p$. Therefore, the eigenvector X^p satisfies the condition (3.1) of the theorem in the neighborhood V_p .

Let V_{p_1}, \dots, V_{p_l} be a finite open covering of the set K of infinite type points by neighborhoods obtained as above. This gives finitely many vector fields X^{p_i} defined on V_{p_i} , for $1 \leq i \leq l$. Let $\{\phi_i\}_{i=1}^l$ be a partition of unity subordinate to V_{p_1}, \dots, V_{p_l} . Let $X = \sum_{i=1}^l \phi_i X_{p_i}$. Since the Levi form is a positive semi-definite Hermitian form, by the Cauchy-Schwarz inequality we have $H_\rho(X_{p_i}, \overline{X_{p_j}}) \leq H_\rho(X_{p_i}, \overline{X_{p_i}})^{1/2} H_\rho(X_{p_j}, \overline{X_{p_j}})^{1/2}$. Therefore, we get

$$\begin{aligned}
H_\rho\left(\sum_{i=1}^l \phi_i X^{p_i}(\xi), \overline{\sum_{i=1}^l \phi_i X^{p_i}(\xi)}\right) &= \left(\sum_{i,j=1}^l \phi_i \overline{\phi_j} H_\rho(X^{p_i}, \overline{X^{p_j}})\right) \\
&\lesssim \sum_{i=1}^l H_\rho(X^{p_i}, \overline{X^{p_i}}) \\
&\lesssim \lambda_0(\xi).
\end{aligned}$$

That is, the vector field X has the property (3.1).

Derridj showed in ([9], Theorem 7.1), that if maximal estimates hold at $p \in \text{b}\Omega$,

and p is a weakly pseudoconvex point, then the Levi form of Ω cannot be diagonalizable near p when Ω is a domain \mathbb{C}^n for $n \geq 3$. Therefore, the cases (1) and (2) are in some sense at opposite ends of the spectrum.

The only part of the proof in Straube's theorem that uses the dimension is the use of the maximal estimates. Therefore, the proof below follows the proof in ([21]) except for the maximal estimates. The method used to obtain estimates here is similar to the methods used in ([8]) to obtain maximal estimates in \mathbb{C}^2 .

Now we give the proof of Theorem 26.

Proof. First we note that we can extend the vector fields Z_p from $\text{b}\Omega$ to a neighborhood of $\text{b}\Omega$ by letting them be constant along the real normal, so that these vector fields, still denoted by Z_p , are complex tangential to the level sets of the boundary distance.

The first part of the proof follows ([21]). We reproduce it here for the reader's convenience. We verify that a compactness estimate (2.3) holds. For a given $\epsilon > 0$ fix j so that $\epsilon_j < \epsilon$. By a standard covering theorem (see [23] Theorem 1.3.1), we may choose a subfamily \mathcal{P} of the closed balls from $\{\overline{B(P, \frac{C_2}{10}(\epsilon_j)^{C_3})} | P \in K\}$ so that \mathcal{P} is pairwise disjoint, and the corresponding closed balls of radius $\frac{C_2}{2}(\epsilon_j)^{C_3}$, and hence the open balls of radius $C_2(\epsilon_j)^{C_3}$, still cover K . Because K is compact, we obtain a finite family of open balls $\{B(P_k, C_2(\epsilon_j)^{C_3}) | 1 \leq k \leq N, P_k \in K\}$ that covers K , with the corresponding balls of radius $\frac{C_2}{10}(\epsilon_j)^{C_3}$ pairwise disjoint. By decreasing C_2 if necessary, we may assume that $\mathcal{F}_{Z_{p_k,j}}^{\epsilon_j}(B(p_k, C_2(\epsilon_j)^{C_3}) \cap K)$ is relatively compact in $\text{b}\Omega \setminus K$ and that the vector field $Z_{p_k,j}$ is defined in V_{p_k} , with $B(p_k, C_2(\epsilon_j)^{C_3}) \subset V_{p_k}$. So there exist open subsets U_k , $1 \leq k \leq N$, of $\overline{\Omega}$, with

$$K \cap B(p_k, C_2(\epsilon_j)^{C_3}) \subseteq U_k \subseteq B(p_k, C_2(\epsilon_j)^{C_3}) \tag{3.2}$$

and

$$\mathcal{F}_{Z_{p_k,j}}^{\epsilon_j}(U_k) \cap K = \emptyset. \quad (3.3)$$

To verify (2.3), let $u \in C_{(0,1)}^\infty(\overline{\Omega}) \cap \text{Dom}(\overline{\partial}^*)$. Then

$$\|u\|_0^2 = \int_{\left(\bigcup_{k=1}^N U_k\right) \cap \Omega} |u|^2 + \int_{\Omega \setminus \left(\bigcup_{k=1}^N U_k\right)} |u|^2. \quad (3.4)$$

Because $\overline{\Omega \setminus \left(\bigcup_{k=1}^N U_k\right)}$ does not intersect K , we can apply subelliptic estimates ([2]) to estimate the second term on the right hand side of (3.4).

So there exist $s > 0$ and $C > 0$ such that the restriction of u to a neighborhood U in Ω of $\overline{\Omega \setminus \left(\bigcup_{k=1}^N U_k\right)}$ belongs to $W_{(0,1)}^s(U)$ and

$$\|u\|_{W_{(0,1)}^s(U)}^2 \leq C(\|\overline{\partial}u\|_0^2 + \|\overline{\partial}^*u\|_0^2).$$

The interpolation inequality for Sobolev norms gives

$$\begin{aligned} \int_{\Omega \setminus \left(\bigcup_{k=1}^N U_k\right)} |u|^2 &\leq \|u\|_{\mathcal{L}_{(0,1)}^2(U)} \\ &\leq \frac{\epsilon}{C} \|u\|_{W_{(0,1)}^s(U)}^2 + C_\epsilon \|u\|_{-1}^2 \\ &\leq \epsilon(\|\overline{\partial}u\|_0^2 + \|\overline{\partial}^*u\|_0^2) + C_\epsilon \|u\|_{-1}^2. \end{aligned} \quad (3.5)$$

We first note that we can extend the fields $Z_{p_k,j}$ from $\text{b}\Omega$ to the inside of Ω by a fixed distance by letting them be constant along the real normal. In order to simplify the notation, we will use Z_k to denote the vector field $Z_{p_k,j}$. To estimate the first term on the right hand side of (3.4), fix k , $1 \leq k \leq N$. Then

$$\begin{aligned}
\int_{U_k \cap \Omega} |u|^2 &= \int_{U_k \cap \Omega} \left| u(\mathcal{F}_{Z_k}^{\epsilon_j}(x)) - \int_0^{\epsilon_j} Z_k u(\mathcal{F}_{Z_k}^t(x)) dt \right|^2 dV(x) \\
&\leq 2 \int_{U_k \cap \Omega} |u(\mathcal{F}_{Z_k}^{\epsilon_j}(x))|^2 dV(x) \\
&\quad + 2 \int_{U_k \cap \Omega} \left| \int_0^{\epsilon_j} Z_k u(\mathcal{F}_{Z_k}^t(x)) dt \right|^2 dV(x).
\end{aligned} \tag{3.6}$$

The first term on the right hand side of (3.6) can be estimated as follows:

$$\begin{aligned}
\int_{U_k \cap \Omega} |u(\mathcal{F}_{Z_k}^{\epsilon_j})|^2 dV(x) &= \int_{\mathcal{F}_{Z_k}^{\epsilon_j}(U_k \cap \Omega)} |u(y)|^2 \det(\partial x / \partial y) dV(y) \\
&\leq 2 \int_{\mathcal{F}_{Z_k}^{\epsilon_j}(U_k \cap \Omega)} |u(y)|^2 dV(y).
\end{aligned} \tag{3.7}$$

We have used here $\det(\partial x / \partial y)$ for the Jacobian of the diffeomorphism $\mathcal{F}_{Z_k}^{-\epsilon_j} : \mathcal{F}_{Z_k}^{\epsilon_j}(U_k \cap \Omega) \rightarrow U_k \cap \Omega$. The uniform bound on the divergence of Z_k , implies that the rate of change of the volume element under the flows generated by the Z_k 's is uniformly bounded. This gives that $\det(\partial x / \partial y) \leq \exp(tC_1) \leq \exp(\epsilon_j C_1) \leq \exp(\epsilon C_1) \leq 2$ for ϵ small enough. We have used this bound in the inequality above.

By (3.3), we can use subelliptic estimates again to estimate the last term of (3.7). Subelliptic estimates, together with the interpolation inequality of Sobolev norms as in (3.5), give

$$\int_{\mathcal{F}_{Z_k}^{\epsilon_j}(U_k \cap \Omega)} |u(y)|^2 dV(y) \leq \frac{\epsilon}{2N} (\|\bar{\partial} u\|_0^2 + \|\bar{\partial}^* u\|_0^2) + C_\epsilon \|u\|_{-1}^2. \tag{3.8}$$

We now estimate the second term on the right hand side of (3.6). Using the

Cauchy-Schwarz inequality, Fubini's theorem, and the bound on $\det(\partial x/\partial y)$, we obtain

$$\begin{aligned}
\int_{U_k \cap \Omega} \left| \int_0^{\epsilon_j} Z_k u(\mathcal{F}_{Z_k}^t(x)) dt \right|^2 dV(x) &\leq \epsilon_j \int_{U_k \cap \Omega} \int_0^{\epsilon_j} |Z_k u(\mathcal{F}_{Z_k}^t(x))|^2 dt dV(x) \\
&= \epsilon_j \int_0^{\epsilon_j} \int_{U_k \cap \Omega} |Z_k u(\mathcal{F}_{Z_k}^t(x))|^2 dV(x) dt \quad (3.9) \\
&= \epsilon_j \int_0^{\epsilon_j} \int_{\mathcal{F}_{Z_k}^t(U_k \cap \Omega)} |Z_k u(y)|^2 \det(\partial x/\partial y) dV(y) dt \\
&\leq 2\epsilon_j \int_0^{\epsilon_j} \int_{\mathcal{F}_{Z_k}^t(U_k \cap \Omega)} |Z_k u(y)|^2 dV(y) dt.
\end{aligned}$$

Combining the estimates (3.6) through (3.9) and summing over k , the first term on the right hand side of (3.4) can be estimated by

$$\begin{aligned}
\int_{\left(\bigcup_{k=1}^N U_k\right) \cap \Omega} |u|^2 &\leq \sum_{k=1}^N \int_{U_k \cap \Omega} |u|^2 \\
&\leq \sum_{k=1}^N \left[\frac{2\epsilon}{N} (\|\bar{\partial} u\|_0^2 + \|\bar{\partial}^* u\|_0^2) + C_\epsilon \|u\|_{-1}^2 \right. \quad (3.10) \\
&\quad \left. + 4\epsilon_j \int_0^{\epsilon_j} \int_{\mathcal{F}_{Z_k}^t(U_k \cap \Omega)} |Z_k u(y)|^2 dV(y) dt \right] \\
&\leq 2\epsilon (\|\bar{\partial} u\|_0^2 + \|\bar{\partial}^* u\|_0^2) + C_\epsilon \|u\|_{-1}^2 \\
&\quad + 4\epsilon_j \int_0^{\epsilon_j} \left(\sum_{k=1}^N \int_{\mathcal{F}_{Z_k}^t(U_k \cap \Omega)} |Z_k u(y)|^2 dV(y) \right) dt.
\end{aligned}$$

The constant C_ϵ is allowed to change its value from one occurrence to the next.

Since the vector fields satisfy $Z_k \in \text{span}_{\mathbb{R}}(\text{Re}X_1, \text{Im}X_1, \dots, \text{Re}X_m, \text{Im}X_m)$, and

$|Z_k| = 1$, we have that

$$\begin{aligned} |Z_k u(y)|^2 &\lesssim m \sum_{i=1}^m 2 \left[|\operatorname{Re} X_i u(y)|^2 + |\operatorname{Im} X_i u(y)|^2 \right] \\ &\lesssim m \sum_{i=1}^m \left(|X_k u(y)|^2 + |\overline{X_k} u(y)|^2 \right). \end{aligned} \quad (3.11)$$

We have used here that the vector fields X_1, \dots, X_m can be extended using a cutoff function to all of $b\Omega$. This will preserve the crucial property (3.1). Then, these vector fields can be extended further to a neighborhood of $b\Omega$ by defining them to be constant along the real normal of $b\Omega$, and then to all of Ω by multiplying again by a suitable cutoff function.

No point of Ω is contained in more than $C(C_2)(\epsilon_j)^{2n-2nC_3}$ of the sets $\mathcal{F}_{Z_k}^t(U_k \cap \Omega)$, $1 \leq k \leq N$, where $C(C_2)$ denotes a constant depending only on C_2 . In fact, if Q is a point of in $\mathcal{F}_{Z_k}^{\epsilon_j}(U_k \cap \Omega) \cap \mathcal{F}_{Z_m}^{\epsilon_j}(U_m \cap \Omega)$, then by the triangle inequality, the distance between $\mathcal{F}_{Z_k}^{-t}(Q)$ and $\mathcal{F}_{Z_m}^{-t}(Q)$ is no more than $2t \leq 2\epsilon_j$. Therefore, $B(p_m, \frac{C_2}{10}(\epsilon_j)^{C_3}) \subseteq B(p_k, 2\epsilon_j + 2C_2(\epsilon_j)^{C_3} + \frac{C_2}{10}(\epsilon_j)^{C_3})$. Since the balls $B(p_m, \frac{C_2}{10}(\epsilon_j)^{C_3})$, $1 \leq m \leq N$, are pairwise disjoint, comparison of volumes gives the upper bound $C(C_2)(\epsilon_j)^{2n-2nC_3}$ on how many of them can be contained in $B(p_k, 2\epsilon_j + 2C_2(\epsilon_j)^{C_3} + \frac{C_2}{10}(\epsilon_j)^{C_3})$. Combining (3.10, 3.11) and the bound on the overlap we have,

$$\begin{aligned}
\int_{\left(\bigcup_{k=1}^N U_k\right) \cap \Omega} |u|^2 &\leq 2\epsilon(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_\epsilon\|u\|_{-1}^2 + 4\epsilon_j \int_0^{\epsilon_j} \left(C(C_2)(\epsilon_j)^{2n-2nC_3} 2m \times \right. \\
&\quad \left. \sum_{i=1}^m \int_{\Omega} \left[|\operatorname{Re}X_i u(y)|^2 + |\operatorname{Im}X_i u(y)|^2 \right] dV(y) \right) dt \\
&\leq 2\epsilon(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_\epsilon\|u\|_{-1}^2 + 8m \epsilon_j \int_0^{\epsilon_j} \left(C(C_2)(\epsilon_j)^{2n-2nC_3} \times \right. \\
&\quad \left. \sum_{i=1}^m \int_{\Omega} \left(|X_k u(y)|^2 + |\bar{X}_k u(y)|^2 \right) dV(y) \right) dt.
\end{aligned}$$

So far the argument has followed ([21]). In \mathbb{C}^2 , the last term on the right hand side of (3.10) can be estimated using maximal estimates. In our situation, we need a different approach.

We show next the following estimate for each $1 \leq i \leq m$,

$$\|\bar{X}_i u\|^2 + \|X_i u\|^2 \leq C \left(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2 \right),$$

for some C independent of ϵ . Fix i and denote X_i by X . It suffices to argue locally, since multiplication by a cutoff function preserves (3.1). We may, therefore, assume that $\frac{\partial\rho}{\partial z_n} \neq 0$. Let

$$L_j = \frac{\partial\rho}{\partial z_j} \frac{\partial}{\partial z_n} - \frac{\partial\rho}{\partial z_n} \frac{\partial}{\partial z_j} \quad \text{for } 1 \leq j \leq n-1 \quad \text{and} \quad L_n = \sum_{j=1}^n \frac{\partial\rho}{\partial \bar{z}_j} \frac{\partial}{\partial z_j}.$$

Then L_j , $1 \leq j \leq n-1$, are complex tangential and $\{L_j\}_{j=1}^n$ is a local basis for the complex vector fields of type (1,0) in a neighborhood of $\text{b}\Omega$. Note that L_1, \dots, L_n are defined in coordinate patches that are independent of ϵ_j ; consequently, the constants involved in patching together these local estimates are independent of ϵ_j .

$\|\bar{X}u\|^2 \lesssim \sum_{j=1}^n \|\bar{L}_j u\|^2 \lesssim \|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2$, by the Kohn-Morrey formula. So it remains to estimate $\|Xu\|^2$. For this we write $X = \sum_{p=1}^n \beta_p \frac{\partial}{\partial z_p}$, where $\beta_p \in C^\infty(\bar{\Omega})$

for all p . Then

$$\|Xu\|^2 = \int_{\Omega} Xu \cdot \overline{Xu} = \int_{\Omega} \sum_{p=1}^n \beta_p \frac{\partial u}{\partial z_p} \cdot \overline{Xu} = \sum_{p=1}^n \int_{\Omega} \frac{\partial u}{\partial z_p} \cdot \overline{\beta_p Xu}.$$

Using integration by parts and the fact that $X\rho = \sum_{p=1}^n \beta_p \frac{\partial \rho}{\partial z_p} = 0$ on $b\Omega$, we have

$$\begin{aligned} \|Xu\|^2 &= - \sum_{p=1}^n \int_{\Omega} u \frac{\partial}{\partial z_p} (\overline{\beta_p Xu}) + \sum_{p=1}^n \int_{b\Omega} u \overline{Xu} \beta_p \frac{\partial \rho}{\partial z_p} \\ &= - \sum_{p=1}^n \left[\int_{\Omega} u \beta_p \frac{\partial}{\partial z_p} (\overline{Xu}) + \int_{\Omega} u \cdot \overline{Xu} \frac{\partial}{\partial z_p} (\beta_p) \right]. \end{aligned} \quad (3.12)$$

Since X is a C^∞ vector field in a neighborhood of $\bar{\Omega}$, we have for some $a \in C^\infty(\bar{\Omega})$

$$\|Xu\|^2 = - \sum_{p=1}^n \int_{\Omega} u \beta_p \frac{\partial}{\partial z_p} (\overline{Xu}) + \int_{\Omega} a u \cdot \overline{Xu}. \quad (3.13)$$

Hence, for a suitable constant C (depending only on the fields X_i):

$$\begin{aligned} \|Xu\|^2 &= - \int_{\Omega} u \cdot X(\bar{X}\bar{u}) + \int_{\Omega} a u \cdot \overline{Xu} \\ &\lesssim \left[\int_{\Omega} u \cdot [\bar{X}, X] \bar{u} - \int_{\Omega} \overline{\bar{u} \cdot X \bar{X} u} \right] + C(\|u\| \|Xu\|) \\ &= \int_{\Omega} u \cdot [\bar{X}, X] \bar{u} + \|\bar{X}u\|^2 + C(\|u\| \|Xu\|) \\ &\lesssim \int_{\Omega} u \cdot [\bar{X}, X] \bar{u} + C(\|\bar{X}u\|^2 + \|u\|^2) + \frac{1}{2}\|Xu\|^2, \end{aligned} \quad (3.14)$$

where we have used $X\rho = 0$ on $b\Omega$, and integration by parts for the third equality.

This gives

$$\begin{aligned} \|Xu\|^2 &\lesssim 2 \int_{\Omega} u \cdot [\bar{X}, X] \bar{u} + C(\|\bar{X}u\|^2 + \|u\|^2) \\ &\lesssim 2 \int_{\Omega} u \cdot [\bar{X}, X] \bar{u} + C(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2), \end{aligned} \quad (3.15)$$

because $\|u\| \lesssim (\|\bar{\partial}u\|_0 + \|\bar{\partial}^*u\|_0)$ and $\|\bar{X}u\|^2 \lesssim (\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2)$. As $\{L_i, \bar{L}_i\}_{i=1}^n$ is

a basis for vector fields near $b\Omega$, we write $[\bar{X}, X]$ in this basis. For this we find,

$$\langle [\bar{X}, X], L_n \rangle = \sum_{p,q} \bar{\beta}_q \frac{\partial \beta_p}{\partial \bar{z}_q} \cdot \frac{\partial \rho}{\partial z_p}, \quad (3.16)$$

$$\langle [\bar{X}, X], \bar{L}_n \rangle = - \sum_{p,q} \beta_p \frac{\partial \bar{\beta}_q}{\partial z_p} \cdot \frac{\partial \rho}{\partial \bar{z}_q}, \quad (3.17)$$

$$a_j := \langle [\bar{X}, X], L_j \rangle = \sum_p \bar{\beta}_p \left[\frac{\partial \beta_n}{\partial \bar{z}_p} \frac{\partial \rho}{\partial \bar{z}_j} - \frac{\partial \beta_j}{\partial \bar{z}_p} \frac{\partial \rho}{\partial \bar{z}_n} \right], \quad (3.18)$$

and

$$b_j := \langle [\bar{X}, X], \bar{L}_j \rangle = \sum_p -\beta_p \left[\frac{\partial \bar{\beta}_n}{\partial z_p} \frac{\partial \rho}{\partial z_j} - \frac{\partial \bar{\beta}_j}{\partial z_p} \frac{\partial \rho}{\partial z_n} \right]. \quad (3.19)$$

Therefore,

$$\begin{aligned} \|Xu\|^2 &\lesssim \int_{\Omega} u \cdot \left(\sum_{p,q} \bar{\beta}_p \beta_q \frac{\partial^2 \rho}{\partial \bar{z}_p \partial z_q} \right) (\bar{L}_n - L_n) \bar{u} \\ &\quad + \sum_{j=1}^{n-1} \left[\int_{\Omega} u \cdot a_j L_j \bar{u} + \int_{\Omega} u \cdot b_j \bar{L}_j \bar{u} \right] \\ &\quad + C (\|\bar{X}u\|^2 + \|u\|^2) \\ &\lesssim \int_{\Omega} u \cdot \left(\sum_{p,q} \bar{\beta}_p \beta_q \frac{\partial^2 \rho}{\partial \bar{z}_p \partial z_q} \right) (\bar{L}_n - L_n) \bar{u} \\ &\quad + \sum_{j=1}^{n-1} \|a_j\|_{\infty} \|u\| \|\bar{L}_j u\| + \sum_{j=1}^{n-1} \int_{\Omega} u \cdot b_j \overline{L_j u} \\ &\quad + C (\|\bar{X}u\|^2 + \|u\|^2). \end{aligned} \quad (3.20)$$

Now we estimate the third term in the inequality above. Using integration by parts we have,

$$\begin{aligned} \int_{\Omega} u \cdot b_j \overline{L_j u} &= \int_{\Omega} u \cdot b_j \left(\frac{\partial \rho}{\partial \bar{z}_j} \frac{\partial \bar{u}}{\partial \bar{z}_n} - \frac{\partial \rho}{\partial \bar{z}_n} \frac{\partial \bar{u}}{\partial \bar{z}_j} \right) \\ &= - \int_{\Omega} b_j \bar{u} \bar{L}_j u + O(\|u\|^2) \end{aligned} \quad (3.21)$$

Since the vector field X is smooth there is a constant C independent of ϵ and k so that $\max_{1 \leq j \leq n-1} \{\|b_j\|_\infty, \|a_j\|_\infty, \|\bar{L}_j b_j\|_\infty\} \leq C$. Therefore,

$$\sum_{j=1}^{n-1} \|a_j\|_\infty \|u\| \|\bar{L}_j u\| + \sum_{j=1}^{n-1} \int_{\Omega} u \cdot b_j \overline{\bar{L}_j u} \lesssim C [\|u\| \|\bar{L}_j u\| + \|u\|^2]. \quad (3.22)$$

Now using this in (3.20) we have,

$$\begin{aligned} \|Xu\|^2 &\lesssim \int_{\Omega} u \cdot H_\rho(X, \bar{X}) \overline{L_n u} + \int_{\Omega} u \cdot H_\rho(X, \bar{X}) \overline{L_n u} \\ &\quad + C \left[\|u\| \sum_{j=1}^{n-1} \|\bar{L}_j u\| + \|u\|^2 \right] + C (\|\bar{X}u\|^2 + \|u\|^2) \\ &\lesssim \int_{\Omega} u \cdot H_\rho(X, \bar{X}) \overline{L_n u} + C [\|u\| \|\bar{L}_n u\|] \\ &\quad + C (\|\bar{X}u\|^2 + \|u\|^2) \\ &\lesssim \int_{\Omega} u \cdot H_\rho(X, \bar{X}) \overline{L_n u} + C (\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2). \end{aligned} \quad (3.23)$$

We now estimate the remaining term.

$$\begin{aligned} \int_{\Omega} H_\rho(X, \bar{X}) u \overline{L_n u} &= - \sum_{j=1}^n \int_{\Omega} \frac{\partial}{\partial \bar{z}_j} \left(H_\rho(X, \bar{X}) u \frac{\partial \rho}{\partial z_j} \right) \cdot \bar{u} \\ &\quad + \sum_{j=1}^n \int_{b\Omega} H_\rho(X, \bar{X}) |u|^2 \frac{\partial \rho}{\partial z_j} \frac{\partial \rho}{\partial \bar{z}_j} \\ &= - \sum_{j=1}^n \int_{\Omega} \frac{\partial u}{\partial \bar{z}_j} H_\rho(X, \bar{X}) \frac{\partial \rho}{\partial z_j} \cdot \bar{u} + \sum_{j=1}^n \int_{\Omega} |u|^2 \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} H_\rho(X, \bar{X}) \\ &\quad - \sum_{j=1}^n \int_{\Omega} |u|^2 H_\rho(X, \bar{X}) \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_j} + \sum_{j=1}^n \int_{b\Omega} H_\rho(X, \bar{X}) |u|^2 \left| \frac{\partial \rho}{\partial z_j} \right|^2 \end{aligned} \quad (3.24)$$

We continue the estimate here.

$$\begin{aligned}
\int_{\Omega} H_{\rho}(X, \bar{X}) u \overline{L_n u} &\lesssim - \int_{\Omega} H_{\rho}(X, \bar{X}) \bar{u} \bar{L}_n u + \sum_{j=1}^n \int_{b\Omega} H_{\rho}(X, \bar{X}) |u|^2 \left| \frac{\partial \rho}{\partial z_j} \right|^2 + C \|u\|^2 \\
&\lesssim C (\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2) + C \int_{b\Omega} |u|^2 H_{\rho}(X, \bar{X}) \\
&\lesssim C (\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2) + C \int_{b\Omega} |u|^2 \lambda_0(\xi) \\
&\lesssim C (\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2) + C \int_{b\Omega} \sum_{k,j} \frac{\partial^2 \rho}{\partial \bar{z}_k \partial z_j} u_j \bar{u}_k \\
&\lesssim C (\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2).
\end{aligned}$$

We have used the assumption $H_{\rho}(X_p(\xi), \overline{X_p(\xi)}) \leq C \lambda_0(\xi)$ above. The Kohn-Morrey formula was used above to estimate terms with $\|\bar{X}u\|^2$ and to estimate the boundary integral in the third inequality by $\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2$. This gives us that $\|Xu\|^2 \leq C (\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2)$, where C is a constant independent of k and ϵ_j .

With this estimate we now have

$$\begin{aligned}
\int_{\left(\bigcup_{k=1}^N U_k\right) \cap \Omega} |u|^2 &\leq 2\epsilon (\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_{\epsilon} \|u\|_{-1}^2 \\
&\quad + 4\epsilon_j C(C_2) (\epsilon_j)^{2n-2nC_3} \int_0^{\epsilon_j} \int_{\Omega} C (\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) dt \\
&\leq 2\epsilon (\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_{\epsilon} \|u\|_{-1}^2 \\
&\quad + 4C(C_2) (\epsilon_j)^{(2n+2)-2nC_3} C (\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2).
\end{aligned}$$

Since the constants $C(C_2)$ and C are independent of ϵ_j , this estimates the first term in the right hand side of (3.4). Combining the estimates for (3.4) and (3.5) shows that there is a constant C independent of ϵ such that for all sufficiently small

$\epsilon > 0$, we have

$$\|u\|_0^2 \leq C(\epsilon + (\epsilon)^{(2n+2)-2nC_3})(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_\epsilon \|u\|_{-1}^2, \quad (3.25)$$

which gives the required compactness estimate. We use the upper bound $(n+1)/n$ of C_3 here, i.e., $2n+2-2nC_3 > 0$. This completes the proof of the theorem. \square

As for the case in \mathbb{C}^2 , the following are corollaries to the theorem above. Again, as in the theorem, we need some additional hypothesis here. We say that $b\Omega \setminus K$ satisfies a complex tangential cone condition if there exists a finite open real cone Γ in $\mathbb{C}^n \approx \mathbb{R}^{2n}$ having the following property. For each $p \in K$ there exists a complex tangential direction so that when Γ is moved by a rigid motion to have vertex at p and axis in that direction, the (open) cone obtained intersects $b\Omega$ in a set contained in $b\Omega \setminus K$.

Corollary 28. *Let Ω be a C^∞ -smooth bounded pseudoconvex domain in \mathbb{C}^n . Denote by K the set of boundary points of infinite type. For all points ξ in a neighborhood of K in $b\Omega$, denote by $\lambda_0(\xi)$ the smallest eigenvalue of the Levi form. Assume that $b\Omega$ satisfies the following conditions. There exist smooth, complex tangential, unit vector fields X_1, \dots, X_m , defined on $b\Omega$ near K so that $H_\rho(X_i(\xi), \overline{X_i(\xi)}) \leq C \lambda_0(\xi)$, for some constant C and all ξ such that the boundary of Ω satisfies a complex tangential cone condition with the axis of the cone in $\text{span}_{\mathbb{R}}(\text{Re}X_1^p, \text{Im}X_1^p, \dots, \text{Re}X_m^p, \text{Im}X_m^p)$, for all $p \in K$. Then the $\bar{\partial}$ -Neumann operator on Ω is compact.*

An example like the one described after the statement of Theorem 26 shows that it is not sufficient to just assume a weak complex tangential cone condition where the axis of the cone at $p \in K$ lies in the eigenspace associated with the minimal eigenvalue of the Levi form at p .

On the other hand, when the Levi form of $b\Omega$ has at most one degenerate

eigenvalue, there is a complex tangential vector field X and a constant such that $H_\rho(X, \bar{X}) \leq C\lambda_0$ near K (λ_0 denotes again the smallest eigenvalue of the Levi form at a point, see remark (3) after the formulation of Theorem 26). This results in a formulation of the corollary that is as simple as the corresponding corollary in the case of \mathbb{C}^2 in ([21]).

Corollary 29. *Let Ω be a smooth bounded pseudoconvex domain in \mathbb{C}^n ; assume that at each boundary point, the Levi form has at most one degenerate eigenvalue. If the set K of boundary points of infinite type satisfies a cone condition with the axis of the cone in the null space of the Levi form, then the $\bar{\partial}$ -Neumann operator on Ω is compact.*

Proof of Corollary 28. Let $Z \in \text{span}_{\mathbb{R}}(\text{Re}X_1^p, \text{Im}X_1^p, \dots, \text{Re}X_m^p, \text{Im}X_m^p)$ be a smooth non-vanishing vector field near each point p , with $Z(p)$ in the direction of the axis of the cone given by the cone condition. There is a smooth change of coordinates F_p (smooth diffeomorphism) (u_1, \dots, u_{2n-1}) centered at p with respect to which $Y = F'_p(p)(Z) = \frac{\partial}{\partial u_1}$ and $F_p(p) = 0$. Let $\Gamma_p = b\Omega \cap \{\text{cone at } p\}$. Then $F_p(\Gamma_p)$ contains a cone in \mathbb{R}^{2n-1} with vertex at 0 and axis u_1 . Therefore, the flow \mathcal{F}_Y^t satisfies the conditions of the theorem with $C_3 = 1$. The pull-back of this flow under F_p is the flow \mathcal{F}_Z^t . Since F_p is a diffeomorphism we also have a $C > 0$ with $\frac{1}{C} \|p - q\| \leq \|F_p(q)\| \leq C \|p - q\|$. Therefore, \mathcal{F}_Z^t also satisfies the conditions of the theorem with $C_3 = 1$. \square

Another corollary to the theorem is the following.

Corollary 30. *Let Ω be a smooth bounded pseudoconvex domain in \mathbb{C}^n , and K the set of infinite type points of the boundary. For all points ξ in a neighborhood of K in $b\Omega$, denote by $\lambda_0(\xi)$ the smallest eigenvalue of the Levi form. Assume that $b\Omega$ satisfies the following conditions. There exist smooth, complex tangential, unit vector*

fields X_1, \dots, X_m , defined near K on $b\Omega$ so that $H_\rho(X_i(\xi), \overline{X_i(\xi)}) \leq C \lambda_0(\xi)$, for some constant C and all ξ , and such that for all $p \in K$ there exist a unit vector field $Z_p \in \text{span}_{\mathbb{R}}(\text{Re}X_1, \text{Im}X_1, \dots, \text{Re}X_m, \text{Im}X_m)$ defined near p , a neighborhood U_p of p , and $\epsilon_p > 0$ with $\mathcal{F}_{Z_p}^t(U_p \cap K) \subseteq b\Omega \setminus K$ for $t \leq \epsilon_p$. Then the $\bar{\partial}$ -Neumann operator on Ω is compact.

A special case of the corollary arises when the set K is contained in a submanifold M of holomorphic dimension zero, provided the vector fields X_1, \dots, X_m exist so that $\text{span}_{\mathbb{R}}(\text{Re}X_1(p), \text{Im}X_1(p), \dots, \text{Re}X_m(p), \text{Im}X_m(p)) \neq 0$ for all $p \in K$. Fix $p \in K$, and choose j so that $X_j(p) \neq 0$. Note that by assumption $H_\rho(X_j, \overline{X_j})(p) = 0$ (since p is a weakly pseudoconvex point). Therefore, because M has holomorphic dimension zero, the real two dimensional plane spanned by $\text{Re}X_j$ and $\text{Im}X_j$ is not tangential to M at p . Choose Z_p in this plane, transverse to $T_p^{\mathbb{R}}(M)$, and $\epsilon_p > 0$ small enough. The assumptions in the corollary are now easily seen to be satisfied. The corollary removes the requirement that K be contained in a smooth submanifold of the boundary (of holomorphic dimension zero).

CHAPTER IV

CONCLUSION

The result considered in this thesis contributes to the understanding of boundary conditions that imply compactness of the $\bar{\partial}$ -Neumann operator N . We have discussed a generalization of a technique introduced by Straube in ([21]) for establishing compactness that does not rely on property(P)/ condition(\tilde{P}). In particular our theory gives examples of domains with compact $\bar{\partial}$ -Neumann operator. The obvious examples arising in this manner also satisfy the well known sufficient conditions for compactness, property(P). Whether our geometric conditions are necessary for compactness of N is not known. Straube showed in ([21]) that these conditions are necessary in \mathbb{C}^2 modulo the radius of certain balls.

Another natural question is whether these geometric conditions in Theorem 26 imply property(P)/ condition(\tilde{P}), or whether there are examples of domains not satisfying condition(\tilde{P}) but satisfying the geometric conditions discussed here. Since an analytic disc in the boundary is enough for property(P)/ condition(\tilde{P}) to fail, one approach to finding such examples is to construct a domain containing an analytic disc but still satisfying the geometric conditions mentioned here. It is known, however, that an analytic disc in the boundary of a domain in \mathbb{C}^2 is enough for compactness to fail. Also, it is known that on locally convexifiable domains, the three conditions compactness of N , property(P)/ condition (\tilde{P}) and absence of discs in the boundary are all equivalent. Therefore, any such example will have to be in \mathbb{C}^n with $n \geq 3$ and not locally convexifiable. Thus many interesting questions about compactness still remain to be investigated.

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