ON THE STRUCTURE OF SOME FREE PRODUCTS OF C^*-ALGEBRAS

A Dissertation

by

NIKOLAY ANTONOV IVANOV

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2007

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Kenneth Dykema
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ABSTRACT

On the Structure of Some Free Products of C^*-Algebras. (August 2007)

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Chair of Advisory Committee: Dr. Kenneth Dykema

The research area of this work is Operator Algebras. Concretelly we study some free products of C^*-algebras. We are concerned with the questions of simplicity, uniqueness of trace, positive cone of K-theory and some others.

In Chapter I we recall the notions of full and reduced free product of C^*-algebras and give some properties of those.

In Chapter II we prove the existence of a six term exact sequence for the K-theory of full amalgamated free product C^*-algebras $A *_C B$, in the case when C is an ideal in both C^*-algebras A and B.

In Chapter III we find a necessary and sufficient conditions for the simplicity and uniqueness of trace for reduced free products of finite families of finite dimensional C^*-algebras with specified traces on them.

In Chapter IV we study some reduced free products of C^*-algebras with amalgamations. We give sufficient conditions for the positive cone of the K_0 group to be the largest possible. We also give sufficient conditions for simplicity and uniqueness of trace.

The research on Operator Algebras was inspired by Quantum Mechanics. The small contribution we made on free products of C^*-algebras helps us to understand these mathematical objects a little bit better.

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CHAPTER I

INTRODUCTION: DEFINITION AND PROPERTIES OF FREE PRODUCTS

A. Introduction

In [39] Voiculescu indroduced the noncommutative probabilistic theory of freeness together with the notion of reduced amalgamated free products of C^*-algebras (W^*-algebras). The simplest case is amalgamation over the complex numbers, which was considered independently by Avitzour in [3]. Since then free probability became an important branch of operator algebra theory. There are many examples of reduced amalgamated free products. Some of the most important ones are the reduced C^*-algebras (W^*-algebras) of amalgams of countable discrete groups. Many properties of those mathematical objects have been, and are, studied. In this report we give small contribution to this research.

One of the important question concerning reduced amalgamated free products is the question of simplicity which usually goes together with the question of uniqueness of trace. Avitzour gave a sufficient condition for simplicity and uniqueness of trace of reduced free products. Avitzour's work is based on the work of Powers [31], in which Powers proved that the reduced C^*-algebra of the free group on two generators is simple and has a unique trace. Subsequently Pashke and Salinas in [28] and Choi in [6] considered other reduced C^*-algebras of amalgams of discrete groups. The most general result for the case of reduced C^*-algebras of amalgams of discrete groups, that generalize Power's result is due to de la Harpe ([19]).

Another important question about reduced (amalgamated) free products is the computation of their K-theory. The K-theory of reduced free products of nuclear

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C^*-algebras was determined by Germain in [17] in terms of the K-theory of the underlying C^*-algebras. He gave partial results in [18] for the K-theory of some reduced amalgamated free products. The question of determinig the K-theory of reduced C^*-algebras of amalgams of discrete groups in terms of the K-theory of the reduced C^*-algebras of the underlying groups was resolved completely by Pimsner in [30].

In [2] Anderson, Blackadar and Haagerup studied the scale and the positive cone of $\mathbf{K_0}$ for the Choi algebras. In [16] Dykema and Rørdam extended their result to the case of reduced free products of C^*-algebras.

One somewhat related C^*-algebra construction is the full amalgamated free product of C^*-algebaras which we mention and compute the K-theory of a very special case.

The structure of this report is as follows:

• In the next section we will briefly recall the notion of reduced free product of C^*-algebras and give some of its properties. After that we will recall the notion of reduced amalgamated free product of a family of C^*-algebras (introduced by Voiculescu in [39]) in more details and explain the actual construction. We will also recall the definition of the full amalgamated free product of C^*-algebras.

• In Chapter II we compute the K-theory of the full amalgamated free product C^*-algebas $A *_B C$ in the case when the C^*-algebra C is an ideal in both of the C^*-algebras A and B.

• In Chapter III we give a necessary and sufficient condition for simplicity and uniqueness of trace of the reduced free product of finite family of finite dimensional C^*-algebras.

• In Chapter IV we give a sufficient condition for simplicity and uniqueness of trace for the reduced amalgamated free products of C^*-algebras. We also give a sufficient condition for the positive cone of the $\mathbf{K_0}$ group to be the largest possible.

B. Definition and Properties of Free Products

We recall the definitions and some of the properties of the reduced free product, the reduced amalgamated free product and the full amalgamated free product of C^*-algebras. We begin by recalling the definition of freeness ([39]).

Definition B.1. *The couple (A, ϕ), where A is a unital C^*-algebra and ϕ a state is called a C^*-noncommutative probability space or C^*-NCPS.*

Definition B.2. *Let (A, ϕ) be a C^*-NCPS and $\{A_i | i \in I\}$ be a family of C^*-subalgebras of A, s.t. $1_A \in A_i$, $\forall i \in I$, where I is an index set. We say that the family $\{A_i | i \in I\}$ is free if $\phi(a_1 ... a_n) = 0$, whenever $a_j \in A_{i_j}$ with $i_1 \neq i_2 \neq ... \neq i_n$ and $\phi(a_j) = 0$, $\forall j \in \{1, ...n\}$. A family of subsets $\{S_i | i \in I\} \subset A$ is $*$-free if $\{C^*(S_i \cup \{1_A\}) | i \in I\}$ is free.*

Let $\{(A_i, \phi_i) | i \in I\}$ be a family of C^*-NCPS such that the GNS representations of A_i associated to ϕ_i are all faithful. Then there is a unique C^*-NCPS $(A, \phi) \overset{def}{=} \underset{i \in I}{*} (A_i, \phi_i)$ with unital embeddings $A_i \hookrightarrow A$ that has the following **properties**:

(1) $\phi|_{A_i} = \phi_i$

(2) the family $\{A_i | i \in I\}$ is free in (A, ϕ)

(3) A is the C^*-algebra generated by $\bigcup_{i \in I} A_i$

(4) the GNS representation of A associated to ϕ is faithful.

And also:

(5) If ϕ_i are all traces then ϕ is a trace too ([39]).

(6) If ϕ_i are all faithful then ϕ is faithful too ([11]).

In the above situation A is called the reduced free product algebra and ϕ is called the free product state. Also the construction of the reduced free product is based on defining a free product Hilbert space, which turns out to be $\mathfrak{H}_A$ - the GNS Hilbert space for A, associated to ϕ (GNS stands for Gel′fand, Naimark, Segal).

Now we will recall the construction of reduced amalgamated free products of C^*-algebras of Voiculescu, following closely [39] and [14, §1].

Definition B.3. *Suppose that we have unital C^*-algebras $1_{\mathfrak{A}} \in \mathfrak{B} \subset \mathfrak{A}$ and conditional expectation $\mathfrak{E} : \mathfrak{A} \to \mathfrak{B}$. Suppose that we have a family $\mathfrak{B} \subset \mathfrak{A}_\iota \subset \mathfrak{A}$, $\iota \in I$ of C^*-subalgebras of $\mathfrak{A}$, all of them containing $\mathfrak{B}$. We say that the family $\{\mathfrak{A}_\iota | \iota \in I\}$ is $\mathfrak{E}$-free if for any elements $a_{\iota_k} \in \mathfrak{A}_{\iota_k}$, $k = 1, \ldots, n$, such that $\iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \ldots \iota_{n-1} \neq \iota_n$ and $\mathfrak{E}(a_{\iota_k}) = 0$, we have $\mathfrak{E}(a_{\iota_1} a_{\iota_2} \cdots a_{\iota_n}) = 0$. We say that the elements $a_\iota \in \mathfrak{A}, \iota \in I$ are $\mathfrak{E}$-free if the family $\{C^*(\mathfrak{B} \cup \{a_\iota\}) | \iota \in I\}$ is $\mathfrak{E}$-free. This includes the case $\mathfrak{B} = \mathbb{C}$ and $\mathfrak{E}$ being a state.*

Let I be a index set, $\mathrm{card}(I) \geq 2$. Let B be a unital C^*-algebra and for each $\iota \in I$ we have a unital C^*-algebra A_ι, which contains a copy of B as a unital C^*-subalgebra. We also suppose that for each $\iota \in I$ there is a conditional expectation $E_\iota : A_\iota \to B$, satisfying

$$\forall a \in A_\iota,\ a \neq 0,\ \exists x \in A_\iota,\ E_\iota(x^* a^* a x) \neq 0. \tag{1.1}$$

The reduced amalgamated free product of (A_ι, E_ι) is denoted by

$$(A, E) = \mathop{*}_{\iota \in I} (A_\iota, E_\iota).$$

The construction in the case $B \neq \mathbb{C}$ depends on some knowledge on Hilbert C^*-modules (see Lance's book [24] for a good exposition).

$M_\iota = L^2(A_\iota, E_\iota)$ will denote the right Hilbert B-module obtained from A_ι by separation and completion with respect to the norm $\|a\| = \|\langle a, a\rangle_{M_\iota}\|^{1/2}$, where $\langle a_1, a_2\rangle_{M_\iota} = E_\iota(a_1^* a_2)$. Then the linear space $\mathcal{L}(M_\iota)$ of all adjointable B-module operators on M_ι is actually a C^*-algebra and we have a representation $\pi_\iota : A_\iota \to \mathcal{L}(M_\iota)$

defined by $\pi_\iota(a)\widehat{a'} = \widehat{aa'}$, where by $\hat{a}$ we denote the element of M_ι, corresponding to $a \in A_\iota$. π_ι is faithful by condition (1.1). Notice that $\pi_\iota|_B : B \to \mathcal{L}(M_\iota)$ makes M_ι a Hilbert $B-B$-bimodule. In this construction we have the specified element $\xi_\iota \overset{def}{=} \widehat{1_{A_\iota}} \in M_\iota$. We call the tripple $(\pi_\iota, M_\iota, \xi_\iota)$ the KSGNS representation of (A_ι, E_ι), i.e. $(\pi_\iota, M_\iota, \xi_\iota)$ =KSGNS(A_ι, E_ι) (KSGNS stands for Kasparov, Steinspring, Gel′fand, Naimark, Segal).

For every right B-module N one has operators $\theta_{x,y} \in \mathcal{L}(N)$ given by $\theta_{x,y}(n) = x\langle y, n\rangle_N$ $(x, y, n \in N)$. The C^*-subalgebra of $\mathcal{L}(N)$ that they generate is actually an ideal of $\mathcal{L}(N)$, which is denoted by $\mathcal{K}(N)$. It is an analogue of the C^*-algebra of all compact operators on a Hilbert space.

Since for every $\iota \in I$, $\theta_{\xi_\iota,\xi_\iota} \in \mathcal{L}(M_\iota)$ is the projection onto the Hilbert $B-B$-subbimodule $\xi_\iota B$ of M_ι it follows that $\xi_\iota B$ is a complemented submodule of M_ι. Therefore if $P_\iota^\circ = 1 - \theta_{\xi_\iota,\xi_\iota}$ then $\pi_\iota(b)P_\iota^\circ = P_\iota^\circ\pi_\iota(b) \in \mathcal{L}(M_\iota)$ for each $b \in B$. We define $M_\iota^\circ \overset{def}{=} P_\iota^\circ M_\iota$. If we view $\xi \overset{def}{=} 1_B$ as an element of the Hilbert $B-B$-bimodule B, we can define

$$M = \xi B \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ \iota_1,\ldots,\iota_n\in I \\ \iota_1\neq\iota_2,\iota_2\neq\iota_3,\ldots,\iota_{n-1}\neq\iota_n}} M_{\iota_1}^\circ \otimes_B M_{\iota_2}^\circ \otimes_B \cdots \otimes_B M_{\iota_n}^\circ, \tag{1.2}$$

where $\otimes_B$ means interior tensor product (see [24]). The Hilbert $B-B$-bimodule M constructed above is called the free product of $\{M_\iota, \iota \in I\}$ with respect to vectors $\{\xi_\iota, \iota \in I\}$ and is denoted by $(M, \xi) = \underset{\iota\in I}{*}(M_\iota, \xi_\iota)$.

For each $\iota \in I$ set

$$M(\iota) = \eta_\iota B \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ \iota_1,\ldots,\iota_n\in I \\ \iota_1\neq\iota_2,\iota_2\neq\iota_3,\ldots,\iota_{n-1}\neq\iota_n \\ \iota_1\neq\iota}} M_{\iota_1}^\circ \otimes_B M_{\iota_2}^\circ \otimes_B \cdots \otimes_B M_{\iota_n}^\circ, \tag{1.3}$$

where $\eta_\iota \stackrel{def}{=} 1_B \in B$. We define a unitary operator

$$V_\iota : M_\iota \otimes_B M(\iota) \to M$$

given on elementary tensors by:

$$[\xi_\iota] \otimes [\eta_\iota] \mapsto \xi,$$

$$[\zeta] \otimes [\eta_\iota] \mapsto \zeta, \text{ where } \zeta \in M_\iota \subset M$$

$$[\xi_\iota] \otimes [\zeta_1 \otimes \cdots \otimes \zeta_n] \mapsto \zeta_1 \otimes \cdots \otimes \zeta_n, \text{where } \zeta_j \in M_{\iota_j} \text{ and } \iota \neq \iota_1, \iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n$$

$$[\zeta] \otimes [\zeta_1 \otimes \cdots \otimes \zeta_n] \mapsto \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n, \text{ where } \zeta \in M_\iota \text{ and }$$
$$\zeta_j \in M_{\iota_j} \text{ with } \iota \neq \iota_1, \iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n.$$

Let $\lambda_\iota : A_\iota \to \mathcal{L}(M)$ be the $*$-homomorphism given by $\lambda_\iota(a) = V_\iota(\pi_\iota(a) \otimes 1)V_\iota^*$. Condition (4.1) implies that λ_ι is injective. Then A is defined as the C^*-subalgebra of $\mathcal{L}(M)$, generated by $\bigcup_{\iota \in I} \lambda_\iota(A_\iota)$, and $E : A \to B$ is the conditional expectation, given by $E(a) = \langle \xi, a(\xi) \rangle_M$. Note that if $b \in B$, then $\lambda_\iota(b) \in \mathcal{L}(M)$ does not depend on ι. $\lambda_\iota(b)$ gives the left action of B on M. Because of condition (4.1) for each $\iota \in I$ we have unital embeddings $A_\iota \hookrightarrow A$, which come from the $*$-homorphisms $\lambda_\iota : A_\iota \to \mathcal{L}(M)$. We will denote by π the representation $\pi : A \to \mathcal{L}(M)$ arising from the reduced amalgamated free product construction. We actually have that $(\pi, M, \xi) = \text{KSGNS}(A, E)$.

Set $A_\iota^\circ = A_\iota \cap \ker(E_\iota)$. For $a_\iota \in A_\iota^\circ$, $\zeta_j \in M_{\iota_j}$ with $\iota_1, \ldots, \iota_n \in I, n \geq 2$, and $\iota_j \neq \iota_{j+1}$ we have

$$\lambda_\iota(a)(\zeta_1 \otimes \cdots \otimes \zeta_n) = \begin{cases} \widehat{a_\iota} \otimes \zeta_1 \otimes \cdots \otimes \zeta_n, & \text{if } \iota \neq \iota_1, \\ (a(\zeta_1) - \xi_{\iota_1}\langle \xi_{\iota_1}, a(\zeta_1)\rangle) \otimes \zeta_2 \otimes \cdots \otimes \zeta_n + & \\ \qquad \pi_{\iota_2}(\langle \xi_{\iota_1}, a(\zeta_1)\rangle)\zeta_2 \otimes \cdots \otimes \zeta_n, & \text{if } \iota = \iota_1. \end{cases} \tag{1.4}$$

We will omit writing λ_ι and π_ι if this leads to no confusion.

Finally let us recall the definition of the full amalgamated free product C^*-algebra.

Definition and Theorem B.4. *Let I be an index family,* $\mathrm{card}(I) \geq 2$ *and suppose for each $\iota \in I$ we have a C^*-algebra A_ι. Then:*

(I) The full free product C^*-algebra of the C^*-algrbras $\{A_\iota | \iota \in I\}$ is the C^*-algebra $\underset{\iota\in I}{*} A_\iota$ obtained by separation and completion of the algebraic free product (over $\mathbb{C}$) with respect to the C^*-semi-norm

$$\|x\| = \sup_{\pi_\iota : A_\iota \to \mathcal{B}(\mathcal{H}_\iota)} \{\|(\underset{\iota\in I}{*}\pi_\iota)(x)| x \in Alg_{\mathbb{C}}(\bigcup_{\iota\in I} A_\iota)\},$$

where the supremum is taken over all $*$-representations $\pi_\iota : A_\iota \to \mathcal{B}(\mathcal{H}_\iota)$. Denote by $j_\iota : A_\iota \to \underset{\iota\in I}{*} A_\iota$ the canonical inclusion.

(II) Let B be a C^*-algebra. Suppose that for each $\iota \in I$ A_ι contains a copy of B, i.e. there is an injective $*$-homomorphism $i_\iota : B \to A_\iota$. The full free product of $\{A_\iota | \iota \in I\}$ amalgamated over B, $\underset{\iota\in I}{*}(A_\iota, B)$, is the quotient of the C^*-algebra $\{A_\iota | \iota \in I\}$ by the ideal generated by

$$\bigcup_{p\neq q} \{j_p \circ i_q(b) - j_q \circ i_p(b) | b \in B\}.$$

We will denote the canonical inclusions $j_\iota : A_\iota \to \underset{\iota\in I}{*}(A_\iota, B)$ in this case too.

The full amalgamated free product $\underset{\iota\in I}{*}(A_\iota, B)$ has the following property:

Proposition B.5. *Let X be a C^*-algebra and let $\alpha_\iota : A_\iota \to X$, $\iota \in I$ be $*$-homomorphisms, such that $\alpha_{\iota_1} \circ i_{\iota_1} = \alpha_{\iota_2} \circ i_{\iota_2}$ for any $\iota_1, \iota_2 \in I$. Then there is a unique $*$-homomorphism $\alpha : \underset{\iota\in I}{*}(A_\iota, B) \to X$ which satisfies $\alpha_\iota = \alpha \circ j_\iota$ for each $\iota \in I$.*

In the case $\mathrm{card}(I) = 2$, i.e. if we have C^*-algebras $A \supset C \subset B$, we will denote the full amalgamated free product by $A *_C B$.

For a good exposition of and many properties of full amalgamated free products of C^*-algebras (pushouts) see[29].

CHAPTER II

ON THE K-THEORY OF FULL FREE PRODUCT C^*-ALGEBRAS WITH AMALGAMATION OVER IDEALS

A. Introduction and Some Definitions

Cuntz conjectured [7, Remark 2] that there is an exact sequence for the K-groups of the amalgamated free product $A *_C B$, where C is a C^*-subalgebra of both the C^*-algebras A and B of the form:

$$\begin{array}{ccccc} \mathbf{K_0}(C) & \longrightarrow & \mathbf{K_0}(A) \oplus \mathbf{K_0}(B) & \longrightarrow & \mathbf{K_0}(A *_C B) \\ \uparrow & & & & \downarrow \\ \mathbf{K_1}(A *_C B) & \longleftarrow & \mathbf{K_1}(A) \oplus \mathbf{K_1}(B) & \longleftarrow & \mathbf{K_1}(C) \end{array} \tag{2.1}$$

For the definition and properties of amalgamated free products (pushouts), see [29]. Here $\mathbf{K}_*$ are the usual K-groups (see [5]).

In [7] Cuntz proved the conjecture for the case when C is a retract in both A and B i.e. there are $*$-homomorphisms $\rho_A : A \to C$ and $\rho_B : B \to C$, s.t. $\rho_A|_C = \rho_B|_C = id|_C$.

In [18] Germain conjectured the existence of a six term exact sequence, similar to the upper one for the Kasparov's KK-groups and proved there that this conjecture is true for the case, where A and B are separable and relatively K-nuclear to C (a notion that he defines there). For the definition and properties of KK-groups, see Kasparov's paper [22] also [5].

In [37] Thomsen proved the exactness of the six-term sequence (conjectured by Germain) for the functors $\mathbf{KK}(D, *)$ and $\mathbf{KK}(*, D)$, for C finite and D separable. In the same paper he proved the exactness of a six-term sequence for the functors $\mathbf{E}(*, D)$ and $\mathbf{E}(D, *)$ for the case when D is separable and either C is nuclear or if there are conditional expectations $E_A : A \to C$ and $E_B : B \to C$. Here $\mathbf{E}$ is the Cones-Higson's $\mathbf{E}$-functor. In all cases A and B have to be separable. For information on the E-groups, see for example [34].

In this paper we prove that the above six term sequence is exact for the $\mathbf{F}$-theory of $A *_C B$ for the case when C is an ideal in both A and B, where $\mathbf{F}$ is a covariant, homotopy-invariant, half-exact, stable functor form the category $\mathbf{C}^*$ of all C^*-algebras to the category $\mathbf{Ab}$ of all abelian groups.

We will be interested only in covariant functors. Note that $\mathbf{KK}(*, D)$ and $\mathbf{E}(*, D)$ are contravariant and the six term sequence is exact for the cases mentioned above with all arrows reversed.

We give some definitions.

Definition A.1. *A covariant functor* $\mathbf{F}$ *from* $\mathbf{C}^*$ *to* $\mathbf{Ab}$ *is called* **stable** *if whenever* $f : A \to A \otimes K$ *is given by* $f(a) = a \otimes e$*, where* e *is a rank 1 projection in* K *- the* C^**-algebra of the compact operators on a separable Hilbert space, and* A *is any* C^**-algebra, then* $\mathbf{F}(f) : \mathbf{F}(A) \simeq \mathbf{F}(A \otimes K)$ *is an isomorphism.*

Definition A.2. *A functor* $\mathbf{F}$ *from* $\mathbf{C}^*$ *to* $\mathbf{Ab}$ *is called* **homotopy invariant** *if whenever* $f_1, f_2 : A \to B$ *are homotopic* $*-$ *homomorphisms (in the topology of pointwise convergence) between* C^**-algebras* A *and* B*, then* $\mathbf{F}(f_1) = \mathbf{F}(f_2)$*.*

Definition A.3. *A covariant functor* $\mathbf{F}$ *from* $\mathbf{C}^*$ *to* $\mathbf{Ab}$ *is called* **half-exact** *if whenever we have a short exact sequence of* C^**-algebras* $0 \to I \to A \to B \to 0$*, then the induced sequence in* $\mathbf{Ab}$ *is exact in the middle term:* $\mathbf{F}(I) \to \mathbf{F}(A) \to \mathbf{F}(B)$*.*

Definition A.4. *A functor* $\mathbf{F}$ *from* $\mathbf{C}^*$ *to* $\mathbf{Ab}$ *is called* **additive** *if whenever* $f_1, f_2 : A \to B$ *are* $*$*-homomorphisms between* C^**-algebras such that* $f_1(a).f_2(b) = 0$ *for every* $a, b \in A$*, then we have* $\mathbf{F}(f_1 + f_2) = \mathbf{F}(f_1) + \mathbf{F}(f_2)$*.*

Examples of covariant, homotopy-invariant, half-exact, stable functors are $\mathbf{K}(*)$ as a functor from $\mathbf{C}^*$ to $\mathbf{Ab}$, $\mathbf{E}(D,*)$ as a functor from $\mathbf{SC}^*$ - the category of all separable C^*-algebras to $\mathbf{Ab}$, where D is separable (see [34], or [5, chapter 25]) and also $\mathbf{KK}(D, * \otimes E)$ as a functor from $\mathbf{C}^*$ to $\mathbf{Ab}$, for D and E nuclear (see [22, §7, Theorem 2]).

With $A * B$ we will denote the free product of A and B with amalgamation over the zero C^*-algebra.

B. Some Results by Cuntz

We will need the following results:

This theorem is due to Cuntz (see [8]):

Theorem B.1. *Let* $\mathbf{F}$ *be a covariant, homotopy-invariant, stable, half-exact functor and let* $\pi_A : A \to A * B$ *and* $\pi_B : B \to A * B$ *be the canonical inclusions. Then* $\mathbf{F}(\pi_A) \oplus \mathbf{F}(\pi_B)$ *is an isomorphism.*

From [8] we have:

Lemma B.2. *Every covariant, stable, half-exact, homotopy invariant functor is additive.*

Now take a short exact sequence $0 \to J \xrightarrow{j} A \xrightarrow{q} B \to 0$. We define $C_q = \{(a, \tilde{b}) \in A \oplus CB \mid q(a) = \tilde{b}(0)\}$ to be the cone of q, where $CB = B \otimes C([0,1))$ is the cone of B and $SB = B \otimes C((0,1))$ is the suspension of B. Let also $S^n B = SS^{n-1}B$ be the n-th suspension of B

We define also $e : J \to C_q$ by $j \mapsto (j, 0)$ and $i : SB \to C_q$ by $\tilde{b} \mapsto (0, \tilde{b})$. It's easy to see that these maps are correctly defined.

Now, using Lemma B.2 and [5, Corollary 21.4.2] we get that $\mathbf{F}(e)$ is an isomorphism for every stable, half-exact, homotopy invariant functor $\mathbf{F}$.

This is from [5, Theorem 21.4.3]. See also [22, §7, Lemma 5]:

Theorem B.3. *Let* $\mathbf{F}$ *be a covariant, additive, homotopy invariant, half-exact functor from* $\mathbf{C}^*$ *to* $\mathbf{Ab}$*. Then* $\mathbf{F}(S^n *)$ *is a homology theory. In other words if* $0 \to J \xrightarrow{j} A \xrightarrow{q} B \to 0$ *is a short exact sequence of* C^**-algebras we have the long exact sequence:*

$$\ldots \overset{\mathbf{F}(Sj)}{\to} \mathbf{F}(SA) \overset{\mathbf{F}(Sq)}{\to} \mathbf{F}(SB) \overset{\partial}{\to} \mathbf{F}(J) \overset{\mathbf{F}(j)}{\to} \mathbf{F}(A) \overset{\mathbf{F}(q)}{\to} \mathbf{F}(B).$$

Here ∂ *is* $\mathbf{F}(e)^{-1} \circ \mathbf{F}(i)$ *with* e *and* i *defined above. Moreover* ∂ *is a natural map.*

Combining [8, Theorem 4.4] and Theorem B.3 we get:

Theorem B.4. *For every covariant, stable, half-exact, homotopy invariant functor* $\mathbf{F}$ *and every short exact sequence of* C^**-algebras* $0 \to J \xrightarrow{j} A \xrightarrow{q} B \to 0$ *the following six term sequence is exact:*

$$\begin{array}{ccccc} \mathbf{F}(J) & \xrightarrow{\mathbf{F}(j)} & \mathbf{F}(A) & \xrightarrow{\mathbf{F}(q)} & \mathbf{F}(B) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\hat{\partial}} \\ \mathbf{F}(SB) & \xleftarrow{\mathbf{F}(Sq)} & \mathbf{F}(SA) & \xleftarrow{\mathbf{F}(Sj)} & \mathbf{F}(SJ) \end{array}$$

Here $\hat{\partial}$ *is the composition of* $S\partial : S^2B \to SJ$ *and the Bott isomorphism* $\mathbf{F}(S^2B) \simeq \mathbf{F}(B)$*.*

We will crucially need the naturality condition (for ∂) from Theorem B.3, so we will give a proof.

Lemma B.5. *The map* ∂ $(\hat{\partial})$ *is a natural map.*

Proof. Suppose we have the following commutative diagram of C*-algebras, where the rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \xrightarrow{j} & A & \xrightarrow{q} & B & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & J' & \xrightarrow{j'} & A' & \xrightarrow{q'} & B' & \longrightarrow & 0 \end{array}$$

We have to prove that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{F}(SB) & \xrightarrow{\partial} & \mathbf{F}(J) \\ {\scriptstyle \mathbf{F}(S\gamma)}\downarrow & & \downarrow{\scriptstyle \mathbf{F}(\alpha)} \\ \mathbf{F}(SB') & \xrightarrow{\partial'} & \mathbf{F}(J') \end{array}$$

But this follows from the following commutative diagram:

$$\begin{array}{ccccc} \partial : \mathbf{F}(SB) & \xrightarrow{\mathbf{F}(i)} & \mathbf{F}(C_q) & \xrightarrow{\mathbf{F}(e)^{-1}} & \mathbf{F}(J) \\ {\scriptstyle \mathbf{F}(S\gamma)}\downarrow & & {\scriptstyle \mathbf{F}(\delta)}\downarrow & & \downarrow{\scriptstyle \mathbf{F}(\alpha)} \\ \partial' : \mathbf{F}(SB') & \xrightarrow{\mathbf{F}(i')} & \mathbf{F}(C_{q'}) & \xrightarrow{\mathbf{F}(e')^{-1}} & \mathbf{F}(J') \end{array}$$

Here δ is the cannonical map from C_q to $C_{q'}$, which, of course, makes the upper diagram commutative.

The result for $\hat{\partial}$ follows from naturality of ∂ and naturality of the Bott periodicity map (see [8, Theorem 4.4]). □

C. Our Notations and Settings

The settings from this section will be used in the consecutive one. Suppose we are given two exact sequences of C^*-algebras ($j = 1, 2$):

$$0 \longrightarrow I \xrightarrow{i_j} A_j \xrightarrow{\pi_j} B_j \longrightarrow 0 \tag{2.2}$$

From this we have the following six term exact sequences ($j = 1, 2$):

$$\begin{array}{ccccc} \mathbf{F}(I) & \xrightarrow{\mathbf{F}(i_j)} & \mathbf{F}(A_j) & \xrightarrow{\mathbf{F}(\pi_j)} & \mathbf{F}(B_j) \\ {\scriptstyle \omega_j}\uparrow & & & & \downarrow{\scriptstyle \varphi_j} \\ \mathbf{F}(SB_j) & \xleftarrow{\mathbf{F}(S\pi_j)} & \mathbf{F}(SA_j) & \xleftarrow{\mathbf{F}(Si_j)} & \mathbf{F}(SI) \end{array} \tag{2.3}$$

By [29, Theorem 9.3] we have the following exact sequence:

$$0 \longrightarrow I \xrightarrow{p} A_1 *_I A_2 \xrightarrow{q} B_1 * B_2 \longrightarrow 0 \tag{2.4}$$

and that the following diagrams commute ($j = 1, 2$):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{i_j} & A_j & \xrightarrow{\pi_j} & B_j & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle s_j} & & \downarrow{\scriptstyle t_j} & & \\ 0 & \longrightarrow & I & \xrightarrow{p} & A_1 *_I A_2 & \xrightarrow{q} & B_1 * B_2 & \longrightarrow & 0 \end{array} \tag{2.5}$$

where s_j and t_j are the cannonical inclusions.

Now applying $\mathbf{F}$ to (3.11) we obtain the following diagram with exact rows:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathbf{F}(A_j) & \xrightarrow{\mathbf{F}(\pi_j)} & \mathbf{F}(B_j) & \xrightarrow{\varphi_j} & \mathbf{F}(SI) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \mathbf{F}(s_j)} & & \downarrow{\scriptstyle \mathbf{F}(t_j)} & & \| & & \\ \dots & \longrightarrow & \mathbf{F}(A_1 *_I A_2) & \xrightarrow{\mathbf{F}(q)} & \mathbf{F}(B_1 * B_2) & \xrightarrow{\partial} & \mathbf{F}(SI) & \longrightarrow & \dots \end{array} \tag{2.6}$$

It follows from Lemma B.5 that (2.6) commutes, so $\varphi_j = \partial \circ \mathbf{F}(t_j)$. Therefore $\varphi_1 + \varphi_2 = \partial \circ (\mathbf{F}(t_1) + \mathbf{F}(t_2))$. This yields

$$\partial = (\varphi_1 + \varphi_2) \circ (\mathbf{F}(t_1) + \mathbf{F}(t_2))^{-1} \tag{2.7}$$

Lemma C.1. *The following six-term sequence is exact:*

$$\begin{array}{ccccc} \mathbf{F}(I) & \xrightarrow{\mathbf{F}(p)} & \mathbf{F}(A_1 *_I A_2) & \xrightarrow{r} & \mathbf{F}(B_1) \oplus \mathbf{F}(B_2) \\ {\scriptstyle \omega_1+\omega_2}\uparrow & & & & \downarrow{\scriptstyle \varphi_1+\varphi_2} \\ \mathbf{F}(SB_1) \oplus \mathbf{F}(SB_2) & \xleftarrow{Sr} & \mathbf{F}(S(A_1 *_I A_2)) & \xleftarrow{\mathbf{F}(Sp)} & \mathbf{F}(SI) \end{array} \tag{2.8}$$

where $r = (\mathbf{F}(t_1) + \mathbf{F}(t_2))^{-1} \circ \mathbf{F}(q)$ *and* $Sr = (\mathbf{F}(St_1) + \mathbf{F}(St_2))^{-1} \circ \mathbf{F}(Sq)$.

Proof. This follows immediatelly from (2.7) and the following six-term exact sequence, corresponding to (3.10):

$$\begin{array}{ccccc} \mathbf{F}(I) & \xrightarrow{\mathbf{F}(p)} & \mathbf{F}(A_1 *_I A_2) & \xrightarrow{\mathbf{F}(q)} & \mathbf{F}(B_1 * B_2) \\ {\scriptstyle \tilde{\partial}}\uparrow & & & & \downarrow{\scriptstyle \partial} \\ \mathbf{F}(S(B_1 * B_2)) & \xleftarrow{\mathbf{F}(Sq)} & \mathbf{F}(S(A_1 *_I A_2)) & \xleftarrow{\mathbf{F}(Sp)} & \mathbf{F}(SI) \end{array} \tag{2.9}$$

For $\tilde{\partial}$ we argue similarly as for ∂. $\square$

D. The Proof of the Main Result

We are now ready to state and prove our main result.

Proposition D.1. *If we suppose everything from the previous section, the following six term sequence is exact:*

$$\begin{array}{ccccc} \mathbf{F}(I) & \xrightarrow{(\mathbf{F}(i_1),-\mathbf{F}(i_2))} & \mathbf{F}(A_1)\oplus\mathbf{F}(A_2) & \xrightarrow{\mathbf{F}(s_1)+\mathbf{F}(s_2)} & \mathbf{F}(A_1 *_I A_2) \\ {\scriptstyle\beta}\uparrow & & & & \downarrow{\scriptstyle\alpha} \\ \mathbf{F}(S(A_1 *_I A_2)) & \xleftarrow{\mathbf{F}(Ss_1)+\mathbf{F}(Ss_2)} & \mathbf{F}(SA_1)\oplus\mathbf{F}(SA_2) & \xleftarrow{(\mathbf{F}(Si_1),-\mathbf{F}(Si_2))} & \mathbf{F}(SI) \end{array} \tag{2.10}$$

where $\alpha : \mathbf{F}(A_1 *_I A_2) \to \mathbf{F}(SI)$ is equal to $\varphi_1' \circ r$, where $\varphi_1' : \mathbf{F}(B_1) \oplus \mathbf{F}(B_2) \to \mathbf{F}(SI)$ is given by $(a_1, a_2) \mapsto \varphi_1(a_1)$ and $\beta : \mathbf{F}(S(A_1 *_I A_2)) \to \mathbf{F}(I)$ is equal to $\omega_1' \circ Sr$, where $\omega_1' : \mathbf{F}(SB_1) \oplus \mathbf{F}(SB_2) \to \mathbf{F}(I)$ is given by $(a_1, a_2) \mapsto \omega_1(a_1)$.

Proof. We have to show exactness only at terms $\mathbf{F}(I)$, $\mathbf{F}(A_1)\oplus\mathbf{F}(A_2)$ and $\mathbf{F}(A_1 *_I A_2)$ and the exactness at the other three terms will follow from the same argument, applied to the functor $\mathbf{F}(S*)$.

(**i**) Exactness at $\mathbf{F}(I)$:

$Im(\beta) = Im(\omega_1' \circ r) = \omega_1'(Im(r)) = \langle$from the exactness of (2.8)$\rangle = \omega_1'(Ker(\omega_1 + \omega_2)) = \omega_1'(\{(a,b) \mid \omega_1(a) = -\omega_2(b)\}) = \{\omega_1(a) \mid \exists b, \omega_1(a) = -\omega_2(b)\} = \{\omega_1(a) \mid \omega_1(a) \in Im(\omega_2)\} = Im(\omega_1) \cap Im(\omega_2) = \langle$from the exactness of (2.3)$\rangle = Ker(\mathbf{F}(i_1)) \cap Ker(\mathbf{F}(i_2)) = Ker((\mathbf{F}(i_1), -\mathbf{F}(i_2))$.

(**ii**) Exactness at $\mathbf{F}(A_1) \oplus \mathbf{F}(A_2)$:

For $i \in I$ we have

$$(\mathbf{F}(s_1) + \mathbf{F}(s_2)) \circ ((\mathbf{F}(i_1), -\mathbf{F}(i_2)))(i) = \mathbf{F}(s_1 \circ i_1)(i) - \mathbf{F}(s_2 \circ i_2)(i) =$$

$$= \langle \text{from the commutativity of (3.11)} \rangle \ = \mathbf{F}(p)(i) - \mathbf{F}(p)(i) \equiv 0.$$

So $Im((\mathbf{F}(i_1), -\mathbf{F}(i_2))) \subseteq Ker(\mathbf{F}(s_1) + \mathbf{F}(s_2))$. Suppose $(a, b) \in \mathbf{F}(A_1) \oplus \mathbf{F}(A_2)$ is such that $\mathbf{F}(s_1)(a) + \mathbf{F}(s_2)(b) = 0$. We will show that $(a, b) \in Im((\mathbf{F}(i_1), -\mathbf{F}(i_2)))$ and this will prove case (**ii**).

We have

$$0 = \mathbf{F}(q \circ s_1)(a) + \mathbf{F}(q \circ s_2)(b) = \ \langle \text{from the commutativity of (3.11)} \rangle \ =$$

$$= \mathbf{F}(t_1 \circ \pi_1)(a) + \mathbf{F}(t_2 \circ \pi_2)(b) = (\mathbf{F}(t_1) + \mathbf{F}(t_2))(\mathbf{F}(\pi_1)(a), \mathbf{F}(\pi_2)(b)).$$

Since $\mathbf{F}(t_1) + \mathbf{F}(t_2)$ is an isomorphism we get $\mathbf{F}(\pi_1)(a) = \mathbf{F}(\pi_2)(b) = 0$. The exactness of (2.3) yields elements $a', b' \in \mathbf{F}(I)$, s.t. $\mathbf{F}(i_1)(a') = a$ and $\mathbf{F}(i_2)(b') = b$. We compute

$$0 = \mathbf{F}(s_1)(a) + \mathbf{F}(s_2)(b) = \mathbf{F}(s_1 \circ i_1)(a') + \mathbf{F}(s_2 \circ i_2)(b') =$$

$$= \ \langle \text{from the commutativity of (3.11)} \rangle \ = \mathbf{F}(p)(a' + b') \ \Rightarrow \ a' + b' \in Ker(\mathbf{F}(p)).$$

Using the exactness of (2.8) we get

$$Ker(\mathbf{F}(p)) = Im(\omega_1 + \omega_2) = \omega_1(SB_1) + \omega_2(SB_2) = \ \langle \text{from the exactness of (2.3)} \rangle \ =$$

$$= Ker(\mathbf{F}(i_1)) + Ker(\mathbf{F}(i_2)) (\text{as subgroups of } \mathbf{F}(I)).$$

Thus we can write $a' + b' = c_1 + c_2 \in \mathbf{F}(I)$, where $c_j \in Ker(\mathbf{F}(i_j))$. Now denote $\gamma \overset{def}{=} a' - c_1 (= -b' + c_2)$. We have $\mathbf{F}(i_1)(\gamma) = \mathbf{F}(i_1)(a' - c_1) = a + 0 = a$ and analogously $\mathbf{F}(i_2)(\gamma) = \mathbf{F}(i_2)(-b' + c_2) = -b + 0 = -b$. So $(a, b) \in Im((\mathbf{F}(i_1), -\mathbf{F}(i_2)))$, just what we needed.

(**iii**) Exactness at $\mathbf{F}(A_1 *_I A_2)$:

For any $a_j \in \mathbf{F}(A_j), j = 1, 2$ we have

$$\varphi_1' \circ r \circ (\mathbf{F}(s_1) + \mathbf{F}(s_2))(a_1, a_2) = \varphi_1' \circ (\mathbf{F}(t_1) + \mathbf{F}(t_2))^{-1} \circ \mathbf{F}(q) \circ (\mathbf{F}(s_1) + \mathbf{F}(s_2))(a_1, a_2) =$$

$$= \varphi_1' \circ (\mathbf{F}(t_1) + \mathbf{F}(t_2))^{-1} \circ \mathbf{F}(q) \circ \mathbf{F}(s_1)(a_1) + \varphi_1' \circ (\mathbf{F}(t_1) + \mathbf{F}(t_2))^{-1} \circ \mathbf{F}(q) \circ \mathbf{F}(s_2)(a_2) =$$

$$= \langle \text{from the commutativity of (3.11)} \rangle =$$

$$= \varphi_1' \circ (\mathbf{F}(t_1) + \mathbf{F}(t_2))^{-1} \circ \mathbf{F}(t_1) \circ \mathbf{F}(\pi_1)(a_1) + \varphi_1' \circ (\mathbf{F}(t_1) + \mathbf{F}(t_2))^{-1} \circ \mathbf{F}(t_2) \circ \mathbf{F}(\pi_2)(a_2) =$$

$$= \varphi_1' \circ (\mathbf{F}(t_1) + \mathbf{F}(t_2))^{-1} \circ (\mathbf{F}(t_1) + \mathbf{F}(t_2))(\mathbf{F}(\pi_1)(a_1), \mathbf{F}(\pi_2)(a_2)) =$$

$$= \varphi_1'(\mathbf{F}(\pi_1)(a_1), \mathbf{F}(\pi_2)(a_2)) = \varphi_1(\mathbf{F}(\pi_1)(a_1) = 0,$$

since $\varphi_1 \circ \mathbf{F}(\pi_1) = 0$, which follows from the exactness of (2.3). So $Im(\mathbf{F}(s_1) + \mathbf{F}(s_2)) \subseteq Ker(\alpha)$.

Suppose now that $\epsilon \in Ker(\alpha)$. First we will show that $r(\epsilon) \in Im(\mathbf{F}(\pi_1)) \oplus Im(\mathbf{F}(\pi_2))$.

From the exactness of (2.8) we see that $r(\epsilon) \in Ker(\varphi_1 + \varphi_2)$, therefore $\exists (b_1, b_2) \in \mathbf{F}(B_1) \oplus \mathbf{F}(B_2)$, s.t. $r(\epsilon) = (b_1, b_2)$ and $\varphi_1(b_1) + \varphi_2(b_2) = 0$. But $\epsilon \in Ker(\varphi_1' \circ r)$ implies $0 = \varphi_1' \circ r(\epsilon) = \varphi_1'((b_1, b_2)) = \varphi_1(b_1)$, so $b_1 \in Ker(\varphi_1)$, so also $b_2 \in Ker(\varphi_2)$. Since (2.3) is exact then $\exists a_j \in \mathbf{F}(A_j), (j = 1, 2)$, s.t. $\mathbf{F}(\pi_j)(a_j) = b_j$, so $(b_1, b_2) = \mathbf{F}(\pi_1)(a_1) + \mathbf{F}(\pi_2)(a_2)$.

We will now show that for an element $\theta \in \mathbf{F}(A_1 *_I A_2)$, $r(\theta) \in Im(\mathbf{F}(\pi_1)) + Im(\mathbf{F}(\pi_2))$ implies $\theta \in Im(\mathbf{F}(s_1) + \mathbf{F}(s_2))$. This will yield $\epsilon \in Im(\mathbf{F}(s_1) + \mathbf{F}(s_2))$, just what we need.

So suppose $\exists (a_1, a_2) \in \mathbf{F}(A_1) \oplus \mathbf{F}(A_2)$, s.t. $r(\theta) = \mathbf{F}(\pi_1)(a_1) + \mathbf{F}(\pi_2)(a_2)$. Then

$$\mathbf{F}(q)(\theta) = (\mathbf{F}(t_1) + \mathbf{F}(t_2)) \circ (\mathbf{F}(\pi)(a_1), \mathbf{F}(\pi)(a_2)) =$$

$$= \mathbf{F}(t_1) \circ \mathbf{F}(\pi_1)(a_1) + \mathbf{F}(t_2) \circ \mathbf{F}(\pi_2)(a_2)$$

and from the commutativity of (3.11) we get $\mathbf{F}(q)(\theta) = \mathbf{F}(q) \circ (\mathbf{F}(s_1)(a_1) + \mathbf{F}(s_2)(a_2))$. So $\mathbf{F}(q)(\theta - \mathbf{F}(s_1)(a_1) - \mathbf{F}(s_2)(a_2)) = 0$. Now from the exactness of (2.9) we get $\exists \delta \in \mathbf{F}(I)$, s.t. $\mathbf{F}(p)(\delta) = \theta - \mathbf{F}(s_1)(a_1) - \mathbf{F}(s_2)(a_2)$ and therefore $\theta = [\mathbf{F}(p)(\delta) + \mathbf{F}(s_1)(a_1)] + [\mathbf{F}(s_2)(a_2)]$. Using that (3.11) is exact gives us $\mathbf{F}(p)(\delta) = \mathbf{F}(s_1) \circ \mathbf{F}(i_1)(\delta)$ so this means $\theta = [\mathbf{F}(s_1) \circ \mathbf{F}(i_1)(\delta) + \mathbf{F}(s_1)(a_1)] + [\mathbf{F}(s_2)(a_2)] = [\mathbf{F}(s_1) \circ (\mathbf{F}(i_1)(\delta) + a_1)] + [\mathbf{F}(s_2)(a_2)]$.

This proves (**iii**) and the proposition. □

CHAPTER III

REDUCED FREE PRODUCTS OF FINITE DIMENSIONAL C^*-ALGEBRAS

A. Introduction

In this section we give a necessary and sufficient condition for simplicity and uniqueness of trace of reduced free products of finite family of finite dimensional C^*-algebras. We will use the properties of reduced free products of C^*-algebras which we gave in Chapter I. Beside the definition and properties of reduced free products we gave in Chapter I we will use the following lemma:

Lemma A.1 ([16]). *Let I be an index set and let (A_i, ϕ_i) be a C^*-NCPS ($i \in I$), where each ϕ_i is faithful. Let (B, ψ) be a C^*-NCPS with ψ faithful. Let*

$$(A, \phi) = \mathop{*}_{i \in I} (A_i, \phi_i).$$

Given unital $*$-homomorphisms, $\pi_i : A_i \to B$, such that $\psi \circ \pi_i = \phi_i$ and $\{\pi_i(A_i)\}_{i \in I}$ is free in (B, ψ), there is a $*$-homomorphism, $\pi : A \to B$ such that $\pi|_{A_i} = \pi$ and $\psi \circ \pi = \phi$.

From now on we will be concerned only with C^*-algebras equipped with tracial states.

We will make use also of the following result due to Avitzour:

Theorem A.2 ([3]). *Let*

$$(\mathfrak{A}, \tau) = (A, \tau_A) * (B, \tau_B),$$

where τ_A and τ_B are traces and (A, τ_A) and (B, τ_B) have faithful GNS representations. Suppose that there are unitaries $u, v \in A$ and $w \in B$, such that $\tau_A(u) = \tau_A(v) = \tau_A(u^*v) = 0$ and $\tau_B(w) = 0$. Then $\mathfrak{A}$ is simple and has a unique trace τ.

Note: It is clear that uw satisfies $\tau((uw)^n) = 0$, $\forall n \in \mathbb{Z}\backslash\{0\}$. Unitaries with this property we define below.

B. Statement of the Main Result and Preliminaries

We adopt the following notation:

If $A_0, \ldots, A_n$ are unital C^*-algebras equipped with traces $\tau_0, \ldots, \tau_n$ respectively, then $A = \underset{\alpha_0}{\overset{p_0}{A_0}} \bigoplus \underset{\alpha_1}{\overset{p_1}{A_1}} \bigoplus \ldots \bigoplus \underset{\alpha_n}{\overset{p_n}{A_n}}$ will mean that the C^*-algebra A is isomorphic to the direct sum of $A_0, \ldots, A_n$, and is such that A_i are supported on the projections p_i. Also A comes with a trace (let's call it τ) given by the formula $\tau = \alpha_0\tau_0 + \alpha_1\tau_1 + \ldots + \alpha_n\tau_n$. Here of course $\alpha_0, \alpha_1, \ldots, \alpha_n > 0$ and $\alpha_0 + \alpha_1 + \ldots + \alpha_n = 1$.

Definition B.1. *If (A, τ) is a C^*-NCPS and $u \in A$ is a unitary with $\tau(u^n) = 0$, $\forall n \in \mathbb{Z}\backslash\{0\}$, then we call u a Haar unitary.*

If $1_A \in B \subset A$ is a unital abelian C^*-subalgebra of A we call B a diffuse abelian C^*-subalgebra of A if $\tau|_B$ is given by an atomless measure on the spectrum of B. We also call B a unital diffuse abelian C^*-algebra.

From [15, Proposition 4.1(i), Proposition 4.3] we can conclude the following:

Proposition B.2. *If (B, τ) is a C^*-NCPS with B-abelian, then B is diffuse abelian if and only if B contains a Haar unitary.*

C^*-algebras of the form $(\underset{\alpha}{\overset{p}{\mathbb{C}}} \bigoplus \underset{1-\alpha}{\overset{1-p}{\mathbb{C}}}) * (\underset{\beta}{\overset{q}{\mathbb{C}}} \bigoplus \underset{1-\beta}{\overset{1-q}{\mathbb{C}}})$ have been described explicitly in [2] (see also [13]):

Theorem B.3. *Let $1 > \alpha \geqq \beta \geqq \frac{1}{2}$ and let*

$$(A, \tau) = (\underset{\alpha}{\overset{p}{\mathbb{C}}} \oplus \underset{1-\alpha}{\overset{1-p}{\mathbb{C}}}) * (\underset{\beta}{\overset{q}{\mathbb{C}}} \oplus \underset{1-\beta}{\overset{1-q}{\mathbb{C}}}).$$

If $\alpha > \beta$ *then*

$$A = \underset{\alpha-\beta}{\overset{p\wedge(1-q)}{\mathbb{C}}} \oplus C([a,b], M_2(\mathbb{C})) \oplus \underset{\alpha+\beta-1}{\overset{p\wedge q}{\mathbb{C}}},$$

for some $0 < a < b < 1$*. Furthermore, in the above picture*

$$p = 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 1,$$

$$q = 0 \oplus \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix} \oplus 1,$$

and the faithful trace τ *is given by the indicated weights on the projections* $p \wedge (1-q)$ *and* $p \wedge q$*, together with an atomless measure, whose support is* $[a, b]$*.*

If $\alpha = \beta > \frac{1}{2}$ *then*

$$A = \{\ f : [0,b] \to M_2(\mathbb{C}) |\ f \text{ is continuous and } f(0) \text{ is diagonal }\} \oplus \underset{\alpha+\beta-1}{\overset{p\wedge q}{\mathbb{C}}},$$

for some $0 < b < 1$*. Furthermore, in the above picture*

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 1,$$

$$q = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix} \oplus 1,$$

and the faithful trace τ *is given by the indicated weight on the projection* $p \wedge q$*, together with an atomless measure on* $[0, b]$*.*

If $\alpha = \beta = \frac{1}{2}$ *then*

$$A = \{\ f : [0,1] \to M_2(\mathbb{C}) |\ f \text{ is continuous and } f(0) \text{ and } f(1) \text{ are diagonal }\}.$$

Furthermore in the above picture

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$q = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix},$$

and the faithful trace τ *is given by an atomless measure, whose support is* $[0,1]$.

The question of describing the reduced free product of a finite family of finite dimensional abelian C^*-algebras was studied by Dykema in [12]. He proved the following theorem:

Theorem B.4 ([12])**.** *Let*

$$(\mathfrak{A},\phi) = (\underset{\alpha_0}{\overset{p_0}{A_0}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus \ldots \oplus \underset{\alpha_n}{\overset{p_n}{\mathbb{C}}}) * (\underset{\beta_0}{\overset{q_0}{B_0}} \oplus \underset{\beta_1}{\overset{q_1}{\mathbb{C}}} \oplus \ldots \oplus \underset{\beta_m}{\overset{q_m}{\mathbb{C}}}),$$

where $\alpha_0 \geq 0$ *and* $\beta_0 \geq 0$ *and* A_0 *and* B_0 *are equipped with traces* $\phi(p_0)^{-1}\phi|_{A_0}$, $\phi(q_0)^{-1}\phi|_{B_0}$ *and* A_0 *and* B_0 *have diffuse abelian* C^**-subalgebras, and where* $n \geq 1$, $m \geq 1$ *(if* $\alpha_0 = 0$ *or* $\beta_0 = 0$*, or both, then, of course, we don't impose any conditions on* A_0 *or* B_0*, or both respectively). Suppose also that* $\dim(A) \geq 2$, $\dim(B) \geq 2$*, and* $\dim(A) + \dim(B) \geq 5$.

Then

$$\mathfrak{A} = \overset{r_0}{\mathfrak{A}_0} \oplus \bigoplus_{(i',j)\in L_+} \underset{\alpha_i+\beta_i-1}{\overset{p_i\wedge q_j}{\mathbb{C}}},$$

where $L_+ = \{(i,j)|1 \leq i \leq n,\ 1 \leq j \leq m \text{ and } \alpha_i + \beta_j > 1\}$*, and where* $\mathfrak{A}_0$ *has a unital, diffuse abelian sublagebra supported on* r_0p_1 *and another one supported on* r_0q_1.

Let $L_0 = \{(i,j)|1 \leq i \leq n,\ 1 \leq j \leq m \text{ and } \alpha_i + \beta_j = 1\}$.

If L_0 *is empty then* $\mathfrak{A}_0$ *is simple and* $\phi(r_0)^{-1}\phi|_{\mathfrak{A}_0}$ *is the unique trace on* $\mathfrak{A}_0$.

If L_0 *is not empty, then for each* $(i,j) \in L_0$ *there is a* $*$*-homomorphism* $\pi_{(i,j)} : \mathfrak{A}_0 \to \mathbb{C}$ *such that* $\pi_{(i,j)}(r_0p_i) = 1 = \pi_{(i,j)}(r_0q_j)$*. Then:*

(1) $\mathfrak{A}_{00} \overset{def}{=} \bigcap_{(i,j)\in L_0} \ker(\pi_{(i,j)})$
is simple and nonunital, and $\phi(r_0)^{-1}\phi|_{\mathfrak{A}_{00}}$ *is the unique trace on* $\mathfrak{A}_{00}$.

(2) For each $i \in \{1, ...n\}$, $r_0 p_i$ *is full in* $\mathfrak{A}_0 \cap \bigcap_{\substack{(i',j)\in L_0 \\ i'\neq i}} \ker(\pi_{(i',j)})$.

(3) For each $j \in \{1, ..., m\}$, $r_0 q_j$ *is full in* $\mathfrak{A}_0 \cap \bigcap_{\substack{(i,j')\in L_0 \\ j'\neq j}} \ker(\pi_{(i,j')})$.

One can define von Neumann algebra free products, similarly to reduced free products of C^*-algebras. We will denote by $\mathbb{M}_n$ the C^*-algebra (von Neumann algebra) of $n \times n$ matrices with complex coefficients.

Dykema studied the case of von Neumann algebra free products of finite dimensional (von Neumann) algebras:

Theorem B.5 ([9])**.** *Let*

$$A = \underset{\alpha_0}{\overset{p_0}{L(F_s)}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{M}_{n_1}}} \oplus ... \oplus \underset{\alpha_k}{\overset{p_k}{\mathbb{M}_{n_k}}}$$

and

$$B = \underset{\beta_0}{\overset{q_0}{L(F_r)}} \oplus \underset{\beta_1}{\overset{q_1}{\mathbb{M}_{m_1}}} \oplus ... \oplus \underset{\beta_l}{\overset{q_l}{\mathbb{M}_{m_l}}},$$

where $L(F_s), L(F_r)$ *are interpolated free group factors,* $\alpha_0, \beta_0 \geq 0$*, and where* $\dim(A) \geq 2$, $\dim(B) \geq 2$ *and* $\dim(A) + \dim(B) \geq 5$*. Then for the von Neumann algebra free product we have:*

$$A * B = L(F_t) \oplus \bigoplus_{(i,j)\in L_+} \underset{\gamma_{ij}}{\overset{f_{ij}}{\mathbb{M}_{N(i,j)}}},$$

where $L_+ = \{(i,j)|1 \leq i \leq k, 1 \leq j \leq l, (\frac{\alpha_i}{n_i^2}) + (\frac{\beta_j}{m_j^2}) > 1\}$, $N(i,j) = max(n_i, m_j)$, $\gamma_{ij} = N(i,j)^2 \cdot (\frac{\alpha_i}{n_i^2} + \frac{\beta_j}{m_j^2} - 1)$*, and* $f_{ij} \leq p_i \wedge q_j$.

Note: t can be determined from the other data, which makes sense only if the interpolated free group factors are all different. We will use only the fact that $L(F_t)$ is a factor. For definitions and properties of interpolated free group factors see [32] and [10].

In this paper we will extend the result of Theorem B.4 to the case of reduced free products of finite dimensional C^*-algebras with specified traces on them. We will prove:

Theorem B.6. *Let*

$$(\mathfrak{A}, \phi) = (\underset{\alpha_0}{\overset{p_0}{A_0}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{M}_{n_1}}} \oplus ... \oplus \underset{\alpha_k}{\overset{p_k}{\mathbb{M}_{n_k}}}) * (\underset{\beta_0}{\overset{q_0}{B_0}} \oplus \underset{\beta_1}{\overset{q_1}{\mathbb{M}_{m_1}}} \oplus ... \oplus \underset{\beta_l}{\overset{q_l}{\mathbb{M}_{m_l}}}),$$

where $\alpha_0, \beta_0 \geq 0$, $\alpha_i > 0$, for $i = 1, .., k$ and $\beta_j > 0$, for $j = 1, ..., l$, and where $\phi(p_0)^{-1}\phi|_{A_0}$ and $\phi(q_0)^{-1}\phi|_{B_0}$ are traces on A_0 and B_0 respectivelly. Suppose that $\dim(A) \geq 2$, $\dim(B) \geq 2$, $\dim(A) + \dim(B) \geq 5$, and that both A_0 and B_0 contain unital, diffuse abelian C^*-subalgebras (if $\alpha_0 > 0$, respectivelly $\beta_0 > 0$). Then

$$\mathfrak{A} = \underset{\gamma}{\overset{f}{\mathfrak{A}_0}} \oplus \bigoplus_{(i,j) \in L_+} \underset{\gamma_{ij}}{\overset{f_{ij}}{\mathbb{M}_{N(i,j)}}},$$

where $L_+ = \{(i,j) | \frac{\alpha_i}{n_i^2} + \frac{\beta_j}{m_j^2} > 1\}$, $N(i,j) = max(n_i, m_j)$, $\gamma_{ij} = N(i,j)^2(\frac{\alpha_i}{n_i^2} + \frac{\beta_j}{m_j^2} - 1)$, $f_{ij} \leq p_i \wedge q_j$. There is a unital, diffuse abelian C^*-subalgebra of $\mathfrak{A}_0$, supported on fp_1 and another one, supported on fq_1.

If $L_0 = \{(i,j) | \frac{\alpha_i}{n_i^2} + \frac{\beta_j}{m_j^2} = 1\}$, is empty, then $\mathfrak{A}_0$ is simple with a unique trace. If L_0 is not empty, then $\forall (i,j) \in L_0$, $\exists \pi_{(i,j)} : \mathfrak{A}_0 \to \mathbb{M}_{N(i,j)}$ a unital $*$-homomorphism, such that $\pi_{(i,j)}(fp_i) = \pi_{(i,j)}(fq_j) = 1$. Then:

(1) $\mathfrak{A}_{00} \overset{def}{=} \bigcap_{(i,j) \in L_0} \ker(\pi_{(i,j)})$ is simple and nonunital, and has a unique trace $\phi(f)^{-1}\phi|_{\mathfrak{A}_{00}}$.

(2) For each $i \in \{1, ..., k\}$, fp_i is full in $\mathfrak{A}_0 \cap \bigcap_{\substack{(i',j) \in L_0 \\ i' \neq i}} \ker(\pi_{(i',j)})$.

(3) For each $j \in \{1, ..., l\}$, fq_j is full in $\mathfrak{A}_0 \cap \bigcap_{\substack{(i,j') \in L_0 \\ j' \neq j}} \ker(\pi_{(i,j')})$.

C. Beginning of the Proof - A Special Case

In order to prove this theorem we will start with a simpler case. We will study first the C^*-algebras of the form $(A,\tau) \overset{def}{=} (\underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus ... \oplus \underset{\alpha_m}{\overset{p_m}{\mathbb{C}}}) * (\mathbb{M}_n, tr_n)$ with $0 < \alpha_1 \leq ... \leq \alpha_m$. We chose a set of matrix units for $\mathbb{M}_n$ and denote them by $\{e_{ij} | i,j \in \{1,...n\}\}$ as usual. Let's take the (trace zero) permutation unitary

$$u \overset{def}{=} \begin{pmatrix} 0 & 1 & ... & 0 \\ . & . & . & . \\ 0 & 0 & ... & 1 \\ 1 & 0 & ... & 0 \end{pmatrix} \in \mathbb{M}_n.$$

We see that $\mathrm{Ad}(u)(e_{11}) = ue_{11}u^* = e_{nn}$ and for $2 \leq i \leq n$, $\mathrm{Ad}(u)(e_{ii}) = ue_{ii}u^* = e_{(i-1)(i-1)}$.

It's clear that

$$A = C^*(\{p_1, ..., p_m\}, \{e_{ii}\}_{i=1}^n, u).$$

Then it is also clear that

$$A = C^*(\{u^i p_1 u^{-i}, ..., u^i p_m u^{-i}\}_{i=0}^{n-1}, \{e_{ij}\}_{i=1}^n, u).$$

We want to show that the family

$$\{\{\mathbb{C} \cdot u^i p_1 u^{-i} \oplus, ..., \oplus \mathbb{C} \cdot u^i p_m u^{-i}\}_{i=0}^{n-1},\ \{\mathbb{C} \cdot e_{11} \oplus ... \oplus \mathbb{C} \cdot e_{nn}\}\}$$

is free.

We will prove something more general. We denote

$$B \overset{def}{=} C^*(\{u^k p_1 u^{-k}, ..., u^k p_m u^{-k}\}_{k=0}^{n-1}, \{e_{11}, ..., e_{nn}\}).$$

Let l be an integer and $l|n$, $1 < l < n$ (if such l exists). Let

$$E \overset{def}{=} C^*(\{\{u^k p_1 u^{-k}, ..., u^k p_m u^{-k}\}_{k=0}^{l-1}, \{e_{11}, ..., e_{nn}\}, \{u^l, u^{2l}, ..., u^{n-l}\}\}).$$

It's easy to see that

$$C^*(\{e_{11}, ..., e_{nn}\}, \{u^l, u^{2l}, ..., u^{n-l}\}) = \underbrace{\mathbb{M}_{\frac{n}{l}} \oplus ... \oplus \mathbb{M}_{\frac{n}{l}}}_{l-times} \subset \mathbb{M}_n.$$

We will adopt the following notation from [13]:

Let (D, φ) be a C^*-NCPS and $1_D \in D_1, ..., D_k \subset D$ be a family of unital C^*-subalgebras of D, having a common unit 1_D. We denote by $D^\circ \overset{def}{=} \{d \in D | \varphi(d) = 0\}$. We denote by $\Lambda^\circ(D_1^\circ, D_2^\circ, ..., D_k^\circ)$ the set of all words of the form $d_1 d_2 \cdots d_j$ and of nonzero length, where $d_t \in D_{i_t}^\circ$, for some $1 \le i_t \le k$ and $i_t \ne i_{t+1}$ for any $1 \le t \le j-1$.

We have the following

Lemma C.1. *If everything is as above, then:*

(i) The family $\{\{u^k p_1 u^{-k}, ..., u^k p_m u^{-k}\}_{k=0}^{n-1}, \{e_{11}, ..., e_{nn}\}\}$ *is free in* (A, τ)*. And more generally if*

$$\omega \in \Lambda^\circ(C^*(p_1, ..., p_m)^\circ, ..., C^*(u^{n-1} p_1 u^{1-n}, ..., u^{n-1} p_m u^{1-n})^\circ, C^*(e_{11}, ..., e_{nn})^\circ),$$

then $\tau(\omega u^r) = 0$ *for all* $0 \le r \le n-1$*.*

(ii) The family $\{\{u^k p_1 u^{-k}, ..., u u^k p_m u^{-k}\}_{k=0}^{l-1}, \{e_{11}, ..., e_{nn}, u^l, u^{2l}, ... u^{n-l}\}\}$ *is free in* (A, τ)*. And more generally if*

$$\omega \in \Lambda^\circ(C^*(p_1, ..., p_m)^\circ, ..., C^*(u^{l-1} p_1 u^{1-l}, ..., u^{l-1} p_m u^{1-l})^\circ,$$

$$C^*(e_{11}, ..., e_{nn}, u^l, ..., u^{n-l})^\circ),$$

then $\tau(\omega u^r) = 0$ *for all* $0 \le r \le l-1$*.*

Proof. Each letter $\alpha \in C^*(\{u^k p_1 u^{-k}, ..., u^k p_m u^{-k}\})$ with $\tau(\alpha) = 0$ can be represented as $\alpha = u^k\alpha' u^{-k}$ with $\tau(\alpha') = 0$, and $\alpha' \in C^*(\{p_1, ..., p_m\})$.

Case (i):

Each

$$\omega \in \Lambda^\circ(C^*(p_1, ..., p_m)^\circ, ..., C^*(u^{n-1}p_1u^{1-n}, ..., u^{n-1}p_mu^{1-n})^\circ, C^*(e_{11}, ..., e_{nn})^\circ)$$

is of one of the four following types:

$$\omega = \alpha_{11}\alpha_{12}\cdots\alpha_{1i_1}\beta_1\alpha_{21}\cdots\alpha_{2i_2}\beta_2\alpha_{31}\cdots\alpha_{t-1i_{t-1}}\beta_{t-1}\alpha_{t1}\cdots\alpha_{ti_t}, \tag{3.1}$$

$$\omega = \beta_1\alpha_{21}\cdots\alpha_{2i_2}\beta_2\alpha_{31}\cdots\alpha_{t-1i_{t-1}}\beta_{t-1}\alpha_{t1}\cdots\alpha_{ti_t}, \tag{3.2}$$

$$\omega = \beta_1\alpha_{21}\cdots\alpha_{2i_2}\beta_2\alpha_{31}\cdots\alpha_{t-1i_{t-1}}\beta_{t-1}, \tag{3.3}$$

$$\omega = \alpha_{11}\alpha_{12}\cdots\alpha_{1i_1}\beta_1\alpha_{21}\cdots\alpha_{2i_2}\beta_2\alpha_{31}\cdots\alpha_{t-1i_{t-1}}\beta_{t-1}, \tag{3.4}$$

where $\alpha_{ij} \in C^*(u^{k_{ij}}p_1u^{k_{ij}}, ..., u^{k_{ij}}p_mu^{k_{ij}})^\circ$ with $0 \leq k_{ij} \leq n-1$, $k_{ij} \neq k_{i(j+1)}$ and $\beta_i \in C^*(e_{11}, ..., e_{nn})^\circ$.

We consider the following two cases:

(a) We look at $\alpha_{ji}\alpha_{ji+1}$ with $\alpha_{jc} \in C^*(\{u^{k_c}p_1u^{-k_c}, ..., u^{k_c}p_mu^{-k_c}\})^\circ$ for $c = i, i+1$. We write $\alpha_{jc} = u^{k_c}\alpha'_{jc}u^{-k_c}$ with $\alpha'_{jc} \in C^*(\{p_1, ..., p_m\})^\circ$ for $c = i, i+1$. So $\alpha_{ji}\alpha_{ji+1} = u^{k_i}\alpha'_{ji}u^{k_{i+1}-k_i}\alpha'_{ji+1}u^{-k_{i+1}}$. Here α'_{ji} and α'_{ji+1} are free from $u^{k_{i+1}-k_i}$ in (A, τ) (Notice that we have $k_{i+1} - k_i \neq 0$).

(b) We look at $\alpha_{ji_j}\beta_j\alpha_{(j+1)1}$ with $\beta \in C^*(\{e_{11}, ..., e_{nn}\})^\circ$, $\alpha_{(j+1)1} \in C^*(\{u^{k_{j+1}}p_1u^{-k_{j+1}}, ..., u^{k_{j+1}}p_mu^{-k_{j+1}}\})^\circ$, $\alpha_{ji_j} \in C^*(\{u^{k_j}p_1u^{-k_j}, ..., u^{k_j}p_mu^{-k_j}\})^\circ$. Now we write $\alpha_{ji_j} = u^{k_j}\alpha'_{ji_j}u^{-k_j}$ and $\alpha_{(j+1)1} = u^{k_{j+1}}\alpha'_{(j+1)1}u^{-k_{j+1}}$ with $\alpha'_{ji_j}, \alpha'_{(j+1)1} \in C^*(\{p_1, ..., p_m\})^\circ$. We see that $\alpha_{ji_j}\beta_j\alpha_{(j+1)1} = u^{k_j}\alpha'_{ji_j}u^{-k_j}\beta_ju^{k_{j+1}}\alpha'_{(j+1)1}u^{-k_{j+1}}$. If $k_j = k_{j+1}$ then $\tau(u^{-k_j}\beta_ju^{k_{j+1}}) = \tau(u^{k_{j+1}}u^{-k_j}\beta_j) = \tau(\beta_j) = 0$ since τ is a trace. If $k_j \neq k_{j+1}$ then $\tau(u^{-k_j}\beta_ju^{k_{j+1}}) = \tau(u^{k_{j+1}}u^{-k_j}\beta_j)$ and $u^{k_{j+1}-k_j}\beta_j \in \mathbb{M}_n$ is a linear combination of off-diagonal elements, so $\tau(u^{k_{j+1}}u^{-k_j}\beta_j) =$

0 also. Notice that α'_{ji_j} and $\alpha'_{(j+1)1}$ are free from $u^{-k_j}\beta_j u^{k_{j+1}}$ in (A,τ).

Now we expand all the letters in the word ω according to the cases (a) and (b). We see that we obtain a word, consisting of letters of zero trace, such that every two consequitive letters come either from $C^*(\{p_1,...,p_m\})$ or from $\mathbb{M}_n$. So $\tau(\omega)=0$. It only remains to look at the case of the word ωu^r which is the word ω, but ending in u^r. There are two principally different cases for ωu^r from the all four possible choices for ω:

In cases (3.1) and (3.2) $\alpha_{ti_t} = u^k\alpha'_{ti_t}u^{-k}$ for some $0\le k\le n-1$ with $\alpha'_{ti_t}\in C^*(\{p_1,...,p_m\})^\circ$. So the word will end in $u^k\alpha'_{ti_t}u^{r-k}$. If $r=k$ then α'_{ti_t} will be the last letter with trace zero and everything else will be the same as for ω, so the whole word will have trace 0. If $k\neq r$ then $\tau(u^{r-k})=0$ and u^{r-k} is free from α'_{ti_t} so the word in this case will be of zero trace too.

In cases (3.3) and (3.4) if $\beta_{t-1}u^r$ is the whole word then $\beta_{t-1}u^r$ is a linear combination of off-diagonal elements of $\mathbb{M}_n$, and so its trace is 0. If not then $\alpha_{(t-1)i_{t-1}} = u^k\alpha'_{(t-1)i_{t-1}}u^{-k}$ with $\alpha'_{(t-1)i_{t-1}}\in C^*(\{p_1,...,p_m\})^\circ$. So the word ends in

$u^k\alpha'_{(t-1)i_{t-1}}u^{-k}\beta_{t-1}u^r$. Similarly as above we see that $\tau(u^{-k}\beta_{t-1}u^r)=0$ for all values of k and r. The rest of the word we treat as above and conclude that it's of zero trace in this case too.

So in all cases $\tau(\omega u^r)=0$ just what we had to show.

Case (ii):

As in case (i)

$$\omega\in\Lambda^\circ(C^*(p_1,...,p_m)^\circ,...,C^*(u^{l-1}p_1u^{1-l},...,u^{l-1}p_mu^{1-l})^\circ,C^*(e_{11},...,e_{nn},u^l,...,u^{n-l})^\circ)$$

is of one of the following types:

$$\omega = \alpha_{11}\alpha_{12}\cdots\alpha_{1i_1}\beta_1\alpha_{21}\cdots\alpha_{2i_2}\beta_2\alpha_{31}\cdots\alpha_{t-1i_{t-1}}\beta_{t-1}\alpha_{t1}\cdots\alpha_{ti_t}, \tag{3.5}$$

$$\omega = \beta_1\alpha_{21}\cdots\alpha_{2i_2}\beta_2\alpha_{31}\cdots\alpha_{t-1i_{t-1}}\beta_{t-1}\alpha_{t1}\cdots\alpha_{ti_t}, \tag{3.6}$$

$$\omega = \beta_1\alpha_{21}\cdots\alpha_{2i_2}\beta_2\alpha_{31}\cdots\alpha_{t-1i_{t-1}}\beta_{t-1}, \tag{3.7}$$

$$\omega = \alpha_{11}\alpha_{12}\cdots\alpha_{1i_1}\beta_1\alpha_{21}\cdots\alpha_{2i_2}\beta_2\alpha_{31}\cdots\alpha_{t-1i_{t-1}}\beta_{t-1}, \tag{3.8}$$

where $\alpha_{ij} \in C^*(u^{k_{ij}}p_1u^{k_{ij}}, ..., u^{k_{ij}}p_mu^{k_{ij}})^\circ$ with $0 \leq k_{ij} \leq l-1$ and $k_{ij} \neq k_{(i+1)j}$ and $\beta_i \in C^*(e_{11}, ..., e_{nn}, u^l, u^{2l}, ..., u^{n-l})^\circ$.

Similarly as case (i) we consider two cases:

(a) We look at $\alpha_{ji}\alpha_{ji+1}$ with $\alpha_{jc} \in C^*(\{u^{k_c}p_1u^{-k_c}, ..., u^{k_c}p_mu^{-k_c}\})$, and $0 \leq k_c \leq l-1$ for $c = i, i+1$. We write $\alpha_{jc} = u^{k_c}\alpha'_{jc}u^{-k_c}$ with $\alpha'_{jc} \in C^*(\{p_1, ..., p_m\})^\circ$ for $c = i, i+1$. It follows $\alpha_{ji}\alpha_{ji+1} = u^{k_i}\alpha'_{ji}u^{k_{i+1}-k_i}\alpha'_{ji+1}u^{-k_{i+1}}$. Here α'_{ji} and α'_{ji+1} are free from $u^{k_{i+1}-k_i}$ in (A, τ) (and again $k_{i+1} - k_i \neq 0$).

(b) We look at $\alpha_{ji_j}\beta_j\alpha_{(j+1)1}$ with $\beta_j \in C^*(\{e_{11}, ..., e_{nn}\}, \{u^l, u^{2l}, ..., u^{n-l}\})^\circ$, $\alpha_{(j+1)1} \in C^*(\{u^{k_{j+1}}p_1u^{-k_{j+1}}, ..., u^{k_{j+1}}p_mu^{-k_{j+1}}\})^\circ$, $\alpha_{ji_j} \in C^*(\{u^{k_j}p_1u^{-k_j}, ..., u^{k_j}p_mu^{-k_j}\})^\circ$, where in this case $k_j, k_{j+1} \in \{0, ..., l-1\}$. Again we write $\alpha_{ji_j} = u^{k_j}\alpha'_{ji_j}u^{-k_j}$ and $\alpha_{(j+1)1} = u^{k_{j+1}}\alpha'_{(j+1)1}u^{-k_{j+1}}$ with $\alpha'_{ji_j}, \alpha'_{(j+1)1} \in C^*(\{p_1, ..., p_m\})^\circ$,. We have $\alpha_{ji_j}\beta_j\alpha_{(j+1)1} = u^{k_j}\alpha'_{ji_j}u^{-k_j}\beta_ju^{k_{j+1}}\alpha'_{(j+1)1}u^{-k_{j+1}}$.
We only need to show that $\tau(u^{-k_j}\beta_ju^{k_{j+1}}) = 0$. $\tau(u^{-k_j}\beta_ju^{k_{j+1}}) = \tau(u^{k_{j+1}}u^{-k_j}\beta_j) = \tau(u^{k_{j+1}-k_j}\beta_j)$. The case $k_{j+1} = k_j$ is clear. Notice that if $k_{j+1} \neq k_j$ then $0 < k_{j+1} - k_j \leq l-1$. Is it clear that $u^{k_{j+1}-k_j}\cdot\mathrm{Span}(\{e_{11}, ..., e_{nn}\}) \subset \mathbb{M}_n$ consists of liner combination of off-diagonal elements. The same is clear for $u^{k_{j+1}-k_j}\cdot\mathrm{Span}(\{u^l, u^{2l}, ..., u^{n-l}\}) \subset \mathbb{M}_n$. It's not difficult to see then that

$$u^{k_{j+1}-k_j}\cdot\mathrm{Alg}(\{e_{11}, ..., e_{nn}\}, \{u^l, u^{2l}, ..., u^{n-l}\})$$

will consist of linear span of the union of the off-diagonal entries among $\{e_{ij}|1 \leq i, j \leq n\}$ present in $u^{k_{j+1}-k_j} \cdot \mathrm{Span}(\{e_{11}, ..., e_{nn}\})$ and the ones present in $u^{k_{j+1}-k_j} \cdot \mathrm{Span}(\{u^l, u^{2l}, ..., u^{n-l}\})$. This shows that $u^{k_{j+1}-k_j}\beta_j$ will be also a linear span of off-diagonal entries in $\mathbb{M}_n$ and will have trace 0. So $\tau(u^{-k_j}\beta_j u^{k_{j+1}}) = 0$. In this case also α'_{ji_j} and $\alpha'_{(j+1)1}$ are free from $u^{-k_j}\beta_j u^{k_{j+1}}$ in (A, τ).

We expand all the letters of the word ω and see that it is of trace 0 similarly as in case (i). For the word ωu^r with $0 \leq r \leq l-1$ we argue similarly as in case (i). Again there are two principally different cases:

In cases (3.5) and (3.6) $\alpha_{ti_t} = u^k \alpha'_{ti_t} u^{-k}$ for some $0 \leq k \leq l-1$ with $\alpha'_{ti_t} \in C^*(\{p_1, ..., p_m\})^\circ$. So the word will end in $u^k \alpha'_{ti_t} u^{r-k}$. If $r = k$ then α'_{ti_t} will be the last letter with trace zero and everything else will be the same as for ω, so the whole word will have trace 0. If $k \neq r$ then $\tau(u^{r-k}) = 0$ and u^{r-k} is free from α'_{ti_t} so the word in this case will be of zero trace too.

In cases (3.7) and (3.8) $\beta_{t-1}u^r$ then this is a linear combination of off-diagonal elements as we showed in case (ii)-(b). If not we write $\alpha_{(t-1)i_{t-1}} = u^k \alpha'_{(t-1)i_{t-1}} u^{-k}$ with $0 \leq k \leq l-1$ and $\alpha'_{(t-1)i_{t-1}} \in C^*(\{p_1, ..., p_m\})^\circ$. So the word that we are looking at will end in $u^k \alpha'_{(t-1)i_{t-1}} u^{-k} \beta_{t-1} u^r$. Since $0 \leq k, r \leq l-1$ similarly as in case (ii)-(b) we see that $\tau(u^{-k}\beta_{t-1}u^r) = 0$. We treat the remaining part of the word as above and conclude that in this case the word has trace 0.

So in all cases $\tau(\omega u^r) = 0$ just what we had to show.

This proves the lemma. $\square$

From properties (5) and (6) of the reduced free product it follows that τ is a faithful trace. From Lemma A.1 it follows that

$$B = (\mathbb{C} \cdot e_{11} \oplus ... \oplus \mathbb{C} \cdot e_{nn}) * (\mathop{*}_{k=0}^{n-1} (\mathbb{C} \cdot u^k p_1 u^{-k} \oplus ... \oplus \mathbb{C} \cdot u^k p_m u^{-k}))$$

$$\cong (\underset{\frac{1}{n}}{\mathbb{C}} \oplus ... \oplus \underset{\frac{1}{n}}{\mathbb{C}}) * (\overset{n-1}{\underset{k=0}{*}} (\underset{\alpha_1}{\mathbb{C}} \oplus ... \oplus \underset{\alpha_m}{\mathbb{C}}))$$

and that

$$E = C^*(\{e_{11}, ..., e_{nn}, u_l, u^{2l}, ..., u^{n-l}\}) * (\overset{l-1}{\underset{k=0}{*}} (\mathbb{C} \cdot u^k p_1 u^{-k} \oplus ... \oplus \mathbb{C} \cdot u^k p_m u^{-k}))$$

$$\cong (\underset{\frac{l}{n}}{\mathbb{M}_{\frac{n}{l}}} \oplus ... \oplus \underset{\frac{l}{n}}{\mathbb{M}_{\frac{n}{l}}}) * (\overset{l-1}{\underset{k=0}{*}} (\underset{\alpha_1}{\mathbb{C}} \oplus ... \oplus \underset{\alpha_m}{\mathbb{C}})).$$

Corollary C.2. *If everything is as above:*

(1) For $b \in B$ and $0 < k \leq n-1$ we have $\tau(bu^k) = 0$, so also $\tau(u^k b) = 0$.

(2) For $e \in E$ and $0 < k \leq l-1$ we have $\tau(eu^k) = 0$, so also $\tau(u^k e) = 0$.

For $(B, \tau|_B)$ and $(E, \tau|_E)$ we have that $\mathfrak{H}_B \subset \mathfrak{H}_E \subset \mathfrak{H}_A$. If $a \in A$ we will denote by $\hat{a} \in \mathfrak{H}_A$ the vector in $\mathfrak{H}_A$, corresponding to a by the GNS construction. We will show that

Corollary C.3. *If everything is as above:*

(1) $u^{k_1}\mathfrak{H}_B \perp u^{k_2}\mathfrak{H}_B$ for $k_1 \neq k_2$, $0 \leq k_1, k_2 \leq n-1$.

(2) $u^{k_1}\mathfrak{H}_E \perp u^{k_2}\mathfrak{H}_E$ for $k_1 \neq k_2$, $0 \leq k_1, k_2 \leq l-1$.

Proof. (1) Take $b_1, b_2 \in B$. We have $\langle u^{k_1}\hat{b_1}, u^{k_2}\hat{b_2}\rangle = \tau(u^{k_2} b_2 b_1^* u^{-k_1}) = \tau(b_2 b_1^* u^{k_2 - k_1}) = 0$, by the above Corollary.

(2) Similarly take $e_1, e_2 \in E$, so $\langle u^{k_1}\hat{e_1}, u^{k_2}\hat{e_2}\rangle = \tau(u^{k_2} e_2 e_1^* u^{-k_1}) = \tau(e_2 e_1^* u^{k_2 - k_1}) = 0$, again by the above Corollary. □

Now $\mathfrak{H}_A$ can be written in the form $\mathfrak{H}_A = \bigoplus_{i=0}^{n-1} u^i \mathfrak{H}_B$ as a Hilbert space because of the Corollary above. Denote by P_i the projection $P_i : \mathfrak{H}_A \to \mathfrak{H}_A$ onto the subspace $u^i\mathfrak{H}_B$. Now it's also true that $A = \bigoplus_{i=0}^{n-1} u^i B$ as a Banach space. To see this we notice that Span$\{u^i B, i = 0, ...n-1\}$ is dense in A, also that $u^i B$, $0 \leq i \leq n-1$ are closed in A. Now take a sequence $\{\sum_{i=0}^{n-1} u^i b_{mi}\}_{m=1}^{\infty}$ converging to an element $a \in A$ ($b_{mi} \in B$). Then for each i we have $\{P_j \sum_{i=0}^{n-1} u^i b_{mi} P_0\}_{m=1}^{\infty} = \{P_j u^j b_{mj} P_0\}_{m=1}^{\infty}$

converges (to P_jaP_0), consequently the sequence $\{b_{mj}\}_{m=1}^{\infty}$ converges to an element b_j in B $\forall 0 \leq j \leq n-1$. So $a = \sum_{i=0}^{n-1} u^i b_i$. Finally the fact that $u^{i_1}B \cap u^{i_2}B = 0$, for $i_1 \neq i_2$ follows easily from $u^{i_1}\mathfrak{H}_B \cap u^{i_2}\mathfrak{H}_B = 0$, for $i_1 \neq i_2$ and the fact that the trace τ is faithful. We also have $A = \bigoplus_{i=0}^{n-1} Bu^i$.

Let C is a C^*-algebra and Γ is a discrete group with a given action $\alpha : \Gamma \to Aut(C)$ on C. By $C \rtimes \Gamma$ we will denote the reduced crossed product of C by Γ. It will be clear what group action we take.

Let's denote by G the multiplicative group, generated by the automorphism $\mathrm{Ad}(u)$ of B. Then $G \cong \mathbb{Z}_n$ and by what we proved above $\mathfrak{H}_A \cong L^2(G, \mathfrak{H}_B)$.

Lemma C.4. $A \cong B \rtimes G$

Proof. We have to show that the action of A on $\mathfrak{H}_A$ "agrees" with the crossed product action. Take $a = \sum_{k=0}^{n-1} b_k u^k \in A$, $b_k \in B, k = 0, 1, ..., n-1$ and take $\xi = \sum_{k=0}^{n-1} u^k \hat{b'_k} \in \mathfrak{H}_A$, $b'_k \in B, k = 0, 1, ..., n-1$. Then

$$a(\xi) = \sum_{k=0}^{n-1}\sum_{m=0}^{n-1} b_k u^k u^m \hat{b'_m} = \sum_{k=0}^{n-1}\sum_{m=0}^{n-1} u^{k+m}.(u^{-k-m} b_k u^{k+m})\hat{b'_m}$$

$$= \sum_{s=0}^{n-1}\sum_{k=0}^{n-1} (u^s.\mathrm{Ad}(u^{-s})(b_k))(\widehat{b'_{s-k(mod\ n)}}).$$

This shows that the action of A on $\mathfrak{H}_A$ is the crossed product action. □

To study simplicity in this situation, we can invoke [26, Theorem 4.2] and [27, Theorem 6.5], or with the same success, use the following result from [23]:

Theorem C.5 ([23])**.** *Let Γ be a discrete group of automorphisms of C^*-algebra $\mathfrak{B}$. If $\mathfrak{B}$ is simple and if each γ is outer for the multiplier algebra $M(\mathfrak{B})$ of $\mathfrak{B}$, $\forall \gamma \in \Gamma \backslash \{1\}$, then the reduced crossed product of $\mathfrak{B}$ by Γ, $\mathfrak{B} \rtimes \Gamma$, is simple.*

An automorphism ω of a C^*-algebra $\mathfrak{B}$, contained in a C^*-algebra $\mathfrak{A}$ is outer for $\mathfrak{A}$, if there doesn't exist a unitary $w \in \mathfrak{A}$ with the property $\omega = \mathrm{Ad}(w)$.

A representation π of a C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$ is called non-degenerate if there doesn't exist a vector $\xi \in \mathfrak{H}$, $\xi \neq 0$, such that $\pi(\mathfrak{A})\xi = 0$.

The idealizer of a C^*-algebra $\mathfrak{A}$ in a C^*-algebra $\mathfrak{B}$ ($\mathfrak{A} \subset \mathfrak{B}$) is the largest C^*-subalgebra of $\mathfrak{B}$ in which $\mathfrak{A}$ is an ideal.

We will not give a definition of multiplier algebra of a C^*-algebra. Instead we will give the following property from [1], which we will use (see [1] for more details on multiplier algebras):

Proposition C.6 ([1])**.** *Each nondegenerate faithful representation* π *of a* C^**-algebra* $\mathfrak{A}$ *extends uniquely to a faithful representation of* $M(\mathfrak{A})$*, and* $\pi(M(\mathfrak{A}))$ *is the idealizer of* $\pi(\mathfrak{A})$ *in its weak closure.*

Suppose that we have a faithful representation $\pi$of a C^* algebra $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$. If confusion is impossible we will denote by $\bar{\mathfrak{A}}$ (in $\mathfrak{H}$) the weak closure of $\pi(\mathfrak{A})$ in $\mathbb{B}(\mathfrak{H})$.

To study uniqueness of trace we invoke a theorem of Bédos from [4].

Let $\mathfrak{A}$ be a simple, unital C^*-algebra with a unique trace φ and let $(\pi_{\mathfrak{A}}, \mathfrak{H}_{\mathfrak{A}}, \widehat{1_{\mathfrak{A}}})$ denote the GNS-triple associated to φ. The trace φ is faithful by the simplicity of $\mathfrak{A}$ and $\mathfrak{A}$ is isomorphic to $\pi_{\mathfrak{A}}(\mathfrak{A})$. Let $\alpha \in Aut(\mathfrak{A})$. The trace φ is α-invariant by the uniqueness of φ. Then α is implemented on $\mathfrak{H}_{\mathfrak{A}}$ by the unitary operator U_α given by $U_\alpha(\hat{a}) = \alpha(a) \cdot \widehat{1_{\mathfrak{A}}}$, $a \in \mathfrak{A}$. Then we denote the extension of α to the weak closure $\bar{\mathfrak{A}}$ (in $\mathfrak{H}_{\mathfrak{A}}$) of $\pi_{\mathfrak{A}}(\mathfrak{A})$ on $\mathbb{B}(\mathfrak{H}_{\mathfrak{A}})$ by $\tilde{\alpha} \overset{def}{=} \mathrm{Ad}(U_\alpha)$. We will say that α is φ-outer if $\tilde{\alpha}$ is outer for $\bar{\mathfrak{A}}$.

Theorem C.7 ([4])**.** *Suppose* $\mathfrak{A}$ *is a simple unital* C^**-algebra with a unique trace* φ *and that* Γ *is a discrete group with a representation* $\alpha : \Gamma \to Aut(\mathfrak{A})$*, such that* α_γ *is* φ*-outer* $\forall \gamma \in \Gamma \backslash \{1\}$*. Then the reduced crossed product* $\mathfrak{A} \rtimes \Gamma$ *is simple with a unique trace* τ *given by* $\tau = \varphi \circ E$*, where* E *is the canonical conditional expectation from*

$\mathfrak{A} \rtimes \Gamma$ *onto* $\mathfrak{A}$.

Let's now return to the C^*-algebra $(A, \tau) = (\underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus ... \oplus \underset{\alpha_m}{\overset{p_m}{\mathbb{C}}}) * (\mathbb{M}_n, tr_n)$, with $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_m$. If $B \subset E \subset A$ are as in the beginning of this section, then the representations of B, E and A on $\mathfrak{H}_A$ are all nondegenerate. Also we have the following:

Lemma C.8. *The weak closure of B in $\mathbb{B}(\mathfrak{H}_B)$ and the one in $\mathbb{B}(\mathfrak{H}_A)$ are the same (or $\bar{B}$ (in $\mathfrak{H}_B$) $\cong \bar{B}$ (in $\mathfrak{H}_A$)). Analoguously, $\bar{E}$ (in $\mathfrak{H}_E$) $\cong \bar{E}$ (in $\mathfrak{H}_A$).*

Proof. For $b \in B \subset A$ we have $b(u^t h) = u^t(\mathrm{Ad}(u^{-t}b))(h)$ for $h \in \mathfrak{H}_B$ and $0 \leq t \leq n-1$. Taking a weak limit in $\mathbb{B}(\mathfrak{H}_B)$ we obtain the same equation $\forall \bar{b} \in \bar{B}$ (in $\mathfrak{H}_B$): $\bar{b}(u^t h) = u^t(\mathrm{Ad}(u^{-t})(\bar{b}))(h)$, which shows, of course, that $\bar{b}$ has a unique extension to $\mathbb{B}(\mathfrak{H}_A)$. Conversely if $\tilde{b} \in \bar{B}$ (in $\mathfrak{H}_A$), then since $\mathfrak{H}_B$ is invariant for B it will be invariant for $\tilde{b}$ also. So the restriction of $\tilde{b}$ to $\mathfrak{H}_B$ is the element we are looking for.

Analoguously if $e \in E$ and if $h_0 + u^l h_1 + ... + u^{n-l} h_{\frac{n}{l}-1} \in \mathfrak{H}_E$, then for $0 \leq t \leq l-1$ we have $e(u^t(h_0 + u^l h_1 + ... + u^{n-l} h_{\frac{n}{l}-1})) = u^t(\mathrm{Ad}(u^{-t})(e))(h_0 + u^l h_1 + ... + u^{n-l} h_{\frac{n}{l}-1})$. And again for an element $\bar{e} \in \bar{E}$ (in $\mathfrak{H}_E$) we see that $\bar{e}$ has a unique extension to an element of $\bar{E}$ (in $\mathfrak{H}_A$). Conversely an element $\tilde{e} \in \bar{E}$ (in $\mathfrak{H}_A$) has $\mathfrak{H}_E$ as an invariant subspace, so we can restrict it to $\mathfrak{H}_E$ to obtain an element in $\bar{E}$ (in $\mathfrak{H}_E$). □

We will state the following theorem from [12], which we will frequently use:

Theorem C.9 ([12])**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras with traces $\tau_{\mathfrak{A}}$ and $\tau_{\mathfrak{B}}$ respectively, whose GNS representations are faithful. Let*

$$(\mathfrak{C}, \tau) = (\mathfrak{A}, \tau_{\mathfrak{A}}) * (\mathfrak{B}, \tau_{\mathfrak{B}}).$$

Suppose that $\mathfrak{B} \neq \mathbb{C}$ and that $\mathfrak{A}$ has a unital, diffuse abelian C^*-subalgebra $\mathfrak{D}$ ($1_{\mathfrak{A}} \in \mathfrak{D} \subseteq \mathfrak{A}$). Then $\mathfrak{C}$ is simple with a unique trace τ.

Using repeatedly Theorem B.4 we see that

$$B = (\mathbb{C} \cdot e_{11} \oplus ... \oplus \mathbb{C} \cdot e_{nn}) * (\mathop{*}_{k=0}^{n-1} (\mathbb{C} \cdot u^k p_1 u^{-k} \oplus ... \oplus \mathbb{C} \cdot u^k p_m u^{-k}))$$

$$\cong (U \oplus \underset{max\{n\alpha_m - n+1,\ 0\}}{\overset{\tilde{p}}{\mathbb{C}}}) * (\underset{\frac{1}{n}}{\overset{e_{11}}{\mathbb{C}}} \oplus ... \oplus \underset{\frac{1}{n}}{\overset{e_{nn}}{\mathbb{C}}}),$$

where U has a unital, diffuse abelian C^*-subalgebra, and where $\tilde{p} = \bigwedge_{i=0}^{n-1} u^i p_m u^{-i}$.

We will consider the following 3 cases, for $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_m$:

(I) $\alpha_m < 1 - \frac{1}{n^2}$.

(II) $\alpha_m = 1 - \frac{1}{n^2}$.

(III) $\alpha_m > 1 - \frac{1}{n^2}$.

We will organize those cases in few lemmas:

(I)

Lemma C.10. *If A is as above, then for $\alpha_m < 1 - \frac{1}{n^2}$ we have that A is simple with a unique trace.*

Proof. We consider:

(1) $\alpha_m \leq 1 - \frac{1}{n}$.

Then $B \cong U * (\underset{\frac{1}{n}}{\overset{e_{11}}{\mathbb{C}}} \oplus ... \oplus \underset{\frac{1}{n}}{\overset{e_{nn}}{\mathbb{C}}})$ with U containing a unital, diffuse abelian C^*-subalgebra (from Theorem B.4). From the Theorem C.9 we see that B is simple with a unique trace.

(2) $1 - \frac{1}{n} < \alpha_m < 1 - \frac{1}{n^2}$.

Then $B \cong (U \oplus \underset{n\alpha_m - n+1}{\overset{\tilde{p}}{\mathbb{C}}}) * (\underset{\frac{1}{n}}{\overset{e_{11}}{\mathbb{C}}} \oplus ... \oplus \underset{\frac{1}{n}}{\overset{e_{nn}}{\mathbb{C}}})$ with U having a unital, diffuse abelian C^*-subalgebra. Using Theorem B.4 one more time we see that B is simple with a unique trace in this case also.

We know that $A = B \rtimes G$, where $G = \langle \mathrm{Ad}(u) \rangle \cong \mathbb{Z}_n$. Since B is unital then the multiplier algebra $M(B)$ coinsides with B. We note also that since $\bar{B}$ (in $\mathfrak{H}_B$) is isomorphic to $\bar{B}$ (in $\mathfrak{H}_A$) to prove that some element of $Aut(B)$ is τ_B-outer it's enough to prove that this automorphism is outer for $\bar{B}$ (in $\mathfrak{H}_A$) (and it will be outer for $M(B) = B$ also). Making these observations and using Theorem C.5 and Theorem C.7 we see that if we prove that $\mathrm{Ad}(u^i)$ is outer for $\bar{B}$ (in $\mathfrak{H}_A$), $\forall 0 < i \leq n-1$, then it will follow that A is simple with a unique trace. We will show that $\mathrm{Ad}(u^i)$ is outer for $\bar{B}$ (in $\mathfrak{H}_A$) (we will write just $\bar{*}$ for $\bar{*}$ (in $\mathfrak{H}_A$) and omit writting $\mathfrak{H}_A$ - all the closures will be in $\mathbb{B}(\mathfrak{H}_{\mathfrak{A}})$) for the case $\alpha_m \leq 1 - \frac{1}{n^2}$.

Fix $0 < k \leq n-1$. Since $u^k \mathfrak{H}_B \perp \mathfrak{H}_B$ it follows that $u^k \notin \bar{B}$ (in $\mathfrak{H}_A$). Suppose $\exists w \in \bar{B}$, such that $\mathrm{Ad}(u^k) = \mathrm{Ad}(w)$ on $\bar{B}$. Then $u^k w u^{-k} = w w w^* = w$ and $u^k w^* u^{-k} = w w^* w^* = w^*$ and this implies that u^k, u^{-k}, w and w^* commute, so it follows $u^k w^*$ commutes with $\overline{C^*(B, u^k)}$, so it belongs to its center. If $k \nmid n$ then $\overline{C^*(B, u^k)} = \bar{A}$ and by Theorem B.5 $\bar{A}$ (in $\mathfrak{H}_A$)is a factor, so $u^k w^*$ is a multiple of 1_A, which contradicts the fact $u^k \notin \bar{B}$. If $k = l \mid n$, then $\overline{C^*(B, u^k)} = \bar{E}$ and $\bar{E}$ (in $\mathfrak{H}_A$) $\cong \bar{E}$ (in $\mathfrak{H}_E$) is a factor too (by Theorem B.5), so this implies again that $u^k w^*$ is a multiple of $1_A = 1_E$, so this is a contradiction again and this proves that $\mathrm{Ad}(u^k)$ are outer for $\bar{B}$, $\forall 0 < k \leq n-1$. This concludes the proof. $\square$

(III)

Lemma C.11. *If A is as above, then for $\alpha_m > 1 - \frac{1}{n^2}$ we have $A = A_0 \oplus \underset{n^2\alpha_m - n^2 + 1}{\mathbb{M}_n}$, where A_0 is simple with a unique trace.*

Proof. In this case $B \cong (U \oplus \underset{n\alpha_m - n+1}{\overset{\tilde{p}}{\mathbb{C}}}) * (\underset{\frac{1}{n}}{\overset{e_{11}}{\mathbb{C}}} \oplus ... \oplus \underset{\frac{1}{n}}{\overset{e_{nn}}{\mathbb{C}}})$, where U has a unital, diffuse abelian C^*-subalgebra. Form Theorem B.4 we see that $B \cong \overset{\tilde{p}_0}{B_0} \oplus \underset{n\alpha_m - n + \frac{1}{n}}{\overset{e_{11}\wedge\tilde{p}}{\mathbb{C}}} \oplus ... \oplus \underset{n\alpha_m - n + \frac{1}{n}}{\overset{e_{nn}\wedge\tilde{p}}{\mathbb{C}}}$ with $\tilde{p}_0 = 1 - e_{11} \wedge \tilde{p} - ... - e_{nn} \wedge \tilde{p}$, and B_0 being a unital, simple and having a unique trace. It's easy to see that $\mathrm{Ad}(u)$ permutes $\{e_{ii} | 1 \leq i \leq n\}$ and that $\mathrm{Ad}(u)$ permutes $\{u^i p_j u^{-i} | 0 \leq i \leq n-1\}$ for each $1 \leq j \leq m$. But since $\tilde{p} = \bigwedge_{i=0}^{n-1} u^i p_m u^{-i}$ we see that $\mathrm{Ad}(u)(\tilde{p}) = \tilde{p}$. This shows that $\mathrm{Ad}(u)$ permutes

$\{e_{ii} \wedge \tilde{p} | 1 \leq i \leq n\}$. This shows that $\mathrm{Ad}(\tilde{p}_0 u)$ is an automorphism of B_0 and that $\mathrm{Ad}((1-\tilde{p}_0)u)$ is an automorphism of $\overset{e_{11}\wedge\tilde{p}}{\mathbb{C}} \oplus ... \oplus \overset{e_{nn}\wedge\tilde{p}}{\mathbb{C}}$. If we denote $G_1 = \langle \mathrm{Ad}(\tilde{p}_0 u) \rangle$ and $G_2 = \langle \mathrm{Ad}((1-\tilde{p}_0)u) \rangle$, then we have $A = B_0 \rtimes G_1 \oplus (\overset{e_{11}\wedge\tilde{p}}{\mathbb{C}} \oplus ... \oplus \overset{e_{nn}\wedge\tilde{p}}{\mathbb{C}}) \rtimes G_2$. Now it's easy to see that $(\overset{e_{11}\wedge\tilde{p}}{\mathbb{C}} \oplus ... \oplus \overset{e_{nn}\wedge\tilde{p}}{\mathbb{C}}) \rtimes G_2 = C^*(\{e_{11} \wedge \tilde{p}, ..., e_{nn} \wedge \tilde{p}\}, (1-\tilde{p}_0)u) = (1-\tilde{p}_0).C^*(\{e_{11}, ..., e_{nn}\}, u) \cong \mathbb{M}_n$ (because $\tilde{p}_0$ is a central projection). To study $A_0 \overset{def}{=} B_0 \rtimes G_1$ we have to consider the automorphisms $\mathrm{Ad}(\tilde{p}_0 u)$. From Lemma C.8 we see that

$$\overline{B_0 \oplus \overset{e_{11}\wedge\tilde{p}}{\mathbb{C}} \oplus ... \oplus \overset{e_{nn}\wedge\tilde{p}}{\mathbb{C}}} \ (in\ \mathfrak{H}_B) \cong \overline{B_0 \oplus \overset{e_{11}\wedge\tilde{p}}{\mathbb{C}} \oplus ... \oplus \overset{e_{nn}\wedge\tilde{p}}{\mathbb{C}}} \ (in\ \mathfrak{H}_A).$$

This implies $\bar{B}_0$ (in $\mathfrak{H}_{B_0}$) $\cong$ $\bar{B}_0$ (in $\mathfrak{H}_{A_0}$). This is because $\mathfrak{H}_{A_0} = \tilde{p}_0 \mathfrak{H}_A$ and $\mathfrak{H}_{B_0} = \tilde{p}_0 \mathfrak{H}_B$ (which is clear, since $\mathfrak{H}_{A_0}$ and $\mathfrak{H}_{B_0}$ are direct summands in $\mathfrak{H}_A$ and $\mathfrak{H}_B$ respectivelly). For some $l|n$ if we denote $E_0 \overset{def}{=} \tilde{p}_0 E$ then by the same reasoning as above

$$E = E_0 \oplus (1-\tilde{p}_0).C^*(\{e_{11}, ..., e_{nn}\}, u^l) \cong E_0 \oplus (\underbrace{\mathbb{M}_{\frac{n}{l}} \oplus ... \oplus \mathbb{M}_{\frac{n}{l}}}_{l-times}).$$

So we similarly have $\bar{E}_0$ (in $\mathfrak{H}_{E_0}$) $\cong$ $\bar{E}_0$ (in $\mathfrak{H}_{A_0}$). We use Theorem B.5 and see that $\bar{A} \cong L(F_t) \oplus \mathbb{M}_n$ and that

$$\bar{E} \cong L(F_{t'}) \oplus (\underbrace{\mathbb{M}_{\frac{n}{l}} \oplus ... \oplus \mathbb{M}_{\frac{n}{l}}}_{l-times}),$$

for some $1 < t, t' < \infty$. This shows that $\bar{A}_0$ and $\bar{E}_0$ are both factors. Now for $\mathrm{Ad}(\tilde{p}_0 u^k)$, $1 \leq k \leq n-1$ we can make the same reasoning as in the case (I) to show that $\mathrm{Ad}(\tilde{p}_0 u^k)$ are all outer for $\bar{B}_0$, $\forall 1 \leq k \leq n-1$. Now we use Theorem C.5 and Theorem C.7 to finish the proof. Notice that the trace of the support projection of $\mathbb{M}_n$, $e_{11} \wedge \tilde{p} + ... + e_{nn} \wedge \tilde{p}$, is $n^2\alpha_m - n^2 + 1$. □

(II)

We already proved that $\mathrm{Ad}(u^k)$ are outer for $\bar{B}$, $\forall 1 \leq k \leq n-1$. Using Theorem B.4 we see $B \cong (U \oplus \underset{1-\frac{1}{n}}{\overset{\tilde{p}}{\mathbb{C}}}) * (\underset{\frac{1}{n}}{\overset{e_{11}}{\mathbb{C}}} \oplus ... \oplus \underset{\frac{1}{n}}{\overset{e_{nn}}{\mathbb{C}}})$ with U having a unital, diffuse abelian C^*-subalgebra. There are $*$-homomorphisms $\pi_i : B \to \mathbb{C}$, $1 \leq i \leq n$ with $\pi_i(\tilde{p}) = \pi_i(e_{ii}) = 1$, and such that $B_0 \overset{def}{=} \bigcap_{i=0}^{n-1} \ker(\pi_i)$ is simple with a unique trace. Now if $1 \leq k \leq n-1$, then $B_0 \bigcap \mathrm{Ad}(u^k)(B_0) =$ either 0 or B_0, because B_0 and $\mathrm{Ad}(u^k)(B_0)$ are simple ideals in B. The first possibility is actually impossible, because of dimension reasons, so this shows that B_0 is invariant for $\mathrm{Ad}(u^k)$, $1 \leq k \leq n-1$. In other words $\mathrm{Ad}(u^k) \in Aut(B_0)$. Similarly as in Lemma C.4 it can be shown that

$$A_0 \overset{def}{=} C^*(B_0 \oplus B_0 u \oplus ... \oplus B_0 u^{n-1}) \cong B_0 \rtimes \{\mathrm{Ad}(u^k) | 0 \leq k \leq n-1\} \subset A.$$

Lemma C.12. *We have a short split-exact sequence:*

$$0 \hookrightarrow A_0 \to A \overset{\curvearrowleft}{\to} \mathbb{M}_n \to 0.$$

Proof. It's clear that we have the short exact sequence

$$0 \to B_0 \hookrightarrow B \overset{\pi}{\longrightarrow} \underbrace{\mathbb{C} \oplus ... \oplus \mathbb{C}}_{n-times} \to 0,$$

where $\pi \overset{def}{=} (\pi_1, ..., \pi_n)$. We think π to be a map from B to $diag(\mathbb{M}_n)$, defined by

$$\pi(b) = \begin{pmatrix} \pi_1(b) & 0 & ... & 0 \\ 0 & \pi_2(b) & ... & 0 \\ . & . & . & . \\ 0 & 0 & ... & \pi_n(b) \end{pmatrix}.$$

Now since $\pi_i(\tilde{p}) = \pi_i(e_{ii}) = 1$ and $\mathrm{Ad}(u)(e_{11}) = ue_{11}u^* = e_{nn}$ and for $2 \leq i \leq n$, $\mathrm{Ad}(u)(e_{ii}) = ue_{ii}u^* = e_{(i-1)(i-1)}$, then $\pi_i \circ \mathrm{Ad}(u)(e_{(i+1)(i+1)}) = \pi_i \circ \mathrm{Ad}(u)(\tilde{p}) = 1$ for $1 \leq i \leq n-1$ and $\pi_n \circ \mathrm{Ad}(u)(e_{11}) = \pi_n \circ \mathrm{Ad}(u)(\tilde{p}) = 1$. So since two $*$-homomorphism of a C^*-algebra, which coinside on a set of generators of the C^*-algebra, are identical, we have $\pi_i \circ \mathrm{Ad}(u) = \pi_{i+1}$ for $1 \le i \le n-1$ and $\pi_n \circ \mathrm{Ad}(u) = \pi_1$. Define $\tilde{\pi} : A \to \mathbb{M}_n$ by $\sum_{k=0}^{n-1} b_k u^k \mapsto \sum_{k=0}^{n-1} \pi(b_k) W^k$ (with $b_k \in B$), where $W \in \mathbb{M}_n$ is represented by the matrix, which represent $u \in \mathbb{M}_n \subset A$, namely

$$W \stackrel{def}{=} \begin{pmatrix} 0 & 1 & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

We will show that if $b \in B$ and $0 \le k \le n-1$, then $\pi(u^k b u^{-k}) = W^k \pi(b) W^{-k}$. For this it's enough to show that $\pi(ubu^{-1}) = W\pi(b)W^{-1}$. For the matrix units $\{E_{ij} | 1 \le i, j \le n\}$ we have as above $WE_{ii}W^* = E_{(i-1)(i-1)}$ for $2 \le i \le n-1$ and $WE_{11}W^* = E_{nn}$. So

$$W \begin{pmatrix} \pi_1(b) & 0 & \dots & 0 \\ 0 & \pi_2(b) & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & \pi_n(b) \end{pmatrix} W^* = \begin{pmatrix} \pi_2(b) & 0 & \dots & 0 \\ 0 & \pi_3(b) & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & \pi_1(b) \end{pmatrix}$$

$$= \begin{pmatrix} \pi_1(\mathrm{Ad}(u)(b)) & 0 & \dots & 0 \\ 0 & \pi_2(\mathrm{Ad}(u)(b)) & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & \pi_n(\mathrm{Ad}(u)(b)) \end{pmatrix} = \pi(\mathrm{Ad}(u)(b)),$$

just what we wanted.

Now for $b \in B$ and $0 \le k \le n-1$ we have

$$\tilde{\pi}((bu^k)^*) = \tilde{\pi}(u^{-k}b^*) = \tilde{\pi}(u^{-k}b^*u^k u^{-k}) = \pi(u^{-k}b^*u^k)W^{-k} = W^{-k}\pi(b^*)W^k W^{-k},$$

$$= W^{-k}\pi(b)^* = (\pi(b)W^k)^* = (\tilde{\pi}(bu^k))^*.$$

Also if $b, b' \in B$ and $0 \le k, k' \le n-1$, then

$$\tilde{\pi}((b'u^{k'}).(bu^k)) = \tilde{\pi}(b'(u^{k'}bu^{-k'})u^{k+k'}) = \pi(b'(u^{k'}bu^{-k'}))W^{k+k'}$$

$$= \pi(b')\pi(u^{k'}bu^{-k'})W^{k+k'} = \pi(b')W^{k'}\pi(b)W^{-k'}W^{k+k'} = \tilde{\pi}(b'u^{k'})\tilde{\pi}(bu^k).$$

This proves that that $\tilde{\pi}$ is a $*$-homomorphism. Continuity follows from continuity of π and the Banach space representation $A = \bigoplus_{i=0}^{n-1} Bu^i$.

Clearly $A_0 = \bigoplus_{i=0}^{n-1} B_0 u^i$ as a Banach space. It's also clear by the definition of $\tilde{\pi}$ that $A_0 \subset \ker(\tilde{\pi})$. Since A_0 has a Banach space codimension n^2 in A, and so does $\ker(\tilde{\pi})$, then we must have $A_0 = \ker(\tilde{\pi})$.

From the construction of the map $\tilde{\pi}$ we see that $\tilde{\pi}(e_{ii}) = E_{ii}$, since $\pi(e_{ii}) = E_{ii}$ and also $\tilde{\pi}(u^k) = W^k$. Since $\{e_{ii} | 1 \le i \le n\} \cup \{W^k | 0 \le k \le n-1\}$ generate $\mathbb{M}_n$, then we have $\tilde{\pi}(e_{ij}) = E_{ij}$, so the inclusion map $s : \mathbb{M}_n \to A$ given by $E_{ij} \mapsto e_{ij}$ is a right inverse for $\tilde{\pi}$. □

From this lemma follows that we can write $A = A_0 \oplus \mathbb{M}_n$ as a Banach space.

Lemma C.13. *If η is a trace on A_0, then the linear functional on A $\tilde{\eta}$, defined by $\tilde{\eta}(a_0 \oplus M) = \eta(a_0) + tr_n(M)$, where $a_0 \in A_0$ and $M \in \mathbb{M}_n$ is a trace and $\tilde{\eta}$ is the unique extension of η to a trace on A (of norm 1).*

Proof. The functional η can be extended in at most one way to a tracial state on A, because of the requirement $\tilde{\eta}(1_A) = 1$, the fact that $\mathbb{M}_n$ sits as a subalgebra in A, and the uniqueness on trace on $\mathbb{M}_n$. Since $\tilde{\eta}(1_A) = 1$, to show that $\tilde{\eta}$ is a trace we need to show that $\tilde{\eta}$ is positive and satisfies the trace property. For the trace property: If $x, y \in A$ then we need to show $\tilde{\eta}(xy) = \tilde{\eta}(yx)$. It is easy to see, that to prove this it's enough to prove that if $a_0 \in A_0$ and $M \in \mathbb{M}_n$, then $\eta(a_0 M) = \eta(M a_0)$. Since η is linear and a_0 is a linear combination of 4 positive elements we can think, without loss of generality, that $a_0 \geq 0$. Then $a_0 = a_0^{1/2} a_0^{1/2}$ and $Ma_0^{1/2}, a_0^{1/2}M \in A_0$, so since η is a trace on A_0, we have $\eta(Ma_0) = \eta((Ma_0^{1/2})a_0^{1/2}) = \eta(a_0^{1/2}(Ma_0^{1/2})) = \eta((a_0^{1/2}M)a_0^{1/2}) = \eta(a_0^{1/2}(a_0^{1/2}M)) = \eta(a_0 M)$. This shows that $\tilde{\eta}$ satisfies the trace property. It remains to show positivity. Suppose $a_0 \oplus M \geq 0$. We must show $\eta(a_0 \oplus M) \geq 0$. Write $M = \sum_{i=0}^{n}\sum_{j=0}^{n} m_{ij}e_{ij}$ and $a_0 = \sum_{i=0}^{n}\sum_{j=0}^{n} e_{ii}a_0 e_{jj}$ Since $\tilde{\eta}$ is a trace if $i \neq j$, then $\tilde{\eta}(e_{ii}a_0e_{jj}) = \tilde{\eta}(e_{jj}e_{ii}a_0) = 0$, so this shows that $\tilde{\eta}(a_0 \oplus M) = \sum_{i=0}^{n}(\frac{m_{ii}}{n} + \eta(e_{ii}a_0e_{ii}))$. Clearly $a_0 \oplus M \geq 0$ implies $\forall 1 \leq i \leq n, e_{ii}(a_0 \oplus M)e_{ii} \geq 0$. So to show positivity we only need to show $\forall 1 \leq i \leq n$ $\tilde{\eta}(e_{ii}(a_0 + M)e_{ii}) \geq 0$, given $\forall 1 \leq i \leq n, m_{ii}e_{ii} + e_{ii}a_0e_{ii} \geq 0$. Suppose that for some i, $m_{ii} < 0$. Then it follows that $e_{ii}a_0e_{ii} \geq -m_{ii}e_{ii}$, so $e_{ii}a_0e_{ii} \in e_{ii}A_0e_{ii}$ is invertible, which implies $e_{ii} \in A_0$, that is not true. So this shows that $m_{ii} \geq 0$, and $m_{ii}e_{ii} \geq -e_{ii}a_0e_{ii}$. If $\{\epsilon_\gamma\}$ is an approximate unit for A_0, then positivity of η implies $1 = \|\eta\| = \lim_{\gamma} \eta(\epsilon_\gamma)$. Since η is a trace we have $\lim_{\gamma} \eta(\epsilon_\gamma e_{ii}) = \frac{1}{n}$. Since $\forall\gamma,\ m_{ii}\epsilon_\gamma^{1/2}e_{ii}\epsilon_\gamma^{1/2} \geq -\epsilon_\gamma^{1/2}e_{ii}a_0e_{ii}\epsilon_\gamma^{1/2}$, then

$$tr_n(m_{ii}e_{ii}) = \frac{m_{ii}}{n} = \lim_{\gamma} \eta(m_{ii}e_{ii}\epsilon_\gamma) = \lim_{\gamma} \eta(m_{ii}\epsilon_\gamma^{1/2}e_{ii}\epsilon_\gamma^{1/2}) \geq \lim_{\gamma} \eta(\epsilon_\gamma^{1/2}e_{ii}a_0e_{ii}\epsilon_\gamma^{1/2})$$

$$= \lim_{\gamma} \eta(e_{ii}a_0e_{ii}\epsilon_\gamma) = \eta(e_{ii}a_0e_{ii}).$$

This finishes the proof of positivity and the proof of the lemma. □

Remark C.14. *We will show below that $\tau|_{A_0}$ is the unique trace on A_0. Since we have $A = A_0 \oplus \mathbb{M}_n$ as a Banach space, then clearly the free product trace τ on A is given by $\tau(a_0 \oplus M) = \tau|_{A_0}(a_0) + tr_n(M)$, where $a_0 \oplus M \in A_0 \oplus \mathbb{M}_n = A$. All tracial positive linear functionals of norm ≤ 1 on A_0 are of the form $t\tau|_{A_0}$, where $0 \leq t \leq 1$. Then there will be no other traces on A then the family $\lambda_t \stackrel{def}{=} t\tau|_{A_0} \oplus tr_n$. To show that these are traces indeed, we can use the above lemma (it is still true, no mater that the norm of $t\tau_{A_0}$ can be less than one), or we can represent them as a convex linear combination $\lambda_t = t\tau + (1-t)\mu$ of the free product trace τ and the trace μ,* *defined by* $\mu(a_0 \oplus M) = tr_n(M) = tr_n(\tilde{\pi}(a_0 \oplus M))$.

Lemma C.15. $\bar{B}_0$ *(in* $\mathfrak{H}_A$*)* $= \bar{B}$ *(in* $\mathfrak{H}_A$*).*

Proof. Let's take $D \overset{def}{=} (\overset{1-\tilde{p}}{\mathbb{C}} \oplus \overset{\tilde{p}}{\mathbb{C}}) * (\overset{e_{11}}{\mathbb{C}} \oplus \overset{e_{22}+...+e_{nn}}{\mathbb{C}}) \subset B$. Denote $D_0 \overset{def}{=} D \cap B_0$. From Theorem B.3 follows that $D \cong \{f : [0,b] \to \mathbb{M}_2 | f$ is continuous and $f(0)$ - diagonal$\} \oplus \overset{\tilde{p}\wedge(1-e_{11})}{\mathbb{C}}$, where $0 < b < 1$ and $\tau|_D$ is given by an atomless measure μ on $\{f : [0,b] \to \mathbb{M}_2 | f$ is continuous and $f(0)$ - diagonal $\}$, $\tilde{p}$ is represented by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 1$, and e_{11} is represented by $\begin{pmatrix} 1-t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix} \oplus 0$. A $*$-homomorphism, defined on the generators of a C^*-algebra can be extended in at most one way to the whole C^*-algebra. This observation, together with $\pi_1(e_{11}) = \pi_1(\tilde{p}) = 1$ and $\pi_i(e_{22} = ... + e_{nn}) = \pi(\tilde{p}) = 1$ implies that $\pi_1|_D(f \oplus c) = f_{11}(0)$ and $\pi_i|_D(f \oplus c) = c$ for $2 \le i \le n-1$. This means that $D_0 = \{f : [0,b] \to \mathbb{M}_2 | f$ is continuous and $f_{11}(0) = f_{12}(0) = f_{21}(0) = 0\} \oplus 0$. Now we see $\bar{D_0}$ (in $\mathfrak{H}_D$) $\cong \mathbb{M}_2 \otimes L^\infty([0,b],\mu) \oplus 0$, so then $e_{11} \in \bar{D_0}$ (in $\mathfrak{H}_D$). So we can find sequence $\{\varepsilon_n\}$ of self-adjoined elements (functions) of D_0, supported on e_{11}, weakly converging to e_{11} on $\mathfrak{H}_D$ and such that $\{\varepsilon_n^2\}$ also converges weakly to e_{11} on $\mathfrak{H}_D$. Then take $a_1, a_2 \in A$. in $\mathfrak{H}_A$ we have $\langle \widehat{a_1}, (\varepsilon_n^2 - e_{11})\widehat{a_2} \rangle = \tau((\varepsilon_n^2 - e_{11})a_2 a_1^*) = \tau((\varepsilon_n - e_{11})a_2 a_1^*(\varepsilon_n - e_{11})) \le 4\|a_2 a_1^*\| \tau(\varepsilon_n^2 - e_{11})$ (The last inequality is obtained by representing $a_2 a_1^*$ as a linear combination of 4 positive elements and using Cauchy-Bounjakovsky-Schwartz inequality). This shows that $e_{11} \in \bar{D_0}$ (in $\mathfrak{H}_A$) $\subset \bar{B_0}$ (in $\mathfrak{H}_A$). Analoguously $e_{ii} \in \bar{B}_0$ (in $\mathfrak{H}_A$), so this shows $\bar{B}_0 = \bar{B}$ (in $\mathfrak{H}_A$). □

It easily follows now that

Corollary C.16. $\bar{A}_0$ *(in* $\mathfrak{H}_A$*)* $= \bar{A}$ *(in* $\mathfrak{H}_A$*).*

The representation of B_0 on $\mathfrak{H}_A$ is faithful and nondegenerate, and we can use

Proposition C.6, together with Theorem C.5 and the fact that $\mathrm{Ad}(u^k)$ are outer for $\bar{B} = \bar{B}_0$ to get:

Lemma C.17. *$A_0 = B_0 \rtimes G$ is simple.*

For the uniqueness of trace we need to modify a little the proof Theorem C.7 (which is [4, Theorem 1], stated for "nontwisted" crossed products).

Lemma C.18. *$A_0 = B_0 \rtimes G$ has a unique trace, $\tau|_{A_0}$.*

Proof. Above we already proved that $\{\mathrm{Ad}(u^k) | 1 \le k \le n-1\}$ are $\tau|_{B_0}$-outer for B_0.

Suppose that η is a trace on A_0. We will show that $\tau|_{A_0} = \eta$. We consider the GNS representation of B, associated to $\tau|_B$. By repeating the proof of Lemma C.13 we see that $\bar{B}_0$ (in $\mathfrak{H}_B$) $= \bar{B}$ (in $\mathfrak{H}_B$). The simplicity of B_0 allows us to identify B_0 with $\pi_{\tau|_B}(B_0)$. We will also identify B_0 with it's canonical copy in A_0. A_0 is generated by $\{b_0 \in B_0\} \cup \{u^k | 0 \le k \le n-1\}$ and $\{\mathrm{Ad}(u^k) | 0 \le k \le n-1\}$ extend to $\bar{B}_0$ (in $\mathfrak{H}_A$), so also to $\bar{B}_0$ (in $\mathfrak{H}_B$) ($\cong \bar{B}$ (in $\mathfrak{H}_A$)). Now we can form the von Neumann algebra crossed product $\tilde{A} \overset{def}{=} \bar{B}_0 \rtimes \{\mathrm{Ad}(u^k) | 0 \le k \le n-1\} \cong \bar{B} \rtimes \{\mathrm{Ad}(u^k) | 0 \le k \le n-1\}$, where the weak closures are in $\mathfrak{H}_B$. Clearly $\tilde{A} \cong \bar{A}$ (in $\mathfrak{H}_A$). Denote by $\widetilde{\tau_{B_0}}$ the extension of $\tau|_{B_0}$ to $\bar{B}_0$ (in $\mathfrak{H}_A$), given by $\widetilde{\tau_{B_0}}(x) = \langle x(\widehat{1_A}), \widehat{1_A} \rangle_{\mathfrak{H}_A}$. By [36, Chapter V, Proposition 3.19], $\widetilde{\tau_{B_0}}$ is a faithful normal trace on $\bar{B}_0$ (in $\mathfrak{H}_A$). Now from the fact that $\bar{B}_0$ (in $\mathfrak{H}_A$) is a factor and using [25, Lemma 1] we get that $\widetilde{\tau_{B_0}}$ is unique on $\bar{B}_0$ (in $\mathfrak{H}_A$). By the same argument we have that the extension $\widetilde{\tau_{A_0}}$ of $\tau|_{A_0}$ to $\bar{A}_0$ (in $\mathfrak{H}_A$) $\cong \bar{A}$ (in $\mathfrak{H}_A$) is unique, since $\bar{A}_0$ (in $\mathfrak{H}_A$) $\cong \bar{A}$ (in $\mathfrak{H}_A$) is a factor.

We take the unique extension of η to A. We will call it again η for convenience. We denote by $\mathfrak{H}'_C$ the GNS Hilbert space for C, corresponding to $\eta|_C$ (for $C = A$, B, B_0, A_0). Since $\eta|_{B_0} = \tau|_{B_0}$ it follows that $\bar{B}_0$ (in $\mathfrak{H}'_{B_0}$) $\cong \bar{B}$ (in $\mathfrak{H}'_B$) and of course $\mathfrak{H}'_{B_0} = \mathfrak{H}'_B$. Then similarly as in Lemma C.12 we get that $\bar{A}_0$ (in $\mathfrak{H}'_{A_0}$) $\cong \bar{A}$ (in $\mathfrak{H}'_A$), so $\mathfrak{H}'_{A_0} = \mathfrak{H}'_A$ (this can be done, since $\tau|_{B_0} = \eta|_{B_0}$). Now again by [36, Chapter V, Proposition 3.19] we have that $\tilde{\eta}(x) \stackrel{def}{=} \langle x(\widehat{1_A}), \widehat{1_A}\rangle_{\mathfrak{H}'_A}$ ($\widehat{1_A}$ is abuse of notation - in this case it's the element, corresponding to 1_A in $\mathfrak{H}'_A$) defines a faithful normal trace on $\overline{\pi'_A(A)}$ (in $\mathfrak{H}'_A$). In particular $\tilde{\eta}|_{\overline{\pi'_A(B)}}$ is a faithful normal trace on $\overline{\pi'_A(B)}$ (in $\mathfrak{H}'_A$). By uniqueness of $\tau|_{B_0}$ we have $\tau|_{B_0} = \eta|_{B_0}$, so for $b_0 \in B_0$ we have $\tilde{\tau}(b_0) = \tau(b_0) = \eta(b_0) = \langle \pi'_A(b_0)(\widehat{1_A}), \widehat{1_A}\rangle_{\mathfrak{H}'_A} = \tilde{\eta}(\pi'_A(b_0))$.

Since B_0 is simple, it follows that $\pi'_A|_{B_0}$ is a $*$-isomorphism from B_0 onto $\pi'_A(B_0)$ and from [20, Exercise 7.6.7] it follows that $\pi'_A|_{B_0}$ extends to a $*$-isomorphism from $\bar{B}_0$ (in $\mathfrak{H}_A$) $\cong \bar{B}$ (in $\mathfrak{H}_A$) onto $\overline{\pi'_A(B_0)}$ (in $\mathfrak{H}'_A$) $\cong \overline{\pi'_A(B)}$ (in $\mathfrak{H}'_A$). We will denote this $*$-isomorphism by θ. We set $w \stackrel{def}{=} \pi'_A(u)$, $\beta \stackrel{def}{=} \theta \mathrm{Ad}(u)\theta^{-1} \in Aut(\overline{\pi'_A(B)}$ (in $\mathfrak{H}'_A$)). For $b_0 \in B_0$ we have $w\pi'_A(b_0)w^* = \pi'_A(ub_0u^*) = \pi'_A((\mathrm{Ad}(u))(b_0)) = \beta(\pi'_A(b_0))$. So by weak continuity follows $\beta = \mathrm{Ad}(w)$ on $\overline{\pi'_A(B)}$ (in $\mathfrak{H}'_A$). Since $\bar{B}$ (in $\mathfrak{H}_A$) is a factor and $\{\mathrm{Ad}(u^k)|1 \le k \le n-1\}$ are all outer, Kallman's Theorem ([21, Corollary 1.2]) gives us that $\{\mathrm{Ad}(u^k)|1 \le k \le n-1\}$ act freely on $\bar{B}$ (in $\mathfrak{H}_A$). Namely if $\bar{b} \in \bar{B}$ (in $\mathfrak{H}_A$), and if $\forall \bar{b}' \in \bar{B}$ (in $\mathfrak{H}_A$), $\bar{b}\bar{b}' = \mathrm{Ad}(u^k)(\bar{b}')\bar{b}$, then $\bar{b} = 0$. Then by the above settings it is clear that $\{\mathrm{Ad}(w^k)|1 \le k \le n-1\}$ also act freely on $\overline{\pi'_A(B)}$ (in $\mathfrak{H}'_A$).

Since $\tilde{\eta}$ is a faithful normal trace on $\overline{\pi'_A(A)}$ (in $\mathfrak{H}'_A$), then by [36, Chapter V, Proposition 2.36] there exists a faithful conditional expectation $P : \overline{\pi'_A(A)} \to \overline{\pi'_A(B)}$ (both weak closures are in $\mathfrak{H}'_A$). $\forall x \in \overline{\pi'_A(B)}$ (in $\mathfrak{H}'_A$), and $\forall 1 \le k \le n-1$, $\mathrm{Ad}(w^k)(x)w^k = w^k x$. Applying P we get $\mathrm{Ad}(w^k)(x)(P(w^k)) = P(w^k)x$, so by the free action of $\mathrm{Ad}(w^k)$ we get that $P(w^k) = 0$, $\forall 1 \le k \le n-1$. It's clear that $\{\overline{\pi'_A(B)}\} \cup \{w^k|1 \le k \le n-1\}$ generates $\overline{\pi'_A(A)}$ (in $\mathfrak{H}'_A$) as a von Neumann algebra. Now we use [35, Proposition 22.2]. It gives us a $*$-isomorphism $\Phi : \overline{\pi'_A(A)}$ (in $\mathfrak{H}'_A$) $\to \bar{B} \rtimes \{\mathrm{Ad}(u^k)|1 \le k \le n-1\} \cong \bar{A}$ (last two weak closures are in $\mathfrak{H}_A$) with $\Phi(\theta(x)) = x$, $x \in \bar{B}$ (in $\mathfrak{H}_A$), $\Phi(w) = u$. So since $\bar{A}$ (in $\mathfrak{H}_A$) is a finite factor, so is $\overline{\pi'_A(A)}$ (in $\mathfrak{H}'_A$), and so it's trace $\tilde{\eta}$ is unique. Hence, $\tilde{\eta} = \tilde{\tau} \circ \Phi$, and so $\forall b \in B$, and $\forall 1 \le k \le n-1$ we have $\eta(bu^k) = \tilde{\eta}(\pi'_A(b)\pi'_A(u^k)) = \tilde{\tau}(\Phi(\pi'_A(b))\Phi(\pi'_A(u^k))) =$

$\tilde{\tau}(\Phi(\theta(b))\Phi(w^k)) = \tilde{\tau}(bu^k) = \tau(bu^k)$. By continuity and linearity of both traces we get $\eta = \tau$, just what we want. □

We conclude this section by proving the following

Proposition C.19. *Let*

$$(A, \tau) \overset{def}{=} (\underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus ... \oplus \underset{\alpha_m}{\overset{p_m}{\mathbb{C}}}) * (\mathbb{M}_n, tr_n),$$

where $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_m$. *Then:*

(I) If $\alpha_m < 1 - \frac{1}{n^2}$, *then* A *is unital, simple with a unique trace* τ.

(II) If $\alpha_m = 1 - \frac{1}{n^2}$, *then we have a short exact sequence* $0 \to A_0 \to A \to \mathbb{M}_n \to 0$, *where* A *has no central projections, and* A_0 *is nonunital, simple with a unique trace* $\tau|_{A_0}$.

(III) If $\alpha_m > 1 - \frac{1}{n^2}$, *then* $A = \underset{n^2 - n^2\alpha_m}{\overset{f}{A_0}} \oplus \underset{n^2\alpha_m - n^2 + 1}{\overset{1-f}{\mathbb{M}_n}}$, *where* $1 - f \leq p_m$, *and where* A_0 *is unital, simple and has a unique trace* $(n^2 - n^2\alpha_m)^{-1}\tau|_{A_0}$.

Let f *means the identity projection for cases (I) and (II). Then in all cases for each of the projections* $fp_1, ..., fp_m$ *we have a unital, diffuse abelian* C^**-subalgebra of* A*, supported on it.*

In all the cases p_m *is a full projection in* A.

Proof. We have to prove the second part of the proposition, since the first part follows from Lemma C.10, Lemma C.11, Lemma C.12, Lemma C.17 and Lemma C.18. From the discussion above we see that in all cases we have $fA = fB \rtimes \{\text{Ad}(fu^kf) | 0 \leq k \leq n-1\}$, where B and $\{\text{Ad}(fu^k) | 0 \leq k \leq n-1\}$ are as above. So the existence of the unital, diffuse abelian C^*-sublagebras follows from Theorem B.4, applied to B.

In the case (I) p_m is clearly full, since A is simple. In the case (III) it's easy to see that $p_m \wedge f \neq 0$ and $p_m \geq (1 - f)$, so since A_0 and $\mathbb{M}_n$ are simple in this case, then p_m is full in A. In case (II) it follows from Theorem B.4 that p_m is full in B, and consequently in A. □

D. The General Case

In this section we prove the general case of Theorem B.6, using the result from the previous section (Proposition C.19). The prove of the general case involves techniques from [12]. So we will need two technical results from there.

The first one is [12, Proposition 2.8] (see also [9]):

Proposition D.1. *Let* $A = A_1 \oplus A_2$ *be a direct sum of unital* C^**-algebras and let* $p = 1 \oplus 0 \in A$*. Suppose* ϕ_A *is a state on* A *with* $0 < \alpha \overset{def}{=} \phi_A(p) < 1$*. Let* B *be a unital* C^**-algebra with a state* ϕ_B *and let* $(\mathfrak{A}, \phi) = (A, \phi_A) * (B, \phi_B)$*. Let* $\mathfrak{A}_1$ *be the* C^**-subalgebra of* $\mathfrak{A}$ *generated by* $(0 \oplus A_2) + \mathbb{C}p \subseteq A$*, toghether with* B*. In other words*

$$(\mathfrak{A}_1, \phi|_{\mathfrak{A}_1}) = (\underset{\alpha}{\overset{p}{\mathbb{C}}} \oplus \underset{1-\alpha}{\overset{1-p}{A_2}}) * (B, \phi_B).$$

Then $p\mathfrak{A}p$ *is generated by* $p\mathfrak{A}_1p$ *and* $A_1 \oplus 0 \subset A$*, which are free in* $(p\mathfrak{A}p, \frac{1}{\alpha}\phi|_{p\mathfrak{A}p})$*. In other words*

$$(p\mathfrak{A}p, \frac{1}{\alpha}\phi|_{p\mathfrak{A}p}) \cong (p\mathfrak{A}_1p, \frac{1}{\alpha}\phi|_{p\mathfrak{A}_1p}) * (A_1, \frac{1}{\alpha}\phi_A|_{A_1}).$$

Remark D.2. *This proposition was proved for the case of von Neumann algebras in [9]. It is true also in the case of* C^**-algebras.*

The second result is [12, Proposition 2.5 (ii)], which is easy and we give its proof also:

Proposition D.3. *Let* A *be a* C^**-algebra. Take* $h \in A, h \geq 0$*, and let* B *be the hereditary subalgebra* $\overline{hAh}$ *of* A *(* $\overline{*}$ *means norm closure). Suppose that* B *is full in* A*. Then if* B *has a unique trace, then* A *has at most one tracial state.*

Proof. It's easy to see that $\text{Span}\{xhahy | a, x, y \in A\}$ is norm dense in A. If τ is a tracial state on A then $\tau(xhahy) = \tau(h^{1/2}ahyxh^{1/2})$. Since $h^{1/2}ahyxh^{1/2} \in B$, τ is uniquely determined by τ_B. □

It is clear that Proposition C.19 agrees with Theorem B.6, so it is a special case.

As a next step we look at a C^*-algebra of the form

$$(M,\tau) = (\underset{\alpha_0'}{\overset{p_0'}{A_0}} \oplus \underset{\alpha_1'}{\overset{p_1'}{\mathbb{M}_{m_1}}} \oplus \ldots \oplus \underset{\alpha_k'}{\overset{p_k'}{\mathbb{M}_{m_k}}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus \ldots \oplus \underset{\alpha_l}{\overset{p_l}{\mathbb{C}}}) * (\mathbb{M}_n, tr_n),$$

where A_0 comes with a specified trace and has a unital, diffuse abelian C^*-subalgebra with unit p_0'. Also we suppose that $\alpha_0' \geq 0$, $0 < \alpha_1' \leq \ldots \leq \alpha_k'$, $0 < \alpha_1 \leq \ldots \leq \alpha_l$, $m_1, \ldots, m_k \geq 2$, and either $\alpha_0' > 0$ or $k \geq 1$, or both. Let's denote $p_0 \overset{def}{=} p_0' + p_1' + \ldots + p_k'$, $B_0 \overset{def}{=} \underset{\alpha_1'}{\overset{p_1'}{\mathbb{M}_{m_1}}} \oplus \ldots \oplus \underset{\alpha_k'}{\overset{p_k'}{\mathbb{M}_{m_k}}}$, and $\alpha_0 \overset{def}{=} \alpha_0' + \alpha_1' + \ldots + \alpha_k' = \tau(p_0)$.

Let's have a look at the C^*-subalgebras N and N' of M given by

$$(N, \tau|_N) = (\underset{\alpha_0}{\overset{p_0}{\mathbb{C}}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus \ldots \oplus \underset{\alpha_l}{\overset{p_l}{\mathbb{C}}}) * (\mathbb{M}_n, tr_n)$$

and

$$(N', \tau|_{N'}) = (\underset{\alpha_0'}{\overset{p_0'}{\mathbb{C}}} \oplus \underset{\alpha_1'}{\overset{p_1'}{\mathbb{C}}} \oplus \ldots \oplus \underset{\alpha_k'}{\overset{p_k'}{\mathbb{C}}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus \ldots \oplus \underset{\alpha_l}{\overset{p_l}{\mathbb{C}}}) * (\mathbb{M}_n, tr_n).$$

We studied the C^*-algebras, having the form of N and N' in the previous section. A brief description is as follows:

If $\alpha_0, \alpha_l < 1 - \frac{1}{n^2}$, then N is simple with a unique trace and N' is also simple with a unique trace. For each of the projections $p_0', p_1', \ldots, p_k', p_1, \ldots, p_l$ we have a unital, diffuse abelian C^*-subalgebra of N', supported on it.

If α_0, or $\alpha_l = 1 - \frac{1}{n^2}$, then N has no central projections, and we have a short exact sequence $0 \to N_0 \to N \to \mathbb{M}_n \to 0$, with N_0 being simple with a unique trace. Moreover p_0 or p_l respectivelly is full in N. For each of the projections $p_0', p_1', \ldots, p_k', p_1, \ldots, p_l$ we have a unital, diffuse abelian C^*-subalgebra of N', supported on it.

If α_0 or $\alpha_l > 1 - \frac{1}{n^2}$, then $N = \overset{q}{N_0} \oplus \mathbb{M}_n$, with N_0 being simple and having a unique trace.

We consider 2 cases:

(I) case: $\alpha_l \geq \alpha_0$.

(1) $\alpha_l < 1 - \frac{1}{n^2}$.

In this case N and N' are simple and has unique traces, and p_0 is full in N and consequently $1_M = 1_N$ is contained in $\langle p_0 \rangle_N$ - the ideal of N, generated by p_0. Since $\langle p_0 \rangle_N \subset \langle p_0 \rangle_M$ it follows that p_0 is full also in M. From Proposition 4.1 we get $p_0 M p_0 \cong (A_0 \oplus B_0) * p_0 N p_0$. Then from Theorem C.9 follows that $p_0 M p_0$ is simple and has a unique trace. Since p_0 is a full projection, Proposition D.3 tells us that M is simple and τ is its unique trace. For each of the projections $p'_0, p'_1, ..., p'_k, p_1, ..., p_l$ we have a unital, diffuse abelian C^*-subalgebra of M, supported on it, and comming from N'.

(2) $\alpha_l = 1 - \frac{1}{n^2}$.

In this case it is also true that for each of the projections $p'_0, p'_1, ..., p'_k, p_1, ..., p_l$ we have a unital, diffuse abelian C^*-subalgebra of M, supported on it, and comming from N'. It is easy to see that M is the linear span of $p_0 M p_0$, $p_0 M (1-p_0) N (1-p_0)$, $(1-p_0) N p_0 M p_0$, $(1-p_0) N p_0 M p_0 N (1-p_0)$ and $(1-p_0) N (1-p_0)$. We know that we have a $*$-homomorphism $\pi : N \to M_n$, such that $\pi(p_l) = 1$. Then it is clear that $\pi(p_0) = 0$, so we can extend π to a linear map $\tilde{\pi}$ on M, defining it to equal 0 on $p_0 M p_0$, $p_0 M (1-p_0) N (1-p_0)$, $(1-p_0) N p_0 M p_0$ and $(1-p_0) N p_0 M p_0 N (1-p_0)$. It is also clear then that $\tilde{\pi}$ will actually be a $*$-homomorphism. Since $\ker(\pi)$ is simple in N and $p_0 \in \ker(\pi)$, then p_0 is full in $\ker(\pi) \subset N$, so by the above representation of M as a linear span we see that p_0 is full in $\ker(\tilde{\pi})$ also. From Proposition D.1 follows that $p_0 M p_0 \cong (A_0 \oplus B_0) * (p_0 N p_0)$. Since $p_0 N p_0$ has a unital, diffuse abelian C^*-subalgebra with unit p_0, it follows from Theorem C.9 that $p_0 M p_0$ is simple and has a unique trace (to make this conclusion we could use Theorem A.2 instead). Now since $p_0 M p_0$ is full and hereditary in $\ker(\tilde{\pi})$, from Proposition D.3 follows that $\ker(\tilde{\pi})$

is simple and has a unique trace.

(3) $\alpha_l > 1 - \frac{1}{n^2}$.

In this case $N = \underset{n^2-n^2\alpha_l}{\overset{q}{N_0}} \oplus \underset{n^2\alpha_l-n^2+1}{\overset{1-q}{\mathbb{M}_n}}$ and also $N' = \underset{n^2-n^2\alpha_l}{\overset{q}{N_0'}} \oplus \underset{n^2\alpha_l-n^2+1}{\overset{1-q}{\mathbb{M}_n}}$ with N_0 and N_0' being simple with unique traces. For each of the projections $qp_0', qp_1', \ldots, qp_k'$, $qp_1, \ldots, qp_l$ we have a unital, diffuse abelian C^*-subalgebra of M, supported on it, and coming from N_0'.

Since $p_0 \leq q$ we can write M as a linear span of p_0Mp_0, $p_0Mp_0N_0(1-p_0)$, $(1-p_0)N_0p_0Mp_0$, $(1-p_0)N_0p_0Mp_0N_0(1-p_0)$, $(1-p_0)N_0(1-p_0)$ and $\mathbb{M}_n$. So we can write $M = \underset{n^2-n^2\alpha_l}{\overset{q}{M_0}} \oplus \underset{n^2\alpha_l-n^2+1}{\overset{1-q}{\mathbb{M}_n}}$, where $M_0 \overset{def}{=} qMq \supset N_0$. We know that p_0 is full in N_0, so as before we can write $1_{M_0} = 1_{N_0} \in \langle p_0 \rangle_{N_0} \subset \langle p_0 \rangle_{M_0}$, so $\langle p_0 \rangle_{M_0} = M_0$. Because of Proposition D.1, we can write $p_0M_0p_0 \cong (A_0 \oplus B_0) * (p_0N_0p_0)$. Since $p_0N_0p_0$ has a unital, diffuse abelian C^*-subalgebra with unit p_0, then from Theorem 3.9 (or from Theorem A.2) it follows that $p_0M_0p_0$ is simple with a unique trace. Since $p_0M_0p_0$ is full and hereditary in M_0, Proposition D.3 yields that M_0 is simple with a unique trace.

(II) $\alpha_0 > \alpha_l$.

(1) $\alpha_0 \leq 1 - \frac{1}{n^2}$.

In this case p_0 is full in N and also in N', so $1_M = 1_N \in \langle p_0 \rangle_N$, which means p_0 is full in M also. p_0Mp_0 is a full hereditary C^*-subalgebra of M and $p_0Mp_0 \cong (A_0 \oplus B_0) * p_0Np_0$ by Proposition D.1. Since p_0Np_0 has a diffuse abelian C^*-subalgebra, Theorem C.9 (or Theorem A.2) shows that p_0Mp_0 is simple with a unique trace and then by Proposition D.3 follows that the same is true for M. For each of the projections $p_0', p_1', ..., p_k', p_1, ..., p_l$ we have a unital, diffuse abelian C^*-subalgebra of M, supported on it, comming from N'.

(2) $\alpha_0 > 1 - \frac{1}{n^2}$.

We have 3 cases:

(2′) $\alpha'_0 > 1 - \frac{1}{n^2}$.

In this case $N \cong \overset{q}{N_0} \oplus \mathbb{M}_n$ and $N' \cong \overset{q'}{N'_0} \oplus \mathbb{M}_n$, where $q \leq q'$, with N_0 and N'_0 being simple and having unique traces. It is easy to see that $p'_1, ..., p'_k, p_1, ..., p_l \leq q'$, so for each of the projections $p'_1, ..., p'_k, p_1, ..., p_l$ we have a unital, diffuse abelian C^*-subalgebra of N', supported on it. So those C^*-subalgebras live in M also. We have a unital, diffuse abelian C^*-subalgebra of A_0, supported on 1_{A_0}, which yields a unital, diffuse abelian C^*-subalgebra on M, supported on p'_0. It is clear that p_0 is full in N, so as before, $1_M = 1_N \in \langle p_0 \rangle_N$, so p_0 is full in M also, so $p_0 M p_0$ is a full hereditary C^*-subalgebra of M. From Proposition D.1 we have $p_0 M p_0 \cong (A_0 \oplus B_0) * (p_0 N_0 p_0 \oplus \mathbb{M}_n)$. It is easy to see that $\mathbb{M}_n$, for $n \geq 2$ contains two tr_n-orthogonal zero-trace unitaries. Since also $p_0 N_0 p_0$ has a unital, diffuse abelian C^*-subalgebra, supported on 1_{N_0}, it is easy to see (using Proposition B.2) that it also contains two $\tau | N_0$-orthogonal, zero-trace unitaries. Then the conditions of Theorem A.2 are satisfied. This means that $p_0 M p_0$ is simple with a unique trace and Proposition D.3 implies that M is simple with a unique trace also.

(2′′) $\alpha'_k > 1 - \frac{1}{n^2}$.

Let's denote

$$N'' = (\underset{\alpha'_0}{\overset{p'_0}{A_0}} \oplus \underset{\alpha'_1}{\overset{p'_1}{\mathbb{M}_{m_1}}} \oplus ... \oplus \underset{\alpha'_{k-1}}{\overset{p'_{k-1}}{\mathbb{M}_{m_{k-1}}}} \oplus \underset{\alpha'_k}{\overset{p'_k}{\mathbb{C}}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus ... \oplus \underset{\alpha_l}{\overset{p_l}{\mathbb{C}}}) * (\mathbb{M}_n, tr_n).$$

Then N'' satisfies the conditions of case (I,3) and so $N'' \cong \overset{q}{N''_0} \oplus \mathbb{M}_n$. Clearly $p'_0, p'_1, ..., p'_{k-1}, p_1, ..., p_l \leq q$, so for each of the projections $p'_0, p'_1, ..., p'_{k-1}, p_1, ..., p_l$ we have a unital, diffuse abelian C^*-subalgebra of N''_0, supported on it. Those C^*-algebras live in M also. From case (I,3) we have that p'_k is full in N'' and as before $1_M = 1_{N''} \in \langle p'_k \rangle_{N''}$ implies that p'_k is full in M also. From Proposition D.1 follows that $p'_k M p'_k \cong (p'_k N''_0 p'_k \oplus \mathbb{M}_n) * \mathbb{M}_{m_k}$. Since N''_0 has a unital, diffuse abelian C^*-

subalgebra, supported on qp'_k, then an argument, similar to the one we made in case (II, 2"), allows to apply Theorem A.2 to get that $p'_k M p'_k$ is simple with a unique trac. By Proposition D.3 follows that the same is true for M. The unital, diffuse abelian C^*-subalgebra of M, supported on p'_k, we can get by applying the note after Theorem A.2 to $p'_k M p'_k \cong (p'_k N''_0 p'_k \oplus \mathbb{M}_n) * \mathbb{M}_{m_k}$.

(2‴) α'_0 and $\alpha'_k \leq 1 - \frac{1}{n^2}$.

In this case $N \cong \overset{q}{N_0} \oplus \mathbb{M}_n$, with N_0 being simple and having a unique trace. Moreover N' has no central projections and for each of the projections $p'_0, p'_1, ..., p'_k, p_1, ..., p_l$ we have a unital, diffuse abelian C^*-subalgebra of N', supported on it. So those C^*-subalgebras live in M also. It is clear that p_0 is full in N, so as before $1_M = 1_N \in \langle p_0 \rangle_N$, so p_0 is full in M also, so $p_0 M p_0$ is a full hereditary C^*-subalgebra of M. From Proposition D.1 we have $p_0 M p_0 \cong (A_0 \oplus B_0) * (p_0 N_0 p_0 \oplus \mathbb{M}_n)$. Since A_0 and $p_0 N_0 p_0$ both have unital, diffuse abelian C^*-subalgebras, supported on their units, it is easy to see (using Proposition B.2), that the conditions of Theorem A.2 are satisfied. This means that $p_0 M p_0$ is simple with a unique trace and Proposition D.3 yields that M is simple with a unique trace also.

We summarize the discussion above in the following

Proposition D.4. *Let*

$$(M,\tau) \overset{def}{=} (\underset{\alpha'_0}{\overset{p'_0}{A_0}} \oplus \underset{\alpha'_1}{\overset{p'_1}{\mathbb{M}_{m_1}}} \oplus ... \oplus \underset{\alpha'_k}{\overset{p'_k}{\mathbb{M}_{m_k}}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus ... \oplus \underset{\alpha_l}{\overset{p_l}{\mathbb{C}}}) * (\mathbb{M}_n, tr_n),$$

where $n \geq 2$, $\alpha'_0 \geq 0$, $\alpha'_1 \leq \alpha'_2 \leq ... \leq \alpha'_k$, $\alpha_1 \leq ... \leq \alpha_l$, $m_1, ..., m_k \geq 2$, and $\overset{p'_0}{A_0} \oplus 0$ has a unital, diffuse abelian C^*-subalgebra, having p'_0 as a unit. Then:

(I) If $\alpha_l < 1 - \frac{1}{n^2}$, then M is unital, simple with a unique trace τ.

(II) If $\alpha_l = 1 - \frac{1}{n^2}$, then we have a short exact sequence $0 \to M_0 \to M \to \mathbb{M}_n \to 0$, where M has no central projections and M_0 is nonunital, simple with a unique trace

$\tau|_{M_0}$.

(III) If $\alpha_l > 1 - \frac{1}{n^2}$, *then* $M = \underset{n^2-n^2\alpha_l}{\overset{f}{M_0}} \oplus \underset{n^2\alpha_l-n^2+1}{\overset{1-f}{\mathbb{M}_n}}$, *where* $1 - f \leq p_l$, *and where* M_0 *is unital, simple and has a unique trace* $(n^2 - n^2\alpha_l)^{-1}\tau|_{M_0}$.

Let f *means the identity projection for cases (I) and (II). Then in all cases for each of the projections* $fp'_0, fp'_1, ..., fp'_k, fp_1, ..., fp_l$ *we have a unital, diffuse abelian* C^**-subalgebra of* M*, supported on it.*

In all the cases p_l *is a full projection in* M.

To prove Theorem B.6 we will use Proposition D.4. First let's check that Proposition D.4 agrees with the conclusion of Theorem B.6. We can write

$$(M, \tau) \overset{def}{=} (\underset{\alpha'_0}{\overset{p'_0}{A_0}} \oplus \underset{\alpha'_1}{\overset{p'_1}{\mathbb{M}_{m_1}}} \oplus ... \oplus \underset{\alpha'_k}{\overset{p'_k}{\mathbb{M}_{m_k}}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{C}}} \oplus ... \oplus \underset{\alpha_l}{\overset{p_l}{\mathbb{C}}}) * \underset{\beta_1}{\overset{q_1}{\mathbb{M}_n}},$$

where $q_1 = 1_M$ and $\beta_1 = 1$. It is easy to see that $L_0 = \{(l,1)|\frac{\alpha_l}{1^2} + \frac{1}{n^2} = 1\} = \{(l,1)|\alpha_l = 1 - \frac{1}{n^2}\}$, which is not empty if and only if $\alpha_l = 1 - \frac{1}{n^2}$. Also $L_+ = \{(l,1)|\frac{\alpha_l}{1^2} + \frac{1}{n^2} > 1\} = \{(l,1)|\alpha_l > 1 - \frac{1}{n^2}\}$, and here L_+ is not empty if and only if $\alpha_l > 1 - \frac{1}{n^2}$. If both L_+ and L_0 are empty, then M is simple with a unique trace. If L_0 is not empty, then clearly L_+ is empty, so we have no central projections and a short exact sequence $0 \to M_0 \to M \to \mathbb{M}_n \to 0$, with M_0 being simple with a unique trace. In this case all nontrivial projections are full in M. If L_+ is not empty, then clearly L_0 is empty and so $M = \underset{n^2-n^2\alpha_l}{\overset{q}{M_0}} \oplus \underset{n^2(\frac{\alpha_l}{1^2}+\frac{1}{n^2}-1)}{\overset{1-q}{\mathbb{M}_n}}$, where M_0 is simple with a unique trace. p_l is full in M.

Proof of Theorem B.6:

Now to prove Theorem B.6 we start with

$$(\mathfrak{A}, \phi) = (\underset{\alpha_0}{\overset{p_0}{A_0}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{M}_{n_1}}} \oplus ... \oplus \underset{\alpha_k}{\overset{p_k}{\mathbb{M}_{n_k}}}) * (\underset{\beta_0}{\overset{q_0}{B_0}} \oplus \underset{\beta_1}{\overset{q_1}{\mathbb{M}_{m_1}}} \oplus ... \oplus \underset{\beta_l}{\overset{q_l}{\mathbb{M}_{m_l}}}),$$

where A_0 and B_0 have unital, diffuse abelian C^*-subalgebras, supported on their units (we allow $\alpha_0 = 0$ or/and $\beta_0 = 0$). The case where $n_1 = ... = n_k = m_1 = ... = m_l = 1$ is treated in Theorem B.5. The case where $\alpha_0 = 0$, $k = 1$, and $n_k > 1$ was treated in Proposition D.4. So we can suppose without loss of generality that $n_k \geq 2$ and either $k > 1$ or $\alpha_0 > 0$ or both. To prove that the conclusions of Theorem B.6 takes place in this case we will use induction on $\text{card}\{i|n_i \geq 2\} + \text{card}\{j|m_j \geq 2\}$, having Theorem B.5 ($\text{card}\{i|n_i \geq 2\} + \text{card}\{j|m_j \geq 2\} = 0$) as first step of the induction. We look at

$$(\mathfrak{B}, \phi|_{\mathfrak{B}}) = (\underset{\alpha_0}{\overset{p_0}{A_0}} \oplus \underset{\alpha_1}{\overset{p_1}{\mathbb{M}_{n_1}}} \oplus ... \oplus \underset{\alpha_{k-1}}{\overset{p_{k-1}}{\mathbb{M}_{n_{k-1}}}} \oplus \underset{\alpha_k}{\overset{p_k}{\mathbb{C}}}) * (\underset{\beta_0}{\overset{q_0}{B_0}} \oplus \underset{\beta_1}{\overset{q_1}{\mathbb{M}_{m_1}}} \oplus ... \oplus \underset{\beta_l}{\overset{q_l}{\mathbb{M}_{m_l}}}) \subset (\mathfrak{A}, \phi).$$

We suppose that Theorem B.6 is true for $(\mathfrak{B}, \phi|_{\mathfrak{B}})$ and we will prove it for $(\mathfrak{A}, \phi)$. This will be the induction step and will prove Theorem B.6.

Denote

$$L_0^{\mathfrak{A}} \overset{def}{=} \{(i,j)| \frac{\alpha_i}{n_i^2} + \frac{\beta_j}{m_j^2} = 1\},$$

$$L_0^{\mathfrak{B}} \overset{def}{=} \{(i,j)|i \leq k-1 \text{ and } \frac{\alpha_i}{n_i^2} + \frac{\beta_j}{m_j^2} = 1\} \cup \{(k,j)| \frac{\alpha_k}{1^2} + \frac{\beta_j}{m_j^2} = 1\}$$

and similarly

$$L_+^{\mathfrak{A}} \overset{def}{=} \{(i,j)| \frac{\alpha_i}{n_i^2} + \frac{\beta_j}{m_j^2} > 1\},$$

and

$$L_+^{\mathfrak{B}} \overset{def}{=} \{(i,j)|i \leq k-1, \text{ and } \frac{\alpha_i}{n_i^2} + \frac{\beta_j}{m_j^2} > 1\} \cup \{(k,j)| \frac{\alpha_k}{1^2} + \frac{\beta_j}{m_j^2} > 1\}.$$

Clearly

$$L_0^{\mathfrak{A}} \cap \{1 \leq i \leq k-1\} = L_0^{\mathfrak{B}} \cap \{1 \leq i \leq k-1\}$$

and similarly

$$L_+^{\mathfrak{A}} \cap \{1 \leq i \leq k-1\} = L_+^{\mathfrak{B}} \cap \{1 \leq i \leq k-1\}.$$

Let $N_{\mathfrak{A}}(i,j) = max(n_i, m_j)$, let $N_{\mathfrak{B}}(i,j) = N_{\mathfrak{A}}(i,j), 1 \leq i \leq k-1$, and let $N_{\mathfrak{B}}(k,j) = m_j$.

By assumption

$$\mathfrak{B} = \overset{g}{\underset{\delta}{\mathfrak{B}_0}} \oplus \bigoplus_{(i,j)\in L_+^{\mathfrak{B}}} \overset{g_{ij}}{\underset{\delta_{ij}}{\mathbb{M}_{N_{\mathfrak{B}}(i,j)}}}.$$

We want to show that

$$\mathfrak{A} = \overset{f}{\underset{\gamma}{\mathfrak{A}_0}} \oplus \bigoplus_{(i,j)\in L_+^{\mathfrak{A}}} \overset{f_{ij}}{\underset{\gamma_{ij}}{\mathbb{M}_{N_{\mathfrak{A}}(i,j)}}}. \tag{3.9}$$

We can represent $\mathfrak{A}$ as the span of $p_k\mathfrak{A}p_k$, $p_k\mathfrak{A}p_k\mathfrak{B}(1-p_k)$, $(1-p_k)\mathfrak{B}p_k\mathfrak{A}p_k$, $(1-p_k)\mathfrak{B}p_k\mathfrak{A}p_k\mathfrak{B}(1-p_k)$, and $(1-p_k)\mathfrak{B}(1-p_k)$. From the fact that $g_{kj} \leq p_k$ and $g_{ij} \leq 1-p_k, \forall 1 \leq i \leq k-1$ we see that $p_k\mathfrak{B}(1-p_k) = p_k\mathfrak{B}_0(1-p_k)$, $(1-p_k)\mathfrak{B}p_k = (1-p_k)\mathfrak{B}_0p_k$, and $(1-p_k)\mathfrak{B}(1-p_k) = (1-p_k)\mathfrak{B}_0(1-p_k) \oplus \bigoplus_{\substack{(i,j)\in L_+^{\mathfrak{B}} \\ i\neq k}} \mathbb{M}_{N(i,j)}$. All this tells us that we can represent $\mathfrak{A}$ as the span of $p_k\mathfrak{A}p_k$, $p_k\mathfrak{A}p_k\mathfrak{B}_0(1-p_k)$, $(1-p_k)\mathfrak{B}_0p_k\mathfrak{A}p_k$, $(1-p_k)\mathfrak{B}_0p_k\mathfrak{A}p_k\mathfrak{B}_0(1-p_k)$, $(1-p_k)\mathfrak{B}_0(1-p_k)$, and $\bigoplus_{\substack{(i,j)\in L_+^{\mathfrak{B}} \\ i\neq k}} \overset{g_{ij}}{\underset{\delta_{ij}}{\mathbb{M}_{N(i,j)}}}$.

In order to show that $\mathfrak{A}$ has the form (4.1), we need to look at $p_k\mathfrak{A}p_k$. From Proposition D.1 we have

$$p_k\mathfrak{A}p_k \cong (p_k\mathfrak{B}p_k) * \mathbb{M}_{n_k} \cong (p_k\overset{g}{\underset{\frac{\delta}{\alpha_k}}{\mathfrak{B}_0}}p_k \oplus \bigoplus_{(k,j)\in L_+^{\mathfrak{B}}} \overset{g_{kj}}{\underset{\frac{\delta_{kj}}{\alpha_k}}{\mathbb{M}_{N(k,j)}}}) * \mathbb{M}_{n_k}.$$

Since by assumption $p_k\mathfrak{B}_0p_k$ has a unital, diffuse abelian C^*-subalgebra, supported on $1_{p_k\mathfrak{B}_0p_k}$, we can use Proposition D.4 to determine the form of $p_k\mathfrak{A}p_k$.

Thus $p_k\mathfrak{A}p_k$:

(i) Is simple with a unique trace if whenever for all $1 \leq r \leq l$ with $N(k,r) = 1$ we have $\frac{\delta_{kr}}{\alpha_k} < 1 - \frac{1}{n_k^2}$.

(ii) Is an extension $0 \to I \to p_k\mathfrak{A}p_k \to \mathbb{M}_{n_k} \to 0$ if $\exists 1 \leq r \leq l$, with $N(k,r) = 1$, and $\frac{\delta_{kr}}{\alpha_k} = 1 - \frac{1}{n_k^2}$. Moreover I is simple with a unique trace and has no central projections.

(iii) Has the form $p_k\mathfrak{A}p_k = I \oplus \underset{n_k^2(\frac{\delta_{kr}}{\alpha_k}-1+\frac{1}{n_k^2})}{\mathbb{M}_{n_k}}$, where I is unital, simple with a unique trace whenever $\exists 1 \leq r \leq l$ with $N(k,r) = 1$, and $\frac{\delta_{kr}}{\alpha_k} > 1 - \frac{1}{n_k^2}$.

By assumption $\delta_{ij} = N(i,j)^2(\frac{\alpha_i}{n_i^2} + \frac{\beta_j}{m_j^2} - 1)$, so when r satisfies the conditions of case (iii) above, then $m_r = 1$ and

$$n_k^2(\frac{\delta_{kr}}{\alpha_k} - 1 + \frac{1}{n_k^2}) = n_k^2(\frac{\alpha_k + \beta_r - 1}{\alpha_k} + \frac{1}{n_k^2} - 1) = \frac{n_k^2}{\alpha_k}(\frac{\alpha_k}{n_k^2} + \frac{\beta_r}{1^2} - 1),$$

just what we needed to show. Defining

$$\mathfrak{A}_0 \overset{def}{=} (1 - (\bigoplus_{(i,j)\in L_+^{\mathfrak{A}}} f_{ij}))\mathfrak{A}(1 - (\bigoplus_{(i,j)\in L_+^{\mathfrak{A}}} f_{ij})),$$

we see that $\mathfrak{A}$ has the form (4.1).

We need to study $\mathfrak{A}_0$ now. Since clearly $g \leq f$, we see that $\mathfrak{A}p_k\mathfrak{B}_0 = \mathfrak{A}p_k g\mathfrak{B}_0 = \mathfrak{A}gp_k\mathfrak{B}_0 = \mathfrak{A}_0 p_k\mathfrak{B}_0$ and similarly $\mathfrak{A}p_k\mathfrak{B}_0 = \mathfrak{A}_0 p_k\mathfrak{B}_0$. From this and from what we proved above follows that:

$$\mathfrak{A}_0 \text{ is the span of } p_k\mathfrak{A}_0 p_k,\ (1-p_k)\mathfrak{B}_0 p_k\mathfrak{A}_0 p_k, \tag{3.10}$$
$$p_k\mathfrak{A}_0 p_k\mathfrak{B}_0(1-p_k),\ (1-p_k)\mathfrak{B}_0 p_k\mathfrak{A}_0 p_k\mathfrak{B}_0(1-p_k),\ \text{and } (1-p_k)\mathfrak{B}_0(1-p_k).$$

We need to show that for each of the projections fp_s, $0 \leq s \leq k$ and fq_t, $1 \leq t \leq l$, we have a unital, diffuse abelian C^*-subalgebra of $\mathfrak{A}_0$, supported on it. The ones, supported on fp_s, $1 \leq s \leq k-1$ come from $(1-p_k)\mathfrak{B}_0(1-p_k)$ by the induction hypothesis. The one with unit fp_k comes from the representation $p_k\mathfrak{A}p_k \cong (p_k\mathfrak{B}p_k) * \mathbb{M}_{n_k}$ and Proposition D.4. For $1 \leq s \leq l$ we have

$$q_s\mathfrak{A}q_s \cong \underset{\frac{\gamma}{\beta_s}}{\overset{fq_s}{q_s\mathfrak{A}_0 q_s}} \oplus \bigoplus_{\substack{(i,s)\in L_+^{\mathfrak{A}} \\ 1\leq i\leq k-1}} \underset{\frac{\gamma_{is}}{\beta_s}}{\overset{f_{is}}{\mathbb{M}_{N_{\mathfrak{A}}(i,s)}}} \oplus \underset{\frac{\gamma_{ks}}{\beta_s}}{\overset{f_{ks}}{\mathbb{M}_{N_{\mathfrak{A}}(k,s)}}} \tag{3.11}$$

and

$$q_s\mathfrak{B}q_s \cong \underset{\frac{\delta}{\beta_s}}{\overset{gq_s}{q_s\mathfrak{B}_0q_s}} \oplus \bigoplus_{\substack{(i,s)\in L_+^{\mathfrak{B}}\\ 1\le i\le k-1}} \underset{\frac{\delta_{is}}{\beta_s}}{\overset{g_{is}}{\mathbb{M}_{N_{\mathfrak{B}}(i,s)}}} \oplus \underset{\frac{\delta_{ks}}{\beta_s}}{\overset{g_{ks}}{\mathbb{M}_{N_{\mathfrak{B}}(k,s)}}}. \tag{3.12}$$

From what we showed above follows that for $1 \le i \le k-1$ we have $\gamma_{is} = \delta_{is}$ and $f_{is} = g_{is}$. If $(k,s) \notin L_+^{\mathfrak{B}}$, (or $\alpha_k < 1 - \frac{\beta_s}{m_s^2}$), then $(k,s) \notin L_+^{\mathfrak{A}}$ and by (3.11) and (3.12) we see that $gq_s = fq_s$ and so in $\mathfrak{A}_0$ we have a unital, diffuse abelian C^*-subalgebra with unit $gq_s = fq_s$, which comes from $\mathfrak{B}_0$. If $(k,s) \in L_+^{\mathfrak{B}}$, then $gq_s \lneqq fq_s$ and since we have a unital, diffuse abelian C^*-subalgebra of $\mathfrak{A}_0$, supported on gq_s, comming from $\mathfrak{B}_0$, we need only to find a unital, diffuse abelian C^*-subalgebra of $\mathfrak{A}_0$, supported on $fq_s - gq_s$ and its direct sum with the one supported on gq_s will be a unital, diffuse abelian C^*-subalgebra of $\mathfrak{A}_0$, supported on fq_s. But from the form (3.11) and (3.12) it is clear that $fq_s - gq_s \le g_{ks}$, since from (3.11) and (3.12) $(f_{1s}+...+f_{(k-1)s})q_s\mathfrak{A}q_s(f_{1s}+...+f_{(k-1)s}) = (g_{1s}+...+g_{(k-1)s})q_s\mathfrak{B}q_s(g_{1s}+...+g_{(k-1)s})$. It is also clear then that $fq_s - gq_s = fg_{ks} \le p_k$, since $gq_s \perp g_{ks}$. We look for this C^*-subalgebra in

$$p_k\mathfrak{A}p_k = \underset{\frac{\gamma}{\alpha_k}}{\overset{fp_k}{p_k\mathfrak{A}_0p_k}} \oplus \bigoplus_{(k,j)\in L_+^{\mathfrak{A}}} \underset{\frac{\gamma_{kj}}{\alpha_k}}{\overset{f_{kj}}{\mathbb{M}_{N_{\mathfrak{A}}(k,j)}}} \cong (p_k\mathfrak{B}p_k) * \mathbb{M}_{n_k}$$

$$\cong \big(\underset{\frac{\delta}{\alpha_k}}{\overset{g}{p_k\mathfrak{B}_0p_k}} \oplus \bigoplus_{(k,j)\in L_+^{\mathfrak{B}}} \underset{\frac{\delta_{kj}}{\alpha_k}}{\overset{g_{kj}}{\mathbb{M}_{N_{\mathfrak{B}}(k,j)}}}\big) * \mathbb{M}_{n_k}.$$

Proposition D.4 gives us a unital, diffuse abelian C^*-subalgebra of $p_k\mathfrak{A}_0p_k$, supported on $(fp_k)g_{ks} = fg_{ks} = fq_s - gq_s$. This proves that we have a unital, diffuse abelian C^*-subalgebra of $\mathfrak{A}_0$, supported on fq_s.

Now we have to study the ideal structure of $\mathfrak{A}_0$, knowing by the induction hypothesis, the form of $\mathfrak{B}$. We will use the "span representation" of $\mathfrak{A}_0$ (3.10).

For each $(i,j) \in L_0^{\mathfrak{B}}$ we know the existance of $*$-homomorphisms $\pi_{(i,j)}^{\mathfrak{B}_0} : \mathfrak{B}_0 \to$

$\mathbb{M}_{N_{\mathfrak{B}}(i,j)}$. For $i \neq k$ we can write those as $\pi^{\mathfrak{B}_0}_{(i,j)} : \mathfrak{B}_0 \to \mathbb{M}_{N_{\mathfrak{A}}(i,j)}$ and since the support of $\pi^{\mathfrak{B}_0}_{(i,j)}$ is contained in $(1-p_k)$, using (3.10), we can extend linearly $\pi^{\mathfrak{B}_0}_{(i,j)}$ to $\pi^{\mathfrak{A}_0}_{(i,j)} : \mathfrak{A}_0 \to \mathbb{M}_{N_{\mathfrak{A}}(i,j)}$, by defining it to be zero on $p_k\mathfrak{A}_0p_k$, $(1-p_k)\mathfrak{B}_0p_k\mathfrak{A}_0p_k$, $p_k\mathfrak{A}_0p_k\mathfrak{B}_0(1-p_k)$, and $(1-p_k)\mathfrak{B}_0p_k\mathfrak{A}_0p_k\mathfrak{B}_0(1-p_k)$. Clearly $\pi^{\mathfrak{A}_0}_{(i,j)}$ is a $*$-homomorphism also.

By the induction hypothesis we know that gp_k is full in $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{B}}\\ i\neq k}} \ker(\pi^{\mathfrak{B}_0}_{(i,j)}) \subset \mathfrak{B}_0$ and by (3.10), and the way we extended $\pi^{\mathfrak{B}_0}_{(i,j)}$, we see that fp_k is full in $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq k}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)}) \subset \mathfrak{A}_0$. Then $p_k\mathfrak{A}_0p_k$ is full and hereditary in $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq k}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$, so by the Rieffel correspondence from [33], we have that $p_k\mathfrak{A}_0p_k$ and $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq k}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$ have the same ideal structure.

Above we saw that

$$\begin{aligned} p_k\mathfrak{A}p_k = \underset{\frac{\gamma}{\alpha_k}}{\overset{fp_k}{p_k\mathfrak{A}_0p_k}} \oplus \bigoplus_{(k,j)\in L_+^{\mathfrak{A}}} \underset{\frac{\gamma_{kj}}{\alpha_k}}{\overset{f_{kj}}{\mathbb{M}_{N_{\mathfrak{A}}(k,j)}}} \cong (p_k\mathfrak{B}p_k) * \mathbb{M}_{n_k} \cong \\ \cong (\underset{\frac{\delta}{\alpha_k}}{\overset{gp_k}{p_k\mathfrak{B}_0p_k}} \oplus \bigoplus_{(k,j)\in L_+^{\mathfrak{B}}} \underset{\frac{\delta_{kj}}{\alpha_k}}{\overset{g_{kj}}{\mathbb{M}_{N_{\mathfrak{B}}(k,j)}}}) * \mathbb{M}_{n_k}. \end{aligned} \tag{3.13}$$

From Proposition D.4 follows that $p_k\mathfrak{A}_0p_k$ is not simple if and only if $\exists 1 \le s \le m$, such that $(k,s) \in L_+^{\mathfrak{B}}, m_s = 1$ with $\frac{\delta_{ks}}{\alpha_k} = 1 - \frac{1}{n_k^2}$, where $\delta_{ks} = \alpha_k + \beta_s - 1$. This means that $\frac{\alpha_k+\beta_s-1}{\alpha_k} = 1 - \frac{1}{n_k^2}$, which is equivalent to $\frac{\beta_s}{1^2} + \frac{\alpha_k}{n_k^2} = 1$, so this implies $(k,s) \in L_0^{\mathfrak{A}}$. If this is the case (4.2), together with Proposition D.4 gives us a $*$-homomorphism $\pi'_{(k,s)} : p_k\mathfrak{A}_0p_k \to \mathbb{M}_{n_k}$, such that $\ker(\pi'_{(k,s)}) \subset p_k\mathfrak{A}_0p_k$ is simple with a unique trace. Using (3.10) we extend $\pi'_{(k,s)}$ linearly to a linear map $\pi^{\mathfrak{A}_0}_{(k,s)} : \mathfrak{A}_0 \to \mathbb{M}_{n_k}$, by defining $\pi^{\mathfrak{A}_0}_{(k,s)}$ to be zero on $(1-p_k)\mathfrak{B}_0p_k\mathfrak{A}_0p_k$, $p_k\mathfrak{A}_0p_k\mathfrak{B}_0(1-p_k)$, $(1-p_k)\mathfrak{B}_0p_k\mathfrak{A}_0p_k\mathfrak{B}_0(1-p_k)$, and $(1-p_k)\mathfrak{B}_0(1-p_k)$. Similarly as before, $\pi^{\mathfrak{A}_0}_{(k,s)}$ turns out to be a $*$-homomorphism. By the Rieffel correspondence of the ideals of $p_k\mathfrak{A}_0p_k$ and $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq k}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$, it is easy to see that the simple ideal $\ker(\pi'_{(k,s)}) \subset p_k\mathfrak{A}_0p_k$ corresponds to the ideal $\bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)}) \subset \bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq k}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$, so $\bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$ is simple. To see that $\bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$ has a unique trace we notice that from the construction of $\pi^{\mathfrak{A}_0}_{(i,j)}$ we have $\ker(\pi'_{(k,s)}) = p_k \ker(\pi^{\mathfrak{A}_0}_{(k,s)})p_k = p_k \bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})p_k$ (the last equality is true because $p_k\mathfrak{A}_0p_k \subset \bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq k}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$). Now we argue similarly as in the proof of Proposition D.3, using the fact that $\ker(\pi'_{(k,s)})$ has a unique trace: Suppose that ρ is a trace on $\bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$. It is easy to see that $\mathrm{Span}\{xp_kap_ky | x, y, a \in \bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)}), a \geq 0\}$ is dense in $\bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$, since $\ker(\pi'_{(k,s)})$ is full in $\bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$. Then since $p_kap_k \geq 0$ we have $\rho(xp_kap_ky) = \rho((p_kap_k)yx) = \rho((p_kap_k)^{1/2}yx(p_kap_k)^{1/2})$ and since $(p_kap_k)^{1/2}yx(p_kap_k)^{1/2}$ is supported on p_k, it follows that $(p_kap_k)^{1/2}yx(p_kap_k)^{1/2} \in p_k \bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})p_k = \ker(\pi'_{(k,s)})$, so ρ is uniquely determined by $\rho|_{\ker(\pi'_{(k,s)})}$ and hence $\bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$ has a unique trace.

If $\nexists 1 \leq s \leq m$ with $(k,s) \in L_0^{\mathfrak{A}}$ it follows from what we said above, that $p_k\mathfrak{A}_0p_k$ is simple with a unique trace. But since $p_k\mathfrak{A}_0p_k$ is full and hereditary in $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq k}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)}) = \bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$ it follows that $\bigcap_{(i,j)\in L_0^{\mathfrak{A}}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$ is simple with a unique trace in this case too.

We showed already that fp_k is full in $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq k}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$. Now let $1 \leq r \leq k-1$. We need to show that fp_r is full in $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq r}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$. From (3.11) and (3.12) follows that $f - g \leq p_k$. So $fp_r = gp_r$ for all $1 \leq r \leq k-1$. From the way we constructed $\pi^{\mathfrak{A}_0}_{(i,j)}$ is clear that $fp_r \in \bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}}\\ i\neq r}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$. It is also true that $fp_r \notin \ker(\pi^{\mathfrak{A}_0}_{(r,j)})$ for any

$1 \leq j \leq l$. So the smallest ideal of $\mathfrak{A}_0$, that contains fp_r, is $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}} \\ i\neq r}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$, meaning that we must have $\langle fp_r\rangle_{\mathfrak{A}_0} = \bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}} \\ i\neq r}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$.

Finally, we need to show that for all $1 \leq s \leq l$ we have that fq_s is full in $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}} \\ j\neq s}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$. Let $(i,j) \in L_0^{\mathfrak{A}}$ with $i \neq k$, $j \neq s$. Since $gq_s \in \ker(\pi^{\mathfrak{B}}_{(i,j)})$ and since $(f-g)q_s \leq p_k$, the way we extended $\pi^{\mathfrak{B}}_{(i,j)}$ to $\pi^{\mathfrak{A}}_{(i,j)}$ shows that $fq_s \in \ker(\pi^{\mathfrak{B}}_{(i,j)})$. Let $(i,s) \in L_0^{\mathfrak{A}}$ and $i \neq k$. Then we know that $gq_s \notin \ker(\pi^{\mathfrak{B}}_{(i,j)})$, which implies $fq_s \notin \ker(\pi^{\mathfrak{A}}_{(i,j)})$. Suppose $(k,s) \in L_0^{\mathfrak{A}}$. Then $m_s = 1$ and (4.2), Proposition D.4, and the way we extended $\pi'_{(k,s)}$ to $\pi^{\mathfrak{A}_0}_{(k,s)}$ show, that $fg_{ks} = fq_s - gq_s$ is full in $p_k\mathfrak{A}_0p_k$, meaning that $fq_s - gq_s$, and consequently fq_s, is not contained in $\ker(\pi^{\mathfrak{A}_0}_{(k,s)})$. Finally let $j \neq s$, and suppose $(k,j) \in L_0^{\mathfrak{A}}$. This means that $(k,j) \in L_+^{\mathfrak{B}}$ and also that the trace of q_j is so big, that $(i,s) \notin L_+^{\mathfrak{B}}$ and $(i,s) \notin L_0^{\mathfrak{B}}$ for any $1 \leq i \leq k$. Then (3.12) shows that $q_s \leq g$. The way we defined $\pi^{\mathfrak{A}_0}_{(k,j)}$ using (4.2) and Proposition D.4 shows us that $\mathfrak{B}_0 \subset \ker(\pi^{\mathfrak{A}_0}_{(k,j)})$ in this case. This shows $q_s = gq_s = fq_s \in \ker(\pi^{\mathfrak{A}_0}_{(k,j)})$. All this tells us that the smallest ideal of $\mathfrak{A}_0$, containing fq_s, is $\bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}} \\ j\neq s}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$, and therefore $\langle fq_s\rangle_{\mathfrak{A}_0} = \bigcap_{\substack{(i,j)\in L_0^{\mathfrak{A}} \\ j\neq s}} \ker(\pi^{\mathfrak{A}_0}_{(i,j)})$.

This concludes the proof of Theorem B.6.

□

CHAPTER IV

ON THE STRUCTURE OF SOME REDUCED AMALGAMATED FREE PRODUCT C^*-ALGEBRAS

A. Introduction

In this Chapter we give give a sufficient condition for simplicity and uniqueness of trace for reduced amalgamated free products of C^*-algebras. We also give a sufficient condition for the positive cone of $\mathbf{K_0}$ to be the largest possible.
We will use the notation from Chapter I.

For an index set I with $\text{card}(I) \geq 2$, let B be a unital C^*-algebra and suppose that for each $\iota \in I$ we have a unital C^*-algebra A_ι, which contains a copy of B as a unital C^*-subalgebra. Also suppose that for each $\iota \in I$ there is a conditional expectation $E_\iota : A_\iota \to B$, satisfying

$$\forall a \in A_\iota,\ a \neq 0,\ \exists x \in A_\iota,\ E_\iota(x^* a^* a x) \neq 0. \tag{4.1}$$

We denote the reduced amalgamated free product of (A_ι, E_ι) by

$$(A, E) = \underset{\iota \in I}{*} (A_\iota, E_\iota).$$

We will use the following notation which is similar to the notation in [13] used for the case of amalgamation over the scalars. If everything is as above by $\Lambda_B^\circ(\{A_\iota^\circ | \iota \in I\})$ we will denote the set of words of the form $a_1 a_2 \cdots a_n$, where $n \geq 1$ and $a_j \in A_{\iota_j}^\circ$ with $\iota_j \neq \iota_{j+1}$ for $1 \leq j \leq n-1$. We will not distinguish between two words from $\Lambda_B^\circ(\{A_\iota^\circ | \iota \in I\})$ which are equal as elements of A. We will denote $\Lambda_B(\{A_\iota^\circ | \iota \in I\}) \overset{def}{=} B \cup \Lambda_B^\circ(\{A_\iota^\circ | \iota \in I\})$. By $\mathbb{C}(A)$ we will denote the span of words from $\Lambda_B(\{A_\iota^\circ | \iota \in I\})$. Notice that $\mathbb{C}(A)$ is norm-dense in A. For a word $a_1 a_2 \cdots a_n \in \Lambda_B^\circ(\{A_\iota^\circ | \iota \in I\})$, where

$n \geq 1$, $a_j \in A^\circ_{\iota_j}$ with $\iota_j \neq \iota_{j+1}$ for $1 \leq j \leq n-1$ we will consider to be of length n. Elements of B we will consider to be of length 0.

We will be mainly interested in the case $\text{card}(I) = 2$ and that there exist states ϕ_ι on A_ι for $\iota = 1, 2$, such that these states are invariant under E_ι, i.e. for $\iota = 1, 2$ and $\forall a_\iota \in A_\iota$ we have $\phi_\iota(a_\iota) = \phi_\iota(E_\iota(a_\iota))$. We also require $\phi_1(b) = \phi_2(b)$ for $b \in B$. $\phi \stackrel{def}{=} \phi_B \circ E$, where $\phi_B \stackrel{def}{=} \phi_1|_B = \phi_2|_B$ is a well defined E-invariant state on $(A, E) = (A_1, E_1) * (A_2, E_2)$. In such case we will write formally

$$(A, E, \phi) = (A_1, E_1, \phi_1) * (A_2, E_2, \phi_2),$$

although the construction of (A, E) does not depend on $\phi_\iota, \iota = 1, 2$.

Remark A.1. *Using the same techniques as in [11] it can be shown that if ϕ_1 and ϕ_2 are faithful traces then ϕ is also a faithful trace.*

Define

$$\Lambda^1_B \stackrel{def}{=} \text{Span}(\bigcup_{k=0}^{\infty} A^\circ_1(A^\circ_2 A^\circ_1)^k) \subset \mathbb{C}(A) \tag{4.2}$$

and

$$\Lambda^2_B \stackrel{def}{=} \text{Span}(\bigcup_{k=0}^{\infty} A^\circ_2(A^\circ_1 A^\circ_2)^k) \subset \mathbb{C}(A). \tag{4.3}$$

Define also

$$\Lambda^{21}_B \stackrel{def}{=} \text{Span}(\bigcup_{k=1}^{\infty} (A^\circ_2 A^\circ_1)^k) \subset \mathbb{C}(A)$$

and

$$\Lambda^{12}_B \stackrel{def}{=} \text{Span}(\bigcup_{k=1}^{\infty} (A^\circ_1 A^\circ_2)^k) \subset \mathbb{C}(A).$$

Some of the most important examples are those of reduced C^*-algebras of amalgams of discrete groups. For each discrete group N we have the canonical tracial state $\tau_N \stackrel{def}{=} \langle \cdot, \widehat{1_H} \rangle_{l^2(H)}$ on $C^*_r(N)$. For each subgroup S of N we have a canonical conditional expectation $E_S^N : C_r^*(N) \to C_r^*(S)$ given on elements $\{\lambda_n, n \in N\}$ by

$$E_S^N(\lambda_n) = \begin{cases} \lambda_n, & \text{if } n \in S, \\ 0, & \text{if } n \notin S. \end{cases}$$

Let $G_1 \supset H \subset G_2$ be two discrete groups, containing a common subgroup (an isomorphic copy of H). Then we have $(C_r^*(G), E_H^G) = (C_r^*(G_1), E_H^{G_1}) * (C_r^*(G_2), E_H^{G_2})$, where $G = G_1 \underset{H}{*} G_2$.

The canonical tracial states $\tau_{G_\iota}, \iota = 1, 2$ and τ_G are invariant under $E_H^{G_\iota}, \iota = 1, 2$ and E_H^G respectivelly and $\tau_G = \tau_H \circ E_H^G$. Thus we can write formally

$$(C_r^*(G), E_H^G, \tau_G) = (C_r^*(G_1), E_H^{G_1}, \tau_{G_1}) * (C_r^*(G_2), E_H^{G_2}, \tau_{G_2}).$$

B. $\mathbf{K_0}^+$

We give the results of Germain and Pimsner first.

Theorem B.1 ([17])**.** *Let $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ is the reduced free product (with amalgamation over $\mathbb{C}$) of the unital, nuclear C^*-algebras A_1 and A_2 with respect to states ϕ_1 and ϕ_2. Then we have the following six term exact sequence:*

$$\begin{array}{ccccc} \mathbb{Z} \cong \mathbf{K_0}(\mathbb{C}) & \xrightarrow{(\mathbf{K_0}(i_1), -\mathbf{K_0}(i_2))} & \mathbf{K_0}(A_1) \oplus \mathbf{K_0}(A_2) & \xrightarrow{\mathbf{K_0}(j_1)+\mathbf{K_0}(j_2)} & \mathbf{K_0}(A) \\ \uparrow & & & & \downarrow \\ \mathbf{K_1}(A) & \xleftarrow{\mathbf{K_1}(j_1)+\mathbf{K_1}(j_2)} & \mathbf{K_1}(A_1) \oplus \mathbf{K_1}(A_2) & \xleftarrow{(\mathbf{K_1}(i_1), -\mathbf{K_1}(i_2))} & \mathbf{K_1}(\mathbb{C}) \cong 0, \end{array}$$

where $i_k : \mathbb{C} \to A_k$ are the the unital $*$-homorphisms and $j_k : A_k \to A$ are the unital embeddings arising from the construction of reduced free product ($k = 1, 2$).

Theorem B.2 ([30])**.** *Suppose that $G_1 \supset H \subset G_2$ are countable, discrete groups. Let*

$G = G_1 \underset{H}{*} G_2$. *Then we have the following six term exact sequence:*

$$\begin{array}{ccccc} \mathbf{K_0}(C_r^*(H)) & \xrightarrow{(\mathbf{K_0}(i_1),-\mathbf{K_0}(i_2))} & \mathbf{K_0}(C_r^*(G_1)) \oplus \mathbf{K_0}(C_r^*(G_2)) & \xrightarrow{\mathbf{K_0}(j_1)+\mathbf{K_0}(j_2)} & \mathbf{K_0}(C_r^*(G)) \\ \uparrow & & & & \downarrow \\ \mathbf{K_1}(C_r^*(G)) & \xleftarrow{\mathbf{K_1}(j_1)+\mathbf{K_1}(j_2)} & \mathbf{K_1}(C_r^*(G_1)) \oplus \mathbf{K_1}(C_r^*(G_2)) & \xleftarrow{(\mathbf{K_1}(i_1),-\mathbf{K_1}(i_2))} & \mathbf{K_1}(C_r^*(H)), \end{array}$$

where $i_k : C_r^*(H) \to C_r^*(G_k)$ *and* $j_k : C_r^*(G_k) \to C_r^*(G)$ *are the canonical inclusion maps* ($k = 1, 2$).

Now suppose that we have unital C^*-algebaras A_ι, $\iota = 1, 2$ and B. Suppose that we have unital inclusions $B \hookrightarrow A_\iota$ and conditional expectations $E_\iota : A_\iota \to B$ that satisfy property (4.1). Suppose also that for $\iota = 1, 2$ we have tracial states τ_ι on A_ι which satisfy $\tau_B \overset{def}{=} \tau_1|_B = \tau_2|_B$ and which are invariant under E_ι, i.e $\tau_\iota(a_\iota) = \tau_\iota(E_\iota(a_\iota))$ for each $a_\iota \in A_\iota$. Let us denote $(A, E, \tau) \overset{def}{=} (A_1, E_1, \tau_1) * (A_2, E_2, \tau_2)$ and let $j_\iota : A_\iota \to A$ are the inclusion maps, coming from the construction of reduced amalgamated free products. Suppose that $\tau \overset{def}{=} \tau_B \circ E$ is a faithful tracial state. Let's define

$$\Gamma \overset{def}{=} \mathbf{K_0}(j_1)(\mathbf{K_0}(A_1)) + \mathbf{K_0}(j_2)(\mathbf{K_0}(A_2)) \subset \mathbf{K_0}(A).$$

Then every element in Γ can be represented as

$$([p_1]_{\mathbf{K_0}(A)} - [q_1]_{\mathbf{K_0}(A)}) + ([p_2]_{\mathbf{K_0}(A)} - [q_2]_{\mathbf{K_0}(A)}),$$

where p_ι, q_ι are projections in some matrix algebras over A_ι for $\iota = 1, 2$. By expanding those matrices and adding zeros we can suppose without loss of generality that p_ι, q_ι are projections from $M_n(A_\iota)$ for some $n \in \mathbb{N}$ for $\iota = 1, 2$. Therefore every element of Γ can be represented in the form

$$\left[\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\right]_{\mathbf{K_0}(A)} - \left[\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}\right]_{\mathbf{K_0}(A)}, \qquad (4.4)$$

where now

$$\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \text{ and } \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \in M_{2n}(A).$$

We want to obtain a sufficient condition so that all elements $\gamma \in \Gamma$ for which $\mathbf{K_0}(\tau)(\gamma) > 0$ come from projections, i.e. $\exists m \in \mathbb{N}$ and $\rho \in M_m(A)$, such that $\gamma = [\rho]_{\mathbf{K_0}(A)}$ in $\mathbf{K_0}(A)$.

By definition the positive cone of $\mathbf{K_0}(A)$ is

$$\mathbf{K_0}(A)^+ = \{x \in \mathbf{K_0}(A) | \exists p \text{ projection in } M_n(A) \text{ for some } n \text{ with } x = [p]_{\mathbf{K_0}(A)}\}.$$

The scale of $\mathbf{K_0}(A)$ is

$$\Sigma(A) = \{x \in \mathbf{K_0}(A) | \exists p \text{ projection in } A \text{ with } x = [p]_{\mathbf{K_0}(A)}\}.$$

Dykema and Rørdam proved the following:

Theorem B.3 ([16]). *Let $(A, \tau) = (A_1, \tau_1) * (A_2, \tau_2)$ be the reduced free product of the unital C^*-algebras A_1 and A_2 with respect to the faithful tracial states τ_1 and τ_2. Suppose that the Avitzour condition holds, namely there exist unitaries $u_1 \in A_1$ and $u_2, u_2' \in A_2$, such that $\tau_1(u_1) = \tau_2(u_2) = \tau_2(u_2') = \tau_2(u_1^* u_1') = 0$. Then we have*

$$\Gamma \cap \mathbf{K_0}(A)^+ = \{\gamma \in \Gamma | \mathbf{K_0}(\tau)(\gamma) > 0\} \cup \{0\}$$

and

$$\Gamma \cap \Sigma(A) = \{\gamma \in \Gamma | 0 < \mathbf{K_0}(\tau)(\gamma) < 1\} \cup \{0, 1\}.$$

Notice that Theorem B.1 implies that if A_1 and A_2 are nuclear then $\Gamma = \mathbf{K_0}(A)$.

Anderson, Blackadar and Haagerup proved this theorem for the case of $A = C_r^*(\mathbb{Z}_n * \mathbb{Z}_m)$ and gave one of the main technical tool for proving Theorem B.3, which we will use here also:

Proposition B.4 ([2]). *Let $\mathfrak{A}$ be a unital C^*-algebra and let ϕ be a faithful state on $\mathfrak{A}$. Suppose that $p, q \in \mathfrak{A}$ are projections that are ϕ-free in $\mathfrak{A}$. If $\phi(p) < \phi(q)$ then $\|p(1-q)\| < 1$ and there is a partial isometry $\nu \in \mathfrak{A}$ such that $\nu\nu^* = p$ and $\nu^*\nu < q$.*

Now we can state and prove our result:

Theorem B.5. *Let A_ι be unital C^*-algebras that contain the unital C^*-algebra B as a unital C^*-subalgebra, i.e. $1_{A_\iota} \in B \subset A_\iota$, $\iota = 1, 2$. Suppose that we have conditional expectations $E_\iota : A_\iota \to B$ and tracial states τ_ι on A_ι for $\iota = 1, 2$ such that $\tau_\iota = \tau_\iota \circ E_\iota$ and $\tau_1|_B = \tau_2|_B$. Form the reduced amalgamated free product $(A, E, \tau) = (A_1, E_1, \tau_1) * (A_2, E_2, \tau_2)$. Suppose that τ_1 and τ_2 are faithful tracial states. Suppose that the following two conditions hold:*

$$\begin{cases} \forall b_1, \ldots, b_l \in B, \text{ with } \tau(b_1) = \cdots = \tau(b_l) = 0, \ \exists m \in \mathbb{N} \text{ and unitaries} \\ \nu_{11}, \ldots, \nu_{1m}, \nu_{21}, \ldots, \nu_{2m} \text{ such that } \nu_{12}, \ldots, \nu_{1m} \in A_1^\circ, \ \nu_{21}, \ldots, \nu_{2(m-1)} \in A_2^\circ, \text{ and:} \\ \ \text{either } \nu_{11} \in A_1^\circ, \ \nu_{2m} \in A_2^\circ \text{ or} \\ \nu_{11} = 1_{A_1}, \ \nu_{2m} \in A_2^\circ, \ \text{or} \\ \nu_{11} \in A_1^\circ, \ \nu_{2m} = 1_{A_2}, \\ \nu_{11} = 1_{A_1}, \ \nu_{2m} = 1_{A_2}, \ k \geq 2 \\ \ \text{with } E((\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m})b_k(\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m})^*) = 0 \text{ for } k = 1, \ldots, l, \\ (\text{i.e. there are unitaries that conjugate } B \ominus \mathbb{C}1_B \text{ out of } B) \end{cases} \tag{4.5}$$

and

$$\exists \text{ unitaries } u_1 \in A_1^\circ, \ u_2, u_2' \in A_2^\circ, \text{ with } E_2(u_2 u_2'^*) = 0. \tag{4.6}$$

Then:

$$\Gamma \cap \mathbf{K_0}(A)^+ = \{\gamma \in \Gamma | \mathbf{K_0}(\tau)(\gamma) > 0\} \cup \{0\}. \tag{4.7}$$

Proof. All elements of Γ have the form (4.4) for some $n \in \mathbb{N}$ and projections p_1, q_1 from $M_n(A_1)$ and p_2, q_2 from $M_n(A_2)$. Denote

$$\gamma = \left[\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\right]_{\mathbf{K_0}(A)} - \left[\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}\right]_{\mathbf{K_0}(A)}.$$

Consider

$$P \stackrel{def}{=} \begin{pmatrix} U_2 & 0 \\ 0 & U_2U_1 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} U_2^* & 0 \\ 0 & U_1^*U_2^* \end{pmatrix} \text{ and}$$

$$Q \stackrel{def}{=} \begin{pmatrix} U_2 & 0 \\ 0 & U_2U_1 \end{pmatrix} \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} U_2^* & 0 \\ 0 & U_1^*U_2^* \end{pmatrix},$$

where $U_1 = \mathrm{diag}(u_1, \ldots, u_1) \in M_n(A_1)$ and $U_2 = \mathrm{diag}(u_2, \ldots, u_2) \in M_n(A_2)$. It is clear that $P, Q \in M_{2n}(\Lambda_B^2 \oplus B1_B)$. For $T \in M_m(A)$ we will denote by T_{ij} the ij-entry of T. Now consider the set of elements $S_P = \{E(P_{ij}) - \tau(P_{ij}) | 1 \le i, j \le 2n\} \cup \{E(u_1P_{ij}u_1^*) - \tau(u_1P_{ij}u_1^*) | 1 \le i, j \le 2n\}$ and the set $S_Q = \{E(Q_{ij}) - \tau(Q_{ij}) | 1 \le i, j \le 2n\} \cup \{E(u_1Q_{ij}u_1^*) - \tau(u_1Q_{ij}u_1^*) | 1 \le i, j \le 2n\}$.

Applying condition (4.5) to the set S_P we obtain unitaries $\nu_{ij}, i = 1, 2, j = 1, \ldots, m_P$.

Set

$$W_P \stackrel{def}{=} \begin{cases} \nu_{11}\nu_{21}\nu_{12}\cdots\nu_{2(m_P-1)}\nu_{1m_P}, & \text{if } \nu_{2m_P} = 1_{A_2},\ \nu_{11} \in A_1^\circ, \\ \nu_{11}\nu_{21}\nu_{12}\cdots\nu_{2(m_P-1)}\nu_{1m_P}\nu_{2m_P}u_1, & \text{if } \nu_{2m_P} \in A_2^\circ,\ \nu_{11} \in A_1^\circ, \\ u_1\nu_{21}\nu_{12}\cdots\nu_{2(m_P-1)}\nu_{1m_P}\nu_{2m_P}u_1, & \text{if } \nu_{2m_P} \in A_2^\circ,\ \nu_{11} = 1_{A_1}. \\ u_1\nu_{21}\nu_{12}\cdots\nu_{2(m_P-1)}\nu_{1m_P}, & \text{if } \nu_{2m_P} = 1_{A_2},\ \nu_{11} = 1_{A_1},\ k \ge 2. \end{cases}$$

Applying condition (4.5) to the set S_Q we obtain unitaries $\nu'_{ij}, i = 1, 2, j = 1, \ldots, m_Q$.

Set

$$W_Q \overset{def}{=} \begin{cases} \nu'_{11}\nu'_{21}\nu'_{12}\cdots\nu'_{2(m_P-1)}\nu'_{1m_P}, & \text{if } \nu'_{2m_P}=1_{A_2},\ \nu'_{11}\in A_1^\circ, \\ \nu'_{11}\nu'_{21}\nu'_{12}\cdots\nu'_{2(m_P-1)}\nu'_{1m_P}\nu'_{2m_P}u_1, & \text{if } \nu'_{2m_P}\in A_2^\circ,\ \nu'_{11}\in A_1^\circ, \\ u_1\nu'_{21}\nu'_{12}\cdots\nu'_{2(m_P-1)}\nu'_{1m_P}\nu'_{2m_P}u_1, & \text{if } \nu'_{2m_P}\in A_2^\circ,\ \nu'_{11}=1_{A_1}. \\ u_1\nu'_{21}\nu'_{12}\cdots\nu'_{2(m_P-1)}\nu'_{1m_P}, & \text{if } \nu'_{2m_P}=1_{A_2},\ \nu'_{11}=1_{A_1},\ k\geq 2. \end{cases}$$

It is easy to see that $W_PPW_P^*,\ W_QQW_Q^*\in M_{2n}(\Lambda_B^1\oplus\mathbb{C}1_B)$.

Now consider the following matrix in $M_{2n}(A)$:

$$U=(\frac{\omega^{ij}}{\sqrt{2n}}u'_2(u_1u_2)^{2ni+j}u'^*_2)_{i,j=1}^{2n},$$

where $\omega=\exp(2\pi\sqrt{-1}/2n)$ is a primitive $2n$-th root of 1. It is clear that $U\in M_{2n}(\Lambda_B^2)$. We will check that U is a unitary matrix:

$$(UU^*)_{ij}=(2n)^{-1}\sum_{k=1}^{2n}\omega^{ik}u'_2(u_1u_2)^{2ni+k}\omega^{-jk}(u_1u_2)^{-2nj-k}u'^*_2=$$

$$(2n)^{-1}\sum_{k=1}^{2n}\omega^{(i-j)k}u'_2(u_1u_2)^{2n(i-j)}u'^*_2=(2n)^{-1}u'_2(u_1u_2)^{2n(i-j)}u'^*_2\sum_{k=1}^{2n}\omega^{(i-j)k}=\delta_{ij}1_A.$$

$$(U^*U)_{ij}=(2n)^{-1}\sum_{k=1}^{2n}\omega^{-ik}u'_2(u_1u_2)^{-2nk-i}\omega^{jk}(u_1u_2)^{2nk+j}u'^*_2=$$

$$(2n)^{-1}\sum_{k=1}^{2n}\omega^{(j-i)k}u'_2(u_1u_2)^{j-i}u'^*_2=(2n)^{-1}u'_2(u_1u_2)^{j-i}u'^*_2\sum_{k=1}^{2n}\omega^{(j-i)k}=\delta_{ij}1_A.$$

Thus $U\in M_{2n}(A)$ is a unitary.

Take $T\in M_{2n}(\Lambda_B^1\oplus\mathbb{C}1_B)$. Then $T=T_0+T_1\otimes 1_A$, with $T_0\in M_{2n}(\Lambda_B^1)$ and $T_1\in M_{2n}(\mathbb{C})$. It is easy to see that $UT_0U^*\in M_{2n}(\Lambda_B^2)$. Now if $T_1=(t_{ij})_{i,j=1}^{2n}$ then for $U(T_1\otimes 1_A)U^*=(s_{ij})_{i,j=1}^{2n}$ we have

$$s_{ij}=(2n)^{-1}\sum_{k=1}^{2n}\sum_{l=1}^{2n}\omega^{ik}u'_2(u_1u_2)^{2ni+k}u'^*_2t_{kl}\omega^{-jl}u'_2(u_1u_2)^{-2nj-l}u'^*_2=$$

$$(2n)^{-1}\sum_{k=1}^{2n}\sum_{l=1}^{2n}t_{kl}\omega^{ik-jl}u_2'(u_1u_2)^{2ni+k-2nj-l}u_2'^*.$$

If $i \neq j$ then $2ni+k-2nj-l \neq 0$ for any $1 \leq k,l \leq 2n$, so in this case $s_{ij} \in \Lambda_B^2$. If $i = j$ then:

$$s_{ii} = (2n)^{-1}\sum_{k=1}^{2n}\sum_{l=1}^{2n}t_{kl}\omega^{i(k-l)}u_2'(u_1u_2)^{k-l}u_2'^* =$$

$$(2n)^{-1}\sum_{\substack{1\leq k,l\leq 2n\\ k\neq l}}t_{kl}\omega^{i(k-l)}u_2'(u_1u_2)^{k-l}u_2'^* + ((2n)^{-1}\sum_{k=1}^{2n}t_{kk})\otimes 1_A.$$

So $s_{ii} = s_{ii}' + \mathrm{tr}_{2n}(T_1)\otimes 1_A$, where $s_{ii}' \in \Lambda_B^2$. All this means that $U(T_1\otimes 1_A)U^* = T_1' + \mathrm{tr}_{2n}(T_1)1_A \otimes 1_{M_{2n}(\mathbb{C})}$, with $T_1' \in M_{2n}(\Lambda_B^2)$, which implies that $UTU^* \in M_{2n}(\Lambda_B^2) \oplus \mathbb{C}1_{M_{2n}(A)}$.

This means that we have

$$P' \overset{def}{=} UW_PPW_P^*U^* \in M_{2n}(\Lambda_B^2) \oplus \mathbb{C}1_{M_{2n}(A)} \tag{4.8}$$

and

$$Q' \overset{def}{=} u_1UW_QQW_Q^*U^*u_1^* \in M_{2n}(\Lambda_B^1) \oplus \mathbb{C}1_{M_{2n}(A)}. \tag{4.9}$$

It is clear that $\mathrm{tr}_{2n}\otimes E(P') = \mathrm{tr}_{2n}\otimes\tau(P')$ and that $\mathrm{tr}_{2n}\otimes E(Q') = \mathrm{tr}_{2n}\otimes\tau(Q')$. Since P' and Q' are nontrivial projections it is also clear that $C^*(\{P', 1_A\})$ and $C^*(\{Q', 1_A\})$ are both 2-dimensional. Therefore for any $p \in C^*(\{P', 1_A\})$ and $q \in C^*(\{Q', 1_A\})$ we have $\mathrm{tr}_{2n}\otimes E(p) = \mathrm{tr}_{2n}\otimes\tau(p)$ and $\mathrm{tr}_{2n}\otimes E(q) = \mathrm{tr}_{2n}\otimes\tau(q)$. Therefore from (4.8), (4.9) and the definition of freeness it follows that P' is both $\mathrm{tr}_{2n}\otimes E$-free and $\mathrm{tr}_{2n}\otimes\tau$-free from Q'.

Since $\mathrm{tr}_{2n}\otimes\tau$ is a faithful tracial state (because of faithfulness of τ_1, τ_2 and Remark A.1) and because

$$\mathrm{tr}_{2n}\otimes\tau(P') = (2n)^{-1}\mathbf{K_0}(\tau)(P) > (2n)^{-1}\mathbf{K_0}(\tau)(Q) = \mathrm{tr}_{2n}\otimes\tau(Q'),$$

we can apply Proposition B.4 and conclude that there is a projection $Q'' < P'$ and a partial isometry ν with $\nu\nu^* = Q'$ and $\nu^*\nu = Q''$. Thus $\gamma = [P' - Q'']_{\mathbf{K_0}(A)}$ in $\mathbf{K_0}(A)$. This proves the theorem. □

Corollary B.6. *Suppose that* $G_1 \supsetneq H \subsetneq G_2$ *are countable discrete groups with* $H \neq \{1\}$. *Suppose that* $\exists \gamma \in G \stackrel{def}{=} G_1 \underset{H}{*} G_2$ *with* $\gamma(H\backslash\{1\})\gamma^{-1} \cap H = \emptyset$. *Suppose also that* $\mathbf{K_1}(C_r^*(H)) = 0$. *Then*

$$\mathbf{K_0}(C_r^*(G))^+ = \{\gamma \in \mathbf{K_0}(C_r^*(G)) | \mathbf{K_0}(\tau_G)(\gamma) > 0\} \cup \{0\}.$$

Proof. Because of the existence of γ we see that condition (4.5) of Theorem B.5 is satisfied. The existence of γ implies also that H is not normal in at least one of the groups G_1 or G_2. Suppose without loss of generality that H is not normal in G_2. Then Index$[G_1 : H] \geq 2$ and Index$[G_2 : H] \geq 3$ so we can find $g_1 \in G_1\backslash H$ and $g_2, g_2' \in G_2\backslash H$ with $g_2 g_2'^{-1} \in G_2\backslash H$. Then condition (4.6) is satisfied with elements $u_1 = \lambda_{g_1}$, $u_2 = \lambda_{g_2}$ and $u_2' = \lambda_{g_2'}$ and therefore we can apply Theorem B.5. From the fact that $\mathbf{K_1}(C_r^*(H)) = 0$ and Theorem B.2 it follows that $\Gamma = \mathbf{K_0}(C_r^*(G))$. This proves the corollary. □

Remark B.7. *Condition (4.6) is an analogue of the Avitzour condition for the case of reduced amalgamated free products. We will use it in the next section to prove simplicity and uniqueness of trace.*

C. Simplicity and Uniqueness of Trace

In this section we will use Power's idea ([31]) to obtain a sufficient condition for simplicity and uniqueness of trace for reduced amalgamated free product C^*-algebras. We will make use the following result (due to Avitzour) and its proof:

Theorem C.1 ([3]). *Let A_1 and A_2 be two unital C^*-algebras and ϕ_1 respectivelly ϕ_2 states on them with faithfil GNS-representations. Suppose that there are unitaries $u_i \in A_i$, $i = 1, 2$ such that ϕ_1 and ϕ_2 are invariant with respect to conjugation by u_1 and u_2 respectivelly and such that $\phi_i(u_i) = 0$ for $i = 1, 2$. Suppose also that there is a unitary $u_2' \in A_2$, such that $\phi_2(u_2') = 0$ and $\phi_2(u_2^* u_2') = 0$. Then:*

(I) $(A, \phi) \overset{def}{=} (A_1, \phi_1) * (A_2, \phi_2)$ is simple.

(II) If ϕ is invariant with respect to conjugation by u_2' then ϕ is the only state on A which is invariant with respect to conjugation by u_1, u_2, u_2'. If ϕ is not invariant with respect to conjugation by u_2' then there is no state on A which is invariant with respect to conjugation by u_1, u_2, u_2'.

The proof of Theorem C.1 uses a lemma of Choi from [6]. We will need the following straightforward generalization of this lemma to the case of Hilbert modules:

Lemma C.2. *Let H_1 and H_2 be right Hilbert B-modules. Let $u_1, \ldots, u_n \in \mathcal{L}(H_1 \oplus H_2)$ be unitaries such that $u_i^* u_j(H_2) \perp H_2$, whenever $i \neq j$. Suppost that $b \in \mathcal{L}(H_1 \oplus H_2)$ is such that $b(H_1) \perp H_1$. Then $\|\frac{1}{n}\sum_{k=1}^{n} u_i^* b u_i\| \leq 2\|b\|/\sqrt{n}$.*

Proof. First assume that

$$b = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} \in \mathcal{L}(H_1 \oplus H_2).$$

If

$$c = \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(H_1 \oplus H_2)$$

then for $x \oplus y \in H_1 \oplus H_2$ we have

$$\begin{bmatrix} c_1 & c_2 \\ b_1 & b_2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 x + c_2 y \\ b_1 x + b_2 y \end{pmatrix}.$$

Then:

$$\left\| \begin{pmatrix} c_1x + c_2y \\ b_1x + b_2y \end{pmatrix} \right\|_B^2 = \|\langle (c_1x + c_2y) \oplus (b_1x + b_2y), (c_1x + c_2y) \oplus (b_1x + b_2y) \rangle_{H_1 \oplus H_2}\|_B =$$

$$= \|\langle c_1x + c_2y, c_1x + c_2y \rangle_{H_1} + \langle b_1x + b_2y, b_1x + b_2y \rangle_{H_2}\|_B \leq$$

$$\|\langle c_1x + c_2y, c_1x + c_2y \rangle_{H_1}\|_B + \|\langle b_1x + b_2y, b_1x + b_2y \rangle_{H_2}\|_B = \|c_1x + c_2y\|_B^2 + \|b_1x + b_2y\|_B^2$$

$$= \left\| \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_B^2 + \left\| \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_B^2 .$$

Taking supremum on both sides over all vectors $x \oplus y$ in the unit ball of $H_1 \oplus H_2$ we get

$$\left\| \begin{bmatrix} c_1 & c_2 \\ b_1 & b_2 \end{bmatrix} \right\|^2 = \|c + b\|^2 \leq \|c\|^2 + \|b\|^2.$$

Now $u_j^* u_i b u_i^* u_j(H_2) \subseteq u_j u_i^* b(H_1) = 0$. So $u_j^* u_i b u_i^* u_j$ has the form $\begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix}$. Now $\|\sum\limits_{i=1}^n u_i b u_i^*\|^2 = \|u_1^*(\sum\limits_{i=1}^n u_i b u_i^*)u_1\|^2 = \|b + \sum\limits_{i=2}^n u_1^* u_i b u_i^* u_1\| \leq \|b\|^2 + \|\sum\limits_{i=2}^n u_1^* u_i b u_i^* u_1\|^2 = \|b\|^2 + \|\sum\limits_{i=2}^n u_i b u_i^*\|^2$. It follows by induction that $\|\sum\limits_{i=1}^n u_i b u_i^*\|^2 \leq n\|b\|^2$. For the general case we represent

$$b = \begin{bmatrix} 0 & b_3 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_3^* & 0 \end{bmatrix}^* .$$

Then

$$\|\sum_{i=1}^n u_i b u_i^*\| \leq \|\sum_{i=1}^n u_i \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} u_i^*\| + \|\sum_{i=1}^n u_i \begin{bmatrix} 0 & 0 \\ b_3^* & 0 \end{bmatrix} u_i^*\| \leq$$

$$\sqrt{n}\left\|\begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix}\right\| + \sqrt{n}\left\|\begin{bmatrix} 0 & 0 \\ b_3^* & 0 \end{bmatrix}\right\| \leq 2\sqrt{n}\|b\|.$$

□

Untill the end of the section we will assume that we have unital C^*-algebras A_1, A_2 that contain the unital C^*-algebra B as a unital C^*-subalgebra. We will also assume that we have condiditonal expectations $E_i : A_i \to B$ for $i = 1, 2$ that have faithful KSGNS-representations (i.e. satisfy condition (4.1)). We now form the reduced amalgamated free product $(A, E) \stackrel{def}{=} (A_1, E_1) * (A_2, E_2)$.

Now we can imitate Avitzour's proof of Theorem C.1 and prove the following version for the case of amalgamation:

Proposition C.3. *Suppose everything is as above and also suppose that there are unitaries $u_1 \in A_1$, $u_2, u_2' \in A_2$ with $E_1(u_1) = 0 = E_2(u_2) = E_2(u_2') = E(u_2 u_2'^*)$. Then if $x \in \Lambda_B^1$ then $0 \in \overline{\text{conv}}\{uxu^* | u \in A \text{ is a unitary }\}$.*

Proof. We will use the notation from section A with $I = \{1, 2\}$. Let $W_0 \subset \mathbb{C}(A)$ be the span of all words from $\Lambda_B(A_1^\circ, A_2^\circ)$ that either begin with an element $a_1 \in A_1^\circ$ or begin with $u_2^* b$ with $b \in B$, or come from B. Let $W_1 \subset \mathbb{C}(A)$ be the span of all words from $\Lambda_B(A_1^\circ, A_2^\circ)$ that begin with an element $a_2 \in A_2^\circ$ satisfying $E_2(u_2 a_2) = 0$. Denote

$$H_i \stackrel{def}{=} \overline{\pi(W_i)\widehat{1_A}} \subset M, \; i = 0, 1$$

We have $M = H_0 \oplus H_1$ (the orthogonality is with respect to $\langle ., . \rangle_M$). To show this notice first that $\text{Span}(W_0 \cup W_1)$ is dense in A. Therefore $M = H_0 + H_1$. For every word $w_0 \in W_0$ and every word $w_1 \in W_1$we have $E(w_0^* w_1) = 0$ which is easy to see by considering the three possible cases for w_0. Thus $H_0 \perp H_1$ by linearity.

We claim that $(u_2^* u_1)^k (H_1) \subseteq H_0$ for $k \neq 0$.

It is enough to prove that $(u_2^* u_1)^k W_1 \subseteq W_0$.

If $k > 0$ then $(u_2^* u_1)^k W_1$ is spanned by words from $\Lambda_B^\circ(A_1^\circ, A_2^\circ)$ starting with u_2^*. If $k < 0$ then take any word $w_1 \in W_1$. Then $w_1 = a_2 w_1'$,where $a_2 \in A_2^\circ$ satisfies $E(u_2 a_2) = 0$ and $w_1' \in \Lambda_B^\circ(A_1^\circ, A_2^\circ)$ starts with an element of A_1°. Then

$$(u_2^* u_1)^k w_1 = (u_1^* u_2)^{-k} a_2 w_1' = (u_1^* u_2)^{-k-1} u_1^* (u_2 a_2) w_1'$$

is a word, starting with $u_1^* \in A_1^\circ$. Thus $(u_2^* u_1)^k W_1 \subseteq W_0$.

Now $u_2'^* x u_2' \in \Lambda_B^2$ and also it is clear that $(u_2'^* x u_2')(W_0) \subseteq W_1$ by considering the three possibilities for W_0 (notice that $E(u_2' u_2^* b) = 0 \; \forall b \in B$). Now we can use Lemma C.2 and get

$$\|\frac{1}{N}\sum_{k=1}^{N}(u_2'^* u_1)^k (u_2'^* x u_2')(u_2'^* u_1)^{-k}\| \leq \frac{2\|x\|}{\sqrt{N}}.$$

This implies that $0 \in \overline{\text{conv}}\{uxu^* | u \in A \text{ is a unitary }\}$. □

We will prove the next technical lemma:

Lemma C.4. *Suppose that everything is as above and suppose that there are states ϕ_i on A_i for $i = 1, 2$ which are invariant with respect to E_i, $i = 1, 2$ and satisfy $\phi_1|_B = \phi_2|_B (\stackrel{def}{=} \phi_B)$, and construct $\phi \stackrel{def}{=} \phi_B \circ E$.*

Suppose that there are two multiplicative sets $1_A \in \tilde{A}_i \subset A_i$ such that $\mathrm{Span}(\tilde{A}_i)$ *is dense in A_i, suppose from $a_i \in \tilde{A}_i$ follows $E_i(a_i)$, $a_i - E_i(a_i)$, $a_i - \phi_i(a_i) \in \tilde{A}_i$, for $i = 1, 2$, and $B \cap \tilde{A}_1 = B \cap \tilde{A}_2 \stackrel{def}{=} \tilde{B}$.*

Suppose also that there are two sets of unitaries $\emptyset \neq W_i \subset \tilde{A}_i \cap A_i^\circ$ such that $(W_i)^* \subset \tilde{A}_i$ for $i = 1, 2$. Let $u_i \in W_i$, $i = 1, 2$ and suppose that ϕ is invariant with respect to conjugation by u_1 and u_2.

Suppose also that the following condition, similar to condition (4.5), holds:

$$\begin{cases} \forall b_1,\dots,b_l \in \tilde{B},\ \text{with } \phi(b_1)=\cdots=\phi(b_l)=0,\ \exists m\in\mathbb{N} \text{ and unitaries} \\ \nu_{11},\dots,\nu_{1m},\nu_{21},\dots,\nu_{2m} \text{ such that } \nu_{12},\dots,\nu_{1m}\in W_1,\ \nu_{21},\dots,\nu_{2(m-1)}\in W_2,\ \text{and:} \\ \text{either } \nu_{11}\in W_1,\ \nu_{2m}\in W_2 \text{ or} \\ \nu_{11}=1_{A_1},\ \nu_{2m}\in W_2,\ \text{or} \\ \nu_{11}\in W_1,\ \nu_{2m}=1_{A_2}, \\ \nu_{11}=1_{A_1},\ \nu_{2m}=1_{A_2},\ k\geq 2 \\ \text{with } E((\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m})b_k(\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m})^*)=0 \text{ for } k=1,\dots,l, \\ (\text{i.e. there are unitaries that conjugate } \tilde{B}\ominus\mathbb{C}1_B \text{ out of } B) \end{cases} \tag{4.10}$$

Suppose finally that there are unitaries $\omega_1 \in W_1$ *and* ω_2 *with* $\omega_2 = 1_A$ *or* $\omega_2 \in W_2$, *such that* $\forall b \in \tilde{B}$, $\exists \omega_1^b \in W_1$, *and* $\omega_2^b \in W_2$ *if* $\omega_2 \in W_2$ *or* $\omega_2^b = 1$ *if* $\omega_2 = 1$ *with* $E((\omega_2^b)^*(\omega_1^b)^* b\omega_1\omega_2) = 0$.

Then given $x \in \mathrm{Alg}(\tilde{A}_1 \cup \tilde{A}_2)$ *with* $\phi(x) = 0$ *there exist unitaries* $\alpha_1, \dots, \alpha_s$ *with* $\alpha_i \in W_{1+(i \mod 2)}$ *such that* $\alpha_1^* \cdots \alpha_s^* x \alpha_s \cdots \alpha_1 \in \Lambda_B^2$.

Proof. Until the end of this proof we will use the following settings:

$$\tilde{\Lambda}_B^1 \overset{def}{=} \mathrm{Span}(\bigcup_{k=0}^{\infty} (A_1^\circ \cap \tilde{A}_1) \cdot [(A_2^\circ \cap \tilde{A}_2) \cdot (A_1^\circ \cap \tilde{A}_1)]^k \subset \mathbb{C}(A),$$

$$\tilde{\Lambda}_B^2 \overset{def}{=} \mathrm{Span}(\bigcup_{k=0}^{\infty} (A_2^\circ \cap \tilde{A}_2) \cdot [(A_1^\circ \cap \tilde{A}_1) \cdot (A_2^\circ \cap \tilde{A}_2)]^k) \subset \mathbb{C}(A),$$

$$\tilde{\Lambda}_B^{21} \overset{def}{=} \mathrm{Span}(\bigcup_{k=1}^{\infty} [(A_2^\circ \cap \tilde{A}_2) \cdot (A_1^\circ \cap \tilde{A}_1)]^k) \subset \mathbb{C}(A),$$

$$\tilde{\Lambda}_B^{12} \overset{def}{=} \mathrm{Span}(\bigcup_{k=1}^{\infty} [(A_1^\circ \cap \tilde{A}_1) \cdot (A_2^\circ \cap \tilde{A}_2)]^k) \subset \mathbb{C}(A).$$

We can write $x = x_B + x_1 + x_2 + x_{12} + x_{21}$, where $x_B \in \text{Span}(\tilde{B})$ with $\phi(x_B) = 0$, $x_1 \in \tilde{\Lambda}_B^1$, $x_2 \in \tilde{\Lambda}_B^2$, $x_{12} \in \tilde{\Lambda}_B^{12}$ and $x_{21} \in \tilde{\Lambda}_B^{21}$. We will be alternativelly conjugating x with unitaries from W_1 and W_2 until we end up with an element of $\tilde{\Lambda}_B^2$. So at the start we call the words from $\tilde{\Lambda}_B^1$ "good words". When we conjugate a word $w_1 \in \tilde{\Lambda}_B^1$ with $a_2 \in W_2$ we end up with a word $a_2 w_1 a_2^* \in \tilde{\Lambda}_B^2$. Now we call the words of $\tilde{\Lambda}_B^2$ "good words". If we now take a word $w_2 \in \tilde{\Lambda}_B^2$ and conjugate it with an element $a_1 \in W_1$ we obtain the word $a_1 w_2 a_1^* \in \tilde{\Lambda}_B^1$ so we can call the words from $\tilde{\Lambda}_B^1$ "good words". We will show that proceeding in this way, i.e. alternativelly conjugating x with elements from W_1 and W_2 we can come to an element $\alpha_1^* \cdots \alpha_s^* x \alpha_s \cdots \alpha_1 \in \tilde{\Lambda}_B^2$ consisting of a linear combination of "good words" from $\tilde{\Lambda}_B^2$. This will prove the lemma.

We have to consider the other 4 possibilies:

(i) Take a word $b \in \tilde{B}$. Suppose that the "good words" are in $\tilde{\Lambda}_B^2$ and we are going to conjugate b with the element $u_1 \in W_1$. Then we obtain

$$u_1 b u_1^* = E(u_1 b u_1^*) + (u_1 b u_1^* - E(u_1 b u_1^*))$$

for which $(u_1 b u_1^* - E(u_1 b u_1^*)) \in \tilde{A}_1 \cap A_1^\circ \subset \tilde{\Lambda}_1$ is a "good word" and the word $E(u_1 b u_1^*) \in \tilde{B}$ satisfies $\phi(E(u_1 b u_1^*)) = \phi(b)$. Analoguous conclusion can be drawn if we suppose that the "good words" are in $\tilde{\Lambda}_B^1$ and we are conjugating with the element $u_2 \in W_2$.

(ii) Take a word $\gamma_1 \cdots \gamma_{2n} \in \tilde{\Lambda}_B^{12}$ ($\gamma_i \in A^\circ_{1+(i-1 \mod 2)} \cap \tilde{A}_{1+(i-1 \mod 2)}$) and conjugate it with a unitary $a_2 \in W_2$ thinking that the "good words" are in $\tilde{\Lambda}_B^1$. We get

$$a_2 \gamma_1 \cdots \gamma_{2n-1} \gamma_{2n} a_2^* = a_2 \gamma_1 \cdots \gamma_{2n-1} E(\gamma_{2n} a_2^*) + a_2 \gamma_1 \cdots \gamma_{2n-1} (\gamma_{2n} a_2^* - E(\gamma_{2n} a_2^*)).$$

The first word is from $\tilde{\Lambda}_B^{21}$ of the same length $2n$ as the word $\gamma_1 \cdots \gamma_{2n-1} \gamma_{2n}$ and the second word is from $\tilde{\Lambda}_B^2$, i.e. a "good word". If we supposed that the good words were in $\tilde{\Lambda}_B^2$ and we were conjugating with a unitary $a_1 \in W_1$ then we would have

$$a_1\gamma_1 \cdots \gamma_{2n-1}\gamma_{2n}a_2^* = E(a_1\gamma_1)\gamma_2 \cdots \gamma_{2n-1}\gamma_{2n}a_1^* + (a_1\gamma_1 - E(a_1\gamma_1))\gamma_2 \cdots \gamma_{2n-1}\gamma_{2n}a_1^*$$

So again we end up with a word from $\tilde{\Lambda}_B^{21}$ of length $2n$ and a "good word" from $\tilde{\Lambda}_B^1$.

(iii) In a similar way we can treat a word $\gamma_2 \cdots \gamma_{2n+1} \in \tilde{\Lambda}_B^{21}$ ($\gamma_i \in A^\circ_{1+(i-1 \mod 2)} \cap \tilde{A}_{1+(i-1 \mod 2)}$). If we conjugate with a unitary $a_2 \in W_2$ knowing that the "good words" are in $\tilde{\Lambda}_B^1$ we end up with

$$a_2\gamma_2\gamma_3 \cdots \gamma_{2n+1}a_2^* = E(a_2\gamma_2)\gamma_3 \cdots \gamma_{2n+1}a_2^* + (a_2\gamma_2 - E(a_2\gamma_2))\gamma_3 \cdots \gamma_{2n+1}a_2^*.$$

The first word is from $\tilde{\Lambda}_B^{12}$ and of the same length $2n$ and the second word is from $\tilde{\Lambda}_B^2$, i.e. a "good word". In the same way if the good words were in $\tilde{\Lambda}_B^2$ and we were conjugating with a unitary $a_1 \in W_1$ we would obtain

$$a_1\gamma_2 \cdots \gamma_{2n}\gamma_{2n+1}a_1^* = a_1\gamma_2 \cdots \gamma_{2n}E(\gamma_{2n+1}a_1^*) + a_1\gamma_2 \cdots \gamma_{2n}(\gamma_{2n+1}a_1^* - E(\gamma_{2n+1}a_1^*)).$$

The first word is from $\tilde{\Lambda}_B^{12}$ of length $2n$ and the second word is from $\tilde{\Lambda}_B^1$, i.e. a "good word".

(iv) Take a word $\gamma_2 \cdots \gamma_{2n} \in \tilde{\Lambda}_B^2$ ($\gamma_i \in A^\circ_{1+(i-1 \mod 2)} \cap \tilde{A}_{1+(i-1 \mod 2)}$). If the "good words" are in $\tilde{\Lambda}_B^1$ and if we conjugate this word with the unitary $u_2 \in W_2$, we get

$$\begin{aligned} u_2\gamma_2\gamma_3 \cdots \gamma_{2n-1}\gamma_{2n}u_2^* \quad &= \quad E(u_2\gamma_2)\gamma_3 \cdots \gamma_{2n-1}E(\gamma_{2n}u_2^*)+ \\ +(u_2\gamma_2 - E(u_2\gamma_2))\gamma_3 \cdots \gamma_{2n-1}E(\gamma_{2n}u_2^*) &+ E(u_2\gamma_2)\gamma_3 \cdots \gamma_{2n-1}(\gamma_{2n}u_2^* - E(\gamma_{2n}u_2^*))+ \\ +(u_2\gamma_2 - E(u_2\gamma_2))&\gamma_3 \cdots \gamma_{2n-1}(\gamma_{2n}u_2^* - E(\gamma_{2n}u_2^*)). \end{aligned}$$

The last word is in $\tilde{\Lambda}_B^2$, so it is a "good word". The second word is in $\tilde{\Lambda}_B^{21}$, the third is in $\tilde{\Lambda}_B^{12}$ and the first one is in $\tilde{\Lambda}_B^1$ but of length $2n-3$. Since ϕ is invariant with respect to conjugation by u_2 we see that $0 = \phi(\gamma_2\gamma_3\cdots\gamma_{2n-1}\gamma_{2n}) = \phi(u_2\gamma_2\gamma_3\cdots\gamma_{2n-1}\gamma_{2n}u_2^*) = \phi(E(u_2\gamma_2)\gamma_3\cdots\gamma_{2n-1}E(\gamma_{2n}u_2^*))$.

Similarly if we have a word $\gamma_1\cdots\gamma_{2n-1} \in \tilde{\Lambda}_B^1$ ($\gamma_i \in A^\circ_{1+(i-1 \mod 2)} \cap \tilde{A}_{1+(i-1 \mod 2)}$) and if the "good words" are in $\tilde{\Lambda}_B^2$ and if we conjugate with the unitary $u_1 \in W_1$ we will get

$$u_1\gamma_1\gamma_2\cdots\gamma_{2n-2}\gamma_{2n-1}u_1^* = E(u_1\gamma_1)\gamma_2\cdots\gamma_{2n-2}E(\gamma_{2n-1}u_1^*)+$$

$$+(u_1\gamma_1-E(u_1\gamma_1))\gamma_2\cdots\gamma_{2n-2}E(\gamma_{2n-1}u_1^*)+E(u_1\gamma_1)\gamma_2\cdots\gamma_{2n-2}(\gamma_{2n-1}u_1^*-E(\gamma_{2n-1}u_1^*))+$$

$$+(u_1\gamma_1 - E(u_1\gamma_1))\gamma_2\cdots\gamma_{2n-2}(\gamma_{2n-1}u_1^* - E(\gamma_{2n-1}u_1^*)).$$

Notice that the last word is from $\tilde{\Lambda}_B^1$, so it is a "good word". The second word is from $\tilde{\Lambda}_B^{12}$ and the third one is from $\tilde{\Lambda}_B^{21}$. The first word is from $\tilde{\Lambda}_B^2$ but with length $2n-3$. In this case we also can conclude that $0 = \phi(\gamma_1\gamma_2\cdots\gamma_{2n-2}\gamma_{2n-1}) = \phi(u_1\gamma_1\gamma_2\cdots\gamma_{2n-2}\gamma_{2n-1}u_1^*) = \phi(E(u_1\gamma_1)\gamma_2\cdots\gamma_{2n-2}E(\gamma_{2n-1}u_1^*))$.

From this we can conclude that if we take the word $\gamma_2\cdots\gamma_{2n} \in \tilde{\Lambda}_B^2$ and if the "good words" are in $\tilde{\Lambda}_B^1$ then $(u_1u_2)\gamma_2\cdots\gamma_{2n}(u_2^*u_1^*)$ will be the span of some "good words", i.e. belonging to $\tilde{\Lambda}_B^1$, some words from $\tilde{\Lambda}_B^{21}$, some words from $\tilde{\Lambda}_B^{12}$, and the word from $\tilde{\Lambda}_B^2$ with length $2n-5$

$$E(u_1E(u_2\gamma_2)\gamma_3)\gamma_4\cdots\gamma_{2n-2}E(\gamma_{2n-1}E(\gamma_{2n}u_2^*)u_1^*) =$$

$$= E(u_1u_2\gamma_2\gamma_3)\gamma_4\cdots\gamma_{2n-2}E(\gamma_{2n-1}\gamma_{2n}u_2^*u_1^*)$$

if $n \geq 3$. Continuing in the same fashion we see that if $l \geq n/2$, $(u_1u_2)^l\gamma_2\cdots\gamma_{2n}(u_2^*u_1^*)$ will be the span of some "good words", i.e. belonging to $\tilde{\Lambda}_B^1$, some words from $\tilde{\Lambda}_B^{21}$, some words from $\tilde{\Lambda}_B^{12}$, and a word $b \in \tilde{B}$. Actually it is easy to see that $b = E((u_1u_2)^l\gamma_2\cdots\gamma_{2n}(u_2^*u_1^*)^l) \in \tilde{B}$ since this is the element which projects onto B under the conditional expectation. Notice that since ϕ is E-invariant and also invariant with respect to conjugation by u_1 and u_2 then $\phi(E((u_1u_2)^l\gamma_2\cdots\gamma_{2n}(u_2^*u_1^*)^l))=0$.

We can now return to the element $x=x_B+x_1+x_2+x_{12}+x_{21}$. Set the words from $\tilde{\Lambda}_B^1$ to be "good words". From the observation above we see that if l is greater that the length of the longest word appearing in x_2, then $(u_1u_2)^lx_2(u_2^*u_1^*)^l$ is the span of some "good words" from $\tilde{\Lambda}_B^1$, some words from $\tilde{\Lambda}_B^{21}$, some words from $\tilde{\Lambda}_B^{12}$, and some words from $\tilde{B}$, each one of them when evaluated on ϕ gives 0. But considering cases (i), (ii) and (iii) we can easily conclude that $x' \overset{def}{=} (u_1u_2)^lx(u_2^*u_1^*)^l$ can be written as $x'=x'_B+x'_1+x'_{12}+x'_{21}$ with x'_B being a span of words from $\tilde{B}$ and satisfying $\phi(x'_B)=0$, x'_1 being a span of "good words" from $\tilde{\Lambda}_B^1$, x'_{12} being a span of words from $\tilde{\Lambda}_B^{12}$ and x'_{21} being a span of words from $\tilde{\Lambda}_B^{21}$.

Let $x'_B=\sum_{i=1}^n\alpha_ib_i$, where $b_i\in\tilde{B}$ and $\alpha_i\in\mathbb{C}$. $0=\phi(x'_B)=\phi(\sum_{i=1}^n\alpha_ib_i)=\sum_{i=1}^n\alpha_i\phi(b_i)$. Thus $x'_B=\sum_{i=1}^n\alpha_i(b_i-\phi(b_i))$ if we set $b'_i=b_i-\phi(b_i)$ for $i=1,\ldots,n$, then $b'_i\in\tilde{B}$ with $\phi(b'_i)=0=\phi(u_2b_iu_2^*)$. So we can apply condition (4.10) to the set of elements $\{b'_1,\ldots,b'_n,E(u_2b'_1u_2^*),\ldots,E(u_2b'_nu_2^*)\}\subset\tilde{B}$. We obtain unitaries $\nu_1,\ldots,\nu_m$. Set

$$u=\begin{cases}\nu_1\cdots\nu_m, & \text{if } \nu_1\in W_2,\nu_m\in W_2\\ u_2\nu_1\cdots\nu_m, & \text{if } \nu_1\in W_1,\nu_m\in W_2,\\ u_2\nu_1\cdots\nu_mu_2, & \text{if } \nu_1\in W_1,\nu_m\in W_1,\\ \nu_1\cdots\nu_mu_2, & \text{if } \nu_1\in W_2,\nu_m\in W_1.\end{cases}$$

Then it is clear that $u^*x'_Bu\in\tilde{\Lambda}_B^2$ and the "good words" are in $\tilde{\Lambda}_B^2$. Then from cases (ii) and (iii) also follows that $x''\overset{def}{=}u^*x'u$ can be represented as $x''=x''_2+x''_{12}+x''_{21}$, where $x''_2\in\tilde{\Lambda}_B^2$ is a span of "good words" and $x''_{12}\in\tilde{\Lambda}_B^{12}$, $x''_{21}\in\tilde{\Lambda}_B^{21}$. Let n be the number of words from $\tilde{\Lambda}_B^{21}$ and from $\tilde{\Lambda}_B^{12}$ that appear in the span of $x''_{12}+x''_{21}$. We will argue by induction on n to conclude the proof of the lemma.

Let $\gamma_1\cdots\gamma_{2l}\in\tilde{\Lambda}_B^{12}$ ($\gamma_i\in A^\circ_{1+(i-1 \mod 2)}\cap\tilde{A}_{1+(i-1 \mod 2)}$) is a word from the span of

x''_{12}. (The case x''_{21} is completely analoguous.) Set

$$\tilde{u} \overset{def}{=} \begin{cases} \omega_1\omega_2(u_1u_2)^{l-1}, & \text{if } \omega_2 \in W_2, \\ \omega_1(u_2u_1)^{l-1}u_2, & \text{if } \omega_2 = 1_A. \end{cases}$$

Let's observe first that if $\alpha_1\cdots\alpha_{2l}, \beta_1\cdots\beta_{2l} \in \tilde{\Lambda}_B^{12}$, then we can write

$$E(\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l}) = E(\beta_{2l}^*\cdots\beta_2^*E(\beta_1^*\alpha_1)\alpha_2\cdots\alpha_{2l}) +$$

$$+ E(\beta_{2l}^*\cdots\beta_2^*(\beta_1^*\alpha_1 - E(\beta_1^*\alpha_1))\alpha_2\cdots\alpha_{2l}) = E(\beta_{2l}^*\cdots\beta_2^*E(\beta_1^*\alpha_1)\alpha_2\cdots\alpha_{2l}).$$

It follows by induction that $E(\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l}) \in \tilde{B}$. Also from

$$\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l} = \beta_{2l}^*\cdots\beta_2^*E(\beta_1^*\alpha_1)\alpha_2\cdots\alpha_{2l} +$$

$$+\beta_{2l}^*\cdots\beta_2^*(\beta_1^*\alpha_1 - E(\beta_1^*\alpha_1))\alpha_2\cdots\alpha_{2l} = \beta_{2l}^*\cdots\beta_2^*E(\beta_1^*\alpha_1)\alpha_2\cdots\alpha_{2l}$$

again by induction follows that $\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l}$ is the span of words from $\tilde{\Lambda}_B^2$ plus the word $E(\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l}) \in \tilde{B}$.

All this implies that $\tilde{u}^*\gamma_1\cdots\gamma_{2l}\tilde{u}$ is a span of "good words" from $\tilde{\Lambda}_B^2$ and the word $E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\tilde{u} \in \tilde{\Lambda}_B^{12}$. Set $\tilde{b} \overset{def}{=} E(\tilde{u}^*\gamma_1\cdots\gamma_{2l}) \in \tilde{B}$ (see the observation above). Now we choose unitaries $\omega_1^{\tilde{b}}, \omega_2^{\tilde{b}}$ as in the statement of the lemma. We have

$$(\omega_2^{\tilde{b}})^*(\omega_1^{\tilde{b}})^*E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\tilde{u}\omega_1^{\tilde{b}}\omega_2^{\tilde{b}} =$$

$$= \begin{cases} (\omega_2^{\tilde{b}})^*(\omega_1^{\tilde{b}})^*E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\omega_1\omega_2(u_1u_2)^{l-1}\omega_1^{\tilde{b}}\omega_2^{\tilde{b}}, & \text{if } \omega_2 \in W_2, \\ (\omega_1^{\tilde{b}})^*E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\omega_1(u_2u_1)^{l-1}u_2\omega_1^{\tilde{b}}, & \text{if } \omega_2 = 1_A. \end{cases}$$

From this and from the choice of $\omega_1^{\tilde{b}}, \omega_2^{\tilde{b}}$ (and from case (i)) it is clear that

$$(\omega_2^{\tilde{b}})^*(\omega_1^{\tilde{b}})^*E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\tilde{u}\omega_1^{\tilde{b}}\omega_2^{\tilde{b}}$$

is a span of "good words".

Since by cases (ii) and (iii) follows that when we alternatively conjugate words from $\tilde{\Lambda}_B^{12}$ and from $\tilde{\Lambda}_B^{21}$ by unitaries from W_1 and W_2 the number of such words doesn't increase, and since we managed to conjugate the word $\gamma_1 \cdots \gamma_{2l}$ to a span of "good words", the induction on n is concluded.

This proves the lemma. □

Combining Proposition C.3 and Lemma C.4 we obtain the following

Theorem C.5. *Assume that we have unital C^*-algebras A_1, A_2 that contain the unital C^*-algebra B as a unital C^*-subalgebra. Also assume that there are condidional expectations $E_i : A_i \to B$ for $i = 1, 2$ that have faithful KSGNS-rapresentations (i.e. satisfy condition (4.1)) and form the reduced amalgamated free product $(A, E) \stackrel{def}{=} (A_1, E_1) * (A_2, E_2)$.*

Suppose that there are states ϕ_i on A_i for $i = 1, 2$ which are invariant with respect to E_i, $i = 1, 2$ and satisfy $\phi_1|_B = \phi_2|_B (\stackrel{def}{=} \phi_B)$. Construct $\phi \stackrel{def}{=} \phi_B \circ E$.
Assume that there are unitaries $u_1 \in A_1$, $u_2, u_2' \in A_2$ with $E_1(u_1) = 0 = E_2(u_2) = E_2(u_2') = E(u_2 u_2'^*)$. (Or assume that there are unitaries $u_1, u_1' \in A_1^\circ$, $u_2 \in A_2^\circ$ with $E(u_1 u_1'^*) = 0$.)

Suppose that there are two multiplicative sets $1_A \in \tilde{A}_i \subset A_i$ such that $\mathrm{Span}(\tilde{A}_i)$ *is dense in A_i, suppose from $a_i \in \tilde{A}_i$ follows $E_i(a_i)$, $a_i - E_i(a_i)$, $a_i - \phi_i(a_i) \in \tilde{A}_i$, for $i = 1, 2$, and $B \cap \tilde{A}_1 = B \cap \tilde{A}_2 \stackrel{def}{=} \tilde{B}$.*

Suppose also that there are two sets of unitaries $\emptyset \neq W_i \subset \tilde{A}_i \cap A_i^\circ$ such that $(W_i)^* \subset \tilde{A}_i$ for $i = 1, 2$. Let $v_i \in W_i$, $i = 1, 2$ and suppose that ϕ is invariant with respect to conjugation by v_1 and v_2.

Suppose that condition (4.10) holds, namely:

$$\begin{cases} \forall b_1,\dots,b_l \in \tilde{B},\ \textit{with } \phi(b_1)=\cdots=\phi(b_l)=0,\ \exists m\in\mathbb{N} \textit{ and unitaries} \\ \nu_{11},\dots,\nu_{1m},\nu_{21},\dots,\nu_{2m} \textit{ such that } \nu_{12},\dots,\nu_{1m}\in W_1,\ \nu_{21},\dots,\nu_{2(m-1)}\in W_2,\ \textit{and:} \\ \textit{either } \nu_{11}\in W_1,\ \nu_{2m}\in W_2 \textit{ or} \\ \nu_{11}=1_{A_1},\ \nu_{2m}\in W_2,\ \textit{or} \\ \nu_{11}\in W_1,\ \nu_{2m}=1_{A_2}, \\ \nu_{11}=1_{A_1},\ \nu_{2m}=1_{A_2},\ k\geq 2 \\ \ \textit{with } E((\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m})b_k(\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m})^*)=0 \textit{ for } k=1,\dots,l, \\ (\textit{i.e. there are unitaries that conjugate } \tilde{B}\ominus\mathbb{C}1_B \textit{ out of } B) \end{cases}$$

Suppose finally that there are unitaries $\omega_1 \in W_1$ and ω_2 with $\omega_2 = 1_A$ or $\omega_2 \in W_2$, such that $\forall b \in \tilde{B}$, $\exists \omega_1^b \in W_1$, and $\omega_2^b \in W_2$ if $\omega_2 \in W_2$ or $\omega_2^b = 1$ if $\omega_2 = 1$ with $E((\omega_2^b)^*(\omega_1^b)^* b\omega_1\omega_2) = 0$.

Then:

(1) If ϕ_B has a faithful GNS-representation then A is simple.

(2) If ϕ is invariant with respect to conjugation by u_1, u_2, u_2' (or by u_1, u_1', u_2) and all the unitaries from W_1 and W_2, then ϕ is the only tracial state on A, invariant with respect to conjugation by all those unitaries.

Proof. (1) Suppose $I \neq 0$ is an ideal of A. Notice that $\mathrm{Alg}(\tilde{A}_1 \cup \tilde{A}_2)$ is dense in A. Take a nonzero element $x \in I$. Because E has a faithful KSGNS-representation it satisfies condition (4.1), i.e. $\exists y \in A$ such that $b \overset{def}{=} E(y^*x^*xy) \neq 0$. Notice that $b^* = b$. Since ϕ has a faithful GNS-representation we can find $b' \in B$ such that $\phi((b')^*bb') \neq 0$. Then

$$\phi((b')^*y^*x^*xyb') = \phi(E((b')^*y^*x^*xyb')) = \phi((b')^*E(y^*x^*xy)b') = \phi((b')^*bb') \neq 0.$$

Then $c \stackrel{def}{=} \phi((b')^*bb')^{-1}(b')^*y^*x^*xyb'$ $(\in I)$ is self-adjoined and satisfies $\phi(c) = 1$. Find $a \in \mathrm{Alg}(\tilde{A}_1 \cup \tilde{A}_2)$ such that $\|a - c\| < \epsilon$. From Lemma C.4 it follows that we can find unitaries $\alpha_1, \ldots, \alpha_m \in W_1 \cup W_2$ such that $(\alpha_1 \cdots \alpha_m)^*(a - \phi(a)1_A)(\alpha_1 \cdots \alpha_m) \in \Lambda_B^1$. Then it follows from Proposition C.3 that we can find unitaries $U_1, \ldots U_N \in A$ that are constructed from u_1, u_2, u_2' and the unitaries from $W_1 \cup W_2$ and are such that

$$\|\sum_{i=1}^{N} \frac{1}{N} U_i^*(\alpha_1 \cdots \alpha_m)^*(a - \phi(a)1_A)(\alpha_1 \cdots \alpha_m)U_i\| < \epsilon.$$

Then

$$\|\sum_{i=1}^{N} \frac{1}{N} U_i^*(\alpha_1 \cdots \alpha_m)^*(a - \phi(a)1_A - c + 1_A)(\alpha_1 \cdots \alpha_m)U_i\| \leq$$

$$\sum_{i=1}^{N} \frac{1}{N} \|U_i^*(\alpha_1 \cdots \alpha_m)^*(a - \phi(a)1_A - c + 1_A)(\alpha_1 \cdots \alpha_m)U_i\| =$$

$$= \sum_{i=1}^{N} \frac{1}{N} \|a - \phi(a)1_A - c + 1_A\| = \|a - \phi(a)1_A - c + 1_A\| =$$

$$= \|(a - c) - \phi(a - c)\| \leq \|a - c\| + \|a - c\| < 2\epsilon.$$

Therefore $\|\sum_{i=1}^{N} \frac{1}{N} U_i^*(\alpha_1 \cdots \alpha_m)^*(c - 1_A)(\alpha_1 \cdots \alpha_m)U_i\| < 3\epsilon$. Set $d \stackrel{def}{=} \sum_{i=1}^{N} \frac{1}{N} U_i^*(\alpha_1 \cdots \alpha_m)^* c (\alpha_1 \cdots \alpha_m)U_i$ $(\in I)$.

Thus $\|d - 1_A\| < 3\epsilon$. Then if we take $\epsilon < \frac{1}{3}$ it would follow that d is invertible, and therefore $I = A$.

(2) Take $0 \neq x \in A$. Then if we argue as in case (1) we can find unitaries $U_1, \ldots, U_N \in \overline{conv}\{u | u$ is a product of unitaries from $W_1 \cup W_2 \cup \{u_1, u_2, u_2'\}\}$ with

$$\|\sum_{i=1}^{N} \frac{1}{N} U_i^*(x - \phi(x)1_A)U_i\| < 3\epsilon.$$

If we take a state ϕ' such that ϕ and ϕ' are invariant with respect to conjugation by

u_1, u_2, u_2' and by all unitaries from $W_1 \cup W_2$ then we will have

$$3\epsilon > |\phi'(\sum_{i=1}^{N} \frac{1}{N} U_i^*(x - \phi(x)1_A)U_i)| = |\sum_{i=1}^{N} \frac{1}{N} \phi'(U_i^* x U_i) - \phi(x)| = |\sum_{i=1}^{N} \frac{1}{N} \phi'(x) - \phi(x)| =$$

$$|\phi'(x) - \phi(x)|.$$

Since this is true for any $\epsilon > 0$ it follows that $\phi' \equiv \phi$. □

Although the statement of Theorem C.5 looks complicated some applications can be given. The next proposition is a slight generalization of the de la Harpe's result from [19].

Corollary C.6. *Suppose that* $G_1 \supsetneq H \subsetneq G_2$ *are discrete groups and suppose that* $H \neq \{1\}$. *Denote* $G \overset{def}{=} G_1 \underset{H}{*} G_2$. *Suppose that for any finitely many* $h_1, \ldots, h_m \in H \backslash \{1\}$ *there is* $g \in G$ *with* $g^{-1} h_i g \notin H$ *for all* $i = 1, \ldots, m$. *Then* $C_r^*(G)$ *is simple with a unique trace.*

Proof. Set $A_i = C_r^*(G_i)$, $i = 1, 2$, $B = C_r^*(H)$ and $A = C_r^*(G)$. Clearly H is not normal in at least one of the groups G_1 or G_2. Without loss of generality suppose that H is not normal in G_1. Then there are $g_1, g_1' \in G_1 \backslash H$ and $g_2 \in G_2 \backslash H$ with $g_1 (g_1')^{-1} \in G_1 \backslash H$. Then set $u_1 = \lambda(g_1)$, $u_1' = \lambda(g_1')$, $u_2 = \lambda(g_2)$. We take $\tilde{A}_i = \{\lambda(c_i) | c_i \in G_i\}$, $i = 1, 2$, $\tilde{B} = \{\lambda(h) | h \in H\}$. Also $W_i = \tilde{A}_i \backslash \tilde{B}$ for $i = 1, 2$. Condition (4.10) is satisfied since for finitely many elements from $H \backslash \{1\}$ we can find an element from G that conjugates them away from H. Finally for the last condition of Theorem C.5 we can set $\omega_1 = u_1$, $\omega_2 = 1$ and for $\lambda(h) \in \tilde{B}$ we set $\omega_1^{\lambda(h)} = h u_1'$. Thus all requirements of Theorem C.5 are met and this finishes the proof. □

We give also an application to HNN extensions of discrete groups. We will use the notion of reduced HNN extensions for C^*-algebras introduced by Ueda in [38]. We will use the following settings:

Let $\{1\} \subsetneq H \subset G$ be countable discrete groups and let $\tilde{\theta} : H \to G$ be an injective group homomorphism. Thus we have that $C_r^*(H) \subset C_r^*(G)$ and that we have a well defined injective $*$-homomorphism $\theta : C_r^*(H) \to C_r^*(G)$. By $E_H^G : C_r^*(G) \to C_r^*(H)$ and $E_{\tilde{\theta}(H)}^G : C_r^*(G) \to C_r^*(\theta(C_r^*(H))$ we will denote the canonical conditional expectations. By τ_G we will denote the canonical trace on $C_r^*(G)$. Let $A_1 = C_r^*(G) \otimes M_2(\mathbb{C})$, $A_2 = C_r^*(H) \otimes M_2(\mathbb{C})$ and $B = C_r^*(H) \oplus C_r^*(H)$. Define the inclusion maps $i_1 : B \to A_1$ and $i_2 : B \to A_2$ as

$$i_1(b_1 \oplus b_2) = \begin{bmatrix} b_1 & 0 \\ 0 & \theta(b_2) \end{bmatrix}, \qquad i_2(b_1 \oplus b_2) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$$

and define the conditional expectations $E_1 : A_1 \to B$ and $E_2 : A_2 \to B$ as

$$E_1 = \begin{bmatrix} E_H^G & 0 \\ 0 & E_{\tilde{\theta}(H)}^G \end{bmatrix}, \qquad E_2 = \begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}.$$

Then let

$$(A, E) = (A_1, E_1) * (A_2, E_2)$$

be the reduced amalgamated free product of (A_1, E_1) and (A_2, E_2) and let

$$(\mathcal{A}, E_{C_r^*(G)}^{\mathcal{A}}, u(\theta)) = (C_r^*(G), E_H^G) \bigstar_{C_r^*(H)} (\theta, E_{\tilde{\theta}(H)}^G)$$

be the reduced HNN extension of $C_r^*(G)$ by θ as in [38]. Also let $i_B : B \to A$ be the canonical inclusion.

From [38, Proposition 2.2] follows that A is isomorphic to $\mathcal{A} \otimes M_2(\mathbb{C})$. Therefore the questions of simplicity and uniqueness of trace for A and for $\mathcal{A}$ are equivalent. The following corollary of Theorem C.5 is true:

Corollary C.7. *In the above settings suppose that $H \subsetneq G$ and $\tilde{\theta}(H) \subsetneq G$. Suppose also that $\forall h \in H \backslash \{1\}$, $\exists n_h \in \mathbb{N}$, such that $\tilde{\theta}^{n_h - 1}(h) \in \tilde{\theta}(H)$ and $\tilde{\theta}^{n_h}(h) \notin \tilde{\theta}(H)$. Then*

A (and therefore $\mathcal{A}$ also) is simple with a unique trace.

Proof. We will show that all the conditions of Theorem C.5 are met.

First the canonical traces τ_i on A_i, $i = 1, 2$ satisfy $\tau_i \circ E_i = \tau_i$ for $i = 1, 2$ and $\tau_1|_B = \tau_2|_B$ ($\overset{def}{=} \tau_B$). We have $\tau = \tau_B \circ E$.

Define

$$\tilde{A}_1 = \mathrm{Span}(\{\lambda(g) \otimes e_{ij} | g \in G,\ 1 \leq i, j \leq 2\})$$

and

$$\tilde{A}_2 = \mathrm{Span}(\{\lambda(h) \otimes e_{ij} | h \in H,\ 1 \leq i, j \leq 2\}),$$

where e_{ij} for $1 \leq i, j \leq 2$ are the matrix units for $M_2(\mathbb{C})$. Then we have $\tilde{A}_1 \cap B = \tilde{A}_2 \cap B (\overset{def}{=} \tilde{B})$. It is also clear that $a_i \in \tilde{A}_i$ implies $E(a_i),\ a_i - E(a_i), a_i - \tau_i(a_i) \in \tilde{A}_i$ for $i = 1, 2$.

Choose $\bar{g}_1 \in G \backslash H,\ \bar{g}_2 \in G \backslash \tilde{\theta}(H)$.

Define the following unitaries from $A_1 \cap \tilde{A}_1$:

$$u_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad u_1' = \begin{bmatrix} \lambda(\bar{g}_1) & 0 \\ 0 & \lambda(\bar{g}_2) \end{bmatrix}, \qquad u_1'' = \frac{1}{\sqrt{2}} \begin{bmatrix} -\lambda(\bar{g}_1) & \lambda(\bar{g}_1) \\ \lambda(\bar{g}_2) & \lambda(\bar{g}_2) \end{bmatrix},$$

and the following unitary from $A_2 \cap \tilde{A}_2$:

$$u_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Set $W_1 = \{u_1, u_1', u_1''\}$, $W_2 = \{u_2\}$.

Set $\omega_1 = u_1$, $\omega_2 = 1_{A_2}$ and for every $b = b_1 \oplus b_2 \in \tilde{B}$ set $\omega_1^b = u_1'$. Then

$$E((u_1')^*(b_1 \oplus b_2)u_1) = E(\begin{bmatrix} \lambda(\bar{g}_1^{-1}) & 0 \\ 0 & \lambda(\bar{g}_2^{-1}) \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & \theta(b_2) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) =$$

$$= E(\begin{bmatrix} 0 & \lambda(\bar{g}_1^{-1})b_1 \\ \lambda(\bar{g}_1^{-1})\theta(b_2) & 0 \end{bmatrix}) = 0.$$

It remains to check that condition (4.10) holds.

For an element $b = b_1 \oplus b_2 \in B$ it is easy to see that

$$u_2^* u_1^* b u_1 u_2 = E(u_2^* E(u_1^* b u_1) u_2) + u_2^*(u_1^* b u_1 - E(u_1^* b u_1)) u_2$$

and that

$$i_B^{-1} \circ E(u_2^* u_1^* b u_1 u_2) = \begin{cases} \theta^{-1}(b_1) \oplus \theta(b_2), & \text{if } b_1 \in \theta(C_r^*(H)),\ b_2 \in C_r^*(H), \\ \theta^{-1}(b_1) \oplus 0, & \text{if } b_1 \in \theta(C_r^*(H)),\ b_2 \notin C_r^*(H), \\ 0 \oplus \theta(b_2), & \text{if } b_1 \notin \theta(C_r^*(H)),\ b_2 \in C_r^*(H), \\ 0 \oplus 0, & \text{if } b_1 \notin \theta(C_r^*(H)),\ b_2 \notin C_r^*(H). \end{cases}$$

Using induction one can show that for any $n \in \mathbb{N}$ we have

$$(u_2^* u_1^*)^n b (u_1 u_2)^{-n} - E((u_2^* u_1^*)^{-n} b (u_1 u_2)^n) \in \Lambda_B^2.$$

Let $\hat{\theta}$ be the linear map which extends of θ to $C_r^*(G)$ by $\hat{\theta}(\lambda(g)) = 0$ for $g \in G \backslash H$. Also let $\hat{\theta^{-1}}$ be the linear map which extends of θ^{-1} to $C_r^*(G)$ by $\hat{\theta^{-1}}(\lambda(g)) = 0$ for $g \in G \backslash \tilde{\theta}(H)$. Then:

$$i_B^{-1} \circ E((u_2^* u_1^*)^{-n} b (u_1 u_2)^n) =$$

$$= \begin{cases} \theta^{-n}(b_1) \oplus \theta^n(b_2), & \text{if } b_1 \in (\hat{\theta})^n(C_r^*(H)),\ b_2 \in (\hat{\theta^{-1}})^{n-1}(C_r^*(H)), \\ \theta^{-n}(b_1) \oplus 0, & \text{if } b_1 \in (\hat{\theta})^n(C_r^*(H)),\ b_2 \notin (\hat{\theta^{-1}})^{n-1}(C_r^*(H)), \\ 0 \oplus \theta^n(b_2), & \text{if } b_1 \notin (\hat{\theta})^n(C_r^*(H)),\ b_2 \in (\hat{\theta^{-1}})^{n-1}(C_r^*(H)), \\ 0 \oplus 0, & \text{if } b_1 \notin (\hat{\theta})^n(C_r^*(H)),\ b_2 \notin (\hat{\theta^{-1}})^{n-1}(C_r^*(H)). \end{cases}$$

If we set $c_1 = \lambda(\bar{g}_1^{-1})(\hat{\theta^{-1}})^n(b_1)\lambda(\bar{g}_1)$ and $c_2 = \lambda(\bar{g}_2^{-1})(\hat{\theta})^{n+1}(b_2)\lambda(\bar{g}_2)$ the we will have

$$i_B^{-1} \circ E(u_2^*(u_1')^*(u_2^*u_1^*)^n b(u_1u_2)^n u_1'u_2) =$$

$$= \begin{cases} \theta^{-1}(c_2) \oplus c_1, & \text{if } c_2 \in \theta(C_r^*(H)),\ c_1 \in C_r^*(H), \\ \theta^{-1}(c_2) \oplus 0, & \text{if } c_2 \in \theta(C_r^*(H)),\ c_1 \notin C_r^*(H), \\ 0 \oplus c_1, & \text{if } c_2 \notin \theta(C_r^*(H)),\ c_1 \in C_r^*(H), \\ 0 \oplus 0, & \text{if } c_2 \notin \theta(C_r^*(H)),\ c_1 \notin C_r^*(H). \end{cases}$$

Now take elements $\tilde{b}_1, \ldots, \tilde{b}_l \in \tilde{B}$ with $\tau_B(\tilde{b}_1) = \cdots = \tau_B(\tilde{b}_l) = 0$. We can write $\tilde{b}_k = \alpha_k + b_{k1} \oplus -\alpha_k + b_{k2}$ for each $k = 1, \ldots, l$ with $b_{kj} \in \mathrm{Span}(\{\lambda(h) | h \in H \backslash \{1\}\})$. Clearly from the statement of the corollary follows that there exists an $N \in \mathbb{N}$ with $E_H^G(\hat{\theta}^N(b_{k2})) = 0$ for each $k = 1, \ldots, l$. Therefore for each $k = 1, \ldots, l$ we have

$$i_B^{-1} \circ E(u_2^*(u_1')^*(u_2^*u_1^*)^{-N} \tilde{b}_k (u_1u_2)^N u_1'u_2) =$$

$$= \begin{cases} \alpha_k \oplus -\alpha_k + c_k, & \text{if } c_k \in C_r^*(H), \\ \alpha_k \oplus -\alpha_k, & \text{if } c_k \notin C_r^*(H), \end{cases}$$

where $c_k = \lambda(\bar{g}_1^{-1})(\hat{\theta^{-1}})^N(b_{k1})\lambda(\bar{g}_1)$, $k = 1, \ldots l$. Now we can find an $M \in \mathbb{N}$ such that $(\hat{\theta})^M(c_k) = 0$ for all $k = 1, \cdots, l$. Then for all $k = 1, \ldots, l$ we have

$$i_B^{-1} \circ E((u_2^*u_1^*)^{-M} u_2^*(u_1')^*(u_2^*u_1^*)^{-N} \tilde{b}_k (u_1u_2)^N u_1'u_2(u_1u_2)^M) = \alpha_k \oplus -\alpha_k.$$

Finally for all $k = 1, \ldots, l$

$$i_B^{-1} \circ E((u_1'')^*(u_2^*u_1^*)^{-M} u_2^*(u_1')^*(u_2^*u_1^*)^{-N} \tilde{b}_k (u_1u_2)^N u_1'u_2(u_1u_2)^M u_1'') = 0.$$

This proves that condition (4.10) holds and thus we can apply Theorem C.5.

This proves the Corollary. □

Remark C.8. *By symmetry it is clear that in the corollary the assumption*

$''\ \forall h \in H\backslash\{1\},\ \exists n_h \in \mathbb{N},$ such that $\tilde{\theta}^{n_h-1}(h) \in \tilde{\theta}(H)$ and $\tilde{\theta}^{n_h}(h) \notin \tilde{\theta}(H)\ ''$

can be replaced by the assumption

$''\ \forall h \in H\backslash\{1\},\ \exists n_h \in \mathbb{N},$ such that $\tilde{\theta}^{-n_h+1}(h) \in H$ and $\tilde{\theta}^{-n_h}(h) \notin H\ ''$.

Examples of HNN extensions of discrete groups which satisfy the assumption of this corollary (and which are therefore simple with a unique trace) are the Baumslag-Solitar groups $BS(n,m)$ for $|n| \neq |m|$ and $|n|, |m| \geq 2$.

CHAPTER V

CONCLUSION

In this report we made some contribution to the Operator Algebra theory and in particular to Free Probability. We briefly describe what we achieved and what further can be researched.

In Chapter I we recalled the notions of full and reduced free product of C^*-algebras and gave some properties of those.

In Chapter II we proved the existence of a six term exact sequence for the K-theory of full amalgamated free product C^*-algebras $A *_C B$, in the case when C is an ideal in both C^*-algebras A and B.

In Chapter III we found a necessary and sufficient conditions for the simplicity and uniqueness of trace for reduced free products of finite families of finite dimensional C^*-algebras with specified traces on them.

In Chapter IV we studied some reduced free products of C^*-algebras with amalgamations. We gave sufficient conditions for the positive cone of the K_0 group to be the largest possible. We also gave sufficient conditions for simplicity and uniqueness of trace.

It would be interesting to know if reduced free products of C^* (W^*) algebras can be used in Physics or some other natural science to explain some phemomena from the nature. One of the main reason for which the von Neumann Algebras and C^* algebras were developed was to explain some Quantum Mechanical phenomena. Thus our question is not unreasonable.

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