ESSAYS IN ASSET PRICING AND PORTFOLIO CHOICE

A Dissertation

by

PHILIPP KARL ILLEDITSCH

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2007

Major Subject: Finance

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ABSTRACT

Essays in Asset Pricing and Portfolio Choice. (August 2007)

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In the first essay, I decompose inflation risk into (i) a part that is correlated with real returns on the market portfolio and factors that determine investor's preferences and investment opportunities and (ii) a residual part. I show that only the first part earns a risk premium. All nominal Treasury bonds, including the nominal money-market account, are equally exposed to the residual part except inflation-protected Treasury bonds, which provide a means to hedge it. Every investor should put 100% of his wealth in the market portfolio and inflation-protected Treasury bonds and hold a zero-investment portfolio of nominal Treasury bonds and the nominal money market account.

In the second essay, I solve the dynamic asset allocation problem of finite lived, constant relative risk averse investors who face inflation risk and can invest in cash, nominal bonds, equity, and inflation-protected bonds when the investment opportunity set is determined by the expected inflation rate. I estimate the model with nominal bond, inflation, and stock market data and show that if expected inflation increases, then investors should substitute inflation-protected bonds for stocks and they should borrow cash to buy long-term nominal bonds.

In the last essay, I discuss how heterogeneity in preferences among investors with external non-addictive habit forming preferences affects the equilibrium nominal term structure of interest rates in a pure continuous time exchange economy and complete securities markets. Aggregate real consumption growth and inflation are exogenously specified and contain stochastic components that affect their means and volatilities. There are two classes of investors who have external habit forming preferences and different local curvatures of their utility functions. The effects of time varying risk aversion and different inflation regimes on the nominal short rate and the nominal market price of risk are explored, and simple formulas for nominal bonds, real bonds, and inflation risk premia that can be numerically evaluated using Monte Carlo simulation techniques are provided.

To my parents, Franka and Karl Illeditsch, who made all this possible.

ACKNOWLEDGEMENTS

I am deeply indebted to my advisor Kerry Back for his mentoring, guidance, and support. I also thank Ekkehart Boehmer, Manfred Deistler, Michael Gallmeyer, Bob Goldstein, Michael Jeckle, Shane Johnson, Gabriel Lee, Todd Milbourn, and Sorin Sorescu for their guidance and support throughout my graduate study. Special thanks go to Andriy Vernydub for encouraging me to pursue a Ph.D. in the United States and for many helpful suggestions and extensive discussions. Thanks also to Anatoliy Belaygorod, Matthias Bühlmaier, Bruce Carlin, Georg Giokas, Jeremy Jackson, Thomas Kutt, Suraj Prasad, Jason Smith, Semih Tartaroğlu, Ariel Viale, Julie Wu, and Zhenxin Yu for many discussions about mathematics, economics, or finance. I also thank Dante DeBlassie and Dmitry Livdan for serving on my committee. Financial support from the Mays Business School is deeply appreciated. Finally, I would like to thank my friends, my family, and my girlfriend, A. Barış Günersel, for giving me the energy to finish this project. Without their understanding, help, and support, this would not have been possible.

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CHAPTER I

INTRODUCTION

This dissertation consists of three essays. The title of the first essay is *"Idiosyncratic Inflation Risk and Inflation-Protected Bonds"*, the title of the second essay is *"Inflation and Asset Allocation"*, and the title of the last essays is *"The Term Structure of Interest Rates with Heterogeneous Habit Forming Preferences"*.

I.1 Idiosyncratic Inflation Risk and Inflation-Protected Bonds

Inflation can affect real security prices through two channels. First, inflation may affect the real economy, meaning the real stochastic discount factor and the real cash flows of positive-net-supply securities. Second, inflation will affect the real cash flows of zero-net-supply securities such as nominal Treasury bonds. I decompose inflation risk into (i) a part that is correlated with real returns on the market portfolio and factors that determine investor's preferences and investment opportunities and (ii) a residual part. I show that only the first part earns a risk premium and investors should seek to avoid exposure to the second part.

I consider an economy with heterogeneous investors who can continuously trade in a frictionless security market and receive labor income that is spanned by real asset returns. The market price of residual inflation risk is zero because only the part of inflation risk that is correlated with factors that determine investor's preferences and investment opportunities and real returns on the market portfolio is priced in equilibrium; i.e. I show that the ICAPM for real asset returns holds. This is true even when the government issues inflation-protected and nominal Treasury bonds and collects nominal lump-sum tax payments from investors to cover the interest payments on the Treasuries outstanding. Moreover, the conclusion that the market price of residual inflation risk is zero does not require complete markets or identical tax payments among investors.

This dissertation follows the style of *Journal of Finance*.

Inflation-protected Treasury bonds provide a means to hedge exposure to residual inflation risk. All nominal bonds, including the nominal money-market account, are equally affected by inflation through the second channel described above. I show (i) there is a real instantaneously risk-free asset consisting of a long position in inflation-protected bonds and a zero-investment portfolio of nominal bonds and the nominal money market account, (ii) the portfolios on the instantaneous mean-variance frontier of risky assets consist of long or short positions in the market portfolio and inflation-protected bonds and zero-investment portfolios of nominal bonds and the nominal money market account, and (iii) the portfolios that hedge changes in the investment opportunity set consist of long or short positions in the market portfolio and inflation-protected bonds and zero-investment portfolios of nominal bonds and the nominal money market account. These facts imply directly that (iv) every investor should put 100% of his wealth in the market portfolio and inflation protected-bonds and hold a zero-investment portfolio of nominal bonds and the nominal money-market account.

Results (i)-(iv) follow from the equal exposure of nominal bonds and the nominal money market account to residual inflation risk. This risk cannot be present in the real locally risk-free asset; thus (i) holds. This risk is not priced; thus, the variance-minimizing portfolio producing a given expected return has no residual inflation risk, producing result (ii). The hedging portfolios are the portfolios maximally correlated with the latent state variables and therefore cannot include residual inflation risk; thus, (iii) holds.

The conclusion that investors in aggregate should hold zero-investment portfolios in nominal bonds and the nominal money market account follows from equilibrium considerations — market clearing for zero-net-supply securities. However, the conclusion here is much stronger: every investor, not just the representative investor, should hold zero-investment portfolio in nominal bonds and the nominal money market account. Moreover, the zero-investment portfolio in nominal bonds should be interpreted as inclusive of the investor's short position in nominal Treasury bonds that corresponds to his position as a taxpayer and inclusive of his short position in nominal corporate bonds that corresponds to his position as a shareholder. In other words, the investor's allocation to corporate bonds versus stocks should be the same as a representative investor, and he should hold enough Treasury bonds to immunize his tax liability.

It is well known since Merton (1971) that the optimal dynamic investment strategy is to hold a linear combination of $(k + 2)$ mutual funds; two funds to form the optimal portfolio on the mean-variance frontier and k funds to hedge changes in investor's preferences and investment opportunities. I show for a broad class of preferences and asset return distributions that the optimal amount of nominal Treasury bonds and the nominal money market account invested in each mutual fund is always zero without explicitly solving for the value function. Moreover, when investors are subject to nominal lump-sum tax payments that are affine functions of the price level, then they should hold an additional fund with exactly enough in Treasury bonds to immunize their tax liabilities.

Fischer (1975), Bodie, Kane, and McDonald (1983), and Viard (1993), assuming a constant investment opportunity set, show that (i) only the part of inflation risk that is correlated with real stock returns should earn a risk premium if the CAPM for real asset returns holds (residual inflation risk is unpriced) and (ii) investors should shun nominal bonds when inflation-protected bonds are available. I show that the second part is no longer true when the real and nominal short rate is stochastic (the nominal money market account and nominal bonds, as well as, the real risk-free asset and inflation-protected bonds aren't perfect substitutes) because in this case investors hold long/short positions in nominal bonds that are financed by an equal amount of other nominal bonds and the nominal money market account when inflation-protected bonds are available. However, I derive the ICAPM for heterogeneous investors with state dependent preferences and investment opportunities and confirm the first result when residual inflation risk is defined as the part of inflation risk that is not only uncorrelated with real stock returns but with real returns on the market portfolio and factors that determine investor's preferences and investment opportunities. Moreover, I show that inflation-protected bonds are used to hedge residual inflation risk (allow investors to create a real risk-free asset) without assuming that the real short rate is constant.

Recent studies on optimal portfolio choice with inflation-protected bonds include Campbell and Viceira (2001) and Campbell, Chan, and Viceira (2003). Campbell and Viceira (2001) and Campbell, Chan, and Viceira (2003) solve the discrete-time dynamic portfolio choice problem of an infinitely-lived investor with Epstein-Zin preferences, who can invest in equity, nominal bonds, and inflation-protected bonds, using a log linear approximation and a Gaussian investment opportunity set. While this paper employs different assumptions and a different solution method – I assume a finite-lived investor, preferences and investment opportunity sets that are described by an exogenously given state vector (this excludes Epstein-Zin preferences), and an exogenously given stochastic discount factor and solve a continuous-time portfolio choice problem) – the principal difference is that the main portfolio choice results are derived when residual inflation risk is unpriced.

This paper is also related to recent papers of Brennan and Xia (2002) and Sangvinatsos and Wachter (2005), who discuss dynamic asset allocation decision with inflation risk and provide closed form solutions. Brennan and Xia (2002) analyze the portfolio problem of a finite-lived investor with power utility who can invest in the stock market, cash, and nominal bonds when the conditional distribution of all asset returns is Gaussian. Sangvinatsos and Wachter (2005) extend their work by adding another state variable to account for time-varying risk premia and explore the resulting predictability of nominal bond returns for portfolio choice. My paper differs from these papers in that I add inflation-indexed bonds to the analysis and consider a broader class of preferences and asset return distributions. Importantly, the fact that residual inflation risk is not priced allows me to determine the optimal investment in nominal bonds and the nominal money market account in each mutual fund without explicitly solving for the value function of the dynamic portfolio choice problem.

My paper is also related to recent studies of inflation-protected bonds by Bodie (1990), Gapen and Holden (2005), Hunter and Simon (2005), Kothari and Shanken (2004), Roll (2004), Brynjolfsson and Fabozzi (1999), Deacon, Derry, and Mirfendereski (2004), and

Benaben (2005). These studies analyze the mean, variance, and correlation of returns on nominal bonds, inflation-protected bonds, and stocks and discuss the welfare gains of adding inflation-protected bonds to standard investment portfolios consisting of nominal bonds and stocks in a static mean-variance framework. The main conclusion is that adding inflation-protected bonds increases the welfare of investors because of the low standard deviation of real returns of inflation-protected bonds and their diversification benefits (the low correlation between inflation-protected bonds and both nominal bonds and stocks). However, the gains are usually found to be quite small for U.S. investors because of the low volatility of inflation risk in the United States.

I.2 Inflation and Asset Allocation

This paper explores the effect of inflation on optimal portfolio choice. The asset classes considered are stocks, nominal Treasury bonds, inflation-protected Treasury bonds, and a nominal money market account. Expected inflation is modelled as a latent state variable. It is assumed that the ICAPM holds for real returns and that all of the above assets other than stocks are in zero-net supply (hence do not appear in the market portfolio). Optimal portfolios for a CRRA investor, consisting as usual of the mean-variance efficient portfolio and hedging portfolios, are computed analytically. The model is calibrated to U.S. data and the sensitivity of optimal portfolios to expected inflation is determined.

Fama and Schwert (1977), using the short rate as a proxy for expected inflation, show that neither stocks nor nominal bonds perform well in inflationary environments: An increase in the short rate lowers the risk premia of stocks and bonds, and may actually reduce their expected returns. This is contrary to the simple view that stocks are claims to cash flows that increase on average at the rate of inflation and hence should be good hedges against inflation. Without using the short rate as a proxy for expected inflation, this paper confirms that stocks are poor investments in high inflation environments. This is true even when hedging demands are considered in addition to the locally mean-variance efficient portfolio. However, I obtain results for nominal bonds that are somewhat at odds with Fama and Schwert's results: The real risk premia of nominal bonds increase with expected inflation, and optimal asset allocations to nominal bonds increase with expected inflation. For inflation-protected bonds, I find that real risk premia decline with expected inflation, yet optimal allocations increase with expected inflation. A rough summary of the paper's results for asset allocation is that when expected inflation increases, investors should substitute inflation-protected bonds for stocks and should borrow at the nominal risk-free rate to buy nominal bonds.

It may seem paradoxical that the risk premia of inflation-protected bonds decline with expected inflation but optimal allocations to inflation-protected bonds increase. The explanation for this result is that in general an investor should hold 100% of his wealth in stocks and inflation-protected bonds and should hold a zero-investment portfolio in nominal bonds and the nominal money market account. This serves to avoid exposure to the part of unanticipated inflation that is uncorrelated to changes in expected inflation.[1] The risk premium of inflation-protected bonds and stocks declines with expected inflation, but the optimal allocation to inflation-protected bonds depends on how much investors would like to allocate to the real risk-free asset, whereas the optimal allocation to stocks depends on how much they would like to allocate to the tangency portfolio. The investor should substitute inflation-protected bonds for stocks when expected inflation increases in order to increase his allocation to the real risk-free asset and reduce his allocation to the tangency portfolio.

I find, in contrast to the popular view, that nominal bond portfolios perform well in inflationary environments. This seems surprising given that the real value of nominal bonds is eroded by unanticipated inflation risk. The explanation for the good performance of nominal bonds in inflationary environments is twofold. First, investors can always avoid unanticipated inflation risk by financing every long/short position in nominal bonds by shorting/buying an equal amount of other nominal bonds or by borrowing/lending at the

[1] The zero-investment portfolio in nominal bonds should be interpreted as inclusive of the investor's short position in nominal government bonds that corresponds to his position as a taxpayer (see Illeditsch (2007b)).

nominal short rate. Second, the real risk premium for nominal bonds is increasing in the expected inflation rate. This makes nominal bonds not only a very attractive investment when expected inflation is high but also a good hedge against changes in the investment opportunity set.

To capture the predictability of excess returns of stocks, inflation-protected bonds, and nominal bonds, I specify the dynamics of the real stochastic discount factor and the price level (the dynamics for the nominal stochastic discount factor follow from no arbitrage) and assume that their drifts and volatility terms are functions of the expected inflation rate that follows a mean reverting Ornstein-Uhlenbeck process. The real short rate is a quadratic function and the market price of risk an affine function of the expected inflation rate. The nominal short rate is a quadratic function of the expected inflation rate, and a simple restriction on the parameter space ensures its positivity. In this case, both inflation-protected bonds and nominal bonds belong to the class of quadratic Gaussian term structure models, and hence the local volatilities are affine functions and the risk premia are quadratic functions of the expected inflation rate. The real stock return has a constant local volatility, but its risk premium depends on expected inflation because the market price of risk is an affine function of the expected inflation rate. Hence, the model belongs to the class of "quadratic asset return" models proposed in Liu (2007), and portfolio demands for CRRA investors can be computed in closed form.[2]

Recent studies on optimal dynamic asset allocation with inflation risk include Brennan and Xia (2002) and Sangvinatsos and Wachter (2005) who provide closed form solutions for portfolio demands of finite lived investors with constant relative risk aversion preferences. Brennan and Xia (2002) discuss optimal portfolios when the nominal bond and equity risk premium is constant and the investment opportunity set is determined by the real risk-free rate and the expected inflation rate that both follow mean reverting Ornstein-

[2] I derive a closed form solution for optimal portfolio demands in Section B.2 of Appendix B for the more general case in which the expected inflation rate is a quadratic function of x and both the local volatility of real stock returns and inflation is an affine function of x. Moreover, it is straightforward to extend the model by considering a k dimensional state vector that follows a multivariate Ornstein-Uhlenbeck process and solving portfolio demands in closed form.

Uhlenbeck processes. Sangvinatsos and Wachter (2005) extend their analysis by adding another Gaussian state variable and account for time varying risk premia of nominal bond returns. My paper differs from these papers along three dimensions: (i) investors can also invest in inflation-protected bonds which provide a perfect hedge against unexpected inflation risk, (ii) both the nominal price of a nominal bond and the real price of an inflation-protected bond belong to the class of quadratic Gaussian term structure models which ensures positivity of the nominal short rate and nominal bond yields and implies that not only the risk premia but also the volatilities of nominal and inflation-protected bonds depend on expected inflation,[3] and (iii) the equity risk premium is not constant but depends on expected inflation which allows me to focus on the effects of expected inflation risk on optimal bond and stock allocations.

This paper is also related to recent papers of Campbell and Viceira (2001) and Campbell, Chan, and Viceira (2003) who discuss optimal dynamic allocations to cash, nominal bonds, equity, and inflation-protected bonds with inflation risk. In both papers the authors solve the discrete time, dynamic portfolio choice problem of an infinite-lived investor with Epstein-Zin preferences using a log linear approximation and a Gaussian investment opportunity set. Campbell and Viceira (2001) assume that the risk premia of all assets are constant and Campbell, Chan, and Viceira (2003) assume that all assets returns are described by a first order VAR in which the nominal Treasury bill rate, the yield spread, and the dividend-price ratio are state variables. While this paper employs different assumptions and a different solution method (I assume a finite-lived investor and solve a continuous time portfolio choice problem in closed form), the principal difference is that in this paper the risk premia of assets depend on expected inflation which allows me to focus on the effects of expected inflation risk on optimal bond and stock allocations.

[3] Brandt and Chapman (2002) show that the three-factor quadratic Gaussian term structure model of Ahn, Dittmar, and Gallant (2002) dominates the three factor essentially affine term structure models of Duffee (2002) at matching economic moments.

I.3 The Term Structure of Interest Rates with Heterogeneous Habit Forming Preferences

This paper discusses the equilibrium term structure of nominal interest rates with heterogenous external habit forming preferences and inflation uncertainty. Aggregate real consumption growth and inflation are exogenously specified and exhibit stochastic means and volatilities. Heterogeneity in preferences leads to countercyclical variations in aggregate risk aversion and external habits imply that the countercyclicality does not vanish in the long run. The equilibrium nominal stochastic discount factor is determined in closed form and the effects of aggregate risk aversion and inflation on the nominal short rate and the nominal market price of risk are explored.

The predictability of nominal bond returns and the high persistence of changes in their yields are important features of U.S. Treasury bonds. While there are many sophisticated reduced form models that are successful in explaining these feature, the economic mechanism behind these empirical stylized facts remains mainly unexplored. Moreover, the change in monetary policy in 1979, the high inflationary period in the 70's, and the ability of the Fed over the last twenty years to keep inflation in check has led to changes in the dynamics of inflation (it seems that the persistence in inflation has decreased over time (Kroszner (2007))) and has affected the evolution of the term structure of interest rates. For instance, unconditional volatilities of changes in yields were decreasing in maturity for the pre Volcker-Greenspan period but are now hump-shaped, the hump occurring for two to three year maturities.

In this paper, I relate the evolution of the nominal term structure to a real business cycle variable and inflation. I consider a pure continuous time exchange economy and a complete securities market. There are two investors with external non-addictive habit forming preferences and different constant local curvature of their utility functions. The assumption that one investor is twice as risk averse as the other leads to a quadratic consumption sharing rule and closed form solutions for the nominal stochastic discount factor. Moreover, I provide simple formulas for nominal bond prices, real bond prices, and the inflation risk premium that can be numerically evaluted using Monte Carlo simulation techniques.

This paper is related to Chan and Kogan (2002) who analyze a general equilibrium exchange economy with a continuum of investors who have also external non-addictive habit forming preferences and differ with respect to the curvature of the utility function. While their focus is to explain empirical stylized facts of real stock market returns the main focus of this paper is to discuss the nominal term structure of interest rates. Moreover, I consider only two investors (one investor is twice as risk averse as the other) but obtain closed form solutions for the nominal stochastic discount factor.[4]

This paper is related to Dumas (1989) and Wang (1996) who consider two investors with different risk aversion coefficients and discuss the term structure of interest rates in a production and exchange economy, respectively. This paper differs from their work along two dimensions. First, while Dumas and Wang do not distinguish between real and nominal prices, I specify dynamics of real aggregate consumption and inflation and discuss the term structure of nominal interest rates. Second, they consider time-separable, state independent utility functions, whereas I consider external habit forming preferences. The habit feature of the model has the advantage that stocks and bonds can have high risk-premia premia but at the same time both the level and the volatility of interest rates rates are low and it ensures stationarity of the economy.

The paper is also related to Campbell and Cochrane (1999), Brandt and Wang (2003), and Wachter (2006). All these papers exogenously specify the local curvature of a representative investor's utility function and choose the sensitivity function that drives the log surplus consumption ratio such that (i) the real interest rate is constant as in Campbell and Cochrane (1999) or (ii) it follows an Ornstein-Uhlenbeck process as in Wachter (2006) and Brandt and Wang (2003). To study the implications for the nominal term structure Wachter (2006) assumes that inflation follows an autoregressive homoscedastic process and

[4]The real stochastic discount factor in Chan and Kogan (2002) is a function of the logarithm of the shadow price of the social planner's resource constraint that satisfies an integral equation.

Brandt and Wang (2003) assume that inflation follows an autoregressive heteroscedastic process. The main difference is that in this paper the countercyclical variation in aggregate risk aversion arises endogenously in equilibrium.

CHAPTER II

IDIOSYNCRATIC INFLATION RISK AND INFLATION-PROTECTED BONDS

II.1 Asset Prices

Let X denote a k-dimensional vector of state variables (factors) that describe investor's preferences and investment opportunity sets and Z a d-dimensional vector of independent Brownian motions. The dynamics of the state vector are

$$dX = \mu_X(X)\,dt + \sigma_X(X)'\,dZ, \tag{II.1}$$

in which $\mu_X(X)$ is k-dimensional and $\sigma_X(X)$ is $d \times k$-dimensional.[1]

Prices in the economy are measured in terms of a basket of real goods. Let π denote the price level, $\mu_\pi(X)$ the expected inflation rate, and $\sigma_\pi(X)$ the d-dimensional volatility vector of π. The dynamics of the price level are

$$\frac{d\pi}{\pi} = \mu_\pi(X)\,dt + \sigma_\pi(X)'\,dZ. \tag{II.2}$$

Assume there is no arbitrage and therefore there exists a strictly positive stochastic discount factor M that determines real prices of all assets in the economy. Let $r(X)$ denote the (shadow) risk-free rate or real short rate and $\Lambda(X)$ the d-dimensional vector of market prices of risk. The dynamics of the real stochastic discount factor are

$$\frac{dM}{M} = -r(X)\,dt - \Lambda(X)'\,dZ. \tag{II.3}$$

The real stochastic discount factor M and the price level π are sufficient to price all assets in the economy. Let M^* denote the the nominal stochastic discount factor that is given by

[1] The covariance matrix of X is not necessarily invertible, e.g. time could be a state variable. An apostrophe denotes the transpose of a vector or matrix.

$M^* = M/\pi$. The dynamics of M^* are

$$\frac{dM^*(\pi)}{M^*(\pi)} = -r^*(X)\,dt - (\Lambda(X) + \sigma_\pi(X))'\,dZ, \tag{II.4}$$

in which

$$r^*(X) = r(X) + \mu_\pi(X) - \Lambda(X)'\sigma_\pi(X) - \sigma_\pi(X)'\sigma_\pi(X). \tag{II.5}$$

The nominal short rate $r^*(X)$ is equal to the sum of the real short rate, the expected inflation rate, an inflation risk premium, and a Jensen inequality term. The Fisher equation for the nominal short rate does not hold unless the term $-\Lambda(X)'\sigma_\pi(X)$ is zero in which case the expected real return of the nominal money market account is equal to the real short rate (see equation (II.11) below).[2]

Let S denote the real price of the market portfolio with dynamics

$$\frac{dS}{S} = \mu_S(X)\,dt + \sigma_S(X)'\,dZ, \tag{II.6}$$

in which $\mu_S(X) = r(X) + \sigma_S(X)'\Lambda(X)$ and $\sigma_S(X)$ is d-dimensional. The market portfolio is the value of the cash flows of all positive-net-supply securities and may consist of stocks, corporate bonds, real estate, etc, but excludes zero-net-supply securities such as the nominal money market account, nominal Treasury bonds, and inflation-protected Treasury bonds.[3] The real stochastic discount factor and the cash flows of (positive-net-supply) assets within the market portfolio may be affected by inflation risk, and hence real returns on the market portfolio may be correlated with inflation.

The state vector X, the market portfolio S, and the consumer price index π form a

[2] A zero inflation risk premium for the nominal money market account does not imply that the inflation risk premium for longer holding periods is zero. Specifically, the τ-year inflation risk premium (the expected real return difference of holding a τ-year nominal zero-coupon bond until maturity and of holding a τ-year real zero-coupon bond until maturity) is in general not zero if $\Lambda(X)'\sigma_\pi(X) = 0$. It is in general not zero even if $\sigma_\pi(X) = 0$.

[3] The cash flows of Treasury bonds are offset by corresponding tax liabilities, rendering the net supply of these cash flows zero.

Markov system with dynamics

$$\begin{pmatrix} dX \\ dS/S \\ d\pi/\pi \end{pmatrix} = \begin{pmatrix} \mu_X(X) \\ \mu_S(X) \\ \mu_\pi(X) \end{pmatrix} dt + \sigma(X)'\, dZ. \tag{II.7}$$

Without loss of generality, one can take X_1 to depend only on the Brownian motion Z_1, X_2 to depend only on Z_1 and Z_2, etc. This means that we can assume $d = k + 2$ and that the $(d \times d)$-dimensional, volatility matrix

$$\sigma(X) = (\sigma_X(X), \sigma_S(X), \sigma_\pi(X)) \tag{II.8}$$

is upper diagonal. Define Z_{k+2} which is the additional shock in $d\pi/\pi$ that is uncorrelated with changes in the state variables and real returns on the market portfolio as residual inflation risk. The Markov system in equation (II.7) is very general. It allows for perfect or imperfect correlations of any variables, and it does not impose an affine or any other structure on the drifts and volatilities.

All bonds considered in this paper are default-free zero-coupon bonds if not explicitly stated otherwise.[4] An inflation-protected bond pays one unit of a basket of real goods at its maturity date T. A nominal bond pays \$1 at its maturity date. Denote real prices of real (inflation-protected) bonds by P, real prices of nominal bonds by B, and the real value of the nominal money market account by R. Asterisks indicate nominal prices ($S^* = S\pi$, $P^* = P\pi$, $B^* = B\pi$, and $R^* = R\pi$).

The real price of an inflation-protected bond and its dynamics are given in the next proposition. Nominal and real prices of nominal bonds and the nominal money market are discussed below.

Proposition II.1 (Inflation-protected bonds)**.** *The real price of an inflation-protected bond maturing at T is only a function of the state vector X and time to maturity $T - t$; i.e.*

[4] The market portfolio may consist of inflation-protected and nominal corporate bonds that are not necessarily default-free.

$P = P(T-t, X)$.[5] *The real return of an inflation-protected bond maturing at* T *is*

$$\frac{dP(T-t,X)}{P(T-t,X)} = \big(r(X) + \sigma_P(T-t,X)'\Lambda(X)\big)\; dt + \sigma_P(T-t,X)'\, dZ, \tag{II.9}$$

in which the d-dimensional local real return volatility vector is

$$\sigma_P(T-t,X) = \sigma_X(X)P_X(T-t,X)/P(T-t,X) \tag{II.10}$$

and $P_X(T-t,X)$ *denotes the gradient of* $P(T-t,X)$. *Moreover,* $\sigma_{Pk+2}(T-t,X) = 0$.[6]

Proof. See Section A.1 of Appendix A. □

Real cash flows of inflation-protected bonds are constant, and hence the real return of inflation-protected bonds may be affected by inflation only through the first channel: the real stochastic discount factor. Specifically, real returns of inflation-protected bonds are only exposed to factor risk. This is in stark contrast to assets such as nominal bonds and the nominal money market account whose real cash flows are affected by inflation risk. Their real returns are given in the next proposition.

Proposition II.2 (Nominal bonds and the nominal money market account)**.** *The nominal value at time* t *of a* \$1 *invested in the nominal money market account at time* 0 *depends on the path of the state vector* X *and time* t, *i.e.* $R^* = R^*(t, \{X(a), 0 \le a \le t\})$. *The real return of the nominal cash or money market account is*

$$\frac{dR(R^*,\pi)}{R(R^*,\pi)} = \big(r(X) - \sigma_\pi(X)'\Lambda(X)\big)\; dt - \sigma_\pi(X)'\, dZ. \tag{II.11}$$

The nominal price of a nominal bond maturing at T *is only a function of the state vector* X *and time to maturity* $T-t$*; i.e.* $B^* = B^*(T-t, X)$.[7] *The real return of a nominal*

[5] Assume that real prices of inflation-protected bonds are sufficiently smooth (see Definition A.1 in Section A.1 of Appendix A).

[6] I denote with v_i the i-th component of the vector v.

[7] Assume that nominal prices of nominal bonds are sufficiently smooth (see Definition A.1 in Section A.1 of Appendix A).

bond maturing at T is

$$\frac{dB(T-t,X,\pi)}{B(T-t,X,\pi)} = \big(r(X) + \sigma_B(T-t,X)'\Lambda(X)\big)\ dt + \sigma_B(T-t,X)'\,dZ, \tag{II.12}$$

in which the d-dimensional local real return volatility vector is

$$\sigma_B(T-t,X) = \sigma_X(X)B^*_X(T-t,X)/B^*(T-t,X) - \sigma_\pi(X) \tag{II.13}$$

and $B^*_X(T-t,X)$ denotes the gradient of $B^*(T-t,X)$.[8] *Moreover, $\sigma_{Bk+2}(T-t,X) = -\sigma_{\pi k+2}(X)$ for all maturities T.*

Proof. See Section A.1 of Appendix A. □

Nominal bonds are claims on a dollar at maturity and their real returns are therefore affected by inflation through the second channel (the price level) and may also be affected by inflation through the first channel (the real stochastic discount factor). Specifically, real returns of nominal bonds and the nominal money market account are exposed to factor and residual inflation risk. Moreover, equation (II.11) implies that real returns of the nominal money market account are perfectly negatively correlated with inflation.

If unanticipated inflation risk is not perfectly correlated with changes in the factors and the real return on the market portfolio (i.e. $\sigma_{\pi k+2}(X) \neq 0$), then the effects of inflation risk (i) on the real cash flows of positive-net-supply securities such as stocks, corporate bonds, real estate, etc. can be distinguished from the effects (ii) on the real cash flows of zero-net-supply securities such as nominal bonds and the nominal money market account. All assets may be affected by inflation risk through the part of unanticipated inflation risk that is correlated with changes in the factors and real returns on the market portfolio but only nominal bonds and the nominal money market account are affected by residual inflation risk. Specifically, all nominal bonds and the nominal money market account have exactly the same exposure to this risk source which is $-\sigma_{\pi k+2}(X)$, as shown in Proposition

[8] The nominal return of a nominal bond is given in equation (A.6) in Section A.1 of Appendix A.

II.2. Hence, it is impossible to have a long or short position in a portfolio consisting solely of nominal bonds and the nominal money market account without having exposure to residual inflation risk. This risk is not priced and investors should avoid it, as the next two sections show.

II.2 Equilibrium

Suppose that there are I individuals in the economy that share the same beliefs and can continuously trade in a frictionless security market. The security market may consist of stocks, inflation-protected and nominal corporate bonds, real estate, inflation-protected Treasury bonds, nominal Treasury bonds, a nominal money market account, etc. Each individual makes investment decisions and consumption choices to maximize

$$\mathrm{E}\left[\int_0^{T^i} u^i(t, c^i(t), X(t))\, dt + U^i(T^i, W^i(T), X(T)) \mid X(0) = x\right] \tag{II.14}$$

for some horizon T^i, utility function u^i, and bequest function U^i.[9] The horizon T^i could be infinite in which case $U = 0$ or it could be random in which case it is assumed to be independent of asset returns.

It is assumed that the labor income of every investor is spanned by real asset returns and hence it can be taken as part of an investor's initial wealth w^i. Moreover, each individual has to continuously pay the nominal lump-sum tax $\tau^{*i}(t)$ until T^i. Real tax payments are denoted without asterisks ($\tau^{*i}(t) = \tau^i(t)\pi(t)$).

Suppose that for any i there exist a stochastic discount factor process $M^i(t)$ such that investor i's static budget constraint can be written as[10]

$$w^i - \mathrm{E}\left[\int_0^{T} M^i(t)\tau^i(t)\, dt\right] \geq \mathrm{E}\left[\int_0^{T} M^i(t)c^i(t)\, dt\right] + \mathrm{E}\left[M^i(T)W^i(T)\right] \tag{II.15}$$

[9] The expectation in equation (II.14) is assumed to be finite and u and U are assumed to fulfill the standard conditions for utility functions (see Karatzas and Shreve (1998)).

[10] It is in general very hard to show existence of $M^i(t)$.

and each investor's initial wealth exceeds his tax liability (the left hand side of equation (II.15) is positive).[11] The equilibrium market price of residual inflation risk when $\tau^* = 0$ and $\tau^* \neq 0$ is determined in Theorem II.1 and II.2 below.

No Taxes ($\tau^* = 0$)

I show in the next theorem that the market price of residual inflation risk is zero when there are no tax liabilities.[12]

Theorem II.1 (ICAPM)**.** *Assume that the nominal money market account, nominal Treasury bonds, and inflation-protected Treasury bonds are in zero-net-supply and investors have homogeneous beliefs, their endowments are spanned by real asset returns, their initial wealth (including the present value of future labor income) is strictly positive, and their tax liabilities are zero. Then the market price of residual inflation risk is zero; i.e.* $\Lambda_{k+2}(X) = 0$.

Proof. See Section A.2 of Appendix A. □

Intuitively, the value function of the representative investor depends on aggregate wealth which is equal to the market portfolio and on the state vector that describes changes in investors's preferences and investment opportunities. The market portfolio (with dynamics given in equation (II.6)) is a value weighted sum of all positive-net-supply securities and hence excludes assets such as inflation-protected Treasury bonds, nominal Treasury bonds, and the nominal money market account. Residual inflation risk is by definition neither correlated with the state vector nor with real returns on the market portfolio and therefore it is not priced.[13]

The conclusion that residual inflation risk is not priced does not require complete markets and homogeneous investors. Specifically, investors can differ with respect to endowments, preferences, and investment horizons.

[11] I define initial wealth for every investor in Theorem II.1 and Theorem II.2 and show that it always exceeds an investor's tax liability.

[12] See Merton (1973) for more details about the ICAPM.

[13] The result that residual inflation risk is unpriced does not depend on a firm's capital structure because nominal and inflation-protected corporate bonds are part of the market portfolio.

Taxes ($\tau^* \neq 0$)

Suppose the investment horizon for every investor is infinite, i.e. $T = \infty$. Each individual can invest in a well diversified asset portfolio (consisting of stocks, inflation-protected and nominal corporate bonds, real estate, etc., but excluding nominal and inflation-protected Treasury bonds) and two Treasury bonds (a real consol that continuously pays the real constant coupon ν and a nominal consol that continuously pays the nominal constant coupon κ^*). Let $S(t)$ denote the real ex-dividend price per share of the asset portfolio and $\delta^*(t)$ the continuous nominal dividend payment per unit of time dt. The total number of shares with price S outstanding is normalized to one. Moreover, denote the real price of the real consol by $P_\nu(t)$ and the real price of the nominal consol by $B_\kappa(t)$. The total real return of $S(t)$ is $(dS(t) + \delta(t)\, dt)/S(t)$, the total real return of the inflation-protected consol is $(dP_\nu(t) + \nu\, dt)/P_\nu(t)$, and the total real return of the nominal consol is $(dB_\kappa(t) + \kappa(t)\, dt)/B_\kappa(t)$. Asterisks indicate nominal dividend or coupon payments ($\delta^*(t) = \delta(t)\pi(t)$, $\nu^*(t) = \nu\pi(t)$, and $\kappa^* = \kappa(t)\pi(t)$).

At any time t the government has one inflation-protected and one nominal consol outstanding and it collects continuously the nominal lump-sum tax $\tau^{*i}(t) = f^i \cdot \tau^*(t)$ from each investor. The constant f^i captures the heterogeneity in tax liabilities across investors and satisfies $\sum_{i=1}^{I} f^i = 1$. Assume that aggregate tax payments are used to pay the interest on both consols; i.e. $\tau^*(t) = \nu\pi(t) + \kappa^*(t)$.

The tax liability of an investor is the present value of his future tax payments. It is determined in the next lemma.

Lemma II.1 (Individual tax liabilities)**.** *The real value of investor i's tax liability is*

$$L^i_\tau(t) = f^i(P_\nu(t) + B_\kappa(t)), \qquad \forall\, 0 \leq t < \infty. \tag{II.16}$$

Proof. See Section A.2 of Appendix A. □

Lemma II.1 implies that every investor can immunize his tax liability by holding a constant share of Treasury consols. Hence, the initial wealth of every investor has to exceed the cost of this strategy; i.e. $w^i > f^i(P_\nu(0) + B_\kappa(0))$. I show in the next theorem that the market price of residual inflation risk is zero.

Theorem II.2 (ICAPM with taxes)**.** *Assume that investors have homogeneous beliefs and their endowments are spanned by real asset returns. Each investor is subject to continuous lump-sum tax payments $f^i\tau^*(t)$ and is initially endowed (including the present value of future labor income) with $\alpha^i_{S0} > 0$ shares of the asset portfolio and f^i shares of both the inflation-protected and nominal consol. Moreover, the aggregate tax payment $\tau^*(t)$ is used by the government to pay the interest on their two Treasury consols outstanding (one inflation-protected and one nominal). Then the market price of residual inflation risk is zero.*

Proof. See Section A.2 of Appendix A. □

The two consols outstanding do not appear in the market portfolio because their positive cash flows are offset by the negative cash flows of investor's tax liabilities. Residual inflation risk is by definition not correlated with real returns on the market portfolio and changes in factors and hence it is not priced.

The conclusion that every investor, not just the representative investor, should hold exactly enough Treasury bonds to cover his tax liability does not require complete markets or homogeneous investors. In particular, investors can be subject to different tax payments.

In the remainder of this paper I make the following assumption.

Assumption II.1 (Residual inflation risk)**.** *The inflation rate is not spanned by the state vector and real returns on the market portfolio, i.e. $\sigma_{\pi k+2}(X) \neq 0$ (residual inflation risk is not zero). Moreover, the real market price of residual inflation risk is zero, i.e. $\Lambda_{k+2}(X) = 0$.*

Assumption $II.1$ implies that neither the price level nor functions of the price level can be part of the state vector, but it doesn't rule out the expected inflation rate and/or the volatility of inflation as state variables. Moreover, it is possible that the price level and functions of it can be correlated with the state variables. It is only being assumed that they are not perfectly correlated with state variables.

Optimal portfolios when the market price of residual inflation risk is zero are determined in the next section.

II.3 Dynamic Portfolio Choice

Consider investors who can continuously trade in a frictionless security market and maximize

$$\mathrm{E}\left[\int_0^T e^{-\int_0^t \beta(X(a))\,da}\, u(c(t), X(t))\, dt + e^{-\int_0^T \beta(X(a))\,da}\, U(W(T), X(T))\right] \tag{II.17}$$

for some investment horizon T, subjective discount factor β, utility function u, and bequest U.[14] All investors have strictly positive initial wealth and receive either no labor income or labor income that is spanned by real asset returns in which case the present value of future labor income is taken to be part of the initial wealth.[15]

The following spanning condition is imposed:

Assumption II.2 (Spanning condition)**.** *Let $X=(U,V)$ in which U is spanned by real returns of inflation-protected bonds and nominal returns of nominal bonds. Either (i) the market is complete, or (ii) the part of inflation risk that is not spanned by U is orthogonal to V and to the real return on the market portfolio.*

Neither condition (i) nor (ii) of Assumption $II.2$ implies the other.[16] Assumption $II.2$

[14] The expectation in equation (II.17) is assumed to be finite and u and U are assumed to fulfill the standard conditions for utility functions (see Karatzas and Shreve (1998)).

[15] The case in which investors are subject to lump-sum tax payments is discussed further below.

[16] It is equivalent to say in Assumption $II.2$ that U is spanned by real returns of inflation-protected bonds and real returns of zero-investment portfolios of nominal bonds and the nominal money market account because the additional exposure of real returns of nominal bonds to (i) residual inflation risk and (ii) to factor risk (if the factor is correlated with inflation) is offset by borrowing/lending in the nominal money market account. A formal discussion of the spanning condition is provided in Proposition A.1 in Section A.3 of Appendix A.

implies that there is a mimicking portfolio for the real risk-free asset.[17] Intuitively, a long position in inflation-protected bonds avoids exposure to residual inflation risk, which is not possible with a long or short position in nominal bonds and the nominal money market account because of their equal exposure to residual inflation risk. On the other hand, the exposure of the long position in inflation-protected bonds to factor risk (components of U) can be hedged, because U is spanned by real returns of inflation protected bonds and real returns of zero investment portfolios of nominal bonds and the nominal money market account. Moreover, every claim that solely depends on the state vector U can be perfectly replicated with a portfolio consisting of inflation-protected bonds and zero-investment portfolios of nominal bonds and the nominal money market account. Hence, Assumption II.2 implies that the nominal and inflation-protected bond market is complete.

The optimal portfolio of an investor who can trade continuously in the nominal money market account, the market portfolio, and nominal and inflation-protected bonds, and who seeks to maximize the utility function in equation (II.14) is given in the next theorem.[18]

Theorem II.3. *Adopt Assumptions II.1 and II.2. Every investor should hold a linear combination of the real risk-free asset, the tangency portfolio, and hedging portfolios. Moreover,*

1. *The mimicking portfolio for the real risk-free asset consists of a long position in inflation-protected bonds and a zero-investment portfolio of nominal bonds and the nominal money market account.*
2. *The tangency portfolio consists of long or short positions in the market portfolio and inflation-protected bonds, and a zero-investment portfolio of nominal bond bonds and the nominal money market account.*
3. *The portfolios that hedge changes in the investment opportunity set consist of long or short positions in the market portfolio and inflation-protected bonds, and zero-investment portfolios of nominal bonds and the nominal money market account.*

[17] The proof is given in Theorem 1.
[18] The value function $J(\cdot)$ is defined in equation (A.43) in Section A.3 of Appendix A.

4. *Investors should put* 100% *of their wealth in the market portfolio and inflation-protected bonds and hold a zero-investment portfolio of nominal bonds and the nominal money market account.*

Proof. See Section A.3 of Appendix A. □

A brief description of the proof is as follows. Assumption $II.2$ implies that there exists a real risk-free asset and hence by the $(k+2)$-fund separation theorem of Merton (1971) the optimal portfolio is a linear combination of the mimicking portfolio for the real risk-free asset, the tangency portfolio, and k portfolios that hedge changes in investor's preferences and investment opportunities. The tangency portfolio is by definition the portfolio with maximal local Sharpe ratio and hence the local volatility vector of this portfolio is proportional to the projection of the market price of risk vector onto the asset space. The hedging portfolios are maximally correlated with the factors and hence determined by projecting the state vector onto the asset space. But the mimicking portfolio of the real-risk free asset is locally riskless, the market price of residual inflation risk and its projection onto the asset space is zero, and the projection of all factors onto the asset space is orthogonal to residual inflation risk, and hence the total investments in nominal bonds and the nominal money market account in the mimicking portfolio for the real risk-free asset, the tangency portfolio, and all hedging portfolios are zero.[19]

The composition of the mimicking portfolios for the real risk-free asset, the tangency portfolio, and the hedging portfolios do not depend on the value function. But to obtain the optimal portfolio (to choose the optimal linear combination of the $(k+2)$ funds) it is necessary to determine the sensitivity of marginal utility of wealth to changes in wealth and to changes in the state variables. Specifically, the optimal point on the local mean-variance frontier depends on the investor's attitude towards risk as measured by the relative risk aversion coefficient $\gamma \equiv -wJ_{ww}/J_w$, whereas the hedging demands depend on the sensitivity of marginal utility of real wealth to changes in the factors measured by

[19]The projector onto the asset space is provided in Lemma A.1 in Section A.3 of Appendix A.

$\Theta \equiv -J_{wX}/(wJ_{ww})$.[20]

Taxes

I now discuss optimal portfolios when investors are subject to nominal lump-sum tax payments. Let $\tau^*(t)$ denote the lump-sum tax investors have to pay continuously until T and $L^*_\tau(T-t, X)$ the total nominal tax liability. Specifically,

$$L^*_\tau(T-t, x) = \mathrm{E}\left[\int_t^T \tau^*(a)M^*(a)/M^*(t)\, du \mid X(t) = x\right] \tag{II.18}$$

The real value of the lump-sum tax payment and the tax liability are $\tau(t) = \tau^*(t)/\pi(t)$ and $L_\tau(t) = L^*_\tau(t)/\pi(t)$.

The following condition on tax payments is imposed.[21]

Assumption II.3 (Tax payments)**.** *The nominal lump-sum tax payment is an affine function of the price level; i.e.* $\tau^*(t) = \kappa^* + \nu\pi(t)$*.*

Optimal portfolios for investors subject to nominal lump-sum tax payments are given in the next theorem.

Theorem II.4. *Adopt Assumptions II.1, II.2, and II.3. Suppose an investor's total tax liability does not exceed his initial wealth. Then an investor should hold an inflation-protected bond that pays continuously the real coupon* ν *and a nominal bond that continuously pays the nominal coupon* κ^* *to immunize his tax liability and should hold a linear combination of the real risk-free asset, the tangency portfolio, and hedging portfolios. The compositions of the real risk-free asset, the tangency portfolio, and all hedging portfolios are given in Theorem II.3.*

Proof. See Section A.3 of Appendix A. □

[20] Illeditsch (2007a) provides closed form solutions for the value function and optimal portfolios when investors have constant relative risk aversion preferences and asset drifts are quadratic and asset volatilities are affine functions of the expected inflation rate that follows a mean reverting Ornstein-Uhlenbeck process. More generally, Liu (2007) solves the dynamic portfolio choice problem of constant relative risk averse investors (up to the solution of a system of ordinary differential equations) when asset returns are quadratic.

[21] The nominal tax payment is an affine function of the price level in Section II.2.

The tax rate $\tau^*(t)$ is exogenous and investors can not refuse to pay taxes. Assumption II.3 implies that investors can always meet their tax obligations by holding nominal and inflation-protected coupon bonds with real value equal to their tax liabilities.[22] The remaining wealth is to a 100% invested in inflation-protected bonds and the market portfolio and a zero investment portfolio in nominal bonds and the nominal money market account.

[22]Coupon bonds pay typically the coupon plus the face value at maturity. Hence, the position in the nominal and inflation-protected coupon bond (with total coupon payments equal to $\tau^*(t)$) should be interpreted as inclusive a short position in an inflation-protected and nominal zero-coupon bond.

CHAPTER III

INFLATION AND ASSET ALLOCATION

III.1 Investment Opportunities

Let x denote the state variable or factor that describes the investment opportunity set and Z a three-dimensional vector of independent Brownian motions. The state variable x follows a mean reverting Ornstein-Uhlenbeck process with mean reversion coefficient κ, long run mean $\bar{x}$, and three-dimensional volatility σ_x.[1] Specifically,

$$dx = \kappa\left(\bar{x} - x\right)\, dt + \sigma_x'\, dZ. \tag{III.1}$$

The dynamics of the price level π are

$$\frac{d\pi}{\pi} = \mu_\pi(x)\, dt + \sigma_\pi'\, dZ, \tag{III.2}$$

in which the three-dimensional volatility of inflation is constant and the expected inflation rate is an affine function of x. Specifically,

$$\mu_\pi(x) = \mu_{\pi 0} + \mu_{\pi x} x. \tag{III.3}$$

Assume that there is no arbitrage and hence there exists a strictly positive stochastic discount factor M that determines real prices of all assets in the economy. The dynamics of M are

$$\frac{dM}{M} = -r(x)\, dt - \Lambda(x)'\, dZ, \tag{III.4}$$

[1] We will see below that the first component of Z describes all the uncertainty in the factor x and the second and third component of Z allows for the possibility that real stock returns are locally not perfectly correlated with the factor and inflation is not perfectly correlated with a linear combination of the factor and real stock returns.

in which the (shadow) real risk-free rate or real short rate is a quadratic function of x and the three dimensional vector of market prices of risk is an affine function of x. Specifically,

$$r(x) = \rho_0 + \rho_x x + \rho_{xx} x^2 \tag{III.5}$$

$$\Lambda(x) = \lambda_0 + \lambda_x x. \tag{III.6}$$

Let S denote the real price of a well diversified equity portfolio or a stock market index with dynamics

$$\frac{dS}{S} = \mu_S(x)\,dt + \sigma_S'\,dZ, \tag{III.7}$$

in which the three-dimensional volatility σ_S is constant and the expected rate of return is a quadratic function of x. Specifically,

$$\mu_S(x) = r(x) + \sigma_S'\Lambda(x). \tag{III.8}$$

The factor x, the real stock price S, and the price level π form a Markov system. Without loss of generality one can take x to depend only on the Brownian motion Z_1, S to depend only on Z_1 and Z_2, and π to depend on Z_1, Z_2, and Z_3, and one can assume that $\bar{x} = 0$ and $\sigma_x = (1, 0, 0)'$.[2] Hence, we can assume without loss of generality that the dynamics of the Markov system x, S, and π are

$$\begin{pmatrix} dx \\ dS/S \\ d\pi/\pi \end{pmatrix} = \begin{pmatrix} -\kappa x \\ \mu_S(x) \\ \mu_\pi(x) \end{pmatrix} dt + \begin{pmatrix} 1 & 0 & 0 \\ \sigma_{S1} & \sigma_{S2} & 0 \\ \sigma_{\pi 1} & \sigma_{\pi 2} & \sigma_{\pi 3} \end{pmatrix} dZ. \tag{III.9}$$

The local covariance of the Markov system in equation (III.9) is non singular if neither real stock returns are perfectly correlated with the factor nor inflation is perfectly correlated

[2] In other words, there is no loss in generality by considering the Cholesky decomposition of the local covariance matrix of x, S, and π as the local volatility matrix. Moreover, there always exists an affine transformation of the latent factor x such that the economies described by x and its affine transformation are informationally equivalent; i.e. the long run mean and local volatility of x are not identified by the data (see Proposition B.1 in Section B.1 of Appendix B for a formal proof).

with a linear combination of real stock returns and the factor; i.e. it is non-singular if and only if $\sigma_{S2} \neq 0$ and $\sigma_{\pi 3} \neq 0$.

Let P denote the real price of an inflation-protected bond that pays one unit of the consumption good at its maturity T. All inflation-protected bonds are default-free zero-coupon bonds. Real prices and returns of inflation-protected bonds are given in the next proposition.

Proposition III.1. *The real price of an inflation-protected bond maturing at T is*

$$P(T-t,x) = e^{a(T-t)+b(T-t)x+c(T-t)x^2}, \tag{III.10}$$

in which $a(T-t)$, $b(T-t)$, and $c(T-t)$ are deterministic functions of time to maturity that solve the ordinary differential equations (B.9), (B.10), and (B.11) given in Section B.1 of Appendix B.

The real return of an inflation-protected bond maturing at T is

$$\frac{dP(T-t,x)}{P(T-t,x)} = \big(r(x) + D(T-t,x)\boldsymbol{e}_1'\Lambda(x)\big)\, dt + D(T-t,x)\boldsymbol{e}_1'\, dZ, \tag{III.11}$$

in which

$$D(T-t,x) = b(T-t) + 2c(T-t)x \tag{III.12}$$

and $\boldsymbol{e}_1 = (1,0,0)'$.

Proof. See Section B.1 of Appendix B. □

Inflation-protected bonds belong to the class of quadratic Gaussian term structure models proposed by Ahn, Dittmar, and Gallant (2002) and hence both the volatility and risk premium of the inflation-protected bond return depends on the state of the economy x.

Let $M^* = M/\pi$ denote the nominal stochastic discount factor, $r^*(x)$ the nominal risk-free rate or nominal short rate, and $\Lambda^*(x)$ the nominal market price of risk. The dynamics of the nominal stochastic discount factor M^* are

$$\frac{dM^*}{M^*} = -r^*(x)\, dt - \Lambda^*(x)'\, dZ. \tag{III.13}$$

The nominal market price of risk is an affine function of x because the volatility of inflation is constant and the real market price of risk is affine in x. Specifically,

$$\Lambda^*(x) = \Lambda(x) + \sigma_\pi. \tag{III.14}$$

The nominal short rate is a quadratic function of x because the real risk-free rate is a quadratic function of x, both the expected inflation rate and the inflation risk premium are affine in x, and the local variance of inflation that represents the Jensen inequality term is constant. Specifically,[3]

$$\begin{aligned} r^*(x) &= r(x) + \mu_\pi(x) - \sigma_\pi'\Lambda(x) - \sigma_\pi'\sigma_\pi \\ &= \delta_0 + \delta_x x + \delta_{xx} x^2. \end{aligned} \tag{III.15}$$

The Fisher equation for the nominal short rate does not hold unless the inflation risk premium, $-\sigma_\pi'\Lambda(x)$, is zero in which case the expected real rate of return of the nominal money market account equals the real short rate (see equation (III.20) below). To assure positivity of the nominal short rate, I impose the parameter restrictions[4]

$$\delta_x^2 = 4\,\delta_0\,\delta_{xx} \quad \text{and} \quad \delta_{xx} > 0. \tag{III.16}$$

Let B denote the real price of a nominal bond maturing at time T and R the real value of the nominal money market account. All nominal bonds are default-free zero-coupon bonds.

[3]The coefficients in the nominal short rate equation (III.15) are

$$\delta_0 = \rho_0 + \mu_{\pi 0} - \sigma_\pi\,(\lambda_0 + \sigma_\pi)\,, \qquad \delta_x = \rho_x + \mu_{\pi x} - \sigma_\pi \lambda_x, \qquad \text{and} \qquad \delta_{xx} = \rho_{xx}.$$

[4]The nominal short rate is zero if $x = -\delta_x/(2\delta_{xx})$ and strictly positive otherwise. However, the probability of x attaining this value is zero because x has continuous support and hence $r^*(x)$ is strictly positive almost surely.

Asterisks denote nominal prices ($B^* = B\pi$ and $R^* = R\pi$). The nominal price of a nominal bond, which belongs also to the class of quadratic Gaussian term structure models, and real returns of nominal bonds and the nominal money market account are given in the next proposition.

Proposition III.2. *The nominal price of a nominal bond maturing at T is*

$$B^*(T-t,x) = e^{a^*(T-t)+b^*(T-t)x+c^*(T-t)x^2}, \tag{III.17}$$

in which $a^*(T-t)$, $b^*(T-t)$, and $c^*(T-t)$ are functions of time to maturity that solve the ordinary differential equations (B.12), (B.13), and (B.14) given in Section B.1 of Appendix B.

The real return of a nominal bond with maturity T is

$$\frac{B(T-t,x)}{B(T-t,x)} = \left(r(x) + (D^*(T-t,x)\boldsymbol{e}_1 - \sigma_\pi)'\Lambda(x)\right)\, dt + (D^*(T-t,x)\boldsymbol{e}_1 - \sigma_\pi)'\, dZ, \tag{III.18}$$

in which

$$D^*(T-t,x) = b^*(T-t) + 2c^*(T-t)x \tag{III.19}$$

and $\boldsymbol{e}_1 = (1,0,0)'$.

The real return of the nominal money market account is

$$\frac{dR}{R} = \left(r(x) - \sigma_\pi'\Lambda(x)\right)\, dt - \sigma_\pi'\, dZ. \tag{III.20}$$

Proof. See Section B.1 of Appendix B. □

Real returns of nominal bonds and the nominal money market account are equally exposed to residual inflation risk (the part of inflation risk that is uncorrelated with changes in the factor and real stock returns) and to residual stock market risk (the part of real stock returns that is uncorrelated with changes in the factor). Real returns of the nominal money market account are perfectly negatively correlated with inflation and hence provide no hedge against inflation risk whereas nominal bonds provide a partial hedge against inflation risk when changes in the factor are correlated with it.

III.2 Dynamic Portfolio Choice

In this section, I derive the optimal dynamic portfolio strategy of finite-lived CRRA investors who face inflation risk and can continuously trade in the nominal money market account, nominal bonds, stocks, and inflation-protected bonds. I show in the next proposition that (i) the nominal money market account, one nominal bond, one inflation-protected bond, and the stock market are non-redundant assets that complete the market and (ii) the value function of investors with unit wealth (the value function is homogenous in wealth) is an exponential quadratic function of the state variable.

Proposition III.3. *The nominal money market account, a nominal bond with maturity T_B, the stock market, and an inflation-protected bond with maturity T_P are non-redundant assets that span all the uncertainty in the economy.*

Moreover, the value function of a power utility investor is

$$J(t,W,x) = \begin{cases} \frac{1}{1-\gamma} W^{1-\gamma} \left(\mathrm{E}\left[(M(T)/M(t))^{\frac{\gamma-1}{\gamma}} \mid x(t) = x \right] \right)^{\gamma} & \text{if} \quad \gamma > 0,\ \gamma \neq 1 \\ \log(W) - \mathrm{E}\left[\log(M(T)/M(t)) \mid x(t) = x \right] & \text{if} \quad \gamma = 1. \end{cases} \tag{III.21}$$

Specifically,

$$\mathrm{E}\left[(M(T)/M(t))^{\frac{\gamma-1}{\gamma}} \mid x(t) = x \right] = e^{h_0(T-t) + h_x(T-t)x + h_{xx}(T-t)x^2}, \tag{III.22}$$

in which $h_{xx}(T-t)$, $h_x(T-t)$, and $h_0(T-t)$ are functions of the remaining investment horizon $T-t$ that solve the ordinary differential equations (B.22), (B.23), and (B.24) in Section B.2 of Appendix B.[5]

Proof. See Section B.2 of Appendix B. □

[5]The solution for the expectation in the log-utility case is not provided but can be obtained from the author upon request.

The market is complete and thus (i) there exists a mimicking portfolio for the real risk-free asset, (ii) the Sharpe ratio of the locally mean-variance efficient portfolio is $\sqrt{\Lambda(x)'\Lambda(x)}$, and (iii) the hedging portfolio is perfectly correlated with the factor. Moreover, three fund separation implies that the optimal portfolio is a linear combination of (i) the mimicking portfolio for the real risk-free asset, (ii) the tangency portfolio, and (iii) the hedging portfolio. The weight on these three portfolios depends on an investors risk aversion and the sensitivity of his marginal value of wealth to changes in the investment opportunity set.

As is well known (and shown in equation (III.21)), the relative risk aversion of the value function equals the relative risk aversion of the utility function. Moreover, it is straightforward to compute the sensitivity of marginal value of wealth to changes in the investment opportunity set from equation (III.21). Specifically,

$$-J_{WX}(t, W, x)/(WJ_{WW}(t, W, x)) = h_x(T - t) + 2h_{xx}(T - t)x. \tag{III.23}$$

To determine the optimal composition of the three mutual funds I follow Illeditsch (2007b) and assume that all asset besides stocks are in zero-net supply and the ICAPM for real stock returns holds. In this case, investors should hold 100% of their wealth in stocks and inflation-protected bonds and a zero investment portfolio in nominal bonds and the nominal money market account. Moreover, the investment in nominal bonds and the nominal money market account in the mutual funds (i)-(iii) named above is always zero.

A brief description of the mutual funds is as follows. The *mimicking portfolio for the real-risk free asset* consists of a long position in the inflation-protected bond and a zero investment portfolio of cash and the nominal bond. Both positions cause only an exposure to factor risk and thus investing the fraction $-D(T_p - t, x)/D^*(T_B - t, x)$ in the nominal bond and financing this investment by borrowing/lending in the nominal money market account neutralizes the exposure of the long position in the inflation-protected bond to factor risk (the local volatility is $D(T_p - t, x)$) and creates a locally risk-free asset.

The *tangency portfolio* is fully invested in equity and consists of a zero investment portfolio of cash and the nominal bond. Specifically, the fraction invested in stocks is chosen to create the optimal exposure to residual stock market risk (the part of real stock returns that is uncorrelated with the factor), whereas the fraction invested in the nominal bond is chosen to create the optimal exposure to factor risk. The investment in the nominal bond is financed by borrowing/lending in the nominal money market account to avoid any exposure to unpriced residual inflation risk.

The *hedging portfolio* for the factor is a zero-investment portfolio in the nominal bond and the nominal money market account because any investment in stocks would lead to an exposure to residual stock market risk and hence would lower the correlation with the factor.[6] Moreover, investing in the nominal bond leads to an exposure to residual inflation risk and to residual stock market risk (if inflation is correlated with this risk source) which would lower the correlation of the hedging portfolio with the factor. Hence, any investment in the nominal bond is offset by borrowing/lending in the nominal money market account.[7]

A formal proof of the optimal portfolio demand is given in Theorem III.1 below. Let x denote the realization of the state variable at time t, $\alpha_S(x)$ the fraction of wealth invested in the stock market at time t, $\alpha_P(x)$ the fraction of wealth invested in the inflation-protected bond at time t, $\alpha_B(t,x)$ the fraction of wealth invested in the nominal bond at time t, and $\alpha_R(t,x)$ denote the fraction of wealth invested in the nominal money market account at time t.

Theorem III.1 (Optimal Portfolio Choice)**.** *The optimal demand of an investor with constant relative risk aversion $\gamma > 0$ who maximizes expected utility of real wealth at date T and can trade continuously in a nominal money market account, a nominal bond with*

[6] A mutual fund should have a positive investment. To accomplish this, one can include an investment in the mimicking portfolio for the real risk-free asset.

[7] The maturity of the nominal and inflation-protected bond is indeterminate within the model. The amount invested in the inflation-protected bond does not depend on its maturity because it coincides with the amount invested in the mimicking portfolio for the real risk-free asset. The exposure of the inflation-protected bond to factor risk depends on its maturity and hence the nominal bond allocation in the mimicking portfolio for the real risk-free asset depends on the maturity of both bonds.

maturity T_B, a stock market index, and an inflation protected bond with maturity T_P is

$$\alpha_S(x) = \frac{\Lambda_2(x)}{\gamma\, \sigma_{S2}} \qquad \alpha_P(x) = 1 - \alpha_S(x), \qquad \alpha_R(t,x) = -\alpha_B(t,x),$$
$$\alpha_B(t,x) = -\frac{D(T_P - t, x)}{D^*(T_B - t, x)}\, \alpha_P(x) + \frac{\Lambda_1(x)/\gamma - \sigma_{S1}\, \alpha_S(x)}{D^*(T_B - t, x)} + \frac{h_x(T-t) + 2h_{xx}(T-t)x}{D^*(T_B - t, x)} \tag{III.24}$$

in which $h_{xx}(T-t)$ and $h_x(T-t)$ are solutions of the ordinary differential equations (B.22) and (B.23) in Section B.2 of Appendix B. Moreover, if $\gamma = 1$ then $h_{xx}(T-t) = h_x(T-t) = 0$.

Proof. See Section B.2 of Appendix B. □

III.3 Model Calibration

I derived in the previous section the optimal portfolio strategy for CRRA investors who can invest in a nominal money market account, stocks, and nominal and inflation-protected bonds when the expected real rate of return and the local variance of real returns of all assets are quadratic functions of the latent factor x. In this section I calibrate the model to match first and second moments of inflation, stock return, and bond return data and discuss the implications of changes in expected inflation on real asset returns.[8]

Data

The data consist of monthly observations of the one-month Treasury bill rate, the five year zero-coupon Treasury bond, the consumer price index (CPI), and the S&P500 stock market index excluding dividends from June 1952 to December 2003. All data are available from CRSP.[9] Historical statistics of all four time series are reported in Table III.1.

Parameters

[8]The econometric identification of the model is based on heuristic arguments. A mathematical rigorous analysis of the identification problem is beyond the scope of this paper.

[9]There is not a sufficiently long time series for inflation-protected bonds available in the US.

Table III.1: Summary Statistics

$r^*(x)$ is the one-month Treasury bill rate, $y_{60}(x)$ is the five year nominal Treasury bond yield, $RS(x)$ is the return of the S&P500, deflated by the CPI, and $i(x)$ is the inflation rate. The historical statistics are based on the June 1952 to December 2003 sample period. All four series are continuously compounded rates and observed at a monthly frequency. Mean $\mathrm{E}[\cdot]$ and standard deviation $\sigma[\cdot]$ are reported in annual terms. $\rho_1[\cdot]$ denotes the one-month autocorrelation and $\mathrm{Corr}[\cdot,\cdot]$ denotes the contemporaneous one-month correlation.

Statistic	Data	Statistic	Data
$\mathrm{E}[r^*(x)]$	5.04%	$\mathrm{E}[y_{60}(x)]$	4.94%
$\sigma[r^*(x)]$	0.81%	$\sigma[y_{60}(x)]$	0.64%
$\rho_1[r^*(x)]$	95.96%	$\rho_1[y_{60}(x)]$	99%
$\mathrm{E}[RS(x)]$	3.68%	$\mathrm{E}[i(x)]$	3.77%
$\sigma[RS(x)]$	14.82%	$\sigma[i(x)]$	1.15%
$\rho_1[RS(x)]$	4.36%	$\rho_1[i(x)]$	54.04%
$\mathrm{Corr}[r^*(x), RS(x)]$	-10.87%	$\mathrm{Corr}[RS(x), i(x)]$	-21.03%
$\mathrm{Corr}[r^*(x), i(x)]$	53.98%		

The economy described in the previous section is completely described by the real stochastic discount factor in equation (III.4) and the Markov system in equation (III.9). The dynamics of both the discount factor and the Markov system depend on the 15-dimensional parameter vector Ψ. Specifically,

$$\Psi = (\kappa, \rho_0, \rho_x, \rho_{xx}, \lambda_{01}, \lambda_{x1}, \lambda_{02}, \lambda_{x2}, \mu_{\pi 0}, \mu_{\pi x}, \sigma_{S1}, \sigma_{S2}, \sigma_{\pi 1}, \sigma_{\pi 2}, \sigma_{\pi 3}) \,. \tag{III.25}$$

Time is measured in months. Let $y_\tau(x) = -1/\tau \log(B^*(\tau, x))$ denote the yield of a nominal zero-coupon bond with τ months to maturity,[10] $RS(x) = \log(S(t+1)/S(t))$ the one-month real (log) stock return, and $i(x) = \log(\pi(t+1)/\pi(t))$ the one-month (log) inflation rate. I use the one month risk-free rate as a proxy for the nominal risk-free rate, the CPI as proxy for the price level, and the S&P500 deflated by the CPI as proxy for the real price of a stock market index. Moreover, I consider the five year nominal Treasury bond yield.[11]

Rather than analyzing the parameter vector Ψ I consider the vector Ψ^* in which I

[10]The positivity of the nominal short rate implies that every nominal bond yield is positive. The proof is a direct consequence of Jensen's inequality.

[11]The joint distribution of r^*, y_τ, RS, and i is unknown.

replace the parameters describing the real term structure ρ_0, ρ_x, ρ_{xx}, λ_{01}, and λ_{x1} with the parameters describing the nominal term structure δ_0, δ_x, δ_{xx}, λ^*_{01}, and λ^*_{x1}. Specifically,

$$\Psi^* = (\kappa, \delta_0, \delta_x, \delta_{xx}, \lambda^*_{01}, \lambda^*_{x1}, \lambda_{02}, \lambda_{x2}, \mu_{\pi 0}, \mu_{\pi x}, \sigma_{S1}, \sigma_{S2}, \sigma_{\pi 1}, \sigma_{\pi 2}, \sigma_{\pi 3}) . \tag{III.26}$$

This does not change the identification and estimation problem, because if one knows Ψ^* it is straightforward to determine Ψ from equation (III.14) and equation (III.15). However, it greatly simplifies the analysis, which now can be conducted in two steps: (i) estimate the first six components of Ψ^* using the one month Treasury bill and the five year nominal Treasury bond yield, and (ii) use these six estimates to estimate the remaining nine components of Ψ^*.

Nominal Short Rate

It is straightforward to compute the steady state mean, variance, and autocovariance for any time lag of the nominal risk-free rate in closed form. Moreover, there exist a unique solution for κ, δ_0, δ_x^2, and δ_{xx} given the sample estimates for the mean, variance, and autocovariance of the one-month risk-free rate that fulfills the parameter restriction in equation (III.16) that ensures positivity of the nominal risk-free rate.[12] Assuming that δ_x is non-negative does not change the distribution of the nominal risk-free rate and hence I restrict $\delta_x \geq 0$.[13] All four parameters are identifiable with this restriction. Their estimates are reported in Table III.2.

Nominal Bond Yield

The coefficients of nominal bond yields depend only on the first six components of Ψ^*. The first four parameters were already estimated from the nominal short rate and it is straightforward to compute the steady mean and the steady state variance of the five year

[12] A formal proof is provided in Proposition B.2 in Section B.3 of Appendix B.

[13] It is well known that the sign of δ_x is not identifiable (see Ahn, Dittmar, and Gallant (2002)).

nominal Treasury bond yield.[14] The resulting two non-linear equations can be numerically solved for the two parameters λ^*_{01} and λ^*_{x1}. Their estimates are reported in Table III.2. The monthly five-year nominal Treasury bond yield can never fall below $6.97bp$. Moreover, the one-month autocorrelation implied by the model – with formula given in equation (B.44) in Section B.3 of Appendix B – is 96%, which is a littler lower than the historical 99% (see Table III.1).

Table III.2: Estimation Results for the Nominal Term Structure
Parameter estimates for the nominal risk-free rate $r^*(x)$ and the nominal market price of factor risk $\Lambda^*_1(x)$ using the one-month Treasury bill rate and the five year nominal Treasury bond yield for the period June 1952 to December 2003. Time is measured in months.

Parameter	Estimate
State variable: $dx = -\kappa x dt + (1,0,0)' dZ$	
κ	0.040736
Nominal risk-free rate: $r^*(x) = \delta_0 + \delta_x x + \delta_{xx} x^2$	
δ_0	0.003859
δ_x	0.000653
δ_{xx}	0.000028
Nominal market price of factor risk: $\Lambda^*_1(x) = \lambda^*_{01} + \lambda^*_{x1} x$	
λ^*_{01}	0.030573
λ^*_{x1}	-0.036820

Real Stock Returns and Inflation

So far I have identified and estimated the first six components of Ψ^* from the one-month Treasury bill and the five year nominal Treasury bond yield. The remaining nine components are summarized in the parameter vector Φ. Specifically,

$$\Phi = (\lambda_{02}, \lambda_{x2}, \sigma_{S1}, \sigma_{S2}, \sigma_{\pi 1}, \sigma_{\pi 2}, \sigma_{\pi 3}, \mu_{\pi 0}, \mu_{\pi x}) \,. \tag{III.27}$$

The parameter vector Φ can in principle be identified from the information in real stock returns, inflation, and the nominal short rate.[15] To estimate the parameter vector Φ I use

[14] For a formal proof see Proposition B.3 in Section B.3 of Appendix B.
[15] The nominal short rate and the five year nominal bond yield are both quadratic functions of x and

the simulated method of moments approach of Duffie and Singleton (1993). Specifically, I simulate the model and determine the mean, standard deviation, and one-month autocorrelation of real stock returns and inflation as well as the contemporaneous correlation between (i) real stock returns and inflation, (ii) the nominal risk-free rate and real stock returns, and (iii) the nominal risk-free rate and inflation. I then compare the simulated moments to their historical counterparts.[16] Specifically, I minimize the sum of squared relative errors (the ratio between the simulated statistic and the historical statistic minus one). The estimation results are reported in Table III.3 and Table III.4.

Table III.3: Estimation Results

The first column presents the statistics – mean $\mathrm{E}[\cdot]$, standard deviation $\sigma[\cdot]$, one-month autocorrelation $\rho_1[\cdot]$, and contemporaneous one-month correlation $\mathrm{Corr}[\cdot,\cdot]$ – that are used to estimate the model. $r^*(x)$ is the one-month Treasury bill rate, $RS(x)$ is the return of the S&P500, deflated by the CPI, and $i(x)$ is the inflation rate. All four series are continuously compounded rates and observed at a monthly frequency. The historical statistics presented in the second column are based on the June 1952 to December 2003 sample period. The model statistics presented in the third column are based on 60,000 months of simulated data. The last column reports the relative error – the ratio between model and historical statistic minus one. The last row reports the sum of squared relative errors.

Statistic	Data	Model	Relative Error
$\mathrm{E}[RS]$	0.003067	0.003067	0.000005
$\mathrm{E}[i]$	0.003139	0.003139	−0.000021
$\sigma[RS]$	0.042775	0.042723	−0.001206
$\sigma[i]$	0.003314	0.003313	−0.000005
$\rho_1[RS]$	0.043572	0.041574	−0.045863
$\rho_1[i]$	0.540401	0.479707	−0.112313
$\mathrm{Corr}[r^*, RS]$	−0.108682	−0.114242	0.051160
$\mathrm{Corr}[r^*, i]$	0.539778	0.618188	0.145263
$\mathrm{Corr}[RS, i]$	−0.210273	−0.211460	0.005645
Sum of (equally weighted) squared errors: 0.03847			

hence either variable can be used to estimate the parameters that determine the correlation of both inflation and real stock returns with the factor x.

[16]The only free parameter describing the dynamics of the factor x – the mean reversion coefficient κ – can be estimated from the nominal risk-free rate and hence it is straightforward to simulate the economy – i.e. to simulate x, Z, $\int_{t-1}^{t} x(u)\,du$, and $\int_{t-1}^{t} x(u)^2\,du$. The state variable x is very persistent and hence I simulate 60,000 months of data ensure that x is in steady state. Moreover, both integrals are approximated by dividing each month in ten equidistant intervals, evaluating the integrand at all ten left end points of the interval, multiplying by 1/10, and summing up – i.e. I simulated a total of 600,000 realizations of x and Z.

Table III.4: Parameter Estimates of the Economy

The model was calibrated to match the historical statistics reported in the second column of Table III.3. The data for the one month Treasury bill $r^*(x)$, the real return on the S&P500, deflated by the CPI, $RS(x)$, and inflation $i(x)$ are continuously compounded rates and observed at a monthly frequency over the sample period June 1952 to December 2003. The simulation is based on $60,000$ observations. Time is measured in months.

State variable: $dx = \kappa(\bar{x} - x)dt + \sigma_x' dZ$		
κ	$\bar{x}$	σ_x
0.040736	0	$(1, 0, 0)$
Real risk-free rate: $r(x) = \rho_0 + \rho_x x + \rho_{xx} x^2$		
ρ_0	ρ_x	ρ_{xx}
0.00076705	-0.00023611	0.0000276
Market price of risk: $\Lambda(x) = \lambda_0 + \lambda_x x$		
λ_0	λ_x	
$(0.0321, 0.0661, 0)'$	$(-0.0368, -0.1139, 0)'$	
Real stock returns: $dS/S = (r(x) + \sigma_S' \Lambda(x))\, dt + \sigma_S'\, dZ$		
σ_S	$(0.0337, 0.0244, 0)'$	
Expected inflation rate: $\mu_\pi(x) = \mu_{\pi 0} + \mu_{\pi x} x$		
$\mu_{\pi 0}$	$\mu_{\pi x}$	
0.00316829 bp	0.00073891 bp	
Realized Inflation: $d\pi/\pi = \mu_\pi(x)\, dt + \sigma_\pi'\, dZ$		
σ_π	$(-0.001485, 0.001799, 0.000016)'$	

Parameter Estimates

Time is in months because all data are observed at a monthly frequency. The expected inflation rate follows a mean reverting Ornstein-Uhlenbeck process with long run mean $\mu_{\pi 0} = 32$bp, mean reversion $\kappa = 4.07\%$, and local volatility $\mu_{\pi x} = 7$bp. Hence, the unconditional mean and standard deviation of expected inflation is 3.8% p.a. and 0.9% p.a., respectively. Moreover, the mean reversion coefficient $\kappa = 4.07\%$ implies a half life of innovations in the expected inflation rate (factor) of 1.4 years. The market price of both expected inflation and residual stock market risk has a positive unconditional mean and is decreasing in the expected inflation rate.

The estimates for the local volatility of inflation σ_π imply that almost all the variation in realized inflation comes from innovations to expected inflation and real stock returns. While I would expect a high correlation between expected and realized inflation, most of the variation in realized inflation (59%) is due to variations in real stock returns, which seems too high. Similarly, the estimate of the local volatility of real stock returns σ_S implies that 66% of the variation in real stock returns comes from innovations in the expected inflation rate, which also seems a little high.

Summary statistics for the real risk-free rate and the square of the maximal Sharpe ratio of all real local asset returns are reported in Table III.5. The parameters ρ_0, ρ_x, and ρ_{xx} imply an unconditional mean and standard deviation for the real risk-free rate of 1.3% p.a. and 0.34% p.a., respectively. Although the expected inflation rate can attain arbitrary negative values, both the nominal and real risk-free rate are non-negative. Specifically, the smallest possible value for the real risk-free rate is 0.31% p.a..

Figure III.1 presents the inflation risk premium and the real and nominal risk-free rate as a function of the expected inflation rate (with domain equal to its long run mean plus/minus twice its standard deviation.) The nominal risk-free rate is strictly increasing with the expected inflation rate.[17] However, the real risk-free rate is strictly decreasing with

[17] The nominal risk-free rate is a decreasing function of the expected inflation rate for very negative values of expected inflation. However, the probability of such realizations of the expected inflation rate is virtually

Table III.5: The Real Risk-free Rate and the Maximal Sharpe Ratio

The real risk-free rate is given in equation (III.5) and the derivation of the square of the maximal Sharpe ratio of real local asset returns is based on the real market price of risk given in equation (III.6). The derivation of the mean $\mathrm{E}[\cdot]$, standard deviation, $\sigma[\cdot]$, one-month autocorrelation $\rho_1[\cdot]$, minimum $\min[\cdot]$, and maximum $\max[\cdot]$ are based on the parameter estimates reported in Table III.4. Time is measured in months.

Statistic	Model	Statistic	Model
$\mathrm{E}\left[r(x)\right]$	11.06bp	$\mathrm{E}\left[\Lambda(x)'\Lambda(x)\right]$	18.13%
$\sigma\left[r(x)\right]$	9.56bp	$\sigma\left[\Lambda(x)'\Lambda(x)\right]$	25.62%
$\rho_1\left[r(x)\right]$	95.04%	$\rho_1\left[\Lambda(x)'\Lambda(x)\right]$	92.39%
$\min\left[r(x)\right]$	2.63bp	$\min\left[\Lambda(x)'\Lambda(x)\right]$	1.04bp
$\max\left[r(x)\right]$	∞	$\max\left[\Lambda(x)'\Lambda(x)\right]$	∞

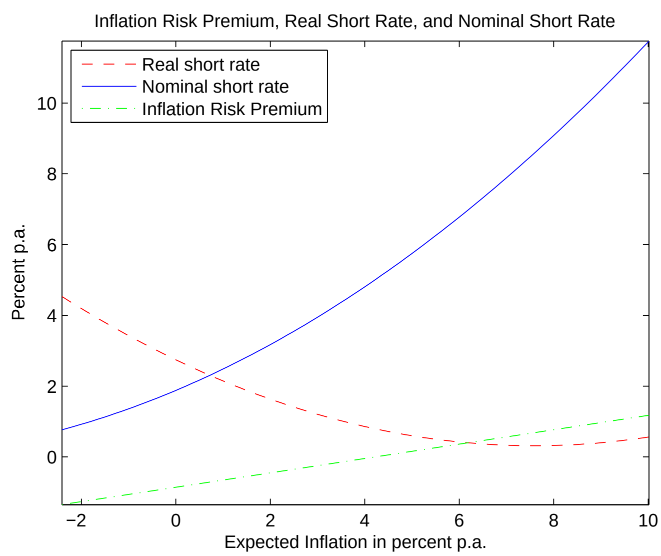

Figure III.1: The Real and Nominal Short Rate

The figure shows the real risk-free rate $r(x)$ (dashed line), the nominal risk-free rate $r^*(x)$ (solid line), and the inflation risk premium $-\sigma_\pi'\Lambda(x)$ (chain dotted line) as a function of the expected inflation rate $\mu_\pi(x)$. The domain is $\mu_{\pi 0} \pm 2\mu_{\pi x}/\sqrt{2\kappa}$.

the expected inflation rate until it hits its minimum of 2.6 bp and after that it is increasing with the expected inflation rate. Moreover, changes in both the nominal and real risk-free rate occur at an increasing rate. The inflation risk premium is linearly increasing in the expected inflation rate. Moreover, it is negative for low values and positive for high values of the expected inflation rate.

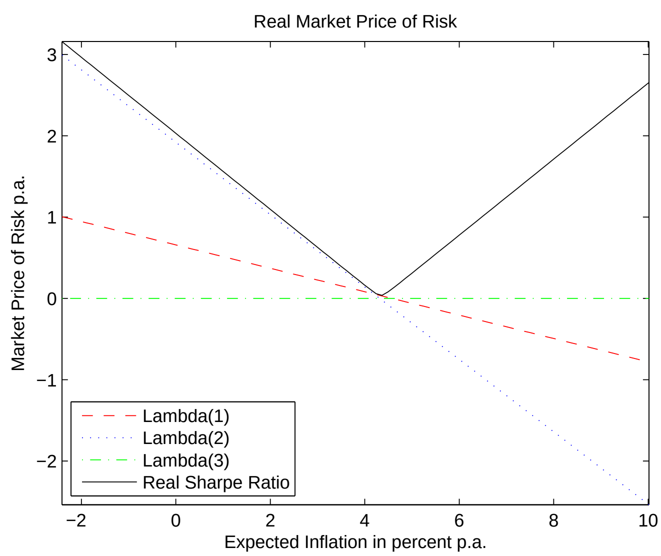

Figure III.2: The Real Market Price of Risk and the Maximal Sharpe Ratio
The figure shows the market price of factor risk (dashed line), the market price of residual stock market risk (dotted line), the market price of residual inflation risk (chain dotted line), and the maximal Sharpe ratio of real local asset returns (solid line) as a function of the expected inflation rate $\mu_\pi(x)$. The domain is $\mu_{\pi 0} \pm 2\mu_{\pi x}/\sqrt{2\kappa}$.

Figure III.2 presents the real market price of risk of all three innovations and the maximal Sharpe ratio of real local asset returns as a function of the expected inflation rate. The square of the maximal Sharpe ratio in the economy $(\Lambda(x)'\Lambda(x))$ is a strictly positive quadratic function of the expected inflation rate that is bounded below by 1 bp. Both the market price of factor risk Z_1 and the market price of residual stock market risk Z_2 are linearly decreasing in the expected inflation rate. Moreover, the maximal Sharpe ratio is larger the more expected inflation differs from its long run mean.

zero.

Figure III.3 presents the risk premia and volatilities of inflation-protected bonds and nominal bonds for different maturities as well as the risk premium of real stock returns and the sensitivity of marginal values of wealth with respect to the factor as a function of the expected inflation rate. Nominal returns of nominal bonds are locally perfectly negatively correlated with expected inflation while real returns of inflation-protected bonds are locally perfectly positively correlated with expected inflation. However, the local volatilities of inflation-protected bond and nominal bond returns are decreasing in the expected inflation rate. The real risk premium of inflation-protected bonds is strictly decreasing with the expected inflation rate for almost the whole domain. On the other hand, the nominal risk premium of nominal bonds is strictly increasing with the expected inflation rate. The real risk premium for real stock returns is strictly decreasing with the expected inflation rate. This is consistent with Fama and Schwert (1977) who show that stock returns are negatively related to expected inflation. Moreover, if the expected inflation rate exceeds 4.4% per annum, then the real risk premium for equity becomes negative.

Finally, the sensitivity of the marginal value of wealth with respect to changes in the investment opportunity set given by $h_x(T-t)+2h_{xx}(T-t)x$ is a strictly decreasing function of the expected inflation rate. Moreover, it starts out to be positive and becomes negative for high realizations of the expected inflation rate.

III.4 Dynamic Portfolio Strategies

In this section I combine the theoretical results of Section III.1 and III.2 with the estimation results of Section III.3 and discuss the implications of changes in expected inflation for optimal portfolio choice. In particular, I analyze dynamic portfolio strategies of investors with different expectations about future inflation, different investment horizons, different risk aversion, and different maturities of nominal and inflation-protected bonds.

I have shown in Section III.2 that the optimal portfolio is a linear combination of the mimicking portfolio for the real risk-free asset, the tangency portfolio, and a hedging portfolio. Specifically, the mimicking portfolio for the real risk free asset is fully invested

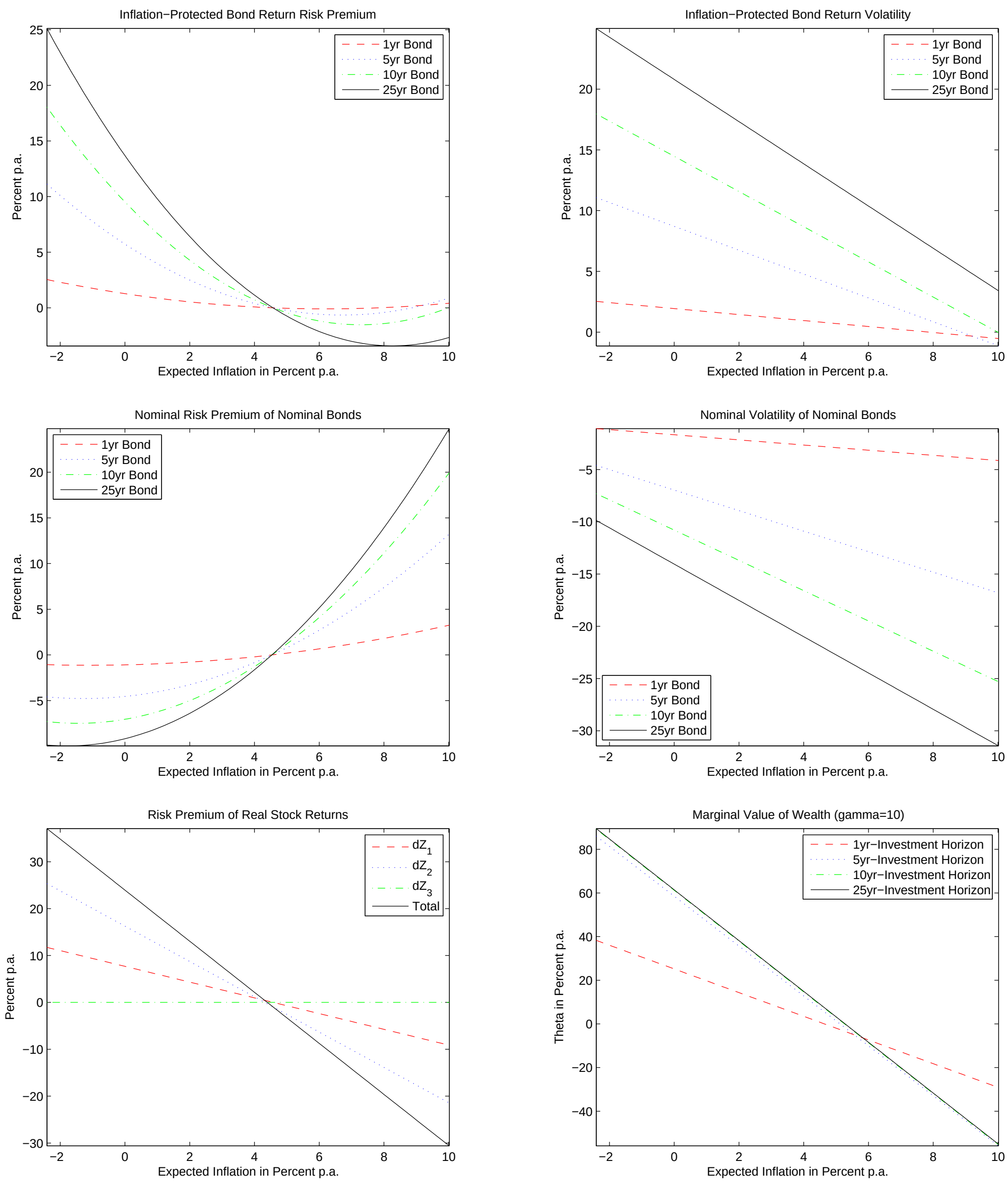

Figure III.3: Asset Returns and the Marginal Value of Wealth
The figure shows the real risk premium of inflation-protected bonds, nominal bonds, and equity as well as the local volatility of inflation-protected and nominal bonds as a function of the expected inflation rate. Moreover, it shows the sensitivity of the marginal value of wealth with respect to the factor as a function of the expected inflation rate. The domain is $\mu_{\pi 0} \pm 2\mu_{\pi x}/\sqrt{2\kappa}$.

in the inflation-protected bond, and the tangency portfolio is fully invested in stocks. Moreover, these portfolios and the hedging portfolio contain zero-investment portfolios of the nominal bond and cash.

Expected inflation affects optimal portfolio strategies in two ways: (i) it affects the decision of investors as to how much of their wealth to allocate to the risk-free asset (the inflation-protected bond) and to the tangency portfolio (stocks) and (ii) it affects the decision of investors as to how much cash to borrow or lend in order to finance the investment in the nominal bond. The first decision is much simpler than the second because it only depends on the risk aversion of the investor and the expected inflation rate while the second decision also depends on the investment horizon and the maturity of the nominal and the inflation protected bond.

Figure III.4 plots the optimal portfolio allocations as a function of the expected inflation rate when the investment horizon coincides with the maturity of both the nominal and inflation-protected bond. In the left panel, risk aversion $\gamma = 4$ and in the right panel, risk aversion $\gamma = 10$. Moreover, the investment horizon is one-year in the top panel, five years in the middle panel, and 25 years in the bottom panel. The optimal stock allocation and hence the optimal fraction of wealth invested in the tangency portfolio is decreasing in the expected inflation rate because the risk premium of real stock returns is negatively related to expected inflation. This negative relation follows from the positive local correlation of real stock returns with expected inflation and the fact that the market price of expected inflation risk (factor risk) and residual stock market risk is a decreasing function of the expected inflation rate (see Figure III.3 of the previous section). Not surprisingly, the optimal stock allocation is more sensitive to expected inflation when risk aversion is lower because less risk averse investors try to time the market more aggressively.

Table III.6 reports optimal portfolio allocations for (i) two different risk aversion coefficients, $\gamma = 4$ and $\gamma = 10$, (ii) four different investment horizons, $T = 1$, $T = 5$, $T = 10$, and $T = 25$, (iii) three different values for expected inflation, $\mu_\pi(x) = 1\%$ p.a., $\mu_\pi(x) = 4\%$ p.a., and $\mu_\pi(x) = 7\%$ p.a., and (iv) four different nominal bond maturities, $T_B = 1$, $T_B = 5$,

$T_B = 10$, and $T_B = 25$. The maturity of the inflation-protected bond is ten, i.e. $T_P = 10$. An investor with risk aversion, $\gamma = 4$ should put 68% of his wealth in stocks, and an investor with risk aversion $\gamma = 10$ should put 27% of his wealth in stocks if the expected inflation rate is equal to its long run mean of 4% per annum. The optimal stock allocation is much higher if the expected inflation rate is one standard deviation lower than its long run mean; i.e. if it is 1% per annum, because in this case the risk-premium for stocks is high. Specifically, investors with risk aversion $\gamma = 4$ should put 476% of their wealth in stocks and investors with risk aversion $\gamma = 10$ should put 190% of their wealth in stocks. On the other hand, if the expected inflation rate is one standard deviation higher than its long run mean; i.e. if it is 7% per annum, then investors with risk aversion $\gamma = 4$ should short 341% of their wealth in stocks and investors with risk aversion $\gamma = 10$ should short 136% of their wealth in stocks, because in this case the equity premium is negative. The huge long (short) positions in stocks if expected inflation is low (high) is financed by borrowing (lending) in the inflation-protected bond.

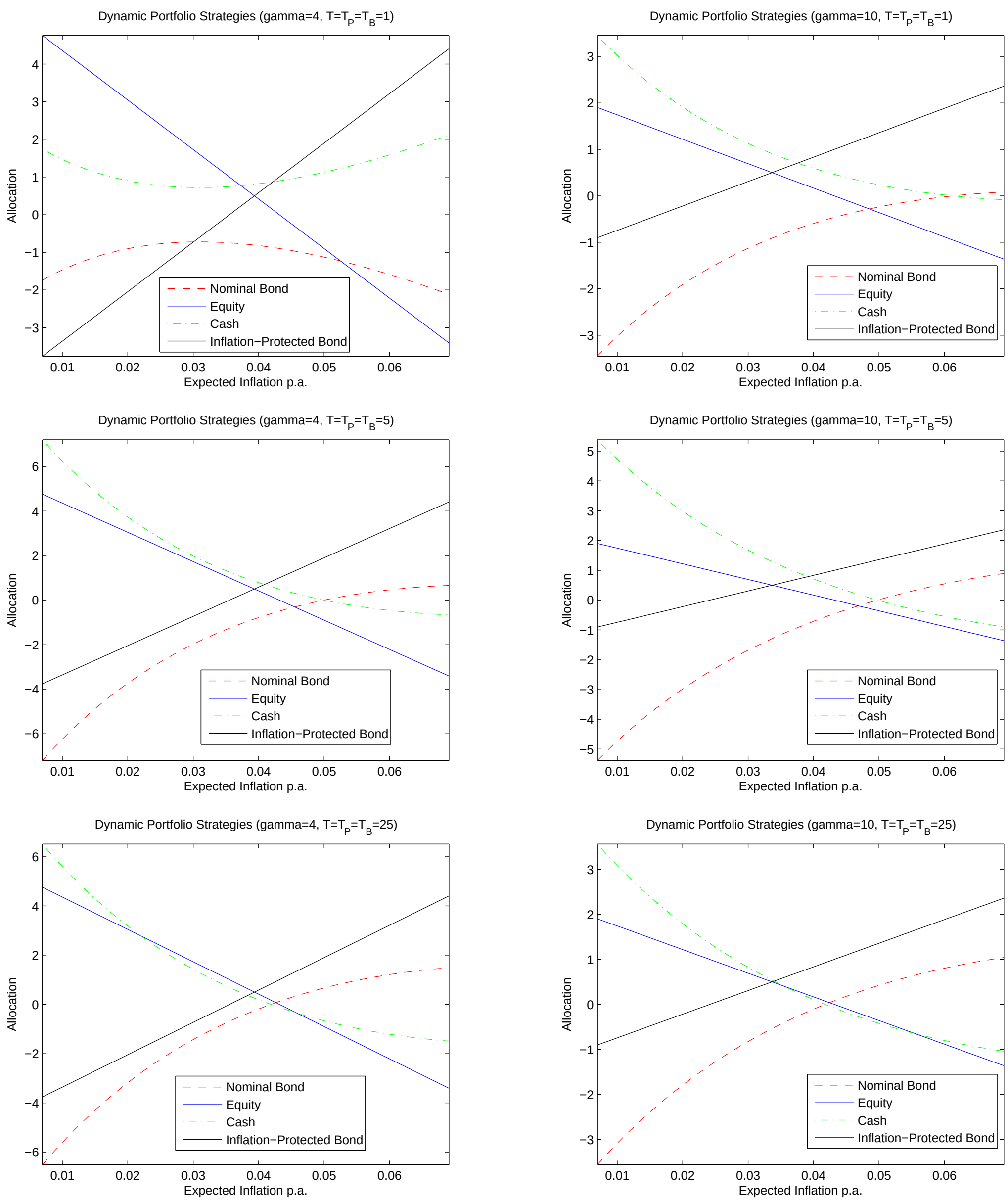

Figure III.4: Dynamic Portfolio Strategies

Shown are the optimal allocations to the nominal money market account, a nominal bond, stocks and a inflation-protected bond as a function of the expected inflation rate when the investment horizon coincides with the maturity of both the nominal and inflation-protected bond. In the left panel, risk aversion $\gamma = 4$ and in the right panel, risk aversion $\gamma = 10$. The investment horizon is one-year in the top panel, five years in the middle panel, and 25 years in the bottom panel.

Table III.6: Dynamic Portfolio Strategies

This table reports optimal portfolio allocations for (i) two different risk aversion coefficients, $\gamma = 4$ and $\gamma = 10$, (ii) four different investment horizons, $T = 1$, $T = 5$, $T = 10$, and $T = 25$, (iii) three different values for expected inflation, $\mu_\pi(x) = 1\%$ p.a., $\mu_\pi(x) = 4\%$ p.a., and $\mu_\pi(x) = 7\%$ p.a., and (iv) four different nominal bond maturities, $T_B = 1$, $T_B = 5$, $T_B = 10$, and $T_B = 25$. The maturity of the inflation-protected bond is ten, i.e. $T_P = 10$. Investment horizon and bond maturities are measured in years. The allocation in the inflation-protected bond is one minus the allocation in equity. The allocation in the nominal money market account is minus the allocation in the nominal bond (see Theorem III.1).

		Risk aversion							
		$\gamma = 4$				$\gamma = 10$			
Expected inflation	Asset	Investment Horizon				Investment Horizon			
		$T = 1$	$T = 5$	$T = 10$	$T = 25$	$T = 1$	$T = 5$	$T = 10$	$T = 25$
7% p.a.	Equity	−3.41	−3.41	−3.41	−3.41	−1.36	−1.36	−1.36	−1.36
	1-yr Nominal Bond	3.41	6.03	5.87	5.86	3.05	5.45	5.08	5.02
	5-yr Nominal Bond	0.84	1.48	1.44	1.44	0.75	1.34	1.25	1.23
	10-yr Nominal Bond	0.55	0.98	0.95	0.95	0.49	0.88	0.82	0.81
	25-yr Nominal Bond	0.44	0.78	0.76	0.76	0.39	0.71	0.66	0.65
4% p.a.	Equity	0.68	0.68	0.68	0.68	0.27	0.27	0.27	0.27
	1-yr Nominal Bond	0.21	−3.54	−3.80	−3.81	1.55	−2.49	−3.22	−3.30
	5-yr Nominal Bond	0.05	−0.86	−0.92	−0.93	0.38	−0.61	−0.78	−0.80
	10-yr Nominal Bond	0.03	−0.57	−0.61	−0.61	0.25	−0.40	−0.51	−0.53
	25-yr Nominal Bond	0.03	−0.45	−0.48	−0.48	0.20	−0.31	−0.41	−0.42
1% p.a.	Equity	4.76	4.76	4.76	4.76	1.90	1.90	1.90	1.90
	1-yr Nominal Bond	−25.66	−41.07	−41.50	−41.51	−9.20	−25.05	−26.44	−26.56
	5-yr Nominal Bond	−6.17	−9.88	−9.98	−9.98	−2.21	−6.02	−6.36	−6.39
	10-yr Nominal Bond	−4.00	−6.40	−6.47	−6.47	−1.43	−3.90	−4.12	−4.14
	25-yr Nominal Bond	−3.09	−4.95	−5.00	−5.00	−1.11	−3.02	−3.19	−3.20

Table III.6 shows that the optimal nominal bond allocation is quite sensitive to risk aversion, the investment horizon, and the expected inflation rate. Moreover, it also depends on which nominal and inflation-protected bond is hold in the investment portfolio. For instance, an investor with risk aversion $\gamma = 10$ and a five year investment horizon who can invest in equity, the nominal money market account, a ten year nominal bond, and a ten year inflation-protected bond should (i) should short 390% of his wealth in the nominal bond if expected inflation is 1% per annum, (ii) short 40% of his wealth in the nominal bond if expected inflation is 4% per annum, and (iii) should put 88% of his wealth in the nominal bond if expected inflation is 7% per annum.

Figure III.4 shows that the optimal nominal bond allocation is strictly increasing with expected inflation (except for the top left panel). Moreover, the increase is more pronounced for investors with risk aversion $\gamma = 10$. The reason for the apparent positive relation between expected inflation and the optimal nominal bond allocation is not immediately obvious because the allocation is composed of three parts: (i) the allocation to the nominal bond in the mimicking portfolio for the real risk-free asset, (ii) the allocation to the nominal bond in the tangency portfolio, and (iii) the allocation to the nominal bond in the hedging portfolio. Table III.7 reports the optimal nominal bond allocations in each of these three categories for (i) two different risk aversion coefficients, $\gamma = 4$ and $\gamma = 10$, (ii) four different investment horizons, $T = 1$, $T = 5$, $T = 10$, and $T = 25$, (iii) three different values for expected inflation, $\mu_\pi(x) = 1\%$ p.a., $\mu_\pi(x) = 4\%$ p.a., and $\mu_\pi(x) = 7\%$ p.a., and (iv) two different nominal bond maturities, $T_B = 1$ and $T_B = 10$. The maturity of the inflation-protected bond is ten, i.e. $T_P = 10$.

Table III.7: Nominal Bond Allocation

This table reports the optimal nominal bond allocations in (i) the mimicking portfolio for the real risk-free asset (RFA), (ii) the tangency portfolio (TP), and (iii) the hedging portfolio (HP) for (i) two different risk aversion coefficients, $\gamma = 4$ and $\gamma = 10$, (ii) four different investment horizons, $T = 1$, $T = 5$, $T = 10$, and $T = 25$, (iii) three different values for expected inflation, $\mu_\pi(x) = 1\%$ p.a., $\mu_\pi(x) = 4\%$ p.a., and $\mu_\pi(x) = 7\%$ p.a., and (iv) two different nominal bond maturities, $T_B = 1$ and $T_B = 10$. The maturity of the inflation-protected bond is ten, i.e. $T_P = 10$.

			Risk aversion							
			$\gamma = 4$				$\gamma = 10$			
Expected inflation	Nominal Bond	Portfolio	Investment Horizon				Investment Horizon			
			$T = 1$	$T = 5$	$T = 10$	$T = 25$	$T = 1$	$T = 5$	$T = 10$	$T = 25$
7% p.a.	$T_B = 1$	RFA	5.83	5.83	5.83	5.83	3.13	3.13	3.13	3.13
		TP	−9.32	−9.32	−9.32	−9.32	−3.73	−3.73	−3.73	−3.73
		HP	6.90	9.52	9.35	9.35	3.65	6.05	5.68	5.62
	$T_B = 10$	RFA	0.95	0.95	0.95	0.95	0.51	0.51	0.51	0.51
		TP	−1.51	−1.51	−1.51	−1.51	−0.60	−0.60	−0.60	−0.60
		HP	1.12	1.54	1.52	1.51	0.59	0.98	0.92	0.91
4% p.a.	$T_B = 1$	RFA	1.11	1.11	1.11	1.11	2.51	2.51	2.51	2.51
		TP	1.97	1.97	1.97	1.97	0.79	0.79	0.79	0.79
		HP	−2.87	−6.62	−6.88	−6.89	−1.75	−5.79	−6.52	−6.60
	$T_B = 10$	RFA	0.18	0.18	0.18	0.18	0.40	0.40	0.40	0.40
		TP	0.31	0.31	0.31	0.31	0.13	0.13	0.13	0.13
		HP	−0.46	−1.06	−1.10	−1.10	−0.28	−0.93	−1.04	−1.05
1% p.a.	$T_B = 1$	RFA	−27.52	−27.52	−27.52	−27.52	−6.62	−6.62	−6.62	−6.62
		TP	22.65	22.65	22.65	22.65	9.06	9.06	9.06	9.06
		HP	−20.79	−36.20	−36.63	−36.65	−11.65	−27.49	−28.89	−29.01
	$T_B = 10$	RFA	−4.29	−4.29	−4.29	−4.29	−1.03	−1.03	−1.03	−1.03
		TP	3.53	3.53	3.53	3.53	1.41	1.41	1.41	1.41
		HP	−3.24	−5.64	−5.71	−5.71	−1.81	−4.28	−4.50	−4.52

Real Risk-Free Asset

Intuitively, if investors are long (short) the real risk-free asset, then they should allocate a positive (negative) amount to the nominal bond in the mimicking portfolio for the real risk-free asset because real returns of inflation-protected bonds are locally perfectly positively correlated with expected inflation while real returns of nominal bonds are negatively correlated with expected inflation.

Hedging Demand

Similarly, the allocation to nominal bonds in the hedging portfolio is negative if expected inflation is low and positive if expected inflation is high because of the negative local correlation of nominal bonds with expected inflation and the fact that the sensitivity of the marginal value of wealth to changes in the investment opportunity set is positive if expected inflation is low and negative if it is high (see Figure III.3)). The intuition for the nominal bond allocation in the mimicking portfolios for the real risk-free asset and the hedging portfolio is confirmed by the quantitative results reported in Table III.7.

Tangency Portfolio

On the other hand, the allocation to nominal bonds in the tangency portfolio is negative for high values of the expected inflation rate and positive for low values. This seems counterintuitive because nominal bonds are cheap when the expected inflation rate is high and are expected to increase in value when expected inflation reverts back to its long run mean. The reason for the results is the large exposure of real stock returns to expected inflation risk. Specifically, assume that real stock returns are locally uncorrelated with expected inflation. In this case the nominal bond allocation would increase with the expected inflation rate, as expected, because the risk premium for nominal bonds is increasing with the expected inflation rate. However, the high local correlation of real stock returns with expected inflation and the fact that the state dependent component of the market price of residual stock market risk ($\lambda_{x2} = -11\%$) is significantly more negative than the state dependent component of the market price of expected inflation risk ($\lambda_{x1} = -4\%$) implies that the exposure of equity to expected inflation risk in the tangency portfolio is to high. Hence, the nominal bond is actually used to reduce this exposure which leads to the counterintuitive result that the allocation to the nominal bond in the tangency portfolio goes down when expected inflation goes up.

CHAPTER IV

THE TERM STRUCTURE OF INTEREST RATES WITH HETEROGENEOUS HABIT FORMING PREFERENCES

IV.1 The Economy

Consider a finite horizon $[0, T]$ pure exchange economy with a single perishable consumption good. Let $c(t)$ denote the aggregate consumption or aggregate endowment process, $\mu_c(t)$ its expected growth rate, and $\sigma_c(t)$ its volatility.[1] The dynamics of aggregate consumption are

$$\frac{dc(t)}{c(t)} = \mu_c(t)\,dt + \sigma_c(t)\,dz_c(t), \tag{IV.1}$$

in which $z_c(t)$ denotes a one-dimensional Brownian motion.

Real prices are measured in units of the consumption good. Let $\pi(t)$ denote the nominal price level, $\mu_\pi(t)$ the expected inflation rate, and $\sigma_\pi(t)$ the volatility of inflation. The dynamics of the price level are

$$\frac{d\pi(t)}{\pi(t)} = \mu_\pi(t)\,dt + \sigma_\pi(t)\,dz_\pi(t), \tag{IV.2}$$

in which $z_\pi(t)$ denotes a one-dimensional Brownian motion that may be correlated with aggregate consumption; i.e. $dz_c(t)dz_\pi(t) = \rho_{\pi c}(t)\,dt$.[2]

Security Market

Consider a frictionless complete security market that operates continuously during the time interval $[0, T]$. Let $r(t)$ denote the real risk-free rate or real short rate and $R(t)$ the nominal risk-free rate or nominal short rate. All bonds considered in this paper are default-free zero-coupon bonds. A real bond pays one unit of the consumption good at its maturity

[1]Bansal and Yaron (2004) consider a model in which aggregate consumption has a persistent time varying expected growth rate and volatility.

[2]Real consumption and inflation are contemporaneously negatively correlated in the U.S. (e.g. Brandt and Wang (2003)).

and a nominal bond pays one unit of currency at its maturity. The market portfolio is in positive net supply normalized to one and the nominal money market account, real bonds, and nominal bonds are in zero net supply.

Preferences

There are two (classes of) agents in the economy that derive all their wealth from investing in the security market and continuously consume part of the aggregate endowment. All agents have the same beliefs but may differ with respect to their (strictly positive) initial wealth and their preferences.[3] Specifically, each agent has habit forming preferences given by

$$U_i(\{c_i(t)\}_{0\leq t\leq T}) = \mathrm{E}\left[\int_0^T e^{-\beta t} u_i(c_i(t), X(t))\, dt\right], \qquad i = 1, 2, \tag{IV.3}$$

in which β denotes the common subjective discount factor and $X(t)$ denotes an exogenous state variable that captures the path dependence of each agent's preferences and is defined below.

Each investor has non-addictive (multiplicative) habit forming preferences.[4] Specifically,

$$u_i(c_i(t), X(t)) = \begin{cases} \frac{1}{1-\gamma_i}(c_i(t)/X(t))^{1-\gamma_i} & c_i > 0 \\ -\infty & \text{otherwise.} \end{cases} \tag{IV.4}$$

The external habit or "catching up with the Joneses" feature of agent's preferences implies that a higher standard of leaving has a complementary effect on current consumption because an increase in $X(t)$ raises marginal utility of consumption. Hence, the relative risk aversion coefficient of each agent has to be greater or equal than one ($\gamma_i \geq 1$) to guarantee that the first derivative of marginal utility with respect to the standard of living

[3]Initial wealth is treated as a free parameter.

[4]There are two different definitions of habit forming preferences in the literature: the additive or addictive preferences considered by Campbell and Cochrane (1999) or the non-addictive or multiplicative habit forming preferences considered by Abel (1990), Abel (1999), or Chan and Kogan (2002). The name non-addictive habit forming preferences is adopted from Detemple and Zapatero (1991) who provide a general discussion of equilibrium outcomes in exchange economies with agents that have habit forming preferences. Campbell, Lo, and MacKinlay (1996) call non-addictive models ratio models and addictive models difference models.

is non-negative.[5]

I follow Chan and Kogan (2002) and assume that the standard of living process $X(t)$ is a weighted "geometric sum" of past realizations of aggregate consumption.[6]

$$\log(X(t)) = \log(X(0))e^{-\delta t} + \delta \int_0^t e^{-\delta(t-a)} \log(c(a))\, da \qquad \delta > 0. \tag{IV.5}$$

Define a new state variable, relative (log) consumption, as $\omega(t) = \log(c(t)/X(t))$.[7] Since $\omega(t)$ is high in good states of the economy and low in bad state of the economy it can be interpreted as a business cycle variable. Moreover, it is straightforward to verify that $\omega(t)$ follows a mean reverting process. Specifically,

$$d\omega(t) = \delta(\bar{\omega}(t) - \omega(t))\, dt + \sigma_c(t)\, dz_c(t) \tag{IV.6}$$

with $\bar{\omega}(t) = (\mu_c(t) - \sigma_c(t)^2/2)/\delta$. The economy will be said to be in a recession if relative consumption is lower than its long run mean ($\bar{\omega}(t) > \omega(t)$) and to be in an expansion if relative consumption is greater than its long run mean ($\omega(t) > \bar{\omega}(t)$).[8]

The parameter δ describes the dependence of $X(t)$ on the history of aggregate consumption. If δ is large, then shocks to relative consumption are transitory and hence the standard of living process resembles closely current consumption; i.e. $\omega(t) \approx 0$. On the other hand, if $\delta \approx 0$, then shocks to relative consumption are persistent and hence past aggregate consumption is heavily weighted by the standard of living process.

[5]Specifically,

$$\frac{\partial^2 u_i(c_i, X)}{\partial c_i \partial X} = (\gamma_i - 1)(c_i/X)^{-\gamma_i} \geq 0.$$

[6]If $\delta = 0$, then the standard of living process is constant and the pricing results are the same as with standard additive preferences. Sundaresan (1989) and Constantinides (1990) also define the standard of living process as a geometric sum of past realizations of aggregate consumption. They consider a slightly more general specification of the standard of living process.

[7]The dynamics of $\log(X(t))$ are $d\log(X(t)) = \delta\omega(t)\, dt$.

[8]Relative consumption is in general not a Markov process.

IV.2 Equilibrium

In this section I derive closed form solutions for the nominal stochastic discount factor, discuss the properties for the nominal short rate and the nominal market price of risk, and provide explicit formulas for nominal bond prices, real bond prices, and the term structure of inflation risk premia. As standard in the literature the equilibrium is determined in three steps: (i) the optimal consumption sharing rule is determined, (ii) each efficient allocation is characterized by a stochastic discount factor process, and (iii) it is shown that the efficient allocations can be achieved by continuously trading in the security market.[9]

The efficient allocations are determined by maximizing the utility function of a representative agent subject to the resource constraint that aggregate consumption does not exceed aggregate endowment. Let $\eta(t)$ denote the shadow price of the resource constraint and $\kappa \in (0,1)$ the social weight. The optimization problem can be solved state by state and hence the value function of the representative agent at each point in time and each state of the world is[10]

$$u^{\kappa}(c,X) = \inf_{\eta}\left(\sup_{\{c_1,c_2\}}\{\kappa u_1(c_1,X) + (1-\kappa)u_2(c_2,X) - \eta(c_1+c_2-c)\}\right). \tag{IV.7}$$

The market is complete and hence the social weight κ is constant. The social weight κ can be uniquely determined form the initial wealth of agents and hence is treated as a free parameter.

Let $\gamma_1 \geq 1$ denote the constant risk aversion coefficient of the first agent, γ_2 the constant risk aversion of the second agent, and assume that the second agent is twice as risk averse as the first agent; i.e. $\gamma_2 = 2\gamma_1$. Moreover, let $\gamma(t)$ denote the aggregate risk aversion of the economy, $\gamma_\omega(t)$ the first derivative, and $\gamma_{\omega\omega}(t)$ the second derivative of $\gamma(t)$

[9]A detailed discussion of steps (i)-(iii) is provided in Section $C.1$ of Appendix C. Wang (1996) conducts a similar equilibrium analysis for two investors with heterogeneous standard time additive preferences and Chan and Kogan (2002) conduct a similar equilibrium analysis for a continuum of investors with heterogeneous habit forming preferences.

[10]I suppress time dependence of all processes for notional convenience.

with respective to relative consumption $\omega(t)$.[11] Aggregate risk aversion and its properties are summarized in the next proposition.

Proposition IV.1 (Aggregate Risk Aversion)**.** *Aggregate risk aversion is*

$$\gamma(t) = \gamma_1 \left(1 + \frac{1}{\sqrt{1 + Ke^{\omega(t)}}}\right) \qquad \text{with} \quad K = 4\left(\frac{\kappa}{1-\kappa}\right)^{\frac{1}{\gamma_1}}. \tag{IV.8}$$

The dynamics of $\gamma(t)$ are

$$d\gamma(t) = \gamma_\omega(t)\, d\omega(t) + \frac{1}{2}\gamma_{\omega\omega}(t)\, (d\omega(t))^2, \tag{IV.9}$$

in which $\gamma_\omega(t) < 0$. Moreover, $\lim_{\kappa\to 1}\gamma(t) = \gamma_1$, $\lim_{\kappa\to 0}\gamma(t) = \gamma_2$, *and* $\gamma_1 \le \gamma(t) \le \gamma_2$.

Proof. See Section C.1 of Appendix C. □

Aggregate risk aversion is countercyclical. In other words, a negative shock to real consumption growth increases the risk aversion of the economy. This results from endogenous changes in the cross sectional distribution of wealth. Specifically, the less risk averse investor is more exposed to aggregate consumption risk because he invests a larger portion of his wealth in risky assets. Hence, a negative shock to aggregate consumption increases aggregate risk aversion because it has a larger negative impact on the wealth of the less risk averse agent.

Let $m(t)$ denote the real stochastic discount factor, $M(t) = m(t)/\pi(t)$ the nominal stochastic discount factor, and $c^*(t) = c(t)\pi(t)$ the nominal price of aggregate consumption. Moreover, let $B_U(t)$ denote the real price and $B_U^*(t) = \pi(t)B_U(t)$ the nominal price of a nominal bond maturing at U.[12] Nominal bond prices are given in the next theorem.

[11] Aggregate risk aversion is equal to the local curvature of the representative investor. Specifically,

$$\gamma(t) = -c(t)\frac{u^\kappa_{cc}(c(t), X(t))}{u^\kappa_c(c(t), X(t))},$$

in which $u^\kappa_c(\cdot)$ denotes the first derivative and $u^\kappa_{cc}(\cdot)$ denotes the second derivative of $u^\kappa(\cdot)$ with respect to aggregate consumption $c(t)$.

[12] The real discount factor $m(t)$ is determined in Lemma C.2 in Section C.1 of Appendix C.

Theorem IV.1 (Nominal bond prices)**.** *The nominal price of a nominal bond maturing at U is*

$$B^*_U(t) = \mathrm{E}_t\left[M(U)/M(t)\right], \qquad \forall\, t \leq U < T \tag{IV.10}$$

with

$$M(t) = e^{-\beta t}\,\frac{\xi(t)}{c^*(t)}\,\frac{c^*(0)}{\xi(0)} \tag{IV.11}$$

and

$$\xi(t) = \frac{2^{\gamma_2}\kappa^2}{1-\kappa}\,\frac{e^{\omega(t)}}{\left(\sqrt{1+\Lambda e^{\omega(t)}}-1\right)^{\gamma_2}}. \tag{IV.12}$$

Proof. See Section C.1 of Appendix C. □

There are no closed form solutions for nominal bond prices but the the nominal stochastic discount factor is given in closed form and hence it is straightforward to calculate nominal bond prices numerically using Monte Carlo simulation techniques.[13]

The effects of heterogenous habit forming preferences on the nominal stochastic discount factor are discussed below. Let $\lambda(t)$ denote the real market price of risk and $\Lambda(t)$ the nominal market price of risk. The dynamics of the nominal stochastic discount factor $M(t)$ given in equation (IV.11) are provided in the next corollary.[14]

Corollary IV.1 (Nominal discount factor dynamics)**.** *The dynamics of $M(t)$ are*

$$\frac{dM(t)}{M(t)} = -R(t)\,dt - \Lambda(t)\,dz_M(t), \tag{IV.13}$$

in which $z_M(t)$ denotes a one-dimensional Brownian motion that satisfies $\Lambda(t)dz_M(t) = \lambda(t)dz_c(t) + \pi(t)dz_\pi(t)$.

[13]See Glasserman (2004) for Monte Carlo simulation techniques.

[14]Chan and Kogan (2002) provide similar expressions for the real market price of risk and the real risk-free rate. Moreover, they perform an asymptotic analysis of the equilibrium (around the "log-investor") and discuss the properties of the Sharpe ratio, the volatility, and excess returns of the stock market. The reader is referred to their working paper (Chan and Kogan (2000)) for details about the asymptotic analysis.

The nominal short rate and the nominal market price of risk are

$$R(t) = r(t) + \mu_\pi(t) - \rho_{\pi c}(t)\lambda(t)\sigma_\pi(t) - \sigma_\pi(t)^2 \tag{IV.14}$$

$$\Lambda(t) = \sqrt{\lambda(t)^2 + 2\rho_{\pi c}(t)\lambda(t)\sigma_\pi(t) + \sigma_\pi(t)^2}. \tag{IV.15}$$

The real short rate and the real market price of risk are

$$r(t) = \beta + \delta\left(\bar{\omega}(t) + (\gamma(t) - 1)(\bar{\omega}(t) - \omega(t))\right) - \frac{1}{2}\left(\gamma(t)^2 - \gamma_\omega(t)\right)\sigma_c(t)^2 \tag{IV.16}$$

$$\lambda(t) = \sigma_c(t)\gamma(t). \tag{IV.17}$$

Proof. See Section C.1 of Appendix C. □

Real short rate

In equilibrium real interest rates are determined such that investors in aggregate are indifferent between consuming today and consuming in the future. Hence, the real short rate depends on the willingness of the representative investor to substitute consumption over time and states, and it depends on the conditional distribution of aggregate consumption which is locally uniquely determined by its mean and variance because aggregate consumption follows a diffusion process.[15] The local curvature of the representative investor's utility function $\gamma(t)$ captures both the willingness to substitute consumption over time and states.[16]

The real short rate consists of three components: the impatience parameter β, the intertemporal substitution component $\delta\left(\bar{\omega}(t) + (\gamma(t) - 1)(\bar{\omega}(t) - \omega(t))\right)$, and the precautionary savings component $(\gamma(t)^2 - \gamma_\omega(t))\sigma_c(t)^2/2$. The impatience parameter β, the ex-

[15]Cochrane (2005) discusses the case of homogeneous standard time additive CRRA preferences and i.i.d. consumption growth.

[16]This link is broken with Epstein-Zin preferences (Epstein and Zin (1989), Duffie and Epstein (1992a), Duffie and Epstein (1992b))), but it is hard to solve for equilibria in heterogenous economies with Epstein-Zin preferences. See Dumas, Uppal, and Wang (2000) for a recent contribution in this direction. There are a lot of other preferences that allow one to break the link between intertemporal substitution and risk aversion see Backus, Routledge, and Zin (2004) for an overview.

pected consumption growth rate $\mu_c(t)$ (through the long run mean of relative consumption $\bar{\omega}(t)$), and the variance of consumption growth have the standard interpretation for the magnitude of the risk-free rate.

The substitution effect is related to the business cycle of the economy. If the economy is in a recession ($\bar{\omega}(t) > \omega(t)$), then investors are eager to consume today, and hence a higher interest rate is required to convince them to save. Similarly, if the economy is in an expansion ($\bar{\omega}(t) < \omega(t)$), then investors are willing to give up consumption today and hence a low interest rate is required to prevent them from saving. Both affects are more pronounced if aggregate risk aversion is very different from one ($\gamma(t) \gg 1$) and the standard of living process tracks closely most recent aggregate consumption realizations ($\delta \gg 0$). The former follows from the fact that investors care more about their habits if $\gamma(t) \gg 1$ and the latter follows from the fact that the standard of living grows faster than aggregate consumption in an expansion and decreases faster in a recession if δ is large.[17]

The precautionary savings component depends not only on the level of aggregate risk aversion but also on its changes with respect to relative consumption when investors differ with respect to the curvature of the utility function. Specifically, a negative shock to aggregate consumption raises the risk-free rate because aggregate risk aversion increases ($\gamma_\omega(t) < 0$). Intuitively, a negative shock to aggregate consumption leads to a redistribution of aggregate wealth from the less risk averse agent, who is a net borrower in equilibrium, to the more risk averse agent, who is a net lender in equilibrium and hence to avoid excess demand for the real risk-free asset the real short rate has to go down.

Nominal short rate

The nominal short rate $R(t)$ is equal to the sum of the real short rate $r(t)$, the expected inflation rate $\mu_\pi(t)$, an inflation risk premium $-\rho_{\pi c}(t)\lambda(t)\sigma_\pi(t)$, and a Jensen inequality term $-\sigma_\pi(t)$. The equilibrium behavior of the real short rate is discussed above, expected inflation is exogenously specified as part of the inflation dynamics that are given in equation

[17] $\delta\bar{\omega}(t) = \mu_c(t) - \sigma_c(t)^2/2$ does not depend on δ.

(IV.2), and the Jensen inequality adjustment arises because $\mathrm{E}[1/\pi] \neq 1/\mathrm{E}[\pi]$. Moreover, the Fisher equation for the nominal short rate does not hold unless inflation is uncorrelated with real consumption.

I determine the real rate of return on the nominal money market account to analyze the local inflation risk premium. The inflation risk premium for longer holding periods is defined and discussed below. Let $B_0(t)$ denote the real and $B_0^*(t) = B_0(t)\pi(t)$ the nominal value of the nominal money market account at time t. Specifically,

$$B_0^*(t) = B_0^*(0)e^{\int_0^t R(a)\, da}.$$

Applying Itô's lemma to $B_0(t)$ leads to the real return of the nominal money market account. Specifically,

$$\frac{dB_0(t)}{B_0(t)} = (r(t) - \rho_{\pi c}(t)\lambda(t)\sigma_\pi(t))\, dt - \sigma_\pi(t)\, dz_\pi(t). \qquad \text{(IV.18)}$$

Investing in the nominal money market account earns an expected real return in excess of the real risk-free rate – i.e. a compensation for inflation risk – if investors prefer assets that pay off when inflation is high over assets that pay of when inflation is low because real returns on the money market account are perfectly negatively correlated with inflation. Hence, the inflation risk premium is positive if inflation is negatively correlated with real consumption and positive if inflation is positively correlated with real consumption.[18]

Real and nominal market price of risk

Aggregate risk aversion and the volatility of aggregate consumption have the standard interpretation for the magnitude of the real market price of risk. Moreover, the market price of risk may be procyclical if the volatility of consumption is highly correlated with realized consumption. Otherwise, it is countercyclical because aggregate risk aversion is

[18]See Fischer (1975) for a discussion of the inflation risk premium when the real and nominal short rate are constant and real stock prices and the price level follow a geometric Brownian motion.

countercyclical. The nominal market price of risk depends on the real market price of risk, the volatility of inflation, and the covariance of inflation with real consumption. Moreover, it is decreasing in the covariance of real consumption with inflation.

Inflation risk premium

Let $P_U(t)$ denote the real and $P_U^*(t) = P_U(t)\pi(t)$ the nominal price of a real (inflation-protected) bond maturing at U, and $I_\tau(t)$ the inflation risk premium for non-negative holding periods τ. If $\tau = 0$, then the inflation risk premium is the difference between the expected real rate of return on the nominal money market account and the real short rate. Specifically, the instantaneous or local inflation risk premium is $I_0(t) = -\rho_{c\pi}(t)\lambda(t)\sigma_\pi(t)$.

If $\tau > 0$, then the inflation risk premium is the difference between the expected real (log) return of a nominal bond with τ years to maturity and a real bond with τ years to maturity. In other words, the inflation risk premium is the expected real (log) return difference of holding a nominal bond and a real bond until maturity.[19] Specifically,

$$I_\tau(t) = \log\left(P_{t+\tau}(t)/B^*_{t+\tau}(t)\right) - \mathrm{E}_t\left[\log(\pi(t+\tau)/\pi(t))\right]. \tag{IV.19}$$

To determine the inflation risk premium one has to know the price of both bonds and one has to form an expectation of future inflation rates.

The real price of a real bond is

$$P_U(t) = \mathrm{E}_t\left[m(U)/m(t)\right], \qquad \forall\, t \leq U < T. \tag{IV.20}$$

There are no closed form solutions for real bond prices but the real stochastic discount factor is given in closed form. Hence, it is straightforward to determine the nominal price of a nominal bond, the real price of a real bond, long run expected inflation and hence the inflation risk premium using Monte Carlo simulation techniques.

[19] The inflation risk premium is continuous in τ, i.e. $\lim_{\tau\to 0} I_\tau(t) = I_0(t)$.

Nominal bond yields

Let $y_B^{(\tau)}(t) = -\frac{1}{\tau}\log\left(B^*_{t+\tau}(t)\right)$ denote the log-yield at time t of a nominal zero-coupon bond with τ years to maturity and $y_P^{(\tau)}(t) = -\frac{1}{\tau}\log\left(P_{t+\tau}(t)\right)$ denote the log-yield at time t of a real zero-coupon bond with τ years to maturity. Every nominal bond yield is the sum of a real bond yield, an expected inflation rate, and an inflation risk premium. Specifically,

$$y_B^{(\tau)}(t) = y_P^{(\tau)}(t) + \frac{1}{\tau}\,\mathrm{E}_t\left[\log(\pi(t+\tau)/\pi(t))\right] + \frac{1}{\tau}\,I_\tau(t). \qquad \text{(IV.21)}$$

CHAPTER V

SUMMARY AND CONCLUSIONS

This dissertation consists of three essays. The title of the first essay is *"Idiosyncratic Inflation Risk and Inflation-Protected Bonds"*, the title of the second essay is *"Inflation and Asset Allocation"*, and the title of the last essays is *"The Term Structure of Interest Rates with Heterogeneous Habit Forming Preferences"*.

V.1 Idiosyncratic Inflation Risk and Inflation-Protected Bonds

In the first essay, I decompose inflation risk into (i) a part that is correlated with real returns on the market portfolio and factors that determine investor's preferences and investment opportunities and (ii) a residual part. I show that only the first part earns a risk premium. Therefore investors should seek to avoid exposure to the second part. All nominal Treasury bonds, including the nominal money-market account, are equally exposed to the residual part except inflation-protected Treasury bonds, which provide a means to hedge it. Every investor should put 100% of his wealth in the market portfolio and inflation-protected Treasury bonds and hold a zero-investment portfolio of nominal Treasury bonds and the nominal money market account.

V.2 Inflation and Asset Allocation

In the second essay, I solve the dynamic asset allocation problem of finite lived, constant relative risk averse investors who face inflation risk and can invest in cash, nominal bonds, equity, and inflation-protected bonds when the investment opportunity set is determined by the expected inflation rate. The instantaneous mean and variance of all asset returns are quadratic functions of the expected inflation rate, and optimal investment strategies are given in closed form. I estimate the model with nominal bond, inflation, and stock market data and find that the equity risk premium is negatively related to the expected inflation rate which is consistent with Fama and Schwert (1977). I show that if expected inflation increases, then investors should substitute inflation-protected bonds for stocks, and they should borrow cash to buy long-term nominal bonds. Moreover, they should buy (short) nominal bonds when expected inflation is high (low) in order to hedge changes in the investment opportunity set. The size of these positions is increasing in the investment horizon.

To discuss the effects of inflation risk on optimal portfolios I specified a one factor model in which the conditional joint distribution of real returns of stocks, the nominal money market account, and inflation-protected and nominal bonds depends on the latent expected inflation rate. While the model is able to fit the correlation of real stock returns and inflation and both the mean and standard deviation of real stock returns and inflation, it is unable to simultaneously fit the autocorrelation of real stock returns and inflation and the correlations between the nominal risk-free rate and real stock returns and inflation (see Table III.3). In particular, the model does not have enough flexibility to match both the autocorrelation of inflation and the correlation of inflation with the nominal risk-free rate.

Inflation has a persistent component and a transitory component. The model increases the persistent inflation component – by increasing $\mu_{\pi x}$ – and reduces the transitory inflation component – by decreasing $\sigma_{\pi 3}$ – to fit the high autocorrelation of inflation in the data. However, increasing the autocorrelation also increases the correlation of inflation with the nominal risk-free rate which is already too high compared to the data. The situation is similar, though less severe, with the correlation between real stock returns and the nominal risk-free rate.

If I underweight the autocorrelation of inflation in the objective function of the estimation, then the model would be able to fit all other statistics very well, and residual inflation risk increases from close to zero to almost 100%. I conclude that residual inflation risk is surprisingly small because the model increases the local correlation of realized inflation with expected inflation and real stock returns to better match the autocorrelation of inflation in the data.[1]

[1] I also regress inflation rates on changes in different nominal Treasury bond yields and real stock returns and find that residual inflation risk accounts for more than 90% of the total.

An increase in the dimension of the state vector may improve the fit of the model without losing the tractability of the portfolio choice problem (it would be still possible to obtain optimal portfolio demands in closed form). However, it would increase the complexity of the estimation problem and more importantly it would be very hard to distinguish the effects of inflation risk on optimal portfolio choice, which is the main focus of the paper, from other effects caused by the increase in the dimension of the state vector.

V.3 The Term Structure of Interest Rates with Heterogeneous Habit Forming Preferences

In the third essay, I derive closed form solution for the nominal stochastic discount factor in a pure continuous time exchange economy with a complete securities market. Aggregate consumption growth and inflation are exogenously specified and contain stochastic components that affect their mean and volatility. There are two classes of investors with external habit forming preferences and different local curvatures of their utility function. Aggregate risk aversion of the economy is countercyclical and the nominal short rate and the nominal market price of risk depend on a real business cycle variable, expected inflation, the volatility of inflation, and the local correlation of inflation with real consumption growth. I show that the inflation risk premium for the nominal money market account is positive (negative) if inflation is negatively (positively) correlated with real consumption growth. Moreover, I derive explicit formulas for nominal bonds and decompose the nominal bond yield in (i) a real bond yield, (ii) an expected inflation rate, and (iii) an inflation risk premium. Each component can be derived using Monte Carlo simulation techniques.

To empirically discuss the effects of heterogenous habit forming preferences on the nominal term structure of interest rate on needs to specify the dynamics of real consumption growth and inflation and estimate the parameters that governs the dynamics. Moreover, one needs to calibrate the model to asset price data to estimate the subjective discount factor β, the history dependence of the standard of living governed by the parameter δ, the risk aversion γ_1, and the social weight κ that determines the initial cross sectional wealth distribution of agents.

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APPENDIX A

IDIOSYNCRATIC INFLATION RISK AND INFLATION-PROTECTED BONDS

A.1 Asset Prices

Definition A.1 (D: Sufficiently smooth). *A function $f(t, X, S, \pi)$ is sufficiently smooth if it is continuously differentiable with respect to time t and all second partial derivatives with respect to X, S, and π exist and are continuous.*

Proof of Proposition II.1 (Inflation-protected bonds). The solution of the stochastic differential equation (II.3) is

$$M(T) = M(t)e^{-\int_t^T \left(r(X(a))+\frac{1}{2}\Lambda(X(a))'\Lambda(X(a))\right)\,da-\int_t^T \Lambda(X(a))'\,dZ(a)}. \tag{A.1}$$

The state vector X is Markov and therefore the conditional distribution of $M(T)/M(t)$ at time t only depends on the value of X at time t and time to maturity $T-t$. Hence, the real price of an inflation-protected bond at time t given by

$$P = \mathrm{E}_t\left[\frac{M(T)}{M(t)}\right] \tag{A.2}$$

depends only on X and $T-t$, i.e. $P = P(T-t, X)$.

$P(T-t, X)$ is sufficiently smooth and hence applying Itô's Lemma to $P(T-t, X)$ and using the continuous time pricing equation $\mathrm{E}[dP/P] - r\;dt = -\mathrm{E}[dP/P\;dM/M]$ for real assets leads to the local return dynamics in equation (II.9) with $\sigma_P(T-t, X)$ given in equation (II.10).

□

Proof of Proposition II.2 (Nominal bonds). The nominal value of a \$1 invested in the nominal money market account at time t is

$$R^* = e^{\int_0^t r^*(X(a))\, da} \tag{A.3}$$

and hence depends on the whole path of the nominal short rate. Specifically, $R^* = R^*(t, \{X(a), 0 \le a \le t\})$.

Applying Itô's Lemma to $R(t, \{X(a) \mid 0 \le a \le t\}, \pi) = R^*(t, \{X(a) \mid 0 \le a \le t\})/\pi$ and using equation (II.5) for the nominal short rate leads to the real return dynamics of R given in equation (II.11).

The solution of the stochastic differential equation (II.4) is

$$M^*(T) = M^*(t) e^{-\int_t^T \left(r^*(X(a)) + \frac{1}{2}\Lambda^*(X(a))'\Lambda^*(X(a))\right) da - \int_t^T \Lambda^*(X(a))'\, dZ(a)}, \tag{A.4}$$

in which $\Lambda^*(X) = \Lambda(X) + \sigma_\pi(X)$ denotes the nominal market price of risk. The state vector X is Markov and therefore the conditional distribution of $M^*(T)/M^*(t)$ at time t only depends on the value of X at time t and time to maturity $T - t$. Hence, the nominal price of a nominal bond at time t is given by

$$B^* = \mathrm{E}_t \left[\frac{M^*(T)}{M^*(t)} \right] \tag{A.5}$$

depends only on X and $T - t$, i.e. $B^* = B^*(T - t, X)$.

$B^*(T - t, X)$ is sufficiently smooth and hence applying Itô's Lemma to $B^*(T - t, X)$ and using the continuous time pricing equation $\mathrm{E}[dB^*/B^*] - r^*\, dt = -\mathrm{E}[dB^*/B^*\, dM^*/M^*]$ for nominal assets leads to the nominal return dynamics of nominal bonds. Specifically,

$$\frac{dB^*(T - t, X, \pi)}{B^*(T - t, X, \pi)} = \left(r^*(X) + \sigma_B^*(T - t, X)'(\Lambda(X) + \sigma_\pi(X))\right) dt + \sigma_B^*(T - t, X)'\, dZ \tag{A.6}$$

with

$$\sigma_B^*(T - t, X) = \frac{\sigma_X(X) B_X^*(T - t, X)}{B^*(T - t, X)}.$$

Applying Itô's Lemma to $B(T-t, X, \pi) = B^*(T-t, X)/\pi$ and using the continuous time pricing equation $\mathrm{E}[dB/B] - r\,dt = -\mathrm{E}[dB/B\; dM/M]$ for real assets leads to the real return dynamics of nominal bonds given in equation (II.12).

Moreover, the upper diagonal form of the volatility matrix $\sigma_X(X)$ (see equation $(II.8)$) implies that the last column of $\sigma_X(X)$ is zero and hence $\sigma_{Pk+2}(T-t, X) = 0$ and $\sigma_{Bk+2}(T-t, X) = -\sigma_{\pi k+2}(X)$.

$\square$

A.2 Equilibrium

In this section I prove Theorem II.1 and Theorem II.2. For $i = 1, \ldots, I$. Optimal consumption of investor i who maximizes (II.14) subject to the static budget constraint (II.15) has to satisfy the first order condition

$$u_c^i(t, c^i(t), X(t)) = \lambda^i M^i(t), \tag{A.7}$$

in which u_c^i denotes the partial derivative of u^i with respect to consumption and λ^i denotes the Lagrange multiplier for the budget constraint (II.15). Let U_W^i denote the partial derivative of U^i with respect to wealth. If the investment horizon is finite, then $M^i(t)$ has to satisfy the FOC, $U_W^i(T, W^i(T), X(T)) = \lambda^i M^i(T)$, and if the investment horizon is infinite, then it has to satisfy, $\lim\limits_{T\to\infty} M^i(T) = 0$.

Proof of Theorem II.1 (ICAPM). Consider a frictionless security market consisting of $N+1$ risky assets, such as stocks, inflation-protected and nominal corporate bonds, real estate, inflation-protected Treasury bonds, nominal Treasury bonds, a nominal money market account, etc. There are no tax liabilities and the nominal money market account and all Treasury bonds are in zero-net-supply. Moreover, assume w.l.o.g. that the number of shares of each positive-net-supply security outstanding is normalized to one.

For $n = 0, 1, ..., N+1$, let $S_0(t)$ denote the real price of the nominal money market account, $S_n(t)$ the real ex-dividend price of risky asset n, $\delta_n(t)$ the real dividend payed by asset n, and $Y_n(t)$ the dividend reinvested price of risky asset n. If asset n doesn't pay dividends (e.g. a nominal zero-coupon bond), then $\delta_n(t) \equiv 0$. The price of the market portfolio is

$$S = \sum_{n \in \{\text{positive-net-supply securities}\}} S_n.$$

Moreover, let $dY(t)/Y(t)$ denote the N-dimensional column vector with $(dS_n(t) + \delta_n(t)\,dt)/S_n(t)$ as its n-th component. The dynamics of all assets are

$$\begin{aligned} \frac{dY(t)}{Y(t)} &= \mu(t)\,dt + \sigma(t)'dZ(t) \\ \frac{dS_0(t)}{S_0(t)} &= \mu_0(t)\,dt + \sigma_0(t)'dZ(t), \end{aligned} \tag{A.8}$$

in which μ_0 is one-dimensional, μ is N-dimensional, σ_0 is d-dimensional, and σ is $(d \times N)$-dimensional.

Let $\alpha_n^i(t)$ the fraction of wealth investor i holds in the n-th risky asset at time t, and $\alpha^i(t)$ denote the column vector with n-th component equal to $\alpha_n^i(t)$. The remaining wealth of investor i is put in the nominal money market account; i.e. $\alpha_0^i(t) = 1 - \mathbf{1}'\alpha^i(t)$.[1]

The intertemporal budget constraint of each investor is

$$dW^i + c^i\,dt = W^i \left(\left(\mu_0 + (\mu - \mu_0\mathbf{1})'\,\alpha^i\right)\,dt + \left(\sigma_0 + \left(\sigma(t) - \sigma_0\mathbf{1}'\right)\alpha^i\right)'\,dZ(t) \right). \tag{A.9}$$

The value function of each investor is

$$\begin{aligned} J^i(t, w^i, x) = \sup_{\{\alpha^i(a), c^i(a) \mid t \le a \le T\}} & \left(\mathrm{E}\left[\int_t^T u^i(a, c^i(a), X(a))\,da \right.\right. \\ & \left.\left. + \; U^i(T, W^i(T), X(T)) \mid W^i(t) = w^i, X(t) = x \right] \right). \end{aligned} \tag{A.10}$$

The envelope condition and the boundary condition of the HJB-equation together with

[1] $\mathbf{1}$ denotes the N-dimensional vector of ones.

the FOC of the static optimization problem in equation (A.7) imply that

$$\lambda^i M^i(t) = J^i_w(t, w^i(t), X(t)) \qquad \forall\, 0 \leq t \leq T,\ \forall\, i = 1, \ldots, I, \tag{A.11}$$

in which $J^i_w(\cdot)$ denotes the partial derivative of investor i's value function w.r.t. his wealth.

Applying Itô's Lemma to equation (A.11) leads to[2]

$$\frac{dM^i}{M^i} - \mathrm{E}\left[\frac{dM^i}{M^i}\right] = -A^i\left(dW^i - \mathrm{E}\left[dW^i\right]\right) - \sum_{l=1}^{k} \Psi^i_l\left(dX_l - \mathrm{E}\left[dX_l\right]\right) \qquad \forall\, i, \tag{A.12}$$

in which $A^i = -J^i_{ww}/J^i_w$ denotes absolute risk aversion of consumer i and $\Psi^i_l = -J^i_{wX_l}/J^i_w$ denotes the sensitivity of the marginal value of wealth with respect to changes in the state vector.

For $i = 1, \ldots, I$ and $n = 1, \ldots, N$. The following pricing equation for asset n has to hold at an optimum for investor i:

$$\begin{aligned} (\mu_n(t) - \mu_0(t))\, dt &= -\left(\frac{dY_n}{Y_n} - \frac{dS_0}{S_0}\right)\frac{dM^i}{M^i} \\ &= \left(\frac{dY_n}{Y_n} - \frac{dS_0}{S_0}\right)\left(A^i dW^i + \sum_{l=1}^{k} \Psi^i_l dX_l\right) \end{aligned} \tag{A.13}$$

Rearranging terms and summing over all investors leads to

$$(\mu_n(t) - \mu_0(t))\, dt = \left(\frac{dY_n}{Y_n} - \frac{dS_0}{S_0}\right)\left(A dW + \sum_{l=1}^{k} \Psi_l dX_l\right) \tag{A.14}$$

in which $W = \sum_{i=1}^{I} W^i$ denotes aggregate wealth, $A = 1/(\sum_{i=1}^{I} 1/A^i)$, and $\Psi = \sum_{i=1}^{I}(A/A^i)\Psi^i$.

Market clearing implies that $S = W$ and hence the market price of residual inflation risk is zero. $\square$

[2] I abuse notation and denote with $\mathrm{E}[dX]$ the drift of the stochastic process X.

Proof of Lemma II.1. For $i = 1, \ldots, I$. The individual real tax liability of investor i is

$$\begin{aligned} L_\tau^i(t) &= \mathrm{E}_t\left[\int_t^\infty \frac{M^i(a)}{M^i(t)}\tau^i(a)\,da\right] \\ &= \mathrm{E}_t\left[\int_t^\infty \frac{M^i(a)}{M^i(t)}f^i(\nu + \kappa^*/\pi(a))\,da\right] \\ &= f^i\mathrm{E}_t\left[\int_t^\infty \frac{M^i(a)}{M^i(t)}\nu\,da\right] + \frac{f^i}{\pi(t)}\mathrm{E}_t\left[\int_t^\infty \frac{M^{*i}(a)}{M^{*i}(t)}\kappa^*\,da\right] \\ &= f^i(P_\nu(t) + B_\kappa(t)), \end{aligned} \tag{A.15}$$

in which $M^{*i}(t) = M^i(t)/\pi(t)$ is investor i's nominal stochastic discount factor. □

Proof of Theorem II.2 (ICAPM with taxes). For each individual $i = 1, \ldots, I$. Let $\alpha_S^i(t)$ denote the number of shares invested in the asset portfolio at time t, $\alpha_P^i(t)$ the number of shares invested in the inflation-protected consol at time t, $\alpha_B^i(t)$ the number of shares invested in the nominal consol at time t, $W^i(t)$ the real wealth at time t, $c^i(t)$ real consumption at time t, and $L_\tau^i(t)$ the real tax liability at time t. Moreover, let $W(t) = \sum_{i=1}^I W^i(t)$ denote aggregate wealth, $c(t) = \sum_{i=1}^I c^i(t)$ aggregate consumption, and $L_\tau(t) = \sum_{i=1}^I L_\tau^i(t)$ the aggregate real tax liability.

Each investor is initially endowed with $\alpha_S^i(0) = \alpha_{S0}^i > 0$ shares of the asset portfolio and f^i shares of the inflation-protected and nominal consol; i.e. $\alpha_P^i(0) = \alpha_B^i(0) = f^i$. Hence, investor i's initial wealth is equal to

$$w^i = \alpha_{S0}^i S(0) + f^i(P_\nu(0) + B_\kappa(0)). \tag{A.16}$$

Lemma II.1 implies that every investor can always immunize his tax liability by holding the constant share f^i in the inflation-protected and nominal consol. This strategy is affordable for every investor because $w^i - L_\tau^i(0) = \alpha_{S0}^i S(0) > 0$.

Moreover, Lemma II.1 implies that

$$L_\tau(t) = P_\nu(t) + B_\kappa(t) \tag{A.17}$$

because $\sum_{i=1}^{I} f^i = 1$.

The intertemporal budget constraint of each investor is

$$dW^i = -c^i\,dt - (dL^i_\tau + \tau^i\,dt) + \alpha^i_S(dS + \delta\,dt) + \alpha^i_P(dP_\nu + \nu\,dt) + \alpha^i_B(dB_\kappa + \kappa\,dt) \tag{A.18}$$

$$W^i(0) = w^i - L^i_\tau(0) \tag{A.19}$$

In equilibrium markets clear. Specifically,

$$\sum_{i=1}^{I} \alpha^i_S(t) = 1, \qquad \sum_{i=1}^{I} \alpha^i_P(t) = 1, \qquad \sum_{i=1}^{I} \alpha^i_B(t) = 1, \qquad c(t) = \delta(t). \tag{A.20}$$

Summing over all individuals in equations (A.18) and (A.19), and using equations (A.17) and (A.20) leads to

$$dW = dS \qquad \text{with} \quad W(0) = S(0). \tag{A.21}$$

Hence, aggregate wealth equals the asset or market portfolio; i.e. $W(t) = S(t)$.

The two consols outstanding do not appear in the market portfolio because their positive cash flows are offset by the negative cash flows of investor's tax liabilities. Only the part of inflation risk that is correlated with factors and real stock returns is priced and hence the market price of residual inflation risk is zero.[3]

□

A.3 Dynamic Portfolio Choice

We know that the expected rate of return of every traded asset in a frictionless economy that allows for continuous trading is equal to the real risk-free rate plus the local volatility

[3] The derivation of the pricing equation (A.14) is similar to the derivation in the proof of Theorem II.2 and thus omitted.

of the asset times the market price of risk and hence every continuously traded asset is uniquely defined by its local volatility vector.[4]

Define local excess returns of all real assets introduced in Section II.1 as the difference between the real local return of an asset minus the real local return of the nominal money market account.[5] Specifically, the real local excess return of a nominal zero-coupon bond maturing at T is[6]

$$\frac{dB_T}{B_T} - \frac{dR}{R} = \sigma_{B_T}' \Lambda \, dt + \sigma_{B_T}' \, dZ. \tag{A.22}$$

The real local excess return of an inflation-protected zero-coupon bond maturing at T is

$$\frac{dP_T}{P_T} - \frac{dR}{R} = (\sigma_{P_T} + \sigma_\pi)' \Lambda \, dt + (\sigma_{P_T} + \sigma_\pi)' \; dZ. \tag{A.23}$$

The real local excess return of the market portfolio is

$$\frac{dS}{S} - \frac{dR}{R} = (\sigma_S + \sigma_\pi)' \Lambda \, dt + (\sigma_S + \sigma_\pi)' \; dZ. \tag{A.24}$$

Let $\Omega(X)$ denote the $(d \times n)$-dimensional local real excess return volatility matrix with $n = h + l + 2$. Specifically, the first h columns of $\Omega(X)$ are real excess returns of nominal bonds $(dB_1/B_1 - dR/R)$, …, $(dB_h/B_h - dR/R)$, the next $l + 1$ columns are real local excess returns of inflation-protected bonds $(dP_1/P_1 - dR/R)$, …, $(dP_{l+1}/P_{l+1} - dR/R)$, and the last column is the local volatility vector of the excess return of the market portfolio $(dS/S - dR/R)$.

Moreover, let $\mathcal{M}(X)$ denote the asset return or asset space that consists of $n+1$ assets: h nominal bonds, $l + 1$ inflation-protected bonds, the market portfolio, and the nominal money market account and $\mathcal{E}(X)$ the excess return space consisting of the same assets. Geometrically, $\mathcal{E}(X)$ is an n-dimensional vector space that is spanned by the columns of

[4] See equation (II.6), Proposition II.1, and Proposition II.2 in Section II.1.

[5] There is no loss in generality to choose the nominal money market account as reference asset. The nominal money market (the nominal risk-free rate) is usually chosen as reference asset in the literature. However, I consider real returns in which case the nominal money market account is in general not risk-free because of its exposure to inflation risk.

[6] I sometimes suppress arguments of functions for notional simplicity.

$\Omega(X)$ and $\mathcal{M}(X) = (-\sigma_P(X), \mathcal{E}(X))$ is an n-dimensional affine space.[7] The dimension of $\mathcal{E}(X)$ and hence $\mathcal{M}(X)$ is equal to the number of non redundant assets (linearly independent columns of $\Omega(X)$). I show in Claim III of Proposition A.1 below that if Assumption II.1 and Assumption II.2 are true and if returns on the market portfolio are not spanned by returns of the nominal money market account, nominal bonds, and inflation-protected bonds, then the number of linearly independent nominal and inflation-protected bonds is equal to the dimension of U and $n = k_1 + 2$, in which k_1 denotes the dimension of U.[8]

Let W denote the real value of a self financing investment portfolio $\alpha \in \mathbb{R}^n$. Specifically, the first h elements of α denote the fraction of W invested in the h nominal bonds, the second $l+1$ elements denote the fraction of W invested in the $l+1$ inflation-protected bonds, the last element denotes the fraction of W invested in the market portfolio, and $1 - \mathbf{1}_n'\alpha$ denotes the fraction of W invested in the nominal money market account.[9] The real local return of the portfolio α is uniquely defined by the local return volatility

$$\sigma_W(X) = -\sigma_\pi(X) + \Omega(X)\alpha. \tag{A.25}$$

Hence, the real local return of W is

$$\frac{dW}{W} = (r(X) + \sigma_W(X)'\Lambda(X))\,dt + \sigma_W(X)'\,dZ \tag{A.26}$$

and the volatility $\sigma_W(X)$ is an element of the affine space $\mathcal{M}(X)$.

We will see below that the geometric interpretation of any self financing portfolio with dynamics given in (A.26) as an element of $\mathcal{M}(X)$ is very useful in determining the optimal investment portfolio. Specifically, I show in the proof of Theorem II.3 that the mimicking portfolio for the real risk-free asset, the tangency portfolio, and the hedging portfolios are uniquely determined by the projection of the null vector, the market price of risk $\Lambda(X)$,

[7] If the nominal money market account is locally riskless, then $\mathcal{E}(X)$ and $\mathcal{M}(X)$ coincide.

[8] If real returns on the market portfolio are spanned by returns of the nominal money market account and nominal and inflation-protected bonds, then we can exclude the market portfolio as an asset and the whole analysis that follows holds true with $n = k_1 + 1$.

[9] $\mathbf{1}_n$ denotes the n-dimensional vector of ones.

and the local covariance matrix of the state vector $\sigma(X)$ onto the asset space $\mathcal{M}(X)$.

Recall that $X = (U, V)$ with $\sigma_X(X) = (\sigma_U(X), \sigma_V(X))$ and let U be k_1- and V be k_2-dimensional. If time t is a state variable, then redefine the state vector X as (t, X). Moreover, exclude any state variable that can be written as a linear combination of other state variables from the definition of the state vector X. Finally, if S is a state variable or can be written as a linear combination of some state variables, then the state of the economy is defined by $Y = (X, S)$. Hence, we can without loss of generality assume that the local covariance matrix of X and S which is $(\sigma_X(X), \sigma_S(X))'(\sigma_X(X), \sigma_S(X))$ has full rank.[10] Implications of the spanning condition of the economy – Assumption II.2 – are provided in the next proposition.

Proposition A.1. *Let $\sigma_\pi^\perp$ denote the part of inflation risk that is not spanned by U. Then,*

$$\sigma_\pi^\perp(X) = (0, \ldots, 0, \sigma_{\pi k_1+1}(X), \ldots, \sigma_{\pi k+2}(X))' . \tag{A.27}$$

Adopt Assumption II.2. Then the following six claims are true.

Claim I: The part of inflation risk that is not spanned by U is orthogonal to V and real returns on the market portfolio if and only if

$$\sigma_V(X)'\sigma_\pi^\perp(X) = \sigma_S(X)'\sigma_\pi^\perp(X) = 0. \tag{A.28}$$

Claim II: The part of inflation risk that is not spanned by U is orthogonal to V if and only if it is orthogonal to X.

Claim III: The part of inflation risk that is not spanned by U is orthogonal to X and real returns on the market portfolio if and only if

$$\sigma_{\pi i} = 0 \qquad \forall i = k_1 + 1, \ldots, k + 1. \tag{A.29}$$

Claim IV: U is spanned by real returns of inflation protected bonds and nominal returns

[10] If Assumption II.1 holds, then the local covariance matrix of the Markov system given in equation (II.8) has full rank.

of nominal bonds if and only if U is spanned by real returns of inflation protected bonds and real returns of zero-investment portfolios of nominal bonds and the nominal money market account.

Claim V: If residual inflation risk is not zero, then real returns of inflation-protected bonds and zero investment portfolios of nominal bonds and the nominal money market account span U if $h + l \geq k_1$. Moreover, the dimension of the asset space is equal to $n = k_1 + 2$.

Claim VI: Neither Condition (i) nor (ii) of Assumption II.2 implies the other.

Proof. It follows directly from the upper diagonal form of the local covariance matrix of the Markov system $\sigma(X)$ given in equation (II.8) in Section II.1 that the k_1 linearly independent columns of $\sigma_U(X)$ span the vector $(\sigma_{\pi 1}(X), \ldots, \sigma_{\pi k_1}(X), 0, \ldots, 0)'$. Hence, the part of inflation risk that is not spanned by U is given in equation (A.27).

Moreover, two stochastic processes are locally uncorrelated if their local volatility vectors are orthogonal to each other and hence the part of inflation risk that is not spanned by V and real returns on the market portfolio is orthogonal to V and real returns on the market portfolio if and only if equation (A.28) holds. This proves Claim I.

Similarly, the upper diagonal form $\sigma(X)$ and equation (A.27) imply that $\sigma_U(X)'\sigma_\pi^\perp(X) = 0$ and hence $\sigma_V(X)'\sigma_\pi^\perp(X) = 0$ if and only if $\sigma_X(X)'\sigma_\pi^\perp(X) = 0$. This proves Claim II.

The "if part" of Claim III follows directly from equation (A.29). For the "only if part" we rewrite condition (A.28) and drop the zero identities. This leads to

$$\begin{pmatrix} \sigma_{V1k_1+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \sigma_{Vk_2k_1+1} & \ddots & \sigma_{Vk_2k} & 0 \\ \sigma_{Sk_1+1} & \cdots & \sigma_{Sk} & \sigma_{Sk+1} \end{pmatrix} \cdot \begin{pmatrix} \sigma_{\pi k_1+1} \\ \vdots \\ \sigma_{\pi k} \\ \sigma_{\pi k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \tag{A.30}$$

The solution given in equation (A.29) is the trivial solution of the system of linear equation given in (A.30). The columns of $\sigma_V(X)$ and $\sigma_S(X)$ are linearly independent implying that the coefficient matrix of the system of linear equations in (A.30) is non-singular and hence the trivial solution is the unique solution. This proves the "only if part" of Claim III.

The span of nominal returns of nominal bonds (see equation (A.6)) coincides with real returns of zero investment portfolios of nominal bonds (see equation (II.12)) and the nominal money market account (see equation (II.11)) because the difference between the local volatility vector of nominal and real returns of every nominal bond and the nominal money market account is equal to the volatility vector σ_π and hence this difference vanishes if the total investment in nominal bonds and the nominal money market account is zero. This proves Claim IV.

Let $\mathcal{E}_{\text{bonds}}(X)$ denote the space spanned by real excess returns of $l+1$ inflation-protected and h nominal bonds. The $h+l+1$ bonds are linearly independent. Moreover, elementary column transformations lead to a set of $h+l+1$ unit vectors $\{e_1, \ldots, e_{h+l}, e_d\}$ which span $\mathcal{E}_{\text{bonds}}(X)$.[11] From the upper diagonal form of the local covariance matrix of the Markov system given in equation (II.7) in Section II.1 follows that the local volatility matrix of U only loads on the first k_1 components and hence it is spanned if $h+l=k_1$. If $h+l>k_1$, then U is still spanned but in this case $h+l-k_1$ bonds are redundant. Moreover, if real returns on the market portfolio are not spanned by real returns of the nominal money market account and inflation-protected and nominal bonds, then the dimension of the asset space is $n=k_1+2$. If they are spanned, then there is no need to add them to the investment opportunity set and hence we drop the last column of $\Omega(X)$ and let $n=k_1+1$. This proves Claim V.

I provide two counter examples to prove Claim VI. Let $k=0$, $\sigma_{\pi 1}\neq 0$, $\sigma_{\pi 2}\neq 0$, and $\sigma_{S1}\neq 0$. Then, the nominal money market account, the market portfolio, and an inflation-protected bond (which is in this case the real risk-free asset) complete the market. But $\sigma_{\pi 1}\neq 0$ and hence part (i) of Assumption II.2 is satisfied but part (ii) is violated.

Assume that part (ii) of Assumption II.2 is satisfied and consider a state variable that is locally not perfectly correlated with real returns on the market portfolio and is not

[11] e_i denotes a d-dimensional vector with i-th component equal to one and remaining components zero. See Lemma A.1 for details on the basis change.

spanned by real returns of inflation protected bonds and nominal returns of nominal bonds – e.g. stochastic volatility of the market portfolio. In this case the market is incomplete and hence part (ii) of Assumption II.2 is satisfied but part (i) is violated. □

Let $\mathcal{P}_{\mathcal{M}}$ denote the projector onto the asset space $\mathcal{M}$ and $\mathcal{P}_{\mathcal{E}}$ the projector onto the excess return space $\mathcal{E}$.[12] Both projectors are given in the next lemma.

Lemma A.1. *[Projector onto the asset space]*

The projector onto the asset space $\mathcal{M}$ is

$$\mathcal{P}_{\mathcal{M}}(X) = -\mathcal{P}_{\mathcal{E}^{\perp}}(X)\sigma_{\pi}(X) + \mathcal{P}_{\mathcal{E}}(X) \tag{A.31}$$

with projector on the excess return space $\mathcal{E}$ and the orthogonal complement of the excess return space $\mathcal{E}^{\perp}$ given by[13]

$$\begin{aligned} \mathcal{P}_{\mathcal{E}}(X) &= \Omega(X)\left(\Omega(X)'\Omega(X)\right)^{-1}\Omega(X)', \\ \mathcal{P}_{\mathcal{E}^{\perp}}(X) &= I_d - \mathcal{P}_{\mathcal{E}}(X), \end{aligned} \tag{A.32}$$

respectively. Specifically, the projection of the d-dimensional vector v onto $\mathcal{M}$ is

$$\mathcal{P}_{\mathcal{M}}(X)v = -\mathcal{P}_{\mathcal{E}^{\perp}}(X)\sigma_{\pi}(X) + \mathcal{P}_{\mathcal{E}}(X)v. \tag{A.33}$$

Adopt Assumption $II.1$ and $II.2$. If the market is complete, then the projector onto $\mathcal{E}$ simplifies to

$$\mathcal{P}_{\mathcal{E}}(X) = I_d. \tag{A.34}$$

If the market is incomplete, then the projector simplifies to

$$\mathcal{P}_{\mathcal{E}}(X) = \mathcal{P}_{\mathcal{E}_{bonds}}(X) + \mathcal{P}_{\mathcal{E}_S}(X), \tag{A.35}$$

[12]See Kreyszig (1989) Chapter 3 and Brockwell and Davis (2006) Chapter 2 for properties of Hilbert spaces and projectors.

[13]Let I_k denote the k-dimensional unit matrix.

in which $\mathcal{P}_{\mathcal{E}_{bonds}}$ denotes the projector onto the space spanned by the real excess returns of the h nominal bonds and the $l+1$ inflation-protected bonds and $\mathcal{P}_{\mathcal{E}_S}(X)$ denotes the projector onto the space spanned by the part of real excess returns on the market portfolio that is uncorrelated with real excess returns of nominal and inflation-protected bonds.[14] *Specifically,*

$$\mathcal{P}_{\mathcal{E}_{bonds}}(X) = \begin{pmatrix} I_{k_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.36}$$

and

$$\mathcal{P}_{\mathcal{E}_S}(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho(X) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.37}$$

in which $\rho(X)$ is the $(k_2+1) \times (k_2+1)$-dimensional matrix with

$$\rho_{ij}(X) = \frac{\sigma_{Sk_1+i}(X)\sigma_{Sk_1+j}(X)}{v_S(X)} \qquad 1 \leq i, j \leq k_2+1 \tag{A.38}$$

and

$$v_S(X) = \sum_{i=1}^{k_2+1} \sigma_{Sk_1+i}(X)^2. \tag{A.39}$$

Proof. If the market is complete, then $\mathcal{R}^d = \mathcal{E}$ and hence $\mathcal{P}_{\mathcal{E}}(X) = I_d$.

If the market is incomplete, then Claim III of Proposition A.1 implies that the volatility of inflations is

$$\sigma_\pi = (\sigma_{\pi 1}, \ldots, \sigma_{\pi k_1}, 0, \ldots, 0, \sigma_{\pi d})'. \tag{A.40}$$

Hence,

[14] In the special case when real returns of the market portfolio are uncorrelated with real returns of nominal and inflation protected bond, then $\mathcal{E}_S$ is spanned by real returns on the market portfolio.

$$\Omega(X)=\left(\begin{array}{ccc|ccc|c}
\sigma_{B1}^{1} & \dots & \sigma_{B1}^{h} & \sigma_{P1}^{1}+\sigma_{\pi 1} & \dots & \sigma_{P1}^{l+1}+\sigma_{\pi 1} & \sigma_{S1}+\sigma_{\pi 1}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\sigma_{Bk_1}^{1} & \dots & \sigma_{Bk_1}^{h} & \sigma_{Pk_1}^{1}+\sigma_{\pi k_1} & \dots & \sigma_{Pk_1}^{l+1}+\sigma_{\pi k_1} & \sigma_{Sk_1}+\sigma_{\pi k_1}\\
\hline
0 & \dots & 0 & 0 & \dots & 0 & \sigma_{Sk_1+1}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots & 0 & 0 & \dots & 0 & \sigma_{Sk}\\
\hline
0 & \dots & 0 & 0 & \dots & 0 & \sigma_{Sk+1}\\
\hline
0 & \dots & 0 & \sigma_{\pi d} & \dots & \sigma_{\pi d} & \sigma_{\pi d}
\end{array}\right),$$

in which the first block of columns denotes the excess return volatility of the h nominal bonds, the second block denotes the excess return volatility of the $l+1$ inflation-protected bonds, and the last column denotes the excess return volatility of the market portfolio. The first block of rows denotes excess return exposure to the first k_1 components of Z, the second block of rows denotes excess return exposure to the next k_2 components of Z, the third row denotes excess return exposure to residual market portfolio risk Z_{k+1}, and the last row denotes excess return exposure to residual inflation risk Z_{k+2}.

The first $n-1$ columns span $\mathcal{E}_{\text{bonds}}$ by definition. Moreover, the h nominal bonds and the $l+1$ inflation-protected bonds are non-redundant and hence elementary column transformations lead to

$$\begin{pmatrix}
I_{k_1} & 0 & 0\\
0 & \vdots & \sigma_{Sk_1+1}\\
\vdots & \vdots & \vdots\\
\vdots & 0 & \sigma_{Sk+1}\\
0 & 1 & 0
\end{pmatrix}. \tag{A.41}$$

The last column which I define as $\bar{\sigma}_S(X)$ is the part of real returns on the market portfolio that is not spanned by real returns of inflation-protected bonds and real returns of zero investment portfolios of nominal bonds and the nominal money market account and hence the vector $\bar{\sigma}_S(X)$ spans $\mathcal{E}_S$. It is clear from equation (A.41) that $\mathcal{E}_{\text{bonds}}$ and $\mathcal{E}_S$ are orthogonal and hence $\mathcal{E}$ can be written as direct sum of the two spaces. Hence, the projector onto $\mathcal{E}$ is equal to the projector onto $\mathcal{E}_{\text{bonds}}$ plus the projector onto $\mathcal{E}_S$.

Moreover, the space $\mathcal{E}_{\text{bonds}}$ is spanned by the $k_1 + 1$ unit vectors $\{e_1, \ldots, e_{k_1}, e_d\}$ and thus $\mathcal{P}_{\mathcal{E}_{\text{bonds}}}(X)$ is given in equation (A.36). The projector onto the space $\mathcal{E}_S(X)$ that is spanned by the vector $\bar{\sigma}_S(X)$ is

$$\mathcal{P}_{\mathcal{E}_S}(X) = \bar{\sigma}_S(X) \left(\bar{\sigma}_S(X)'\bar{\sigma}_S(X)\right)^{-1} \bar{\sigma}_S(X)'. \tag{A.42}$$

Straightforward algebra lead to equation (A.37). □

Proof of Theorem II.3. The value function of investors who can continuously trade in the nominal money market account, h nominal zero-coupon bonds, $l+1$ inflation-protected zero-coupon bonds, and the market portfolio and who seek to maximize the utility function in equation (II.17) is

$$\begin{aligned} J(t, W, X) = \sup_{\{c(s),\alpha(s)|t \le s \le T\}} \mathrm{E}\Bigg[& \int_t^T e^{-\int_t^b \beta(X(a))\,da}\, u(c(b), X(b))\, db \\ & + e^{-\int_t^T \beta(X(a))\,da}\, U(W(T), X(T)) \mid W(t) = W, X(t) = X \Bigg]. \end{aligned} \tag{A.43}$$

Assume that the value function satisfies all regularity condition. Hence, the value function $J(t, W, X)$ solves the HJB equation

$$\sup_{c>0, \alpha\in\mathbb{R}^n} \left(\mathcal{A}^\alpha J(t, W, X)\right) = 0, \qquad J(T, W(T), X(T)) = U(W(T), X(T)), \tag{A.44}$$

in which the characteristic operator is given by[15]

$$\begin{aligned} \mathcal{A}^\alpha J = {} & J_t + J_X'\mu_X + \left(rW + W\sigma_W(\alpha)'\Lambda - c\right) J_W + \frac{1}{2}\text{trace}\left(J_{XX}\sigma_X'\sigma_X\right) \\ & + \sigma_W(\alpha)'\sigma_X W J_{WX} + \frac{1}{2}\sigma_W(\alpha)'\sigma_W(\alpha) W^2 J_{WW} + u - \beta J. \end{aligned} \tag{A.45}$$

[15]I sometimes suppress arguments for notional convenience.

If the investment horizon is infinite, then the value function does not depend on time t and hence $J_t = 0$.

Investors prefer more to less and are strictly risk averse which implies that $J_W > 0$ and $J_{WW} < 0$. Hence, the characteristic operator given in equation (A.45) can be rewritten as

$$\mathcal{A}^\alpha J = W^2 J_{WW} \cdot \frac{1}{2} \left\| \sigma_W(\alpha) - \left(\frac{1}{\gamma} \Lambda + \sigma_X \Theta \right) \right\|^2 + K, \tag{A.46}$$

in which $\gamma = -W J_{WW}/J_W$ denotes the relative risk aversion coefficient, $\Theta = -J_{WX}/(W J_{WW})$ denotes the sensitivity of the marginal utility of real wealth with respect to changes in the state vector, $\|\cdot\|$ denotes the Euclidian norm, and K is given by

$$K = J_t + J_X' \mu_X + (rW - c) J_W + \frac{1}{2} \text{trace}\left(J_{XX} \sigma_X' \sigma_X \right) - \frac{1}{2} W^2 J_{WW} \left\| \frac{1}{\gamma} \Lambda + \sigma_X \Theta \right\|^2 + u - \beta J \tag{A.47}$$

and hence does not depend on the portfolio weight α.

The local volatility of the real wealth portfolio is $\sigma_W(\alpha) = -\sigma_\pi + \Omega\alpha$ and $W^2 J_{WW} < 0$ and hence the optimal portfolio demand α^* of the maximization problem given in equation (A.44) is

$$\alpha^* = \underset{\alpha \in \mathbb{R}^n}{\text{argmin}} \left(\frac{1}{2} \left\| \sigma_W(\alpha) - \left(0 + \frac{1}{\gamma} \Lambda + \sigma_X \Theta \right) \right\|^2 \right). \tag{A.48}$$

Hence, the solution of the quadratic optimization problem in equation (A.48) is given by the projection of $\left(0 + \frac{1}{\gamma}\Lambda + \sigma_X \Theta\right)$ onto the asset space $\mathcal{M}$.[16] Specifically, (i) the projection of 0 onto $\mathcal{M}$ is the portfolio with minimum distance to the origin – i.e. the minimum variance portfolio which in this case is equal to the mimicking portfolio of the real risk-free asset because 0 is spanned by real asset returns ($0 \in \mathcal{M}$), (ii) the projection of $\Lambda(X)$ onto $\mathcal{M}$ is the portfolio with maximum local Sharpe ratio – i.e. the tangency portfolio, and (iii) the projection of $\sigma_X(X)$ onto $\mathcal{M}$ are the portfolios that are maximally correlated with the state variables – i.e. the hedging portfolios.

Let $\hat{\Lambda}(X) \equiv \mathcal{P}_\mathcal{M}(X)\Lambda(X)$ and $\hat{\sigma}_X(X) \equiv \mathcal{P}_\mathcal{M}(X)\sigma_X(X)$. The market price of residual

[16] See Bertsekas, Nedić, and Ozdaglar (2003) chapter 2.2 for applications of the projection theorem to quadratic optimization problems.

inflation risk is zero – i.e. $\Lambda_d(X) = 0$ – and the state variables are uncorrelated with residual inflation risk – i.e. $\sigma_{Xd\cdot}(X) = 0$ – and hence it follows from Lemma A.1 that $\hat{\Lambda}_d(X) = 0$ and $\hat{\sigma}_{Xd\cdot}(X) = 0$.[17] Moreover, real returns of inflation-protected bonds and the market portfolio are not exposed to residual inflation risk and real returns of nominal bonds and the nominal money market account have exactly the same exposure to this risk source and hence the total investment in nominal bonds and the nominal money market account in (i) the mimicking portfolio for the real risk-free asset, (ii) the tangency portfolio, and (iii) the hedging portfolio is zero.[18] $\square$

Proof of Theorem II.4. Let $w > 0$ denote the initial wealth and $\hat{w} = w - L_\tau(T-t, X)$ the initial wealth reduced by the tax liability. Every investor has to continuously pay the nominal lump-sum tax payment $\tau^*(t)$ given by Assumption II.3. The tax liability in this case is equal to

$$\begin{aligned} L^*_\tau(T-t, X) &= \kappa^* \mathrm{E}_t \left[\int_t^T M^*(a)/M^*(t)\, da \right] + \delta\pi(t) \mathrm{E}_t \left[\int_t^T M(a)/M(t)\, da \right] \\ &= B^*_\kappa(T-t, X) + P^*_\delta(T-t, X), \end{aligned} \tag{A.51}$$

in which $B^*_\kappa(T-t, X)$ denotes the nominal price of a nominal bond that continuously pays the nominal coupon κ^* until T and $P^*_\delta(T-t, X)$ denotes the nominal price of an inflation-protected bond that continuously pays the real coupon δ until T.

Not paying taxes results in an utility realization of minus infinite and hence every investor should hold a portfolio that covers all future tax payments. The cost of this portfolio

[17] v_i denotes the i-th element of the vector v and $M_{d\cdot}$ denotes the i-th row of the matrix M.

[18] The optimal demand α^* given in equation (A.48) is the solution of the system of linear equations

$$\Omega(X)\alpha^* = \sigma_\pi(X) + \frac{\mathcal{P}_\mathcal{E}(X)\Lambda(X)}{\gamma(W,X)} + \mathcal{P}_\mathcal{E}(X)\sigma_X(X)\Theta(W,X), \tag{A.49}$$

with $\mathcal{P}_\mathcal{E}(X)$ given in Lemma A.1. One could get α^* directly from the first order condition of the HJB equation. Specifically,

$$\alpha^* = \left(\Omega(X)'\Omega(X)\right)^{-1}\Omega(X)'\left(\sigma_\pi(X) + \frac{\Lambda(X)}{\gamma(W,X)} + \sigma_X(X)\Theta(W,X)\right). \tag{A.50}$$

It is straightforward to verify that the solution for equation (A.49) and (A.50) are the same by multiplying both sides of equation (A.49) with $(\Omega(X)'\Omega(X))^{-1}\Omega(X)'$ and using the general formula for $\mathcal{P}_\mathcal{E}(X)$ given in equation (A.32).

is equal to the total tax liability given in equation (A.51). To ensure that all tax payments are affordable we need to impose $\hat{w} > 0$. Hence, investors hold just enough nominal bonds to cover their tax liabilities and $\hat{w}$ is invested as there were no tax obligations. $\square$

APPENDIX B

INFLATION AND ASSET ALLOCATION

B.1 Investment Opportunities

I will show below that the real price of an inflation-protected bond, the nominal price of a nominal bond, and the value function for a power utility investor with unit wealth are solutions of the Heat Equation (B.1) with solution given in the next lemma.

Lemma B.1 (PDE). *Let $G(t,x)$ denote a continuous function from $[0,T]\times\mathbb{R}$ into $\mathbb{R}^+$ that is differentiable with respect to t, twice differentiable with respect to x and solves the pde*

$$\begin{aligned} G_t(t,x) + fG_{xx}(t,x) + (d+ex)\,G_x(t,x) + \left(a+bx+cx^2\right)G(t,x) &= 0 \\ G(T,x) &= 1 \qquad \forall\, x, \end{aligned} \tag{B.1}$$

in which a, b, c, d, e, and f are real constants.

The solution to this pde is

$$G(t,x) = e^{\alpha(T-t)+\beta(T-t)x+\gamma(T-t)x^2}. \tag{B.2}$$

Let $\tau \equiv T-t$. The functions $\gamma(\tau)$, $\beta(\tau)$, and $\alpha(\tau)$ solve the ordinary differential equations[1]

$$\begin{aligned} \dot{\gamma}(\tau) &= 4f\gamma(\tau)^2 + 2e\gamma(\tau) + c \\ \gamma(0) &= 0, \end{aligned} \tag{B.3}$$

$$\begin{aligned} \dot{\beta}(\tau) &= (4f\gamma(\tau)+e)\,\beta(\tau) + 2d\gamma(\tau) + b \\ \beta(0) &= 0, \end{aligned} \tag{B.4}$$

[1]It is straightforward to solve the ode's (B.3)-(B.5) analytically or numerically using the Matlab function "*ode45.m*".

and

$$\begin{aligned}\dot{\alpha}(\tau) &= f\beta(\tau)^2 + d\beta(\tau) + 2f\gamma(\tau) + a \\ \alpha(0) &= 0.\end{aligned} \tag{B.5}$$

Proof. Take derivatives of $G(t,x)$ given in equation (B.2) with respect to t, x, and twice x and plug them back into the pde (B.1). The resulting equation has to hold for all t, x, and x^2 and hence setting the coefficients of t, x, and x^2 equal to zero leads to the odes (B.3)-(B.5). □

Proof of Proposition III.1 and III.2. The real price of an inflation-protected and the nominal price of a nominal zero-coupon bond maturing at T are

$$\begin{aligned}P(T-t,x) &= \mathrm{E}_t\left[M(T)/M(t)\right] = \mathrm{E}^{\mathbb{Q}}\left[-\int_t^T r(a)\,da \mid x(t)=x\right] \\ B^*(T-t,x) &= \mathrm{E}_t\left[M^*(T)/M^*(t)\right] = \mathrm{E}^{\mathbb{Q}^*}\left[-\int_t^T r^*(a)\,da \mid x(t)=x\right],\end{aligned} \tag{B.6}$$

in which $\mathrm{E}[\cdot]$ denotes the expectation under the data generating measure $\mathbb{P}$, $\mathrm{E}^{\mathbb{Q}}[\cdot]$ denotes the expectation under the real risk-neutral measure $\mathbb{Q}$, and $\mathrm{E}^{\mathbb{Q}^*}[\cdot]$ denotes the expectation under the nominal risk-neutral measure $\mathbb{Q}^*$. Specifically, the real and nominal risk neutral measures are defined by

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} &= \frac{M(T)}{M(t)}\,e^{\int_t^T r(a)\,da} \\ \frac{d\mathbb{Q}^*}{d\mathbb{P}} &= \frac{M^*(T)}{M^*(t)}\,e^{\int_t^T r^*(a)\,da}.\end{aligned} \tag{B.7}$$

The real risk-free rate $r(x)$ given in equation (III.5) and the nominal risk-free rate $r^*(x)$ given in equation (III.15) are quadratic functions of the state vector x.

Let $\bar{x}=0$, $\sigma_x=(1,0,0)'$, and $\sigma_\pi(x)=\sigma_{\pi 0}+\sigma_{\pi x}x$.[2] The dynamics of the state variable

[2]The analysis in the text is restricted to the case $\sigma_{\pi x}=0$.

under the real risk-neutral measure and the nominal risk-neutral measure are

$$
\begin{aligned}
dx &= -\left(\lambda_{01} + (\kappa + \lambda_{x1})x\right)\ dt + dZ_1^{\mathbb{Q}} \\
dx &= -\left(\lambda_{01} + \sigma_{\pi 01} + (\kappa + \lambda_{x1} + \sigma_{\pi x1})x\right)\ dt + dZ_1^{\mathbb{Q}^*}
\end{aligned}
\tag{B.8}
$$

in which $Z_1^{\mathbb{Q}}$ denotes the first component of the Brownian motion Z under the real risk-neutral measure and $Z_1^{\mathbb{Q}^*}$ denotes the first component of Z under the nominal risk-neutral measure.

It is straightforward to show that the fundamental pde for the real price of an inflation-protected zero-coupon bond and the nominal price of a zero-coupon nominal bond has the same form as the pde in Lemma B.1.[3] Specifically, let $\tau = T - t$. Then, the deterministic functions $a(\tau)$, $b(\tau)$, and $c(\tau)$ in the inflation-protected bond price equation (III.10) solve the odes

$$\dot{a}(\tau) = \frac{1}{2}b(\tau)^2 - \lambda_{01}\, b(\tau) + c(\tau) - \rho_0 \qquad a(0) = 0 \tag{B.9}$$

$$\dot{b}(\tau) = \left(2c(\tau) - (\lambda_{x1} + \kappa)\right) b(\tau) - 2\lambda_{01} c(\tau) - \rho_x \qquad b(0) = 0 \tag{B.10}$$

$$\dot{c}(\tau) = 2c(\tau)^2 - 2\left(\lambda_{x1} + \kappa\right) c(\tau) - \rho_{xx} \qquad c(0) = 0 \tag{B.11}$$

and the deterministic functions $a^*(\tau)$, $b^*(\tau)$, and $c^*(\tau)$ in the nominal bond price equation (III.17) solve the odes

$$\dot{a^*}(\tau) = \frac{1}{2}b^*(\tau)^2 - \left(\lambda_{01} + \sigma_{\pi 01}\right) b^*(\tau) + c^*(\tau) - \delta_0 \qquad a^*(0) = 0 \tag{B.12}$$

$$\dot{b^*}(\tau) = \left(2c^*(\tau) - (\lambda_{x1} + \sigma_{\pi x1} + \kappa)\right) b^*(\tau) - 2(\lambda_{01} + \sigma_{\pi 01}) c^*(\tau) - \delta_x \qquad b^*(0) = 0 \tag{B.13}$$

$$\dot{c^*}(\tau) = 2c^*(\tau)^2 - 2\left(\lambda_{x1} + \sigma_{\pi x1} + \kappa\right) c^*(\tau) - \delta_{xx} \qquad c^*(0) = 0. \tag{B.14}$$

Taking the first derivative of the real price of an inflation-protected bond given in equation (III.10) and the nominal price of a nominal bond given in equation (III.17) with respect

[3]See Ahn, Dittmar, and Gallant (2002) for details on the derivation of bond prices when the risk-free rate is a quadratic function of Gaussian state variables.

to x leads to equation (III.12) and (III.19). Then, applying Itô's lemma to P, $B = B^*/\pi$, and $R = R^*/\pi$ leads to the local return dynamics given in equations (III.11), (III.18), and (III.20). □

Proposition B.1. *Let $y(x) = -a/b + x/b$ be an affine transformation of the state vector x with $b \neq 0$. Then the economy described by the factor x with dynamics*

$$dx = \kappa(\bar{x} - x)\,dt + \sigma dZ_1 \tag{B.15}$$

is informationally equivalent to an economy that depends on $y(x)$. Moreover, if $b = \sigma$ and $a = \bar{x}$, then

$$dy = -\kappa y\,dt + dZ_1. \tag{B.16}$$

Proof. All functions of the state variable x are either affine or quadratic functions of x. Let $\phi(x) = \phi_0 + \phi_x x$ denote an affine function of x and $\theta(x) = \theta_0 + \theta_x x + \theta_{xx} x^2$ a quadratic function of x with ϕ_0, ϕ_x, θ_0, θ_x, $\theta_{xx} \in \mathbb{R}^3$. Then, the transformations

$$\begin{aligned} (\phi_0, \phi_x) &\longleftrightarrow (\phi_0 + a\phi_x, b\phi_x) \\ (\theta_0, \theta_x, \theta_{xx}) &\longleftrightarrow (\theta_0 + a\theta_x + a^2\theta_{xx}, b\theta_x + 2ab\theta_{xx}, b^2\theta_{xx}), \end{aligned} \tag{B.17}$$

allow to switch between two different state space representations. Hence, $y(x)$ describes an economy that is informationally equivalent to the economy described by x. Moreover, if $b = \sigma$ and $a = \bar{x}$, then $dy = -\kappa y\,dt + dZ_1$. □

B.2 Dynamic Asset Allocation

Proof of Proposition III.3 and Theorem III.1. It is straightforward to verify that the four assets are non-redundant and complete the market. It is also well known that a dynamic portfolio choice problem can be transformed in a static portfolio choice problem.[4] Moreover, the solution is provided in example 6.6 for the log-utility case and example 6.7 for the power

[4] See Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987), and He and Pearson (1991).

utility case on pages $104-106$ in Karatzas and Shreve (1998).

Hence, it remains to determine the expectation in equation (III.22). Consider

$$F(t,x) = \mathrm{E}\left[(M(T)/M(t))^{\frac{\gamma-1}{\gamma}} \mid x(t) = x\right]. \tag{B.18}$$

It follows that the function $G(t,x) = M(t)^{\frac{\gamma-1}{\gamma}} F(t,x)$ is a local martingale and hence the drift of $G(t,x)$ is zero.

Applying Itô's lemma to $G(t,x)$ and using the fact that $G(t,x)$ is a local martingale leads to a pde for the function $F(t,x)$. Specifically,

$$\begin{aligned} &F_t(t,x) + \frac{1}{2}\sigma_x'\sigma_x F_{xx}(t,x) + \left(\kappa(\bar{x}-x) + \frac{1-\gamma}{\gamma}\sigma_x'\Lambda(x)\right) F_x(t,x) \\ &+ \left(\frac{1-\gamma}{\gamma} r(x) + \frac{1-\gamma}{2\gamma^2}\Lambda(x)'\Lambda(x)\right) F(t,x) = 0 \qquad \text{with} \quad F(T,x) = 1. \end{aligned} \tag{B.19}$$

Let $\sigma_x'\sigma_x = 1$, $\bar{x} = 0$, $\sigma_x'\Lambda(x) = \lambda_{01} + \lambda_{x1}x$, $r(x) = \rho_0 + \rho_x x + \rho_{xx}x^2$, and $\Lambda(x)'\Lambda(x) = l_0 + 2l_x x + l_{xx}x^2$ with $l_0 = \lambda_0'\lambda_0$, $l_x = \lambda_0'\lambda_x$, and $l_0 = \lambda_x'\lambda_x$. It follows that the pde in equation (B.19) has the same form as the pde in Lemma B.1. Specifically,

$$\begin{aligned} a &= \frac{1-\gamma}{\gamma}\,\rho_0 + \frac{1-\gamma}{2\gamma^2}\,l_0 & d &= \frac{1-\gamma}{\gamma}\lambda_{01} \\ b &= \frac{1-\gamma}{\gamma}\,\rho_x + \frac{1-\gamma}{\gamma^2}\,l_x & e &= \frac{1-\gamma}{\gamma}\,\lambda_{x1} - \kappa \\ c &= \frac{1-\gamma}{\gamma}\,\rho_{xx} + \frac{1-\gamma}{2\gamma^2}\,l_{xx} & f &= \frac{1}{2}. \end{aligned} \tag{B.20}$$

Let $\tau = T-t$ denote the remaining investment horizon. The solution of the pde in equation (B.19) is

$$F(\tau,x) = e^{h_0(\tau)+h_x(\tau)x+h_{xx}(\tau)x^2}, \tag{B.21}$$

in which $h_{xx}(\tau)$, $h_x(\tau)$, $h_0(\tau)$ solve the odes

$$\dot{h}_{xx}(\tau) = 2h_{xx}(\tau)^2 + 2\left(\frac{1-\gamma}{\gamma}\,\lambda_{x1} - \kappa\right)h_{xx}(\tau) + \frac{1-\gamma}{\gamma}\,\rho_{xx} + \frac{1-\gamma}{2\gamma^2}\,l_{xx} \tag{B.22}$$

$$\dot{h}_x(\tau) = \left(2h_{xx}(\tau) + \frac{1-\gamma}{\gamma}\,\lambda_{x1} - \kappa\right)h_x(\tau) + 2\frac{1-\gamma}{\gamma}\,\lambda_{01}h_{xx}(\tau) + \frac{1-\gamma}{\gamma}\,\rho_x + \frac{1-\gamma}{\gamma^2}\,l_x \tag{B.23}$$

$$\dot{h}_0(\tau) = \frac{1}{2}h_x(\tau)^2 + \frac{1-\gamma}{\gamma}\,\lambda_{01}\,h_x(\tau) + h_{xx}(\tau) + \frac{1-\gamma}{\gamma}\,\rho_0 + \frac{1-\gamma}{2\gamma^2}\,l_0 \tag{B.24}$$

with $h_{xx}(0) = h_x(0) = h_0(0) = 0$. This proves Proposition III.3.

To derive the optimal demands in Theorem III.1 consider the real wealth dynamics of a self-financing portfolio $\alpha(t,x) = (\alpha_B(t,x), \alpha_S(t,x), \alpha_P(t,x))$ with $\alpha_R(t,x) = 1 - \mathbf{1}_3'\alpha(t,x)$. Specifically,

$$\frac{dW}{W} = \left[r(x) + (-\sigma_\pi(x) + \Omega(t,x)\alpha(t,x))'\,\Lambda(x)\right]\,dt + (-\sigma_\pi(x) + \Omega(t,x)\alpha(t,x))'\,dZ, \tag{B.25}$$

in which

$$\Omega(t,x) = \begin{pmatrix} D^*(t,x) & \sigma_{S1}(x) + \sigma_{\pi1}(x) & D(t,x) + \sigma_{\pi1}(x) \\ 0 & \sigma_{S2}(x) + \sigma_{\pi2}(x) & \sigma_{\pi2}(x) \\ 0 & \sigma_{\pi3}(x) & \sigma_{\pi3}(x) \end{pmatrix}. \tag{B.26}$$

The optimal demand $\alpha^*(t,x)$ can be determined from the FOC of the HJB equation (B.27). Specifically, the value function $J(t,W,x)$ given in equation (III.21) solves the HJB equation

$$\sup_{\alpha\in\mathbb{R}^3}\left(\mathcal{A}^\alpha J(t,W,x)\right) = 0, \qquad J(T,W(T),x(T)) = \frac{1}{1-\gamma}W(T)^{1-\gamma}, \tag{B.27}$$

in which the characteristic operator is given by[5]

$$\begin{aligned}\mathcal{A}^\alpha J = {} & J_t - \kappa x J_x + \left(r + (-\sigma_\pi + \Omega\alpha)'\,\Lambda\right)W J_W + (-\sigma_\pi + (1,0,0)\Omega\alpha)\,W J_{Wx} \\ & + \frac{1}{2}\,(-\sigma_\pi + \Omega\alpha)'\,(-\sigma_\pi + \Omega\alpha)\,W^2 J_{WW} + \frac{1}{2}J_{xx}.\end{aligned} \tag{B.28}$$

[5]I suppress arguments for notional convenience.

The FOC leads to

$$\Omega(t,x)'\Omega(t,x)\alpha^*(t,x) = \Omega(t,x)'\left(\sigma_\pi(x) + \Lambda(x)/\gamma + (h_x(t) + 2h_{xx}(t)x)(1,0,0)'\right). \quad \text{(B.29)}$$

The matrix $\Omega(t,x)$ is almost surely non singular and hence the optimal demand α^* given in equation (III.24) is the solution of the system of linear equations

$$\Omega(t,x)\alpha^*(t,x) = \sigma_\pi(x) + \Lambda(x)/\gamma + (h_x(t) + 2h_{xx}(t)x)(1,0,0)'. \quad \text{(B.30)}$$

This proves Theorem III.1. □

B.3 Model Calibration

Lemma B.2. *Let $f(x) = f_0 + f_x x + f_{xx} x^2$ be a quadratic function of x which follows the mean reverting Ornstein-Uhlenbeck process*

$$dx = \kappa\left(\bar{x} - x\right)\, dt + \sigma\, dz, \quad \text{(B.31)}$$

in which z denotes a one dimensional Brownian motion and f_0, f_x, and f_{xx} are constants.

The steady state mean, variance, and autocovariance of $f(x)$ are given by

$$\mathrm{E}[f(x)] = f_0 + f_x\,\bar{x} + f_{xx}\,\left(\bar{x}^2 + v\right) \quad \text{(B.32)}$$

$$\mathrm{V}[f(x)] = f_x^2\, v + 4 f_x f_{xx}\,\bar{x} v + f_{xx}^2\,\left(4\bar{x}^2 v + 2v^2\right) \quad \text{(B.33)}$$

$$\mathrm{Cov}[f(T), f(S)] = 2v^2 f_{xx}^2\, e^{-2\kappa(T-S)} + v e^{-\kappa(T-S)}\left(f_x^2 + 4 f_x f_{xx}\bar{x} + 4 f_{xx}^2 \bar{x}^2\right) \quad \text{(B.34)}$$

in which $v = \sigma^2/(2\kappa)$ denotes the steady state variance of x.

Proof. The steady state distribution of x is normal with mean $\bar{x}$ and variance $v = \sigma^2/(2\kappa)$. Moreover, the steady state autocorrelation of $x(T)$ and $x(S)$ is $ve^{-\kappa(T-S)}$. The mean, variance, and the autocovariance of $f(x(t))$ are functions of the first four moments of the normal distribution (the steady state distribution of x). Tedious algebra leads to results given in equations (B.32)-(B.34). □

Proposition B.2. *The steady state mean, variance, and autocovariance of the nominal risk-free rate* $r^*(x)$ *in equation (III.15) are*

$$m_r \equiv \mathrm{E}[r^*(x)] = \delta_0 + \frac{\delta_{xx}}{2\kappa} \tag{B.35}$$

$$v_r \equiv \mathrm{V}[r^*(x)] = 2\left(\frac{\delta_{xx}}{2\kappa}\right)^2 + \frac{\delta_x^2}{2\kappa} \tag{B.36}$$

$$c_r \equiv \mathrm{Cov}[r^*(t+\tau), r^*(x)] = 2\, e^{-2\kappa\tau} \left(\frac{\delta_{xx}}{2\kappa}\right)^2 + e^{-\kappa\tau}\, \frac{\delta_x^2}{2\kappa} \tag{B.37}$$

If m_r*,* v_r*, and* c_r *are known and the infimum of the nominal risk-free rate is zero, then there exists a solution for the parameters* $(\kappa, \delta_0, \delta_x, \delta_{xx})$ *that is except for the sign of* δ_x *unique. Specifically,*

$$\delta_0 = \sqrt{m_r^2 - v_r/2} \tag{B.38}$$

$$\kappa = -\frac{1}{\tau}\log\left(\frac{-\delta_0 + \sqrt{m_r^2 + (c_r - v_r)/2}}{m_r - \sqrt{m_r^2 - v_r/2}}\right) \tag{B.39}$$

$$\delta_{xx} = 2\kappa\left(m_r - \sqrt{m_r^2 - v_r/2}\right) \tag{B.40}$$

$$\delta_x = \pm\sqrt{4\delta_0\delta_{xx}} \tag{B.41}$$

Proof. The first part follows directly from Lemma B.2 with $\delta_0 = f_0$, $\delta_x = f_x$, $\delta_{xx} = f_{xx}$, $\bar{x} = 0$, and $\sigma = 1$. The assumption of a zero infimum for the nominal risk-free rate leads to the additional equation $\delta_x^2 = 4\delta_{xx}\delta_0$ and the restriction $\delta_{xx} > 0$. Solving the the four equations for the four parameters κ, δ_0, δ_x, and δ_{xx} and excluding the solutions that are not feasible leads to the four equations (B.38)-(B.41). □

Proposition B.3. *The steady state mean, variance, and autocovariance of every nominal* *bond yield* $y_\tau(x)$ *are*

$$m_y = \mathrm{E}[y_\tau(x)] = \bar{A}(\tau) + \bar{C}(\tau)\, v \tag{B.42}$$

$$v_y = \mathrm{V}[y_\tau(x)] = 2v^2\bar{C}(\tau)^2 + v\bar{B}(\tau)^2 \tag{B.43}$$

$$c_y = \mathrm{Cov}[y_\tau(t+T), y_\tau(x)] = 2v^2\bar{C}(\tau)^2\, e^{-2\kappa T} + ve^{-\kappa T}\bar{B}(\tau)^2, \tag{B.44}$$

in which $v = 1/(2\kappa)$, $\bar{A}(\tau) = -A(\tau)/\tau$, $\bar{B}(\tau) = -B(\tau)/\tau$, *and* $\bar{C}(\tau) = -C(\tau)/\tau$.

Proof. Follows directly from Lemma B.2 with $\bar{A} = f_0$, $\bar{B} = f_x$, and $\bar{C} = f_{xx}$, $\bar{x} = 0$, and $\sigma = 1$. □

APPENDIX C

THE TERM STRUCTURE OF INTEREST RATES WITH HETEROGENEOUS HABIT FORMING PREFERENCES

C.1 Competitive Equilibrium

In this section I solve for the competitive equilibrium and prove Proposition IV.1, Theorem IV.1, and Corollary IV.1. The competitive equilibrium is determined in three steps: (i) the optimal consumption sharing rule is determined in Lemma C.1, (ii) each efficient allocation is supported as an Arrow-Debreu equilibrium that is characterized by a stochastic discount factor process in Lemma C.2, and (iii) it is shown that the Arrow-Debreu equilibrium can be achieved by continuously trading in a security market.[1]

Step 1*: Pareto efficient allocations*

There are two investors in the economy with utility function $U_1(\cdot)$ and $U_2(\cdot)$ given in equation (IV.3). Aggregate consumption is distributed among both agents such that the resulting consumption allocation is Pareto efficient. Specifically, the social planner assigns the social weight $\kappa \in (0, 1)$ to the first agent, $(1 - \kappa)$ to the second agent, and seeks to maximize

$$\mathrm{E}\left[\int_0^T e^{-\beta t} \left\{\kappa u_1(c_1(t), X(t)) + (1 - \kappa)u_2(c_2(t), X(t))\right\}\, dt\right] \tag{C.1}$$

subject to the resource constraint

$$c_1(t) + c_2(t) \leq c(t) \qquad \forall\, 0 \leq t \leq T. \tag{C.2}$$

The market is complete (see step three below) and therefore the social weight κ is constant and can be uniquely determined from the initial wealth of both agents.

There are no intertemporal transfers of resources in an exchange economy and each agent has no control over the standard of living process $X(t)$ and hence the optimization

[1] See Karatzas and Shreve (1998)' textbook.

problem (C.1)-(C.2) can by solved state by state.[2] Specifically, the social planner seeks to maximize

$$\kappa u_1(c_1(t), X(t)) + (1-\kappa) u_2(c_2(t), X(t)) \tag{C.3}$$

subject to the resource constraint (C.2). The social weight satisfies $0 < \kappa < 1$ and $u_1(c_1(t), X(t))$ and $u_2(c_2(t), X(t))$ are strictly increasing in $c_1(t)$ and $c_2(t)$, respectively, and hence the resource constraint (C.2) is binding.

Let $\eta(t)$ denote the (strictly positive) shadow price of the resource constraint (C.2). The optimization problem can be written in the form given in equation (IV.7) and the Pareto efficient allocations when one investor is twice as risk averse as the other ($\gamma_2 = 2\gamma_1$) are summarized in the next lemma.[3]

Lemma C.1. *(***Pareto optimal allocations***)*

The optimal consumption sharing rules $(\hat{c}_1(t), \hat{c}_2(t))$ *for each* $\kappa \in (0,1)$ *are*

$$\begin{aligned} \hat{c}_1(t) &= c(t) - \hat{c}_2(t), \qquad & K &= 4\left(\frac{\kappa}{1-\kappa}\right)^{\frac{1}{\gamma_1}} \\ \hat{c}_2(t) &= c(t) f_2(t), \qquad & f_2(t) &= \frac{2}{1+\sqrt{1+Ke^{\omega(t)}}} \end{aligned} \tag{C.4}$$

Proof. Let $d_1 = c_1/X$, $d_2 = c_2/X$, and $d = c/X$. First order conditions for the social planner's optimization problem lead to the equation

$$d_1 = \left(\frac{\kappa}{1-\kappa}\right)^{\frac{1}{\gamma_1}} d_2^2. \tag{C.5}$$

The resource constraint is binding and hence $c_1(t) = c(t) - c_2(t)$. This implies that $d_1 = d - d_2$. Plugging in for d_1 in equation (C.5), solving the resulting quadratic equation for d_2, and ignoring the infeasible (negative) solution leads to the optimal consumption sharing rule in equation (C.4). □

[2] In the case of internal habit forming preferences or if each agent considers the other agents' consumption stream as the standard of living (Catching up with the Joneses), then time separation is no longer possible because the standard of living process is endogenous now.

[3] The analysis is similar to Dumas (1989) and Wang (1996) who considers standard time additive CRRA preferences.

Step 2: Arrow-Debreu equilibrium

The Pareto efficient consumption allocation $(\hat{c}_1(t), \hat{c}_2(t))$ given in equation (C.4) can be supported as an equilibrium consumption allocation in which agents have access to a complete set of Arrow-Debreu securities.[4] The marginal utility of the representative investor is equal to the shadow price of the resource constraint discounted by the subjective discount factor β; i.e. it is $e^{-\beta t}\eta(t)$. The shadow price $\eta(t)$ is equal to the first derivative of the social planner's value function with respect to aggregate consumption (Envelope Theorem). The real stochastic discount factor $m(t)$ is the price (in terms of consumption at time zero $c(0)$) of an Arrow-Debreu security at time t for a particular state. It is equal to the discounted, normalized shadow price of the social planner's budget constraint – i.e. $m(t) = e^{-\beta t}\eta(t)/\eta(0)$ – and is determined in the next lemma.

Lemma C.2. *For each Pareto optimal allocation* $(\hat{c}_1(t),\ \hat{c}_2(t))$ *given in equation (C.4) with* $\kappa \in (0,1)$ *there exists a stochastic discount factor process*

$$m(t) = e^{-\beta t}\,\frac{\xi(t)}{c(t)}\,\frac{c(0)}{\xi(0)}, \tag{C.6}$$

in which

$$\xi(t) = \frac{2^{\gamma_2}\kappa^2}{1-\kappa}\,\frac{e^{\omega(t)}}{\left(\sqrt{1+\Lambda e^{\omega(t)}}-1\right)^{\gamma_2}}. \tag{C.7}$$

Proof. Substituting the optimal consumption sharing rule $(\hat{c}_1(t),\ \hat{c}_2(t))$ given in equation (C.4) into the value function of the social planner given in equation (IV.7) leads to

$$u^{\kappa}(c(t), X(t)) = \frac{\kappa}{1-\gamma_1}\left(\frac{c(t)-\hat{c}_2(t)}{X(t)}\right)^{1-\gamma_1} + \frac{1-\kappa}{1-\gamma_2}\left(\frac{\hat{c}_2(t)}{X(t)}\right)^{1-\gamma_2}. \tag{C.8}$$

[4] See Duffie (1996)'s textbook. An Arrow-Debreu securities pays of one unit of real consumption at a particular state of the economy and zero otherwise. A set of Arrow-Debreu securities is complete if there is an Arrow-Debreu security for each state of the economy.

Differentiating (C.8) with respect to aggregate consumption leads to

$$\begin{aligned}\eta(t) &= u_c^\kappa(c(t), X(t)) \\ &= \frac{2^{\gamma_2}\kappa^2}{1-\kappa}\,\frac{e^{\omega(t)}/c(t)}{\left(\sqrt{1+\Lambda e^{\omega(t)}}-1\right)^{\gamma_2}}.\end{aligned} \tag{C.9}$$

Multiplying $\eta(t)$ by $e^{-\beta t}$ and dividing by $\eta(0)$ leads to equation (C.6) for the real stochastic discount factor.

□

Step 3: Sequential trade equilibrium

It remains to show that the equilibrium characterized by the real stochastic discount factor process determined in the previous lemma can be supported by an equilibrium in which investors continuously trade a few securities (see Duffie and Huang (1985)). This is very hard to show for heterogenous preferences and general dynamics of aggregate consumption growth and inflation. However, it is easy to ensure market completeness by introducing a sufficient number of zero net supply securities with unit volatility (see Karatzas, Lehoczky, and Shreve (1990) for a formal discussion of these securities).[5]

Finally, I show that the social weight κ is constant and can be determined from the initial wealth distribution. Let W_0^i denote the initial wealth of agent i. The static budget constraint is

$$W_0^i = \mathrm{E}\left[\int_0^T m(t)\hat{c}_i(t)\,dt\right] \qquad \forall i = 1, 2. \tag{C.10}$$

The social weight κ can be determined from equation (C.10) as a function of the initial wealth of the first or second agent. This completes step $1-3$ and hence the derivation of the competitive equilibrium.

In the remainder of the section: I prove Proposition IV.1, Theorem IV.1, and Corollary IV.1 and I derive the dynamics of the real stochastic discount factor in Proposition C.1.

[5]Chan and Kogan (2002) make the same argument to ensure market completeness when aggregate consumption follows a GBM.

Proof of Proposition IV.1. Aggregate risk aversion is equal to the local curvature of the representative investor. Specifically,

$$\gamma(t) = -c(t)\frac{u^{\kappa}_{cc}(c(t), X(t))}{u^{\kappa}_{c}(c(t), X(t))}. \tag{C.11}$$

The first derivative of $u^{\kappa}(c(t), X(t))$ with respect to $c(t)$ is given in equation (C.9). Taking the second derivative of $u^{\kappa}(c(t), X(t))$ with respect to $c(t)$, plugging back into equation (C.11) and tedious algebra leads to

$$\gamma(t) = \gamma_1 \left(1 + \frac{1}{\sqrt{1 + Ke^{\omega(t)}}}\right) \qquad \text{with} \quad K = 4\left(\frac{\kappa}{1-\kappa}\right)^{\frac{1}{\gamma_1}}. \tag{C.12}$$

Moreover, $\lim_{\kappa\to 1} K(\kappa) = \infty$ and hence $\lim_{\kappa\to 1}\gamma(t) = \gamma_1$. Similarly, $\lim_{\kappa\to 0} K(\kappa) = 0$ and hence $\lim_{\kappa\to 0}\gamma(t) = 2\gamma_1 = \gamma_2$.

Taking the first and second derivative of $\gamma(t)$ w.r.t. $\omega(t)$ leads to

$$\gamma_{\omega}(t) = -\frac{\gamma_1}{2}\frac{Ke^{\omega(t)}}{\left(1 + Ke^{\omega(t)}\right)^{\frac{3}{2}}} \tag{C.13}$$

$$\gamma_{\omega\omega}(t) = -\frac{\gamma_1}{2}\frac{Ke^{\omega(t)}}{\left(1 + Ke^{\omega(t)}\right)^{\frac{5}{2}}}\left(1 - \frac{K}{2}e^{\omega(t)}\right) \tag{C.14}$$

Aggregate risk aversion is countercyclical because $\gamma_{\omega}(t) < 0$. $\gamma_{\omega\omega}(t)$ can not be signed because when $\omega(t)$ approaches minus infinity, then $\gamma_{\omega\omega}(t) < 0$ and when $\omega(t)$ approaches plus infinity, then $\gamma_{\omega\omega}(t) > 0$.

Moreover, $\lim_{\omega(t)\to-\infty}\gamma(t) = \gamma_2$ and $\lim_{\omega(t)\to\infty}\gamma(t) = \gamma_1$. This and the fact that $\gamma(t)$ is monotonically decreasing in $\omega(t)$ implies that $\gamma_1 \leq \gamma(t) \leq \gamma_2$. Applying Itô's lemma to $\gamma(t)$ given in equation (C.12) leads to the dynamics given in equation (IV.9).

□

Proof of Theorem IV.1. A nominal zero-coupon bond pays one unit of currency – i.e.

$1/\pi(U)$ units of consumption – at maturity U and hence its real price is

$$B_U(t) = \mathrm{E}_t\left[\frac{m(U)}{m(t)}\frac{1}{\pi(U)}\right]. \tag{C.15}$$

Multiplying equation (C.15) with $\pi(t)$ and using $M(t) = m(t)/\pi(t)$ and $B_U^*(t) = B_U(t)\pi(t)$ leads to the nominal price of a nominal zero-coupon bond maturing at U given in equation (IV.10). Equation (IV.11) and (IV.12) follow directly from Lemma C.2 and $M(t) = m(t)/\pi(t)$. □

Proposition C.1 (Real stochastic discount factor dynamics)**.** *The dynamics of $m(t)$ are*

$$\frac{dm(t)}{m(t)} = -r(t)\,dt - \lambda(t)\,dz_c(t). \tag{C.16}$$

The real short rate and the real market price of risk are

$$r(t) = \beta + \delta\left(\bar{\omega}(t) + (\gamma(t)-1)(\bar{\omega}(t) - \omega(t))\right) - \frac{1}{2}\left(\gamma(t)^2 - \gamma_\omega(t)\right)\sigma_c(t)^2 \tag{C.17}$$

$$\lambda(t) = \sigma_c(t)\gamma(t). \tag{C.18}$$

Proof. The real stochastic discount factor is

$$m(t) = m(0)e^{-\beta t}u_c^\kappa(c(t), X(t)) \tag{C.19}$$

with $m(0) = 1$. Applying Itô's lemma to $m(t)$ given in equation (C.19) leads to

$$\frac{dm(t)}{m(t)} = -r(t) - \lambda(t)\,dz_c(t)$$

with

$$\lambda(t) = \sigma_c(t)\left(-c(t)\,\frac{u_{cc}^\kappa(c(t), X(t))}{u_c^\kappa(c(t), X(t))}\right) = \gamma(t)\sigma_c(t) \tag{C.20}$$

and

$$r(t) = \beta + \mu_c(t)\left(-c(t)\,\frac{u^{\kappa}_{cc}(\cdot)}{u^{\kappa}_{c}(\cdot)}\right) + \delta\omega(t)\left(-X(t)\,\frac{u^{\kappa}_{cX}(\cdot)}{u^{\kappa}_{c}(\cdot)}\right) - \frac{\sigma_c(t)^2}{2}\left(c(t)^2\,\frac{u^{\kappa}_{ccc}(\cdot)}{u^{\kappa}_{c}(\cdot)}\right). \tag{C.21}$$

Let $\Omega(t) = e^{\omega(t)}$. Tedious algebra leads to

$$-X(t)\frac{u_{cX}(c(t),X(t))}{u_c(c(t),X(t))} = \left(1 + \Omega(t)\frac{u_{\Omega\Omega}(c(t),X(t)}{u_{\Omega}(c(t),X(t))}\right) = 1 - \gamma(t) \tag{C.22}$$

$$\begin{aligned} c^2\frac{u_{ccc}(c(t),X(t))}{u_c(c(t),X(t))} &= \Omega(t)^2\frac{u_{\Omega\Omega\Omega}(c(t),X(t))}{u_{\Omega}(c(t),X(t))} \\ &= \gamma(t)(1+\gamma(t)) - \Omega(t)\gamma_{\Omega}(t). \end{aligned} \tag{C.23}$$

Plugging back into equation (C.21) and using equation (C.22) and (C.23) leads to

$$r(t) = \beta + \gamma(t)\mu_c(t) + (1-\gamma(t))\delta\omega(t) - \{\gamma(t)(1+\gamma(t)) - \Omega(t)\gamma_{\Omega}(t)\}\,\frac{\sigma_c(t)^2}{2} \tag{C.24}$$

Moreover, $\gamma_{\Omega}(t)\Omega(t) = \gamma_{\omega}(t)$ which leads to the real short rate given in equation (IV.16).

□

Proof of Corollary IV.1. The dynamics of the real stochastic discount factor are given in equation (C.19) and the dynamics of the price level are given in equation (IV.2). Applying Itô's lemma to $M(t) = m(t)/\pi(t)$ leads to

$$\frac{dM(t)}{M(t)} = -R(t)\,dt - \lambda(t)dz_c(t) - \sigma_{\pi}(t)\,dz_{\pi}(t) \tag{C.25}$$

with $R(t)$ given in equation (IV.14) and $\lambda(t)$ given in equation (IV.17). Defining a new Brownian motion that satisfies $\Lambda(t)dz_M(t) = \lambda(t)dz_c(t) + \sigma_{\pi}(t)dz_{\pi}(t)$ with $\Lambda(t)$ given in equation (IV.15) completes the proof. □

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