GEOMETRY AND CONSTRUCTIONS OF FINITE FRAMES

A Thesis

by

NATHANIEL KIRK STRAWN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 2007

Major Subject: Mathematics

GEOMETRY AND CONSTRUCTIONS OF FINITE FRAMES

A Thesis

by

NATHANIEL KIRK STRAWN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Approved by:

Chair of Committee,	Kenneth Dykema
Committee Members,	David Larson
	Catherine Yan
	Nancy Amato
Head of Department,	Albert Boggess

May 2007

Major Subject: Mathematics

ABSTRACT

Geometry and Constructions of Finite Frames.

(May 2007)

Nathaniel Kirk Strawn, B.S., Texas A&M University

Chair of Advisory Committee: Kenneth Dykema

Finite frames are special collections of vectors utilized in Harmonic Analysis and Digital Signal Processing. In this thesis, geometric aspects and construction techniques are considered for the family of k-vector frames in $\mathbb{F}^n = \mathbb{R}^n$ or $\mathbb{C}^n$ sharing a fixed frame operator (denoted $\mathcal{F}^k(E, \mathbb{F}^n)$, where E is the Hermitian positive definite frame operator), and also the subfamily of this family obtained by fixing a list of vector lengths (denoted $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$, where μ is the list of lengths).

The family $\mathcal{F}^k(E, \mathbb{F}^n)$ is shown to be diffeomorphic to the Stiefel manifold $V_n(\mathbb{F}^k)$, and $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$ is shown to be a smooth manifold if the list of vector lengths μ satisfy certain conditions. Calculations for the dimensions of these manifolds are also performed. Finally, a new construction technique is detailed for frames in $\mathcal{F}^k(E, \mathbb{F}^n)$ and $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$.

To my Mother and Father.

ACKNOWLEDGMENTS

Any endeavor requires the direct and indirect support of an entire community. I express my gratitutde to the vast spectrum of seemingly invisible hands that make even the most minor triumphs of civilization possible.

Though I have just thanked everyone in one fell swoop, there are some very special individuals that I single out for thanks. In particular, I thank David Larson for providing me with the opportunity to cut my teeth on research. I thank David Larson, Kenneth Dykema, and Catherine Yan for their mentorship, support, encouragement, and for providing me with opportunities to attain my potential. I also thank Nancy Amato for her time and patience.

The main results obtained here are direct extensions of results obtained by Pete Cassaza, Keri Kornelson, Kenneth Dykema, and David Larson. I thank these people for their direct contribution.

My family and friends have made this process possible. Thank you all. Also, I thank Elizabeth Fraser. She provided the soundtrack for this work.

TABLE OF CONTENTS

Page

Page

LIST OF FIGURES

1. INTRODUCTION

A finite frame is essentially a (possibly overcomplete) basis for a vector space. Frame Theory is an indispensable tool for numerous pursuits including Harmonic Analysis, Internet Coding, Digital Signal Processing, and Quantum Theory to name only a few. This document explores geometric aspects of the family of k-member real and complex frames that share a fixed frame operator, as well as the subfamily obtained by specifying a list of vector lengths for frames in this family. These families are denoted $\mathcal{F}^k(E, \mathbb{F}^n)$ and $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$ respectively. Also, encyclopedic construction techniques are detailed for both of these families.

In the Geometry section of this thesis, a diffeomorphism between $\mathcal{F}^k(E, \mathbb{F}^n)$ and the Stiefel manifold $V_n(\mathbb{F}^k)$ is produced when $n \leq k$. An alternate proof elucidating the manifold structure of $\mathcal{F}^k(E, \mathbb{F}^n)$ is also produced, and an adaptation of the technique utilized in this proof demonstrates that $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$ is a manifold when the list of lengths μ satisfy certain constraints imposed by E. These results generalize a number of results obtained in Dykema et al. [6].

Constructions for frames have been considered in a number of settings. A constructive existence proof for frames in $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$ was first demonstrated by Cassaza and Leon [2], Dhillon et al. [4, 5] detail a number of methods for computing frames in $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$, and the frame potential introduced by Benedetto and Fickus [1] provides method for computing tight and near tight frames from a given vector sequence. The available methods for constructing arbitrary frames in $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$ are either exhaustive in scope but indiscriminant with respect to vector position, or precise with

---

The journal model is *Applied and Computational Harmonic Analysis.*

respect vector position but narrow in scope. In this thesis, an encyclopedic construction method is detailed. Moreover, this method allows one to hand pick frame vectors. This does not come without it's caveats, as the process is computationally expensive. Mitigating the computational expenses of this construction technique is also addressed.

This paragraph discusses the organization of this thesis. In the second section, preliminary notions and notation from Matrix Analysis, Frame Theory, and Differential Geometry are presented. The results summarized above are then presented in sections three and four, corresponding to geometry and construction results. Section five concludes the thesis by summarizing the results and describing avenues of research opened up, but not explored by this thesis. Finally, the Appendix contains all of the MATLAB code produced by the author.

2. PRELIMINARIES

This section details general notational conventions and pertinent results from Matrix Analysis, Frame Theory, and elementary Differential Geometry. Throughout this thesis, $[n]$ is used to denote the set $\{1, \ldots, n\}$.

2.1. Matrix Analysis

What follows is a survey of basic notions and notation from Matrix Analysis. The main reference for this section is Horn and Johnson [8].

For positive integers n and k, $\mathbf{M}_{n\times k}(\mathbb{F})$, $\mathbf{M}_n(\mathbb{F})$, F^*, and $\mathbf{H}^n(\mathbb{F})$ denote the n by k matrices with entries in the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the n by n matrices, the Hermitian adjoint of a matrix $F \in \mathbf{M}_{n\times k}(\mathbb{F})$, and the set of all self adjoint n by n matrices respectively. Finally, I_n denotes the n by n identity matrix.

For a given $H \in \mathbf{H}^n(\mathbb{F})$, it is a standard result that $H = U\Lambda U^*$ where U is such that $UU^* = I_n$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. If the eigenvalues $\{\lambda_i\}_{i=1}^n$ are ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then $\lambda_i(H) = \lambda_i$ and $\lambda(H) = \{\lambda_i(H)\}_{i=1}^n$ is said to be *spectrum* of H.

Given a matrix $E \in \mathbf{M}_n(\mathbb{F})$ such that $0 \leq \langle Ex, x\rangle$ for all $x \in \mathbb{F}^n$, E is said to be positive semidefinite and one writes $0 \preceq E$. When this inequality is strict for all $x \in \mathbb{F}^n$, E is said to be positive definite and one writes $0 \prec E$. Futhermore, the relations $E \preceq F$ and $E \prec F$ are written whenever $0 \preceq F - E$ and $0 \prec F - E$.

Let $\mathbb{R}^n_{+\geq}$ denote the set of all strictly positive non-increasing sequences of real numbers with n entries. Note that E is positive definite if and only if $\lambda(E) \in \mathbb{R}^n_{+\geq}$.

For two sequences $\lambda, \mu \in \mathbb{R}^n_{+\geq}$, λ is said to majorize μ if

$$\sum_{i=1}^{m} \mu_i \leq \sum_{i=1}^{m} \lambda_i \tag{2.1.1}$$

for all $m \in [n-1]$ and $\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \lambda_i$. If λ majorizes μ, we write $\mu \preceq \lambda$. A useful generalization of majorization is n-majorization (see Dahl and Margot [3]). A sequence $\lambda \in \mathbb{R}^n_{+\geq}$ is said to n-majorize a sequence $\mu \in \mathbb{R}^k_{+\geq}$ if (2.1.1) holds for all $m \in [n-1]$ and $\sum_{i=1}^{k} \mu_i = \sum_{i=1}^{n} \lambda_i$. When λ n-majorizes μ, one writes $\mu \preceq_n \lambda$.

The *trace* of a matrix $E \in \mathbf{M}_n(\mathbb{F})$ is the sum of the diagonal entries of E. The trace of a matrix E is denoted $\mathrm{tr}(E)$. Corollary 2.5 of Cassaza and Leon [2] will be utilized extensively in the fourth section, so we state it here for reference.

Proposition 2.1.1. *Suppose $E \in \mathbf{H}^n(\mathbb{F})$ is such that $\lambda(E) \in \mathbb{R}^n_{+\geq}$ and further suppose that P is a rank m projection on $\mathbb{F}^n$. Then*

$$tr(PEP) \leq \sum_{i=1}^{m} \lambda_i(E).$$

2.2. Frame Theory

A *frame* is a collection of vectors $F = \{f_i\}_{i \in I}$ in some Hilbert space $\mathcal{H}$ satisfying

$$A\|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq B\|x\|^2$$

for all $x \in \mathcal{H}$, where $0 < A \leq B$ are constants. When such constants exist, the supremum over applicable A and infimum over applicable B produce the *frame bounds* of F. If the index set I is finite, then $\mathcal{H}$ must be finite dimensional and it is trivial that the members of F span $\mathcal{H}$.

Throughout this document, $\mathcal{H}$ is finite dimensional and any frame will contain only finitely many members. Under these circumstances a frame may be identified with the matrix $F = [f_1 \cdots f_k] \in \mathbf{M}_{n \times k}(\mathbb{F})$, where the columns of F are k vectors that form a frame for $\mathbb{F}^n$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. As an operator from $\mathbb{F}^k$ to $\mathbb{F}^n$, the matrix F is called the *synthesis operator* and its Hermitian adjoint F^* is called the *analysis operator*. Furthermore, the n by n matrix given by

$$E = FF^* = \sum_{i=1}^{k} f_i f_i^*$$

is called the *frame operator* of F. The rightmost expression in this equation harkens back to the rank one decompositions of Kornelson and Larson [11]. Note that the frame operator of a frame F is always Hermitian positive definite. The maps $\Phi_n^k : \mathbf{M}_{n\times k}(\mathbb{F}) \to \mathbf{H}^n(\mathbb{F})$, $F \mapsto FF^*$ receive a great deal of attention in the third section, so these are called the *frame operator maps* and the indices are omitted when there is no danger of ambiguity.

Given a positive definite $E \in \mathbf{H}^n(\mathbb{F})$, define $\mathcal{F}^k(E, \mathbb{F}^n) \subset \mathbf{M}_{n\times k}(\mathbb{F})$ to be the family of frames in $\mathbb{F}^n$ with k members and frame operator E. It is worth noting that $(\Phi_n^k)^{-1}(\{E\}) = \mathcal{F}^k(E, \mathbb{F}^n)$ whenever $E \in \mathbf{H}^n(\mathbb{F})$ is positive definite. For any $\mu \in \mathbb{R}^k_{+\geq}$, denote the embedded product of spheres

$$\mathbb{S}_\mu^k(\mathbb{F}^n) = \{F = [f_1 \cdots f_k] \in \mathbf{M}_{n\times k}(\mathbb{F}) | \|f_i\|^2 = \mu_i, \forall i \in [k]\},$$

Whenever $E \in \mathbf{H}^n(\mathbb{F})$ is positive definite and $\mu \in \mathbb{R}^k_{+\geq}$, define $\mathcal{F}_\mu^k(E, \mathbb{F}^n) = \mathcal{F}^k(E, \mathbb{F}^n) \cap \mathbb{S}_\mu^k(\mathbb{F}^n)$. Moreover, by restricting the domain of the frame operator map Φ_n^k to $\mathbb{S}_\mu^k(\mathbb{F}^n)$, one obtains $(\Phi_n^k)^{-1}(\{E\}) = \mathcal{F}_\mu^k(E, \mathbb{F}^n)$ whenever E is positive definite.

Theorem 2.1 of Cassaza and Leon [2] is an essential ingredient throughout this paper, so a translated version shall be presented as a reference for the reader.

Theorem 2.2.1. *(Cassaza and Leon) Let* $E \in \mathbf{H}^n(\mathbb{F})$ *be such that* $\lambda(E) \in \mathbb{R}^n_{+\geq}$ *and suppose that* $\sum_{i=1}^k \mu_i = \sum_{i=1}^n \lambda_i(E)$ *for some* $\mu \in \mathbb{R}^k_{+\geq}$ *with* $k \geq n$*. Then* $\mathcal{F}_\mu^k(E, \mathbb{F}^n)$ *is nonempty if and only if* $\mu \preceq_n \lambda$*.*

2.3. Basic Differential Geometry

It will be assumed that the reader is familiar with the basic theory of $\mathcal{C}^\infty$-manifolds. The notions pertinent to our discussion are tangent spaces, smooth maps between manifolds, and regular points and values of smooth maps between manifolds. For an elementary overview of these notions, see Milnor [12]. In particular, the following proposition is used in the second section.

Proposition 2.3.1. *If $f : \mathcal{M}^m \to \mathcal{N}^n$ is a smooth map between manifolds of dimension $m \geq n$ and $y \in \mathcal{N}$ is a regular value of f, then the set $f^{-1}(\{y\})$ is a smooth submanifold of $\mathcal{M}$ with dimension $m - n$.*

The families of all subsets of in $\mathbb{F}^k$ composed of n orthonormal vectors (Stiefel manifolds) contribute to the disussion in Section 2. Concretely, these are the sets $V_n(\mathbb{F}^k) = \{F \in \mathbf{M}_{k\times n}(\mathbb{F}) | F^*F = I_n\}$.

3. GEOMETRY

This section begins with the proof that the family of k-member frames sharing a given frame operator is a manifold. The main result of this chapter, however, is that the intersection of this manifold with a product of spheres gives rise to a manifold when certain restrictions are imposed upon the radii of the spheres. Explicit calculations for the dimensions of these manifolds are also performed.

3.1. The Manifold Structure of $\mathcal{F}^k(E, \mathbb{F}^n)$

In this subsection, the manifold structure of $\mathcal{F}^k(E, \mathbb{F}^n)$ is elucidated. First, it will be shown that $\mathcal{F}^k(E, \mathbb{F}^n)$ may be identified with the Stiefel manifold $V_n(\mathbb{F}^k)$. In the sequel, a constructive proof is presented.

3.1.1. Identification with Stiefel manifolds and remarks

A brief remark by Dhillon et al. [5] drew a connection between Stiefel manifolds and frames. Indeed, it is a trivial observation that the map $F \mapsto \sqrt{E}F^*$ is a diffeomorphism between the Stiefel manifold $V_n(\mathbb{F}^k)$ and $\mathcal{F}^k(E, \mathbb{F}^n)$ whenever E is Hermitian and positive definite. The immediate benefit of this connection is that efficient non-linear optimization techniques may be brought to bear on these manifolds. The dissertation of S. Smith [13] details these techniques.

3.1.2. An alternate, generalizable proof

This proof might seem superfluous in light of the previous observation. Nevertheless, it generalizes to the case $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$.

Proposition 3.1.1. *Let $n \leq k$ be integers and suppose that $E_r \in \mathbf{H}^n(\mathbb{R})$ and $E_c \in \mathbf{H}^n(\mathbb{C})$ are both positive definite. Then*

(i) $\mathcal{F}^k(E_r, \mathbb{R}^n)$ *is a smooth submanifold of* $\mathbf{M}_{n\times k}(\mathbb{R})$ *with dimension*

$$dim\left(\mathcal{F}^k(E_r, \mathbb{R}^n)\right) = nk - \binom{n+1}{2};$$

(ii) $\mathcal{F}^k(E_c, \mathbb{C}^n)$ *is a smooth submanifold of* $\mathbf{M}_{n\times k}(\mathbb{C})$ *with dimension*

$$dim\left(\mathcal{F}^k(E_c, \mathbb{C}^n)\right) = 2nk - \binom{n+1}{2} - \binom{n}{2}.$$

Proof. First consider the complex case. Set $E = E_c$ and let $\Phi_n^k : \mathbf{M}_{n\times k}(\mathbb{C}) \to \mathbf{H}^n(\mathbb{C})$ be the frame operator map. For the remainder of the proof, this map is just Φ. It is immediately clear that $\mathcal{F}^k(E, \mathbb{C}^n) = \Phi^{-1}(\{E\})$. Note that $\mathbf{M}_{n\times k}(\mathbb{C})$ and $\mathbf{H}^n(\mathbb{C})$ are smooth manifolds of dimension $2nk$ and $\binom{n+1}{2} + \binom{n}{2}$ respectively, so E a regular value of Φ implies the result by Proposition 2.3.1.

Let $F = [f_1 \cdots f_k]$ be such that $\Phi(F) = E$. For $G = [g_1 \cdots g_n] \in \mathbf{M}_{n\times k}(\mathbb{C}) \equiv T_F\mathbf{M}_{n\times k}(\mathbb{C})$, we have that $\alpha(0) = F$ and $\alpha'(0) = G$ when $\alpha(t) = F + tG$, so application of the differential map yields

$$\begin{aligned}
d\Phi_F(G) &= \left.\frac{d}{dt}\right|_{t=0} \sum_{i=1}^{k} (f_i + tg_i)(f_i + tg_i)^* \\
&= \left.\frac{d}{dt}\right|_{t=0} \sum_{i=1}^{k} f_i f_i^* + t(f_i g_i^* + g_i f_i^*) + t^2 g_i g_i^* \\
&= \sum_{i=1}^{k} f_i g_i^* + g_i f_i^*.
\end{aligned}$$

Since F is a frame, $g_i = \sum_{j=1}^k \gamma_{ij} f_j$ for some choice of γ_{ij}, so

$$
\begin{aligned}
d\Phi_F(G) &= \sum_{i=1}^k f_i \left(\sum_{j=1}^k \gamma_{ij} f_j\right)^* + \left(\sum_{j=1}^k \gamma_{ij} f_j\right) f_i^* \\
&= \sum_{i=1}^k \sum_{j=1}^k \overline{\gamma_{ij}} f_i f_j^* + \gamma_{ij} f_j f_i^* \\
&= \sum_{i=1}^k (\gamma_{ii} + \overline{\gamma_{ii}}) f_i f_i^* \\
&\quad + \sum_{1\le i<j\le k} (\gamma_{ji} + \overline{\gamma_{ij}}) f_i f_j^* + (\overline{\gamma_{ji}} + \gamma_{ij}) f_j f_i^*. \qquad (3.1.1)
\end{aligned}
$$

Now consider a given $H \in T_E\mathbf{H}^n(\mathbb{C}) \equiv \mathbf{H}^n(\mathbb{C})$. If $\{e_i\}_{i=1}^n$ is the canonical orthonormal basis of $\mathbb{C}^n$, then H has an expansion

$$
H = \sum_{i=1}^n h_{ii} e_i e_i^* + \sum_{1\le i<j\le n} \left(h_{ij} e_i e_j^* + \overline{h_{ij}} e_j e_i^*\right)
$$

where h_{ii} is real for all $i \in [n]$. Since F is a frame, it spans $\mathbb{C}^n$ and hence $e_i = \sum_{j=1}^k e_i^j f_j$ for some choice of coefficients e_i^j whenever $i \in [n]$. $\{f_i f_j^*\}_{(i,j)\in[n]^2}$. A computation then yields

$$
\begin{aligned}
H &= \sum_{i=1}^n h_{ii} \left(\sum_{j=1}^k e_i^j f_j\right)\left(\sum_{j=1}^k e_i^j f_j\right)^* + \sum_{1\le i<j\le n} h_{ij} \left(\sum_{l=1}^k e_i^l f_l\right)\left(\sum_{l=1}^k e_j^l f_l\right)^* \\
&\quad + \sum_{1\le i<j\le n} \overline{h_{ij}} \left(\sum_{l=1}^k e_j^l f_l\right)\left(\sum_{l=1}^k e_i^l f_l\right)^* \\
&= \sum_{i=1}^k \left(\sum_{j=1}^n h_{jj} e_j^i \overline{e_j^i}\right) f_i f_i^* + \sum_{i=1}^k \left(\sum_{1\le j<l\le n} h_{jl} e_j^i \overline{e_l^i} + \overline{h_{jl}} e_l^i \overline{e_j^i}\right) f_i f_i^* \\
&\quad + \sum_{1\le i<j\le k} \left(\left(\sum_{l=1}^n h_{ll} e_l^i \overline{e_l^j}\right) f_i f_j^* + \left(\sum_{l=1}^n h_{ll} e_l^j \overline{e_l^i}\right) f_j f_i^*\right) \\
&\quad + \sum_{1\le i<j\le k} \left(\left(\sum_{1\le l<m\le n} h_{lm} e_l^i \overline{e_m^j}\right) f_i f_j^* + \left(\sum_{1\le l<m\le n} h_{lm} e_l^j \overline{e_m^i}\right) f_j f_i^*\right) \\
&\quad + \sum_{1\le i<j\le k} \left(\left(\sum_{1\le l<m\le n} \overline{h_{lm}} e_m^i \overline{e_l^j}\right) f_i f_j^* + \left(\sum_{1\le l<m\le n} \overline{h_{lm}} e_m^j \overline{e_l^i}\right) f_j f_i^*\right).
\end{aligned}
$$

This simplifies to $H = \sum_{i=1}^{k} h'_{ii} f_i f_i^* + \sum_{1 \le i < j \le k} h'_{ij} f_i f_j^* + \sum_{1 \le i < j \le k} \overline{h'_{ij}} f_j f_i^*$ where

$$h'_{ii} \;=\; \left[\left(\sum_{j=1}^{n} h_{jj} |e_j^i|^2 \right) + 2 \left(\sum_{1 \le j < l \le n} \Re \left(h_{jl} e_j^i \overline{e_l^i} \right) \right) \right]$$

for all $i \in [k]$ and

$$h'_{ij} \;=\; \left[\left(\sum_{l=1}^{n} h_{ll} e_l^i \overline{e_l^j} \right) + \left(\sum_{1 \le l < m \le n} h_{lm} e_l^i \overline{e_m^j} + \overline{h_{lm}} e_m^i \overline{e_l^j} \right) \right]$$

for all $i, j \in [k]$ with $i < j$. Noting that h'_{ii} is real, we may simply assume that

$$H = \sum_{i=1}^{k} h_{ii} f_i f_i^* + \sum_{1 \le i < j \le k} h_{ij} f_i f_j^* + \overline{h_{ij}} f_j f_i^*$$

where h_{ii} is real for each $i \in [k]$. Choosing G so that $\gamma_{ji} + \overline{\gamma_{ij}} = h_{ij}$ and $\gamma_{ii} + \overline{\gamma_{ii}} = h_{ii}$ for all $i, j \in [k]$ then implies $d\Phi_F(G) = H$. H and F were arbitrary, so $d\Phi_F$ is onto for all F and hence E is a regular value of Φ. Thus, the result holds in the complex case. The real case follows from a similar proof with marginally less complexity. $\square$

3.2. The Manifold Structure of $\mathcal{F}_\mu^k(E, \mathbb{F}^n)$

In this subsection, it is shown that $\mathcal{F}_\mu^k(E, \mathbb{F}^n)$ is a manifold under suitable conditions. First, redistributions on node-weighted graphs are introduced to provide some essential machinery. After defining the notion of orthodecomposability, the results concerning redistributions are applied to show that a frame F is a regular point of the (restricted) map Φ if and only if F is not orthodecomposable. Finally, characterizations are provided for all pairs μ and E such that $\mathcal{F}_\mu^k(E, \mathbb{F}^n)$ contains no orthodecomposable frame.

3.2.1. Redistributions on a node-weighted graph

Definition 3.2.1. Suppose that $\Gamma = (V, E)$ is a loopless oriented graph and that V is finite. A skew symmetric function $\delta : V \times V \to \mathbb{R}$ is called a *redistribution flow* on Γ if $\delta(v_\alpha, v_\beta) = 0$ whenever $(v_\alpha, v_\beta) \notin E$ and $(v_\beta, v_\alpha) \notin E$. If Γ is equipped with a node weighting function $w : V \to \mathbb{R}$, any redistribution flow induces a new node weighting function by setting

$$w_\delta(v) = w(v) + \sum_{v_\alpha \in V} \delta(v_\alpha, v)$$

for all $v \in V$. Such a w_δ is called a *redistribution* of w on Γ.

Redistribution flows are very intuitive objects. Given a loopless oriented graph Γ, one may imagine that the vertices are asset traders and that the edges are transaction interfaces. If δ is a redistribution flow on Γ, then $\delta(v_\alpha, v_\beta)$ can represent the net amount of assets passed from the trader v_α to the trader v_β over the course of a day. Of course, one should expect that the amount of assests passed from trader v_β to v_α should be negative, and skew symmetry of δ ensures that this occurs. For a node weighting function w on Γ, we may think of a given node's weight as the respective trader's assets before a day's trading commences. The redistribution w_δ then reflects the new distribution of assets among the traders at the end of the day.

It is clear that redistribution flows induce redistributions, but what range of node weighting functions arise in this manner? Intuitively, a redistribution should conserve the sum of initial weights. Indeed, $\sum_{v_\alpha \in V} w_\delta(v_\alpha) = \sum_{v_\alpha \in V} w(v_\alpha)$ follows immediately from the constraints imposed upon a redistribution flow δ and its corresponding redistribution w_δ. If w' is some node weighting function on a loopless, oriented, connected Γ with some node weighting function w, then the conservation condition is also a sufficient condition for the existence of a redistribution flow δ such that

$w' = w_\delta$. Given a w' satisfying the conservation condition imposed by w, we will present an algorithm for constructing a redistribution flow δ so that $w' = w_\delta$. The process is simple, but notation obfuscates the simplicity. To aid comprehension, we present an example.

Example 3.2.2. Consider the following connected graph Γ. A rooted tree is obtained from Γ by finding a spanning tree T and declaring v_1 to be the root of this tree. This is shown in Figure 1.

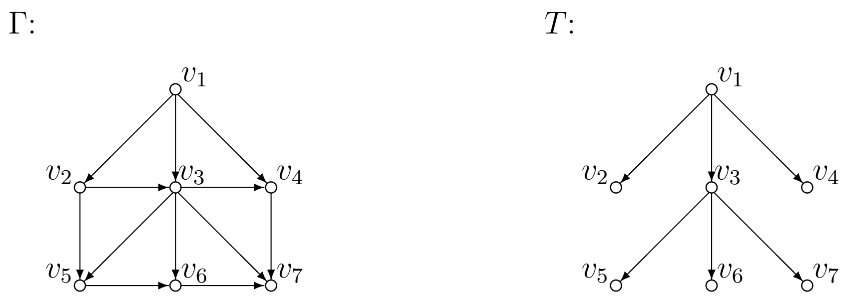

Fig. 1. Extracting a rooted tree from Γ.

Now suppose that w is a node weighting function on Γ and that w' satisfies

$$\sum_{v_i \in V} w'(v_i) = \sum_{v_i \in V} w(v_i).$$

We will use $T = (V, E_T)$ to inductively define a redistribution flow δ so that $w' = w_\delta$.

First, set $\delta(v_i, v_j) = 0$ for all i and j such that $(v_i, v_j) \notin E_T$ and $(v_j, v_i) \notin E_T$. We now turn our attention to the youngest generation of nodes in the tree, $W_2 = \{v_5, v_6, v_7\}$. Set $\delta(v_3, v_i) = w'(v_i) - w(v_i)$ for $v_i \in W_2$. Since no other vertices are adjacent to any $v_i \in W_2$, this assignment of values immediately implies

$$w'(v_i) = w(v_i) + \sum_{j=1}^{7} \delta(v_j, v_i)$$

for all $v_i \in W_2$. We now consider the first generation nodes $W_1 = \{v_2, v_3, v_4\}$. Setting $\delta(v_1, v_i) = w'(v_i) - w(v_i)$ ensures that the above equation is also satisfied for $i = 2, 4$. However, there is now a net flow that has accumulated at v_3, so we must set

$$\delta(v_1, v_3) = w'(v_3) - w(v_3) - \sum_{v_i \in W_2} \delta(v_i, v_3)$$

to ensure that the equation holds when $i = 3$. At this point, δ has been completely defined but we must still verify that the equation holds for $i = 1$. A computation yields

$$\begin{aligned}
w(v_1) + \sum_{i=1}^{7} \delta(v_i, v_1) &= w(v_1) + \sum_{v_i \in W_1} \delta(v_i, v_1) = w(v_1) - \sum_{v_i \in W_1} \delta(v_1, v_i) \\
&= w(v_1) - \sum_{v_i \in W_1} \left(w'(v_i) - w(v_i) - \sum_{j=2}^{7} \delta(v_j, v_i) \right) \\
&= \sum_{i=1}^{4} w(v_i) - \sum_{i=2}^{4} w'(v_i) + \sum_{v_i \in W_2} \delta(v_i, v_3) \\
&= \sum_{i=1}^{4} w(v_i) - \sum_{i=2}^{4} w'(v_i) - \sum_{v_i \in W_2} \delta(v_3, v_i) \\
&= \sum_{i=1}^{4} w(v_i) - \sum_{i=2}^{4} w'(v_i) - \sum_{v_i \in W_2} w'(v_i) - w(v_i) \\
&= \sum_{i=1}^{7} w(v_i) - \sum_{i=2}^{7} w'(v_i) \\
&= w'(v_1),
\end{aligned}$$

and so we may conclude that $w' = w_\delta$. Note how the conservation of total weight was utilized at the very last equality.

We now proceed to demonstrate this algorithm in full generality.

Lemma 3.2.3. *Let Γ and w satisfy the hypothesis of Definition 3.2.1. All functions $w' : V \to \mathbb{R}$ satisfying $\sum_{i \in V} w'(i) = \sum_{i \in V} w(i)$ are redistributions of w on Γ if and only if Γ is connected.*

Proof. If Γ is connected and finite, then it contains a finite spanning tree $T = (V, E_T)$. Convert this to a rooted tree by distinguishing some $v_0 \in V$. Let W_k denote the k^{th}-generation descendants of the root v_0, set $V_k = \bigcup_{i=0}^{k} W_i$ and $V_k^* = V_k \setminus \{v_0\}$, and let n be the maximum depth of the tree. Note that each $v_\alpha \in V$ has exactly one parent when $v_\alpha \neq v_0$. This parent will be denoted v_{β_α}.

Choose an arbitrary w' satisfying the hypothesis. We will define the redistribution flow corresponding to w' via the rooted tree T. First, set $\delta(v_\alpha, v_\beta) = 0$ for all edges (v_α, v_β) that are not contained in E_T. For all $v_\alpha \in W_n$, set $\delta(v_{\beta_\alpha}, v_\alpha) = w'(v_\alpha) - w(v_\alpha)$, and impose skew-symmetry. Next, for all $v_\alpha \in W_{n-1}$, set

$$\delta(v_{\beta_\alpha}, v_\alpha) = w'(v_\alpha) - w(v_\alpha) - \sum_{v_\beta \in W_n} \delta(v_\beta, v_\alpha).$$

Imposing skew-symmetry again and continuing this process inductively yields

$$\delta(v_{\beta_\alpha}, v_\alpha) \quad = \quad w'(v_\alpha) - w(v_\alpha) - \sum_{v_\beta \in W_{k+1}} \delta(v_\beta, v_\alpha)$$

for all $v_\alpha \in W_k$ and all $k \in [n]$. By construction, $w'(v_\alpha) = w(v_\alpha) + \sum_{v_\beta \in V} \delta(v_\beta, v_\alpha)$ for all $v_\alpha \in W_k$ and all $k \in [n]$. All that remains is to show that this holds for v_0. We have that

$$\begin{aligned}
w(v_0) + \sum_{v_\alpha \in V} \delta(v_\alpha, v_0) &= w(v_0) + \sum_{v_\alpha \in W_1} \delta(v_\alpha, v_0) = w(v_0) - \sum_{v_\alpha \in W_1} \delta(v_\alpha, v_0) \\
&= w(v_0) + \sum_{v_\alpha \in W_1} \left(w(v_\alpha) - w'(v_\alpha) + \sum_{v_\beta \in W_2} \delta(v_\beta, v_\alpha) \right) \\
&= \sum_{v_\alpha \in V_1} w(v_\alpha) - \sum_{v_\alpha \in V_1^*} w'(v_\alpha) + \sum_{v_\alpha \in W_1} \sum_{v_\beta \in W_2} \delta(v_\beta, v_\alpha) \\
&= \sum_{v_\alpha \in V_1} w(v_\alpha) - \sum_{v_\alpha \in V_1^*} w'(v_\alpha) - \sum_{v_\alpha \in W_2} \delta(v_{\beta_\alpha}, v_\alpha).
\end{aligned}$$

Applying induction produces

$$\begin{aligned}
w(v_0) + \sum_{v_\alpha \in V} \delta(v_\alpha, v_0) &= \sum_{v_\alpha \in V_{n-1}} w(v_\alpha) - \sum_{v_\alpha \in V_{n-1}^*} w'(v_\alpha) - \sum_{v_\alpha \in W_n} \delta(v_{\beta_\alpha}, v_\alpha) \\
&= \sum_{v_\alpha \in V_{n-1}} w(v_\alpha) - \sum_{v_\alpha \in V_{n-1}^*} w'(v_\alpha) - \sum_{v_\alpha \in W_n} (w(v_\alpha) - w'(v_\alpha)) \\
&= \sum_{v_\alpha \in V} w(v_\alpha) - \sum_{v_\alpha \in V \setminus \{v_0\}} w'(v_\alpha) \\
&= w'(v_0),
\end{aligned}$$

where the last equality holds by hypothesis. This line of reasoning shows that connectivity of Γ is a sufficient condition.

Necessity will follow by demonstration of the contrapositive. Suppose, now, that Γ is not connected. Then Γ contains two nonempty disjoint connected components, $\Gamma_i = (V_i, E_i)$ for $i = 1, 2$. If w' and δ constitute a network redistribution of w on Γ, then skew-symmetry implies $\sum_{v_\alpha \in V_i} \sum_{v_\beta \in V_i} \delta(v_\beta, v_\alpha) = 0$, and so disjointness implies

$$\begin{aligned}
\sum_{v_\alpha \in V_i} w'(v_\alpha) &= \sum_{v_\alpha \in V_i} w(v_\alpha) + \sum_{v_\alpha \in V_i} \sum_{v_\beta \in V} \delta(v_\beta, v_\alpha) \\
&= \sum_{v_\alpha \in V_i} w(v_\alpha) + \sum_{v_\alpha \in V_i} \sum_{v_\beta \in V_i} \delta(v_\beta, v_\alpha) \\
&= \sum_{v_\alpha \in V_i} w(v_\alpha)
\end{aligned}$$

for $i = 1, 2$. Thus, any admissible redistribution must conserve the total weight on each disjoint connected component. Choose $v_1 \in V_1$ and $v_2 \in V_2$. Define w' by setting $w'(v_\alpha) = w(v_\alpha)$ for all $v_\alpha \neq v_1$ and $v_\alpha \neq v_2$. Set $w'(v_1) = w(v_1) + 1$ and $w'(v_2) = w(v_2) - 1$. Clearly, w' satisfies the hypothesis but fails to conserve the weight on the disjoint components. Consequently, w' cannot be a redistribution of w on Γ and the connectivity's necessity follows. $\square$

3.2.2. Orthodecomposablity and regular points of Φ

Capitalizing on the previous technical lemma, a connection is drawn between the notion of orthodecomposability introduced by Dykema et al. [6] and the regular points of the frame operator map. First, we introduce the general definition of orthodecomposability.

Definition 3.2.4. Let $F = [f_i]_{i \in I}$ be a frame for some Hilbert space $\mathcal{H}$ and suppose that none of the frame vectors are zero. We say F is *orthodecomposable* if there is a proper, nonempty subset $A \subset I$ such that $[f_i]_{i \in A}$ is a frame for some subspace $\mathcal{V}$ of $\mathcal{H}$ and $[f_i]_{i \in A^c}$ is a frame for $\mathcal{V}^\perp$. In addition, we define the *correlation network* of F to be the oriented graph on I labeled vertices where an edge (v_i, v_j) is in the edge set if and only if $\langle f_i, f_j \rangle \neq 0$ and $i < j$. This network is denoted Γ_F.

An elementary lemma now demonstrates the relationship between connectivity the correlation network of a frame F and orthodecomposability of F.

Lemma 3.2.5. *Let $\mathcal{H}$ be a Hilbert space. A frame $F = [f_i]_{i \in I}$ for $\mathcal{H}$ is orthodecomposable if and only if its correlation network Γ_F is disconnected.*

Proof. F is orthodecomposable if and only if there is a proper, nonempty subset $A \subset I$ such that $[f_i]_{i \in A}$ is a frame for some subspace $\mathcal{V}$ of $\mathcal{H}$ and $[f_i]_{i \in A^c}$ is a frame for $\mathcal{V}^{\perp}$. This is equivalent to the statement $\langle f_i, f_j \rangle = 0$ for all $i \in A$ and all $j \in A^c$ which is in turn equivalent to the statement that all vertices of Γ_F with labels in A are not connected to any of the vertices of Γ_F with labels in A^c. $\square$

Lastly, restrictions must be placed upon the map Φ in order to generalize the result obtained in the first section. Both the domain and range of the map must be slightly altered. Choose $\mu \in \mathbb{R}^k_{+\geq}$ and suppose that $\sum_{i=1}^k \mu_i = c$. Let $\mathbf{H}^n_c(\mathbb{F})$ denote the space of all Hermitian matrices with trace equal to c. Since $\operatorname{tr}(\sum_{i=1}^k f_i f_i^*) = \sum_{i=1}^k \|f_i\|^2$, we then have that $\Phi : \mathbb{S}^k_\mu(\mathbb{F}^n) \to \mathbf{H}^n_c(\mathbb{F})$ is well defined. Moreover,

$$T_F\mathbb{S}^k_\mu(\mathbb{F}^n) \equiv \{G = [g_1 \cdots g_k] \in \mathbf{M}_{k\times n}(\mathbb{F}) | \Re(\langle f_i, g_i \rangle) = 0, \forall i \in [k]\}$$

and $T_E\mathbf{H}^n_c(\mathbb{F}) \equiv \mathbf{H}^n_0(\mathbb{F})$ for all appropriate F and E. The stage is now set to demonstrate the theorem.

Theorem 3.2.6. *Suppose that $F \in \mathcal{F}^k_\mu(E, \mathbb{F}^n)$ for some $\mu \in \mathbb{R}^k_{+\geq}$ and a positive definite $E \in \mathbf{H}^n_c(\mathbb{F})$. Further suppose that $\sum_{i=1}^k \mu_i = c$. Then F is a regular point of $\Phi : \mathbb{S}^k_\mu(\mathbb{F}^n) \to \mathbf{H}^n_c(\mathbb{F})$ if and only if F is not orthodecomposable.*

Proof. Let $\mathbb{F} = \mathbb{C}$ and suppose $H \in T_E\mathbf{H}^n_c(\mathbb{C}) \equiv \mathbf{H}^n_0(\mathbb{C})$. As in Proposition 1, H admits an expansion $H = \sum_{i=1}^k h_{ii} f_i f_i^* + \sum_{1\leq i<j\leq k} h_{ij} f_i f_j^* + \overline{h_{ij}} f_j f_i^*$. The trace condition on H is equivalent to $\sum_{i=1}^k h_{ii}\|f_i\|^2 + \sum_{1\leq i<j\leq k} h_{ij} \langle f_i, f_j \rangle + \overline{h_{ij}} \langle f_j, f_i \rangle = 0$, which simplifies to

$$\frac{1}{2}\sum_{i=1}^k h_{ii}\|f_i\|^2 + \sum_{1\leq i<j\leq k} \Re\left(h_{ij} \langle f_i, f_j \rangle\right) = 0 \tag{3.2.1}$$

upon further inspection.

Let $G = [g_1 \cdots g_k] \in T_F\mathbb{S}^k_\mu(\mathbb{C}^n)$. There is an expansion $g_i = \sum_{j=1}^k \gamma_{ij} f_j$ for each $i \in [k]$ since F is a frame. By (3.1.1) of Proposition 3.1.1, application of the differential map yields

$$d\Phi_F(G) \;=\; \sum_{i=1}^{k}(\gamma_{ii}+\overline{\gamma_{ii}})f_if_i^* + \sum_{1\le i<j\le k}(\gamma_{ji}+\overline{\gamma_{ij}})f_if_j^* + (\overline{\gamma_{ji}}+\gamma_{ij})f_jf_i^*$$

Equating this expansion with the expansion of H, we arrive at the system of equations $\gamma_{ji}+\overline{\gamma_{ij}} = h_{ij}$ for all $1\le i\le j\le k$. Thus, the γ_{ij}'s we seek must satisfy two conditions:

i. $\gamma_{ji}+\overline{\gamma_{ij}} = h_{ij}$ for all $1\le i\le j\le k$.

ii. $\Re\left(\sum_{j=1}^{k}\overline{\gamma_{ij}}\,\langle f_i,\ f_j\rangle\right) = 0$ for all $i\in[k]$,

where (i) ensures that $d\Phi(G) = H$ and (ii) is equivalent to $G\in T_F\mathbb{S}^k_\mu(\mathbb{F}^n)$. Note that (i) implies $\overline{\gamma_{ij}} = h_{ij}-\gamma_{ji}$ for all $i<j$, and $\Re(\gamma_{ii}) = \frac{1}{2}h_{ii}$ for all $i\in[k]$. Thus, assuming (i) holds, mass substitution into (ii) produces

$$\sum_{j<i}\Re\left(\overline{\gamma_{ij}}\,\langle f_i,\ f_j\rangle\right) + \frac{1}{2}h_{ii}\|f_i\|^2 + \sum_{i<j}\Re\left((h_{ij}-\gamma_{ji})\,\langle f_i,\ f_j\rangle\right) = 0$$

for all $i\in[k]$. Rearrangement and simplification yield

$$\frac{1}{2}h_{ii}\|f_i\|^2 + \sum_{j<i}\Re\left(h_{ji}\,\langle f_i,\ f_j\rangle\right) = -\sum_{j<i}\Re\left(\gamma_{ij}\,\langle f_j,\ f_i\rangle\right) + \sum_{i<j}\Re\left(\gamma_{ji}\,\langle f_i,\ f_j\rangle\right) \tag{3.2.2}$$

for all $i\in[k]$. Hence, the existence of a solution to the system (3.2.2) is a necessary condition for any G satisfying (i) and (ii). It is also a sufficient condition since (3.2.2) depends only upon $\{\gamma_{ij}\}_{i<j}$ ($\{\gamma_{ij}\}_{j\le i}$ may then be chosen so that (i) is satisfied, and subsequent reversal of the mass substitution implies (ii)). Thus, $d\Phi_F$ is surjective if and only if there exists a solution to (3.2.2) for all possible H.

Let $\Gamma_F = (V_F, E'_F)$ be the correlation network of F and equip Γ_F with the weight function $w\equiv 0$. Given any $H\in\mathbf{H}^n_0(\mathbb{C})$ expanded as before, set $w'_H(v_i) = \sum_{j=1}^{i}\Re\left(h_{ji}\,\langle f_i,\ f_j\rangle\right)$ for all $i\in[k]$ vertices. For any $w' : V_F\to\mathbb{R}$ such that $\sum_{v_i\in V_F}w'(v_i) = \sum_{v_i\in V_F}w(v_i) = 0$, there is an H such that $w' = w'_H$. To see this, simply set $h_{ij} = 0$ for $i\ne j$ and set $h_{ii} = w(v_i)/\|f_i\|^2$ in the previous expansion of H. It is clear that (ii) is satisfied by H, so $H \in \mathbf{H}_0^n(\mathbb{C})$. It is also clear that $w'(v_i) = w'_H(v_i)$.

It will now be shown that there is a solution to (3.2.2) for all $H \in \mathbf{H}_0^n(\mathbb{C})$ if and only if all functions $w' : V_F \to \mathbb{R}$ satisfying $\sum_{i=1}^k w'(v_i) = 0$ are redistributions of $w \equiv 0$ on Γ_F. First, suppose that there is a solution to (3.2.2) for all $H \in \mathbf{H}_0^n(\mathbb{C})$. Let w' satisfy the hypothesis, and identify w' with w'_H constructed above. Let $\{\gamma_{ji}\}_{i<j}$ be a solution to (3.2.2) for the H obtained in this construction. Setting $\delta(v_j, v_i) = \Re(\gamma_{ji} \langle f_i, f_j \rangle)$ for $i < j$ and imposing skew symmetry then produces a redistribution flow that makes w' a redistribution of w on Γ_F. On the other hand, if all admissible functions w' are redistributions of w on Γ_F, let H be given and let δ be the redistribution flow corresponding to w'_H. Setting $\gamma_{ji} = \delta(v_j, v_i)/\langle f_i, f_j \rangle$ for all $i < j$ such that $\langle f_i, f_j \rangle \neq 0$ and $\gamma_{ij} = 0$ otherwise then produces a solution to (3.2.2) for H.

Putting this all together, we have that $d\Phi_F$ is surjective if and only if there is a solution to (3.2.2) for all $H \in \mathbf{H}_0^n(\mathbb{C})$. We have just seen that there is a solution to (3.2.2) for all $H \in \mathbf{H}_0^n(\mathbb{C})$ if and only if all functions $w' : V_F \to \mathbb{R}$ satisfying $\sum_{i=1}^k w'(v_i) = 0$ are redistributions of $w \equiv 0$ on Γ_F. By Lemma 2, the latter condition is equivalent to connectivity of Γ_F which, by Lemma 1, is equivalent to the fact that F is not orthodecomposable. Consequently, $d\Phi_F$ is surjective if and only if F is not orthodecomposable and the theorem holds in the complex case. The proof for the real case proceeds similarly, but again with added simplicity. $\square$

3.2.3. Redundant n-majorization and regular values of Φ

This previous proposition implies that, for $\mu \in \mathbb{R}^k_{+\geq}$ and a positive definite $E \in \mathbf{H}_c(\mathbb{F})$ with $c = \sum_{i=1}^k \mu_k$, $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$ will be a smooth manifold if it contains no frame F which is orthodecomposable. Since orthodecomposability depends on the existence of frames for subspaces, Theorem 2.2.1 will be relied upon in an essential way to characterize pairs μ and E for which $\mathcal{F}_\mu^k(E,\mathbb{F}^n)$ has no orthodecomposable member. In this vein, the notion of redundant n-majorization is introduced.

Definition 3.2.7. Suppose $a \leq b$, $\beta \in \mathbb{R}^b_{+\geq}$, $\alpha \in \mathbb{R}^a_{+\geq}$, and $\beta \preceq_b \alpha$. We say α *redundantly a-majorizes* β if there are proper nonempty subsets $A \subset [a]$ and $B \subset [b]$ such that $\{\alpha_i\}_{i\in A}$ $|A|$-majorizes $\{\beta_i\}_{i\in B}$ and $\{\alpha_i\}_{i\in A^c}$ $|A^c|$-majorizes $\{\beta_i\}_{i\in B^c}$.

Any positive definite $E \in \mathbf{H}^n(\mathbb{F})$ has an orthonormal set of eigenvectors $\{e_i\}_{i=1}^n$ corresponding to the eigenvalues given by some $\lambda \in \mathbb{R}^n_{+\geq}$. Let $\mu \in \mathbb{R}^k_{+\geq}$ be redundantly n-majorized by λ with $A \subset [n]$ and $B \subset [k]$ the proper nonempty sets such that $\{\lambda_i\}_{i\in A}$ $|A|$-majorizes $\{\mu_i\}_{i\in B}$ and $\{\lambda_i\}_{i\in A^c}$ $|A^c|$-majorizes $\{\mu_i\}_{i\in B^c}$. Let $\mathcal{V}_\mathcal{A} \subset \mathbb{F}^n$ be the subspace spanned by the eigenvectors with indices in A and set $E_A = \sum_{i\in A}\lambda_i e_i e_i^*$. It is then true that $\mathcal{V}_A^\perp = \mathcal{V}_{A^c}$ and that $E = E_A + E_{A^c}$ where $E_{A^c} = \sum_{i\in A^c}\lambda_i e_i e_i^*$. Setting $\mu_B = \{\mu_i \in \mu | i \in B\}$, the theorem of Cassaza and Leon then implies the existence of $F_\mathcal{V} \in \mathcal{F}^{|B|}_{\mu_B}(E_A,\mathcal{V}_A)$ and $F_{\mathcal{V}^\perp} \in \mathcal{F}^{|B^c|}_{\mu_{B^c}}(E_{A^c},\mathcal{V}_{A^c})$. Concatenation of these two frames then produces $F \in \mathcal{F}^k_\mu(E,\mathbb{F}^n)$ which is, by construction, orthodecomposable and hence a critical point of the map Φ by Theorem 3.2.6. The following theorem illustrates the converse.

Theorem 3.2.8. *Suppose the positive definite matrices $E_r \in \mathbf{H}^n(\mathbb{R})$ and $E_c \in \mathbf{H}^n(\mathbb{C})$ are such that $\lambda(E_r) = \lambda(E_c) \in \mathbb{R}^n_{+\geq}$. Suppose further that $\mu \in \mathbb{R}^k_{+\geq}$ is n-majorized by $\lambda(E_c)$, but that $\lambda(E_c)$ does not redundantly n-majorize μ. Then*

(i) $\mathcal{F}^k_\mu(E_r,\mathbb{R}^n)$ is a smooth submanifold of $\mathbf{M}_{n\times k}(\mathbb{R})$ with dimension

$$dim\left(\mathcal{F}^k_\mu(E_r,\mathbb{R}^n)\right) = k(n-1) - \binom{n+1}{2} + 1;$$

(ii) $\mathcal{F}^k_\mu(E_c,\mathbb{C}^n)$ *is a smooth submanifold of* $\mathbf{M}_{n\times k}(\mathbb{C})$ *with dimension*

$$dim\left(\mathcal{F}^k_\mu(E_c,\mathbb{C}^n)\right)=k(2n-1)-\binom{n+1}{2}-\binom{n}{2}+1.$$

Proof. Let $E=E_c$ or $E=E_r$. Suppose the hypotheses hold and further suppose (by way of contradiction) that some $F\in\mathcal{F}^k_\mu(E,\mathbb{F}^n)$ is orthodecomposable into $F_\mathcal{V}\in\mathcal{F}^{k_\mathcal{V}}_{\mu_\mathcal{V}}(E_\mathcal{V},\mathcal{V})$ and $F_{\mathcal{V}^\perp}\in\mathcal{F}^{k_{\mathcal{V}^\perp}}_{\mu_{\mathcal{V}^\perp}}(E_{\mathcal{V}^\perp},\mathcal{V}^\perp)$ where $k_\mathcal{V}+k_{\mathcal{V}^\perp}=k$ and $1\leq k_\mathcal{V},k_{\mathcal{V}^\perp}$. Let $\{a_i\}_{i=1}^{k_\mathcal{V}}$ and $\{\alpha_i\}_{i=1}^{k_\mathcal{V}}$, and $\{b_i\}_{i=1}^{k_{\mathcal{V}^\perp}}$ and $\{\beta_i\}_{i=1}^{k_{\mathcal{V}^\perp}}$ be the respective eigenvectors and eigenvalues of $E_\mathcal{V}$ and $E_{\mathcal{V}^\perp}$ respectively. If $\{e_i\}_{i=1}^{k_\mathcal{V}}$ and $\{\lambda_i\}_{i=1}^n$ are the eigenvectors and eigenvalues of E, it is clear that $\{e_i\}_{i=1}^n=\{a_i\}_{i=1}^{dim(\mathcal{V})}\cup\{b_i\}_{i=1}^{dim(\mathcal{V}^\perp)}$ and also that $\{\lambda_i\}_{i=1}^n=\{\alpha_i\}_{i=1}^{dim(\mathcal{V})}\cup\{\beta_i\}_{i=1}^{dim(\mathcal{V}^\perp)}$ since $\mathcal{V}$ and $\mathcal{V}^\perp$ are orthogonal subspaces. Let $A\subset[n]$ be such that $\{\lambda_i\}_{i\in A}=\{\alpha_i\}_{i=1}^{dim(\mathcal{V})}$ and note that $\{\lambda_i\}_{i\in A^c}=\{\beta_i\}_{i=1}^{dim(\mathcal{V}^\perp)}$. Theorem 2 of Cassaza and Leon then implies that $\mu_\mathcal{V}$ is $|\mu_\mathcal{V}|$-majorized by $\{\lambda_i\}_{i\in A}$ and that $\mu_{\mathcal{V}^\perp}$ is $|\mu_{\mathcal{V}^\perp}|$-majorized by $\{\lambda_i\}_{i\in A^c}$. The concatenation of $\mu_\mathcal{V}$ and $\mu_{\mathcal{V}^\perp}$ yield μ which is thus redundantly n-majorized by $\{\lambda_i\}_{i=1}^n$. This contradicts the hypotheses. Thus, no $F\in\mathcal{F}^k_\mu(E,\mathbb{F}^n)$ is orthodecomposable and application of Proposition 2 gives us that E is a regular value of $\Phi:\mathbb{S}^k_\mu(\mathbb{F}^n)\to\mathbf{H}^n_c(\mathbb{F})$ where $c=\sum_{i=1}^k\mu_i$. We may then conclude that $\Phi^{-1}(E)=\mathcal{F}^k_\mu(E,\mathbb{F}^n)$ is a smooth manifold.

In the real case, the domain of Φ is as smooth manifold of dimension $k(n-1)$ and the range is a smooth manifold of dimension $\binom{n+1}{2}-1$. In the complex case, the domain is a smooth manifold of dimension $k(2n-1)$ and the range is a smooth manifold of dimension $\binom{n+1}{2}+\binom{n}{2}-1$. These observations conclude the result. □

Noting that $\mathcal{F}^k_{\mathbf{1}_k}(\frac{k}{n}I,\mathbb{F}^n)$ denotes the space of k-member unit-norm tight frames when I is the $n\times n$ identity matrix and setting $\mathbf{1}_k=\{1\}_{i=1}^k$, we obtain the following generalization of Theorem 4.3 of Dykema et al. [6].

Corollary 3.2.9. *If* n *and* k *are relatively prime, then* $\mathcal{F}^k_{\mathbf{1}_k}(\frac{k}{n}I,\mathbb{F}^n)$ *is a smooth man-*

ifold of dimension $k(n-1)-\binom{n+1}{2}+1$ *when* $\mathbb{F}=\mathbb{R}$ *and dimension* $k(2n-1)-\binom{n+1}{2}-\binom{n}{2}+1$ *when* $\mathbb{F}=\mathbb{C}$.

Proof. If $\mathbf{1}_k=\{1\}_{i=1}^k$ is redundantly n-majorized by $\{\frac{k}{n}\}_{i=1}^n$, then there are partitions $k=k_1+k_2$ and $n=n_1+n_2$ with $1\leq k_i$ and $1\leq n_i$ for $i=1,2$ so that $\frac{k}{n}=\frac{k_i}{n_i}$ for $i=1,2$. This then implies that $kn_i=k_in$. Since n and k are relatively prime, k must divide k_i. This contradicts the fact that $k_i<k$. The result then follows from the previous Theorem. $\square$

4. CONSTRUCTIONS

Numerous algorithmic constructions of finite frames have been discovered over the past decade, but the current constructions methods often lack control or scope over the constructed frames. In this section, a very general framework for the construction of finite frames with a given frame operator will be presented. Application of these techniques allows one to hand-pick frame vectors in an iterative fashion so that the final collection is a frame with a desired frame operator.

In the first subsection, the theory behind these construction tools will be explained. The second and third subsection apply these tools to obtain constructions in $\mathcal{F}^k(E, \mathbb{C}^n)$ and $\mathcal{F}^k_\mu(E, \mathbb{C}^n)$ respectively. This section then concludes with an overview of techniques that are useful in implementing these results.

4.1. The Main Construction Tools

This section provides the main results justifying the iterative constructions. First, a lemma will demonstrate that $E - ff^*$ has a determinant with a very simple expansion. This expansion implies an ellipsoidal condition on f ensuring that $E - ff^*$ is positive definite, as well as a hyperbolic condition on f ensuring that $E - ff^*$ has an eigenvalue greater than some $c \in \mathbb{R}$.

4.1.1. The determinant of $E - ff^*$

The determinant of $E - ff^*$ has a very pleasant expansion which yields a number of helpful techniques.

Lemma 4.1.1. *Suppose that $E \in \mathbf{M}_n(\mathbb{C})$ is Hermitian, and that $f \in \mathbb{C}^n$. Let $\{\lambda_i\}_{i=1}^n$ be the set of eigenvalues of E and let $f = \sum_{i=1}^n f^i e_i$ be the expansion of f in terms* *of the corresponding orthonormal eigenvectors of E. Then*

$$det(E \pm ff^*) = \prod_{i=1}^{n} \lambda_i \pm \sum_{i=1}^{n} \left(\prod_{j \neq i} \lambda_j \right) |f^i|^2. \tag{4.1.1}$$

Proof. Without loss of generality, we may assume that $E = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ since E is Hermitian and hence unitarily equivalent to a diagonal matrix. Let $\mathfrak{S}_n$ be the group of all permutations on the set $[n]$. The determinant of this matrix has the form

$$\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} \alpha_{i\sigma(i)}$$

where $\alpha_{ii} = \lambda_i \pm |f^i|^2$ and $\alpha_{ij} = \pm f^i \overline{f^j}$ when $i \neq j$, and $\mathrm{sgn}(\sigma)$ is ± 1 depending upon whether the signature of the permutation σ is even or odd. Since any nontrivial permutation must rearrange at least two indices, the only permutation from which a contribution to the term $\prod_{i=1}^{n} \lambda_i$ arises in this sum is the trivial one. Likewise, a contribution to each of the terms $\left(\prod_{j \neq i} \lambda_j \right) |f^i|^2$ is only produced when the trivial permutation is encountered in the sum, and the coefficient will be $\pm$. These terms are exactly those in the right-hand side of (4), so we need only show that the remaining terms cancel in the sum.

The products which will contribute to the term $\prod_{i \in A} \lambda_i \prod_{i \in A^c} |f^i|^2$ when $A \subset [n]$ and $|A| \leq n-2$ arise from permutations such that $\sigma(i) = i$ for all $i \in A$. For each such σ, there is exactly one contribution to this term since the summand at such a σ has the form $\prod_{i \in B_\sigma} (\lambda_i \pm |f^i|^2) \prod_{i \in B_\sigma^c} \pm f^i \overline{f^{\sigma(i)}}$ where $A \subset B_\sigma = \{i \in [n] | \sigma(i) = i\}$. Noting that $i \in B_\sigma^c$ implies the existence of $j \in B_\sigma^c$ such that $\sigma(j) = i$, the summand becomes $(\pm 1)^{|B_\sigma^c|} \prod_{i \in B_\sigma} (\lambda_i \pm |f^i|^2) \prod_{i \in B_\sigma^c} |f^i|^2$. Choosing the λ_i such that $i \in A$ from the first part of the product, and then choosing the remaining $|f^i|^2$ from the first and last part of the product, the term $(\pm 1)^{|A^c|} \prod_{i \in A} \lambda_i \prod_{i \in A^c} |f^i|^2$ is formed. Summing over all σ such that $A \subset B_\sigma$, we see that this term has the coefficient

$$(\pm 1)^{|A^c|} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ A \subset B_\sigma}} \mathrm{sgn}(\sigma).$$

Since $\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)$ for any permutations $\sigma, \tau \in \mathfrak{S}_n$, for every even permutation in this sum there is a corresponding odd permutation since we may multiply by any fixed transposition disjoint from A and any such multiplication induces a bijection between even and odd signature permutations. Consequently, these sums vanish and the result follows. $\square$

4.1.2. The ellipsoidal and hyperbolic conditions

The ellipsoidal condition supplies a precise constraint on f that implies $0 \preceq E - ff^*$ when E is Hermitian positive semidefinite. It also indicates exactly when $E - ff^*$ has rank lower than the rank of E.

Theorem 4.1.2. *Let $E \in \mathbf{H}^n(\mathbb{C})$ be positive semidefinite with orthonormal eigenvectors $\{e_i\}_{i=1}^n$ and corresponding eigenvalues $\{\lambda_i\}_{i=1}^n$. Set $A = \{i \in [n] | \lambda_i \neq 0\}$, fix $f \in \mathbb{C}^n$, and write $f = \sum_{i=1}^n f^i e_i$. Then*

(i) When $0 \prec E$, $0 \preceq E - ff^*$ if and only if $\sum_{i=1}^n \frac{|f^i|^2}{\lambda_i} \leq 1$;

(ii) $0 \preceq E - ff^*$ if and only if $\sum_{i \in A} \frac{|f^i|^2}{\lambda_i} \leq 1$ and $f^i = 0$ for all $i \in A^c$;

(iii) $0 \preceq E - ff^*$ and rank$(E - ff^*) =$ rank$(E) - 1$ if and only if $\sum_{i \in A} \frac{|f^i|^2}{\lambda_i} = 1$ and $f^i = 0$ for all $i \in A^c$.

Proof. i. Without loss of generality, $E = \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. $\Lambda - ff^*$ has at most one non-positive eigenvalue, and hence $0 \preceq \Lambda - ff^*$ if and only if

$0 \leq \det(\Lambda - ff^*)$. Applying Lemma 1,

$$\begin{aligned} 0 \leq \det(\Lambda - ff^*) &\Leftrightarrow 0 \leq \prod_{i=1}^{n} \lambda_i - \sum_{i=1}^{n} \left(\prod_{j \neq i} \lambda_j \right) |f^i|^2 \\ &\Leftrightarrow \sum_{i=1}^{n} \left(\prod_{j \neq i} \lambda_j \right) |f^i|^2 \leq \prod_{i=1}^{n} \lambda_i \\ &\Leftrightarrow \sum_{i=1}^{n} \frac{|f^i|^2}{\lambda_i} \leq 1. \end{aligned}$$

ii. Assuming that $0 \preceq \Lambda$ and $0 \preceq \Lambda - ff^*$, then

$$\langle (\Lambda - ff^*)e_i, e_i \rangle = -|\langle f, e_i \rangle|^2 = -|f^i|^2,$$

whenever $i \in A$, so $f^i = 0$ for all $i \in A^c$. Thus, in this instance, we need only consider the behavior of $\Lambda - ff^*$ on $V = \text{span}\{e_i\}_{i \in A}$. Λ is of full rank on V and $f \in V$, so (i) implies that $\sum_{i \in A} |f^i|^2/\lambda_i \leq 1$. Conversely, suppose that $\sum_{i \in A} |f^i|^2/\lambda_i \leq 1$ and $f^i = 0$ for all $i \in A^c$. The latter condition implies that we need only consider the behavior of $\Lambda - ff^*$ on V. Since $\sum_{i \in A} |f^i|^2/\lambda_i \leq 1$, (i) implies the result.

iii. By a line of reasoning similar to the proof in (ii), we may assume that Λ is of full rank without loss of generality. $\Lambda - ff^*$ has at least rank $n-1$. Therefore, $\Lambda - ff^*$ has rank $n-1$ if and only if $0 = \det(\Lambda - ff^*)$. Replacing '$\leq$' with '$=$' in the chain of equivalences produced in the proof of (i) then leads to this equivalence.

□

Then next theorem is useful when one desires to control the eigenvalues of $E - ff^*$, or at least ensure that the greatest eigenvalue of $E - ff^*$ is greater than some constant $c \in \mathbb{R}$. In certain instances, this is not even concern. This is implicit in the hypothesis $\lambda_2(E - ff^*) < c \leq \lambda_1(E - ff^*)$.

Theorem 4.1.3. *Let $E \in \mathbf{M}_n(\mathbb{C})$ be Hermitian positive definite with eigenvalues $\{\lambda_i\}_{i=1}^n$ non-increasing as the index increases and corresponding orthonormal eigenvectors $\{e_i\}_{i=1}^n$. Write $f \in \mathbb{C}^n$ as $f = \sum_{i=1}^n f^i e_i$ and suppose that $c \in \mathbb{R}^+$.*

(i) Whenever $\lambda_2 < c < \lambda_1$, $E - ff^* \prec cI_n$ if and only if $\sum_{i=1}^n \frac{|f^i|^2}{\lambda_i - c} > 1$;

(ii) Whenever $c = \lambda_1$, $E - ff^* \prec cI_n$ if and only if $0 < |f^1|^2$.

Proof. i. $cI - \Lambda + ff^*$ has at most one negative eigenvalue, so $0 \prec cI - \Lambda + ff^*$ if and only if $0 \leq \det(cI - \Lambda + ff^*)$. As in the previous theorem, we have

$$\begin{aligned} 0 < \det(cI - \Lambda + ff^*) &\Leftrightarrow 0 < \prod_{i=1}^n (c - \lambda_i) + \sum_{i=1}^n \left(\prod_{j \neq i} (c - \lambda_j) \right) |f^i|^2 \\ &\Leftrightarrow -\sum_{i=1}^n \left(\prod_{j \neq i} (c - \lambda_j) \right) |f^i|^2 < \prod_{i=1}^n (c - \lambda_i) \\ &\Leftrightarrow -\sum_{i=1}^n \frac{|f^i|^2}{c - \lambda_i} > 1 \\ &\Leftrightarrow \sum_{i=1}^n \frac{|f^i|^2}{\lambda_i - c} > 1. \end{aligned}$$

ii. If $c = \lambda_1$, then right hand side of the first equivalence in the preceding chain reduces to

$$0 < \left(\prod_{j \neq 1} (c - \lambda_j) \right) |f^1|^2.$$

The term $\left(\prod_{j \neq 1} (c - \lambda_j) \right)$ is positive since $\lambda_j < c$ for all $j \neq 1$. Thus, the equivalence follows.

$\square$

Note that the proofs of these conditions both relied upon the determinant expansion in Lemma 4.1.1.

4.2. Constructing Frames in $\mathcal{F}^k(E, \mathbb{C}^n)$

This section outlines two methods for constructing frames in $\mathcal{F}^k(E, \mathbb{C}^n)$. The first method is computationally inexpensive, but is essentially blind with respect to the frame vectors that are produced. The second method amends this deficiency, but is far more computationally intensive.

4.2.1. The fast blind method

Utilizing the diffeomorphism $F \mapsto \sqrt{E}F^*$, a frame in $\mathcal{F}^k(E, \mathbb{C}^n)$ arises whenever a frame in $V_n(\mathbb{C}^k)$ is constructed. Thus, a random member of $\mathcal{F}^k(E, \mathbb{C}^n)$ can be obtained by constructing a random member of $V_n(\mathbb{C}^k)$. A random member of $\mathrm{St}(n, k)$ can be constructed in a very simple manner. First, a point x_1 is chosen from the unit sphere in $\mathbb{C}^k$. It is then true that $I_n - x_1x_1^*$ is a projection on the the subspace orthogonal to x_1. After x_1 is fixed, x_2 is chosen to lie in both the unit sphere and the kernel of $x_1x_1^*$. Thus, x_2 is orthogonal to x_1. Continuing this process inductively, x_{j+1} is chosen to lie in the kernel of $\sum_{i=1}^{j} x_ix_i^*$ and also in the unit sphere. Clearly, this process produces an $X = [x_1 \cdots x_n] \in V_n(\mathbb{C}^k)$, so $\sqrt{E}X^* \in \mathcal{F}^k(E, \mathbb{C}^n)$.

4.2.2. The interactive method

This next procedure allows one to iteratively choose k vectors interactively and in such a way that the resulting vector collection is a frame with frame operator E. At each step, Theorem 4.1.2 is used to produce the precise region of admissible vectors.

Let $E \in \mathbf{H}^n(\mathbb{C})$ be positive definite. First, one computes E^{-1}. Theorem 4.1.2 ensures that region bounded by the ellipsoid $\{x \in \mathbb{C}^n | \langle E^{-1}x, x\rangle = 1\}$ contains all admissible candidates for f_1. There are two distinct cases to deal with. If f_1 is chosen from this interior of this region, (iii) of Theorem 4.1.2 implies that $E - f_1f_1^*$ is positive definite, and hence invertible. In this case, $(E - f_1 f_1^*)^{-1}$ is calculated, and an f_2 is chosen from the region bounded by the ellipsoid

$$\{x \in \mathbb{C}^n | \left\langle (E - f_1 f_1^*)^{-1} x, x \right\rangle = 1\}.$$

On the other hand, f_1 may also be chosen from the boundary of the ellipsoid. Part (iii) of Theorem 4.1.2 then implies that $E - f_1 f_1^*$ is of rank $n-1$ and hence cannot be inverted. Nevertheless f_2 may still be chosen from the region

$$\{x \in \mathbb{C}^n | \left\langle (E - f_1 f_1^*)^{\dagger} x, x \right\rangle \leq 1, E^{\dagger} E x = x\}$$

by part (ii) of Theorem 4.1.2. Since $E^{-1} = E^{\dagger}$ in the case that E is positive definite, the $(j+1)^{th}$ vector may generally be chosen from the region

$$\left\{ x \in \mathbb{C}^n \middle| \left\langle \left(E - \sum_{i=1}^{j} f_i f_i^* \right)^{\dagger} x, x \right\rangle \leq 1, E^{\dagger} E x = x \right\}$$

so long as the inequality $j < k - \operatorname{rank}(E - \sum_{i=1}^{j} f_i f_i^*)$ holds. If this inequality fails to hold, the only admissible choices for f_{j+1} lie in the region

$$\left\{ x \in \mathbb{C}^n \middle| \left\langle \left(E - \sum_{i=1}^{j} f_i f_i^* \right)^{\dagger} x, x \right\rangle = 1, E^{\dagger} E x = x \right\}.$$

since the subtraction of a rank-one matrix only reduces the rank of another matrix by at most one.

Example 4.2.1. This example shows a progression of three dimensional ellipsoidal regions produced by implementing the described construction method. The construction produces a frame in $\mathcal{F}^6(25I_3, \mathbb{R}^3)$. The frame

$$F = \begin{bmatrix} \frac{4\sqrt{3}}{3} & \frac{4\sqrt{3}}{3} & \frac{4\sqrt{3}}{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} \\ -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \sqrt{3} & -\frac{2\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3} & \frac{4\sqrt{3}}{3} \\ -2\sqrt{2} & -2\sqrt{2} & 0 & -\frac{3\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} & 0 \end{bmatrix}$$

has been chosen. Figure 2 shows the progression of ellipsoidal regions as the frame vectors of F are added column by column.

Note that the final three vectors are chosen from the surface of the ellipsoid, and that the ellipsoids become progressively more degenerate as these vectors are chosen.

4.3. Constructing Frames in $\mathcal{F}_\mu^k(E, \mathbb{C}^n)$

This section is devoted to specializing the interactive iterative method presented in the previous section to produce many members of $\mathcal{F}_\mu^k(E, \mathbb{C}^n)$ whenever $\mu \in \mathbb{R}^k_{+\geq}$, $E \in \mathbf{H}^n(\mathbb{C})$ is positive definite, and $\mu \preceq_n \lambda$, and where $\lambda \in \mathbb{R}^n_{+\geq}$ are the eigenvalues of E. Unfortunately, this process is more delicate than the previous process. It is explained why this is the case and then some computationally inexpensive remedies are proposed.

4.3.1. Ensuring n-majorization

Supposing one begins choosing vectors of prescribed lengths based on the interactive method from 4.2.2. This may prove unstable since $\mathcal{F}_\mu^{k-m}(E - \sum_{i=1}^j f_i f_i, \mathbb{C}^n)$ is only nonempty for particular choices of $\mu \in \mathbb{R}^{k-j}_{+\geq}$. More concretely, suppose that we begin choosing vectors in the same manner as the previous algorithm while also taking care to insure that $\|f_j\|^2 = \mu_j$. Assuming that this process does not fail at the j^{th} step, let $\mu^{(j)} \in \mathbb{R}^{(k-j)}_{+\geq}$ and $\lambda^{(j)} \in \mathbb{R}^n_{+\geq}$ denote the list of remaining squared lengths and the eigenvalues of $E^{(j)} = E - \sum_{i=1}^j f_i f_i^*$ respectively. By the theorem of Cassaza and

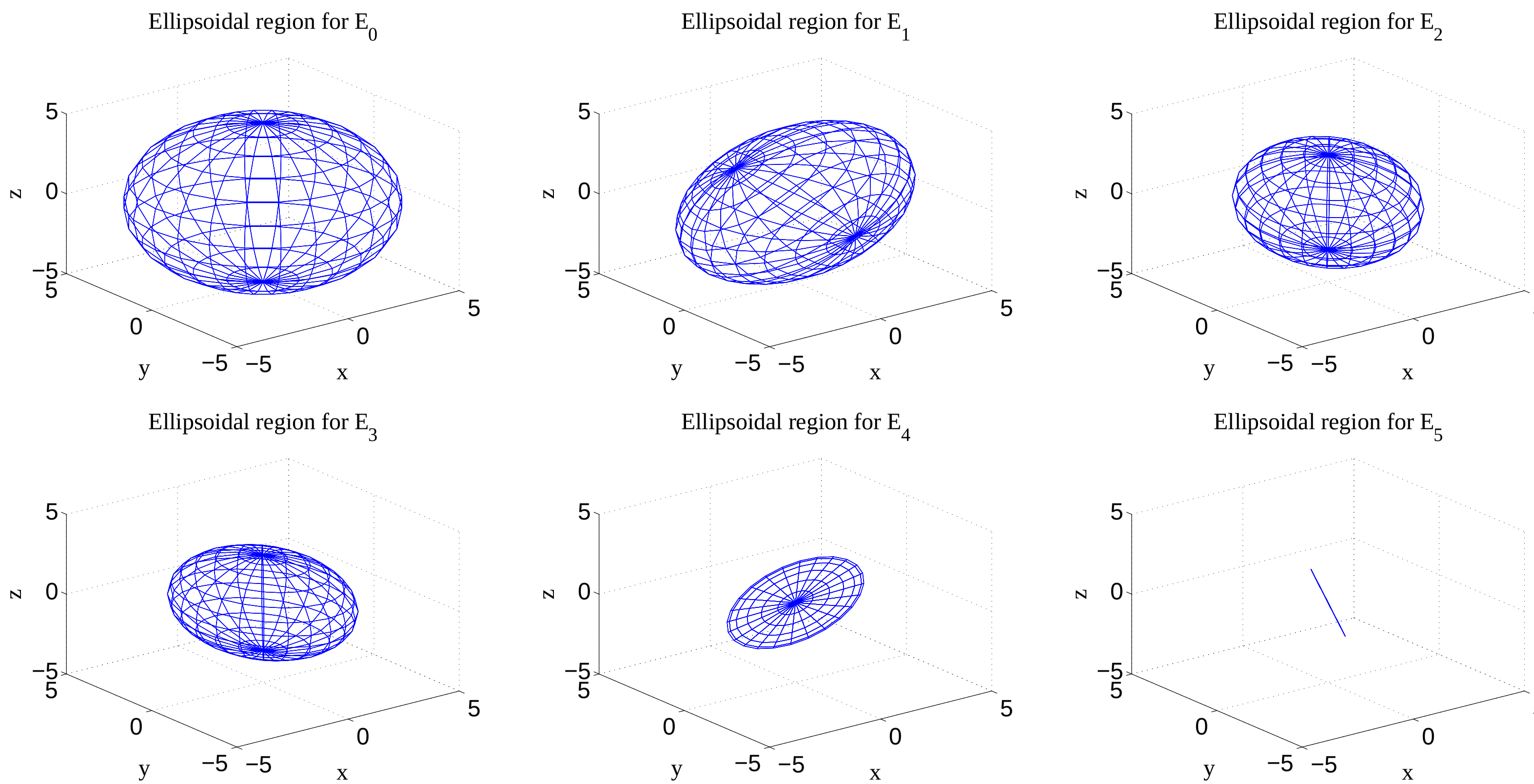

Fig. 2. The progression of ellipsoidal regions from Example 4.2.1.

Leon, $\mathcal{F}^{k-j}_{\mu^{(j)}}(E^{(j)}, \mathbb{C}^n)$ is nonempty if and only if $\mu^{(j)} \preceq_n \lambda^{(j)}$. Thus, if n-majorization fails to hold at this step, we can be certain that the algorithm will also fail.

Let us now turn our attention towards the first of the n-majorization inequalities. In practice, one would like to choose f_{j+1} with $\|f_{j+1}\|^2 = \mu_1^{(j)}$ so that $\mu_1^{(j+1)} \leq \lambda_1^{(j+1)}$. Setting $\mu_1^{(j+1)} = c$, there are two cases to consider if one assumes that $\mu^{(j)} \preceq_n \lambda^{(j)}$. First, if $c \leq \lambda_2^{(j)}$, then any choice of f_{j+1} will work by the interlacing inequalities for eigenvalues. On the other hand, if $\lambda_2^{(j)} < c < \lambda_1^{(j)}$, application of Theorem 4.1.3 produces the region from which f_{j+1} may be chosen. In this instance, the theorem essentially states that all eigenvalues of $E - \sum_{j=1}^{i} f_j f_j^*$ are less than c if and only if

$$\sum_{i=1}^{n} \frac{|f_{j+1}^i|^2}{\lambda_i^{(j)} - c} > 1.$$

Since we seek an eigenvalue greater than c, Theorem 4.1.3 ensures that this will occur when

$$\sum_{i=1}^{n} \frac{|f_{j+1}^i|^2}{\lambda_i^{(j)} - c} \leq 1,$$

so f_{j+1} is chosen to satisfy this inequality.

Example 4.3.1. Let $\mu = (7/3, 7/3, 1, 1/3)$ and suppose $E = \mathrm{diag}(3, 2, 1)$. First, we find all f such that $E - ff^*$ is positive definite with largest eigenvalue greater than $\mu_2 = 7/3$. The ellipsoidal describes the first region, and since $2 < 7/3 < 3$, the hyperbolic condition describes the second region. Figure 3 depicts the region from which f may be chosen. f_1 must also be chosen so that $\|f_1\|^2 = \mu_1 = 7/3$, and the region of admissible f_1 is seen in the above figure.

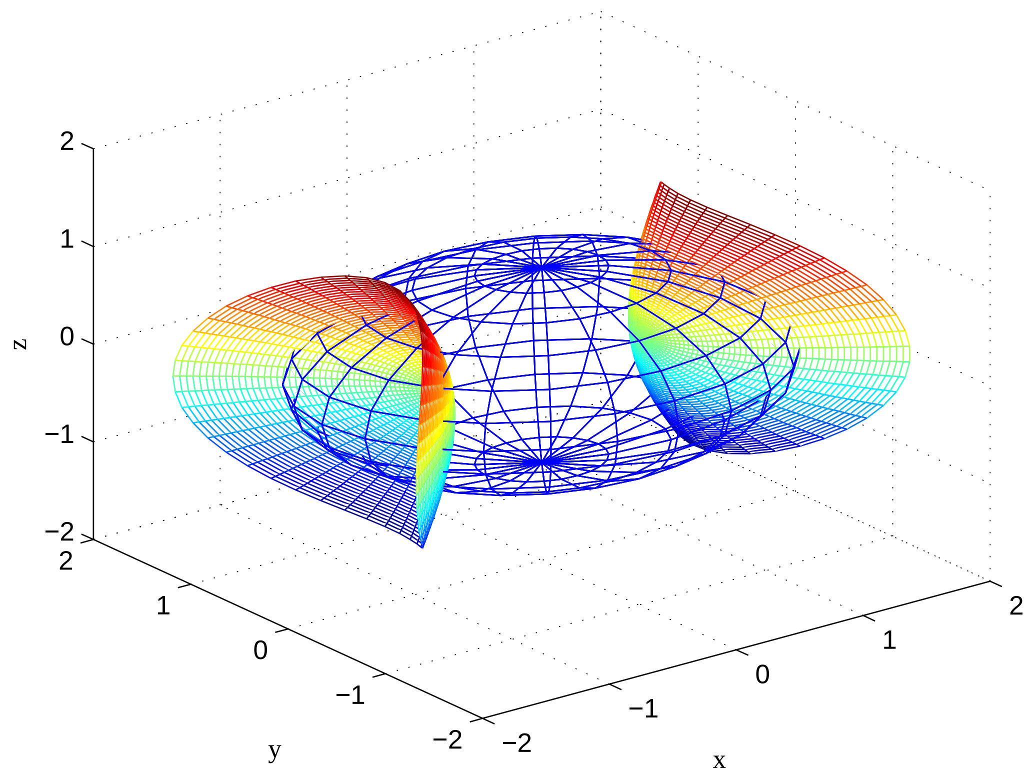

Fig. 3. The region satisfying the ellipsoidal and hyperbolic conditions.

The hyperbolic condition supplies a precise way of choosing f_j so that the first n-majorization inequality holds between $\mu^{(j)}$ and $\lambda^{(j)}$. For the remaining inequalities there are two distinguishable cases. In the case that

$$\sum_{i=1}^{m} \mu_{i+1}^{(j)} = \sum_{i=1}^{m} \mu_i^{(j+1)} \leq \sum_{i=1}^{m} \lambda_{i+1}^{(j)},$$

the interlacing inequalities for eigenvalues immediately imply

$$\sum_{i=1}^{m} \mu_i^{(j+1)} \leq \sum_{i=1}^{m} \lambda_{i+1}^{(j)} \leq \sum_{i=1}^{m} \lambda_i^{(j+1)},$$

and so f_{j+1} may be chosen indiscriminately. However, if

$$\sum_{i=1}^{m} \lambda_{i+1}^{(j)} < \sum_{i=1}^{m} \mu_{i+1}^{(j)} = \sum_{i=1}^{m} \mu_i^{(j+1)},$$

then f_j must be chosen with a fair amount of caution.

There is a very simple, but computationally unadvisable approach to constructing frames in $\mathcal{F}_\mu^k(E, \mathbb{F}^n)$. One may simply choose a vector f_j, compute the eigenvalues of $E^{(j)}$ (perhaps with the aid of Lemma 4.1.1), and then perform a test to determine if n-majorization holds. However, the region of viable choices for f may be quite small, and repeatedly computing the eigenvalues of a matrix proves expensive. The next few conditions will provide a way to sidestep this problem, but implementing these conditions imposes a slight restriction upon the frames that may be obtained via the specialized interactive method.

4.3.2. The cylindrical and directional conditions

The cylindrical and directional conditions are computationally inexpensive, but inexact conditions ensuring that the m^{th} majorization inequality is satisfied between some $\{\mu_{i+1}\}_{i=1}^{k-1} \in \mathbb{R}_{+\geq}^{k-1}$ and $\lambda(E - ff^*)$ whenever

$$\sum_{i=1}^{m} \lambda_{i+1}(E) < \sum_{i=1}^{m} \mu_{i+1}.$$

Along the way to these conditions, some convenient objects are defined.

Given a positive definite $E \in \mathbf{H}^n(\mathbb{C})$ with an orthonormal basis of eigenvectors $\{e_i\}_{i=1}^n \subset \mathbb{C}^n$ in correspondence with the eigenvalues $\lambda(E) \in \mathbb{R}_{+\geq}^n$, define the family of projections $\{P_m\}_{m=1}^{n-1}$ where P_m is the subspace spanned by $\{e_i\}_{i=1}^m$ for $m \in [n-1]$. Based on these projections, define the family of seminorms $\{\|\cdot\|_m\}_{m=1}^{n-1}$ by setting $\|f\|_m = \sqrt{\langle P_m f, P_m f\rangle}$. This family of seminorms leads to the cylindrical condition.

Proposition 4.3.2. *Let $E \in \mathbf{H}^n(\mathbb{C})$ be positive definite with eigenvalues $\lambda(E) \in \mathbb{R}_{+\geq}^n$ and corresponding orthonormal eigenvectors $\{e_i\}_{i=1}^n$. Let $f \in \mathbb{C}^n$ be such that $E - ff^*$ is positive semidefinite, and write $f = \sum_{i=1}^n f^i e_i$. Suppose $\mu \in \mathbb{R}_{+\geq}^k$ is such that $\mu \preceq_n \lambda(E)$. If $\|f\|_m^2 \leq \sum_{i=1}^m \lambda_i(E) - \sum_{i=1}^m \mu_{j+1}$, then $\sum_{i=1}^m \mu_{i+1} \leq \sum_{i=1}^m \lambda_i(E - ff^*)$.*

Proof. As before, assume $E = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. The argument will follow by simply examining the trace and invoking the fact that

$$\mathrm{tr}(PHP) \leq \sum_{i=1}^{m} \lambda_i(H)$$

whenever H is Hermitian positive semidefinite, and P is any rank m projection. Let P_m^Λ be the projection onto the subspace spanned by $\{e_i\}_{i=1}^m$. Then

$$\begin{aligned} \mathrm{tr}(P_m(\Lambda - ff^*)P_m) &= \mathrm{tr}(P_m \Lambda P_m) - \mathrm{tr}(P_m ff^* P_m) \\ &= \sum_{i=1}^{m} \lambda_i - \|f\|_m^2. \end{aligned}$$

By hypothesis, $\|f\|_m^2 \leq \sum_{i=1}^m \lambda_i(E) - \sum_{i=1}^m \mu_{j+1}$, so we also have that $\sum_{i=1}^m \mu_{j+1} \leq$

$\sum_{i=1}^{m} \lambda_i(E) - \|f\|_m^2$. It then follows that

$$\sum_{i=1}^{m} \mu_{j+1} \leq \sum_{i=1}^{m} \lambda_i(\Lambda) - \|f\|_m^2 = \mathrm{tr}(P_m(\Lambda - ff^*)P_m) \leq \sum_{i=1}^{m} \lambda_i(\Lambda - ff^*).$$

$\square$

The inequality $\sum_{i=1}^{m} \lambda_{i+1}(E) < \sum_{i=1}^{m} \mu_{i+1}$ was not utilized to demonstrate the previous result. Nevertheless, application of this result is superfluous if this inequality does not hold.

In the event that this inequality does hold and $\sum_{i=1}^{m} \lambda_i(E) - \sum_{i=1}^{m} \mu_{j+1} < \|f\|_m^2$, the cylindrical condition does not indicate when the m^{th} majorization inequality holds. In search of another condition, define another family of seminorms $\{\|\cdot\|_{E,m}\}_{m=1}^{n-1}$ by setting $\|f\|_{E,m} = \sqrt{\langle EP_m f, P_m f\rangle}$ for all $f \in \mathbb{C}^n$. Also define the *trace of E orthogonal to f on the range of P_m* by setting

$$\mathrm{tr} f_m^{\perp}(E) = tr(P_m E P_m) - \|f\|_{E,m}^2 / \|f\|_m^2$$

for all $f \in \mathbb{C}^n$ with $0 < \|f\|_m^2$. Let $P_{f,m}$ denote the projection onto the subspace spanned by $P_m f$, and note that $P_{f,m} = P_m f(P_m f)^* / \|f\|_m^2$. For brevity, denote $Q_{f,m} = P_m - P_{f,m}$. It is then true that

$$\mathrm{tr}(Q_{f,m} E Q_{f,m}) = \mathrm{tr}(P_m E P_m) - \mathrm{tr}(P_{f,m} E P_m) - \mathrm{tr}(P_m E P_{f,m}) + \mathrm{tr}(P_{f,m} E P_{f,m}).$$

Since E commutes with P_m by construction and $\mathrm{tr}(P_m f(P_m f)^* E) = \mathrm{tr}(EP_m f(P_m f)^*) = \|f\|_{E,m}^2$ by expanding $P_m f$ in terms of the eigenvectors of E, one acquires

$$\begin{aligned} \mathrm{tr}(Q_{f,m} E Q_{f,m}) &= \mathrm{tr}(P_m E P_m) - \|f\|_{E,m}^2 / \|f\|_m^2 \\ &= \mathrm{tr} f_m^{\perp}(E). \end{aligned}$$

It is clear that $\mathrm{tr}(Q_{f,m}ff^*Q_{f,m}) = 0$ by construction, so it is also true that

$$\mathrm{tr}(Q_{f,m}(E - ff^*)Q_{f,m}) = \mathrm{tr} f_m^{\perp}(E).$$

These observations lead to a directional condition on f that implies the m^{th} majorization inequality whenever f fails the cylindrical condition.

Proposition 4.3.3. *Let $E \in \mathbf{H}^n(\mathbb{C})$ be positive definite with eigenvalues $\lambda(E) \in \mathbb{R}^n_{+\geq}$ and corresponding orthonormal eigenvectors $\{e_i\}_{i=1}^n$. Suppose $\mu \in \mathbb{R}^k_{+\geq}$ is such that $\mu \preceq_n \lambda(E)$. Let $f \in \mathbb{C}^n$ be such that $E - ff^*$ is positive semidefinite, write $f = \sum_{i=1}^n f^i e_i$, and suppose that $0 < \sum_{i=1}^m \lambda_i(E) - \sum_{i=1}^m \mu_{i+1} < \|f\|_m^2$. If*

$$\frac{\|f\|_{E,m}^2}{\|f\|_m^2} \leq \sum_{i=1}^{m+1} \lambda_i(E) - \sum_{i=1}^{m} \mu_{i+1},$$

then $\sum_{i=1}^m \mu_{i+1} \leq \sum_{i=1}^m \lambda_i(E - ff^*)$.

Proof. First, we will show that

$$\frac{\|f\|_{E,m+1}^2}{\|f\|_{m+1}^2} \leq \frac{\|f\|_{E,m}^2}{\|f\|_m^2}. \tag{4.3.1}$$

A quick computation yields

$$\lambda_{m+1}\|f\|_m^2 = \sum_{i=1}^m \lambda_{m+1}|f^i|^2 \leq \sum_{i=1}^m \lambda_i |f^i|^2 = \|f\|_{E,m}^2,$$

so we also have

$$\|f\|_{E,m}^2\|f\|_m^2 + \lambda_{m+1}\|f\|_m^2|f^{m+1}|^2 \leq \|f\|_{E,m}^2\|f\|_m^2 + \|f\|_{E,m}^2|f^{m+1}|^2.$$

Factoring both sides of this inequality yields

$$\|f\|_{E,m+1}^2\|f\|_m^2 \leq \|f\|_{E,m}^2\|f\|_{m+1}^2.$$

By hypothesis, $0 < \|f\|_m^2 \leq \|f\|_{m+1}^2$, so division produces the desired inequality.

Applying this inequality to the hypothesis, we obtain

$$\frac{\|f\|_{E,m+1}^2}{\|f\|_{m+1}^2} \leq \frac{\|f\|_{E,m}^2}{\|f\|_m^2} \leq \sum_{i=1}^{m+1} \lambda_i(E) - \sum_{i=1}^{m} \mu_{i+1}$$

and rearrangement yields

$$\sum_{i=1}^{m} \mu_{i+1} \leq \sum_{i=1}^{m+1} \lambda_i(E) - \frac{\|f\|_{E,m+1}^2}{\|f\|_{m+1}^2} = \mathrm{tr} f_{m+1}^{\perp}(E - ff^*).$$

Since $Q_{f,m+1}$ is a projection of rank m,

$$\mathrm{tr} f_{m+1}^{\perp}(E) = \mathrm{tr}(Q_{f,m+1}(E - ff^*)Q_{f,m+1}) \leq \sum_{i=1}^{m} \lambda_i(E - ff^*),$$

and thus the result holds. $\square$

Example 4.3.4. Again, suppose $E = \mathrm{diag}(3, 2, 1)$. Assume that $E - ff^*$ needs to be such that $\frac{13}{3} \leq \lambda_1(E - ff^*) + \lambda_2(E - ff^*)$. Figure 4 depicts the region which is accepted by the cylindrical condition. Any vector lying in or on the cylinder satisfies the desired constraints. Figure 5 depicts an elliptical band and cylindrical band arising from the directional condition. Since the elliptical band is outside of the cylindrical band, no f satisfies the directional condition. Nevertheless, there are many more vectors that satisfy the desired constraints.

4.3.3. The quasihyperbolic condition

In the instance that the cylindrical and directional conditions both fail, there is still one final condition that will guarantee the m^{th} majorization inequality. The f that satisfy this final condition are bounded by a figure resembling a hyperboloid in $\mathbb{C}^n$.

Proposition 4.3.5. *Let $E \in \mathbf{H}^n(\mathbb{R})$ be positive definite with eigenvalues $\lambda(E) \in \mathbb{R}^n_{+\geq}$ and corresponding eigenvectors $\{e_i\}_{i=1}^n$. Suppose $\mu \in \mathbb{R}^k_{+\geq}$ is such that $\mu \preceq_n \lambda$.*

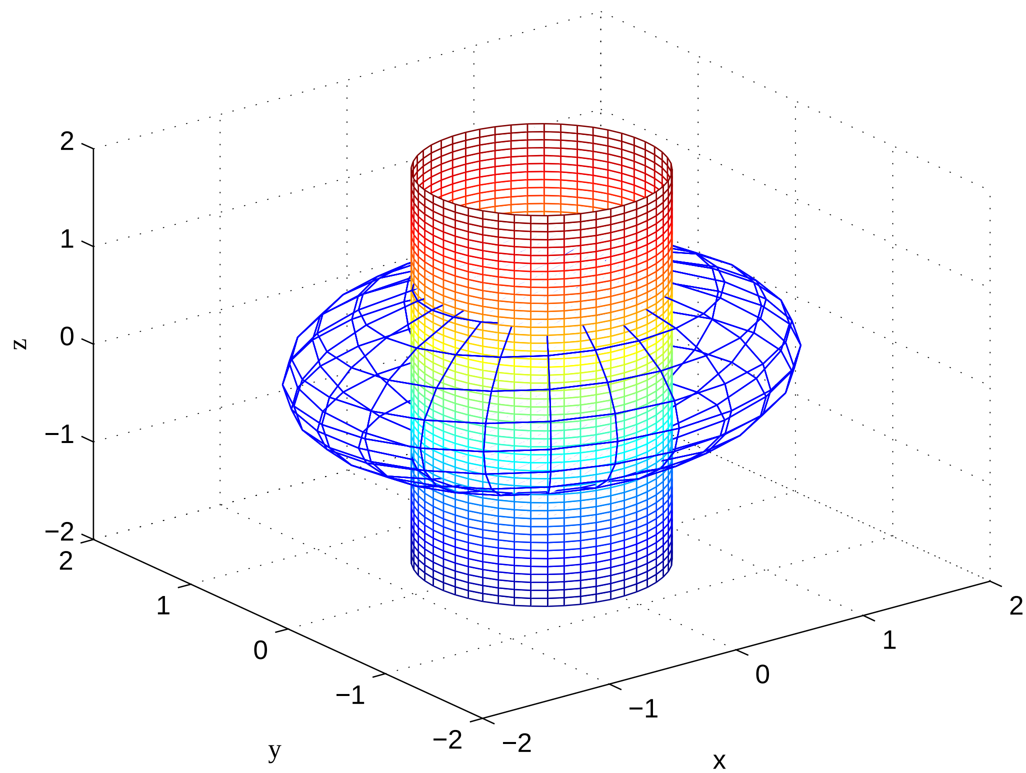

Fig. 4. The cylindrical condition.

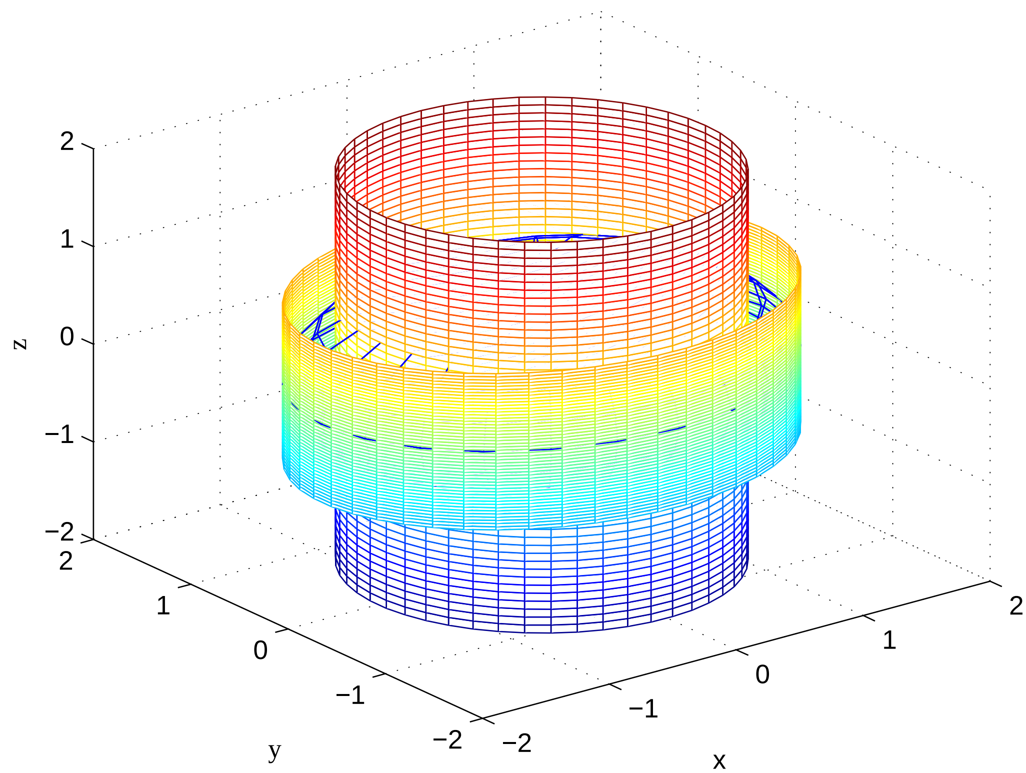

Fig. 5. The directional condition.

Further suppose that

$$\sum_{i=1}^{m+1} \lambda_i(E) - \sum_{i=1}^{m} \mu_{i+1} < \frac{\|f\|_{E,m}^2}{\|f\|_m^2} \quad and \quad 0 < \sum_{i=1}^{m} \lambda_i(E) - \sum_{i=1}^{m} \mu_{i+1} < \|f\|_m^2$$

for some $f = \sum_{i=1}^n f^i e_i \in \mathbb{C}^n$ *and some* $m \in [n-1]$*. If*

$$\frac{\|f\|_m^2}{\sum_{i=1}^m \lambda_i - \sum_{i=1}^m \mu_{i+1}} + \sum_{i=m+1}^{n} \frac{|f^i|^2}{\lambda_i + trf_m^\perp(E) - \sum_{j=1}^m \mu_{j+1}} \leq 1, \tag{4.3.2}$$

then $\sum_{i=1}^m \mu_{i+1} \leq \sum_{i=1}^m \lambda_i(E - ff^*)$.

Proof. As before, $E = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. By hypothesis, $0 < \|f\|_m^2$, so $P_{f,m}$ is not the zero matrix. Consequently, $Q_{f,m} = P_m - P_{f,m}$ is a projection of rank $m-1$. For any normalized $z \in \mathbb{C}^n$ such that $Q_{f,m}z = 0$, it then follows that $P = Q_{f,m} + zz^*$ is a projection of rank m. Moreover,

$$\begin{aligned} \mathrm{tr}(P(\Lambda - ff^*)P) &= \mathrm{tr}(Q_{f,m}\Lambda Q_{f,m}) + \mathrm{tr}(zz^*\Lambda Q_{f,m}) + \mathrm{tr}(Q_{f,m}\Lambda zz^*) + \mathrm{tr}(zz^*\Lambda zz^*) \\ &\quad -\mathrm{tr}(Q_{f,m}ff^*Q_{f,m}) - \mathrm{tr}(zz^*ff^*Q_{f,m}) - \mathrm{tr}(Q_{f,m}ff^*zz^*) \\ &\quad -\mathrm{tr}(zz^*ff^*zz^*) \\ &= \mathrm{tr}(Q_{f,m}\Lambda Q_{f,m}) + (z^*\Lambda z)\,\mathrm{tr}(zz^*) - (z^*ff^*z)\,\mathrm{tr}(zz^*) \\ &= \mathrm{tr}f_m^\perp(\Lambda) + \langle(\Lambda - ff^*)z, z\rangle \end{aligned}$$

Thus, if there is such a z that also satisfies $\sum_{i=1}^m \mu_{i+1} - \mathrm{tr}f_m^\perp(\Lambda) \leq \langle(\Lambda - ff^*)z, z\rangle$, we will have

$$\begin{aligned} \sum_{i=1}^m \mu_{i+1} &= \sum_{i=1}^m \mu_{i+1} - \mathrm{tr}f_m^\perp(\Lambda) + \mathrm{tr}f_m^\perp(\Lambda) \\ &\leq \mathrm{tr}f_m^\perp(\Lambda) + \langle(\Lambda - ff^*)z, z\rangle \\ &= \mathrm{tr}(P(\Lambda - ff^*)P) \\ &\leq \sum_{i=1}^m \lambda_i(\Lambda - ff^*). \end{aligned}$$

In pursuit of such a z, set $\hat{\Lambda} = \operatorname{diag}\left(\|f\|^2_{\Lambda,m}/\|f\|^2_m, \lambda_{m+1}, \dots, \lambda_n\right)$ and set $\hat{f} = \|f\|_m \hat{e}_m + \sum_{i=m+1}^n f^i \hat{e}_i$ where $\{\hat{e}_i\}_{i=m}^n$ is the canonical orthonormal basis for $\mathbb{C}^{n-j+1}$. Given any $z = z^m P_m f/\|f\|_m + \sum_{i=m+1}^n z^i e_i$, it is true that $Q_{f,m} z = 0$. For each such z, note that there is a corresponding $\hat{z} = \sum_{i=m}^n z^i \hat{e}_i \in \mathbb{C}^{n-j+1}$. Let $\hat{\Lambda}' = \hat{\Lambda} - \hat{f}\hat{f}^*$ and observe that

$$\begin{aligned}
\left\langle \hat{\Lambda}' \hat{z}, \hat{z} \right\rangle_{\mathbb{C}^{n-m+1}} &= \left\langle \hat{\Lambda} \hat{z}, \hat{z} \right\rangle_{\mathbb{C}^{n-m+1}} - \left| \left\langle \hat{f}, \hat{z} \right\rangle_{\mathbb{C}^{n-m+1}} \right|^2 \\
&= \frac{\|f\|^2_{\Lambda,m}}{\|f\|^2_m} |z^m|^2 + \sum_{i=m+1}^n \lambda_i |z^i|^2 - \left| \|f\|_m \overline{z^m} + \sum_{i=m+1}^n f^i \overline{z^i} \right|^2 \\
&= \sum_{i=1}^m \lambda_i \left| \frac{z^m f^i}{\|f\|_m} \right|^2 + \sum_{i=m+1}^n \lambda_i |z^i|^2 - \left| \sum_{i=1}^m f^i \frac{\overline{z^m f^i}}{\|f\|_m} + \sum_{i=m+1}^n f^i \overline{z^i} \right|^2 \\
&= \langle \Lambda z, z \rangle_{\mathbb{C}^n} - | \langle f, z \rangle_{\mathbb{C}^n} |^2 \\
&= \langle (\Lambda - f f^*) z, z \rangle_{\mathbb{C}^n} .
\end{aligned}$$

Thus, the desired z exists if and only if there is a $\hat{z} \in \mathbb{C}^{n-m+1}$ such that $\sum_{i=1}^m \mu_{i+1} - \operatorname{tr} f_m^\perp(\Lambda) \leq \left\langle \hat{\Lambda}' \hat{z}, \hat{z} \right\rangle_{\mathbb{C}^{n-m+1}}$. We will now show that such a $\hat{z}$ exists by demonstrating that $\hat{\Lambda}'$ has an eigenvalue bounded below by $\sum_{i=1}^m \mu_{i+1} - \operatorname{tr} f_m^\perp(\Lambda)$.

By hypothesis, we have that

$$\sum_{i=1}^{m+1} \lambda_i - \sum_{i=1}^m \mu_{i+1} < \frac{\|f\|^2_{\Lambda,m}}{\|f\|^2_m}.$$

Rearranging this inequality yields

$$\lambda_{m+1} < \sum_{i=1}^m \mu_{i+1} - \sum_{i=1}^m \lambda_i + \frac{\|f\|^2_{\Lambda,m}}{\|f\|^2_m} = \sum_{i=1}^m \mu_{i+1} - \operatorname{tr} f_m^\perp(\Lambda).$$

Note that

$$\sum_{i=1}^m \mu_{i+1} - \operatorname{tr} f_m^\perp(\Lambda) < \sum_{i=1}^m \lambda_i - \operatorname{tr} f_m^\perp(\Lambda) = \frac{\|f\|^2_{\Lambda,m}}{\|f\|^2_m},$$

and hence

$$\lambda_{m+1} < \sum_{i=1}^{m} \mu_{i+1} - \mathrm{tr} f_m^{\perp}(\Lambda) < \frac{\|f\|_{\Lambda,m}^2}{\|f\|_m^2}.$$

The stage is now set to apply Theorem 4.1.3. Setting $c = \sum_{i=1}^{m} \mu_{i+1} - \mathrm{tr} f_m^{\perp}(\Lambda)$ and noting that $\hat{f}$ satisfies the hyperboloidal inequality by hypothesis, we have that $\hat{\Lambda} - \hat{f}\hat{f}^*$ has an eigenvalue greater than $\sum_{i=1}^{m} \mu_{i+1} - \mathrm{tr} f_m^{\perp}(\Lambda)$. Retracing our steps, we have that the m^{th} majorization condition holds. □

Even if the quasihyperbolic condition fails, there is still a chance that the m^{th} majorization inequality holds. Indeed, the quasihyperbolic condition becomes more conservative as one strays from the eigenvectors of a given E and the range of P_m. Nevertheless, implementing the quasihyperbolic condition is computationally inexpensive when compared to computing eigenvalues. It should also be noted that the quasihyperbolic condition is actually stronger than both the cylindrical and directional condition. Still, one must compute the quantities utilized in both of these conditions to evaluate the quasihyperbolic condition. Thus, one might as well check that these hold before applying the quasihyperbolic condition.

Example 4.3.6. As in Example 4.3.4, suppose $E = \mathrm{diag}(3, 2, 1)$ and assume that $E - ff^*$ needs to be such that $\frac{13}{3} \leq \lambda_1(E - ff^*) + \lambda_2(E - ff^*)$. Figure 6 illustrates the region of vectors satisfying the quasihypberbolic condition. Note that the region satisfying the quasihyperbolic condition contains the regions satisfying both of the other conditions.

4.4. Other Utile Machinery

In this subsection, a number of useful techniques are surveyed. Implementation of the presented construction techniques with computational thrift proves challenging.

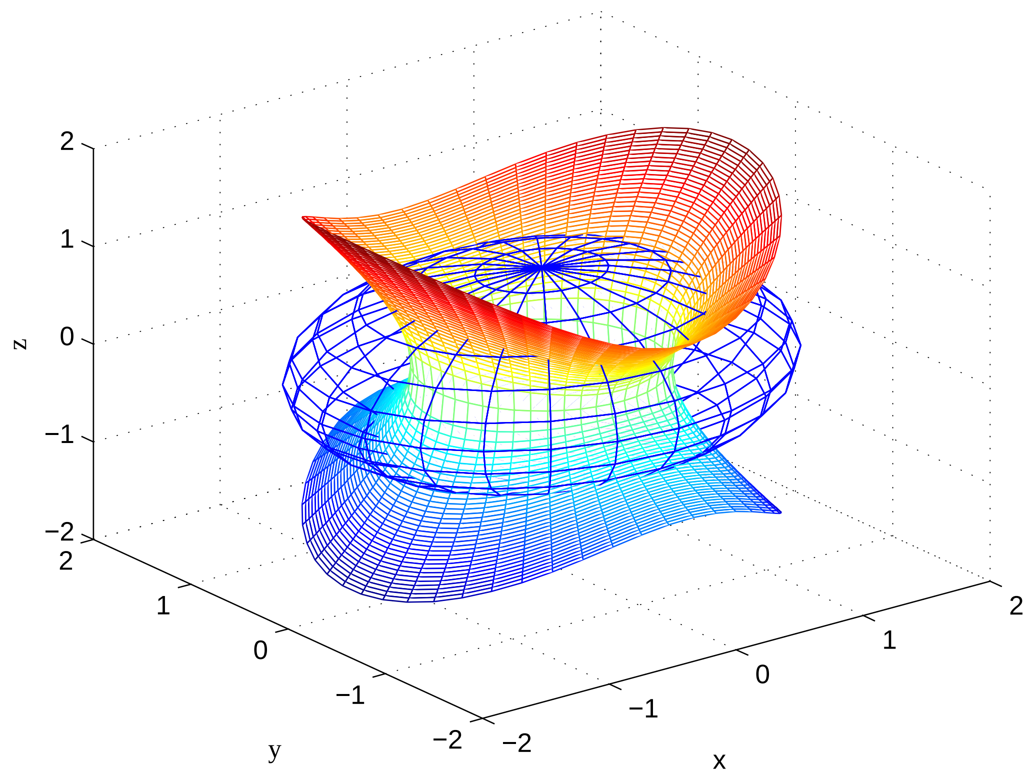

Fig. 6. The quasihyperbolic condition.

Sampling $\mathcal{F}^k(E, \mathbb{F}^n)$ uniformly or almost uniformly is also pertinent consideration. The author found the following results useful when implementing these constructions.

4.4.1. The Sherman-Morrison formula

The Sherman-Morrison formula provides a computationally inexpensive procedure for computing the inverse of $E - fg^*$ when $E \in \mathbf{M}_n(\mathbb{C})$ is invertible, $f, g \in \mathbb{C}^n$, and $E - fg^*$ is invertible. The Sherman-Morrison formula is given by

$$(E - fg^*)^{-1} = E^{-1} - \frac{E^{-1} f g^* E^{-1}}{1 + \langle E^{-1} f, g \rangle}.$$

For our purposes, $f = g$. It should also be noted that a determinant formula for $E - fg^*$ arises from the Sherman-Morrison (see Kéri [10]). This provides another proof of Lemma 4.1.1.

4.4.2. The rank reduced psuedoinverse formula

The Sherman-Morrison formula provides a simple expression for the inverse of $E - ff^*$ when this matrix is invertible. By the ellipsoidal condition, if

$$\sum_{i=1}^{n} \frac{|f^i|^2}{\lambda_i(E)} = 1,$$

then this matrix is not invertible. Nevertheless, it is convenient to have an analogous formula for the pseudoinverse of $E - ff^*$ in this instance. In the case that E is Hermitian and positive definite, the formula is given by

$$(E - ff^*)^\dagger = E^{-1} - \frac{E^{-2} f f^* E^{-1} + E^{-1} f f^* E^{-2}}{\langle E^{-2} f, f \rangle} + \left\langle E^{-3} f, f \right\rangle \frac{E^{-1} f f^* E^{-1}}{\langle E^{-2} f, f \rangle^2}.$$

4.4.3. Uniformly sampling S^n and $V_k(\mathbb{F}^n)$

One may generate random points uniformly on the n-sphere by generating a sample from a normal distribution on $(n+1)$-dimensional dimensional space. Normalizing this sample then produces a point on the sphere. Since Gaussians are isotropic, the distribution obtained in this manner is uniform. To sample from an $(n+1)$-dimensional normal distribution, one need only obtain samples from $n+1$ one dimensional normal distributions since the $n+1$ dimensional normal distribution is the joint distribution arising from $n+1$ normally distributed independent variables. To complexify, one simply multiplies each entry by a random complex number with unit modulus.

Sampling from $V_k(\mathbb{F}^n)$ is possible once one is able to sample from the n-sphere. First, a sample is drawn from the n-sphere. Let this sample be u_1. The next sample (u_2) is then drawn from the $(n-1)$-sphere lying in the subspace orthogonal to u_1. This process continues iteratively until u_k has been chosen.

5. CONCLUSION

This thesis has addressed the manifold structure of both $\mathcal{F}^k(E,\mathbb{F}^n)$ and $\mathcal{F}^k_\mu(E,\mathbb{F}^n)$. In particular, an intimate link between the spaces $\mathcal{F}^k(E,\mathbb{F}^n)$ and Stiefel manifolds has been drawn, and the pairs E, μ for which $\mathcal{F}^k_\mu(E,\mathbb{F}^n)$ is a manifold have been characterized. An interactive method for designing frames contained in these spaces has also been introduced. By considering a causally connected sequence of ellipsoidal regions, any frame in $\mathcal{F}^k(E,\mathbb{F}^n)$ can be constructed vector by vector. This process is specialized to design frames in $\mathcal{F}^k_\mu(E,\mathbb{F}^n)$, but the mitigation of computational costs leads to a causally connected sequence of hyperbolic, cylindrical, directional, and quasihyperbolic regions. Now that these techniques are firmly established, a number of new directions immediately present themselves.

A number of questions may still be asked about the geometry of $\mathcal{F}^k_\mu(E,\mathbb{F}^n)$. In Dykema et al. [6], the connectivity of $\mathcal{F}^{n+2}_{\mathbf{1}_{n+2}}(\frac{n+2}{n}I_n,\mathbb{F}^n)$ was demonstrated and it was conjectured that $\mathcal{F}^k_{\mathbf{1}_k}(\frac{k}{n}I_n,\mathbb{F}^n)$ is connected for all $k \geq n+2$. Verification of this conjecture would be an immediate corrollary if a characterization could be obtained for all the spaces $\mathcal{F}^k_\mu(E,\mathbb{F}^n)$ which are connected. Furthermore, global or even local parameterizations of $\mathcal{F}^k_\mu(E,\mathbb{F}^n)$ would prove useful. It would also be interesting to see if inifinite dimensional analogs held for the results obtained in Section 3.

The promise of numerical verification of Nik Weaver's reformulation of the Kadison-Singer conjecture (see Weaver [14], Kadison and Singer [9]) provided the primary motivation for the construction techniques that have been detailed. The Kadison-Singer conjecture is a forty year old problem in C^*-algebras that has eluded a number of brilliant mathematicians. Though the results obtained in this thesis do not offer a direct resolution of this conjecture, they do present a method by which one may either empirically verify the Kadison-Singer conjecture or ferret out some badly behaved sequence leading to a counterexample. Adapting the constructions to acquire numerical data on this problem will be considered in future publications.

Taken in totality, this thesis has provided apparatuses that pave the way for optimization on these spaces. Since $\mathcal{F}^k(E, \mathbb{F}^n)$ is diffeomorphic to a homogeneous space, a host of nonlinear optimization techniques may be applied to acquire desirable frames. For particular applications, it is often of interest to design frames in $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$ that satisfy further constraints (for example the Grassmanian frames of Heath and Stromer [7]). The presented construction techniques provide a method by which random frames in $\mathcal{F}^k_\mu(E, \mathbb{F}^n)$ may be sampled. Utilizing this sampling method and nonlinear optimization techniques, globally optimal frames may be found with relative ease.

In summary, there are numerous fruitful research directions emerging from this thesis. A number of applications immediately present themselves and shall be considered in due course.

REFERENCES

[1] J.J. Benedetto, M. Fickus, Finite normalized tight frames, Adv. Comp. Math. 18 (2003) 357-358.

[2] P.G. Casazza, M. Leon, Frames with a given frame operator, unpublished manuscript (2002) http://www.math.missouri.edu/%7epete/pdf/PM2.pdf.

[3] G. Dahl, F. Margot, Weak k-majorization and polyhedra, Math. Prog. 81 (1998) 37-53.

[4] I.S. Dhillon, R. W. Heath Jr., J.A. Tropp, Finite-step algorithms for constructing optimal CDMA signature sequences, IEEE Trans. Inform. Theory 50 (2004) 2916-2921.

[5] I. S. Dhillon, R. W. Heath Jr., J.A. Tropp, T. Strohmer, Designing structured tight frames via alternating projection, IEEE Trans. Inform. Theory 51 (2005) 188-209.

[6] K. Dykema, N. Strawn, Manifold structure of spaces of spherical tight frames, Int. Jour. Pure Appl. Math. 28 (2006) 217-256.

[7] R. W. Heath Jr., T. Strohmer, Grassmannian frames with applications to coding and communications, Appl. Comp. Harmon. Anal. 14 (2003) 257-275.

[8] R. Horn, C. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1999.

[9] R. Kadison, I. Singer, Extensions of pure states, Amer. J. Math. 81 (1959) 547-564.

[10] G. Kéri, The Sherman-Morrison formula for the determinant and its application for optimizing quadratic functions on condition sets given by extreme generators, in: F. Giannessi, P.M. Pardalos, Tams Rapcsk (Eds), Optimization Theory: Recent Developments from Matrahaza, Kluwer Academic Publishers, Boston, 2001, pp. 119-122.

[11] K. Kornelson, D. Larson, Rank-one decomposition of operators and construction of frames, in: C. Heil, P. E.T. Jorgensen, D. R. Larson (Eds), Wavelets, Frames, and Operator Theory, Amer. Math. Soc., Providence, 2004, pp. 203-214.

[12] John W. Milnor, Topology from the Differential Viewpoint, Princeton University Press, Princeton, 1997.

[13] S.T. Smith, Geometric optimization methods for adaptive filtering, Ph.D. dissertation, Harvard University, Cambridge, MA., 1993.

[14] N. Weaver, The Kadison-Singer problem in discrepancy theory, Discrete Math. 278 (2004) 227-239.

APPENDIX

MATLAB ROUTINES

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                      %
%   Program: unirandsphere.m                                           %
%   Programmer: Nathaniel Strawn                                       %
%   Contact: nate.strawn@gmail.com                                     %
%   Date: 03/17/07                                                     %
%                                                                      %
%                                                                      %
%   Description:                                                       %
%                                                                      %
%       This routine uniformly samples a sphere in $n$ dimen-          %
%       sional space.                                                  %
%                                                                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function S=unirandsphere(n, cmplx)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   A uniform sample drawn from the real $n$    %
%   cube, converted to a sample drawn from an   %
%   $n$ dimensional gaussian via the inverse    %
%   error function, and then normalized.        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    S=2*rand(n,1)-1;
    S=erfinv(S);
    S=S./norm(S);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   If $cmplx = 1$ then the sample is comple-   %
%   xified randomly.                            %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    if (cmplx == 1)

        theta=rand(n,1);
        C=cos(theta)+i*sin(theta);
        S=S.*C;

    end

return
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                    %
%   Program: SampleStiefel.m                                         %
%   Programmer: Nathaniel Strawn                                     %
%   Contact: nate.strawn@gmail.com                                   %
%   Date: 03/17/07                                                   %
%                                                                    %
%                                                                    %
%   Description:                                                     %
%                                                                    %
%       This routine produces a random sample from the real          %
%       or complex Stiefel manifold of $n$ orthonormal frames        %
%       in $k$ dimensional space.                                    %
%                                                                    %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function V=SampleStiefel(n,k,cmplx)

    count=0;
    V=zeros(k,n);
    P=eye(k);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   During each iteration of the following      %
%   loop, a sample is drawn from the $k$ sph-   %
%   ere, projected onto the space orthonormal   %
%   to the previous samples, normalized, and    %
%   then added to $V$,                          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


    while (count < n)

        count=count+1;

        v=P*unirandsphere(k,cmplx);
        v=v/norm(v);

        V(:,count)=v;
        P=P-v*conj(v');

    end

return
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                    %
%   Program: InterSample.m                                           %
%   Programmer: Nathaniel Strawn                                     %
%   Contact: nate.strawn@gmail.com                                   %
%   Date: 03/17/07                                                   %
%                                                                    %
%                                                                    %
%   Description:                                                     %
%                                                                    %
%       This routine produces a random sample from the inter-        %
%       section of a unit sphere and the ellipsoid induced by        %
%       the Hermitian positive semidefinite matrix $E$.              %
%                                                                    %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function f=InterSample(Etilde,ALIVE,cmplx)

    tol=10^(-12);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   $Etilde$ is the psuedoinverse of $E$.       %
%   $Atilde$ is the projection of $Etilde$ on   %
%   the range of $E$.                           %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    Atilde=(conj(ALIVE)')*Etilde*ALIVE;
    Atilde=(Atilde+conj(Atilde)')/2;

    [n,m]=size(Atilde);
    [V,Lambda]=eig(Atilde);

    lambda=diag(Lambda);

    problem=zeros(n,1);
    dead=zeros(n,1);
    a_up=zeros(n,1);
    a_down=zeros(n,1);
    a_on=zeros(n,1);


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   This determines the indices for which the   %
%   eigenvalues of $Atilde$ are negative,       %
%   zero, between zero and one, one, and gre-   %
%   ater than one.  Let $lambda_i$ be the $i$   %
%   eigenvalue of $Atilde$ then                 %
%                                               %
%       dead(i,1)=1 if abs(lambda_i)<tol        %
%       problem(i,1)=1 if lambda_i<0            %
%       a_up(i,1)=1 if lambda_i>1               %
```

```
%       a_on(i,1)=1 if lambda_i=1               %
%       a_down(i,1)=1 if 0<lambda_i<1           %
%                                               %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    for index=1:n

        if (abs(lambda(index,1)) < tol)

            dead(index,1)=1;

        else

            if (lambda(index,1) <= 0)

                problem(index,1)=1;

            else

                if (lambda(index,1) < 1)

                    a_down(index,1)=1;

                elseif (lambda(index,1) == 1)

                    a_on(index,1)=1;

                else

                    a_up(index,1)=1;

                end

            end

        end

    end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   A random vector is sampled, split into      %
%   parts that are manipulated so that          %
%                                               %
%            <Atilde*f,f>=<f,f>=1               %
%                                               %
%   and then converted lie in the range of      %
%   $E$.                                        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    erratic=rand(n,1);
```

```
    lamb=lambda-1;

    f_up=a_up.*erratic;
    f_down=a_down.*erratic;
    f_on=a_on.*erratic;

    f_up=f_up/sqrt((conj(f_up)')*(lamb.*f_up));
    f_down=f_down/sqrt((conj(f_down)')*(-lamb.*f_down));


    f=f_up+f_down+f_on;
    f=V*(f/norm(f));
    f=ALIVE*f;

return
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                    %
%   Program: SampleUTF.m                                             %
%   Programmer: Nathaniel Strawn                                     %
%   Contact: nate.strawn@gmail.com                                   %
%   Date: 03/17/07                                                   %
%                                                                    %
%                                                                    %
%   Description:                                                     %
%                                                                    %
%       This routine produces a random uniform tight frame           %
%       with $k$ vectors in $n$ dimensional space.                   %
%                                                                    %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    function F=SampleUTF(n,k,cmplx)

    dim=n;
    count=0;
    tol=10^(-12);

    F=zeros(n,k);
    E=diag((k/n)*ones(1,n));
    Etilde=diag((n/k)*ones(1,n));
    ALIVE=eye(n);
    DEAD=[];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   This first loop samples from the interse-   %
%   ction of an ellpsoid bounded region with    %
%   the unit sphere.  Vectors are added to      %
%   the frame in this manner until all remai-   %
%   ning vectors must lie on the surface of     %
%   the ellipsoid.                              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    while (dim < k-count)

        accepted=0;

        while (accepted == 0)


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   A vector on the $n$-sphere is randomly      %
%   chosen.  The ellipsoidal condition is       %
%   then calculated.  If the vector satisfies   %
%   the ellipsoidal condition, then it is put   %
%   in the frame and updates occur for the      %
%   next iteration.  If it does not, then a     %
%   vector is chosen from the intersection of   %
```

```
%   the unit sphere with the ellipsoid.        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

            f=ALIVE*unirandsphere(dim,cmplx);
            f=f/norm(f);

            flipse=1-(conj(f)')*(Etilde*f);

            if (flipse >= 0)

             accepted = 1;
                count=count+1;
                F(:,count)=f;

                if (flipse <= tol)

                    dim=dim-1;

                    ffstar=f*(conj(f)');

                    Etilf=Etilde*f;

                    Etil2=norm(Etilf)^2;
                    Etil3=(conj(Etilf)')*Etilde*Etilf;

                    EtfftE=Etilf*(conj(Etilf)');
                    E2tfftE=Etilde*EtfftE;
                    Etf2ftE=E2tfftE+conj(E2tfftE)';


% Update E and Etilde

                    E=E-ffstar;
                Etilde=Etilde-Etf2ftE/Etil2+Etil3*EtfftE/(Etil2^2);

                    DEAD=[DEAD; Etilf'];
                    ALIVE=null(DEAD);

                else

                    ffstar=f*(conj(f)');

                    Etilf=Etilde*f;
                    EtfftE=Etilf*(conj(Etilf)');

                    E=E-ffstar;
                    Etilde=Etilde+EtfftE/flipse;

                end

            else
```

```
                dim=dim-1;

                f=InterSample(Etilde,ALIVE,cmplx);

                count=count+1;

                F(:,count)=f;

                ffstar=f*(conj(f)');

                Etilf=Etilde*f;

                Etil2=norm(Etilf)^2;
                Etil3=(conj(Etilf)')*Etilde*Etilf;
                EtfftE=Etilf*(conj(Etilf)');
                E2tfftE=Etilde*EtfftE;
                Etf2ftE=E2tfftE+conj(E2tfftE)';

                E=E-ffstar;
                Etilde=Etilde-Etf2ftE/Etil2+Etil3*EtfftE/(Etil2^2);

                DEAD=[DEAD; Etilf'];
                ALIVE=null(DEAD);

                accepted=1;

            end

        end

    end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   This last loop samples exclusively from     %
%   unit sphere - ellipsoid surface intersec-   %
%   tion.                                       %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    while (count < k-1)

        f=InterSample(Etilde,ALIVE,cmplx);

        count=count+1;
        F(:,count)=f;

        ffstar=f*(conj(f)');

        Etilf=Etilde*f;

        Etil2=norm(Etilf)^2;
```

```
        Etil3=(conj(Etilf)')*Etilde*Etilf;

        EtfftE=Etilf*(conj(Etilf)');
        E2tfftE=Etilde*EtfftE;
        Etf2ftE=E2tfftE+conj(E2tfftE)';

        E=E-ffstar;
        Etilde=Etilde-Etf2ftE/Etil2+Etil3*EtfftE/(Etil2^2);

        DEAD=[DEAD; Etilf'];
        ALIVE=null(DEAD);

        dim=dim-1;

    end

    F(:,k)=ALIVE;

return
```

VITA

Name:	Nathaniel Kirk Strawn
Address:	P.O. Box 11122, College Station, TX, 77842-1122
Email Address:	nate.strawn@gmail.com
Education:	B.S., Mathematics, Texas A&M University, 2005
	M.S., Mathematics, Texas A&M University, 2007

The typist for this thesis was Nathaniel Strawn.