MODEL FOR A FUNDAMENTAL THEORY WITH SUPERSYMMETRY

A Dissertation

by

SEIICHIRO YOKOO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2006

Major Subject: Physics

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ABSTRACT

Model for a Fundamental Theory with Supersymmetry. (December 2006)

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Physics in the year 2006 is tightly constrained by experiment, observation, and mathematical consistency. The Standard Model provides a remarkably precise description of particle physics, and general relativity is quite successful in describing gravitational phenomena. At the same time, it is clear that a more fundamental theory is needed for several distinct reasons. Here we consider a new approach, which begins with the unusually ambitious point of view that a truly fundamental theory should aspire to explaining the origins of Lorentz invariance, gravity, gauge fields and their symmetry, supersymmetry, fermionic fields, bosonic fields, quantum mechanics and spacetime. The present dissertation is organized so that it starts with the most conventional ideas for extending the Standard Model and ends with a microscopic statistical picture, which is actually the logical starting point of the theory, but which is also the most remote excursion from conventional physics.

One motivation for the present work is the fact that a Euclidean path integral in quantum physics is equivalent to a partition function in statistical physics. This suggests that the most fundamental description of nature may be statistical. This dissertation may be regarded as an attempt to see how far one can go with this premise in explaining the observed phenomena, starting with the simplest statistical picture imaginable. It may be that nature is richer than the model assumed here, but the present results are quite suggestive, because, with a set of assumptions that are not unreasonable, one recovers the phenomena listed above. At the end, the present theory leads back to conventional physics, except that Lorentz invariance

and supersymmetry are violated at extremely high energy. To be more specific, one obtains local Lorentz invariance (at low energy compared to the Planck scale), an $SO(N)$ unified gauge theory (with $N = 10$ as the simplest possibility), supersymmetry of Standard Model fermions and their sfermion partners, and other familiar features of standard physics. Like other attempts at superunification, the present theory involves higher dimensions and topological defects.

To my parents, Yuichi and Machiko Yokoo

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CHAPTER I

INTRODUCTION

Physics in the year 2006 is tightly constrained by experiment, observation, and mathematical consistency. The Standard Model provides a remarkably precise description of particle physics, and general relativity is quite successful in describing gravitational phenomena. At the same time, it is clear that a more fundamental theory is needed for several distinct reasons. (1) The Standard Model (SM) contains many unexplained features and parameters. (2) It is now known that neutrinos have small masses, and such masses cannot be accommodated in the Standard Model: A Dirac mass would require an extra field in each generation of fermions, and a Majorana mass would break conservation of lepton number. On the other hand, both types of masses fit naturally into a grand unified theory (GUT) like $SO(10)$.(3) Calculations of the running coupling constants for the three forces of the Standard Model show that they converge at high energy if one extends the Standard Model to include both grand unification and supersymmetry (SUSY). At the same time, SUSY eliminates a problem posed by the quadratic divergence of the Higgs mass in the SM. (4) Quantum field theory, which is the basis of the SM (and its extensions) appears to be inconsistent with general relativity. (5) Standard physics fails to account for the observations of dark matter, dark energy, scale invariance of fluctuations in the cosmic microwave background, and preponderance of matter over antimatter. One is then faced with the need for a more fundamental theory, but also with the fact that such a theory must reproduce the very tightly-knit structure of standard physics in the regimes where standard physics has been tested. Formulating a candidate for a

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fundamental theory is then a rather imposing task, with many potential routes to failure when one compares with the extremely precise tests of certain aspects of standard physics. This may account for the common statement that superstring theory is the only viable candidate for a fundamental theory. On the other hand, superstring theory has a weak record of testable predictions, despite 30 years of intense effort by a large community of brilliant mathematical physicists [1, 2, 3].

Here we consider an alternative and very different model for a fundamental theory, which actually has much more ambitious goals than superstring theory, since it begins with the point of view that a truly fundamental theory should aspire to explaining the origins of

- Lorentz invariance
- gravity
- gauge fields and their symmetry
- supersymmetry
- fermionic fields
- bosonic fields
- quantum mechanics
- spacetime.

This dissertation essentially follows the order above, although the logical development of the theory essentially follows the reverse order: In Chapter V, we introduce the fundamental statistical picture, in which both spacetime coordinates and quantum fields are defined in terms of the occupancies of states. In Chapter IV, a primitive supersymmetry is first obtained between the initial bosonic and fermionic fields, and then the more usual form of SUSY is obtained. In Chapter III, gauge fields and gravity are found to follow from the assumption of a specific model for the behavior of the fields in both four-dimensional external spacetime and a d-dimensional internal space. (The present theory is similar to superstring theory in that it contains higher dimensions, SUSY, and topological defects, but in other respects it is quite different.) Finally, in this reversal of the order of presentation within the dissertation, Lorentz invariance is derived as a low energy symmetry, together with the potential for Lorentz violation at higher energies.

Before beginning the presentation of these novel elements of our theory, in the next chapter, we establish a foundation by reviewing the most relevant aspects of standard physics. In deciding how much of this introductory material to put in appendices, and how much to include in the Introduction itself, we were guided by the need for continuity in the presentation: The present theory predicts an $SO(N)$ grand unified gauge group, with $N = 10$ suggested by experiment, so it seems essential that the various ideas for grand unification be reviewed. On the other hand, any GUT is basically a generalization of the Standard Model (SM). For this reason, we begin the Introduction with the SM, then pass to GUTs, then to SUSY, then to radiative corrections with SUSY.

We relegate the following topics to 3 appendices: notation and conventions (for gamma matrices etc.); complex representations; and two-component spinor algebra. The motivation for both these appendices and the introductory material in the main text is that we wish the dissertation to be readable by anyone has had a first course in field theory, rather than just experts in particle physics.

Before considering the Standard Model (SM) and its extensions, it is worthwhile to consider in a little more detail why these extensions are required. The SM is very successful. For example, it predicted the existence of the W and Z bosons, gluons, the charm quark, and the top quark, and the masses of the W and Z gauge bosons [4], see Table I. However, as mentioned above, the SM is clearly not complete:

Table I. Experimental and theoretical values of mass of W and Z bosons.

	Experiment (GeV)	SM Calculated (GeV)
Mass of W boson	80.454 ± 0.059 (UA2, CDF, and DO) 80.412 ± 0.042 (LEP2)	80.390 ± 0.018
Mass of Z boson	91.1876 ± 0.0021	91.1874 ± 0.0021

- There are no masses for the (left-handed) neutrinos. Neutrino oscillations have been observed by the Super-Kamiokande experiment and others, and these oscillations require that the neutrinos have masses which are very small ($\ll 1$ eV in the most plausible models).

- The energy scale difference between the SM scale (100 GeV) and the GUT scale ($10^{14} - 10^{15}$ GeV) is enormous, and it is not natural that there should be nothing between the two scales.

- 23% of the energy density of the universe is apparently cold dark matter, which must be stable and interact only weakly with ordinary matter. The SM provides no such candidate.

- Radiative corrections to the masses of the $SU(2)$ Higgs bosons diverge quadratically.

- There is no detailed mechanism to produce a negative mass-squared term, which is required for $SU(2)$ Higgs fields to acquire a nonzero vacuum expectation value.

- The gravitational field escapes unification.

- The SM cannot explain why there are 3 generations.
- The SM model cannot explain the quark mixing and neutrino mixing mass matrices.

Therefore, when we aspire to a fundamental theory, it is required not only to reproduce the successful parts of the SM at the electroweak energy scale, but also to resolve at least some of these problems.

For example, an appropriate GUT yields natural neutrino masses; SUSY protects the Higgs mass from a quadratic divergence; SUSY and GUTs lead to a beautiful unification of coupling constants, as can be seen in Fig. 1; SUSY breaking can be treated in a supergravity (SUGRA) model; and there are rich predictions concerning dark matter, proton decay etc. The SuperKamiokande proton decay experiment has determined the lower limit of the lifetime of the proton, at the 90% confidence level, to be 2.3×10^{33}, 1.3×10^{33}, and 1.0×10^{33} years for the $p \to \bar{\nu} K^+$, $p \to \mu^+ K^0$, and $p \to e^+ K^0$ modes, respectively, so the minimal SUSY $SU(5)$ is excluded [6], with limits of 2.6×10^{33} and 2.1×10^{33} years for $p \to e^+ \pi^0$ and $p \to \mu^+ \pi^0$ [7]. Nevertheless, future proton decay and dark matter experiments will increasingly move into regimes where SUSY and GUT predictions may lead to experimental verification.

In Appendix A, we introduce our notations and conventions. In Section A of this Introduction, we review the SM; in Section B, the gauge unification groups $SU(N)$ and $SO(N)$; in Section C, SUSY.

A. Review of the Standard Model

1. Main Ideas

The fields of the SM are summarized in Table II.

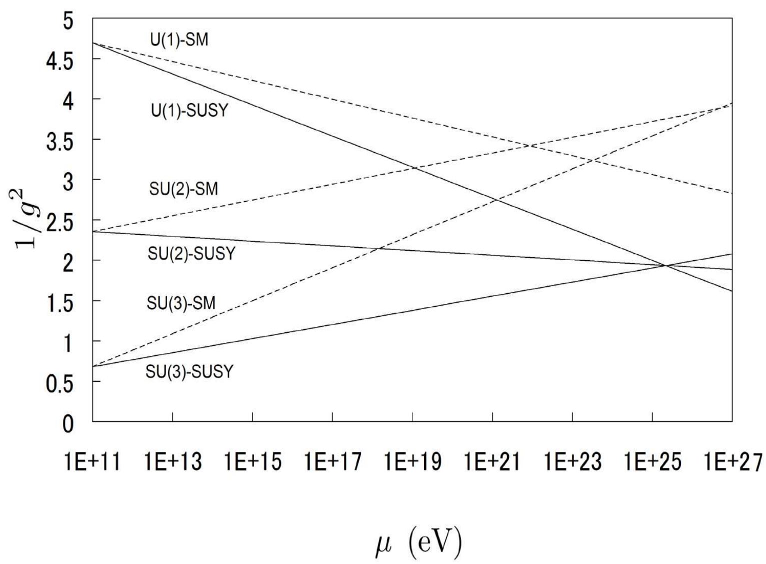

Fig. 1. Running gauge coupling constants in the SM and SUSY-SM. The vertical axis is the inverse of the square of the gauge coupling constant, and the horizontal axis is the energy scale. The broken line is for the SM, and the solid line is for the SUSY-SM. The SM has only one Higgs doublet, but The SUSY-SM calculation involves not only the SM fields and their superpartners but two Higgs doublets [5].

Table II. Field content of the Standard Model.

lepton doublet (left chiral)	quark doublet (left chiral)
$f_{L-l_1} = \begin{pmatrix} f_{\nu_e} \text{ (}e\text{-neutrino)} \\ f_e \text{ (electron)} \end{pmatrix}$	$f_{L-q_1} = \begin{pmatrix} f_u \quad \text{(up)} \\ f_d \quad \text{(down)} \end{pmatrix}$
$f_{L-l_2} = \begin{pmatrix} f_{\nu_\mu} \text{ (}\mu\text{-neutrino)} \\ f_\mu \ (\mu) \end{pmatrix}$	$f_{L-q_2} = \begin{pmatrix} f_c \quad \text{(charm)} \\ f_s \quad \text{(strange)} \end{pmatrix}$
$f_{L-l_3} = \begin{pmatrix} f_{\nu_\tau} \text{ (}\tau\text{-neutrino)} \\ f_\tau \ (\tau) \end{pmatrix}$	$f_{L-q_3} = \begin{pmatrix} f_t \quad \text{(top)} \\ f_b \quad \text{(bottom)} \end{pmatrix}$
lepton singlet (right chiral)	**quark singlet**
$f_{R-l_1} = E_e$ (electron)	$f_{R-q_1} = U, D$ (up, down)
$f_{R-l_2} = E_\mu$ (μ)	$f_{R-q_2} = C, S$ (charm, strange)
$f_{R-l_3} = E_\tau$ (τ)	$f_{R-q_3} = T, B$ (top, bottom)

gauge fields	
B^μ	$(U(1)_Y)$
A^μ	$(SU(2)_L)$
G^μ	$(SU(3)_C)$

Higgs doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{and} \quad \phi^c = \begin{pmatrix} \bar{\phi}^0 \\ \phi^- \end{pmatrix} = i\sigma_2\phi^*$$

The Standard Model is based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, where C, L, and Y stand for color, left-handed, and hypercharge. The gauge interactions are introduced by the covariant derivative, and the fermion-scalar interactions are introduced by the Yukawa terms. The Lagrangian for the Standard Model is given by

$$\mathcal{L}_{SM} = \mathcal{L}_{ew} + \mathcal{L}_{QCD}, \tag{1.1}$$

where $\mathcal{L}_{QCD}$ contains the $SU(3)_C$ physics and $\mathcal{L}_{ew}$ the $SU(2)_L \times U(1)_Y$ physics, and

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^{\alpha}_{\mu\nu} G^{\alpha\mu\nu} + \sum_k i\bar{q}_k \gamma^\mu D^{QCD}_\mu q_k \tag{1.2}$$

with

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu, \tag{1.3}$$

$$D^{QCD}_\mu q_k = \left[\partial_\mu + i g_s \left(\lambda_a/2\right) G^a_\mu\right] q_k, \tag{1.4}$$

$$q = \begin{pmatrix} q_{\mathrm{red}} \\ q_{\mathrm{green}} \\ q_{\mathrm{blue}} \end{pmatrix}, \tag{1.5}$$

where the generators are given by

$$\lambda_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad Tr\left\{\lambda_i \lambda_j\right\} = \frac{1}{2}\delta_{ij}, \quad (1.6)$$

and

$$\mathcal{L}_{ew} = \mathcal{L}_{SU(2)_L \times U(1)_Y} + \mathcal{L}_{fermion} + \mathcal{L}_{scalar}, \qquad (1.7)$$

where

$$\mathcal{L}_{SU(2)_L \times U(1)_Y} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad (1.8)$$

$$\mathcal{L}_{fermion} = \sum_f \left[i \bar{f}_L \gamma^\mu D^{ew}_\mu f_L + i \bar{f}_R \gamma^\mu D^{ew}_\mu f_R \right], \qquad (1.9)$$

$$\begin{aligned} \mathcal{L}_{scalar} &= D^{ew}_\mu \phi^\dagger D^{ew\mu} \phi - m^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 \\ &+ G_{Yf}\left[\bar{f}_L \phi f_R + \bar{f}_R \phi^\dagger f_L\right]. \end{aligned} \qquad (1.10)$$

$W^a_{\mu\nu}$ and $F_{\mu\nu}$ are the $SU(2)_L$ and $U(1)_Y$ field strengths, f is a fermion field (lepton or quark), ϕ is the Higgs doublet, G_{Yf} is a Yukawa coupling, and

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu, \tag{1.11}$$

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{1.12}$$

$$D^{ew}_\mu f_L = \left[\partial_\mu + ig\left(\tau_a/2\right) W^a_\mu + ig'\left(Y/2\right) B_\mu\right] f_L, \tag{1.13}$$

$$D^{ew}_\mu f_R = \left[\partial_\mu + ig'\left(Y/2\right) B_\mu\right] f_R, \tag{1.14}$$

$$D^{ew}_\mu \phi = \left[\partial_\mu + ig\left(\tau_a/2\right) W^a_\mu + ig'\left(Y/2\right) B_\mu\right] \phi, \tag{1.15}$$

in the case of leptons. For quarks, however, the ∂_μ term is to be omitted because it is already included in (1.4). Here g and g' are the $SU(2)_L$ and $U(1)_Y$ gauge coupling constants, τ_a is a Pauli matrix, and Y is the hypercharge.

The Higgs doublet is

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{where} \quad \begin{array}{c} \phi^+ \equiv \left(\phi_1 + i\phi_2\right)/\sqrt{2} \\ \phi^0 \equiv \left(\phi_3 + i\phi_4\right)/\sqrt{2} \end{array}, \tag{1.16}$$

$$\phi^c = \begin{pmatrix} \bar{\phi}^0 \\ \phi^- \end{pmatrix} = i\sigma_2 \phi^*, \tag{1.17}$$

where the vacuum expectation value (V.E.V.) of the ϕ and ϕ^c produce masses for the second component and the first component of the fermion doublet, respectively. The potential $V\left(\phi\right) = -m^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2$ has a minimum at $\phi^\dagger\phi = \frac{m^2}{2\lambda}$, and we choose our vacuum expectation values as

$$\langle\phi_i\rangle = 0, \text{ for } i = 1, 2, 4, \tag{1.18}$$

$$\langle\phi_3\rangle = \upsilon = \sqrt{m^2/\lambda}, \tag{1.19}$$

so that

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \upsilon \end{pmatrix}. \tag{1.20}$$

We call the quantum fluctuations of ϕ_3 about the value υ $H(x)$:

$$H(x) = \phi_3 - \upsilon. \tag{1.21}$$

Then the mass of a fermion is

$$m_f = \frac{G_{Yf}\upsilon}{\sqrt{2}}. \tag{1.22}$$

Next we determine the masses of the gauge bosons:

$$\begin{aligned}
\left| \left(ig \underbrace{\frac{\tau_a}{2}}_{\equiv T_a} W_\mu^a + ig' \frac{Y}{2} B_\mu \right) \langle \phi \rangle \right|^2 &= \frac{1}{8} \left| \begin{pmatrix} gW_\mu^3 + g' B_\mu & g\left(W_\mu^1 - iW_\mu^2\right) \\ g\left(W_\mu^1 + iW_\mu^2\right) & -gW_\mu^3 + g' B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \upsilon \end{pmatrix} \right|^2 \\
&= \left(\frac{1}{2} \upsilon g \right)^2 W_\mu^+ W^{-\mu} + \frac{1}{8} \upsilon^2 \left(-gW_\mu^3 + g' B_\mu \right)^2 + 0 \left(gW_\mu^3 + g' B_\mu \right)^2 \\
&= \left(\frac{1}{2} \upsilon g \right)^2 W_\mu^+ W^{-\mu} + \frac{1}{8} \upsilon^2 \left(g^2 + g'^2 \right) Z_\mu Z^\mu + 0 A_\mu A^\mu,
\end{aligned} \tag{1.23}$$

where $W_\mu^\pm \equiv \left(W_\mu^1 \mp iW_\mu^2\right)/\sqrt{2}$, $Z_\mu \equiv W_\mu^3 \cos\theta_w - B_\mu \sin\theta_w$, and $A_\mu \equiv W_\mu^3 \sin\theta_w + B_\mu \cos\theta_w$. θ_w is the Weinberg (or weak) angle and is defined by

$$\cos\theta_w = \frac{g}{\left(g^2 + g'^2\right)^{1/2}}, \qquad \sin\theta_w = \frac{g'}{\left(g^2 + g'^2\right)^{1/2}}. \tag{1.24}$$

Therefore,

$$M_W = \frac{1}{2} \upsilon g, \quad M_Z = \frac{1}{2} \upsilon \left(g^2 + g'^2\right)^{1/2}, \quad M_A = 0. \tag{1.25}$$

The V.E.V. υ of the Higgs field is calculated as

$$\upsilon = \frac{2M_W}{g} = \frac{1}{\left(\sqrt{2} G_F\right)^{1/2}} = 246.2\,(\mathrm{GeV}) \qquad \text{where } G_F = 1.16637\,(2) \times 10^{-5}\ \mathrm{GeV}^{-2}. \tag{1.26}$$

The quantum numbers of the fields are given in Table III.

Table III. SU(2) and U(1) quantum numbers of Standard Model matter fields.

	T	T_3	Y	Q
ν	$1/2$	$1/2$	-1	0
e_L^-	$1/2$	$-1/2$	-1	-1
e_R^-	0	0	-2	-1
u_L	$1/2$	$1/2$	$1/3$	$2/3$
d_L	$1/2$	$-1/2$	$1/3$	$-1/3$
u_R	0	0	$4/3$	$2/3$
d_R	0	0	$-2/3$	$-1/3$

The electroweak gauge interaction Lagrangian density becomes

$$\begin{aligned}
\mathcal{L}_{\text{ew-int.}} &= -gJ_a^\mu W_\mu^a - g'\frac{1}{2}J_Y^\mu B_\mu \\
&= -g\bar{\Psi}_L\gamma^\mu T_a\psi_L W_\mu^a - g'\frac{1}{2}\bar{\Psi}\gamma^\mu Y\Psi B_\mu \\
&= -\frac{g}{\sqrt{2}}\bar{\Psi}_L\gamma^\mu\left(T_1+iT_2\right)\psi_L\left(\frac{W_\mu^1-iW_\mu^2}{\sqrt{2}}\right) - \frac{g}{\sqrt{2}}\bar{\Psi}_L\gamma^\mu\left(T_1-iT_2\right)\psi_L\left(\frac{W_\mu^1+iW_\mu^2}{\sqrt{2}}\right) \\
&- \left(g^2+g'^2\right)^{1/2}\bar{\Psi}\gamma^\mu\left(\frac{g^2}{g^2+g'^2}T_3 - \frac{g'^2}{g^2+g'^2}\frac{Y}{2}\right)\psi\left(\frac{g}{\left(g^2+g'^2\right)^{1/2}}W_\mu^3 - \frac{g'}{\left(g^2+g'^2\right)^{1/2}}B_\mu\right) \\
&- \frac{gg'}{\left(g^2+g'^2\right)^{1/2}}\bar{\Psi}\gamma^\mu\left(T_3+\frac{Y}{2}\right)\psi\left(\frac{g'}{\left(g^2+g'^2\right)^{1/2}}W_\mu^3 + \frac{g}{\left(g^2+g'^2\right)^{1/2}}B_\mu\right) \\
&= -\frac{g}{\sqrt{2}}J_{\text{charged}}^\mu W_\mu^+ - \frac{g}{\sqrt{2}}J_{\text{charged}}^{\mu\dagger}W_\mu^- - \left(g^2+g'^2\right)^{1/2}J_{\text{neutral}}^\mu Z_\mu - \underbrace{\frac{gg'}{\left(g^2+g'^2\right)^{1/2}}}_{\equiv e}J_{\text{em}}^\mu A_\mu,
\end{aligned} \tag{1.27}$$

where

$$J^{\mu}_{\text{charged}} = \bar{\Psi}_L \gamma^{\mu} \left(T_1 + iT_2\right) \psi_L, \tag{1.28}$$

$$J^{\mu\dagger}_{\text{charged}} = \bar{\Psi}_L \gamma^{\mu} \left(T_1 - iT_2\right) \psi_L, \tag{1.29}$$

$$J^{\mu}_{\text{neutral}} = \bar{\Psi} \gamma^{\mu} \left(\frac{g^2}{g^2 + g'^2} T_3 - \frac{g'^2}{g^2 + g'^2} \frac{Y}{2}\right) \psi, \tag{1.30}$$

$$J^{\mu}_{\text{em}} = \bar{\Psi} \gamma^{\mu} \underbrace{\left(T_3 + \frac{Y}{2}\right)}_{\equiv Q} \psi, \tag{1.31}$$

with $T_i \psi_R = 0$. Therefore, the electric charge e and the charge operator Q are

$$e = \frac{gg'}{\left(g^2 + g'^2\right)^{1/2}}, \tag{1.32}$$

$$Q = T_3 + \frac{Y}{2}. \tag{1.33}$$

2. A Way to Obtain the $SU(N)$ Generators

To obtain the generators of $SU(N)$, we first determine the Cartan generators. As the $SU(N)$ group is rank $N-1$, there are $N-1$ traceless diagonal real Cartan generators H_a and they are taken to satisfy

$$Tr\left(H_a H_b\right) = \frac{1}{2} \delta_{ab}. \tag{1.34}$$

The general way to get the Cartan generators is

$$\left[H_a\right]_{ij} = \frac{1}{\sqrt{2a\left(a+1\right)}} \left(\sum_{k=1}^{a} \delta_{ik} \delta_{jk} - a \delta_{i,a+1} \delta_{j,m+1}\right) \qquad \text{where } a = 1, \cdots, N-1, \tag{1.35}$$

which means that the H_a has 1 in the first a diagonal elements, and to make it traceless the $(a+1)$-th diagonal component must be $-a$, with the rest of the diagonal components being 0:

$$H_a = \frac{1}{\sqrt{2a\,(a+1)}} \begin{pmatrix} 1_{1,1} & & & & & & \\ & \ddots & & & & & \\ & & 1_{a,a} & & & & \\ & & & -a_{a+1,a+1} & & & \\ & & & & 0_{a+2,a+2} & & \\ & & & & & \ddots & \\ & & & & & & 0_{N,N} \end{pmatrix}. \tag{1.36}$$

The other adjoint representation states, which are not the Cartan generators, have weight vector α satisfy

$$[H_a, E_\alpha] = \alpha_a E_\alpha, \tag{1.37}$$

where α_a is the component of the weight vector α. Then we can determine E_α. As E_α can be understood as the raising and lowing operator, it can be related to the $SU\,(N)$ generator T_j with $j = N, \cdots, N^2 - 1$ as

$$E_{\pm\alpha} = \frac{1}{\sqrt{2}}\left(T_j \pm iT_{j+1}\right), \tag{1.38}$$

where T_i and T_{i+1} are given by

$$T_j = \frac{1}{\sqrt{2}}\left(E_{+\alpha} + E_{-\alpha}\right), \tag{1.39}$$

$$T_{j+1} = -\frac{i}{\sqrt{2}}\left(E_{+\alpha} - E_{-\alpha}\right). \tag{1.40}$$

The Cartan generators H_a are related to the $SU\,(N)$ generator T as

$$T_j = H_j \qquad \text{with } j = 1, \cdots, N-1. \tag{1.41}$$

3. Example: $SU(2)$

The rank of $SU(2)$ is 1, and there is one Cartan generator which is given by

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.42}$$

The eigenvectors and the associated weights are

$$H_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underset{\text{weight}}{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.43}$$

$$H_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underset{\text{weight}}{-\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.44}$$

As the weights differ by ± 1, we have one component root vectors α given by

$$\alpha = \pm 1. \tag{1.45}$$

Because of the relation

$$[H_1, E_{\pm 1}] = \pm 1 E_{\pm 1}, \tag{1.46}$$

$E_{\pm 1}$ is determined to be

$$E_{+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{1.47}$$

$$E_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.48}$$

where the factor $1/\sqrt{2}$ is to satisfy the relation

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H. \tag{1.49}$$

$SU\left(2\right)$ has $2^2-1=3$ generators. One of then corresponds to the Cartan generator,

$$\begin{aligned} T_1 &= H_1 \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{1.50}$$

and the others, T_2 and T_3, are related to $E_{\pm 1}$ as

$$E_{\pm 1} = \frac{1}{\sqrt{2}}\left(T_2 \pm iT_3\right). \tag{1.51}$$

Therefore we obtain T_2 and T_3 as

$$\begin{aligned} T_2 &= \frac{1}{\sqrt{2}}\left(E_{+1}+E_{-1}\right) \\ &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{1.52}$$

$$\begin{aligned} T_3 &= -i\left(E_{+1}-E_{-1}\right) \\ &= \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \end{aligned} \tag{1.53}$$

and we can derive the Pauli matrices as essentially the $SU\left(2\right)$ generators.

B. Gauge Unification

1. $SU\left(N\right)$ and $SO\left(N\right)$ Groups

a. $SU(N)$

When the generators of the $SU\left(N\right)$ group are T, the special unitary operator U is written as

$$U\left(\alpha\right) = e^{i\alpha_a T_a}, \tag{1.54}$$

where α_a is a parameter and T_a is required to be Hermitian because $UU^\dagger = U^\dagger U = 1$. From the $\det(U) = 1$ condition we obtain

$$\det(U(\alpha)) = \det\left(e^{i\alpha_a T_a}\right) = \det\left(e^{iD}\right) = \prod_i e^{iD_{ii}} = e^{Tr[iD]} = e^{Tr[i\alpha_a T_a]}, \tag{1.55}$$

where $D = X(\alpha_i T_i) X^\dagger$ with an operator X which satisfies $\det(X) = \det\left(X^\dagger\right) = 1$ and $XX^\dagger = X^\dagger X = 1$. The trace of D is

$$\begin{aligned} Tr(D) &= Tr\left(X(\alpha_a T_a) X^\dagger\right) = X_{ij}(\alpha_a T_a)_{jk} X^\dagger_{kl} = X^\dagger_{kl} X_{ij}(\alpha_a T_a)_{jk} \\ &= Tr\left(\underbrace{X^\dagger X}_{=1}(\alpha_a T_a)\right) = Tr(\alpha_a T_a). \end{aligned} \tag{1.56}$$

Therefore, to satisfy $\det(U) = 1$ it is required that

$$Tr(\alpha_a T_a) = 0 \underset{\text{to be satisfied for arbitrary } \alpha_a}{\longrightarrow} Tr(T_a) = 0 \tag{1.57}$$

Then the requirement for the generators T of a special unitary operator is:

T_a is traceless and Hermitian.

The number of generators corresponds to the number of independent variables in the matrix. Since T_a is Hermitian, the N diagonal components of the matrix are real, and only half of the off-diagonal complex components are independent. Since T_a is traceless, the independent variables are reduced by one, and the order, which is the number of generators, is given by

$$\text{Order} = \frac{N^2 - N}{2} 2 + N \underbrace{-1}_{\text{traceless}} = (N+1)(N-1). \tag{1.58}$$

Since the generators are traceless and Hermitian, we can produce a diagonal matrix by

$$[H_a]_{ij} = \frac{1}{\sqrt{2a\,(a+1)}} \left(\sum_{k=1}^{a} \delta_{ik}\delta_{jk} - a\delta_{i,a+1}\delta_{j,m+1} \right) \qquad \text{where } a = 1, \cdots, N-1, \tag{1.59}$$

as we have already seen in the preceding section. Therefore the rank, which is the number of diagonal matrices (Cartan generators), is given by

$$\text{rank} = N - 1. \tag{1.60}$$

For the definition of a complex representation, please see Appendix B.

b. $SO\,(N)$

If the generators of the $SO\,(N)$ group are represented by M, a special orthogonal operator O is written as

$$O\,(\omega) = e^{i\omega_q M_q}, \tag{1.61}$$

where ω_q is a parameter, and M_q is required to be anti-symmetric because $O^T O = OO^T = 1$. From the $\det(O) = 1$ condition we obtain

$$\det\left(O\,(\omega)\right) = \det\left(e^{i\omega_q M_q}\right) = \det\left(e^{iD}\right) = \prod_i e^{iD_{ii}} = e^{Tr[iD]} = e^{Tr[i\omega_q M_q]}, \tag{1.62}$$

where $D = Y\left(\omega_{ab}M_{ab}\right)Y^T$, with an operator Y which satisfies $\det(Y) = \det\left(Y^T\right) = 1$ and $YY^T = Y^T Y = 1$. The trace of D is

$$\begin{aligned} Tr\,(D) = Tr\left(Y\left(\omega_{ab}M_{ab}\right)Y^T\right) &= Y_{ij}\left(\omega_{ab}M_{ab}\right)_{jk} Y^T_{ki} = Y^T_{ki} Y_{ij}\left(\omega_{ab}M_{ab}\right)_{jk} \\ &= Tr\left(Y^T Y\left(\omega_{ab}M_{ab}\right)\right) = Tr\left(\omega_{ab}M_{ab}\right). \end{aligned} \tag{1.63}$$

Therefore, to satisfy $\det(O) = 1$ it is required that

$$Tr\left(\omega_q M_q\right) = 0 \underset{\text{to be satisfied for arbitrary } \omega_{ab}}{\longrightarrow} Tr\left(M_q\right) = 0. \tag{1.64}$$

Therefore the requirement for a special orthogonal operator is:

M_q is traceless and antisymmetric.

Since M_q is antisymmetric, all of the diagonal components are zero, and only half of the off-diagonal components are independent. Therefore the order, which is the number of generators, is given by

$$\text{Order} = \underbrace{\frac{N^2 - N}{2}}_{\text{\# of independent variables off-diag.}} \quad \text{for } SO\,(N)\,. \tag{1.65}$$

Because the $SO\,(N)$ generators are anti-symmetric matrices and the diagonal components are zero, we cannot produce mutually commuting matrices by diagonalization. To determine the rank of $SO\,(N)$ we go back to the basic idea that $SO\,(N)$ describes a rotation in coordinate space. Then the generators carry two vector indices and can be written as $M_{\mu\nu}$, which means that the generator rotates the vector index ν into μ, and this corresponds to the angular momentum operator. Therefore $M_{\mu\nu}$ is antisymmetric under $\mu \leftrightarrow \nu$. The commutator of these generators is found to be

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\left(\delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} + \delta_{\mu\sigma}M_{\nu\rho} - \delta_{\nu\sigma}M_{\mu\rho}\right), \tag{1.66}$$

where the indices on the right-hand side indicate that one obtains a minus sign under $\mu \leftrightarrow \nu$ or $\rho \leftrightarrow \sigma$, and under $\mu \leftrightarrow \rho$ or $\nu \leftrightarrow \sigma$. The "$-i$" on the right-hand side is needed to satisfy the relation under Hermitian conjugation. The rank is the number of mutually commuting generators. From the right hand side of (1.66) the commutator vanishes when the indices μ, ν, ρ, and σ all have different values. Then, when the rank is N, we need $2N$ different indices, and the size of the space is required to be $2N$ or $2N+1$ dimensional:

$$\text{rank} = N \qquad \text{for } SO\,(2N) \text{ and } SO\,(2N+1)\,. \tag{1.67}$$

Table IV. Summary of the simple Lie groups.

Group	Rank	Order	Complex Rep.
$SU(N)$	$N-1$	$(N+1)(N-1)$	$N \geq 3$
$SO(2N)$	N	$N(2N-1)$	$N = 5, 7, 9, \cdots$
$SO(2N+1)$	N	$N(2N+1)$	No
$Sp(2N)$	N	$N(2N+1)$	No
G_2	2	14	No
F_4	4	52	No
E_6	6	78	Yes
E_7	7	133	No
E_8	8	248	No

.

2. Summary of Simple Lie Groups

Following Collins et al. [8] for the simple Lie groups other than $SU(N)$ and $SO(N)$, we summarize their properties in Table IV.

3. Choice of the Grand Unified Group G

As discussed in Collins et al. [8], when we consider any candidate G for the grand unified group, it must satisfy the following requirements:

- G contains the Standard Model, $SU(3)_C \times SU(2)_L \times U(1)_Y$, so it must have a rank of at least 4.
- G must have complex representations (see Appendix B) because parity violation

requires that left- and right-handed fermions must belong to different representations of the gauge group. E.g. in $SU(3)$, the 3 transforms differently from $\bar{3} \equiv 3^*$.

- G should have a single gauge coupling so that all the interactions are unified, and it should be a simple group. A simple group is a nontrivial group whose only normal subgroups are the trivial group and the group itself, where a normal subgroup N of a group G with elements g is defined by $g^{-1}Ng \subseteq N$ for all g.
- The known fermions should fit economically into representations of G, and since the unified gauge theory should be renormalizable, it must be free of anomalies.

The groups which satisfy the above requirements are $SU(N)$, $SO(2N)$, or E_6, and among them the minimum choice is $SU(5)$, $SO(10)$, or E_6. Here we review $SU(5)$ and $SO(10)$.

4. SU(5)

In the Standard Model, the 3 gauge coupling constants are different. To see the meaning of unification, we have to go to a unified theory which has $SU(3)_C \times SU(2)_L \times U(1)_Y$ as a subgroup. Since the rank of $SU(3)_C$, $SU(2)_L$, and $U(1)_Y$ is 2, 1, and 1 respectively, the unified group G must be at least rank 4. The rank of $SU(N)$ is $N - 1$, so $SU(5)$ can work for unification.

We can accommodate the 15 left-handed fermions (3 colors for u, d quarks and antiquarks (12), electron and antielectron (2), and left-handed neutrino (1)) in a representation of $SU(5)$, but not in a single irreducible representation. The $SU(3)$,

$SU(2)$ content is written as

$$\overbrace{(\bar{3},1)+(1,2)}^{\bar{5}} \quad + \quad \overbrace{(3,2)+(\bar{3},1)+(1,1)}^{10} \tag{1.68}$$
$$d_i^c \qquad (\nu,e^-) \qquad\qquad (u_i,d_i) \qquad u_i^c \qquad\qquad e^c$$

where each multiplet satisfies $\sum Q = 0$.

The family of left-handed fermions fits into the 5 and 10 of $SU(5)$ as follows. First we have

$$\psi_L = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu \end{pmatrix}_L = \bar{5}, \qquad \text{or} \qquad \psi_R^c = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ -\bar{\nu} \end{pmatrix}_R = 5. \tag{1.69}$$

Then we consider the $SU(5)$ product representation:

$$\begin{aligned} 5\times 5 &= [(3,1)+(1,2)]\times[(3,1)+(1,2)] \\ &= [(6,1)+(3,2)+(1,3)]_S + [(\bar{3},1)+\underbrace{(3,2)}_{\equiv 10}+(1,1)]_A, \end{aligned} \tag{1.70}$$

and the 10 is written in antisymmetric form as

$$\chi_{ab} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -e^+ \\ d_1 & d_2 & d_3 & e^+ & 0 \end{pmatrix}_L . \tag{1.71}$$

The gauge fields involve the adjoint representation, $\bar{5}\times 5 = 24+1$, and there are 24 generators. The $SU(3), SU(2)$ decomposition of the 24-dimensional representation is

$$24 = \underset{gluons\ g}{(8,1)} + \underbrace{\underset{W_i}{(1,3)} + \underset{B}{(1,1)}}_{W^{\pm},Z^0,\gamma} + \underset{X,Y}{(\bar{3},2)} + \underset{\bar{X},\bar{Y}}{(3,2)} \tag{1.72}$$

The 5×5 traceless matrix of gauge fields A^{μ} is

$$A \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} & & & \bar{X}_1 & \bar{Y}_1 \\ & SU(3) & & \bar{X}_2 & \bar{Y}_2 \\ & & & \bar{X}_3 & \bar{Y}_2 \\ X_1 & X_2 & X_3 & & \\ Y_1 & Y_2 & Y_3 & \multicolumn{2}{c}{SU(2)} \end{pmatrix} + \frac{1}{2\sqrt{15}} \begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix} B. \tag{1.73}$$

Since X transforms an anti-down quark into an electron and Y transforms an anti-down quark into a neutrino, the charges of the new gauge bosons are

$$Q_X = 4/3, \qquad Q_Y = 1/3 \tag{1.74}$$

The Lagrangian is

$$\mathcal{L} = i \left(\bar{\psi}^c_R\right)_a \gamma_\mu \left(\delta_{ab} \partial^\mu + ig \left(\sum_{j=1}^{24} \frac{1}{2} \lambda_j A^\mu_j \right)_{ab} \right) \left(\psi^c_R\right)_b + i \bar{\chi}_{ac} \gamma^\mu \left(D_\mu \chi\right)_{ac}, \tag{1.75}$$

where

$$\left(D_\mu \chi\right)_{ac} = \partial_\mu \chi_{ac} + ig \left(\frac{1}{2} \lambda \cdot A_\mu \right)_{ad} \chi_{dc} + ig \left(\frac{1}{2} \lambda \cdot A_\mu \right)_{cd} \chi_{ad}, \tag{1.76}$$

and

$$\lambda_j = \begin{pmatrix} \lambda_{SU(3)j} & 0 \\ 0 & 0 \end{pmatrix} \qquad j = 1, \cdots, 8, \text{ where } \lambda_{SU(3)j} \text{ is } SU(3)\ \lambda.$$

$$\lambda_9 = \begin{pmatrix} & & & 1 & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \end{pmatrix} \qquad \lambda_{10} = \begin{pmatrix} & & & -i & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & 0 \end{pmatrix}.$$

The $\lambda_{11}, \lambda_{12}, \cdots, \lambda_{20}$ are obtained by continuing to put 1 and $\mp i$ in the same pattern in the off diagonal blocks.

$$\lambda_j = \begin{pmatrix} 0 & 0 \\ 0 & \tau_i \end{pmatrix} \qquad i = 1, 2, 3, \text{ where } \tau_i \text{ is a Pauli matrix.}$$

$$\lambda_{24} = \frac{1}{\sqrt{15}} \begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}. \tag{1.77}$$

Now let us consider gauge symmetry breaking. Two multiplets of the Higgs scalar fields participate in the $SU(5)$ model. One is a complex 5 dimensional representation H_a (where the first 3 components are a color triplet and the last two components are a color singlet - the $SU(2)$ the Higgs doublet which we saw in $SU(2)$ symmetry breaking). The other is a real adjoint 24 dimensional representation Φ_a, which breaks $SU(5)$ into $SU(3) \times SU(2) \times U(1)$.

For $SU(5)$ symmetry breaking, we must not break the $SU(3)$ and $SU(2)$. The covariant derivative of the real adjoint 24 dimensional Higgs is

$$\begin{aligned} D^{\mu}\Phi &= \partial^{\mu}\Phi + ig\sum_{i,j}\left(T_i A_i^{\mu}\right)\Phi \\ &= \partial^{\mu}\Phi + ig\sum_{i,jk} ic_{ijk}\lambda_k A_i^{\mu}\Phi_j/\sqrt{2} \\ &= \partial^{\mu}\Phi_l + ig\left(A^{\mu}\Phi - \Phi A^{\mu}\right), \end{aligned} \tag{1.78}$$

where $\Phi \equiv \sum\limits_{j=1}^{24}\Phi_j\lambda_j/\sqrt{2}$ and $A^{\mu} \equiv \sum\limits_{j=1}^{24}\lambda_j A_j^{\mu}/2$. When the V.E.V. of Φ commutes with the $SU(3)$ and $SU(2)$ parts of A^{μ}, we can break only the GUT gauge X and Y symmetries. In general, the scalar fields self-coupling potential $V(\Phi)$ is

$$V(\Phi) = -\frac{1}{2}M^2 Tr\left(\Phi^2\right) + \frac{1}{4}a\left[Tr\left(\Phi^2\right)\right]^2 + \frac{1}{2}bTr\left(\Phi^4\right), \tag{1.79}$$

where we have required symmetry under $\Phi \to -\Phi$. Then from $\partial V(\langle\Phi\rangle)/\partial\Phi = 0$ and traceless conditions, the V.E.V. of Φ is

$$\langle 0|\Phi|0\rangle = \upsilon_0\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3/2 & \\ & & & & -3/2 \end{pmatrix} \equiv -\frac{\sqrt{15}}{2}\upsilon_0\lambda_{24}, \tag{1.80}$$

where $\upsilon_0^2 = 2M^2/(15a + 7b)$. Then the masses of X and Y bosons are

$$\begin{aligned} \mathcal{L}_{mass-X,Y} &= \frac{g^2}{2}Tr\left[\left(A\langle\Phi\rangle - \langle\Phi\rangle A\right)^2\right] \\ &= \frac{1}{2}g^2\upsilon_0^2\frac{25}{8}A_i^{\mu}A_{i\mu} \equiv \frac{1}{2}M_i^2 A_i^{\mu}A_{i\mu}, \end{aligned} \tag{1.81}$$

$$\to M_i^2 = g^2\upsilon_0^2\frac{25}{8}. \tag{1.82}$$

Next we consider fermion masses. The Higgs V.E.V. $\langle 0|\Phi|0\rangle$ is GUT scale

($10^{14} \sim 10^{16}$ GeV) and these Higgses are not suitable for the origin of the fermion masses. Therefore we consider a $SU(5)$ invariant from the 5 Higgs,

$$H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{with} \quad \langle 0|\, H\, |0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \upsilon \end{pmatrix}. \tag{1.83}$$

The invariant is produced from $\bar{5} \times 10 \times \bar{5}$ and $10 \times 10 \times 5$, and the fermion mass terms are

$$\begin{aligned}(\mathcal{L}_Y)_{mass} &= \frac{\upsilon G_D}{\sqrt{2}} \left(\bar{\psi}^c_R\right)_a (\chi_L)_{a5} + \frac{\upsilon G_U}{\sqrt{2}} \epsilon_{\alpha\beta\gamma} \left(\bar{\chi}^c_R\right)_{\alpha\beta} (\chi_L)_{\gamma 4} + h.c. \\ &= -\frac{1}{2}\upsilon G_D \left(\bar{e}e + \bar{d}_\alpha d_\alpha\right) - \frac{\upsilon G_U}{\sqrt{2}} \left(\bar{u}_\alpha u_\alpha\right),\end{aligned} \tag{1.84}$$

and

$$m_e = m_d = \frac{1}{2}\upsilon G_D, \qquad m_u = \frac{1}{\sqrt{2}}\upsilon G_U. \tag{1.85}$$

5. SO(10)

In SU(5), the representation of $SU(5)$ is not irreducible because the $\bar{5}$ and 10 are in different representations. $SO(2N)_{N=5}$ contains a 16-dimensional (2^{N-1}) irreducible representation. $SO(10)$ also contains $SU(5)$ as a subgroup, and the representation decomposes into

$$16 = \underbrace{10 + \bar{5}}_{SU(5)} + \underset{\text{right handed neutrino}}{1} \tag{1.86}$$

For the spinorial representation, discussed in Refs. [9]-[10], we first consider a Clifford algebra: $2n$ Hermitian matrices $\gamma_i^{(n)}$, $i = 1, \cdots, 2n$, which are $2^n \times 2^n$, satisfy

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}. \tag{1.87}$$

We can construct the $\gamma^{(n)}$ matrices in a systematic way by iteration: For $n = 1$, they are Pauli matrices

$$\gamma_1^{(1)} \equiv \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2^{(1)} \equiv \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_3^{(1)} \equiv \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.88}$$

When $\gamma_i^{(n)}$ is given, we can construct $\gamma_i^{(n+1)}$ as

$$\gamma_i^{(n+1)} = \gamma_i^{(n)} \otimes \tau_3, \tag{1.89}$$

$$\gamma_{2n+1}^{(n+1)} = 1 \otimes \tau_1, \tag{1.90}$$

$$\gamma_{2n+2}^{(n+1)} = 1 \otimes \tau_2, \tag{1.91}$$

where $i = 1, \cdots, 2n$ and $\gamma_i^{(0)} = 0$. The γ_{FIVE} matrix is defined by

$$\gamma_{FIVE} = (-i)^n \left(\gamma_1 \gamma_2 \cdots \gamma_{2n}\right), \tag{1.92}$$

and γ_{FIVE} anticommutes with all the other γ_i, $\{\gamma_{FIVE}, \gamma_i\} = 0$, with $\gamma_{FIVE}^{(n+1)} = \gamma_{FIVE}^{(n)} \otimes \tau_3$. The rotations in the $i - j$ plane are given by $\Sigma_{ij} = \frac{1}{2} i \left[\gamma_i, \gamma_j\right]$, and the 2^n spinor ψ transforms under $SO\left(2n\right)$ as

$$\psi \to \exp\left(i\omega_{jk}\Sigma_{jk}\right)\psi. \tag{1.93}$$

Since $\Sigma_{jk}\gamma_{FIVE} = \gamma_{FIVE}\Sigma_{jk}$, the rotation does not change the eigenvalues of γ_{FIVE} and we can reduce ψ into $\gamma_{FIVE}\psi = \pm\psi$ states. Then the chiral states

$$\psi_\pm = \frac{1}{2}\left(1 \pm \gamma_{FIVE}\right)\psi \tag{1.94}$$

transform irreducibly, and these states give a $2^n/2 = 2^{n-1}$ dimensional irreducible representation. When $n = 5$, we obtain a 16 irreducible representation, in which all 16 fermion fields fit.

The $SO(10)$ invariant Lagrangian describing the interaction of Yang-Mills fields with a multiplet of massless fermions is

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}^{jk}\right)^2 + i\bar{\psi}_+\gamma_\mu\left(\partial^\mu - igA_\mu^{jk}\Sigma_{jk}\right)\psi_+. \tag{1.95}$$

Now let us turn to the physical particle states. The states are summarized in Table V. In an $SU(5)$ representation the hypercharge Y is given by

$$Y = 2/3\left(\sum \text{color spins}\right) - \left(\sum \text{weak spins}\right), \tag{1.96}$$

and, because of the sign difference, $SU(5)$ rotations raise (or lower) a color index and lower (or raise) a weak index. Since $SO(10)$ includes $SU(5)$ as subgroup, $SO(10)$ rotations other than those in the $SU(5)$ raise or lower any two color spins or two weak spins.

Fermion masses again result from Yukawa couplings of the fermions to the Higgs fields. Now neutrinos couple to the Higgs fields and can obtain masses. However, the coupling of the Higgses depends on how $SO(10)$ breaks into a subgroup, e.g. $SU(5)\times U(1)$, or $SU(4)\times SU(2)\times SU(2)$ etc.

C. Supersymmetry (SUSY)

Here we review the rudiments of SUSY. For a more detailed discussion, please see Ref. [11].

1. Brief History of SUSY

Supersymmetry was originally motivated by interest in a possible symmetry between fermions and bosons. To the author's knowledge, the oldest physically motivated SUSY is by H. Miyazawa in 1966 [12]. It uses a kind of superalgebra related to

Table V. The states of the SO(10) matter fields.The SU(5) 5 and 10 representations are part of the SO(10) 16 representation.

State				Y
$u_L^i =$	$\vert+--++\rangle_r\,,$	$\vert+-+-+\rangle_g\,,$	$\vert+-++-\rangle_b$	$1/3$
$d_L^i =$	$\vert-+-++\rangle_r\,,$	$\vert-++-+\rangle_g\,,$	$\vert-+++-\rangle_b$	$1/3$
$u_L^{ic} =$	$\vert+++--\rangle_r\,,$	$\vert++-+-\rangle_g\,,$	$\vert++--+\rangle_b$	$-4/3$
$d_L^{ic} =$	$\vert--+--\rangle_r\,,$	$\vert---+-\rangle_g\,,$	$\vert----+\rangle_b$	$2/3$
$\nu_L =$	$\vert+----\rangle$			-1
$e_L^- =$	$\vert-+---\rangle$			-1
$e_L^+ =$	$\vert--+++\rangle$			2
$N_L =$	$\vert+++++\rangle$			0

.

$$\underset{SU(5)}{\bar{5}} \qquad \underset{SO(10)\backslash SU(5)}{\leftrightarrow} \qquad \underset{SU(5)}{10} \quad .$$

$$d_i^c \quad (\nu, e^-) \qquad\qquad (u_i, d_i) \quad u_i^c \quad e^c$$

internal symmetry.

From the string theory side, the notion of a symmetry between fermionic and bosonic modes started in 1971 [13]-[15]. However, these are two dimensional theories.

$N = 1$ supersymmetry was first proposed and formulated as a graded Lie algebra by Y. A. Golfand and E. P. Likhtman in 1971 [16].

J. Wess and B. Zumino [17], and A. Salam and J. Strathdee [18], constructed field theories with supersymmetry in 1974.

R. Haag, J. Lopuszanski, and M. Sohnius [19] proposed that supersymmetry is the only possible symmetry between particles with different spins in which the S-matrix is consistent with relativistic quantum field theory.

2. R-parity

In the minimal supersymmetric standard model (MSSM), we assume a new symmetry which conserves B and L (baryon number and lepton number) in the renormalizable superpotential. However, this symmetry cannot distinguish between the particle and the SUSY partner because they have same quantum numbers except spin. Therefore, we also include spin angular momentum conservation in the symmetry. This new symmetry is called "R-parity" and defined as

$$P_R = (-1)^{3(B-L)+2s} . \tag{1.97}$$

P_R for the MSSM fields are given in Table VI (where the upper sign is for the particle and the lower sign is for the antiparticle).

R-parity stabilizes the lightest sparticle and is thus important for SUSY to provide a candidate for dark matter.

Table VI. R-parity of the fields of minimal supersymmetric standard model.

	Lepton	Quark	Gauge	Higgs	S-lepton	S-quark	Gaugino	Higgsino
$B-L$	∓ 1	$\pm 1/3$	0	0	∓ 1	$\pm 1/3$	0	0
s	1/2	1/2	1	0	0	0	1/2	1/2
P_R	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1

,

3. Neutralino as Dark Matter Candidate

The phenomenological requirements for dark matter are:

- It has to be electrically neutral and non-baryonic, and it can interact only weakly with ordinary matter.
- It has to be stable (or have a cosmologically long lifetime).

Because of R-parity, sparticles are always created or destroyed in pairs, and the lightest sparticle is stable. The lightest sparticle is now required to be electrically neutral and also to have no color charge.

The WMAP data has determined that $\Omega_{CDM} = 0.23 \pm 0.04$. Here CDM means "cold dark matter" and the density parameter Ω is defined by $\Omega \equiv \dfrac{\rho}{\rho_c}$, where $\rho_c \equiv \dfrac{3H^2}{8\pi G} = 9.47 \times 10^{-27} \left[\text{kg/m}^3\right] = 5.32 \left[\text{GeV/m}^3\right]$ is the critical density. We have also used the currently standard value of the Hubble parameter $H = 71$ km/s/Mpc=

$2.3 \times 10^{-18}\,[\mathrm{s}^{-1}]$. Therefore the density of the dark matter ρ_{DM} is

$$\begin{aligned}\rho_{DM} &= \Omega_{CDM} \cdot \rho_c \\ &= 0.19 \times 5.32 \sim 0.27 \times 5.32 \left[\mathrm{GeV/m^3}\right] \\ &= 1.01 \sim 1.44 \left[\mathrm{GeV/m^3}\right], \end{aligned} \tag{1.98}$$

and the density of any dark matter candidate needs to be equal to or smaller than the value.

The photino, the zino, and the neutral Higgsinos, or a neutralino which is a mixture of these, satisfy the above requirements and the lightest neutralino is a good candidate for the dark matter. The density of the lightest neutralino is required to be at most in the range

$$\rho_{\text{neutralino}} = 1.01 \sim 1.44 \left[\mathrm{GeV/m^3}\right], \tag{1.99}$$

and this condition is one of the constraints on the SUSY parameters.

4. Fermions (spin 1/2) and Sfermions (spin 0)

The minimum fermion content in four dimensions is a single left-handed two-component Weyl fermion ψ. Since the fermion field is complex, the superpartner scalar sfermion field is also complex. The action without any interaction is

$$S = \int d^4x \left(\mathcal{L}_{scalar} + \mathcal{L}_{fermion}\right), \tag{1.100}$$

$$\mathcal{L}_{scalar} = -\partial^\mu \phi^* \partial_\mu \phi; \qquad \mathcal{L}_{fermion} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi. \tag{1.101}$$

This is called the massless non-interacting Wess-Zumino model. This action is invariant under the supersymmetry transformation

$$\delta\phi = \epsilon\psi; \qquad \delta\phi^* = \epsilon^\dagger \psi^\dagger, \tag{1.102}$$

$$\delta\psi_\alpha = -i\left(\sigma^\mu\epsilon^\dagger\right)_\alpha \partial_\mu\phi; \qquad \delta\psi^\dagger_{\dot{\alpha}} = i\left(\epsilon\sigma^\mu\right)_{\dot{\alpha}} \partial_\mu\phi^*. \tag{1.103}$$

However, when we calculate the commutator of two supersymmetry transformations, it turns out that the algebra does not close:

$$\left[\delta_{\epsilon_2}, \delta_{\epsilon_1}\right]\phi = -i\left(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger\right)\partial_\mu\phi, \tag{1.104}$$

$$\begin{aligned}\left[\delta_{\epsilon_2}, \delta_{\epsilon_1}\right]\psi_\alpha &= -i\left(\epsilon_1\sigma^\mu\epsilon_2^\dagger - \epsilon_2\sigma^\mu\epsilon_1^\dagger\right)\partial_\mu\psi_\alpha \\ &\quad + i\epsilon_{1\alpha}\epsilon_2^\dagger\bar{\sigma}^\mu\partial_\mu\psi - i\epsilon_{2\alpha}\epsilon_1^\dagger\bar{\sigma}^\mu\partial_\mu\psi.\end{aligned} \tag{1.105}$$

For $\left[\delta_{\epsilon_2}, \delta_{\epsilon_1}\right]\psi_\alpha$, the second and third terms vanish when we apply the equation of motion, but the algebra closes only classically. We want the algebra to close quantum mechanically (i.e., without the equation of motion), and the trick invented to accomplish this is the introduction of auxiliary fields.

Since the number of off-shell degrees of freedom for a fermion or the sfermion is 4 or 2, respectively, to match the degrees of freedom we introduce a complex bosonic auxiliary field, with 2 more degrees of freedom. Then the action including the auxiliary field F is

$$S_{chiral} = \int d^4x\left(-\partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + F^*F\right). \tag{1.106}$$

This action is invariant under

$$\delta\phi = \epsilon\psi; \qquad \delta\phi^* = \epsilon^\dagger\psi^\dagger,$$

$$\delta\psi_\alpha = -i\left(\sigma^\mu\epsilon^\dagger\right)_\alpha\partial_\mu\phi - \epsilon_\alpha F; \qquad \delta\psi^\dagger_{\dot{\alpha}} = i\left(\epsilon\sigma^\mu\right)_{\dot{\alpha}}\partial_\mu\phi^* - \epsilon^\dagger_{\dot{\alpha}}F^*, \tag{1.107}$$

$$\delta F = i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi; \qquad \delta F^* = -i\partial_\mu\psi^\dagger\bar{\sigma}^\mu\epsilon, \tag{1.108}$$

and now the SUSY algebra is closed even off-shell.

5. Gauge Bosons (spin 1) and Gauginos (spin 1/2)

The action of a gauge supermultiplet is

$$S_{gauge} = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i\lambda^{\dagger a} \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a \right), \tag{1.109}$$

where again to close the algebra an auxiliary field D is introduced. The d.o.f. (number of degrees of freedom) of each gauge field is 3, the d.o.f. of each gaugino field is 4, and each auxiliary field is a d.o.f.=1 real field. The action is invariant under a supersymmetry transformation given by

$$\begin{aligned}
\delta A^a_\mu &= -\frac{1}{\sqrt{2}} \left[\epsilon^\dagger \bar{\sigma}_\mu \lambda^a + \lambda^{\dagger a} \bar{\sigma}_\mu \epsilon \right], \\
\delta \lambda^a_\alpha &= -\frac{i}{2\sqrt{2}} \left(\sigma^\mu \bar{\sigma}^\nu \epsilon \right)_\alpha F^a_{\mu\nu} + \frac{1}{\sqrt{2}} \epsilon_\alpha D^a, \\
\delta D^a &= \frac{i}{\sqrt{2}} \left[\epsilon^\dagger \bar{\sigma}^\mu D_\mu \lambda^a - D_\mu \lambda^{\dagger a} \bar{\sigma}^\mu \epsilon \right],
\end{aligned} \tag{1.110}$$

with

$$\left[\delta_{\epsilon_2}, \delta_{\epsilon_1} \right] X = i \left(\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger \right) D_\mu X$$

where X corresponds to any of the gauge covariant fields $F^a_{\mu\nu}, \lambda^a, \lambda^{\dagger a}, D^a$. D_μ is the covariant derivative.

The action is also invariant under the gauge transformation

$$\begin{aligned}
\delta_{gauge} A^a_\mu &= -\partial_\mu \Lambda^a + g f^{abc} A^b_\mu \Lambda^c, \\
\delta_{gauge} \lambda^a &= g f^{abc} \lambda^b \Lambda^c.
\end{aligned} \tag{1.111}$$

6. Supersymmetric Gauge Interaction

The gauge interaction is introduced via a covariant derivative as usual. The supersymmetric action is

$$\begin{aligned} S = \int d^4x (&\mathcal{L}_{gauge} + \mathcal{L}_{chiral} \\ &- \sqrt{2} g \left[\left(\phi^* T^a \psi \right) \lambda^a + \lambda^{\dagger a} \left(\psi^\dagger T^a \phi \right) \right] \\ &+ g \left(\phi^* T^a \phi \right) D^a), \end{aligned} \tag{1.112}$$

and the additional supersymmetry transformation is now given by

$$\begin{aligned} \delta \phi_i &= \epsilon \psi_i \\ \delta \left(\psi_i \right)_\alpha &= -i \left(\sigma^\mu \epsilon^\dagger \right)_\alpha D_\mu \phi_i - \epsilon_\alpha F_i \\ \delta F_i &= i \epsilon^\dagger \bar{\sigma}^\mu D_\mu \psi_i - \sqrt{2} g \left(T^a \phi \right)_i \epsilon^\dagger \lambda^{\dagger a}. \end{aligned} \tag{1.113}$$

7. Chiral Interaction from Superpotential

To introduce interactions among the scalar and spinor fields, we require a superpotential W which is given by

$$W = \frac{1}{2} M^{ij} \phi_i \phi_j + \frac{1}{6} y^{ijk} \phi_i \phi_j \phi_k. \tag{1.114}$$

Then the interaction terms are written as

$$\mathcal{L}_{int} = -\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i + c.c., \tag{1.115}$$

where $W^i \equiv \frac{\delta}{\delta \phi_i} W$ and $W^{ij} \equiv \frac{\delta^2}{\delta \phi_i \delta \phi_j} W$. The first term produces Yukawa interactions, and also spinor mass terms for those spinors which can have mass even before the Higgs mechanism. (Supersymmetry of course requires that a scalar and spinor in the same multiplet must have the same mass.)

The scalar potential $V(\phi, \phi^*)$ is

$$V(\phi, \phi^*) = F^{*i} F_i + \frac{1}{2} \sum_a D^a D^a. \tag{1.116}$$

D. Higgs Mass Radiative Correction

In the Standard Model, the scalar mass term is quadratically divergent when one includes radiative corrections. However, with SUSY, the quadratically divergent Higgs mass radiative corrections turn out to cancel because the bosonic and fermionic loops have opposite signs. Here we will consider the one loop correction.

1. Radiative Correction from Chiral Field

When an $SU(2)$ Higgs field acquires a vacuum expectation value, with $\phi = (h + \upsilon) / \sqrt{2}$ where $\upsilon = \sqrt{2} \langle \phi \rangle$, the interaction Lagrangian for the chiral field and Higgs field is

$$\begin{aligned}
\mathcal{L}_{int} &= -\lambda_S \phi \phi \tilde{\phi}_s^\dagger \tilde{\phi}_s - \lambda_f \left[\phi \psi_L^\dagger \psi_R + \phi \psi_R^\dagger \psi_L \right] \\
&= -\lambda_s \frac{h^2}{2} \tilde{\phi}_s^\dagger \tilde{\phi}_s - \sqrt{2} \upsilon \lambda_s h \tilde{\phi}_s^\dagger \tilde{\phi}_s - \underbrace{\lambda_s \frac{\upsilon^2}{2} \tilde{\phi}_s^\dagger \tilde{\phi}_s}_{= -m_{\tilde{f}}^2 \tilde{\phi}_s^\dagger \tilde{\phi}_s} \\
&\quad - \frac{\lambda_f}{\sqrt{2}} \left[h \psi_L^\dagger \psi_R + h \psi_R^\dagger \psi_L \right] - \underbrace{\frac{\lambda_f}{\sqrt{2}} \left[\upsilon \psi_L^\dagger \psi_R + \upsilon \psi_R^\dagger \psi_L \right]}_{= -\left[m_f \psi_L^\dagger \psi_R + m_f \psi_R^\dagger \psi_L \right]},
\end{aligned} \tag{1.117}$$

where λ_f and λ_s are coupling parameters, h is defined by $\phi = (h + \upsilon) / \sqrt{2}$, ψ is a fermion, $\tilde{\phi}_s$ is a sfermion, and m_f and $m_{\tilde{f}}$ are the fermion and sfermion masses respectively. The 1-loop corrections come from the diagrams given in Figure 2.

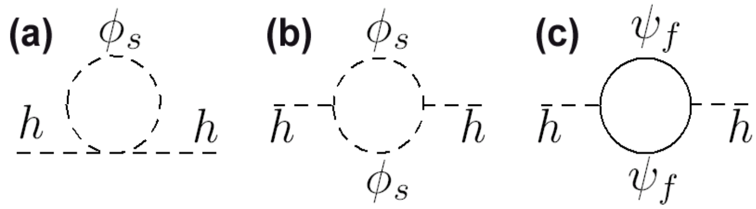

Fig. 2. 1-loop radiative correction of the Higgs mass from sfermion and fermion. (a) and (b) are sfermion loops and (c) is the fermion loop. To show that the logarthmic divergence also cancels under supersymmetry, diagram (b) is required.

a. Sfermion

First we calculate the mass correction from the sfermion loops. The 1-loop radiative correction of the Higgs mass from Figure 2 (a) is

$$\begin{aligned}
\left[\Delta m_h^2\right]_{(a)} &= (-1)\,(2)\,(1)\,(1)\int \frac{d^4k}{(2\pi)^4}\left(-\frac{\lambda_s}{2}\right)\left[\frac{i}{k^2-m_{\tilde{f}}^2}\right] \\
&\underset{\text{Wick rotate}}{=} -\lambda_s\frac{2\pi^2}{\Gamma(2)}\int\frac{dk_E}{(2\pi)^4}\frac{-k_E^3}{k_E^2+m_{\tilde{f}}^2} \\
&= \lambda_s\frac{2\pi^2}{\Gamma(2)}\int\frac{dk_E}{(2\pi)^4}\left[k_E-\frac{m_{\tilde{f}}^2k_E}{k_E^2+m_{\tilde{f}}^2}\right] \\
&= \frac{\lambda_s}{8\pi^2}\left[\frac{k_E^2}{2}-\frac{m_{\tilde{f}}^2}{2}\ln\left(k_E^2+m_{\tilde{f}}^2\right)\right]_0^\Lambda \\
&\simeq \frac{\lambda_s}{16\pi^2}\left[\Lambda^2-2m_{\tilde{f}}^2\ln\left(\frac{\Lambda}{m_{\tilde{f}}}\right)\right],
\end{aligned} \tag{1.118}$$

where we have used

$$\int d^D k_E = \frac{2\pi^{D/2}}{\Gamma\left(D/2\right)} \int k_E^{D-1} dk_E, \tag{1.119}$$

$$\Gamma\left(n\right) = \left(n-1\right)\Gamma\left(n-1\right) = \left(n-1\right)! \quad \text{with } \Gamma\left(1/2\right) = \sqrt{\pi},$$

and

$$\frac{\lambda_s}{2}\upsilon^2 = m_{\tilde{f}}^2. \tag{1.120}$$

The 1-loop radiative correction of the Higgs mass from Figure 2 (b) is

$$\begin{aligned}
\left[\Delta m_h^2\right]_{(b)} &= \left(\frac{-i}{2!}\right)\left(1\right)\left(1\right)\left(1\cdot 1\right)\int \frac{d^4k}{\left(2\pi\right)^4}\left(-\sqrt{2}\upsilon\lambda_s\right)\left(-\sqrt{2}\upsilon\lambda_s\right)\left[\left(\frac{i}{k^2 - m_{\tilde{f}}^2}\right)^2\right] \\
&= -\left(\upsilon\lambda_s\right)^2 \frac{2\pi^2}{\Gamma\left(2\right)}\int \frac{dk_E}{\left(2\pi\right)^4}\frac{k_E^3}{\left(k_E^2 + m_{\tilde{f}}^2\right)^2} \\
&= -\lambda_s \frac{2\pi^2}{\Gamma\left(2\right)}\int \frac{dk_E}{\left(2\pi\right)^4}\left[\frac{\upsilon^2\lambda_s k_E}{k_E^2 + m_{\tilde{f}}^2} - \frac{\upsilon^2\lambda_s m_{\tilde{f}}^2 k_E}{\left(k_E^2 + m_{\tilde{f}}^2\right)^2}\right] \\
&= -\frac{\lambda_s}{8\pi^2}\left[\frac{\upsilon^2\lambda_s}{2}\ln\left(k_E^2 + m_{\tilde{f}}^2\right) + \frac{\upsilon^2\lambda_s m_{\tilde{f}}^2}{2\left(k_E^2 + m_{\tilde{f}}^2\right)}\right]_0^\Lambda \\
&\approx -\frac{\lambda_s}{16\pi^2}\left[4m_{\tilde{f}}^2 \ln\left(\frac{\Lambda}{m_{\tilde{f}}}\right) - 2m_{\tilde{f}}^2\right].
\end{aligned} \tag{1.121}$$

b. Fermion

Next we calculate the 1-loop radiative correction of the Higgs mass from Figure 2 (c).

$$
\begin{aligned}
\left[\Delta m_h^2\right]_{(c)} &= \left(\frac{-i}{2!}\right)(1)(1)(1\cdot 1)\int \frac{d^4k}{(2\pi)^4}\left(-\frac{\lambda_f}{\sqrt{2}}\right)\left(-\frac{\lambda_f}{\sqrt{2}}\right) \\
&\quad \cdot tr\left[\frac{i\left(\gamma^\nu k_\nu + m_f\right)}{k^2-m_f^2}\frac{i\left(\gamma^\mu k_\mu + m_f\right)}{k^2-m_f^2}\right]\{(-1)(2)\} \\
&= -i2\lambda_f^2\int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{k^2-m_f^2}+\frac{2m_f^2}{\left(k^2-m_f^2\right)^2}\right] \\
&= 2\lambda_f^2\frac{2\pi^2}{\Gamma(2)}\int \frac{k_E^3 dk_E}{(2\pi)^4}\left[\frac{-1}{k_E^2+m_f^2}+\frac{2m_f^2}{\left(k_E^2+m_f^2\right)^2}\right] \\
&= 2\lambda_f^2\frac{2\pi^2}{\Gamma(2)}\int \frac{dk_E}{(2\pi)^4}\left[-k_E+\frac{3m_f^2k_E}{k_E^2+m_f^2}-\frac{2m_f^4k_E}{\left(k_E^2+m_f^2\right)^2}\right] \\
&= -\frac{\lambda_f^2}{4\pi^2}\left[\frac{k_E^2}{2}-\frac{3}{2}m_f^2\ln\left(k_E^2+m_f^2\right)-\frac{m_f^4}{k_E^2+m_f^2}\right]_0^\Lambda \\
&\simeq -\frac{\lambda_f^2}{16\pi^2}\left[2\Lambda^2-12m_f^2\ln\left(\frac{\Lambda}{m_f}\right)+4m_f^2\right], \qquad (1.122)
\end{aligned}
$$

where the factor of -1 in $\{\cdots\}$ of the 2nd line arises because of the anticommutation of the fermionic field, and the factor of 2 in $\{\cdots\}$ arises because

$$
\begin{aligned}
\mathcal{L}_{int}(x_1)\,\mathcal{L}_{int}(x_2) &= \Big(m_f\psi_L^\dagger(x_1)\,\psi_R(x_1)+m_f\psi_R^\dagger(x_1)\,\psi_L(x_1)\Big) \\
&\quad \cdot\Big(m_f\psi_L^\dagger(x_2)\,\psi_R(x_2)+m_f\psi_R^\dagger(x_2)\,\psi_L(x_2)\Big) \\
&\quad \to m_f^2\psi_L^\dagger(x_1)\,\psi_R(x_1)\,\psi_R^\dagger(x_2)\,\psi_L(x_2) \\
&\quad + m_f^2\psi_R^\dagger(x_1)\,\psi_L(x_1)\,\psi_L^\dagger(x_2)\,\psi_R(x_2),
\end{aligned}
$$

where the first term and the second term give the same contribution and thus produce a factor of 2.

When the contributions from the sfermion and fermion are added together, the radiative corrections from chiral multiplet sum to give

$$\begin{aligned}
\Delta m_h^2 &= 2\times\left(\left[\Delta m_h^2\right]_{(a)}+\left[\Delta m_h^2\right]_{(b)}\right)+\left[\Delta m_H^2\right]_{(c)} \\
&= 2\frac{\lambda_s}{16\pi^2}\left[\Lambda^2+6m_{\tilde f}^2\ln\left(\frac{\Lambda}{m_{\tilde f}}\right)+2m_{\tilde f}^2\right]-\frac{\lambda_f^2}{16\pi^2}\left[2\Lambda^2-12m_f^2\ln\left(\frac{\Lambda}{m_f}\right)+4m_f^2\right] \\
&= \frac{\Lambda^2}{8\pi^2}\left(\lambda_s-\lambda_f^2\right)-\frac{3}{4\pi^2}\left(\lambda_s m_{\tilde f}^2\ln\left(\frac{\Lambda}{m_{\tilde f}}\right)-\lambda_f^2 m_f^2\ln\left(\frac{\Lambda}{m_f}\right)\right) \\
&\qquad +\frac{1}{4\pi^2}\left(\lambda_s m_{\tilde f}^2-\lambda_f^2 m_f^2\right) \\
&= 0 \qquad\qquad \text{when } \lambda_s=\lambda_f^2 \text{ and } m_{\tilde f}^2=\frac{\lambda_s}{2}\upsilon^2=\frac{\lambda_f^2}{2}\upsilon^2=m_f^2. \qquad (1.123)
\end{aligned}$$

The factor of 2 is included because there are R and L sfermions. Then the 1-loop Higgs mass radiative correction from the chiral multiplet cancels exactly under SUSY. Not only the quadratic divergence, but also the logarithmic divergence cancels when $\lambda_s = \lambda_f^2$ and $m_{\tilde f} = m_f$. Note again that we need to include the diagram as in Figure 2 (b) to obtain the cancellation of the logarithmic divergence.

2. Radiative Corrections from the Gauge and Gaugino and the Higgs and Higgsino Fields

Now we consider the Higgs mass radiative corrections from the $SU(2)\times U(1)$ gauge, gaugino, and Higgs fields, where the Higgs field contributions originate from the D-term of (1.112). (There can be also corrections from the Higgs and Higgsino loops, but these are treated exactly the same as in the fermion and sfermion case.). The model now has **two** left chiral $SU(2)_L$ doublet Higgs fields (up type and down type). Since the mass terms only contribute to a logarithmic divergence, and are thus not important for a check of the quadratic divergence cancellation, we take all fields to be massless. We start with the Lagrangian [20]

Table VII. Summary of the fields in the $SU(2)\times U(1)$ Lagrangian density Eq. (1.124).

	particles	SUSY-particles
gauge	$W^{\mu,\pm}, W^{\mu,3}, B^{\mu}$	$\lambda^{\pm}, \lambda^{3}, \lambda_0$
up type Higgs ($Y_u = +1$)	$\phi_u \equiv \binom{\phi_u^+}{\phi_u^0}$	$\psi_{uL} \equiv \binom{\psi_{uL}^+}{\psi_{uL}^0}$
down type Higgs ($Y_d = -1$)	$\phi_d \equiv \binom{\phi_d^0}{\phi_d^-}$	$\psi_{dL} \equiv \binom{\psi_{dL}^0}{\psi_{dL}^-}$

.

$$
\begin{aligned}
\mathcal{L}_{SU(2)\times U(1)} = & -\frac{1}{4}\vec{W}_{\mu\nu}\vec{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{i}{2}\left(\overline{\vec{\lambda}}\gamma^{\mu}\partial_{\mu}\vec{\lambda} - g_2\overline{\vec{\lambda}}\gamma^{\mu}\vec{W}_{\mu}\vec{\lambda}\right) + \frac{i}{2}\overline{\lambda}_0\gamma^{\mu}\partial_{\mu}\lambda_0 \\
& + i\sum_{r=u,d}\overline{\psi_{rL}}\gamma^{\mu}\Delta^{21}_{\mu}\psi_{rL} + \sum_{r=u,d}\left|\Delta^{21\mu}\phi_r\right|^2 - \frac{1}{2}g_2^2\left(\sum_{r=u,d}\phi_r^{\dagger}\frac{\vec{\tau}}{2}\phi_r\right)^2 \\
& - \frac{1}{2}g_Y^2\left(\frac{1}{2}\sum_{r=u,d}Y_r\phi_r^{\dagger}\phi_r\right)^2 - \sqrt{2}g_2\sum_{r=u,d}\left(\phi_r^{\dagger}\overline{\vec{\lambda}}\frac{\vec{\tau}}{2}\psi_{rL} + \overline{\psi_{rL}}\frac{\vec{\tau}}{2}\vec{\lambda}\phi_r\right) \\
& - \sqrt{2}g_Y\sum_{r=u,d}\left[\frac{Y_r}{2}\left(\phi_r^{\dagger}\overline{\lambda}_0\psi_{rL} + \overline{\psi_{rL}}\lambda_0\phi_r\right)\right],
\end{aligned}
\tag{1.124}
$$

where $\Delta^{21}_{\mu} \equiv \partial_{\mu} + ig_2\vec{W}_{\mu}\frac{\vec{\tau}}{2} + ig_Y B_{\mu}\frac{Y}{2}$, Y is the hypercharge, $W^{\mu,\pm} = \left(W_1^{\mu} \mp iW_2^{\mu}\right)/\sqrt{2}$, and $\lambda^{\pm} = \left(\lambda_1 \mp i\lambda_2\right)/\sqrt{2}$.

The propagators of the fields involved in (1.124) are summarized in Table VIII.

Table VIII. The propagators of fields in Eq. (1.124). We choose the Feynman gauge ($\xi = 1$) for the calculation of radiative corrections in this section. C is the charge conjugation operator.

$B_\mu B_\nu$ $W^\pm_\mu W^\pm_\nu, W^3_\mu W^3_\nu$	$\frac{-ig_{\mu\nu}}{p^2}$ $\frac{-i}{p^2}\left(g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$
$\lambda_D \bar{\lambda}_D, \lambda_M \bar{\lambda}_M$	$\frac{i\gamma^\mu p_\mu}{p^2}$
$\lambda_M \lambda_M$	$\frac{iC^{-1}\gamma^\mu p_\mu}{p^2}$
$\bar{\lambda}_M \bar{\lambda}_M$	$\frac{-i\gamma^\mu p_\mu C}{p^2}$
$\phi\phi^*$	$\frac{i}{p^2}$
$\psi_L \overline{\psi_L}$	$\frac{1-\gamma_5}{2}\frac{i\gamma^\mu p_\mu}{p^2}\frac{1+\gamma_5}{2}$

The terms which contribute to the ϕ_u radiative corrections (1-loop) are

$$
\begin{aligned}
\mathcal{L} = & \frac{i}{2}[\sqrt{2}g_2\partial^\mu\phi_u^{+*}W_\mu^+\phi_u^0 - \sqrt{g_2^2+g_Y^2}\partial^\mu\phi_u^{0*}Z_\mu\phi_u^0 + \sqrt{2}g_2\partial^\mu\phi_u^{0*}W_\mu^-\phi_u^+ \\
& - \sqrt{2}g_2\phi_u^{+*}W_\mu^+\partial^\mu\phi_u^0 + \sqrt{g_2^2+g_Y^2}\phi_u^{0*}Z_\mu\partial^\mu\phi_u^0 - \sqrt{2}g_2\phi_u^{0*}W_\mu^-\partial^\mu\phi_u^+] \\
& + \phi_u^{0*}[\frac{g_Y^2+g_2^2}{4}Z_\mu Z^\mu + \frac{g_2^2}{2}W_\mu^- W^{+\mu} \\
& - \frac{g_2^2}{8}(2\phi_u^{+*}\phi_u^+ - 2\phi_d^{0*}\phi_d^0 + 2\phi_d^{-*}\phi_d^- + \phi_u^{0*}\phi_u^0) \\
& - \frac{g_Y^2}{8}(2\phi_u^{+*}\phi_u^+ - 2\phi_d^{0*}\phi_d^0 - 2\phi_d^{-*}\phi_d^- + \phi_u^{0*}\phi_u^0)]\phi_u^0 \\
& - g_2\left(\phi_u^{0*}\bar{\lambda}_-\psi_{uL}^+ + \overline{\psi_{uL}^+}\lambda_+\phi_u^0\right) + \frac{1}{\sqrt{2}}\sqrt{g_Y^2+g_2^2}\left(\phi_u^{0*}\overline{\lambda}_Z\psi_{uL}^0 + \overline{\psi_{uL}^0}\lambda_Z\phi_u^0\right),
\end{aligned}
\tag{1.125}
$$

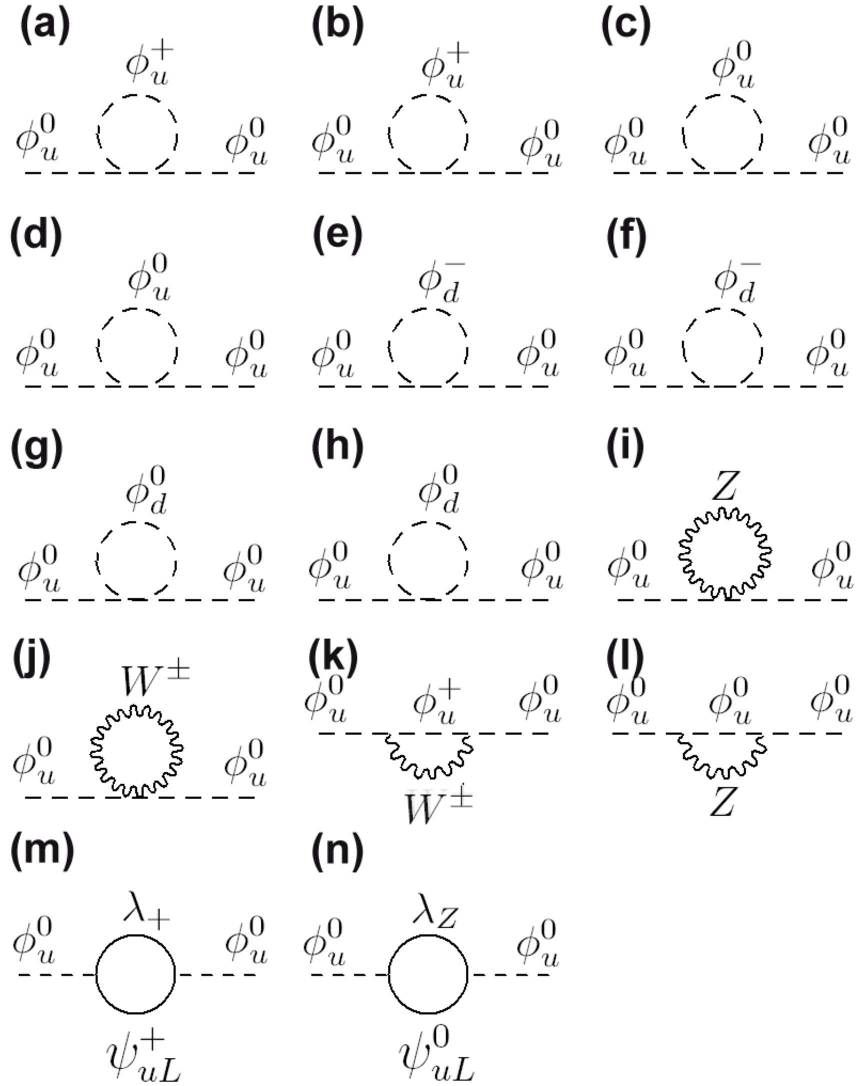

Fig. 3. 1-loop radiative corrections to the Higgs mass from the Higgses, gauge bosons, and gauginos. To show the cancellation of the quadratic divergence, not only gauge and gaugino + Higgsino diagrams but also those for the Higgses are required. This is because the diagrams with the Higgs loops are introduced by the D-term. Although (a) & (b), (c) & (d), (e) & (f), and (g) & (h) appear to be the same diagrams, their couplings are different: $(\text{coupling})_{\text{(a), (b)}} = -g_Y^2/4$, $-g_2^2/4$; $(\text{coupling})_{\text{(c), (d)}} = -g_Y^2/8$, $-g_2^2/8$; $(\text{coupling})_{\text{(e), (f)}} = g_Y^2/4$, $-g_2^2/4$; $(\text{coupling})_{\text{(g), (h)}} = g_2^2/4$, $g_Y^2/4$.

where

$$\lambda_+ = \frac{1}{\sqrt{2}}\left(\lambda_1 - i\lambda_2\right), \tag{1.126}$$

$$\lambda_- = \frac{1}{\sqrt{2}}\left(\lambda_1 + i\lambda_2\right), \tag{1.127}$$

$$\lambda_Z = \frac{1}{\sqrt{g_Y^2 + g_2^2}}\left(g_2\lambda_3 - g_Y\lambda_0\right). \tag{1.128}$$

Now we calculate the 1-loop radiative corrections depicted in Figure 3. There are 14 diagrams, and some of the diagrams are identical except for the couplings, with the same mathematics. However, we will explicitly write each calculation for each diagram.

The mass correction from Figure 3 (a) is

$$\begin{aligned}
\Delta m^2_{(a)} &= (-1)(1)(1)(1)\int \frac{d^4p}{(2\pi)^4}\left(-\frac{g_Y^2}{4}\right)\left[\frac{i}{p^2}\right] \\
&= \frac{g_Y^2}{4}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2} \\
&= \frac{g_Y^2}{64\pi^2}\Lambda^2.
\end{aligned} \tag{1.129}$$

The correction from Figure 3 (b) is

$$\begin{aligned}
\Delta m^2_{(b)} &= (-1)(1)(1)(1)\int \frac{d^4p}{(2\pi)^4}\left(-\frac{g_2^2}{4}\right)\left[\frac{i}{p^2}\right] \\
&= \frac{g_2^2}{4}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2} \\
&= \frac{g_2^2}{64\pi^2}\Lambda^2.
\end{aligned} \tag{1.130}$$

The contribution from Figure 3 (c) is

$$\begin{aligned}\Delta m^2_{(c)} &= (-1)(2)(2)(1)\int \frac{d^4p}{(2\pi)^4}\left(-\frac{g_Y^2}{8}\right)\left[\frac{i}{p^2}\right]\\ &= \frac{g_Y^2}{2}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2}\\ &= \frac{g_Y^2}{32\pi^2}\Lambda^2. \end{aligned} \tag{1.131}$$

The contribution from Figure 3 (d) is

$$\begin{aligned}\Delta m^2_{(d)} &= (-1)(2)(2)(1)\int \frac{d^4p}{(2\pi)^4}\left(-\frac{g_2^2}{8}\right)\left[\frac{i}{p^2}\right]\\ &= \frac{g_2^2}{2}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2}\\ &= \frac{g_2^2}{32\pi^2}\Lambda^2. \end{aligned} \tag{1.132}$$

The correction from Figure 3 (e) is

$$\begin{aligned}\Delta m^2_{(e)} &= (-1)(1)(1)(1)\int \frac{d^4p}{(2\pi)^4}\left(\frac{g_Y^2}{4}\right)\left[\frac{i}{p^2}\right]\\ &= -\frac{g_Y^2}{2}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2}\\ &= -\frac{g_Y^2}{64\pi^2}\Lambda^2. \end{aligned} \tag{1.133}$$

The correction from Figure 3 (f) is

$$\begin{aligned}\Delta m^2_{(f)} &= (-1)(1)(1)(1)\int \frac{d^4p}{(2\pi)^4}\left(-\frac{g_2^2}{4}\right)\left[\frac{i}{p^2}\right]\\ &= \frac{g_2^2}{4}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2}\\ &= \frac{g_2^2}{64\pi^2}\Lambda^2. \end{aligned} \tag{1.134}$$

The correction from Figure 3 (g) is

$$\begin{aligned}\Delta m^2_{(g)} &= (-1)(1)(1)(1)\int \frac{d^4p}{(2\pi)^4}\left(\frac{g_2^2}{4}\right)\left[\frac{i}{p^2}\right] \\ &= -\frac{g_2^2}{4}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2} \\ &= -\frac{g_2^2}{64\pi^2}\Lambda^2. \end{aligned} \tag{1.135}$$

The correction from Figure 3 (h) is

$$\begin{aligned}\Delta m^2_{(h)} &= (-1)(1)(1)(1)\int \frac{d^4p}{(2\pi)^4}\left(\frac{g_Y^2}{4}\right)\left[\frac{i}{p^2}\right] \\ &= -\frac{g_Y^2}{2}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2} \\ &= -\frac{g_Y^2}{64\pi^2}\Lambda^2. \end{aligned} \tag{1.136}$$

The correction from Figure 3 (i) is

$$\begin{aligned}\Delta m^2_{(i)} &= (-1)(1)(1)(1)\int \frac{d^4p}{(2\pi)^4}\left(\frac{g_Y^2+g_2^2}{4}g_{\mu\nu}\right)\left[\frac{-ig^{\mu\nu}}{p^2}\right] \\ &= \left(g_Y^2+g_2^2\right)\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2+m^2} \\ &= \frac{g_Y^2+g_2^2}{16\pi^2}\Lambda^2. \end{aligned} \tag{1.137}$$

The correction from Figure 3 (j) is

$$\begin{aligned}\Delta m^2_{(j)} &= (-1)(1)(1)(1)\int \frac{d^4p}{(2\pi)^4}\left(\frac{g_2^2}{2}g_{\mu\nu}\right)\left[\frac{-ig^{\mu\nu}}{p^2}\right] \\ &= 2g_2^2\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{1}{p_E^2+m^2} \\ &= \frac{g_2^2}{8\pi^2}\Lambda^2. \end{aligned} \tag{1.138}$$

The correction from Figure 3-(k) is

$$\begin{aligned}
\Delta m^2_{(k)} &= \left(\frac{-i}{2!}\right)(1)(1)(1\cdot 1)\int \frac{d^4p}{(2\pi)^4}\left(-\frac{g_2}{\sqrt{2}}p_\mu\right) \\
&\quad \cdot\left(-\frac{g_2}{\sqrt{2}}p_\nu\right)\left[\frac{-ig^{\mu\nu}}{p^2}\frac{i}{p^2}\right]\{2\} \\
&= -\frac{g_2^2}{2}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{p_E^2}{p_E^4} \\
&= -\frac{g_2^2}{32\pi^2}\Lambda^2,
\end{aligned} \tag{1.139}$$

where the factor of $\{2\}$ arises for the same reason as in (1.122). The correction from Figure 3 (l) is

$$\begin{aligned}
\Delta m^2_{(l)} &= \left(\frac{-i}{2!}\right)(1)(1)(1\cdot 1)\int \frac{d^4p}{(2\pi)^4}\left(\frac{\sqrt{g_Y^2+g_2^2}}{2}p_\mu\right) \\
&\quad \cdot\left(\frac{\sqrt{g_Y^2+g_2^2}}{2}p_\nu\right)\left[\frac{-ig^{\mu\nu}}{p^2}\frac{i}{p^2}\right]\{2\} \\
&= -\frac{g_Y^2+g_2^2}{4}\frac{2\pi^2}{\Gamma(2)}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{p_E^2}{p_E^4} \\
&= -\frac{g_Y^2+g_2^2}{64\pi^2}\Lambda^2,
\end{aligned} \tag{1.140}$$

where the factor $\{2\}$ has the same origin as in (1.139). The correction from Figure 3 (m) is

$$\begin{aligned}
\Delta m^2_{(m)} &= \left(\frac{-i}{2!}\right)(1)(1)(1\cdot 1)\int \frac{d^4p}{(2\pi)^4}(-g_2)(-g_2) \\
&\quad \cdot tr\left[\frac{i\gamma^\mu p_\mu}{p^2}\frac{1-\gamma_5}{2}\frac{i\gamma^\nu p_\nu}{p^2}\frac{1+\gamma_5}{2}\right]\{(-1)(2)\} \\
&= -2g_2^2\underbrace{\frac{2\pi^2}{\Gamma(2)}}_{\equiv 1}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{p_E^2}{p_E^4} \\
&= -\frac{g_2^2}{8\pi^2}\Lambda^2,
\end{aligned} \tag{1.141}$$

where the factor of -1 in $\{\cdots\}$ is from the anticommutation of the fermionic fields, and the factor of 2 in $\{\cdots\}$ arises for the same reason as in (1.140). Finally, the correction from Figure 3 (n) is

$$
\begin{aligned}
\Delta m^2_{(n)} = & \left(\frac{-i}{2!}\right)(1)(1)(1\cdot 1)\int \frac{d^4p}{(2\pi)^4}\left(\sqrt{\frac{g_Y^2+g_2^2}{2}}\right)\left(\sqrt{\frac{g_Y^2+g_2^2}{2}}\right) \\
& \cdot tr\left[\frac{i\gamma^\mu p_\mu}{p^2}\frac{1-\gamma_5}{2}\frac{i\gamma^\nu p_\nu}{p^2}\frac{1+\gamma_5}{2}\right]\{(-1)(2)\} \\
& = -\left(g_Y^2+g_2^2\right)\underbrace{\frac{2\pi^2}{\Gamma(2)}}_{\equiv 1}\int \frac{p_E^3 dp_E}{(2\pi)^4}\frac{p_E^2}{p_E^4} \\
& = -\frac{g_Y^2+g_2^2}{16\pi^2}\Lambda^2, \qquad (1.142)
\end{aligned}
$$

where the factor $\{(-1)(2)\}$ was explained immediately above (1.141). Then the total 1-loop radiative correction to the ϕ_u^0 Higgs mass is

$$
\begin{aligned}
\Delta m^2_{\text{total}} = & \frac{g_Y^2}{64\pi^2}\Lambda^2+\frac{g_2^2}{64\pi^2}\Lambda^2+\frac{g_Y^2}{32\pi^2}\Lambda^2+\frac{g_2^2}{32\pi^2}\Lambda^2 \\
& -\frac{g_Y^2}{64\pi^2}\Lambda^2+\frac{g_2^2}{64\pi^2}\Lambda^2-\frac{g_2^2}{64\pi^2}\Lambda^2-\frac{g_Y^2}{64\pi^2}\Lambda^2 \\
& +\frac{g_Y^2+g_2^2}{16\pi^2}\Lambda^2+\frac{g_2^2}{8\pi^2}\Lambda^2-\frac{g_2^2}{32\pi^2}\Lambda^2-\frac{g_Y^2+g_2^2}{64\pi^2}\Lambda^2 \\
& -\frac{g_2^2}{8\pi^2}\Lambda^2-\frac{g_Y^2+g_2^2}{16\pi^2}\Lambda^2 \\
& = 0, \qquad (1.143)
\end{aligned}
$$

and the quadratic divergence thus cancels when all terms are included.

3. MSSM

The MSSM is an minimal extension of the Standard Model. Here **minimal** means:

- The only fields are those of the SM and their supersymmetric partners.
- All $SU(3)_C \times SU(2)_L \times U(1)_Y$ invariant renormalizable interaction terms are allowed.

- Two Higgs doublets are required because of the supersymmetry [11].

We saw above that when we have a superpotential we obtain all interactions required in the model. The superpotential for the MSSM is given by

$$W_{MSSM} = \bar{u} y_u Q H_u - \bar{d} y_d Q H_d - \bar{e} y_e L H_d + \mu H_u H_d. \tag{1.144}$$

$H_u, H_d, Q, L, \bar{u}, \bar{d}, \bar{e}$ are chiral superfields corresponding to the chiral supermultiplets of the Higgs which couples to up-type fermions, the Higgs which couples to down-type fermions, the left hand quark doublet, the left hand lepton doublet, the right hand up-type quark, the right hand down-type quark, and the right hand electron, respectively. The last term is the supersymmetric version of the Higgs boson mass, which guarantees that when one of the Higgs fields acquires a V.E.V. the other does also.

CHAPTER II

POTENTIAL VIOLATIONS OF LORENTZ INVARIANCE

A. Lorentz Symmetry

1. Lorentz Transformation of the Coordinates

First let us review the basic ideas of Lorentz invariance, which is assumed in all of standard physics and even in superstring theory. In special relativity, or in a locally inertial coordinate system, the coordinate x^u is transformed under a Lorentz transformation according to

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu. \tag{2.1}$$

The interval

$$ds = \left(\eta^{\mu\nu} x_\mu x_\nu\right)^{1/2}, \tag{2.2}$$

is required to be invariant under this transformation:

$$\begin{aligned} ds' &= \left(\eta^{\mu\nu} x'_\mu x'_\nu\right)^{1/2} \\ &= \left(\eta^{\mu\nu} \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma x_\rho x_\sigma\right)^{1/2} \\ &\overset{!}{=} ds, \end{aligned} \tag{2.3}$$

where "$\overset{!}{=}$" means "required", and

$$\eta^{\mu\nu} \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma = \eta^{\rho\sigma}, \tag{2.4}$$

$$\begin{aligned} \eta^{\mu\nu} \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\sigma\kappa} &= \eta^{\rho\sigma}\eta_{\sigma\kappa}, \\ \Longrightarrow \Lambda^{\nu\rho}\Lambda_{\nu\kappa} &= \delta^\rho_\kappa. \end{aligned} \tag{2.5}$$

2. Lorentz Transformation of a Scalar Field

A Lorentz scalar is invariant:

$$\begin{aligned} \phi'(x') &= U(\Lambda)\,\phi(x)\,U^{\dagger}(\Lambda) \\ &= \phi\left(\Lambda^{-1}x\right). \end{aligned} \tag{2.6}$$

3. Lorentz Transformation of a Vector Field

A vector field A_μ transforms exactly like the coordinates x_μ:

$$\begin{aligned} A'_\mu(x') &= U(\Lambda)\,A_\mu(x)\,U^{\dagger}(\Lambda) \\ &= \left(\Lambda^{-1}\right)_\mu{}^\rho A_\rho\left(\Lambda^{-1}x\right). \end{aligned} \tag{2.7}$$

4. Lorentz Transformation of a Spinor Field

The mathematical tools which relate the vector index and spinor index are the Pauli matrices and gamma matrices for two and four component spinors, respectively. Here we consider only the transformation of two-component spinors, since these provide the fundamental description of fermions in SUSY, GUTs, and even the SM.

First we need to obtain the generators of Lorentz transformations for spinors. Since a spinor is transformed into a spinor, a generator is required to have two spinor indices. Each might be dotted or undotted at this point. (See Appendix A for the meaning of dotted and undotted indices. A 2-component Weyl spinor with dotted index transforms under a Lorentz transformation as right-handed, and one with undotted index as left-handed.) If one is dotted and the other undotted, a right-handed field is transformed into left-handed or vice-versa. When the field is massless, however, this is not possible, since a Lorentz transformation cannot change the spin in this case. Therefore, the generator is required to have either two undotted or two dotted spinor indices.

We define the transformations of the 2-component spinors by

$$\begin{aligned}\psi'_\alpha(x') &= U(\Lambda)\,\psi_\alpha(x)\,U^\dagger(\Lambda)\\ &\approx \left(\delta_\alpha{}^\beta + \frac{i}{2}\omega_{\mu\nu}\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta\right)\psi_\beta\left(\Lambda^{-1}x\right),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\psi'^\dagger_{\dot\alpha}(x') &= U(\Lambda)\,\psi^\dagger_{\dot\alpha}(x)\,U^\dagger(\Lambda)\\ &\approx \psi^\dagger_{\dot\beta}\left(\Lambda^{-1}x\right)\left(\delta^{\dot\beta}{}_{\dot\alpha} - \frac{i}{2}\omega_{\mu\nu}\left(\sigma^{\dagger\mu\nu}\right)^{\dot\beta}{}_{\dot\alpha}\right),\end{aligned} \tag{2.9}$$

where $\omega_{\mu\nu}$ is an infinitesimal parameter which is antisymmetric under $\mu \leftrightarrow \nu$. $\omega_{\mu\nu}$ is related to $\Lambda_{\mu\nu}$ and $\eta_{\mu\nu}$ by

$$\Lambda_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{symmetric}} + \underbrace{\omega_{\mu\nu}}_{\text{anti-symmetric}}. \tag{2.10}$$

To have a non-vanishing transformation, $\sigma^{\mu\nu}$ in (2.9) has to be also antisymmetric. We now set out to derive this generator $\sigma^{\mu\nu}$.

The Lagrangian density is given by

$$\mathcal{L} = -i\psi^\dagger_{\dot\alpha}(x)\left(\bar\sigma^\nu\right)^{\dot\alpha\beta}\partial_\nu\psi_\alpha(x), \tag{2.11}$$

and this transforms as

$$
\begin{aligned}
\mathcal{L} \rightarrow & -i\psi'^{\dagger}_{\dot{\alpha}}\left(x'\right)\left(\bar{\sigma}^{\nu}\right)^{\dot{\alpha}\alpha}\partial'_{\nu}\psi'_{\alpha}\left(x'\right) \\
& = -i\psi^{\dagger}_{\dot{\beta}}\left(\Lambda^{-1}x\right)\left(\delta^{\dot{\beta}}{}_{\dot{\alpha}} - \frac{i}{2}\omega_{\xi\kappa}\left(\sigma^{\dagger\xi\kappa}\right)^{\dot{\beta}}{}_{\dot{\alpha}}\right)\left(\bar{\sigma}^{\nu}\right)^{\dot{\alpha}\alpha} \\
& \cdot\left(\Lambda^{-1}\right)^{\chi}{}_{\nu}\partial_{\chi}\left(\delta_{\alpha}{}^{\beta} + \frac{i}{2}\omega_{\xi\kappa}\left(\sigma^{\xi\kappa}\right)_{\alpha}{}^{\beta}\right)\psi_{\beta}\left(\Lambda^{-1}x\right) \\
& \approx -i\psi^{\dagger}_{\dot{\beta}}\left(\Lambda^{-1}x\right)\left(\left(\bar{\sigma}^{\nu}\right)^{\dot{\beta}\beta} - \frac{i}{2}\omega_{\xi\kappa}\left[\left(\sigma^{\dagger\xi\kappa}\bar{\sigma}^{\nu} - \bar{\sigma}^{\nu}\sigma^{\xi\kappa}\right)\right]^{\dot{\beta}\beta}\right) \\
& \qquad\qquad \cdot\left(\Lambda^{-1}\right)^{\chi}{}_{\nu}\partial_{\chi}\psi_{\beta}\left(\Lambda^{-1}x\right) \\
& \stackrel{!}{=} -i\psi^{\dagger}_{\dot{\beta}}\left(\Lambda^{-1}x\right)\left(\bar{\sigma}^{\nu}\right)^{\dot{\beta}\beta}\partial_{\nu}\psi_{\beta}\left(\Lambda^{-1}x\right),
\end{aligned}
\tag{2.12}
$$

where the ω^2 term is ignored as usual. Then, from this requirement, we get

$$
\left[\left(\bar{\sigma}^{\nu}\right)^{\dot{\beta}\beta} - \frac{i}{2}\omega_{\xi\kappa}\left[\left(\sigma^{\dagger\xi\kappa}\bar{\sigma}^{\nu} - \bar{\sigma}^{\nu}\sigma^{\xi\kappa}\right)\right]^{\dot{\beta}\beta}\right]\left(\Lambda^{-1}\right)^{\chi}{}_{\nu} = \left(\bar{\sigma}^{\chi}\right)^{\dot{\beta}\beta}, \tag{2.13}
$$

$$
\rightarrow \left[\left(\bar{\sigma}^{\nu}\right)^{\dot{\beta}\beta} - \frac{i}{2}\omega_{\xi\kappa}\left[\left(\sigma^{\dagger\xi\kappa}\bar{\sigma}^{\nu} - \bar{\sigma}^{\nu}\sigma^{\xi\kappa}\right)\right]^{\dot{\beta}\beta}\right]\Lambda^{\lambda}{}_{\chi}\left(\Lambda^{-1}\right)^{\chi}{}_{\nu} = \left(\bar{\sigma}^{\chi}\right)^{\dot{\beta}\beta}\Lambda^{\lambda}{}_{\chi}
$$

$$
\rightarrow \left[\left(\bar{\sigma}^{\lambda}\right)^{\dot{\beta}\beta} - \frac{i}{2}\omega_{\xi\kappa}\left[\left(\sigma^{\dagger\xi\kappa}\bar{\sigma}^{\lambda} - \bar{\sigma}^{\lambda}\sigma^{\xi\kappa}\right)\right]^{\dot{\beta}\beta}\right] = \left(\bar{\sigma}^{\chi}\right)^{\dot{\beta}\beta}\Lambda_{\eta\chi}\eta^{\lambda\eta}.
$$

By using (2.10) on the right hand side, we have

$$
\left[\left(\bar{\sigma}^{\lambda}\right)^{\dot{\beta}\beta} - \frac{i}{2}\omega_{\xi\kappa}\left[\left(\sigma^{\dagger\xi\kappa}\bar{\sigma}^{\lambda} - \bar{\sigma}^{\lambda}\sigma^{\xi\kappa}\right)\right]^{\dot{\beta}\beta}\right] = \left(\bar{\sigma}^{\chi}\right)^{\dot{\beta}\beta}\left(g_{\eta\chi} + \omega_{\eta\chi}\right)\eta^{\lambda\eta}, \tag{2.14}
$$

$$
\rightarrow -\frac{i}{2}\omega_{\xi\kappa}\left[\left(\sigma^{\dagger\xi\kappa}\bar{\sigma}^{\lambda} - \bar{\sigma}^{\lambda}\sigma^{\xi\kappa}\right)\right]^{\dot{\beta}\beta} = \left(\bar{\sigma}^{\chi}\right)^{\dot{\beta}\beta}\omega_{\eta\chi}\eta^{\lambda\eta},
$$

$$
\rightarrow -\frac{i}{2}\left[\left(\sigma^{\dagger\xi\kappa}\bar{\sigma}^{\lambda} - \bar{\sigma}^{\lambda}\sigma^{\xi\kappa}\right)\right]\omega_{\xi\kappa} = \bar{\sigma}^{\kappa}\eta^{\lambda\xi}\omega_{\xi\kappa}. \tag{2.15}
$$

Recall that $\sigma^{\xi\kappa}$ is antisymmetric under $\xi \longleftrightarrow \kappa$, and the spinor indices are $\left(\sigma^{\xi\kappa}\right)_{\alpha}{}^{\beta}$. Therefore, we expect that $\left(\sigma^{\xi\kappa}\right)_{\alpha}{}^{\beta}$ has the form

$$
\left(\sigma^{\xi\kappa}\right)_{\alpha}{}^{\beta} = A\left[\left(\sigma^{\xi}\right)_{\alpha\dot{\alpha}}\left(\bar{\sigma}^{\kappa}\right)^{\dot{\alpha}\beta} - \left(\sigma^{\kappa}\right)_{\alpha\dot{\alpha}}\left(\bar{\sigma}^{\xi}\right)^{\dot{\alpha}\beta}\right],
$$

where the coefficient A is determined from (2.15) by using (C.20). We then obtain

$$\sigma^{\xi\kappa} = \frac{i}{2}\left[\sigma^{\xi}\bar{\sigma}^{\kappa} - \sigma^{\kappa}\bar{\sigma}^{\xi}\right], \tag{2.16}$$

$$\begin{aligned}\bar{\sigma}^{\xi\kappa} &= \sigma^{\dagger\xi\kappa} \\ &= \frac{i}{2}\left[\bar{\sigma}^{\xi}\sigma^{\kappa} - \bar{\sigma}^{\kappa}\sigma^{\xi}\right].\end{aligned} \tag{2.17}$$

In summary, the Lorentz transformation of the spinor field is given by

$$\begin{aligned}\psi'_{\alpha}\left(x'\right) &= U\left(\Lambda\right)\psi_{\alpha}\left(x\right)U^{\dagger}\left(\Lambda\right) \\ &\approx \left(\delta_{\alpha}{}^{\beta} + \frac{i}{2}\omega_{\mu\nu}\left(\sigma^{\mu\nu}\right)_{\alpha}{}^{\beta}\right)\psi_{\beta}\left(\Lambda^{-1}x\right),\end{aligned} \tag{2.18}$$

$$\begin{aligned}\psi'^{\dagger}_{\dot{\alpha}}\left(x'\right) &= U\left(\Lambda\right)\psi^{\dagger}_{\dot{\alpha}}\left(x\right)U^{\dagger}\left(\Lambda\right) \\ &= \psi^{\dagger}_{\dot{\beta}}\left(\Lambda^{-1}x\right)\left(\delta^{\dot{\beta}}{}_{\dot{\alpha}} - \frac{i}{2}\omega_{\mu\nu}\left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\beta}}{}_{\dot{\alpha}}\right).\end{aligned} \tag{2.19}$$

B. Tests of Lorentz Symmetry

During the past few years there has been increasingly widespread interest in possible violations of Lorentz invariance [21]-[48]. There are several motivations for this interest.

1. Theoretical

Every current candidate for a superunified theory contains some potential for Lorentz violation, and the same is true for more restricted theories which attempt to treat quantum gravity alone. (By a "superunified theory" we mean one which includes all known physical phenomena, and which is valid up to the Planck energy.) Theories with potential for Lorentz violation include superstring/M/brane theories, canonical and loop quantum gravity, noncommutative spacetime geometry, nontrivial spacetime topology, discrete spacetime structure at the Planck length, a variable speed of light or variable physical constants, various other *ad hoc* theories, including one that specifically addresses the GZK cutoff [22], and the theory presented in this dissertation. Even in a theory which has Lorentz invariance at the most fundamental level, this symmetry can be spontaneously broken if some field acquires a vacuum expectation value which breaks rotational invariance or invariance under a boost. (It should be mentioned that cosmology already provides a preferred frame of reference – namely a comoving frame, in which the cosmic background radiation does not have a dipole anisotropy – but this is not considered to be a breaking of Lorentz symmetry, since the vacuum is still Lorentz invariant.) A second mechanism for Lorentz violation is the "quantum foam" of Hawking and Wheeler, originally envisioned in the context of canonical or path-integral quantization of Einstein gravity, but now generalized to other theories with quantum gravity. A third possibility is a theory in which Lorentz invariance is not postulated to be an exact fundamental symmetry, but instead emerges as a low-energy symmetry, and that is the possibility explored in this dissertation.

2. Experimental

Both terrestrial [23]-[34] and space-based [35]-[40] experiments have been designed with exquisite precision which would permit detection of even tiny deviations from certain aspects of Lorentz invariance. The systems include atoms, charged particles in traps, masers, cavity-stabilized oscillators, muons, neutrons, kaons, and other neutral mesons.

3. Observational

Particles traveling over cosmological distances from bright sources (including pulsars, supernovae, blazars, and gamma ray bursters) allow long-baseline tests which are again sensitive to even tiny deviations from particular forms of Lorentz violation [41]-[46].

Recall that Lorentz invariance in the context of general relativity means *local* Lorentz invariance, or an invariance of the action under rotations and boosts involving locally inertial frames of reference. There is clearly a connection with the equivalence principle, which can also be tested in, e.g., space-based experiments. There is a close connection with CPT invariance as well: According to the CPT theorem, Lorentz invariance implies CPT invariance (with the supplementary assumptions of unitarity and locality). It follows that CPT violation implies Lorentz violation, although the reverse is not necessarily true. Finally, there is a connection to the spin-statistics theorem, which follows from Lorentz invariance and microcausality.

We know that P (in the 1950s) and CP (in the 1960s) have previously been found not to be inviolate symmetries, for reasons that are now understood in terms of the standard electroweak theory and the CKM matrix. Perhaps CPT and Lorentz symmetry are also not inviolate.

The most extensive theoretical program for systematizing potential forms of Lorentz violation and their experimental signals is that of Kostelecký and coworkers [23],[24],[29]-[38],[40],[46]. Their philosophy is to add small phenomenological Lorentz-violating terms to the Lagrangian of the Standard Model, and then interact with a wide variety of experiments that can detect such deviations from exact Lorentz or CPT invariance. The point of view of this group is rather conservative: The fundamental theory (e.g., string theory) is pictured as Lorentz-invariant, with Lorentz or CPT violation arising from some form of symmetry-breaking – for example, with a vector field or more general tensor field acquiring a vacuum expectation value. Their work has stimulated a considerable amount of experimental activity, with further experiments planned for both terrestrial and space-based laboratories.

So far there is no undisputed evidence for Lorentz violation, and the only solid results from both experiment and observation are strong constraints on particular ways in which this symmetry might be broken. As an example of an astrophysical constraint, we mention a recent paper by Stecker and Glashow [43], in which they conclude "We use the recent reanalysis of multi-TeV [up to 20 TeV] gamma-ray observations of [the blazar] Mrk 501 to constrain the Lorentz invariance breaking parameter involving the maximum electron velocity. Our limit is two orders of magnitude better than that obtained from the maximum observed cosmic-ray electron energy." Their analysis involves the processes

$$\gamma + \gamma_{infrared} \rightarrow e^{+} + e^{-} \quad \text{if} \quad c_e > c_\gamma \tag{2.20}$$

which can lead to inconsistency with the observation of 20 TeV photons and

$$\gamma \rightarrow e^{+} + e^{-} \quad \text{if} \quad c_e < c_\gamma \tag{2.21}$$

which can lead to inconsistency with the observation of 50 TeV photons.

Another example of astrophysical constraints is the series of analyses by Jacobson et al. [41]-[44]. In Ref. [42], Jacobson, Liberati, Mattingly, and Stecker state "We strengthen the constraints on possible Lorentz symmetry violation (LV) of order E/M_{Planck} for electrons and photons in the framework of effective field theory (EFT). The new constraints use (i) the absence of vacuum birefringence in the recently observed polarization of MeV emission from a gamma ray burst and (ii) the absence of vacuum Čerenkov radiation from the synchrotron electrons in the Crab nebula, improving the previous bounds by eleven and four orders of magnitude respectively."

Jacobson, Liberati, and Mattingly [41] have obtained a very strong constraint on a dispersion relation with a cubic term in the expression for E^2:

$$E^2 = p^2 + p^3/M. \tag{2.22}$$

However, the constraint is less stringent for what may be the more natural form with a quartic term:

$$E^2 = p^2 + p^4/M^2. \tag{2.23}$$

Below we will derive the dispersion relation for a fundamental Lorentz-violating theory [21, 47, 48] and will find that it is easily consistent with these constraints, since it has a form quite different from either of those above.

Coleman and Glashow [22] proposed that the limiting velocity of protons, electrons, etc. may be very slightly different from the speed of light. (See also Ref. [44].) This is an *ad hoc* proposal, motivated by the apparent absence of a Greisen-Zatsepin-Kuz'min (GZK) cutoff: Ultrahigh energy cosmic ray protons colliding with the cosmic microwave background radiation should produce pions,

$$p + \gamma_{cmb} \rightarrow p + \pi^0. \tag{2.24}$$

There should consequently be a cutoff in the spectrum of observed protons at about 50 EeV (or 5×10^7 TeV), if they were created in processes at distances of more than about 100 Mpc. But up to 300 EeV cosmic rays (presumably protons) appear to be observed, although this is not entirely certain [58], and there are also theoretical ideas for a closer origin [56].

We conclude by mentioning some reviews of terrestrial and space-based experiments.

Two reviews of atomic experiments to test both Lorentz and CPT symmetries, by Bluhm [34], describe the following: (1) Penning trap experiments with electrons and positrons, and with protons and antiprotons, which look for differences in frequencies or sidereal time variations; (2) clock comparison experiments, with clock frequencies typically those of hyperfine or Zeeman transitions; (3) hydrogen and antihydrogen experiments involving ground-state Zeeman hyperfine transitions (at Harvard) or 1S-2S transitions (proposed at CERN); (4) a spin-polarized torsion pendulum experiment (at the University of Washington); (5) muon and muonium experiments.

Two reviews by Russell [38] discuss clock-based experiments to test Lorentz and CPT invariance in space. Such experiments will probe the effects of variations in both orientation and velocity. Among the systems are H masers, laser-cooled Cs and Rb clocks, and superconducting microwave cavity oscillators. A number of specific space missions have been planned or proposed.

Finally, a review by Kostelecký [46] contains a discussion of experiments involving neutral meson (e.g. kaon) oscillations, a dual nuclear Zeeman He-Xe maser, and cosmological birefringence, in addition to the systems mentioned above.

C. Review of Lorentz Violation Effects on Kinematics

We would like to use the cosmological observation results to see the Lorentz symmetry violation effect in our theory, and we review the GZK cutoff, the vacuum pair production, and vacuum Cerenkov radiation.

1. GZK Cutoff

Proton with sufficiently high energy will lose energy from inelastic collisions with cosmic microwave background radiation (CBR) photons. This gives rise to the GZK cutoff, and protons with energy $E > 5 \times 10^{19}$ eV should not reach us from further away than $\sim$ 50 Mpc (where 1 pc= 1 parsec= 3.26 light years). Ultrahigh energy cosmic ray protons colliding with the CBR should produce pions:

$$p + \gamma_{CBR} \rightarrow p + \pi^0 \tag{2.25}$$

However, up to 3×10^{20} eV cosmic rays appear to be observed experimentally.

The incoming photon has energy ω and momentum $(-\omega\cos\theta, -\omega\sin\theta, 0)$, and the incoming fermion has mass m_a, energy E, and momentum $(p, 0, 0)$. The outgoing fermion (excited state) has mass $m_b^2 = m_a^2 + \triangle m^2$, energy $E + \omega$, and momentum $(p - \omega\cos\theta, -\omega\sin\theta, 0)$. Then from kinematics, we obtain

$$2\omega p\,(1 + \cos\theta) = \triangle m^2 \tag{2.26}$$

$$\rightarrow \cos\theta = \frac{\triangle m^2}{2\omega p} - 1 \leq 1$$

which is possible for

$$p \geq \frac{\triangle m^2}{4\omega} \qquad \text{(GZK cut off)} \tag{2.27}$$

$$\sim 1 \times 10^{20}\ \text{eV} = 100\ \text{EeV} \tag{2.28}$$

with a 2.7 K CBR and $\triangle m^2 = 2m_p m_\pi + m_\pi^2$.

As the density of the CBR is $n_\gamma = 550$ photons/cm^3 and the cross section is $\sigma = 200\ \mu$b, the mean path for interaction is $(n_\gamma\sigma)^{-1} = 9 \times 10^{24}$ cm. Then the rough distance scale L for loss of energy is

$$L = (E/\triangle E)\,(n_\gamma\sigma)^{-1} \tag{2.29}$$

where E is the initial energy of the proton, and $\triangle E$ is the energy loss per collision, with $\triangle E/E \approx 20\%$ [49]. Therefore

$$\begin{aligned} L &\approx 5 \times 9 \times 10^{24} \text{ cm} \\ &= 4.5 \times 10^{25} \text{ cm} \\ &\simeq 5 \times 10^{7} \text{ light years.} \end{aligned} \tag{2.30}$$

2. Vacuum Pair Production

The process of vacuum pair production is

$$\gamma \rightarrow e^{-} + e^{+}. \tag{2.31}$$

When the 4-momenta of the photon, electron, and positron are

$$\begin{aligned} &(\omega, 0, 0, \omega), \\ &\left(E = \sqrt{p^2 + m^2}, \cos\theta\ p, 0, \sin\theta\ p\right), \\ &\left(E' = \sqrt{p'^2 + m^2}, \cos\theta' p', 0, \sin\theta' p'\right), \end{aligned}$$

respectively, energy and momentum conservation give

$$\omega = E + E', \tag{2.32}$$

$$\omega = \sin\theta\ p + \sin\theta' p', \tag{2.33}$$

$$0 = \cos\theta\ p + \cos\theta' p'. \tag{2.34}$$

From the 2nd and 3rd equations we get

$$p'^2 = p^2 + \omega^2 - 2\sin\theta\ \omega p, \tag{2.35}$$

and by using this p'^2 in the 1st equation, we have

$$\left(\omega - \sqrt{p^2 + m^2}\right)^2 = p^2 + \omega^2 - 2\sin\theta\ \omega p + m^2, \tag{2.36}$$

so by rearranging this equation we obtain

$$\frac{\sqrt{p^2+m^2}}{p} = \sin\theta. \tag{2.37}$$

This relation can never be satisfied if the electron is massive.

However, when the electron's energy-momentum dispersion relation is modified to

$$E^2 = p^2 + m^2 + a_n p^n \qquad \text{where } n > 2, \tag{2.38}$$

we may be able to find a solution for vacuum pair production. The kinematics give us

$$\left(\omega - \sqrt{p^2+m^2+a_n p^n}\right)^2 = p^2 + \omega^2 - 2\sin\theta\omega p + m^2 + a_n p^n, \tag{2.39}$$

and we obtain

$$\frac{\sqrt{p^2+m^2+a_n p^n}}{p} = \sin\theta < 1. \tag{2.40}$$

Then when

$$a_n p^n < -m^2 \tag{2.41}$$

we have vacuum pair production.

We can also have vacuum pair production if the maximum speed of the electron c_e is different from the speed of light c_γ:

$$\omega = E + E', \tag{2.42}$$

$$\frac{\omega}{c_\gamma} = \sin\theta p + \sin\theta' p', \tag{2.43}$$

$$0 = \cos\theta p + \cos\theta' p'. \tag{2.44}$$

From (2.43) and (2.44), by solving for p'^2 and then inserting it into (2.42)), we obtain

$$2c_\gamma^2\sqrt{c_e^2 p^2 + c_e^4 m^2} - \left(c_\gamma^2 - c_e^2\right)\omega = 2pc_\gamma c_e^2 \sin\theta < 2pc_\gamma c_e^2 \tag{2.45}$$

and at $p = 0$ there is a solution when

$$\left(c_\gamma^2 - c_e^2\right)\omega > 2c_\gamma^2 c_e^2 m. \tag{2.46}$$

Since the right hand side is positive when $c_\gamma > c_e$, there can be a solution in this case. When the photon's energy ω is $> 2c_\gamma^2 c_e^2 m/\left(c_\gamma^2 - c_e^2\right)$, it decays into electron-positron pairs.

As photons with energies up to 50 TeV are observed, and the electron mass is $mc_e^2 = 0.51$ MeV, the difference of the maximum speed of the photon and electron needs to satisfy

$$\frac{c_\gamma^2}{c_\gamma^2 - c_e^2} > \frac{\omega}{2mc_e^2} \sim 5 \times 10^7. \tag{2.47}$$

When δ is defined by $c_e = c_\gamma\left(1 - |-\delta|\right)$ with $c_\gamma > c_e$, we obtain

$$|-\delta| \lesssim 1 \times 10^{-8}. \tag{2.48}$$

3. Vacuum Cerenkov Radiation

The process of vacuum Cerenkov radiation is

$$\chi \to \chi + \gamma. \tag{2.49}$$

where χ is a charged fermion. If the 4-momenta of the incoming χ, the outgoing χ, and the photon are

$$\left(E = \sqrt{p^2 + m_\chi^2}, p, 0, p\right)$$
$$\left(E' = \sqrt{p'^2 + m_\chi^2}, \cos\theta' p', 0, \sin\theta' p'\right)$$
$$(\omega, \cos\theta\omega, 0, \sin\theta\omega)$$

respectively, energy and momentum conservation give us

$$E = \omega + E', \tag{2.50}$$

$$p = \sin\theta\omega + \sin\theta' p', \tag{2.51}$$

$$0 = \cos\theta\omega + \cos\theta' p'. \tag{2.52}$$

From the 2nd and 3rd equations it follows that

$$p'^2 = p^2 + \omega^2 - 2\sin\theta\omega p,$$

and by using this p'^2 in the 1st equation, we have

$$\left(\omega - \sqrt{p^2 + m_\chi^2}\right)^2 = p^2 + \omega^2 - 2\sin\theta\omega p + m_\chi^2.$$

Rearranging this equation we obtain

$$\frac{\sqrt{p^2 + m_\chi^2}}{p} = \sin\theta. \tag{2.53}$$

This relation can never be satisfied if χ is massive. The reason that this result is same as that for vacuum pair creation is the crossing symmetry.

However, when the electron's energy-momentum dispersion relation is modified to

$$E^2 = p^2 + m_\chi^2 + a_n p^n \qquad \text{where } n > 2, \tag{2.54}$$

we may be able to find a solution for vacuum pair production. The kinematics give us

$$\left(\omega - \sqrt{p^2 + m_\chi^2 + a_n p^n}\right)^2 = p^2 + \omega^2 - 2\sin\theta\omega p + m_\chi^2 + a_n p^n,$$

and we obtain

$$\frac{\sqrt{p^2 + m_\chi^2 + a_n p^n}}{p} = \sin\theta \leq 1. \tag{2.55}$$

When

$$a_n p^n < -m^2, \tag{2.56}$$

we then have vacuum pair production.

We can also have vacuum pair production if the maximum speed of the charged particle χ, c_χ, is different from the speed of the light c_γ:

$$E = \omega + E', \tag{2.57}$$

$$p = \sin\theta \frac{\omega}{c_\gamma} + \sin\theta' p', \tag{2.58}$$

$$0 = \cos\theta \frac{\omega}{c_\gamma} + \cos\theta' p'. \tag{2.59}$$

Then we obtain

$$2c_\gamma^2 \sqrt{c_\chi^2 p^2 + c_\chi^4 m_\chi^2} - \left(c_\gamma^2 - c_\chi^2\right)\omega = 2pc_\gamma c_\chi^2 \sin\theta < 2pc_\gamma c_\chi^2, \tag{2.60}$$

and at $\omega = 0$,

$$p^2\left(c_\chi^2 - c_\gamma^2\right) > 2c_\gamma^2 c_\chi^2 m_\chi^2. \tag{2.61}$$

When $c_\chi > c_\gamma$ and the charged particle energy $E = \sqrt{c_\chi^2 p^2 + c_\chi^4 m_\chi^2}$ is $> m_\chi c_\chi^2 \sqrt{\frac{c_\chi^2 + c_\gamma^2}{c_\chi^2 - c_\gamma^2}}$, the charged particle spontaneously emits photons until the energy becomes $< m_\chi c_\chi^2 \sqrt{\frac{c_\chi^2 + c_\gamma^2}{c_\chi^2 - c_\gamma^2}}$.

Since electrons with energy up to 100 TeV are experimentally observed, and the electron mass is $m_e c_e^2 = 0.51$ MeV, the difference between the maximum speeds of the photon and electron needs to satisfy

$$\frac{c_e^2 + c_\gamma^2}{c_e^2 - c_\gamma^2} > \left(\frac{E}{m_e c_e^2}\right)^2 \sim 4 \times 10^{16}. \tag{2.62}$$

When δ is defined by $c_e = c_\gamma\left(1 + |\delta|\right)$, with $c_\chi > c_\gamma$, we obtain

$$|\delta| \lesssim 10^{-17}. \tag{2.63}$$

D. Lorentz Symmetry Violation in Our Model

We start with the Lorentz-violating action of a right-hand field, and obtain the corresponding left-hand field by using a well-known procedure. Once we have both the right and left hand fields, we introduce fermion mass terms in the usual Yukawa form. Then, by using the Euler-Lagrange equation, we obtain the energy-momentum dispersion relations, and we study the kinematics in conjunction with the observational data.

Our action has a Lorentz-violating term, and we will explicitly show that the term is invariant under rotations but not under boosts. Finally, since Lorentz violation does not necessarily mean violation of CPT, we will consider the behavior under CPT, and will obtain the interesting result that our Lorentz-violating term also violates CPT. It is, in fact, odd under CPT.

1. Lorentz-violating Lagrangian and Its Kinematics

Now let us turn to our specific Lorentz-violating theory and some of its new predictions. We begin with the action for a single initially massless Weyl fermion field [47], and with the coupling to gauge fields and variations in e_{μ}^{α} neglected:

$$S_1 = \int d^4x \, \mathcal{L}_1 \tag{2.64}$$

$$\mathcal{L}_1 = \frac{1}{2} e \, \psi_1^{\dagger} \left(\frac{1}{2M} \eta^{\mu\nu} \partial_\nu \partial_\mu + i e_{\alpha}^{\mu} \sigma^{\alpha} \partial_\mu \right) \psi_1 + h.c. \tag{2.65}$$

Here M is a fundamental mass which is comparable to the Planck mass, $\eta^{\mu\nu} = diag(-1,1,1,1)$ is the Minkowski metric tensor, σ^k is a Pauli matrix, and σ^0 is the 2×2 identity matrix. Also, e_{α}^{μ} is the gravitational vierbein, which determines the gravitational metric tensor $g_{\mu\nu}$ through the relations

$$g_{\mu\nu} = \eta_{\alpha\beta} e^{\alpha}_{\mu} e^{\beta}_{\nu} \quad , \quad e^{\mu}_{\alpha} e^{\alpha}_{\nu} = \delta^{\mu}_{\nu}. \tag{2.66}$$

A factor of $e^{-1/2}$ has been absorbed in ψ_1, where

$$e = \det\, e^{\alpha}_{\mu} = \left(-\det\, g_{\mu\nu}\right)^{1/2}. \tag{2.67}$$

Fundamental units are used as always, with $\hbar = c = 1$. Finally, "$h.c.$" means "Hermitian conjugate", and $\mathcal{L}_1$ has been written in its more fundamental and manifestly Hermitian form. The action (2.65) is invariant under a rotation, but it is not invariant under a Lorentz boost because of the first term. (Recall that the transformation matrix $\Lambda_{1/2}$ is unitary for a rotation and not for a boost [50].) At low energies, however, this term is negligible and full Lorentz invariance is regained.

As before, we choose the directions of the spacetime coordinate axes to be such that all the e^{μ}_{α} are positive. If the term involving M is neglected, $\mathcal{L}_1$ has the form appropriate for a right-handed field. I.e., in order for S_1 to be invariant under local Lorentz transformations at low energy, all the fundamental fermionic fields ψ_1 must be taken to transform as right-handed spinors. This is the reverse of the usual convention in grand-unified theories, where they are all taken to be left-handed. However, we can convert ψ_1 to a left-handed field through the following well-known procedure [50]-[52], which is based on the fact that $\left(\sigma^2\right)^2 = 1$, $\left(\sigma^2\right)^{\dagger} = \sigma^2$, $\left(\sigma^2\right)^{*} = -\sigma^2$, and

$$\sigma^2 \sigma^k \sigma^2 = -\left(\sigma^k\right)^{*}. \tag{2.68}$$

Let

$$\psi_L = \sigma^2 \psi_1^{*} \quad \text{or} \quad \psi_1 = \left(\sigma^2 \psi_L\right)^{*} \tag{2.69}$$

and substitute into (2.64), using (in the fourth step below) the fact that Grassmann fields like ψ_L anticommute:

$$\begin{aligned}
\mathcal{L}_1 &= \frac{1}{2}e\ \left[\left(\sigma^2\psi_L\right)^*\right]^\dagger \left(\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu + ie^\mu_\alpha\sigma^\alpha\partial_\mu\right)\left(\sigma^2\psi_L\right)^* + h.c. \\
&= \frac{1}{2}e\ \left[\left(\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu + ie^\mu_\alpha\sigma^\alpha\partial_\mu\right)\left(\sigma^2\psi_L\right)^*\right]^\dagger \left(\sigma^2\psi_L\right)^* + h.c. \\
&= \frac{1}{2}e\ \left[\left(\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu + ie^\mu_\alpha\sigma^\alpha\partial_\mu\right)^*\left(\sigma^2\psi_L\right)\right]^T \left(\sigma^2\psi_L\right)^* + h.c. \\
&= -\frac{1}{2}e\ \left[\left(\sigma^2\psi_L\right)^*\right]^T \left[\left(\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu - ie^\mu_\alpha\left(\sigma^\alpha\right)^*\partial_\mu\right)\left(\sigma^2\psi_L\right)\right] + h.c \\
&= \frac{1}{2}e\,\psi_L^\dagger\left(\sigma^2\right)^\dagger \left[\left(-\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu + ie^\mu_\alpha\left(\sigma^\alpha\right)^*\partial_\mu\right)\left(\sigma^2\psi_L\right)\right] + h.c. \\
&= \frac{1}{2}e\,\psi_L^\dagger \left[\left(-\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu + ie^\mu_\alpha\left(\sigma^2\sigma^\alpha\sigma^2\right)^*\partial_\mu\right)\psi_L\right] + h.c. \\
&= \frac{1}{2}e\,\psi_L^\dagger \left[\left(-\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu + ie^\mu_\alpha\overline{\sigma}^\alpha\partial_\mu\right)\psi_L\right] + h.c., \qquad (2.70)
\end{aligned}$$

where $\overline{\sigma}^0 = \sigma^0$ and $\overline{\sigma}^k = -\sigma^k$. Then ψ_L has the Lagrangian appropriate for a left-handed field (when the term containing M is neglected), and the definition (2.69) implies that it transforms as a left-handed field if ψ_1 is required to transform as a right-handed field [50]-[52].

If ψ_L corresponds to a particle with a Dirac mass m, it is coupled through this mass to a right-handed field ψ_R. (The origin of this mass – i.e., the coupling to a Higgs field which acquires a V.E.V. – is not considered here.) The Lagrangian density for this pair of fields is then given by

$$\begin{aligned}
e^{-1}\mathcal{L} = \psi_R^\dagger &\left(\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu + ie^\mu_\alpha\sigma^\alpha\partial_\mu\right)\psi_R \\
&+ \psi_L^\dagger\left(-\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu + ie^\mu_\alpha\overline{\sigma}^\alpha\partial_\mu\right)\psi_L \\
&- m\psi_R^\dagger\psi_L - m\psi_L^\dagger\psi_R \qquad (2.71)
\end{aligned}$$

after an integration by parts to get the more familiar form. The resulting equations of motion can be written as

$$\left[\frac{1}{2M}\left(-e^0_\alpha e^\alpha_0\partial_0\partial_0+e^k_\alpha e^\alpha_l\partial_l\partial_k\right)+ie^\mu_\alpha\sigma^\alpha\partial_\mu\right]\psi_R-m\psi_L=0 \tag{2.72}$$

$$\left[-\frac{1}{2M}\left(-e^0_\alpha e^\alpha_0\partial_0\partial_0+e^k_\alpha e^\alpha_l\partial_l\partial_k\right)+ie^\mu_\alpha\overline{\sigma}^\alpha\partial_\mu\right]\psi_L-m\psi_R=0 \tag{2.73}$$

with $k,l=1,2,3$. For simplicity, let us assume spatial isotropy and write

$$e^k_\alpha=\lambda\delta^k_\alpha \quad , \quad e^\alpha_k=\lambda^{-1}\delta^\alpha_k=\lambda^{-2}e^k_\alpha \tag{2.74}$$

$$e^0_\alpha=\lambda_0\delta^0_\alpha \quad , \quad e^\alpha_0=\lambda_0^{-1}\delta^\alpha_0=\lambda_0^{-2}e^0_\alpha. \tag{2.75}$$

After transforming to a locally inertial frame of reference, in which $e^\mu_\alpha=\delta^\mu_\alpha$, we have

$$\left[\left(-\beta\partial_0\partial_0+\alpha\partial_k\partial_k\right)+i\left(\sigma^0\partial_0+\sigma^k\partial_k\right)\right]\psi_R-m\psi_L=0 \tag{2.76}$$

$$\left[-\left(-\beta\,\partial_0\partial_0+\alpha\partial_k\partial_k\right)+i\left(\sigma^0\partial_0-\sigma^k\partial_k\right)\right]\psi_L-m\psi_R=0 \tag{2.77}$$

where

$$\alpha=\left(2\lambda^2M\right)^{-1} \quad , \quad \beta=\left(2\lambda_0^2M\right)^{-1}. \tag{2.78}$$

At fixed energy E and 3-momentum $\overrightarrow{p}$, these become

$$\overrightarrow{\sigma}\cdot\overrightarrow{p}\,\psi_R=\left[\left(\beta E^2-\alpha p^2\right)+E\right]\psi_R-m\psi_L \tag{2.79}$$

$$\overrightarrow{\sigma}\cdot\overrightarrow{p}\,\psi_L=\left[\left(\beta E^2-\alpha p^2\right)-E\right]\psi_L+m\psi_R \tag{2.80}$$

where p is the magnitude of $\overrightarrow{p}$, or, since $\left(\overrightarrow{\sigma}\cdot\overrightarrow{p}\right)^2=p^2$,

$$\begin{aligned}&\left[\left(p^2+m^2\right)-\left[\left(\beta E^2-\alpha p^2\right)+E\right]^2\right]\psi_R\\&\qquad=-2m\left(\beta E^2-\alpha p^2\right)\psi_L\end{aligned} \tag{2.81}$$

$$\begin{aligned}&\left[\left(p^2+m^2\right)-\left[\left(\beta E^2-\alpha p^2\right)-E\right]^2\right]\psi_L\\&\qquad=2m\left(\beta E^2-\alpha p^2\right)\psi_R.\end{aligned} \tag{2.82}$$

We then obtain

$$A_+ A_- = -\left[2m\left(\beta E^2 - \alpha p^2\right)\right]^2 \tag{2.83}$$

$$A_+ = \left(p^2 + m^2\right) - \left[\left(\beta E^2 - \alpha p^2\right) + E\right]^2 \tag{2.84}$$

$$A_- = \left(p^2 + m^2\right) - \left[\left(\beta E^2 - \alpha p^2\right) - E\right]^2 \tag{2.85}$$

and (discarding the unphysical root)

$$E^2 = \left(p^2 + m^2\right) + \left(\beta E^2 - \alpha p^2\right)\left[2\left(E^2 - m^2\right)^{1/2} - \left(\beta E^2 - \alpha p^2\right)\right]. \tag{2.86}$$

There are four solutions to this equation:

$$E_1^2 = \frac{1 - 2\beta p + 2\alpha\beta p^2 - \sqrt{1 - 4\beta^2 m^2 - 4\beta p + 4\alpha\beta p^2}}{2\beta^2}, \tag{2.87}$$

$$E_2^2 = \frac{1 - 2\beta p + 2\alpha\beta p^2 + \sqrt{1 - 4\beta^2 m^2 - 4\beta p + 4\alpha\beta p^2}}{2\beta^2}, \tag{2.88}$$

$$E_3^2 = \frac{1 + 2\beta p + 2\alpha\beta p^2 - \sqrt{1 - 4\beta^2 m^2 + 4\beta p + 4\alpha\beta p^2}}{2\beta^2}, \tag{2.89}$$

$$E_4^2 = \frac{1 + 2\beta p + 2\alpha\beta p^2 + \sqrt{1 - 4\beta^2 m^2 + 4\beta p + 4\alpha\beta p^2}}{2\beta^2}. \tag{2.90}$$

When $p \ll \frac{1}{\alpha}, \frac{1}{\beta}$, E_1^2 and E_3^2 behave like the normal solution $E^2 \simeq p^2 + m^2$. Although we have the exact solutions, it is not easy to use them directly and we will make the approximation that the energy is large compared to the rest mass energy. If m^2 is totally neglected, for the moment, there are eight solutions

$$\begin{aligned} E &= \mp\frac{1}{2\beta} \pm \frac{1}{2\beta}\left[1 + 4\beta\left(\alpha p^2 \pm p\right)\right]^{1/2} \\ &= \mp\frac{1}{2\beta} \pm \frac{1}{2\beta}\left[(1 \pm 2\beta p)^2 + 4\beta\gamma p^2\right]^{1/2} \end{aligned} \tag{2.91}$$

where $\gamma = \alpha - \beta$ and the signs are independent.

The various solutions lead to interesting possibilities for new physics which will be considered in detail elsewhere. For the moment, however, consider only the normal branch, for which the first sign is $-$ and the last two signs are both $+$. The velocity is then

$$v = \partial E/\partial p \tag{2.92}$$

$$= \left[(1+2\beta p)^2 + 4\beta\gamma p^2\right]^{-1/2}(1+2\beta p+2\gamma p)$$

$$= \left[1 + 4\gamma \frac{p+\alpha p^2}{1+4\beta p+4\beta\alpha p^2}\right]^{1/2}. \tag{2.93}$$

It follows that

$$v > 1 \quad \text{if} \quad \alpha > \beta \quad \text{and} \quad v < 1 \quad \text{if} \quad \alpha < \beta. \tag{2.94}$$

As we will find below, the first possibility would imply vacuum Čerenkov radiation, and the second pair production in vacuum, so the only plausible possibility is

$$\alpha = \beta = \frac{1}{\bar{m}} \quad \text{which implies that} \quad v = 1. \tag{2.95}$$

(In the present paper we do not try to explain the origin of this condition, but simply accept it as a phenomenological constraint on a cosmological scale, far from local gravitational sources.) Then (2.91) becomes

$$E = \frac{\overline{m}}{2}\left[\mp 1 \pm \left(1 \pm \frac{2}{\overline{m}}p\right)\right] \tag{2.96}$$

$$= p, -p, -\overline{m}+p, -\overline{m}-p, \overline{m}+p, \overline{m}-p, p, -p$$

where

$$\overline{m} = \beta^{-1}. \tag{2.97}$$

All massless particles thus travel at the speed of light $c = 1$. As usual, the destruction operators for negative energies are reinterpreted as creation operators for antiparticles with positive energies [47]. The implications of negative group velocities for particles and antiparticles will be considered elsewhere, and the existence of very high-energy branches in the dispersion relation will be discussed below.

As E_1^2 and E_3^2 are the dispersion relations which are $E^2 \simeq p^2+m^2$ when $p \ll \frac{1}{\alpha}, \frac{1}{\beta}$, we work on these two possibilities. For a nonzero mass, but with $\beta = \alpha$, (2.87) and (2.89) gives

$$E_{1,3}^2 = \frac{1 \mp 2\alpha p + 2\alpha^2 p^2 - \sqrt{(1 \mp 2\alpha p)^2 - 4\alpha^2 m^2}}{2\alpha^2} \tag{2.98}$$

where the upper sign is for E_1^2 and lower sign is for E_3^2. We are primarily interested in particles with large energy, for which m^2 (or more precisely m^2/p^2) can be treated as a perturbation:

$$E^2 = \left[E^2\right]_{m^2=0} + \left[\partial E^2/\partial m^2\right]_{m^2=0} m^2. \tag{2.99}$$

From (2.98) we obtain

$$\begin{aligned} \partial E_{1,3}^2/\partial m^2 &= -\frac{1}{2\alpha^2} \frac{(-4\alpha^2)}{2\sqrt{(1 \mp 2\alpha p)^2 - 4\alpha^2 m^2}} \\ &= \frac{1}{\sqrt{(1 \mp 2\alpha p)^2 - 4\alpha^2 m^2}}, \end{aligned} \tag{2.100}$$

and

$$\left[\frac{\partial E_{1,3}^2}{\partial m^2}\right]_{m^2=0} = \left[1 \mp 2\alpha p\right]^{-1} \tag{2.101}$$

or

$$E_{1,3}^2 = p^2 + \frac{m^2}{1 \mp 2\alpha p} \tag{2.102}$$

to lowest order in m^2/p^2, which reproduces the usual result $E^2 = p^2 + m^2$ as $\alpha p \to 0$.

The particle velocity is then $v = \partial E/\partial p = \left(\partial E^2/\partial p\right)/\left(2E\right)$. or

$$\begin{aligned} v_{1,3} &= \left[1 \pm \frac{\alpha m^2}{p\left(1 \mp 2\alpha p\right)^2}\right]\left[1 + \frac{m^2}{p^2\left(1 \mp 2\alpha p\right)}\right]^{-1/2} \\ &\approx 1 - \frac{m^2}{2p^2}\frac{1 \mp 4\alpha p}{\left(1 \mp 2\alpha p\right)^2} \\ &= 1 - \frac{m^2}{2p^2}\left(1 - \frac{1}{\left(\left(2\alpha p\right)^{-1} \mp 1\right)^2}\right), \end{aligned} \tag{2.103}$$

so that

$$v \to 1 \text{ as } p \to \infty \tag{2.104}$$

and

$$v < 1 \text{ for } \begin{array}{ll} p < \overline{m}/4 & \text{for } E_1^2 \\ \text{any } p & \text{for } E_3^2 \end{array} . \tag{2.105}$$

Furthermore, it is easy to see that particles with $p > \overline{m}/4$ for E_1^2 will be superluminal by only an extremely small amount except when p lies in a narrow range of energies near $p = \overline{m}/2$ (i.e., $\alpha p = 1/2$): Letting $\alpha p = 1/2 + \delta$ in (2.103), we obtain

$$v - 1 \approx \frac{1}{2}\frac{m^2}{\overline{m}^2}\frac{1}{\delta^2}. \tag{2.106}$$

For example, if m is ~ 1 GeV and $\overline{m}$ were $\sim 10^{10}$ TeV, then $\delta \sim 10^{-4}$ would imply that $(v-1) \sim 10^{-18}$, and the deviation falls like $1/\delta^2$. However, it should also be emphasized that superluminal velocities of any size are not a violation of causality in the present theory, because all signals still propagate forward in time in the initial (preferred) frame of reference.

We have other two energy-momentum dispersion relation E_2^2 and E_4^2, and similarly we expand them with respect to m^2.

$$E_{2,4}^2 = \frac{1 \mp 2\alpha p + 2\alpha^2 p^2 + \sqrt{1 - 4\alpha^2 m^2 \mp 4\alpha p + 4\alpha^2 p^2}}{2\alpha^2} \tag{2.107}$$

$$\underset{m \to 0}{\to} \left(\alpha^{-1} \mp p\right)^2 . \tag{2.108}$$

where the upper sign is for E_2^2 and lower sign for E_4^2.

$$\frac{\partial E_{2,4}^2}{\partial m^2} = \frac{-1}{\sqrt{1 - 4\alpha^2 m^2 \mp 4\alpha p + 4\alpha p^2}} \underset{m \to 0}{\to} \frac{-1}{1 \mp 2\alpha p}. \tag{2.109}$$

Therefore, we then have

$$E_{2,4}^2 \approx \left(\overline{m} \mp p\right)^2 - \frac{m^2}{1 \mp 2p/\overline{m}}. \tag{2.110}$$

$$\begin{aligned} v_{2,4} &= \left[\mp \left(\overline{m} \mp p\right) + \frac{\mp m^2}{\overline{m}\left(1 \mp 2p/\overline{m}\right)^2}\right] \left[\left(\overline{m} \mp p\right)^2 - \frac{m^2}{1 \mp 2p/\overline{m}}\right]^{-1/2} \\ &\approx \mp \left[1 + \frac{\left(3\overline{m} \mp 4p\right)}{2\overline{m}\left(\overline{m} \mp p\right)^2 \left(1 \mp 2p/\overline{m}\right)^2}\right], \end{aligned} \tag{2.111}$$

so

$$v_{2,4} \to \mp 1 \text{ as } p \to \infty \tag{2.112}$$

and

$$v_{2,4} \to \mp \left(1 + \frac{3}{2}\frac{m^2}{\overline{m}^2}\right) \equiv v_0 \text{ as } p \to 0. \tag{2.113}$$

The particles with E_4^2 are then slightly superluminal. For example, if m is ~ 1 GeV and $\overline{m}$ were $\sim 10^{10}$ TeV, then $v_0 - 1$ would be $\sim 10^{-26}$. Again, however, a superluminal velocity of any size in the present theory does not imply a violation of causality.

Now let us turn to the GZK cutoff, [22],[53]-[58] which results from collision of a charged fermion with a blackbody photon. The incoming photon has energy ω and momentum $(-\omega \cos\theta, -\omega \sin\theta, 0)$ in units with $\hbar = c = 1$. The incoming fermion has mass m_a, energy E, and momentum $(p, 0, 0)$. The outgoing fermion has mass m_b, energy $E+\omega$, and momentum $(p-\omega\cos\theta, -\omega\sin\theta, 0)$. If ω is small (as it is for a blackbody photon), it is valid to use

$$\Delta E = \frac{\partial E}{\partial p_x}\Delta p_x + \frac{\partial E}{\partial p_y}\Delta p_y + \frac{\partial E}{\partial m^2}\Delta m^2 \tag{2.114}$$

with $\partial E/\partial p_k = v\,p_k/p$ and $v = \partial E/\partial p$, so that

$$1 + v\cos\theta = \frac{\partial E}{\partial m^2}\frac{\Delta m^2}{\omega} \tag{2.115}$$

and the threshold is for a head-on collision. Consider the normal branch of the dispersion relation, described by (2.101), (2.102), and (2.103). With $\partial E/\partial m^2 = \partial E^2/\partial m^2/\left(2E\right)$, (2.115) becomes

$$2\left(1+v\cos\theta\right)\left(1\mp 2\alpha p\right)p = \Delta m^2/\omega \tag{2.116}$$

where m^2 has been neglected in comparison to p^2. This quadratic equation in p has a solution only if

$$\begin{array}{ll} 2\left(1+v\cos\theta\right) > 8\alpha\,\Delta m^2/\omega & \text{for } E_1^2 \\ 2\left(1+v\cos\theta\right) > -8\alpha\,\Delta m^2/\omega & \text{for } E_3^2 \end{array}, \tag{2.117}$$

or

$$\begin{array}{ll} \overline{m} > 8\left(\Delta m^2/4\omega\right) & \text{for } E_1^2 \\ \text{any value of } \bar{m} & \text{for } E_3^2 \end{array}, \tag{2.118}$$

where again $\alpha^{-1} = \overline{m}$. The E_3 branch has a GZK cutoff at

$$p_{\mathrm{GZK}} = \left[-1+\sqrt{1+8\alpha\Delta m^2/\left(4\omega\right)}\right]/4\alpha \underset{\alpha\to 0}{\to} \Delta m^2/\left(4\omega\right).$$

Therefore, the E_3 branch is a modified version of the usual physical branch, but the E_1 branch can be interpreted as a totally new physical branch, because of its stronger Lorentz violation.

If $\overline{m}$ is lower than eight times the standard GZK cutoff energy, therefore, the present theory implies that the GZK cutoff is eliminated for one of the physical branches (the E_1 branch). The reason for this is that the $(1 - 2p/\overline{m})$ factor in (2.102) and (2.116) tends to push the cutoff up to higher energies even if $\overline{m}$ is large, and completely eliminates it if $\overline{m}$ falls below $2\,\Delta m^2/\omega$.

Finally, let us return to the standard astrophysical threat to a Lorentz-violating theory, that it may lead to disagreement with the observations of high-energy matter particles or photons, including prediction of new processes in the vacuum which are not observed. An example is vacuum Čerenkov radiation. Conservation of energy and momentum implies that

$$-\omega = \Delta E = \frac{\partial E}{\partial p_x}\Delta p_x + \frac{\partial E}{\partial p_y}\Delta p_y = \frac{\partial E}{\partial p}\left(-\omega \cos\theta\right)$$

so this process can occur if

$$v = 1/\cos\theta \geq 1. \tag{2.119}$$

If we were to have $\beta < \alpha$, the particle velocity at high momentum would be greater than the velocity of light, and there would be a radiation of photons in vacuum which is in conflict with observation [22].

Next consider the process *photon* $\rightarrow e^+e^-$, which will occur if

$$2E\left(p\right) = \omega = 2p\cos\theta. \tag{2.120}$$

The normal branch for $E\left(p\right)$ corresponds to the choice of signs $-, +, +$ in (2.91). For 20 or 50 TeV photons, it is reasonable to assume $\alpha p, \beta p \ll 1$, and keep only the terms of first and second order in α and β. Then (2.91) gives $E_{1,3}\left(p\right) \approx p \mp \gamma p^2$. When the mass term in $E\left(p\right)^2$ is also treated only to lowest order in α and β, it is simply m^2. (E.g., see (2.102).) For a massive particle, therefore, $E\left(p\right)$ becomes

$$E_{1,3}\left(p\right) \approx \left[\left(p \mp \gamma p^2\right)^2 + m^2\right]^{1/2} \approx p \mp \gamma p^2 + m^2/2p$$

and the condition for vacuum pair production is

$$1 \mp \gamma p + m^2/2p^2 = \cos\theta. \tag{2.121}$$

This will have a solution if

$$p^3 > m^2/2|\gamma| \text{ when } \gamma > 0 \text{ for } E_1 \text{ and } \gamma < 0 \text{ for } E_3. \tag{2.122}$$

Since observations indicate that 20 TeV photons do not decay in vacuum, $|\gamma|^{-1}$ must lie above the Planck energy.

If $\beta = \alpha$, or $\gamma = 0$, the unphysical processes considered above do not occur. More broadly, since many features of Lorentz invariance are retained in the present theory (including rotational invariance and the same velocity c for all massless particles) it appears that the theory is consistent with experiment and observation. The theory is also fundamental, rather than *ad hoc*, and it leads to various new predictions. Here we have emphasized one feature: the behavior of fermions at extremely high energy, and the possible implications for the GZK cutoff.

2. Lorentz Symmetry Violation Term

Here we study in more detail the Lorentz-violating term

$$\mathcal{L}_{\text{violation}} = \psi_L^\dagger \left(-\frac{1}{2M} \eta^{\mu\nu} \partial_\nu \partial_\mu \right) \psi_L.$$

To use (2.18) and (2.19), we write the spinor indices explicitly as

$$\mathcal{L}_{\text{violation}} = \psi_{L\dot{\alpha}}^\dagger \left(-\frac{1}{2M} \sigma^{0\dot{\alpha}\alpha} \eta^{\mu\nu} \partial_\nu \partial_\mu \right) \psi_{L\alpha}. \tag{2.123}$$

Then under a Lorentz transformation,

$$\begin{aligned}\mathcal{L}_{\text{violation}} &\to \psi'^{\dagger}_{L\dot{\alpha}}\left(x'\right)\left(-\frac{1}{2M}\bar{\sigma}^{0\dot{\alpha}\alpha}\eta^{\mu\nu}\partial'_{\nu}\partial'_{\mu}\right)\psi'_{L\alpha}\left(x'\right)\\ &= \left(-\frac{1}{2M}\right)\psi^{\dagger}_{\dot{\beta}}\left(\Lambda^{-1}x\right)\left(\delta^{\dot{\beta}}{}_{\dot{\alpha}}-\frac{i}{2}\omega_{\mu\nu}\left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\beta}}{}_{\dot{\alpha}}\right)\sigma^{0\dot{\alpha}\alpha}\\ &\cdot\eta^{\mu\nu}\left(\Lambda^{-1}\right)^{\xi}{}_{\nu}\partial_{\xi}\left(\Lambda^{-1}\right)^{\kappa}{}_{\mu}\partial_{\kappa}\left(\delta_{\alpha}{}^{\beta}+\frac{i}{2}\omega_{\mu\nu}\left(\sigma^{\mu\nu}\right)_{\alpha}{}^{\beta}\right)\psi_{\beta}\left(\Lambda^{-1}x\right)\\ &= \left(-\frac{1}{2M}\right)\psi^{\dagger}_{\dot{\beta}}\left(\Lambda^{-1}x\right)\left[\bar{\sigma}^{0\dot{\beta}\beta}-\frac{i}{2}\omega_{\mu\nu}\left(\bar{\sigma}^{\mu\nu}\bar{\sigma}^{0}-\bar{\sigma}^{0}\sigma^{\mu\nu}\right)^{\dot{\beta}\beta}\right]\eta^{\kappa\xi}\partial_{\xi}\partial_{\kappa}\psi_{\beta}\left(\Lambda^{-1}x\right),\end{aligned} \tag{2.124}$$

where we have used (2.4). Therefore, if the second term in $[\cdots]$ vanishes, the action is invariant.

a. Rotation

When the transformation is a rotation, we can take $\mu,\nu \to i.j$, where i and j are space coordinate indices. Then the second term is

$$\begin{aligned}\omega_{\mu\nu}\left(\bar{\sigma}^{\mu\nu}\sigma^{0}-\sigma^{0}\sigma^{\mu\nu}\right) &\to \omega_{ij}\left(\bar{\sigma}^{ij}\bar{\sigma}^{0}-\bar{\sigma}^{0}\sigma^{ij}\right)\\ &= \omega_{ij}\frac{i}{2}\left[\bar{\sigma}^{i}\sigma^{j}-\bar{\sigma}^{j}\sigma^{i}\right]\bar{\sigma}^{0}-\omega_{ij}\frac{i}{2}\bar{\sigma}^{0}\left[\sigma^{i}\bar{\sigma}^{j}-\sigma^{j}\bar{\sigma}^{i}\right]\\ &= i\omega_{ij}\left[-\sigma^{i}\sigma^{j}+\sigma^{i}\sigma^{j}\right]\\ &= 0.\end{aligned} \tag{2.125}$$

Therefore, the Lorentz-symmetry violating term is invariant under rotations.

b. Boost

When the transformation is a boost, we can take $\mu \to 0$ and $\nu \to i$. Then the second term is

$$\begin{aligned}
\omega_{\mu\nu}\left(\bar{\sigma}^{\mu\nu}\sigma^0 - \sigma^0\sigma^{\mu\nu}\right) &\to \omega_{0i}\left(\bar{\sigma}^{0i}\bar{\sigma}^0 - \bar{\sigma}^0\sigma^{0i}\right) \\
&= \omega_{0i}\frac{i}{2}\left[\bar{\sigma}^0\sigma^i - \bar{\sigma}^i\sigma^0\right]\bar{\sigma}^0 - \omega_{ij}\frac{i}{2}\bar{\sigma}^0\left[\sigma^0\bar{\sigma}^i - \sigma^i\bar{\sigma}^0\right] \\
&= 2i\omega_{0i}\sigma^i \\
&\neq 0. \qquad (2.126)
\end{aligned}$$

Therefore, the Lorentz-violating term is not invariant under a boost.

3. CPT Violation in Our Model

We saw that Lorentz symmetry is broken because of non-invariance under a boost (although there is invariance under a rotation). The Lorentz-symmetry violating term is

$$\psi_R^\dagger \frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu\psi_R - \psi_L^\dagger\frac{1}{2M}\eta^{\mu\nu}\partial_\nu\partial_\mu\psi_L, \qquad (2.127)$$

where each term is independently Lorentz-violating. Even if Lorentz symmetry is broken, this does not mean that CPT is also violated. Therefore, we now consider whether our Lorentz-violating terms are CPT conserving.

It is convenient to momentarily change from 2 component notation to 4 component notation, with the Lorentz-violating terms rewritten as

$$\begin{aligned}
\psi_R^\dagger\partial^\mu\partial_\mu\mathbf{1}_{2\times2}\psi_R - \psi_L^\dagger\partial^\mu\partial_\mu\mathbf{1}_{2\times2}\psi_L &= \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix}\partial^\mu\partial_\mu\begin{pmatrix} \mathbf{1}_{2\times2} & 0 \\ 0 & -\mathbf{1}_{2\times2}\end{pmatrix}\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix} \\
&= \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix} g^{\mu\nu}\partial_\mu\partial_\nu\gamma^5\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}. \qquad (2.128)
\end{aligned}$$

We will study the behavior of this term under parity, time reversal, and charge conjugation operations. We follow Ref. [59] in treating P, T, C for a 4 component spinor.

a. Parity

Parity operation for a 4 component spinor is given by

$$P\Psi\left(t,\vec{x}\right)P=\gamma^{0}\Psi\left(t,-\vec{x}\right), \tag{2.129}$$

so

$$\begin{aligned}P\Psi^{\dagger}\left(t,\vec{x}\right)g^{\mu\nu}\partial_{\nu}\partial_{\mu}\gamma^{5}\Psi\left(t,\vec{x}\right)P &= \Psi^{\dagger}\left(t,-\vec{x}\right)\gamma^{0}\left(-1\right)^{\nu}\left(-1\right)^{\mu}g^{\mu\nu}\partial_{\nu}\partial_{\mu}\gamma^{5}\gamma^{0}\Psi\left(t,-\vec{x}\right)\\ &= -\Psi^{\dagger}\left(t,-\vec{x}\right)\left(-1\right)^{\nu}\left(-1\right)^{\mu}g^{\mu\nu}\partial_{\nu}\partial_{\mu}\gamma^{5}\Psi\left(t,-\vec{x}\right)\\ &= -\Psi^{\dagger}\left(t,-\vec{x}\right)g^{\mu\nu}\partial_{\nu}\partial_{\mu}\gamma^{5}\Psi\left(t,-\vec{x}\right),\end{aligned} \tag{2.130}$$

where $(-1)^{\mu}\equiv 1$ for $\mu=0$ and $(-1)^{\mu}\equiv -1$ for $\mu=1,2,3$. This term is **odd** under P.

b. Time Reversal

Time reversal for a 4 component spinor is given by

$$T\Psi\left(t,\vec{x}\right)T=\left(\gamma^{1}\gamma^{3}\right)\Psi\left(-t,\vec{x}\right), \tag{2.131}$$

so

$$\begin{aligned}T\Psi^{\dagger}\left(t,\vec{x}\right)\partial_{\nu}\partial_{\mu}\gamma^{5}\Psi\left(t,\vec{x}\right)T &= \Psi^{\dagger}\left(-t,\vec{x}\right)\left(\gamma^{3}\gamma^{1}\right)\left(-\left(-1\right)^{\nu}\right)\left(-\left(-1\right)^{\mu}\right)\\ &\quad\cdot g^{\mu\nu}\partial_{\nu}\partial_{\mu}\gamma^{5}\left(\gamma^{1}\gamma^{3}\right)\Psi\left(-t,\vec{x}\right)\\ &= \Psi^{\dagger}\left(-t,\vec{x}\right)g^{\mu\nu}\partial_{\nu}\partial_{\mu}\gamma^{5}\Psi\left(-t,\vec{x}\right),\end{aligned} \tag{2.132}$$

and it is **even** under T.

c. Charge Conjugation

Charge conjugation for a 4 component spinor is given by

$$C\Psi\left(t,\vec{x}\right)C = -i\gamma^2\Psi^*\left(t,\vec{x}\right), \tag{2.133}$$

so

$$\begin{aligned}
C\Psi^\dagger\left(t,\vec{x}\right)\partial_\nu\partial_\mu\gamma^5\Psi\left(t,\vec{x}\right)C &= \Psi^T\left(t,\vec{x}\right)\gamma^2 i g^{\mu\nu}\partial_\nu\partial_\mu\gamma^5\left(-i\right)\gamma^2\Psi^*\left(t,\vec{x}\right) \\
&= -\Psi^T\left(t,\vec{x}\right)g^{\mu\nu}\partial_\nu\partial_\mu\gamma^5\Psi^*\left(t,\vec{x}\right) \\
&= g^{\mu\nu}\partial_\nu\partial_\mu\Psi^\dagger\left(t,\vec{x}\right)\gamma^5\Psi\left(t,\vec{x}\right) \\
&\rightarrow \Psi^\dagger\left(t,\vec{x}\right)g^{\mu\nu}\partial_\nu\partial_\mu\gamma^5\Psi\left(t,\vec{x}\right),
\end{aligned} \tag{2.134}$$

and it is **even** under C.

Therefore, CPT yields for this term

$$\left(-1\right)\cdot\left(+1\right)\cdot\left(+1\right) = -1, \tag{2.135}$$

and it turns out that the Lorentz-violating term is CPT **odd**.

The behavior of this term under P, T, C, and CPT is summarized in Table IX.

Table IX. P, T, C, and CPT of our Lorentz-symmetry violating term.

	$\Psi^{\dagger}(t,\vec{x})\, g^{\mu\nu}\partial_{\nu}\partial_{\mu}\gamma^{5}\Psi(t,\vec{x})$
P	-1
T	$+1$
C	$+1$
CPT	-1

.

CHAPTER III

ORIGIN OF GRAVITATIONAL AND GAUGE INTERACTIONS

A. Introduction

In this chapter, we derive both gauge and gravitational interactions from the following fundamental action, which is itself derived from a microscopic statistical picture in Chapters IV and V (see (4.117)):

$$S = \int d^D x \left[\frac{1}{2m} h^{MN} \partial_M \Psi^\dagger \partial_N \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} b \left(\Psi^\dagger \Psi \right)^2 \right] \tag{3.1}$$

with

$$\Psi = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} \quad , \quad z = \begin{pmatrix} z_b \\ z_f \end{pmatrix} . \tag{3.2}$$

Here $h^{MN} = \delta^{MN}$ is the initial metric tensor in a flat D-dimensional Euclidean space, as discussed in Chapter V. This action has a "primitive supersymmetry", in the sense that the initial bosonic fields z_b and fermionic fields z_f are treated in exactly the same way. The only difference is that the z_b are ordinary complex numbers whereas the z_f are anticommuting Grassmann numbers. (Here, as in Ref. [21], "supersymmetry" is taken to have its general definition [60],[61]: An action is supersymmetric if it is invariant under a transformation which converts fermions to bosons and vice-versa.) We will argue that standard physics can emerge from (3.1) at energies that are far below the Planck scale, provided that specific kinds of topological defects are included in the theory. For example, one can obtain an $SO(10)$ grand-unified theory, containing both the Standard Model and a natural mechanism for small neutrino masses [8], [52],[62]-[71].

B. Canonical Quantization in Lorentzian Spacetime

Functional-integral quantization can ordinarily be replaced by canonical quantization, or vice-versa [72], through a procedure that is similar to that for a single particle. In the present theory, whether this can be done consistently is a nontrivial issue, because the resulting field theory has some very unconventional features. In the present section it will simply be assumed that one can define quantized fields $\hat{\Psi}$ etc. in the usual way [72]-[80].

After a change from functional-integral to canonical quantization, and an inverse Wick rotation from Euclidean to Lorentzian time (with $S_L = iS$), the action (3.1) becomes

$$\hat{S}_L = -\int d^D x \left[\frac{1}{2m} \eta^{MN} \partial_M \hat{\Psi}_L^{\dagger} \partial_N \hat{\Psi}_L - \mu \hat{\Psi}_L^{\dagger} \hat{\Psi}_L + \frac{1}{2} b \left(\hat{\Psi}_L^{\dagger} \hat{\Psi}_L \right)^2 \right] \tag{3.3}$$

where $\eta^{MN} = diag(-1, 1, ..., 1)$. (We intend to consider the philosophical problem of transformation from Euclidean to Lorentzian time elsewhere. Here we adopt the point of view that one is allowed simply to perform mathematical transformations starting with an abstract initial theory, as long as the transformations are mathematically consistent and the final version of the theory correctly describes our observed physical reality. It is important to recognize, however, that **all** Euclidean times are effectively mapped into each single Lorentzian time via an inverse Wick rotation.) The notation of (3.3) is rather awkward, however, so for the remainder of the paper we will let

$$\hat{S}_L \to S, \ \hat{\Psi}_L \to \Psi \tag{3.4}$$

with the understanding that these are now quantized operators in Lorentzian spacetime. It is also understood that raising and lowering of indices is now done with the

Minkowski metric tensor:

$$A^{\mu}B_{\mu} = \eta^{\mu\nu}A_{\mu}B_{\nu} \quad \text{or in } D \text{ dimensions} \quad A^{M}B_{M} = \eta^{MN}A_{M}B_{N}. \tag{3.5}$$

Later in this section we will introduce the metric tensor associated with gravity and general coordinate transformations. To avoid confusion, this metric tensor $g_{\mu\nu}$ will always be shown explicitly, and simple raising and lowering of indices will always have the interpretation (3.5).

With the above change of notation, and after an integration by parts, (3.3) becomes

$$S = -\int d^{D}x \left[-\frac{1}{2m}\Psi^{\dagger}\partial^{M}\partial_{M}\Psi - \mu\Psi^{\dagger}\Psi + \frac{1}{2}b\left(\Psi^{\dagger}\Psi\right)^{2}\right]. \tag{3.6}$$

The resulting equation of motion is

$$\left[-\frac{1}{2m}\partial^{M}\partial_{M} - \mu + V_{vac} + b\Delta\left(\Psi^{\dagger}\Psi\right)\right]\Psi = 0 \quad , \quad V_{vac} = b\left\langle\Psi^{\dagger}\Psi\right\rangle_{vac} \tag{3.7}$$

where $\langle\cdots\rangle_{vac}$ represents a vacuum expectation value, and

$$\Psi^{\dagger}\Psi = \left\langle\Psi^{\dagger}\Psi\right\rangle_{vac} + \Delta\left(\Psi^{\dagger}\Psi\right). \tag{3.8}$$

For the remainder of this section, we will consider either the vacuum or a noninteracting free field in the vacuum. We then have

$$\left(-\frac{1}{2m}\partial^{M}\partial_{M} - \mu + V_{vac}\right)\Psi_{b} = 0 \quad , \quad \left(-\frac{1}{2m}\partial^{M}\partial_{M} - \mu + V_{vac}\right)\Psi_{f} = 0. \tag{3.9}$$

It will be assumed that the physical vacuum contains a condensate whose order parameter

$$\Psi_{cond} = \langle\Psi_{b}\rangle_{vac} \tag{3.10}$$

has the form

$$\Psi_{cond} = U\, n_{cond}^{1/2} \eta_0 \tag{3.11}$$

$$U^\dagger U = \eta_0^\dagger \eta_0 = 1. \tag{3.12}$$

(As discussed below Ψ_{cond} is dominantly due to a GUT field that condenses in the very early universe. In the present theory, it is not static, but instead exhibits rotations in space and time that are described by U. Other vacuum fields and physical fields are viewed as "moving with the condensate", in essentially the same way that particles in an ordinary superfluid flow together. In the analogy of a superfluid, the order parameter rotates in the complex plane, and this rotation gives the superfluid velocity.) It will also be assumed that the order parameter can be written in the form

$$\Psi_{cond} = \Psi_{c-ext}\left(x^\mu\right)\; \Psi_{c-int}\left(x^m, x^\mu\right) \tag{3.13}$$

$$\Psi_{c-ext}\left(x^\mu\right) = U_{ext}\left(x^\mu\right)\; n_{ext}^{1/2}\left(x^\mu\right)\eta_{ext} \tag{3.14}$$

$$\Psi_{c-int}\left(x^m, x^\mu\right) = U_{int}\;\left(x^m, x^\mu\right) n_{int}^{1/2} \eta_{int} \tag{3.15}$$

where η_{ext} and η_{int} are constant vectors. Let us define external and internal "superfluid velocities" by

$$m v_M = -iU^{-1}\partial_M U \tag{3.16}$$

or

$$m v_\mu = -iU_{ext}^{-1}\partial_\mu U_{ext} - iU_{int}^{-1}\partial_\mu U_{int} \tag{3.17}$$

$$m v_m \;= -iU_{int}^{-1}\partial_m U_{int}. \tag{3.18}$$

The fact that U is unitary implies that $\partial_M U^\dagger U = -U^\dagger \partial_M U$ with $U^\dagger = U^{-1}$, or

$$mv_M = i\partial_M U^\dagger U \tag{3.19}$$

so that

$$v_M^\dagger = v_M \,. \tag{3.20}$$

Here we will initially consider the case that

$$\partial_\mu U_{int} = 0 \tag{3.21}$$

in which case there are separate equations of motion for external and internal space-time:

$$\left(-\frac{1}{2m}\partial^\mu \partial_\mu - \mu_{ext}\right)\Psi_{c-ext} = 0 \tag{3.22}$$

$$\left(-\frac{1}{2m}\partial^m \partial_m - \mu_{int} + V_{vac}\right)\Psi_{c-int} = 0 \tag{3.23}$$

with $\mu_{int} = \mu - \mu_{ext}$. The quantities V_{vac}, μ_{int}, and Ψ_{int} are allowed to have a slow parametric dependence on x^μ, as long as $\partial^\mu \partial_\mu \Psi_{int}$ is negligible.

When (3.14), (3.17), and (3.21) are used in (3.22), we obtain

$$\eta_{ext}^\dagger n_{ext}^{1/2}[\left(\frac{1}{2}mv^\mu v_\mu - \frac{1}{2m}\partial^\mu \partial_\mu - \mu_{ext}\right) - i\left(\frac{1}{2}\partial^\mu v_\mu + v^\mu \partial_\mu\right) n_{ext}^{1/2}\eta_{ext}] = 0, \tag{3.24}$$

and its Hermitian conjugate

$$\eta_{ext}^\dagger n_{ext}^{1/2}[\left(\frac{1}{2}mv^\mu v_\mu - \frac{1}{2m}\partial^\mu \partial_\mu - \mu_{ext}\right) + i\left(\frac{1}{2}\partial^\mu v_\mu + v^\mu \partial_\mu\right) n_{ext}^{1/2}\eta_{ext}] = 0. \tag{3.25}$$

Subtraction gives the equation of continuity

$$\partial_\mu j^\mu_{ext} = 0 \quad , \quad j^\mu_{ext} = \eta^\dagger_{ext}\, n_{ext} v^\mu \eta_{ext} \tag{3.26}$$

and addition gives the Bernoulli equation of the condensate

$$\frac{1}{2} m \bar{v}^2_{ext} + P_{ext} = \mu_{ext} \tag{3.27}$$

where

$$\bar{v}^2_{ext} = \eta^\dagger_{ext}\, v^\mu v_\mu\, \eta_{ext} \tag{3.28}$$

$$P_{ext} = -\frac{1}{2m} n^{-1/2}_{ext} \partial^\mu \partial_\mu n^{1/2}_{ext}. \tag{3.29}$$

In the present theory, the order parameter in external spacetime, Ψ_{ext}, has the symmetry group $U(1) \times SU(2)$. The "superfluid velocity" in external spacetime, v_μ, can then be written in terms of the identity matrix σ^0 and Pauli matrices σ^a :

$$v^\mu = v^\mu_\alpha \sigma^\alpha \quad , \quad \mu, \alpha = 0, 1, 2, 3. \tag{3.30}$$

It is assumed that the basic texture of the order parameter is such that

$$v^0_k = v^a_0 = 0 \quad , \quad k, a = 1, 2, 3 \tag{3.31}$$

to a good approximation, yielding the simplification

$$\frac{1}{2} m v^{\alpha\mu} v^\alpha_\mu + P_{ext} = \mu_{ext}. \tag{3.32}$$

Letting Ψ_a represent either the general bosonic field Ψ_b or the general fermionic field Ψ_f, which interacts only with the condensate and other vacuum fields, (3.6) gives

$$S_a = -\int d^D x\, \Psi^\dagger_a \left(-\frac{1}{2m} \partial^M \partial_M - \mu + V_{vac} \right) \Psi_a. \tag{3.33}$$

Since Ψ_a satisfies a linear equation involving a Hermitian operator, it can be written in the form

$$\Psi_a\left(x^\mu, x^m\right) = \widetilde{\psi}_a^r\left(x^\mu\right)\psi_r^{int}\left(x^m\right) \tag{3.34}$$

with a summation implied over repeated indices, as usual. The $\widetilde{\psi}_a^r$ are field operators and the ψ_r^{int} are a complete set of basis functions in the internal space, which are required to be orthonormal,

$$\int d^{D-4}x\, \psi_r^{int\dagger}\left(x^m\right)\psi_{r'}^{int}\left(x^m\right) = \delta_{rr'}\,, \tag{3.35}$$

and to satisfy the internal equation of motion

$$\left(-\frac{1}{2m}\partial^m\partial_m - \mu_{int} + V_{vac}\right)\psi_r^{int}\left(x^m\right) = \varepsilon_r\psi_r^{int}\left(x^m\right). \tag{3.36}$$

(The ψ_r^{int} are allowed to have a slow parametric dependence on x^μ, as long as $\partial^\mu\partial_\mu\psi_r^{int}$ is negligible.) As usual, only the zero modes with $\varepsilon_r = 0$ will be kept, since the higher energies involve nodes in the internal space and are comparable to m_P. When (3.34)-(3.36) are used in (3.33), the result is

$$S_a = -\int d^4x\, \widetilde{\psi}_a^\dagger\left(-\frac{1}{2m}\partial^\mu\partial_\mu - \mu_{ext}\right)\widetilde{\psi}_a \tag{3.37}$$

where $\widetilde{\psi}_a$ is the vector with components $\widetilde{\psi}_a^r$.

C. Origin of Gravitational Interaction

Let $\widetilde{\psi}_a$ be rewritten in the form

$$\widetilde{\psi}_a\left(x^\mu\right) = U_{ext}\left(x^\mu\right)\psi_a\left(x^\mu\right). \tag{3.38}$$

(The 2×2 matrix U_{ext} multiplies each of the 2-component operators $\widetilde{\psi}_a^r$.) Here ψ_a has a simple interpretation: It is the field seen by an observer in the frame of reference that is moving with the condensate of the external space. In the present theory, the

GUT condensate Ψ_{cond} forms in the very early universe, and the other bosonic and fermionic fields Ψ_a are subsequently born into it. It is therefore natural to view them from the perspective of the condensate.

Equation (3.38) is, in fact, exactly analogous to rewriting the wavefunction of a particle in an ordinary superfluid moving with velocity v_s: $\psi'_p(x) = \exp(iv_s x)\,\psi_p(x)$. Here ψ_p and ψ'_p are the wavefunctions before and after a Galilean boost to the superfluid's frame of reference.

When (3.38) is substituted into (3.37), the result is

$$S_a = -\int d^4x\ \psi_a^\dagger\Big[(\frac{1}{2}mv^\mu v_\mu - \frac{1}{2m}\partial^\mu\partial_\mu) - \mu_{ext} - i\left(\frac{1}{2}\partial^\mu v_\mu + v^\mu\partial_\mu\right)\psi_a\Big]. \tag{3.39}$$

If n_s and v_μ are slowly varying, so that P_{ext} and $\partial^\mu v_\mu$ can be neglected, (3.32) yields the simplification

$$S_a = \int d^4x\ \psi_a^\dagger\left(\frac{1}{2m}\partial^\mu\partial_\mu + iv^\mu_\alpha\sigma^\alpha\partial_\mu\right)\psi_a. \tag{3.40}$$

In the present theory, the gravitational vierbein is interpreted as the "superfluid velocity" associated with the GUT condensate Ψ_{cond}:

$$e^\mu_\alpha = v^\mu_\alpha. \tag{3.41}$$

The form of the action of the bosonic fields and the fermionic fields are same. When $p^\mu \ll \bar{m}$, the first term of the action is negligible and we obtain

$$S_a \to \int d^4x\ \psi_a^\dagger\left(ie^\mu_\alpha\sigma^\alpha\partial_\mu\right)\psi_a. \tag{3.42}$$

Then at low energy, we obtain the standard spin 1/2 fermionic action which interacts with the gravitational field e^μ_α. On the other hand, we also initially obtain spin 1/2 bosons which will be considered below (and later transformed to scalar bosons and auxiliary fields).

D. Origin of Gauge Interaction

Let us now relax assumption (3.21) and allow U_{int} to vary with the external coordinates x^{μ}. It is convenient to write

$$\begin{aligned}\Psi_{c-int}\left(x^{m}\right) &= \widetilde{U}_{int}\left(x^{\mu},x^{m}\right)\bar{\Psi}_{c-int}\left(x^{m}\right) \\ &= \widetilde{U}_{int}\left(x^{\mu},x^{m}\right)\bar{U}_{int}\left(x^{m}\right)n_{int}^{1/2}\left(x^{m}\right)\eta_{int}\end{aligned} \tag{3.43}$$

where $n_{int}\left(x^{m}\right)=\bar{\Psi}^{\dagger}_{c-int}\left(x^{m}\right)\bar{\Psi}_{c-int}\left(x^{m}\right)$ and $\bar{\Psi}_{int}$ still satisfies the internal equation of motion

$$\left(-\frac{1}{2m}\partial^{m}\partial_{m}-\mu_{int}+V_{vac}\right)\bar{\Psi}_{int}\left(x^{m}\right)=0. \tag{3.44}$$

This is a nonlinear equation because V_{vac} is largely determined by n_{int}.

The internal basis functions satisfy (3.36) with $\varepsilon_r=0$:

$$\left(-\frac{1}{2m}\partial^{m}\partial_{m}-\mu_{int}+V_{vac}\right)\psi_{r}^{int}\left(x^{m}\right)=0. \tag{3.45}$$

This is a linear equation because $V_{vac}\left(x^{m}\right)$ is now regarded as a known function.

If the vacuum of the internal space had a trivial topology, the solutions to (3.44) and (3.45) would be trivial, and the resulting universe would presumably not support nontrivial structures such as intelligent life. The full path integral involving (1.1) contains all configurations of the fields, however, including those with nontrivial topologies. In the present theory, the “geography” of the universe inhabited by human beings involves an internal instanton in

$$d=D-4 \tag{3.46}$$

dimensions which is analogous to a $U(1)$ vortex in 2 dimensions or an $SU(2)$ instanton in 4 Euclidean dimensions. The standard features of four-dimensional physics – including gauge symmetries and chiral fermions – arise from the presence of this instanton.

In the following, it is not necessary to have a detailed knowledge of the internal instanton. The only property required is a d-dimensional spherical symmetry for the internal condensate, and, as a result, for the functions $\widetilde{\psi}_r^{int}$ defined by

$$\psi_r^{int} = \bar{U}_{int} \widetilde{\psi}_r^{int}. \tag{3.47}$$

To be specific, it is required that

$$K_i \, \widetilde{\psi}_r^{int} = 0 \tag{3.48}$$

where

$$K_i = K_i^n \partial_n \tag{3.49}$$

is a Killing vector associated with the spherical symmetry of the internal metric tensor g_{mn} defined below. When K_i corresponds to the generators of group $SO\,(N)$, i and n are $i = 1, \cdots, N\,(N-1)\,/2$ and $n = 1, \cdots, N$. At a given point, the derivatives of (3.49) involve only the $(d-1)$ angular coordinates, and not the radial coordinate r, so (3.48) states that n_{int} and the $\widetilde{\psi}_r^{int}$ are functions only of r.

The vierbein e_α^μ of external spacetime was defined in (3.41). It is convenient to define the remaining components of the vielbein in a slightly different way, by representing mv_M in terms of a set of matrices σ^A,

$$v_M = v_{MA}\sigma^A = v_{M\alpha}\sigma^\alpha + v_{Mc}\sigma^c, \tag{3.50}$$

and letting

$$e_{Mc} = -v_{Mc} \quad , \quad M = 0, 1, ..., D-1 \quad , \quad c \geq 4. \tag{3.51}$$

(The σ^α are associated with U_{ext}, and the σ^c with U_{int}. Since (3.18) implies that $v_{m\alpha} = 0$, all the nonzero e_{MA} have now been specified.) When (3.21) holds, the only nonzero components of the metric tensor are

$$g^{\mu\nu} = \eta^{\alpha\beta} e^\mu_\alpha e^\nu_\beta. \tag{3.52}$$

and

$$g_{mn} = e_{mc} e_{nc} \tag{3.53}$$

which are respectively associated with external spacetime and the internal space. More generally, however, mv_μ contains a contribution

$$mv_{\mu c}\sigma^c = -i\widetilde{U}_{int}^{-1}\left(x^\mu, x^m\right)\partial_\mu \widetilde{U}_{int}\left(x^\mu, x^m\right) \tag{3.54}$$

so that $e_{\mu c}$ is nonzero and the metric tensor has off-diagonal components

$$g_{\mu m} = e_{\mu c} e_{mc}. \tag{3.55}$$

In the present theory, just as in classic Kaluza-Klein theories, it is appropriate to write

$$e_{\mu c} = A^i_\mu K^n_i v_{nc} \quad , \quad g_{\mu m} = A^i_\mu K^n_i g_{mn} \tag{3.56}$$

or, for later convenience,

$$mv_{\mu c}\sigma^c = -A^i_\mu \sigma_i \tag{3.57}$$

$$\sigma_i = mK^n_i v_{nc}\sigma^c. \tag{3.58}$$

For simplicity of notation, let

$$\langle r|Q|s\rangle = \int d^d x\, \psi^{int\dagger}_r Q \psi^{int}_s \quad \text{with} \quad \langle r|s\rangle = \delta_{rs} \tag{3.59}$$

for any operator Q, so that (3.47)-(3.49) and (3.18) give

$$\langle r|\left(-iK_i\right)|s\rangle = \langle r|\left(-iK_i^n\right)\left(imv_n\right)|s\rangle = \langle r|\sigma_i|s\rangle\,. \tag{3.60}$$

With the definition

$$t_i^{rs} = \langle r|\left(-iK_i\right)|s\rangle \tag{3.61}$$

we then have

$$\langle r|\sigma_i|s\rangle = t_i^{rs}. \tag{3.62}$$

The Killing vectors have an algebra

$$K_iK_j - K_jK_i = -c_{ij}^k K_k \tag{3.63}$$

or

$$\left(-iK_i\right)\left(-iK_j\right) - \left(-iK_j\right)\left(-iK_i\right) = ic_{ij}^k\left(-iK_k\right) \tag{3.64}$$

so the same is true of the matrices t_i^{rs}:

$$t_it_j - t_jt_i = ic_{ij}^k t_k. \tag{3.65}$$

With the more general version of (3.34) and (3.38),

$$\Psi_a\left(x^\mu, x^m\right) = U_{ext}\left(x^\mu\right)\widetilde{U}_{int}\left(x^\mu, x^m\right)\psi_a^r\left(x^\mu\right)\psi_r^{int}\left(x^m\right), \tag{3.66}$$

we have

$$\partial_\mu\Psi_a = U_{ext}\left(x^\mu\right)\widetilde{U}_{int}\left(x^\mu, x^m\right)\left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha + imv_{\mu c}\sigma^c\right)\psi_a^r\psi_r^{int} \tag{3.67}$$

and

$$\begin{aligned}
&\int d^dx\, \Psi_a^\dagger\, \partial^\mu \partial_\mu\, \Psi_a \\
&= \int d^dx\, \psi_r^{int\dagger} \psi_a^{r\dagger} \eta^{\mu\nu} \left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha + imv_{\mu c}\sigma^c\right) \left(\partial_\nu + imv_{\nu\beta}\sigma^\beta + imv_{\nu d}\sigma^d\right) \psi_a^s \psi_s^{int} \\
&= \psi_a^{r\dagger}\, \eta^{\mu\nu} \langle r| \left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha + imv_{\mu c}\sigma^c\right) \sum_t |t\rangle\langle t|\, \left(\partial_\nu + imv_{\nu\beta}\sigma^\beta + imv_{\nu d}\sigma^d\right) |s\rangle\, \psi_a^s \\
&= \psi_a^{r\dagger}\, \eta^{\mu\nu} \langle r| \left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha + imv_{\mu c}\sigma^c\right) \sum_t |t\rangle\langle t|\, \left(\partial_\nu + imv_{\nu\beta}\sigma^\beta + imv_{\nu d}\sigma^d\right) |s\rangle\, \psi_a^s \\
&= \psi_a^{r\dagger}\, \eta^{\mu\nu} \left[\delta_{rt} \left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha\right) - iA_\mu^i t_i^{rt}\right] \left[\delta_{ts} \left(\partial_\nu + imv_{\nu\beta}\sigma^\beta\right) - iA_\nu^j t_j^{ts}\right] \psi_a^s \\
&= \psi_a^\dagger\, \eta^{\mu\nu} \left[\left(\partial_\mu - iA_\mu^i t_i\right) + imv_{\mu\alpha}\sigma^\alpha\right] \left[\left(\partial_\nu - iA_\nu^j t_j\right) + imv_{\nu\beta}\sigma^\beta\right] \psi_a
\end{aligned} \tag{3.68}$$

where (3.35), (3.31), and (3.62) have been used. The action (3.33) then becomes

$$\begin{aligned}
S_a = \int d^4x\, {\psi_a}^\dagger \Big(&\frac{1}{2m} D^\mu D_\mu + \frac{1}{2} i v_\alpha^\mu \sigma^\alpha D_\mu \\
&+ \frac{1}{2} D_\mu i v_\alpha^\mu \sigma^\alpha - \frac{1}{2} m v^{\alpha\mu} v_\mu^\alpha + \mu_{ext}\Big)\psi_a
\end{aligned} \tag{3.69}$$

after (3.36) is used, where

$$D_\mu = \partial_\mu - iA_\mu^i t_i. \tag{3.70}$$

With the approximations above (3.40), (3.32) and (3.41) imply that

$$S_a = \int d^4x\, {\psi_a}^\dagger \left(\frac{1}{2m} D^\mu D_\mu + i e_\alpha^\mu \sigma^\alpha D_\mu\right) \psi_a. \tag{3.71}$$

This is in fact the generalization of (3.40) when the "internal order parameter" is permitted to vary as a function of the external coordinates x^μ.

As in Ref. [21], let us postulate a cosmological model in which

$$e_\alpha^\mu = \lambda \delta_\alpha^\mu \equiv \widetilde{e}_\alpha^\mu. \tag{3.72}$$

In this case (3.44) can be rewritten as

$$S_a = \int d^4x \, \widetilde{g} \, \bar{\psi}_a{}^\dagger \left(\bar{m}^{-1} \widetilde{g}^{\mu\nu} D_\mu D_\nu + i e^\mu_\alpha \sigma^\alpha D_\mu \right) \bar{\psi}_a \tag{3.73}$$

where

$$\widetilde{g}^{\mu\nu} \equiv \eta^{\alpha\beta} \widetilde{e}^\mu_\alpha \widetilde{e}^\nu_\beta \quad , \quad \bar{m} = \lambda^2 m \tag{3.74}$$

$$\widetilde{g} = \left(-\det \widetilde{g}_{\mu\nu} \right)^{1/2} = \lambda^{-4} \quad , \quad \bar{\psi}_a = \lambda^2 \psi_a . \tag{3.75}$$

(The tilde is a reminder that the above form is not general, and that $\widetilde{g}^{\mu\nu}$ is not a dynamical quantity.) In a locally inertial coordinate system with $e^\mu_\alpha = \delta^\mu_\alpha$, this becomes

$$S_a = \int d^4x \; \psi_a{}^\dagger \left((2\bar{m})^{-1} \eta^{\mu\nu} D_\mu D_\nu + i \sigma^\mu D_\mu \right) \psi_a \tag{3.76}$$

where the bar has been removed from ψ_a for simplicity, so the fermionic and bosonic actions are respectively

$$S_f = \int d^4x \; \psi_f{}^\dagger \left((2\bar{m})^{-1} \eta^{\mu\nu} D_\mu D_\nu + i \sigma^\mu D_\mu \right) \psi_f \tag{3.77}$$

and

$$S_b = \int d^4x \, \phi_b{}^\dagger \left(\eta^{\mu\nu} D_\mu D_\nu + 2 i \bar{m} \sigma^\mu D_\mu \right) \phi_b \tag{3.78}$$

where now

$$\phi_b = \psi_b / \left(2\bar{m} \right)^{1/2} . \tag{3.79}$$

Again, one regains the usual bosonic action excluding the gravitational interaction at high energy,

$$S_b \;\rightarrow\; \int d^4x \; \phi_b^\dagger \eta^{\mu\nu} D_\mu D_\nu \phi_b \quad \text{for } p^\mu \gg \bar{m}, \tag{3.80}$$

and the usual fermionic action including the gravitational interaction at low energy,

$$S_f \;\rightarrow\; \int d^4x \; \psi_f^\dagger \, i \sigma^\mu \, D_\mu \psi_f \quad \text{for } p^\mu \ll \bar{m}, \tag{3.81}$$

where the expressions now include gauge couplings and are written in a locally inertial coordinate system. In the chapter on the supersymmetrization of our theory, by using a new method, we will recover the usual Lorentz symmetry and the usual gravitational interaction of the bosonic fields at the low energy.

CHAPTER IV

SUPERSYMMETRY OF OUR THEORY

A. Supersymmetric Functional Integration and Supersymmetry Algebra

1. Functional Integral Invariance

Invariance of the functional integral under a supersymmetric transformation requires that both the action and the functional integral volume element, or measure, are left invariant. When there is no auxiliary field the action is given by

$$S = \int d^4x \left(\phi^* \partial^\mu \partial_\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi\right), \tag{4.1}$$

and invariance of the action requires

$$\begin{aligned}
\delta_\epsilon S &= \int d^4x \left(\delta_\epsilon \phi^* \partial^\mu \partial_\mu \phi + \phi^* \partial^\mu \partial_\mu \delta_\epsilon \phi + i\delta_\epsilon \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \delta_\epsilon \psi\right) \\
&= \int d^4x (\psi^\dagger_{\dot{\alpha}} f^\dagger \left(\epsilon\right)^{\dot{\alpha}} \partial^\mu \partial_\mu \phi + \phi^* \partial^\mu \partial_\mu \left(f\left(\epsilon\right)^\alpha \psi_\alpha\right) \\
&\qquad\qquad + i\phi^* g^\dagger \left(\epsilon\right)_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \psi_\alpha + i\psi^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} g\left(\epsilon\right)_\alpha \partial_\mu \phi) \\
&= \int d^4x [\psi^\dagger_{\dot{\alpha}} \left(f^\dagger \left(\epsilon\right)^{\dot{\alpha}} \partial^\mu + i\bar{\sigma}^{\mu\dot{\alpha}\alpha} g\left(\epsilon\right)_\alpha\right) \partial_\mu \phi \\
&\qquad\qquad + \phi^* \left(f\left(\epsilon\right)^\alpha \partial^\mu + ig^\dagger \left(\epsilon\right)_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha}\right) \partial_\mu \psi_\alpha] \\
&\equiv 0,
\end{aligned} \tag{4.2}$$

where the surface terms are assumed to vanish and we have also assumed

$$\delta_\epsilon \phi = f\left(\epsilon\right)^\alpha \psi_\alpha, \quad \delta_\epsilon \phi^* = \psi^\dagger_{\dot{\alpha}} f^\dagger \left(\epsilon\right)^{\dot{\alpha}}, \tag{4.3}$$

$$\delta_\epsilon \psi_\alpha = g\left(\epsilon\right)_\alpha \phi, \quad \delta_\epsilon \psi^\dagger_{\dot{\alpha}} = g^\dagger \left(\epsilon\right)_{\dot{\alpha}} \phi^*. \tag{4.4}$$

The new fields ϕ' and ψ'_α are written as

$$\begin{pmatrix} \phi' \\ \psi'_\alpha \end{pmatrix} = \begin{pmatrix} 1 & f(\epsilon)^\alpha \\ g(\epsilon)_\alpha & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi_\alpha \end{pmatrix} = \tilde{M} \begin{pmatrix} \phi \\ \psi_\alpha \end{pmatrix}, \tag{4.5}$$

where $\tilde{M}$ is a supermatrix given by

$$\tilde{M} = \begin{pmatrix} 1 & f(\epsilon)^\alpha \\ g(\epsilon)_\alpha & 1 \end{pmatrix},$$

and $f(\epsilon)^\alpha$ and $g(\epsilon)_\alpha$ are Grassmann spinor functions. From (4.2), we obtain

$$f^\dagger(\epsilon)^{\dot{\alpha}} \partial^\mu + i\bar{\sigma}^{\mu\dot{\alpha}\alpha} g(\epsilon)_\alpha = 0, \tag{4.6}$$

$$f(\epsilon)^\alpha \partial^\mu + i g^\dagger(\epsilon)_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} = 0. \tag{4.7}$$

Energy-momentum fixed-k functional-integral volume-element invariance requires that

$$d\phi(k)\, d\psi(k) = d\phi'(k)\, d\psi'(k)\, \mathrm{sdet}\left(\tilde{M}\right) = d\phi'(k)\, d\psi'(k) \tag{4.8}$$

and

$$\mathrm{sdet} \begin{pmatrix} 1 & f(\epsilon)^\alpha \\ g(\epsilon)_\alpha & 1 \end{pmatrix} = \det\left(1 - f(\epsilon)^\alpha g(\epsilon)_\alpha\right) \left(\det(1)\right)^{-1} \equiv 1, \tag{4.9}$$

where we have used $\mathrm{sdet} \begin{pmatrix} A & C \\ D & B \end{pmatrix} = \det(A - CB^{-1}D)\left(\det(B)\right)^{-1}$. Then we obtain

$$f(\epsilon)^\alpha g(\epsilon)_\alpha = 0. \tag{4.10}$$

The requirement of the invariance of both the action and the functional integral volume element cannot be simultaneously satisfied unless

$$f(\epsilon)^\alpha = g(\epsilon)_\alpha = 0, \tag{4.11}$$

and we could not have supersymmetry. We therefore need to introduce a bosonic auxiliary field F, with the action given by

$$S = \int d^4x \left(\phi^* \partial^\mu \partial_\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + F^* F \right). \tag{4.12}$$

The supersymmetry transformation is then

$$\delta_\epsilon \phi = f(\epsilon)^\alpha \psi_\alpha, \quad \delta_\epsilon \phi^* = \psi^\dagger_{\dot{\alpha}} f^\dagger (\epsilon)^{\dot{\alpha}} \tag{4.13}$$

$$\delta_\epsilon \psi_\alpha = g(\epsilon)_\alpha \phi + h(\epsilon)_\alpha F, \quad \delta_\epsilon \psi^\dagger_{\dot{\alpha}} = g^\dagger (\epsilon)_{\dot{\alpha}} \phi^* + h^\dagger (\epsilon)_{\dot{\alpha}} F^* \tag{4.14}$$

$$\delta_\epsilon F = j(\epsilon)^\alpha \psi_\alpha, \quad \delta_\epsilon F^* = \psi^\dagger_{\dot{\alpha}} j^\dagger (\epsilon)^{\dot{\alpha}} \tag{4.15}$$

where $f(\epsilon)$, $g(\epsilon)$, $h(\epsilon)$, and $j(\epsilon)$ are anticommuting SUSY spinorial functions. Invariance of the action requires

$$\begin{aligned}
\delta_\epsilon S = \int d^4x (&\delta_\epsilon \phi^* \partial^\mu \partial_\mu \phi + \phi^* \partial^\mu \partial_\mu \delta_\epsilon \phi + i\delta_\epsilon \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi \\
&+ i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \delta_\epsilon \psi + \delta_\epsilon F^* F + F^* \delta_\epsilon F) \\
= \int d^4x (&\psi^\dagger_{\dot{\alpha}} f^\dagger (\epsilon)^{\dot{\alpha}} \partial^\mu \partial_\mu \phi + \phi^* \partial^\mu \partial_\mu \left(f(\epsilon)^\alpha \psi_\alpha \right) \\
&+ i \left(\phi^* g^\dagger (\epsilon)_{\dot{\alpha}} + F^* h^\dagger (\epsilon)_{\dot{\alpha}} \right) \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \psi_\alpha \\
&+ i\psi^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \left(g(\epsilon)_\alpha \partial_\mu \phi + h(\epsilon)_\alpha \partial_\mu F \right) \\
&+ \psi^\dagger_{\dot{\alpha}} j^\dagger (\epsilon)^{\dot{\alpha}} F + F^* j(\epsilon)^\alpha \psi_\alpha) \\
= \int d^4x [&\psi^\dagger_{\dot{\alpha}} \left(f^\dagger (\epsilon)^{\dot{\alpha}} \partial^\mu \partial_\mu + i\bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu g(\epsilon)_\alpha \right) \phi \\
&+ \phi^* \left(f(\epsilon)^\alpha \partial^\mu \partial_\mu + i g^\dagger (\epsilon)_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \right) \psi_\alpha \\
&+ \psi^\dagger_{\dot{\alpha}} \left(i\bar{\sigma}^{\mu\dot{\alpha}\alpha} h(\epsilon)_\alpha \partial_\mu + j^\dagger (\epsilon)^{\dot{\alpha}} \right) F \\
&+ F^* \left(j(\epsilon)^\alpha + i h^\dagger (\epsilon)_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \right) \psi_\alpha] \\
\equiv 0, &
\end{aligned} \tag{4.16}$$

where the surface terms are assumed to vanish, and we obtain

$$f^{\dagger}\left(\epsilon\right)^{\dot{\alpha}}\partial^{\mu}\partial_{\mu}+i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_{\mu}g\left(\epsilon\right)_{\alpha}=0\quad\rightarrow\quad g\left(\epsilon\right)_{\alpha}=-i\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu}f^{\dagger}\left(\epsilon\right)^{\dot{\alpha}},\tag{4.17}$$

$$f\left(\epsilon\right)^{\alpha}\partial^{\mu}\partial_{\mu}+ig^{\dagger}\left(\epsilon\right)_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_{\mu}=0\quad\rightarrow\quad g^{\dagger}\left(\epsilon\right)_{\dot{\alpha}}=-if\left(\epsilon\right)^{\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu},\tag{4.18}$$

$$i\bar{\sigma}^{\mu\dot{\alpha}\alpha}h\left(\epsilon\right)_{\alpha}\partial_{\mu}+j^{\dagger}\left(\epsilon\right)^{\dot{\alpha}}=0\quad\rightarrow\quad j^{\dagger}\left(\epsilon\right)^{\dot{\alpha}}=-i\bar{\sigma}^{\mu\dot{\alpha}\alpha}h\left(\epsilon\right)_{\alpha}\partial_{\mu},\tag{4.19}$$

$$j\left(\epsilon\right)^{\alpha}+ih^{\dagger}\left(\epsilon\right)_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_{\mu}=0\quad\rightarrow\quad j\left(\epsilon\right)^{\alpha}=-ih^{\dagger}\left(\epsilon\right)_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_{\mu},\tag{4.20}$$

where we have used $\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\beta}}+\bar{\sigma}^{\nu\dot{\alpha}\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}=-2\eta^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}}$, and $\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta}+\sigma^{\nu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}=-2\eta^{\mu\nu}\delta^{\beta}_{\alpha}$, and therefore $\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\beta}}\partial_{\mu}\partial_{\nu}=\left[\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^{\nu}_{\alpha\dot{\beta}}+\bar{\sigma}^{\nu\dot{\alpha}\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\right)/2\right]\partial_{\mu}\partial_{\nu}=-\delta^{\dot{\alpha}}_{\dot{\beta}}\partial^{\mu}\partial_{\mu}$.

The fixed-k functional integration is given by

$$\int d\Phi'\left(k\right)d\Phi'^{\dagger}\left(k\right)e^{iS\left[\phi',\phi'^{*},F',F'^{*},\psi',\psi'^{\dagger}\right]},\tag{4.21}$$

where

$$d\Phi'\left(k\right)\equiv d\phi'\left(k\right)dF'\left(k\right)d\psi'\left(k\right),\tag{4.22}$$

and the fields are transformed as

$$\begin{pmatrix}\phi'\\F'\\\psi'_{\alpha}\end{pmatrix}=\begin{pmatrix}1&0&f\left(\epsilon\right)^{\alpha}\\0&1&j\left(\epsilon\right)^{\alpha}\\g\left(\epsilon\right)_{\alpha}&h\left(\epsilon\right)_{\alpha}&1\end{pmatrix}\begin{pmatrix}\phi\\F\\\psi_{\alpha}\end{pmatrix}\equiv\tilde{M}\begin{pmatrix}\phi\\F\\\psi_{\alpha}\end{pmatrix}.\tag{4.23}$$

Since the functional volume element is transformed as

$$d\Phi'^{\dagger}d\Phi'=d\Phi^{\dagger}d\Phi\,\mathrm{sdet}\left(\tilde{M}^{\dagger}\right)\mathrm{sdet}\left(\tilde{M}\right)\tag{4.24}$$

the condition for sdet$\left(\tilde{M}\right)$ =sdet$\left(\tilde{M}^{\dagger}\right) = 1$ is

$$\begin{aligned}
&\mathrm{sdet}\begin{pmatrix} 1 & 0 & f(\epsilon)^{\alpha} \\ 0 & 1 & j(\epsilon)^{\alpha} \\ g(\epsilon)_{\alpha} & h(\epsilon)_{\alpha} & 1 \end{pmatrix} \\
&= \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} f(\epsilon)^{\alpha} \\ j(\epsilon)^{\alpha} \end{pmatrix}\begin{pmatrix} g(\epsilon)_{\alpha} & h(\epsilon)_{\alpha} \end{pmatrix}\right)(\det(1))^{-1} \\
&= \det\begin{pmatrix} 1 - f(\epsilon)^{\alpha} g(\epsilon)_{\alpha} & -f(\epsilon)^{\alpha} h(\epsilon)_{\alpha} \\ -j(\epsilon)^{\alpha} g(\epsilon)_{\alpha} & 1 - j(\epsilon)^{\alpha} h(\epsilon)_{\alpha} \end{pmatrix} \\
&= \left(1 - f(\epsilon)^{\alpha} g(\epsilon)_{\alpha}\right)\left(1 - j(\epsilon)^{\alpha} h(\epsilon)_{\alpha}\right) \\
&\qquad - \left(f(\epsilon)^{\alpha} h(\epsilon)_{\alpha}\right)\left(j(\epsilon)^{\alpha} g(\epsilon)_{\alpha}\right) \\
&= 1 - \left(f(\epsilon)^{\alpha} g(\epsilon)_{\alpha} + j(\epsilon)^{\alpha} h(\epsilon)_{\alpha}\right) \\
&\qquad + \left(f(\epsilon)^{\alpha} g(\epsilon)_{\alpha}\right)\left(j(\epsilon)^{\alpha} h(\epsilon)_{\alpha}\right) \\
&\qquad - \left(f(\epsilon)^{\alpha} h(\epsilon)_{\alpha}\right)\left(j(\epsilon)^{\alpha} g(\epsilon)_{\alpha}\right) \\
&\equiv 1,
\end{aligned} \tag{4.25}$$

and by ignoring the 4th power infinitesimal terms, we obtain

$$f(\epsilon)^{\alpha} g(\epsilon)_{\alpha} + j(\epsilon)^{\alpha} h(\epsilon)_{\alpha} = 0. \tag{4.26}$$

Similarly, the requirement for sdet$\left(\tilde{M}^{\dagger}\right) = 1$ is

$$f^{\dagger}(\epsilon)_{\dot{\alpha}}\, g^{\dagger}(\epsilon)^{\dot{\alpha}} + j^{\dagger}(\epsilon)_{\dot{\alpha}}\, h^{\dagger}(\epsilon)^{\dot{\alpha}} = 0. \tag{4.27}$$

With the use of (4.17) and (4.20), it can be rewritten as

$$\begin{aligned} f(\epsilon)^{\alpha} g(\epsilon)_{\alpha} + j(\epsilon)^{\alpha} h(\epsilon)_{\alpha} &= -f(\epsilon)^{\alpha} i\sigma^{\nu}_{\alpha\dot{\alpha}} \partial_{\nu} f^{\dagger}(\epsilon)^{\dot{\alpha}} - ih^{\dagger}(\epsilon)_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_{\mu} h(\epsilon)_{\alpha} \\ &= -if(\epsilon)^{\alpha} \sigma^{\nu}_{\alpha\dot{\alpha}} f^{\dagger}(\epsilon)^{\dot{\alpha}} \partial_{\nu} + ih(\epsilon)^{\alpha} \sigma^{\nu}_{\alpha\dot{\alpha}} h^{\dagger}(\epsilon)^{\dot{\alpha}} \partial_{\nu} \\ &\equiv 0, \end{aligned} \tag{4.28}$$

where we have used $\chi^{\dagger}(\epsilon)_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \eta(\epsilon)_{\alpha} = -\eta(\epsilon)^{\alpha} \sigma^{\mu}_{\alpha\dot{\alpha}} \chi^{\dagger}(\epsilon)^{\dot{\alpha}}$ in the second term of the second line. We then obtain

$$f(\epsilon)^{\alpha} = h(\epsilon)^{\alpha}. \tag{4.29}$$

2. Closure of Algebra

In this subsubsection, we will show that the requirements of action and functional integral invariance which we have obtained in the previous subsubsection guarantee closure of the algebra. The commutator of the supersymmetric transformations for each field are

$$\begin{aligned} [\delta_{\epsilon_2}, \delta_{\epsilon_1}] \phi &= (\delta_{\epsilon_2}\delta_{\epsilon_1} - \delta_{\epsilon_1}\delta_{\epsilon_2}) \phi \\ &= \delta_{\epsilon_2} (f(\epsilon_1)^{\alpha} \psi_{\alpha}) - \delta_{\epsilon_1} (f(\epsilon_2)^{\alpha} \psi_{\alpha}) \\ &= f(\epsilon_1)^{\alpha} (g(\epsilon_2)_{\alpha} \phi + h(\epsilon_2)_{\alpha} F) - f(\epsilon_2)^{\alpha} (g(\epsilon_1)_{\alpha} \phi + h(\epsilon_1)_{\alpha} F) \\ &= [f(\epsilon_1)^{\alpha} g(\epsilon_2)_{\alpha} \underset{(a)}{-} f(\epsilon_2)^{\alpha} g(\epsilon_1)_{\alpha}]\phi + [f(\epsilon_1)^{\alpha} h(\epsilon_2)_{\alpha} \underset{(b)}{-} f(\epsilon_2)^{\alpha} h(\epsilon_1)_{\alpha}]F \end{aligned} \tag{4.30}$$

$$
\begin{aligned}
\left[\delta_{\epsilon_2},\delta_{\epsilon_1}\right]\psi_\alpha &= \left(\delta_{\epsilon_2}\delta_{\epsilon_1}-\delta_{\epsilon_1}\delta_{\epsilon_2}\right)\psi_\alpha \\
&= \delta_{\epsilon_2}\left(g\left(\epsilon_1\right)_\alpha\phi+h\left(\epsilon_1\right)_\alpha F\right)-\delta_{\epsilon_1}\left(g\left(\epsilon_2\right)_\alpha\phi+h\left(\epsilon_2\right)_\alpha F\right) \\
&= \Big(g\left(\epsilon_1\right)_\alpha f\left(\epsilon_2\right)^\beta\psi_\beta+h\left(\epsilon_1\right)_\alpha j\left(\epsilon_2\right)^\beta\psi_\beta\Big) \\
&\qquad\qquad -\Big(g\left(\epsilon_2\right)_\alpha f\left(\epsilon_1\right)^\beta\psi_\beta+h\left(\epsilon_2\right)_\alpha j\left(\epsilon_1\right)^\beta\psi_\beta\Big) \\
&= \Big[g\left(\epsilon_1\right)_\alpha f\left(\epsilon_2\right)^\beta-g\left(\epsilon_2\right)_\alpha f\left(\epsilon_1\right)^\beta+h\left(\epsilon_1\right)_\alpha j\left(\epsilon_2\right)^\beta-h\left(\epsilon_2\right)_\alpha j\left(\epsilon_1\right)^\beta\Big]\psi_\beta \\
&= -\Big(\psi^\beta g\left(\epsilon_1\right)_\beta\Big)f\left(\epsilon_2\right)_\alpha-\Big(g\left(\epsilon_1\right)^\beta f\left(\epsilon_2\right)_\beta\Big)\psi_\alpha+\Big(\psi^\beta g\left(\epsilon_2\right)_\beta\Big)f\left(\epsilon_1\right)_\alpha \\
&\quad +\Big(g\left(\epsilon_2\right)^\beta f\left(\epsilon_1\right)_\beta\Big)\psi_\alpha+\Big[h\left(\epsilon_1\right)_\alpha j\left(\epsilon_2\right)^\beta-h\left(\epsilon_2\right)_\alpha j\left(\epsilon_1\right)^\beta\Big]\psi_\beta \\
&= \Big[g\left(\epsilon_2\right)^\beta f\left(\epsilon_1\right)_\beta \underset{(a)}{-} g\left(\epsilon_1\right)^\beta f\left(\epsilon_2\right)_\beta\Big]\psi_\alpha+\{-\Big(\psi^\beta g\left(\epsilon_1\right)_\beta\Big)f\left(\epsilon_2\right)_\alpha \\
&+\Big(\psi^\beta g\left(\epsilon_2\right)_\beta\Big)f\left(\epsilon_1\right)_\alpha+\Big[h\left(\epsilon_1\right)_\alpha \underset{(b)}{} j\left(\epsilon_2\right)^\beta-h\left(\epsilon_2\right)_\alpha j\left(\epsilon_1\right)^\beta\Big]\}\psi_\beta
\end{aligned}
\tag{4.31}
$$

where in the 5th line we have used the Fierz identity $(\xi\eta)\,\chi_\alpha=-(\eta\chi)\,\xi_\alpha-(\chi\xi)\,\eta_\alpha$. Then

$$
\begin{aligned}
\left[\delta_{\epsilon_2},\delta_{\epsilon_1}\right]F &= \left(\delta_{\epsilon_2}\delta_{\epsilon_1}-\delta_{\epsilon_1}\delta_{\epsilon_2}\right)F \\
&= \delta_{\epsilon_2}\left(j\left(\epsilon_1\right)^\alpha\psi_\alpha\right)-\delta_{\epsilon_1}\left(j\left(\epsilon_2\right)^\alpha\psi_\alpha\right) \\
&= j\left(\epsilon_1\right)^\alpha\left(g\left(\epsilon_2\right)_\alpha\phi+h\left(\epsilon_2\right)_\alpha F\right)-j\left(\epsilon_2\right)^\alpha\left(g\left(\epsilon_1\right)_\alpha\phi+h\left(\epsilon_1\right)_\alpha F\right) \\
&= [j\left(\epsilon_1\right)^\alpha h\left(\epsilon_2\right)_\alpha \underset{(a)}{-} j\left(\epsilon_2\right)^\alpha h\left(\epsilon_1\right)_\alpha]F+[j\left(\epsilon_1\right)^\alpha g\left(\epsilon_2\right)_\alpha \underset{(b)}{-} j\left(\epsilon_2\right)^\alpha g\left(\epsilon_1\right)_\alpha]\phi.
\end{aligned}
\tag{4.32}
$$

Closure of the algebra means that all of the fields in the same multiplet satisfy the same algebra, $[\delta_{\epsilon_2}, \delta_{\epsilon_1}] X = (\cdots) X$, where $X = \phi$, ψ_α, or F. Therefore, we require

$$(4.30-b) = f(\epsilon_1)^\alpha h(\epsilon_2)_\alpha - f(\epsilon_2)^\alpha h(\epsilon_1)_\alpha = 0 \tag{4.33}$$

$$(4.32-b) = j(\epsilon_1)^\alpha g(\epsilon_2)_\alpha - j(\epsilon_2)^\alpha g(\epsilon_1)_\alpha = 0 \tag{4.34}$$

$$(4.31-b) = 0 \tag{4.35}$$

$$(4.30-a) = (4.31-a) = (4.32-a)\,. \tag{4.36}$$

(4.33) is satisfied by using (4.29) since

$$\begin{aligned}(4.30-b) &= f(\epsilon_1)^\alpha h(\epsilon_2)_\alpha - f(\epsilon_2)^\alpha h(\epsilon_1)_\alpha \\ &= h(\epsilon_1)^\alpha h(\epsilon_2)_\alpha - h(\epsilon_2)^\alpha h(\epsilon_1)_\alpha = 0,\end{aligned} \tag{4.37}$$

and (4.34) is satisfied since

$$\begin{aligned}(4.32-b) &= j(\epsilon_1)^\alpha g(\epsilon_2)_\alpha - j(\epsilon_2)^\alpha g(\epsilon_1)_\alpha \\ &= ih^\dagger(\epsilon_1)_{\dot\alpha} \bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu i\sigma^\nu_{\alpha\dot\alpha}\partial_\nu f^\dagger(\epsilon_2)^{\dot\alpha} \\ &\qquad - ih^\dagger(\epsilon_2)_{\dot\alpha} \bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu i\sigma^\nu_{\alpha\dot\alpha}\partial_\nu f^\dagger(\epsilon_1)^{\dot\alpha} \\ &= ih^\dagger(\epsilon_1)_{\dot\alpha} \bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu i\sigma^\nu_{\alpha\dot\alpha}\partial_\nu f^\dagger(\epsilon_2)^{\dot\alpha} \\ &\qquad - ih^\dagger(\epsilon_2)_{\dot\alpha} \bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu i\sigma^\nu_{\alpha\dot\alpha}\partial_\nu f^\dagger(\epsilon_1)^{\dot\alpha} \\ &= f^\dagger(\epsilon_1)_{\dot\alpha} f^\dagger(\epsilon_2)^{\dot\alpha}\partial^\mu\partial_\mu - f^\dagger(\epsilon_1)_{\dot\alpha} f^\dagger(\epsilon_2)^{\dot\alpha}\partial^\mu\partial_\mu \\ &= 0,\end{aligned} \tag{4.38}$$

where we have used (4.17), (4.20), and (4.29). (4.35) is satisfied since

$$\begin{aligned}
(4.31-b) &= -\left(\psi^{\beta} g\left(\epsilon_1\right)_{\beta}\right) f\left(\epsilon_2\right)_{\alpha} + \left(\psi^{\beta} g\left(\epsilon_2\right)_{\beta}\right) f\left(\epsilon_1\right)_{\alpha} \\
&\qquad + \left[h\left(\epsilon_1\right)_{\alpha} j\left(\epsilon_2\right)^{\beta} - h\left(\epsilon_2\right)_{\alpha} j\left(\epsilon_1\right)^{\beta}\right] \psi_{\beta} \\
&= \varepsilon^{\beta\gamma}\left(i\sigma^{\nu}_{\gamma\dot{\alpha}}\partial_{\nu} f^{\dagger}\left(\epsilon_1\right)^{\dot{\alpha}}\right)\psi_{\beta} f\left(\epsilon_2\right)_{\alpha} - \varepsilon^{\beta\gamma}\left(i\sigma^{\nu}_{\gamma\dot{\alpha}}\partial_{\nu} f^{\dagger}\left(\epsilon_2\right)^{\dot{\alpha}}\right)\psi_{\beta} f\left(\epsilon_1\right)_{\alpha} \\
&\qquad + \left[h\left(\epsilon_1\right)_{\alpha}\left(-ih^{\dagger}\left(\epsilon_2\right)_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_{\mu}\right) - h\left(\epsilon_2\right)_{\alpha}\left(-ih^{\dagger}\left(\epsilon_1\right)_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_{\mu}\right)\right]\psi_{\beta} \\
&= \left(\partial_{\nu}\psi^{\beta} i\sigma^{\nu}_{\beta\dot{\alpha}} h^{\dagger}\left(\epsilon_1\right)^{\dot{\alpha}}\right) h\left(\epsilon_2\right)_{\alpha} - \left(\partial_{\nu}\psi^{\beta} i\sigma^{\nu}_{\beta\dot{\alpha}} h^{\dagger}\left(\epsilon_2\right)^{\dot{\alpha}}\right) h\left(\epsilon_1\right)_{\alpha} \\
&\qquad + h\left(\epsilon_1\right)_{\alpha}\left(\partial_{\nu}\psi^{\beta} i\sigma^{\nu}_{\beta\dot{\alpha}} h^{\dagger}\left(\epsilon_2\right)^{\dot{\alpha}}\right) - h\left(\epsilon_2\right)_{\alpha}\left(\partial_{\nu}\psi^{\beta} i\sigma^{\nu}_{\beta\dot{\alpha}} h^{\dagger}\left(\epsilon_1\right)^{\dot{\alpha}}\right) \\
&= 0.
\end{aligned} \tag{4.39}$$

Next we will prove (4.36).

$$\begin{aligned}
(4.30-a) &= f\left(\epsilon_1\right)^{\alpha} g\left(\epsilon_2\right)_{\alpha} - f\left(\epsilon_2\right)^{\alpha} g\left(\epsilon_1\right)_{\alpha} \\
&= -h\left(\epsilon_1\right)^{\alpha} i\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu} f^{\dagger}\left(\epsilon_2\right)^{\dot{\alpha}} + h\left(\epsilon_2\right)^{\alpha} i\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu} f^{\dagger}\left(\epsilon_1\right)^{\dot{\alpha}} \\
&= -ih\left(\epsilon_1\right)^{\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu} h^{\dagger}\left(\epsilon_2\right)^{\dot{\alpha}} + ih\left(\epsilon_2\right)^{\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu} h^{\dagger}\left(\epsilon_1\right)^{\dot{\alpha}}.
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
(4.32-a) &= j\left(\epsilon_1\right)^{\alpha} h\left(\epsilon_2\right)_{\alpha} - j\left(\epsilon_2\right)^{\alpha} h\left(\epsilon_1\right)_{\alpha} \\
&= -ih^{\dagger}\left(\epsilon_1\right)_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_{\mu} h\left(\epsilon_2\right)_{\alpha} + ih^{\dagger}\left(\epsilon_2\right)_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_{\mu} h\left(\epsilon_1\right)_{\alpha} \\
&= -ih\left(\epsilon_1\right)^{\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu} h^{\dagger}\left(\epsilon_2\right)^{\dot{\alpha}} + ih\left(\epsilon_2\right)^{\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu} h^{\dagger}\left(\epsilon_1\right)^{\dot{\alpha}} \qquad (4.41) \\
&\equiv (4.40).
\end{aligned}$$

$$\begin{aligned}
(4.31-a) &= g\left(\epsilon_2\right)^{\beta} f\left(\epsilon_1\right)_{\beta} - g\left(\epsilon_1\right)^{\beta} f\left(\epsilon_2\right)_{\beta} \\
&= -\varepsilon^{\beta\alpha}\left(i\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu} f^{\dagger}\left(\epsilon_2\right)^{\dot{\alpha}}\right) f\left(\epsilon_1\right)_{\beta} + \varepsilon^{\beta\alpha}\left(i\sigma^{\nu}_{\alpha\dot{\alpha}}\partial_{\nu} f^{\dagger}\left(\epsilon_1\right)^{\dot{\alpha}}\right) f\left(\epsilon_2\right)_{\beta} \\
&= -ih\left(\epsilon_1\right)^{\beta}\sigma^{\nu}_{\beta\dot{\alpha}}\partial_{\nu} h^{\dagger}\left(\epsilon_2\right)^{\dot{\alpha}} + ih\left(\epsilon_2\right)^{\beta}\sigma^{\nu}_{\beta\dot{\alpha}}\partial_{\nu} h^{\dagger}\left(\epsilon_1\right)^{\dot{\alpha}} \qquad (4.42) \\
&\equiv (4.40).
\end{aligned}$$

and we have thus proved (4.36).

As a result, when the action and the functional integral volume element are invariant under the supersymmetry transformation, it is guaranteed that the supersymmetry algebra is closed.

Although we have started with a specific shape of the action as in (4.12), we next consider the more general case. The expected supersymmetric action with a minimum number of fields (one spinor fermion, one complex scalar boson, and one real auxiliary field) would be given by

$$S = \int \frac{d^4p}{(2\pi)^4} \left[\phi^* O_\phi \phi + \psi^\dagger_{\dot{\alpha}} O_\psi^{\dot{\alpha}\alpha} \psi_\alpha + F^* O_F F \right], \tag{4.43}$$

where the operators O_ϕ and O_F, which are scalars, and O_ψ, which is a matrix, are chosen so that $\delta_\epsilon S = 0$, under

$$\delta_\epsilon \phi = f(\epsilon)^\alpha \psi_\alpha, \quad \delta_\epsilon \phi^* = \psi^\dagger_{\dot{\alpha}} f^\dagger (\epsilon)^{\dot{\alpha}} \tag{4.44}$$

$$\delta_\epsilon \psi_\alpha = g(\epsilon)_\alpha \phi + h(\epsilon)_\alpha F, \quad \delta_\epsilon \psi^\dagger_{\dot{\alpha}} = g^\dagger (\epsilon)_{\dot{\alpha}} \phi^* + h^\dagger (\epsilon)_{\dot{\alpha}} F^* \tag{4.45}$$

$$\delta_\epsilon F = j(\epsilon)^\alpha \psi_\alpha, \quad \delta_\epsilon F^* = \psi^\dagger_{\dot{\alpha}} j^\dagger (\epsilon)^{\dot{\alpha}}. \tag{4.46}$$

To obtain the constraint on the choice of O_ϕ, O_ψ, and O_F, we calculate $\delta_\epsilon S$:

$$
\begin{aligned}
\delta_\epsilon S = \int & \frac{d^4p}{(2\pi)^4}[(\delta_\epsilon\phi^*) O_\phi\phi + \phi^* O_\phi (\delta_\epsilon\phi) + \left(\delta_\epsilon\psi^\dagger_{\dot\alpha}\right) O_\psi^{\dot\alpha\alpha}\psi_\alpha \\
& + \psi^\dagger_{\dot\alpha} O_\psi^{\dot\alpha\alpha}(\delta_\epsilon\psi_\alpha) + (\delta_\epsilon F^*) O_F F + F^* O_F (\delta_\epsilon F)] \\
& = \int \frac{d^4p}{(2\pi)^4}[\left(\psi^\dagger_{\dot\alpha} f^\dagger(\epsilon)^{\dot\alpha}\right) O_\phi\phi + \phi^* O_\phi \left(f(\epsilon)^\alpha \psi_\alpha\right) + \left(g^\dagger(\epsilon)_{\dot\alpha}\phi^* + h^\dagger(\epsilon)_{\dot\alpha} F^*\right) O_\psi^{\dot\alpha\alpha}\psi_\alpha \\
& + \psi^\dagger_{\dot\alpha} O_\psi^{\dot\alpha\alpha}\left(g(\epsilon)_\alpha\phi + h(\epsilon)_\alpha F\right) + \left(\psi^\dagger_{\dot\alpha} j^\dagger(\epsilon)^{\dot\alpha}\right) O_F F + F^* O_F \left(j(\epsilon)^\alpha\psi_\alpha\right)] \\
& = \int \frac{d^4p}{(2\pi)^4}[\psi^\dagger_{\dot\alpha}\left(f^\dagger(\epsilon)^{\dot\alpha} O_\phi + O_\psi^{\dot\alpha\alpha} g(\epsilon)_\alpha\right)\phi + \phi^*\left(O_\phi f(\epsilon)^\alpha + g^\dagger(\epsilon)_{\dot\alpha} O_\psi^{\dot\alpha\alpha}\right)\psi_\alpha \\
& + \psi^\dagger_{\dot\alpha}\left(O_\psi^{\dot\alpha\alpha} h(\epsilon)_\alpha + j^\dagger(\epsilon)^{\dot\alpha} O_F\right) F + F^*\left(O_F j(\epsilon)^\alpha + h^\dagger(\epsilon)_{\dot\alpha} O_\psi^{\dot\alpha\alpha}\right)\psi_\alpha] \\
& \equiv 0,
\end{aligned} \tag{4.47}
$$

and we obtain

$$f^\dagger(\epsilon)^{\dot\alpha} O_\phi + O_\psi^{\dot\alpha\alpha} g(\epsilon)_\alpha = 0 \;\rightarrow\; f^\dagger(\epsilon)^{\dot\alpha} = -\frac{O_\psi^{\dot\alpha\alpha}}{O_\phi} g(\epsilon)_\alpha, \tag{4.48}$$

$$O_\phi f(\epsilon)^\alpha + g^\dagger(\epsilon)_{\dot\alpha} O_\psi^{\dot\alpha\alpha} = 0 \;\rightarrow\; f(\epsilon)^\alpha = -g^\dagger(\epsilon)_{\dot\alpha}\frac{O_\psi^{\dot\alpha\alpha}}{O_\phi}, \tag{4.49}$$

$$O_\psi^{\dot\alpha\alpha} h(\epsilon)_\alpha + j^\dagger(\epsilon)^{\dot\alpha} O_F = 0 \;\rightarrow\; j^\dagger(\epsilon)^{\dot\alpha} = -\frac{O_\psi^{\dot\alpha\alpha}}{O_F} h(\epsilon)_\alpha, \tag{4.50}$$

$$O_F j(\epsilon)^\alpha + h^\dagger(\epsilon)_{\dot\alpha} O_\psi^{\dot\alpha\alpha} = 0 \;\rightarrow\; j(\epsilon)^\alpha = -h^\dagger(\epsilon)_{\dot\alpha}\frac{O_\psi^{\dot\alpha\alpha}}{O_F}. \tag{4.51}$$

By using (4.49) and (4.51) to eliminate $f(\epsilon)^\alpha$ and $j(\epsilon)^\alpha$ in (4.26), we obtain

$$g^\dagger(\epsilon)_{\dot\alpha}\frac{O_\psi^{\dot\alpha\alpha}}{O_\phi} g(\epsilon)_\alpha + h^\dagger(\epsilon)_{\dot\alpha}\frac{O_\psi^{\dot\alpha\alpha}}{O_F} h(\epsilon)_\alpha = 0, \tag{4.52}$$

$$\rightarrow \begin{pmatrix} g^\dagger(\epsilon)_{\dot\alpha} & h^\dagger(\epsilon)_{\dot\alpha} \end{pmatrix} \begin{pmatrix} O_\psi^{\dot\alpha\alpha}/O_\phi & 0 \\ 0 & O_\psi^{\dot\alpha\alpha}/O_F \end{pmatrix} \begin{pmatrix} g(\epsilon)_\alpha \\ h(\epsilon)_\alpha \end{pmatrix} = 0$$

and when

$$\left(\begin{array}{cc} g^{\dagger}(\epsilon)_{\dot{\alpha}} & h^{\dagger}(\epsilon)_{\dot{\alpha}} \end{array} \right) \left(\begin{array}{cc} O_{\psi}^{\dot{\alpha}\alpha}/O_{\phi} & 0 \\ 0 & O_{\psi}^{\dot{\alpha}\alpha}/O_{F} \end{array} \right) = \left(\begin{array}{cc} h(\epsilon)^{\alpha} & -g(\epsilon)^{\alpha} \end{array} \right), \tag{4.53}$$

so (4.52) is satisfied. From (4.53), we have

$$h(\epsilon)^{\alpha} = g^{\dagger}(\epsilon)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_{\phi}}, \tag{4.54}$$

$$g(\epsilon)^{\alpha} = -h^{\dagger}(\epsilon)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_{F}}, \tag{4.55}$$

and from (4.49) and (4.54)

$$f(\epsilon)^{\alpha} = -h(\epsilon)^{\alpha}. \tag{4.56}$$

As we have already seen in (4.30)∼(4.32), the commutator of the supersymmetry transformation are

$$[\delta_{\epsilon_2}, \delta_{\epsilon_1}]\,\phi = [\underset{(a)}{f(\epsilon_1)^{\alpha} g(\epsilon_2)_{\alpha} - f(\epsilon_2)^{\alpha} g(\epsilon_1)_{\alpha}}]\phi + [\underset{(b)}{f(\epsilon_1)^{\alpha} h(\epsilon_2)_{\alpha} - f(\epsilon_2)^{\alpha} h(\epsilon_1)_{\alpha}}]F$$

$$\begin{aligned}[\delta_{\epsilon_2}, \delta_{\epsilon_1}]\,\psi_{\alpha} = {} & \Big[\underset{(a)}{g(\epsilon_2)^{\beta} f(\epsilon_1)_{\beta} - g(\epsilon_1)^{\beta} f(\epsilon_2)_{\beta}}\Big]\psi_{\alpha} + \{-\left(\psi^{\beta} g(\epsilon_1)_{\beta}\right) f(\epsilon_2)_{\alpha} \\ & + \left(\psi^{\beta} g(\epsilon_2)_{\beta}\right) f(\epsilon_1)_{\alpha} + \Big[\underset{(b)}{h(\epsilon_1)_{\alpha} j(\epsilon_2)^{\beta} - h(\epsilon_2)_{\alpha} j(\epsilon_1)^{\beta}}\Big]\}\psi_{\beta}\end{aligned}$$

$$[\delta_{\epsilon_2}, \delta_{\epsilon_1}]\,F = [\underset{(a)}{j(\epsilon_1)^{\alpha} h(\epsilon_2)_{\alpha} - j(\epsilon_2)^{\alpha} h(\epsilon_1)_{\alpha}}]F + [\underset{(b)}{j(\epsilon_1)^{\alpha} g(\epsilon_2)_{\alpha} - j(\epsilon_2)^{\alpha} g(\epsilon_1)_{\alpha}}]\phi.$$

By using (4.48)∼(4.51), (4.54)∼(4.56) we obtain

$$\begin{aligned}(4.30)\text{-}(a) &= f(\epsilon_1)^{\alpha} g(\epsilon_2)_{\alpha} - f(\epsilon_2)^{\alpha} g(\epsilon_1)_{\alpha} \\ &= -h(\epsilon_1)^{\alpha} g(\epsilon_2)_{\alpha} + h(\epsilon_2)^{\alpha} g(\epsilon_1)_{\alpha} \\ &= h^{\dagger}(\epsilon_2)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_F} h(\epsilon_1)_{\alpha} - h^{\dagger}(\epsilon_1)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_F} h(\epsilon_2)_{\alpha},\end{aligned}$$

$$(4.31)\text{-}(a) = (4.30)\text{-}(a),$$

$$\begin{aligned}(4.32)\text{-}(a) &= j\left(\epsilon_1\right)^{\alpha} h\left(\epsilon_2\right)_{\alpha} - j\left(\epsilon_2\right)^{\alpha} h\left(\epsilon_1\right)_{\alpha} \\ &= -h^{\dagger}\left(\epsilon_1\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_F} h\left(\epsilon_2\right)_{\alpha} + h^{\dagger}\left(\epsilon_2\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_F} h\left(\epsilon_1\right)_{\alpha},\end{aligned}$$

and therefore

$$(4.30)\text{-}(a) = (4.31)\text{-}(a) = (4.32)\text{-}(a). \tag{4.57}$$

The others are

$$\begin{aligned}(4.30)\text{-}(b) &= f\left(\epsilon_1\right)^{\alpha} h\left(\epsilon_2\right)_{\alpha} - f\left(\epsilon_2\right)^{\alpha} h\left(\epsilon_1\right)_{\alpha} \\ &= -h\left(\epsilon_1\right)^{\alpha} h\left(\epsilon_2\right)_{\alpha} + h\left(\epsilon_2\right)^{\alpha} h\left(\epsilon_1\right)_{\alpha} \\ &= 0,\end{aligned}$$

$$\begin{aligned}(4.31)\text{-}(b) &= -\left(\psi^{\beta} g\left(\epsilon_1\right)_{\beta}\right) f\left(\epsilon_2\right)_{\alpha} + \left(\psi^{\beta} g\left(\epsilon_2\right)_{\beta}\right) f\left(\epsilon_1\right)_{\alpha} \\ &+ \left[h\left(\epsilon_1\right)_{\alpha} j\left(\epsilon_2\right)^{\beta} - h\left(\epsilon_2\right)_{\alpha} j\left(\epsilon_1\right)^{\beta}\right] \psi_{\beta} \\ &= -h^{\dagger}\left(\epsilon_1\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\beta}}{O_F} \psi_{\beta} h\left(\epsilon_2\right)_{\alpha} + h^{\dagger}\left(\epsilon_2\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\beta}}{O_F} \psi_{\beta} h\left(\epsilon_1\right)_{\alpha} \\ &+ [-h\left(\epsilon_1\right)_{\alpha} h^{\dagger}\left(\epsilon_2\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\beta}}{O_F} + h\left(\epsilon_2\right)_{\alpha} h^{\dagger}\left(\epsilon_1\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\beta}}{O_F}] \psi_{\beta} \\ &= 0,\end{aligned}$$

$$\begin{aligned}(4.32)\text{-}(b) &= j\left(\epsilon_1\right)^{\alpha} g\left(\epsilon_2\right)_{\alpha} - j\left(\epsilon_2\right)^{\alpha} g\left(\epsilon_1\right)_{\alpha} \\ &= h^{\dagger}\left(\epsilon_1\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_F} \epsilon_{\alpha\beta} \left(h^{\dagger}\left(\epsilon_2\right)_{\dot{\beta}} \frac{O_{\psi}^{\dot{\beta}\beta}}{O_F}\right) \\ &\qquad - h^{\dagger}\left(\epsilon_2\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_F} \epsilon_{\alpha\beta} \left(h^{\dagger}\left(\epsilon_1\right)_{\dot{\beta}} \frac{O_{\psi}^{\dot{\beta}\beta}}{O_F}\right) \\ &= 0.\end{aligned}$$

Therefore,

$$\left[\delta_{\epsilon_2}, \delta_{\epsilon_1}\right] X = \left(h^{\dagger}\left(\epsilon_2\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_F} h\left(\epsilon_1\right)_{\alpha} - h^{\dagger}\left(\epsilon_1\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_F} h\left(\epsilon_2\right)_{\alpha}\right) X \tag{4.58}$$

where $X = \phi$, ψ_α, or F. We have thus proved that when both the action and the functional integral volume element are invariant under supersymmetry, the algebra closes in general for the matter supermultiplet.

a. Claim 1

When the action and the functional integral volume element are both invariant under a supersymmetry transformation, the algebra of the supersymmetry is always closed. (It is possible that this might be generalized to other symmetries.)

B. SUSY of Matter Field

1. Conventional Spin 1/2 Fermion and Spin 0 Boson

Although we can introduce any number of bosons and fermions in a multiplet in general, here we consider the minimum case, which means the minimum number of fields which make the action and the functional integration invariant under the SUSY transformation. As we saw in the previous subsubsection, we need at least two boson fields and one fermion field, and the minimum free field action was given by

$$S = \int \frac{d^4 p}{(2\pi)^4} \left[\phi^* O_\phi \phi + \psi^{\dagger}_{\dot{\alpha}} O_{\psi}^{\dot{\alpha}\alpha} \psi_\alpha + F^* O_F F\right]. \tag{4.43}$$

From the conditions of invariance of the action and the functional integral volume element, we obtained (4.54):

$$h\left(\epsilon\right)^{\alpha} = g^{\dagger}\left(\epsilon\right)_{\dot{\alpha}} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_\phi}, \tag{4.54}$$

and by eliminating $g^{\dagger}(\epsilon)_{\dot{\alpha}}$ and using (4.55),

$$h(\epsilon)^{\alpha} = -\varepsilon_{\dot{\alpha}\dot{\beta}} \frac{O_{\psi}^{\dot{\beta}\eta}}{O_F} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_{\phi}} h(\epsilon)_{\eta}$$
$$= -\varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon_{\eta\zeta} \frac{O_{\psi}^{\dot{\beta}\eta}}{O_F} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_{\phi}} h(\epsilon)^{\zeta}. \tag{4.59}$$

This is satisfied when

$$-\varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon_{\eta\zeta} \frac{O_{\psi}^{\dot{\beta}\eta}}{O_F} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_{\phi}} = \delta_{\zeta}^{\alpha}, \tag{4.60}$$

and multiplying δ_{α}^{ζ} on the both sides give

$$\frac{1}{2} \varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon_{\alpha\eta} \frac{O_{\psi}^{\dot{\beta}\eta}}{O_F} \frac{O_{\psi}^{\dot{\alpha}\alpha}}{O_{\phi}} = 1. \tag{4.61}$$

The left hand side of (4.61) is nothing but $\det(O_{\psi})/O_F O_{\phi}$ and we finally obtain

$$\frac{\det(O_{\psi})}{O_F O_{\phi}} = 1. \tag{4.62}$$

This is an important result for showing that the functional integral with SUSY is constant. We will prove this next.

The functional integral Z with SUSY and at a fixed energy-momentum k, with an action is given by (4.43), is

$$Z\left[\Phi'(k), \Phi'^{\dagger}(k)\right] = \int d\Phi'(k)\, d\Phi'^{\dagger}(k)\, e^{i\int \frac{d^4p}{(2\pi)^4}\left[\phi'^{*} O_{\phi}\phi' + \psi'^{\dagger}_{\dot{\alpha}} O_{\psi}^{\dot{\alpha}\alpha}\psi'_{\alpha} + F'^{*} O_F F'\right]} \tag{4.63}$$

where we have defined

$$d\Phi' \equiv d\phi' d\phi'^{*} dF' dF'^{*} d\psi' d\psi'^{\dagger}. \tag{4.64}$$

The functional integral of the bosonic part is

$$Z_{b-\phi}(k) = \int d\phi'^{*}(k)\, d\phi'(k)\, e^{i\int \frac{d^4p}{(2\pi)^4}\left[\phi'^{*} O_{\phi}\phi'\right]}$$
$$= \frac{\pi (2\pi)^4}{-iO_{\phi}(k)}, \tag{4.65}$$

$$
\begin{aligned}
Z_{b-F}(k) &= \int dF'^*(k)\, dF'(k)\, e^{i\int \frac{d^4p}{(2\pi)^4}[F'^* O_F F']} \\
&= \frac{\pi (2\pi)^4}{-iO_F(k)}.
\end{aligned} \tag{4.66}
$$

The functional integral of the fermionic part is

$$
\begin{aligned}
Z_f(k) = &\int d\psi'^\dagger(k)\, d\psi'(k)\, e^{i\int \frac{d^4p}{(2\pi)^4}\left[\psi'^\dagger_{\dot\alpha} O_\psi^{\dot\alpha\alpha} \psi'_\alpha\right]} \\
&= \int d\psi'^\dagger_{\dot 1}(k)\, d\psi'_1(k)\, d\psi'^\dagger_{\dot 2}(k)\, d\psi'_2(k) \\
&\cdot (1 + i\int \frac{d^4p}{(2\pi)^4}\left[\psi'^\dagger_{\dot 1} O_\psi^{\dot 1 1}\psi'_1 + \psi'^\dagger_{\dot 2} O_\psi^{\dot 2 2}\psi'_2 + \psi'^\dagger_{\dot 2} O_\psi^{\dot 2 1}\psi'_1 + \psi'^\dagger_{\dot 1} O_\psi^{\dot 1 2}\psi'_2\right] \\
&+ \frac{1}{2!}\int \frac{d^4p}{(2\pi)^4}\left[\psi'^\dagger_{\dot 1} iO_\psi^{\dot 1 1}\psi'_1 + \psi'^\dagger_{\dot 2} iO_\psi^{\dot 2 2}\psi'_2 + \psi'^\dagger_{\dot 2} iO_\psi^{\dot 2 1}\psi'_1 + \psi'^\dagger_{\dot 1} iO_\psi^{\dot 1 2}\psi'_2\right] \\
&\cdot \int \frac{d^4p'}{(2\pi)^4}\left[\psi'^\dagger_{\dot 1} iO_\psi^{\dot 1 1}\psi'_1 + \psi'^\dagger_{\dot 2} iO_\psi^{\dot 2 2}\psi'_2 + \psi'^\dagger_{\dot 2} iO_\psi^{\dot 2 1}\psi'_1 + \psi'^\dagger_{\dot 1} iO_\psi^{\dot 1 2}\psi'_2\right]) \\
&= +\frac{1}{2!}\int d\psi'^\dagger_{\dot 1}(k)\, d\psi'_1(k)\, d\psi'^\dagger_{\dot 2}(k)\, d\psi'_2(k) \int \frac{d^4p d^4p'}{(2\pi)^8} \\
&\cdot [2\psi'^\dagger_{\dot 1}(p)\, iO_\psi^{\dot 1 1}\psi'_1(p)\, \psi'^\dagger_{\dot 2}(p')\, iO_\psi^{\dot 2 2}\psi'_2(p') \\
&\qquad + 2\psi'^\dagger_{\dot 2}(p)\, iO_\psi^{\dot 2 1}\psi'_1(p)\, \psi'^\dagger_{\dot 1}(p')\, iO_\psi^{\dot 1 2}\psi'_2(p')] \\
&= \left[-O_\psi^{\dot 1 1}(k)\, O_\psi^{\dot 2 2}(k) + O_\psi^{\dot 2 1}(k)\, O_\psi^{\dot 1 2}(k)\right]\frac{1}{(2\pi)^8} \\
&= \det\left(-i\frac{O_\psi(k)}{(2\pi)^4}\right).
\end{aligned} \tag{4.67}
$$

Then the total functional integral is

$$
\begin{aligned}
Z[k] &= Z_{b-\phi}(k)\, Z_{b-F}(k)\, Z_f(k) \\
&= \frac{\pi^2 \det\left(-i\frac{O_\psi(k)}{(2\pi)^4}\right)(2\pi)^4 (2\pi)^4}{(-i)\, O_\phi(k)\, (-i)\, O_F(k)} \\
&= \pi^2 = k\text{-independent constant},
\end{aligned} \tag{4.68}
$$

where we have used (4.62). The constant is unimportant because when a physical value is calculated, it is divided by Z as in (4.116). Therefore, we have proved that the functional integration of the matter field SUSY is always constant for the minimal number of fields.

a. Claim 2

For the general free field matter SUSY action (4.43), with the minimal number of fields (one spinor fermion field, one complex scalar boson field, and one complex bosonic auxiliary field), the fixed-k functional integral $Z(k)$ is equal to a constant. (This will work not only for the field theory but also for the supersymmetrization of a statistical model.)

2. Unconventional Spin 1/2 Fermion and Spin 1/2 Boson

Here we will check whether the **Claim 2** is true also for the spin 1/2 fermion and spin 1/2 boson SUSY. We start with one spinor fermion and one spinor boson, and we will check whether this is the minimal number of fields to have the action and functional integral volume element invariance. The action of this model is

$$S = \int d^4x \left[i\tilde{\psi}^{\dagger}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \tilde{\psi}_\alpha + i\psi^{\dagger}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \psi_\alpha \right], \tag{4.69}$$

where $\tilde{\psi}_\alpha$ and ψ_α are the spin 1/2 boson and spin 1/2 fermion, respectively.

As both of the fields are spinor, the SUSY transformation parameter function is scalar:

$$\delta_\theta \tilde{\psi}_\alpha = a(\theta)\, \psi_\alpha, \quad \delta_\theta \tilde{\psi}^{\dagger}_{\dot{\alpha}} = \psi^{\dagger}_{\dot{\alpha}} a^*(\theta), \tag{4.70}$$

$$\delta_\theta \psi_\alpha = b(\theta)\, \tilde{\psi}_\alpha, \quad \delta_\theta \psi^{\dagger}_{\dot{\alpha}} = \tilde{\psi}^{\dagger}_{\dot{\alpha}} b^*(\theta), \tag{4.71}$$

where $a(\theta)$ and $b(\theta)$ are anticommuting scalar functions, and θ is a scalar SUSY parameter. The invariance of the action requires

$$\begin{aligned}
\delta_\theta S = & \int d^4x[i\left(\delta_\theta\tilde{\psi}^\dagger_{\dot{\alpha}}\right)\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\tilde{\psi}_\alpha + i\tilde{\psi}^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\left(\delta_\theta\tilde{\psi}_\alpha\right) \\
& + i\left(\delta_\theta\psi^\dagger_{\dot{\alpha}}\right)\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha + i\psi^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\left(\delta_\theta\psi_\alpha\right)] \\
= & \int d^4x[i\left(\psi^\dagger_{\dot{\alpha}}a^*\left(\theta\right)\right)\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\tilde{\psi}_\alpha + i\tilde{\psi}^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\left(a\left(\theta\right)\psi_\alpha\right) \\
& + i\left(\tilde{\psi}^\dagger_{\dot{\alpha}}b^*\left(\theta\right)\right)\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha + i\psi^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\left(b\left(\theta\right)\tilde{\psi}_\alpha\right)] \\
= & \int d^4x[i\psi^\dagger_{\dot{\alpha}}\left(a^*\left(\theta\right) + b\left(\theta\right)\right)\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\tilde{\psi}_\alpha \\
& i\tilde{\psi}^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\left(a\left(\theta\right) + b^*\left(\theta\right)\right)\partial_\mu\left(\delta_\theta\tilde{\psi}_\alpha\right)] \\
\equiv & \; 0,
\end{aligned} \tag{4.72}$$

and we obtain

$$b\left(\theta\right) = -a^*\left(\theta\right), \tag{4.73}$$

$$a\left(\theta\right) = -b^*\left(\theta\right). \tag{4.74}$$

The SUSY transformation matrix is given as

$$\begin{pmatrix} \tilde{\psi}_\alpha \\ \tilde{\psi}_\alpha \end{pmatrix} = \begin{pmatrix} 1 & a\left(\theta\right) \\ b\left(\theta\right) & 1 \end{pmatrix}\begin{pmatrix} \tilde{\psi}'_\alpha \\ \tilde{\psi}'_\alpha \end{pmatrix} = \tilde{M}\begin{pmatrix} \tilde{\psi}'_\alpha \\ \tilde{\psi}'_\alpha \end{pmatrix}, \tag{4.75}$$

and the invariance of the functional integral volume element requires

$$D\tilde{\psi}_\alpha D\psi_\alpha = D\tilde{\psi}'_\alpha D\psi'_\alpha \text{sdet}\left(\tilde{M}\right) = D\tilde{\psi}'_\alpha D\psi'_\alpha, \tag{4.76}$$

which means

$$\begin{aligned}
\mathrm{sdet}\left(\tilde{M}\right) &= \mathrm{sdet}\left(\left(\begin{array}{cc} 1 & a\left(\theta\right) \\ b\left(\theta\right) & 1 \end{array}\right)\right) \\
&= \det\left(1 - a\left(\theta\right) b\left(\theta\right)\right)\left(\det\left(1\right)\right)^{-1} \\
&\equiv 1, \qquad (4.77)
\end{aligned}$$

so we obtain

$$a\left(\theta\right) b\left(\theta\right) = 0 \quad \rightarrow \quad a\left(\theta\right) = b\left(\theta\right). \qquad (4.78)$$

From (4.73) and (4.78), we have

$$a\left(\theta\right) = b\left(\theta\right) = i\theta, \qquad (4.79)$$

where θ is a real Grassmann scalar, and we have obtained SUSY for the spin 1/2 fermion and spin 1/2 boson, in the form

$$\delta_\theta \tilde{\psi}_\alpha = i\theta\psi_\alpha, \quad \delta_\theta \tilde{\psi}^\dagger_{\dot{\alpha}} = -i\psi^\dagger_{\dot{\alpha}}\theta, \qquad (4.80)$$

$$\delta_\theta \psi_\alpha = i\theta\tilde{\psi}_\alpha, \quad \delta_\theta \psi^\dagger_{\dot{\alpha}} = -i\tilde{\psi}^\dagger_{\dot{\alpha}}\theta. \qquad (4.81)$$

The functional integral of this model is

$$\begin{aligned}
Z\left(k\right) &= \int d\tilde{\psi}\left(k\right) d\psi\left(k\right) d\tilde{\psi}^\dagger\left(k\right) d\psi^\dagger\left(k\right) e^{i\int \frac{d^4p}{(2\pi)^4}\left[-\tilde{\psi}^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}p_\mu\tilde{\psi}_\alpha - \psi^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}p_\mu\psi_\alpha\right]} \\
&= \frac{\det\left(i\frac{\bar{\sigma}^{\mu\dot{\alpha}\alpha}k_\mu}{(2\pi)^4}\right)}{\det\left(i\frac{\bar{\sigma}^{\mu\dot{\alpha}\alpha}k_\mu}{(2\pi)^4}\right)}\pi^2 \\
&= \pi^2 = k\text{-constant}, \qquad (4.82)
\end{aligned}$$

where the mathematical details to show

$$\int d\tilde{\psi}\left(k\right) d\tilde{\psi}^\dagger\left(k\right) e^{i\int \frac{d^4p}{(2\pi)^4}\left[-\tilde{\psi}^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}p_\mu\tilde{\psi}_\alpha\right]} = \frac{\pi^2}{\det\left(i\frac{\bar{\sigma}^{\mu\dot{\alpha}\alpha}k_\mu}{(2\pi)^4}\right)}$$

will be given below. Then **Claim 2** is satisfied.

a. Closure of Algebra Check

According to **Claim 1**, since the action and functional integral volume element are invariant, the algebra should be closed. We find

$$\begin{aligned}\left[\delta_{\theta_2}, \delta_{\theta_1}\right] \tilde{\psi}_\alpha &= \delta_{\theta_2}\left(i\theta_1\psi_\alpha\right) - \delta_{\theta_1}\left(i\theta_2\psi_\alpha\right) \\ &= -\left(\theta_1\theta_2 - \theta_2\theta_1\right)\tilde{\psi}_\alpha,\end{aligned} \tag{4.83}$$

$$\begin{aligned}\left[\delta_{\theta_2}, \delta_{\theta_1}\right] \psi_\alpha &= \delta_{\theta_2}\left(i\theta_1\tilde{\psi}_\alpha\right) - \delta_{\theta_1}\left(i\theta_2\tilde{\psi}_\alpha\right) \\ &= -\left(\theta_1\theta_2 - \theta_2\theta_1\right)\psi_\alpha,\end{aligned} \tag{4.84}$$

and therefore the algebra is in fact closed.

3. Unconventional Spin 0 Fermion and Spin 0 Boson

Here we will check whether **Claim 2** is true also for a spin 0 fermion and spin 0 boson SUSY. We start with one scalar fermion and one scalar boson, and we will check whether this is the minimal number of fields to have the action and functional integral volume element invariance. The action of this model is

$$S = \int d^4x \left[\phi^* \partial^\mu \partial_\mu \phi + \tilde{\phi}^* \partial^\mu \partial_\mu \tilde{\phi}\right], \tag{4.85}$$

where ϕ and $\tilde{\phi}$ are the spin 0 boson and spin 0 fermion, respectively.

As both of the fields are scalar, the SUSY transformation parameter function is scalar:

$$\delta_\theta \phi = c(\theta)\,\tilde{\phi}, \quad \delta_\theta \phi^* = \tilde{\phi}^* c^*(\theta)\,, \tag{4.86}$$

$$\delta_\theta \tilde{\phi} = d(\theta)\,\phi, \quad \delta_\theta \tilde{\phi}^* = \phi^* d^*(\theta)\,, \tag{4.87}$$

where $c(\theta)$ and $d(\theta)$ are anticommuting scalar functions, and θ is a scalar SUSY parameter. The invariance of the action requires

$$\begin{aligned}
\delta_\theta S = &\int d^4x[(\delta_\theta \phi^*)\,\partial^\mu \partial_\mu \phi + \phi^* \partial^\mu \partial_\mu (\delta_\theta \phi) \\
&\qquad + \left(\delta_\theta \tilde{\phi}^*\right) \partial^\mu \partial_\mu \tilde{\phi} + \tilde{\phi}^* \partial^\mu \partial_\mu \left(\delta_\theta \tilde{\phi}\right)] \\
= &\int d^4x[\left(\tilde{\phi}^* c^*(\theta)\right) \partial^\mu \partial_\mu \phi + \phi^* \partial^\mu \partial_\mu \left(c(\theta)\,\tilde{\phi}\right) \\
&\qquad + (\phi^* d^*(\theta))\,\partial^\mu \partial_\mu \tilde{\phi} + \tilde{\phi}^* \partial^\mu \partial_\mu (d(\theta)\,\phi)] \\
= &\int d^4x[\tilde{\phi}^* [c^*(\theta) + d(\theta)]\,\partial^\mu \partial_\mu \phi + \phi^* [d^*(\theta) + c(\theta)]\,\partial^\mu \partial_\mu \tilde{\phi}] \\
\equiv\ & 0,
\end{aligned} \tag{4.88}$$

and we obtain

$$d(\theta) = -c^*(\theta)\,, \tag{4.89}$$

$$c(\theta) = -d^*(\theta)\,. \tag{4.90}$$

The SUSY transformation matrix is given as

$$\begin{pmatrix} \tilde{\psi}_\alpha \\ \tilde{\psi}_\alpha \end{pmatrix} = \begin{pmatrix} 1 & c(\theta) \\ d(\theta) & 1 \end{pmatrix} \begin{pmatrix} \tilde{\psi}'_\alpha \\ \tilde{\psi}'_\alpha \end{pmatrix} = \tilde{M}' \begin{pmatrix} \tilde{\psi}'_\alpha \\ \tilde{\psi}'_\alpha \end{pmatrix}, \tag{4.91}$$

and the invariance of the functional integral volume element requires

$$D\phi D\tilde{\phi} = D\phi' D\tilde{\phi}' \mathrm{sdet}\left(\tilde{M}'\right) = D\phi' D\tilde{\phi}', \tag{4.92}$$

which means

$$\begin{aligned}
\mathrm{sdet}\left(\tilde{M}\right) &= \mathrm{sdet}\left(\left(\begin{array}{cc} 1 & c(\theta) \\ d(\theta) & 1 \end{array}\right)\right) \\
&= \det\left(1 - c(\theta)\, d(\theta)\right) \left(\det(1)\right)^{-1} \\
&\equiv 1, \qquad (4.93)
\end{aligned}$$

so we obtain

$$c(\theta)\, d(\theta) = 0 \quad \rightarrow \quad c(\theta) = d(\theta)\,. \qquad (4.94)$$

From (4.89) and (4.94), we have

$$c(\theta) = d(\theta) = i\theta, \qquad (4.95)$$

where θ is a real Grassmann scalar, and we have obtained SUSY for the spin 0 fermion and spin 0 boson, which has the form

$$\delta_\theta \phi = i\theta\tilde{\phi}, \quad \delta_\theta \phi^* = -i\tilde{\phi}^*\theta, \qquad (4.96)$$

$$\delta_\theta \tilde{\phi} = i\theta\phi, \quad \delta_\theta \tilde{\phi}^* = -i\phi^*\theta. \qquad (4.97)$$

The functional integral of this model is

$$\begin{aligned}
Z(k) &= \int d\phi^*(k)\, d\phi(k)\, d\tilde{\phi}^*(k)\, d\tilde{\phi}(k)\, e^{i\int \frac{d^4p}{(2\pi)^4}\left[-\phi^* p^\mu p_\mu \phi - \tilde{\phi}^* p^\mu p_\mu \tilde{\phi}\right]} \\
&= \frac{\det\left(i\frac{k^\mu k_\mu}{(2\pi)^4}\right)}{\det\left(i\frac{k^\mu k_\mu}{(2\pi)^4}\right)}\pi \\
&= \pi = k\text{-independent}, \qquad (4.98)
\end{aligned}$$

and **Claim 2** is satisfied in this case also.

a. Closure of Algebra Check

According to **Claim 1**, since the action and functional integral volume element are invariant, the algebra should be closed.

$$\begin{aligned}\left[\delta_{\theta_2}, \delta_{\theta_1}\right] \phi &= \delta_{\theta_2} \left(i\theta_1 \tilde{\phi}\right) - \delta_{\theta_1} \left(i\theta_2 \tilde{\phi}\right) \\ &= -\left(\theta_1\theta_2 - \theta_2\theta_1\right) \phi,\end{aligned} \tag{4.99}$$

$$\begin{aligned}\left[\delta_{\theta_2}, \delta_{\theta_1}\right] \tilde{\phi} &= \delta_{\theta_2} \left(i\theta_1 \phi\right) - \delta_{\theta_1} \left(i\theta_2 \phi\right) \\ &= -\left(\theta_1\theta_2 - \theta_2\theta_1\right) \tilde{\phi},\end{aligned} \tag{4.100}$$

and therefore the algebra is closed.

C. SUSY of Gauge Field

We saw that the fixed-k functional integral of the free matter field supermultiplets is k-independent. Here we will extend the argument to the gauge supermultiplet. First, we will calculate the functional integral of the gauge field, and we consider the action of the non-interacting gauge field given by

$$\begin{aligned}
S &= \int d^4x - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} && (4.101)\\
&= \int d^4x - \frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\\
&= \int d^4x \frac{1}{2} g^{\nu\xi} A_\nu \partial_\mu \partial^\mu A_\xi + \frac{1}{2} \left(g^{\mu\xi} \partial_\mu A_\xi\right)^2\\
&= \int d^4x \int \frac{d^4p d^4p'}{(2\pi)^8} e^{i(p'+p)\cdot x} \Bigg[-\frac{1}{2} g^{\nu\xi} A_\nu (p') p_\mu p^\mu A_\xi (p)\\
&\qquad - \frac{1}{2} \left(g^{\mu\xi} p_\mu A_\xi (p')\right) \left(g^{\nu\rho} p_\nu A_\rho (p)\right)\Bigg]\\
&= \int \frac{d^4p}{(2\pi)^4} \left[-\frac{1}{2} g^{\nu\xi} A_\nu (-p) p_\mu p^\mu A_\xi (p) - \frac{1}{2} \left(-g^{\mu\xi} p_\mu A_\xi (-p)\right) \left(g^{\nu\rho} p_\nu A_\rho (p)\right)\right]\\
&= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \left[-g^{\nu\xi} A_\nu (p) p_\mu p^\mu A_\xi (p) + \left(g^{\mu\xi} p_\mu A_\xi (p)\right) \left(g^{\nu\rho} p_\nu A_\rho (p)\right)\right]\\
&= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \end{pmatrix} O_A \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}, && (4.102)
\end{aligned}$$

where in the 6th line we have used $A_\nu (-p) = A_\nu (p)$, which follows from

$$A_\nu (p) = \int \frac{d^4x}{(2\pi)^2} A_\nu (x) e^{-ip\cdot x}.$$

The complex conjugate becomes

$$\begin{aligned}
A_\nu (p) &= \int \frac{d^4x}{(2\pi)^2} A_\nu (x) e^{ip\cdot x}\\
&= A_\nu (-p).
\end{aligned}$$

O_A in (4.102) is given by

$$O_A = \begin{pmatrix} p_1^2 + p_2^2 + p_3^2 & -p_0 p_1 & -p_0 p_2 & -p_0 p_3 \\ -p_1 p_0 & p_0^2 - p_2^2 - p_3^2 & p_1 p_2 & p_1 p_3 \\ -p_2 p_0 & p_2 p_1 & p_0^2 - p_1^2 - p_3^2 & p_2 p_3 \\ -p_3 p_0 & p_3 p_1 & p_2 p_1 & p_0^2 - p_1^2 - p_2^2 \end{pmatrix}$$

Since O_A is Hermitian, there exists a unitary operator to diagonalize it, and we can do the fixed-k functional integration.

$$\begin{aligned} Z\left(k\right) &= \int dA_0\left(k\right) dA_1\left(k\right) dA_2\left(k\right) dA_3\left(k\right) e^{iS} \\ &= \frac{\pi^2}{\left(\det\left(-iO_A / \left[2\left(2\pi\right)^4\right]\right)\right)^{1/2}}, \end{aligned}$$

where we have used the fact that the determinant is invariant under the unitary transformation. However, when we calculate the determinant, it turns out that

$$\det O_A = 0,$$

and we cannot define the functional integral for the gauge field when we use the action given by (4.101)).

To define the functional integral for the gauge field, we introduce a gauge fixing term into the action:

$$\begin{aligned} S_\alpha &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)\right] \\ &= \frac{1}{2} \int \frac{d^4p}{\left(2\pi\right)^4} \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \end{pmatrix} O_{A\alpha} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}. \end{aligned}$$

where we have defined

$$O_{A\alpha} = \begin{pmatrix} p_1^2+p_2^2+p_3^2-\frac{1}{\alpha}p_0^2 & -p_0p_1+\frac{1}{\alpha}p_0p_1 & -p_0p_2+\frac{1}{\alpha}p_0p_2 & -p_0p_3+\frac{1}{\alpha}p_0p_3 \\ -p_1p_0+\frac{1}{\alpha}p_1p_0 & p_0^2-p_2^2-p_3^2-\frac{1}{\alpha}p_1^2 & p_1p_2-\frac{1}{\alpha}p_1p_2 & p_1p_3-\frac{1}{\alpha}p_1p_3 \\ -p_2p_0+\frac{1}{\alpha}p_2p_0 & p_2p_1-\frac{1}{\alpha}p_2p_1 & p_0^2-p_1^2-p_3^2-\frac{1}{2\alpha}p_2^2 & p_2p_3-\frac{1}{\alpha}p_2p_3 \\ -p_3p_0+\frac{1}{\alpha}p_3p_0 & p_3p_1-\frac{1}{\alpha}p_3p_1 & p_2p_1-\frac{1}{\alpha}p_2p_1 & p_0^2-p_1^2-p_2^2-\frac{1}{\alpha}p_3^2 \end{pmatrix}.$$

The fixed-k functional integration with the gauge fixing term yields

$$\begin{aligned} Z_A(k) &= \int dA_0(k)\, dA_1(k)\, dA_2(k)\, dA_3(k)\, e^{iS_\alpha} \\ &= \frac{\pi^2}{\left(\det\left(-iO_{A\alpha}(k)/\left[2(2\pi)^4\right]\right)\right)^{1/2}} \\ &= \frac{4\sqrt{-\alpha}\pi^2(2\pi)^8}{\left(-k_0^2+k_1^2+k_2^2+k_3^2\right)^2}, \end{aligned}$$

since

$$\det\left(O_{A\alpha}(k)/\left[2(2\pi)^4\right]\right) = -\frac{1}{16(2\pi)^{16}\alpha}\left(-k_0^2+k_1^2+k_2^2+k_3^2\right)^4.$$

With the gauge fixing term, we can define the functional integral.

The fixed-k functional integral for a spin 1/2 gaugino is

$$\begin{aligned} Z_\lambda(k) &= \int d\lambda^\dagger(k)\, d\lambda(k)\, e^{i\int d^4x\left(i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda\right)} \\ &= \int d\lambda^\dagger(k)\, d\lambda(k)\, e^{i\int \frac{d^4p}{(2\pi)^4}\left(-\lambda^\dagger\bar{\sigma}^\mu p_\mu\lambda\right)} \\ &= \det\left(i\frac{\bar{\sigma}^\mu k_\mu}{(2\pi)^4}\right) \\ &= \left(-k_0^2+k_1^2+k_2^2+k_3^2\right)\frac{1}{(2\pi)^8}. \end{aligned}$$

The fixed-k functional integral of a free ghost is

$$\begin{aligned} Z_c(k) &= \int d\bar{c}(k)\, dc(k)\, e^{i\int d^4x\left(-\bar{c}\partial^\mu\partial_\mu c\right)} \\ &= \int d\bar{c}(k)\, dc(k)\, e^{i\int \frac{d^4p}{(2\pi)^4}\left(\bar{c}p^\mu p_\mu c\right)} \\ &= -i\left(-k_0^2+k_1^2+k_2^2+k_3^2\right)\frac{1}{(2\pi)^4}. \end{aligned}$$

Then the overall fixed-k functional integral $Z(k)$ is

$$\begin{aligned} Z(k) &= Z_A(k)\, Z_\lambda(k)\, Z_c(k) \\ &= \frac{4\sqrt{-\alpha}\pi^2(2\pi)^8\left(-k_0^2+k_1^2+k_2^2+k_3^2\right)(-i)\left(-k_0^2+k_1^2+k_2^2+k_3^2\right)}{\left(-k_0^2+k_1^2+k_2^2+k_3^2\right)^2(2\pi)^4(2\pi)^8} \\ &= (-i)\frac{4}{(2\pi)^4}\sqrt{-\alpha}\pi^2. \end{aligned}$$

The i comes from $Z_c(k)$, and the non-cancellation of an overall i can be interpreted to mean that we failed to include all of the degrees of freedom. To cancel i, we introduce one complex bosonic auxiliary field or two real bosonic auxiliary fields. In addition, the functional integral is required to be independent of the gauge-fixing constant α. Then the auxiliary field action can be written as

$$S_{\text{auxiliary}} = \int \frac{d^4p}{(2\pi)^4}\left[\alpha B^2 + D^2\right],$$

where B can be an auxiliary field associated with the BRST symmetry and D can be the auxiliary field of SUSY. Then the auxiliary fields' functional integral is

$$\begin{aligned} Z_{\text{auxiliary}}(k) &= \int dB(k)\, dD(k)\, e^{i\int \frac{d^4p}{(2\pi)^4}\left[\alpha B^2+D^2\right]} \\ &= \frac{\sqrt{\pi}}{\sqrt{-i\left(\alpha/(2\pi)^4\right)}}\frac{\sqrt{\pi}}{\sqrt{-i/(2\pi)^4}} \\ &= \frac{\pi(2\pi)^4}{-i\sqrt{\alpha}}. \end{aligned}$$

The overall fixed-k functional integral including the auxiliary fields becomes

$$
\begin{aligned}
Z(k) &= Z_A(k)\, Z_\lambda(k)\, Z_c(k)\, Z_{\text{auxiliary}}(k) \\
&= (-i)\,\frac{4}{(2\pi)^4}\sqrt{-\alpha}\pi^2\frac{\pi\,(2\pi)^4}{-i\sqrt{\alpha}} \\
&= 4\pi^2\sqrt{-1} = k \text{ independent.}
\end{aligned}
$$

Therefore, according to our proposed definition of SUSY, which is the requirement that $Z(k)$ be k independent, SUSY for the gauge field requires the action to be

$$
\begin{aligned}
S = \int d^4x[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\left(\partial^\mu A_\mu\right)^2 + i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda \\
&- \bar{c}\partial^\mu\partial_\mu c + \alpha B^2 + D^2].
\end{aligned}
$$

D. Primitive Supersymmetry and Standard Supersymmetry in the Present Theory

In the next chapter, we will start with a very simple microscopic (Planck-scale) statistical picture and will obtain the following purely bosonic Euclidean action, given in the next chapter as (5.40):

$$
\bar{S}_E\left[\Psi_b, \Psi_b^\dagger\right] = \int d^Dx\left(\frac{1}{2m}\partial^M\Psi_b^\dagger\partial_M\Psi_b - \mu\,\Psi_b^\dagger\Psi_b + i\widetilde{V}\,\Psi_b^\dagger\Psi_b\right). \tag{4.103}
$$

If F is a physical quantity determined by the observable fields Ψ_b,$\Psi_b^\dagger$ and the random potential $i\tilde{V}$, its average value is given by

$$
\langle F\rangle = \left\langle\frac{\int D\Psi_b D\Psi_b^\dagger F\left[\Psi_b, \Psi_b^\dagger, \tilde{V}\right]e^{-\bar{S}_E\left[\Psi_b,\Psi_b^\dagger\right]}}{\int D\Psi_b' D\Psi_b'^\dagger e^{-\bar{S}_E\left[\Psi_b',\Psi_b'^\dagger\right]}}\right\rangle \tag{4.104}
$$

where $\langle\cdots\rangle$ here means an average over the postulated random imaginary potential $i\tilde{V}$ of (5.40), which has a Gaussian distribution and satisfies

$$
\left\langle\tilde{V}\right\rangle = 0, \quad \left\langle\tilde{V}(x)\,\tilde{V}(x')\right\rangle = b\delta(x-x')\,. \tag{4.105}
$$

The presence of the denominator makes it difficult to perform this average, but we replace the degrees of freedom in the denominator with fermionic degrees of freedom Ψ_f by using a mathematical trick which is standard for treating random or disordered system in condensed matter physics:

$$\frac{1}{\int D\Psi'_b D\Psi'^{\dagger}_b e^{-\bar{S}_E\left[\Psi_b,\Psi_b^{\dagger}\right]}} = \det A = \int D\Psi_f D\Psi_f^{\dagger} e^{-\bar{S}_E\left[\Psi_f,\Psi_f^{\dagger}\right]}, \tag{4.106}$$

where $-\bar{S}_E\left[\Psi_b,\Psi_b^{\dagger}\right] \equiv \Psi_b^{\dagger} A \Psi_b$. Then we obtain

$$\begin{aligned}\langle F\rangle &= \left\langle \int D\Psi_b D\Psi_b^{\dagger} D\Psi_f D\Psi_f^{\dagger}\ F\left[\Psi_b,\Psi_b^{\dagger},\tilde{V}\right] e^{-\bar{S}_E\left[\Psi_b,\Psi_b^{\dagger}\right]} \int D\Psi_f D\Psi_f^{\dagger} e^{-\bar{S}_E\left[\Psi_f,\Psi_f^{\dagger}\right]} \right\rangle \\ &= \left\langle \int D\Psi D\Psi^{\dagger}\ F\left[\Psi_b,\Psi_b^{\dagger},\tilde{V}\right] e^{-\bar{S}_E\left[\Psi,\Psi^{\dagger}\right]} \right\rangle. \end{aligned} \tag{4.107}$$

where we have grouped the bosonic and fermionic fields in vector form:

$$\Psi = \begin{pmatrix} \Psi_b \\ \Psi_f \end{pmatrix}. \tag{4.108}$$

The Euclidean action with both bosons and fermions still has the basic form of (4.103):

$$\bar{S}_E\left[\Psi,\Psi^{\dagger}\right] = \int d^D x \left[\frac{1}{2m}\partial^M \Psi^{\dagger}\partial_M \Psi - \mu\Psi^{\dagger}\Psi + i\tilde{V}\Psi^{\dagger}\Psi\right]. \tag{4.109}$$

For a Gaussian random variable υ whose mean is zero, the result

$$\left\langle e^{-i\upsilon}\right\rangle = e^{-\frac{1}{2}\left\langle \upsilon^2\right\rangle}, \tag{4.110}$$

where

$$\begin{aligned}\left\langle e^{-i\upsilon}\right\rangle &= \sqrt{\frac{a}{\pi}}\int d\upsilon e^{-i\upsilon}e^{-a\upsilon^2} \\ &= \sqrt{\frac{a}{\pi}}\int d\upsilon e^{-a\left(\upsilon+\frac{i}{2a}\right)^2-\frac{1}{4a}} \\ &= e^{-\frac{1}{4a}}\end{aligned}$$

and

$$
\begin{aligned}
e^{-\frac{1}{2}\langle v^2\rangle} &= \exp\left(-\frac{1}{2}\sqrt{\frac{a}{\pi}}\int dv v^2 e^{-av^2}\right)\\
&= \exp\left(-\frac{1}{2}\sqrt{\frac{a}{\pi}}\left[\frac{-1}{2a}ve^{-av^2}\right]_{-\infty}^{\infty} + \frac{1}{2}\sqrt{\frac{a}{\pi}}\left(\frac{-1}{2a}\right)\int dv e^{-av^2}\right)\\
&= \exp\left(-\frac{1}{4a}\right),
\end{aligned}
$$

implies that

$$
\begin{aligned}
\left\langle e^{-\int d^Dx i\tilde{V}\Psi^\dagger\Psi}\right\rangle &= e^{-\frac{1}{2}\int d^Dx d^Dx' \Psi^\dagger(x)\Psi(x)\left\langle \tilde{V}(x)\tilde{V}(x')\right\rangle \Psi^\dagger(x')\Psi(x')}\\
&= e^{-\frac{1}{2}b\int d^Dx\left[\Psi^\dagger(x)\Psi(x)\right]^2}, \qquad (4.111)
\end{aligned}
$$

where we have also used (4.105). Then (4.107) can be rewritten as

$$
\langle F\rangle = \int D\Psi D\Psi^\dagger F e^{-S} \qquad (4.112)
$$

with

$$
S = \int d^Dx\left[\frac{1}{2m}\partial^M\Psi^\dagger\partial_M\Psi - \mu\Psi^\dagger\Psi + \frac{1}{2}b\left(\Psi^\dagger\Psi\right)^2\right]. \qquad (4.113)
$$

This action clearly has a primitive supersymmetry, under a global rotation of Ψ which transforms bosons into fermions and vice-versa. The functional integral Z is

$$
Z = \int D\Psi D\Psi^\dagger e^{-S}, \qquad (4.114)
$$

and according to (4.104) with $F = 1$, we have just

$$
Z = 1. \qquad (4.115)
$$

To make the expression for $\langle F\rangle$ independent of how the measure is defined in the functional integral, we can rewrite (4.112) as

$$
\langle F\rangle = \frac{1}{Z}\int D\Psi D\Psi^\dagger F e^{-S}. \qquad (4.116)
$$

Notice that the fermionic variables Ψ_f represent true degrees of freedom, and that they originate from the bosonic variables Ψ_b', which is introduced in the denominator of (4.104). The coupling between the fields Ψ_b and Ψ_f (or Ψ_b') is due to the random perturbing potential $i\tilde{V}$.

E. Emergence of the Usual SUSY at Low Energy

After a transformation to Lorentzian spacetime (see the earlier comments on this transformation), the Euclidean action of (4.113) becomes

$$S_L = \int d^D x \left[\frac{1}{2m} \Psi^\dagger \partial^M \partial_M \Psi + \mu^2 \Psi^\dagger \Psi - \frac{1}{2} b \left(\Psi^\dagger \Psi \right)^2 \right] \tag{4.117}$$

where ∂^M is now defined by $\partial^M = \eta^{MN} \partial_N$, with η^{MN} the D-dimensional Minkowski metric tensor. Then if we choose specific fields ψ_b and ψ_f instead of the full set of fields Ψ_b and Ψ_f, from the results of Chapter III, the free field action can be reduced to

$$S_f = \int d^4 x \ {\psi_f}^\dagger \left((2\bar{m})^{-1} \eta^{\mu\nu} \partial_\mu \partial_\nu + i\sigma^\mu D_\mu \right) \psi_f. \tag{4.118}$$

$$S_b = \int d^4 x \, {\psi_b}^\dagger \left((2\bar{m})^{-1} \eta^{\mu\nu} \partial_\mu \partial_\nu + i\sigma^\mu D_\mu \right) \psi_b. \tag{4.119}$$

At low energy, $\bar{m} \gg p^\mu \sim \mathcal{O}\left(TeV\right)$, the first term is very small and we obtain

$$S_f \simeq \int d^4 x \ \left[{\psi_f}^\dagger i \sigma^\mu \partial_\mu \psi_f \right]. \tag{4.120}$$

$$S_b \simeq \int d^4 x \ \left[{\psi_b}^\dagger i \sigma^\mu \partial_\mu \psi_b \right]. \tag{4.121}$$

The functional integral of the bosonic part is

$$Z_b = \int D\psi_b D\psi_b^\dagger e^{i \int d^4 x \, {\psi_b}^\dagger i \sigma^\mu \partial_\mu \psi_b}. \tag{4.122}$$

To perform this functional integration we need to diagonalize the operator. But since

$i\sigma^\mu\partial_\mu$ is Hermitian, there exists a unitary operator U which diagonalizes it. And since $\det\left(U\sigma^\mu\partial_\mu U^\dagger\right) = \det(U)\det(\sigma^\mu\partial_\mu)\det\left(U^\dagger\right) = \det(\sigma^\mu\partial_\mu)$, it follows that

$$\begin{aligned} Z_b(x_i) &= \int d\psi_b(x_i)\, D\psi_b^\dagger(x_i)\, e^{i\int d^4x\, \psi_b{}^\dagger i\sigma^\mu\partial_\mu\psi_b} \\ &= \frac{\pi^2}{\det(\sigma^\mu\partial_\mu)}. \end{aligned}$$

We now consider this process in more detail, with an explicit expression for

$$\begin{aligned} U &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a_R + ia_I & b_R + ib_I \\ c_R + ic_I & d_R + id_I \end{pmatrix}. \end{aligned} \tag{4.123}$$

The unitary condition requires that

$$b = -c^*, \tag{4.124}$$

$$d = a^*, \tag{4.125}$$

$$1 = |a|^2 + |c|^2, \tag{4.126}$$

and

$$U = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix}. \tag{4.127}$$

Now, after we transform into momentum space, with

$$\int d^4x\, \psi_b{}^\dagger i\sigma^\mu\partial_\mu\psi_b = \int \frac{d^4p}{(2\pi)^4}\, \psi_b{}^\dagger\left(-\sigma^\mu p_\mu\right)\psi_b,$$

we diagonalize $\sigma^\mu p_\mu$ by using the unitary operator:

$$U\left(-\sigma^\mu p_\mu\right)U^\dagger = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix} \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} \begin{pmatrix} a^* & c^* \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix} \begin{pmatrix} \left(p^0 - p^3\right)a^* & \left(p^0 - p^3\right)c^* \\ +\left(p^1 - ip^2\right)c & -\left(p^1 - ip^2\right)a \\ -\left(p^1 + ip^2\right)a^* & -\left(p^1 + ip^2\right)c^* \\ -\left(p^0 + p^3\right)c & +\left(p^0 + p^3\right)a \end{pmatrix}$$

$$= \begin{pmatrix} \left(p^0 - p^3\right)|a|^2 + \left(p^0 + p^3\right)|c|^2 & -2p^3ac^* \\ +\left(p^1 - ip^2\right)ac + \left(p^1 + ip^2\right)a^*c^* & -\left(p^1 - ip^2\right)a^2 + \left(p^1 + ip^2\right)c^{*2} \\ -2p^3a^*c & \left(p^0 + p^3\right)|a|^2 + \left(p^0 - p^3\right)|c|^2 \\ +\left(p^1 - ip^2\right)c^2 - \left(p^1 + ip^2\right)a^{*2} & -\left(p^1 - ip^2\right)ac - \left(p^1 + ip^2\right)a^*c^* \end{pmatrix}. \tag{4.128}$$

Since the off-diagonal components must vanish, we obtain

$$-\left(p^1 - ip^2\right)a^2 + \left(p^1 + ip^2\right)c^{*2} - 2p^3ac^* = 0, \tag{4.129}$$

$$\left(p^1 - ip^2\right)c^2 - \left(p^1 + ip^2\right)a^{*2} - 2p^0a^*c = 0. \tag{4.130}$$

From (4.129)/ac^* we have

$$\left(p^1 - ip^2\right)\frac{a}{c^*} - \left(p^1 + ip^2\right)\frac{c^*}{a} + 2p^3 = 0, \tag{4.131}$$

and the solution for $\frac{a}{c^*}$ is either

$$\frac{a}{c^*} = \frac{p^1 + ip^2}{\left(p^1\right)^2 + \left(p^2\right)^2}\left[-p^3 - \sqrt{\left(p^1\right)^2 + \left(p^2\right)^2 + \left(p^3\right)^2}\right] \tag{4.132}$$

or else

$$\frac{a}{c^*} = \frac{p^1 + ip^2}{\left(p^1\right)^2 + \left(p^2\right)^2}\left[-p^3 + \sqrt{\left(p^1\right)^2 + \left(p^2\right)^2 + \left(p^3\right)^2}\right]. \tag{4.133}$$

Here we choose

$$\frac{a}{c^*} = \frac{p^1 + ip^2}{(p^1)^2 + (p^2)^2} \left[-p^3 - \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right] \tag{4.134}$$

and we obtain

$$a = \frac{p^1 + ip^2}{(p^1)^2 + (p^2)^2} \left[-p^3 - \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right] c^*, \tag{4.135}$$

$$a^* = \frac{p^1 + ip^2}{(p^1)^2 + (p^2)^2} \left[-p^3 - \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right] c. \tag{4.136}$$

Then

$$|a|^2 = a^* a = \frac{\left[p_3 + \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right]^2}{(p^1)^2 + (p^2)^2} |c|^2 . \tag{4.137}$$

Since $|a|^2 + |c|^2 = 1$, we then obtain

$$|c|^2 = \frac{p_1^2 + p_2^2}{2 \left[(p^1)^2 + (p^2)^2 + (p^3)^2 + p_3 \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right]}, \tag{4.138}$$

and

$$|a|^2 = \frac{\left[p_3 + \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right]^2}{2 \left[(p^1)^2 + (p^2)^2 + (p^3)^2 + p_3 \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right]}. \tag{4.139}$$

Since the diagonal components of $U\bar{\sigma}^\mu p_\mu U^\dagger$ involve ac and a^*c^*, from (4.135)$\times c$ and (4.138) we obtain

$$\begin{aligned} ac &= \frac{p^1 + ip^2}{p_1^2 + p_2^2} \left[-p^3 - \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right] |c|^2 \\ &= \frac{\left[-p^3 - \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right]}{2 \left[(p^1)^2 + (p^2)^2 + (p^3)^2 + p_3 \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \right]} \left(p^1 + ip^2 \right) \end{aligned} \tag{4.140}$$

$$a^* c^* = \frac{\left[-p^3 - \sqrt{p_1^2 + p_2^2 + p_3^2}\right]}{2\left[(p^1)^2 + (p^2)^2 + (p^3)^2 + p^3\sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}\right]} \left(p^1 - ip^2\right). \tag{4.141}$$

Therefore, the $(1,1)$ component of $U\sigma^\mu p_\mu U^\dagger$ becomes

$$\begin{aligned}\left[U\bar{\sigma}^\mu p_\mu U^\dagger\right]_{(1,1)} &= \left(p^0 - p^3\right)|a|^2 + \left(p^0 + p^3\right)|c|^2 + \left(p^1 - ip^2\right) ac + \left(p^1 + ip^2\right) a^* c^* \\ &= p^0 - \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \\ &= p^0 - |\vec{p}|, \end{aligned} \tag{4.142}$$

and the $(2,2)$ component of $U\sigma^\mu p_\mu U^\dagger$ becomes

$$\begin{aligned}\left[U\sigma^\mu p_\mu U^\dagger\right]_{(2,2)} &= \left(p^0 + p^3\right)|a|^2 + \left(p^0 - p^3\right)|c|^2 - \left(p^1 - ip^2\right) ac - \left(p^1 + ip^2\right) a^* c^* \\ &= p^0 + \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \\ &= p^0 + |\vec{p}|. \end{aligned} \tag{4.143}$$

The diagonalized operator has thus turned out to be

$$U\left(-\sigma^\mu p_\mu\right) U^\dagger = \begin{pmatrix} p^0 - |\vec{p}| & 0 \\ 0 & p^0 + |\vec{p}| \end{pmatrix} \tag{4.144}$$

with

$$U = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix}, \tag{4.145}$$

where a and c satisfy

$$|a|^2 = \frac{\left[p^3 + \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}\right]^2}{2\left[(p^1)^2 + (p^2)^2 + (p^3)^2 + p^3\sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}\right]}, \tag{4.146}$$

$$|c|^2 = \frac{(p^1)^2 + (p^2)^2}{2\left[(p^1)^2 + (p^2)^2 + (p^3)^2 + p^3\sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}\right]}. \tag{4.147}$$

Therefore we obtain

$$\begin{aligned}
\int d\tilde{\psi}^{\dagger}(k)\, d\tilde{\psi}(k)\, e^{i\int d^4x\left(i\tilde{\psi}^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\tilde{\psi}\right)} &= \int d\tilde{\psi}^{\dagger}(k)\, d\tilde{\psi}(k) \det\left(U^{\dagger}\right)\det\left(U\right) \\
&\quad \cdot \exp\left[-i\int \frac{d^4p}{(2\pi)^4}\tilde{\psi}'^{\dagger}\left(U\bar{\sigma}^{\mu}p_{\mu}U^{\dagger}\right)\tilde{\psi}'\right] \\
&= \frac{\pi\,(2\pi)^4}{\det\left[i\left(p^0-|\vec{p}|\right)\right]}\frac{\pi\,(2\pi)^4}{\det\left[i\left(p^0+|\vec{p}|\right)\right]} \\
&= \frac{\pi^2\,(2\pi)^8}{\det\left[-i\left(\left(p^0\right)^2-|\vec{p}|^2\right)\right]\det\left(-i\right)} \\
&= \frac{\pi^2}{\det\left(ip^{\mu}p_{\mu}\right)\det\left(-i\right)}, \qquad (4.148)
\end{aligned}$$

where $\tilde{\psi}' = U\tilde{\psi}$ and $\det(U) = \det\left(U^{\dagger}\right) = 1$.

It is consistent with our earlier Claim 2 to interpret the extra factor of $\det(-i)$ as reflecting the need for an auxiliary field F. We therefore rewrite (4.148) as

$$\begin{aligned}
\int d\tilde{\psi}^{\dagger} d\tilde{\psi} e^{i\int d^4x\left(i\tilde{\psi}^{\dagger}\sigma^{\mu}\partial_{\mu}\tilde{\psi}\right)} &= \frac{1}{\det\left(ip^{\mu}p_{\mu}\right)\det\left(-i\right)} \\
&= \int d\phi d\phi^* dF dF^* e^{i\int \frac{d^4p}{(2\pi)^4}\left[-\phi^* p^{\mu}p_{\mu}\phi + F^*F\right]} \\
&= \int d\phi d\phi^* dF dF^* e^{i\int d^4x\left[\phi^*\partial^{\mu}\partial_{\mu}\phi + F^*F\right]}. \qquad (4.149)
\end{aligned}$$

The primitive spin 1/2 boson $\tilde{\psi}$ has thus been transformed into a spin 0 boson ϕ plus an auxiliary field F.

Originally there were 4 degrees of freedom $\tilde{\psi}$, now transformed into 2 degrees of freedom ϕ and 2 degrees of freedom F. Conservation of the number of degrees of freedom, and the correct form of the action with ϕ and F, was achieved through a straightforward mathematical transformation.

We have therefore obtained

$$S = \int d^4x\left[\phi^*\partial^{\mu}\partial_{\mu}\phi + F^*F + {\psi_f}^{\dagger} i\sigma^{\mu}\partial_{\mu}\psi_f\right] \qquad (4.150)$$

and standard SUSY has emerged as a low energy approximation within the present fundamental theory.

F. Interpretation Using a Matrix Transformation

Our philosophy above (and throughout this dissertation) is that we are allowed to perform any mathematical transformations, starting with the original theory in its most primitive form, and ending with a theory that correctly describes experimental observations, as long as the transformations are mathematically consistent and the predictions for physical quantities $\langle F \rangle = Z^{-1} \int D\Psi^\dagger D\Psi \; F e^{iS}$ are left unchanged. As emphasized above, this means any functional integral is left unchanged.

If we assume a stable vacuum with no negative-energy bosonic states, the transformation from $\tilde{\psi}$ to ϕ and F can be treated more explicitly. First, as we showed above, the operator $-\sigma^\mu p_\mu$ can be diagonalized by a unitary matrix U:

$$\begin{aligned} S &= \int \frac{d^4p}{(2\pi)^4} \left[-\tilde{\psi}^\dagger \sigma^\mu p_\mu \tilde{\psi} \right] \\ &= \int \frac{d^4p}{(2\pi)^4} \left[-\tilde{\psi}^\dagger U^\dagger U \sigma^\mu p_\mu U^\dagger U \tilde{\psi} \right] \\ &= \int \frac{d^4p}{(2\pi)^4} \left[\tilde{\psi}'^\dagger \begin{pmatrix} p^0 - |\vec{p}| & 0 \\ 0 & p^0 + |\vec{p}| \end{pmatrix} \tilde{\psi}' \right], \end{aligned} \tag{4.151}$$

where

$$\tilde{\psi}' = U\tilde{\psi}. \tag{4.152}$$

with

$$\tilde{\psi}' = \begin{pmatrix} \tilde{\psi}'_1 \\ \tilde{\psi}'_2 \end{pmatrix}. \tag{4.153}$$

Now we relate $\tilde{\psi}'$ to ϕ and F through

$$\begin{pmatrix} \phi \\ F \end{pmatrix} = M \begin{pmatrix} \tilde{\psi}'_1 \\ \tilde{\psi}'_2 \end{pmatrix}. \tag{4.154}$$

(This transformation has determinant 1, and thus keeps the functional integral invariant, even though the trace is not conserved.) Then the action can be rewritten as

$$\begin{aligned} S &= \int \frac{d^4p}{(2\pi)^4} \left[\tilde{\psi}'^{\dagger} \begin{pmatrix} p^0 - |\vec{p}| & 0 \\ 0 & p^0 + |\vec{p}| \end{pmatrix} \tilde{\psi}' \right] \\ &= \int \frac{d^4p}{(2\pi)^4} \left[\tilde{\psi}'^{\dagger} M M^{-1} \begin{pmatrix} p^0 - |\vec{p}| & 0 \\ 0 & p^0 + |\vec{p}| \end{pmatrix} M^{-1} M \tilde{\psi}' \right] \\ &= \int \frac{d^4p}{(2\pi)^4} \left[\begin{pmatrix} \phi^* & F^* \end{pmatrix} \begin{pmatrix} (p^0)^2 - |\vec{p}|^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ F \end{pmatrix} \right] \\ &= \int d^4x \left[\begin{pmatrix} \phi^* & F^* \end{pmatrix} \begin{pmatrix} \partial^\mu \partial_\mu & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ F \end{pmatrix} \right], \end{aligned} \tag{4.155}$$

where

$$M = \begin{pmatrix} \frac{1}{\sqrt{p^0 + |\vec{p}|}} & 0 \\ 0 & \sqrt{p^0 + |\vec{p}|} \end{pmatrix}, \tag{4.156}$$

$$M^{-1} = \begin{pmatrix} \sqrt{p^0 + |\vec{p}|} & 0 \\ 0 & \frac{1}{\sqrt{p^0 + |\vec{p}|}} \end{pmatrix}, \tag{4.157}$$

and we have shown that $\tilde{\psi}$ is transformed into ϕ and F by $O = MU$. Since both $\det M$ and $\det U$ are 1, the functional integration stays the same.

Next we transform the source term for $\tilde{\psi}$ into the source terms for ϕ and F. The free field Lagrangian density of $\tilde{\psi}$ with a source term is written as

$$\int d^4x\mathcal{L} = \int \frac{d^4p}{(2\pi)^4}\left[\tilde{\psi}^\dagger\left(-\sigma^\mu p_\mu\right)\tilde{\psi} + J_{\tilde{\psi}}\tilde{\psi} + J^\dagger_{\tilde{\psi}}\tilde{\psi}^\dagger\right]. \tag{4.158}$$

When $\tilde{\psi}$ is transformed into ϕ and F by using the matrix $O = MU$, the Lagrangian density is rewritten as

$$\begin{aligned}
\int d^4x\mathcal{L} &= \int \frac{d^4p}{(2\pi)^4}\left[\tilde{\psi}^\dagger\left(-\sigma^\mu p_\mu\right)\tilde{\psi} + J_{\tilde{\psi}}\tilde{\psi} + \tilde{\psi}^\dagger J^\dagger_{\tilde{\psi}}\right] \\
&= \int \frac{d^4p}{(2\pi)^4}\left[-\phi^* p^\mu p_\mu \phi + F^* F\right] \\
&\quad + \int \frac{d^4p}{(2\pi)^3}\left[J_{\tilde{\psi}}\left(U^\dagger M^{-1}\right)MU\tilde{\psi} + \tilde{\psi}^\dagger U^\dagger M\left(M^{-1}U\right)J^\dagger_{\tilde{\psi}}\right] \\
&= \int \frac{d^4p}{(2\pi)^4}\left[-\phi^* p^\mu p_\mu \phi + F^* F + J_\phi \phi + \phi^* J^*_\phi + J_F F + F^* J^*_F\right],
\end{aligned} \tag{4.159}$$

where we have defined

$$J_{\tilde{\psi}}U^\dagger M^{-1} = \begin{pmatrix} J_\phi & J_F \end{pmatrix}. \tag{4.160}$$

G. Introduction of Gauge Fields with Supersymmetry

In Chapter III, we introduced the gauge field and gravitational vierbein simultaneously. However, in that chapter we saw that the action for the bosonic fields is Lorentz-violating. In this chapter, we have shown that Lorentz invariance and coupling to the gravitational vierbein are recovered at low energy, and that even standard SUSY is recovered, but the argument above did not include coupling to the gauge fields. We now set out to obtain the coupling of the transformed scalar boson fields ϕ to the gauge fields (as opposed to the coupling of the original primitive bosonic fields $\tilde{\psi}$ were obtained earlier).

We will show that the gauge fields can be introduced either before or after the primitive spin 1/2 bosons are transformed into spin 0 bosons.

1. Gauge Fields Introduced Before Spin 1/2 → Spin 0 Boson Transformation

With gauge fields present, there is no longer a separation of the Fourier-transformed fields:

$$
\begin{aligned}
&\int d^4x\ \tilde{\psi}^{\dagger}(x)\,\sigma^{\mu}A_{\mu}(x)\,\tilde{\psi}(x) \\
&= \int d^4x \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\frac{d^4p''}{(2\pi)^4} e^{-ix\cdot(p-p'-p'')}\tilde{\psi}^{\dagger}(p)\,\sigma^{\mu}A_{\mu}(p')\,\tilde{\psi}(p'') \\
&= \int d^4p\ \frac{d^4p'}{(2\pi)^4}\frac{d^4p''}{(2\pi)^4}\delta(p-p'-p'')\,\tilde{\psi}^{\dagger}(p)\,\sigma^{\mu}A_{\mu}(p')\,\tilde{\psi}(p'') \\
&= \int \frac{d^4p'}{(2\pi)^4}\frac{d^4p''}{(2\pi)^4}\ \tilde{\psi}^{\dagger}(p'+p'')\,\sigma^{\mu}A_{\mu}(p')\,\tilde{\psi}(p'') \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\ \tilde{\psi}^{\dagger}(p)\,\sigma^{\mu}A_{\mu}(p-p')\,\tilde{\psi}(p')\,.
\end{aligned}
$$

However, since

$$
A_{\mu}(p-p') = \int d^4x\ e^{-i(p-p')\cdot x}A_{\mu}(x)
$$

and

$$
\begin{aligned}
A_{\mu}^{\dagger}(p'-p) &= \int d^4x\ e^{i(p'-p)\cdot x}A_{\mu}^{\dagger}(x) \\
&= \int d^4x\ e^{-i(p-p')\cdot x}A_{\mu}(x) \\
&= A_{\mu}(p-p')
\end{aligned}
$$

the matrix $A_{\mu}(p,p') = A_{\mu}(p-p')$ with μ fixed is Hermitian and can be diagonalized through a uniary transformation. (We have used the fact that $A_{\mu}(x)$ is Hermitian for the original gauge fields before symmetry-breaking. Note that $\tilde{\psi}$ now has $2N$ rather than 2 components for an N-dimensional nonabelian gauge representation, and that A_{μ} is $N \times N$. Nevertheless, one can diagonalize $A_{\mu}(p,p')$ in both momentum space and in the $N \times N$ gauge representation.) Then we can write

$$\begin{aligned}
&\int d^4p' d^4p''\ \tilde{\psi}^{\dagger}\left(p'+p''\right)\sigma^{\mu}\left[\delta_{p'+p'',p''}p''_{\mu} - A_{\mu}\left(p'\right)\right]\tilde{\psi}\left(p''\right) \\
&= \int d^4p\ d^4p''\ \tilde{\psi}^{\dagger}\left(p\right)\sigma^{\mu}\left[\delta_{p,p''}p''_{\mu} - A_{\mu}\left(p-p''\right)\right]\tilde{\psi}\left(p''\right) \\
&= \int d^4p\ d^4p'\ \tilde{\psi}'^{\dagger}\left(p\right)\sigma^{\mu}\delta_{p,p'}\left[p'_{\mu} - A'_{\mu}\left(p\right)\right]\tilde{\psi}'\left(p'\right) \\
&= \int d^4p\ \tilde{\psi}'^{\dagger}\left(p\right)\sigma^{\mu}\left[p_{\mu} - A'_{\mu}\left(p\right)\right]\tilde{\psi}'\left(p\right).
\end{aligned}$$

(For simplicity, we have surpressed the index in the gauge representation, but it is understood that A'_{μ} is diagonal also in this representation.) Then the arguments from (4.123) to (4.149) still hold with $p_{\mu} \rightarrow p_{\mu} - A'_{\mu}(p)$, and after undoing the diagonalization and Fourier transform, we obtain (4.150) with $\partial_{\mu} \rightarrow D_{\mu}$.

2. Gauge Fields Introduced After Spin 1/2 → Spin 0 Boson Transformation

In an alternative approach, we start with the fundamental action

$$S = -\int d^D x \left[-\frac{1}{2m}\tilde{\Psi}^{\dagger}\partial^M\partial_M\tilde{\Psi} - \mu\tilde{\Psi}^{\dagger}\tilde{\Psi} + \frac{1}{2}b\left(\tilde{\Psi}^{\dagger}\tilde{\Psi}\right)^2\right]. \tag{4.161}$$

When only the gravitational vierbein is introduced initially, the fields are written as

$$\tilde{\Psi}_a\left(x^{\mu},x^m\right) = U_{ext}\left(x^{\mu}\right)\tilde{\psi}^r_a\left(x^{\mu}\right)\tilde{\psi}^{int}_r\left(x^m\right), \tag{4.162}$$

and we have

$$\partial_{\mu}\tilde{\Psi}_a = U_{ext}\left(x^{\mu}\right)\left(\partial_{\mu} + imv_{\mu\alpha}\sigma^{\alpha}\right)\tilde{\psi}^r_a\tilde{\psi}^{int}_r \tag{4.163}$$

so

$$\int d^{D-4}x\,\tilde{\Psi}_a^\dagger\,\partial^\mu\partial_\mu\,\tilde{\Psi}_a = \int d^{D-4}x\,\tilde{\psi}_r^{int\dagger}\tilde{\psi}_a^{r\dagger}\eta^{\mu\nu}\left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha\right)$$

$$\cdot\left(\partial_\nu + imv_{\nu\beta}\sigma^\beta\right)\tilde{\psi}_a^s\tilde{\psi}_s^{int}$$

$$= \tilde{\psi}_a^{r\dagger}\,\eta^{\mu\nu}\langle r|\left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha\right)\sum_t |t\rangle\langle t|\,\left(\partial_\nu + imv_{\nu\beta}\sigma^\beta\right)|s\rangle\,\tilde{\psi}_a^s$$

$$= \tilde{\psi}_a^{r\dagger}\,\eta^{\mu\nu}\delta_{rt}\left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha\right)\delta_{ts}\left(\partial_\nu + imv_{\nu\beta}\sigma^\beta\right)\tilde{\psi}_a^s \qquad (4.164)$$

where (3.35) has been used. The action (4.161) then becomes

$$S_a = \int d^4x\,\tilde{\psi_a}^\dagger\delta_{rt}\delta_{ts}(\frac{1}{2m}\partial^\mu\partial_\mu + \frac{1}{2}iv_\alpha^\mu\sigma^\alpha\partial_\mu$$

$$+ \frac{1}{2}\partial_\mu iv_\alpha^\mu\sigma^\alpha - \frac{1}{2}mv^{\alpha\mu}v_\mu^\alpha + \mu_{ext})\tilde{\psi}_a, \qquad (4.165)$$

instead of (3.68). The approximations above (3.40), (3.32), and (3.41) imply that

$$S_a = \int d^4x\,\tilde{\psi_a}^\dagger\delta_{rt}\delta_{ts}\left(\frac{1}{2m}\partial^\mu\partial_\mu + ie_\alpha^\mu\sigma^\alpha\partial_\mu\right)\tilde{\psi}_a$$

$$\underset{\text{when } p\ll m}{\longrightarrow} \int d^4x\,\tilde{\psi_a}^\dagger\delta_{rt}\delta_{ts}ie_\alpha^\mu\sigma^\alpha\partial_\mu\tilde{\psi}_a. \qquad (4.166)$$

The functional integration with respect to the fundamental boson is

$$
\begin{aligned}
Z(x_i) &= \int d\tilde{\psi}_a^\dagger(x_i)\, d\tilde{\psi}_a(x_i)\, e^{i\int d^4x\, \tilde{\psi}_a{}^\dagger(-i)\delta_{rt}\delta_{ts}e^\mu_\alpha\sigma^\alpha\partial_\mu\tilde{\psi}_a} \\
&= \frac{\pi^2}{\det\left(-\delta_{rt}\delta_{ts}e^\mu_\alpha\sigma^\alpha\partial_\mu\right)} \\
&= \frac{\pi^2}{\det\left(-i\delta_{rt}\delta_{ts}\eta^{\alpha\beta}e^\mu_\alpha\partial_\mu e^\nu_\beta\partial_\nu\right)\det\left(-i\delta_{rt}\delta_{ts}\right)} \\
&\approx \frac{\pi^2}{\det\left(-i\delta_{rt}\delta_{ts}g^{\mu\nu}\partial_\mu\partial_\nu\right)\det\left(-i\delta_{rt}\delta_{ts}\right)} \\
&= \int d\phi_a^*(x_i)\, d\phi_a(x_i)\, dF_a^*(x_i)\, dF_a(x_i) \\
&\qquad \cdot e^{i\int d^4x\,\left[\phi_a^*\delta_{rt}\delta_{ts}g^{\mu\nu}\partial_\mu\partial_\nu\phi_a(p)+F_a^*\delta_{rt}\delta_{ts}F_a\right]},
\end{aligned}
\tag{4.167}
$$

where we have assumed that $\partial_\mu e^\nu_\alpha$ is negligible as in (3.40). Then the action is rewritten as

$$
S_a = \int d^4x\left[\phi_a^* g^{\mu\nu}\delta_{rt}\delta_{ts}\partial_\mu\partial_\nu\phi_a + F_a^*\delta_{rt}\delta_{ts}F_a\right]. \tag{4.168}
$$

The internal rotation matrix $\tilde{U}_{int}(x^\mu, x^m)$ has not been shown explicitly above. First, in the trivial case when $\tilde{U}_{int}$ is a function of only the internal coordinates x^m, we would just have

$$
\begin{aligned}
S_a &= \int d^4x\left[\phi_a^* g^{\mu\nu}\delta_{rt}\delta_{ts}\partial_\mu\partial_\nu\phi_a + F_a^*\delta_{rt}\delta_{ts}F_a\right] \\
&= \int d^4x\left[\phi_a^{r*} g^{\mu\nu}\left(\delta_{rt}\partial_\mu\right)\left(\delta_{ts}\partial_\nu\right)\phi_a^s + F_a^{r*}\delta_{rt}\delta_{ts}F_a^s\right] \\
&= \int d^4x[\phi_a^{r*} g^{\mu\nu}\langle r|\left(\partial_\mu\right)\sum_t |t\rangle\langle t|\left(\partial_\nu\right)|s\rangle\phi_a^s \\
&\qquad\qquad + F_a^{r*}\sum_t\langle r|1|t\rangle\langle t|1|s\rangle F_a^s] \\
&= \int d^Dx\left[\phi_r^{int*}\phi_a^{r*}g^{\mu\nu}\left(\partial_\mu\right)\left(\partial_\nu\right)\phi_a^s\phi_s^{int} + F_r^{int*}F_a^{r*}F_a^sF_s^{int}\right] \\
&= \int d^Dx\left[\Phi_a^* g^{\mu\nu}\left(\partial_\mu\right)\left(\partial_\nu\right)\Phi_a + \boldsymbol{F}_a^*\boldsymbol{F}_a\right],
\end{aligned}
\tag{4.169}
$$

where

$$\int d^{D-4}x\phi_r^{int*}\phi_s^{int} = \langle r|s\rangle_\phi = \delta_{rs}, \tag{4.170}$$

$$\int d^{D-4}xF_r^{int*}F_s^{int} = \langle r|s\rangle_F = \delta_{rs}, \tag{4.171}$$

and

$$\Phi_a\left(x^\mu, x^m\right) = \phi_a^s\phi_s^{int}, \tag{4.172}$$

$$\boldsymbol{F}_a = F_a^sF_s^{int}. \tag{4.173}$$

Now consider the nontrivial internal rotation matrix $\widetilde{U}_{int}\left(x^\mu, x^m\right)$ which we already considered in Chapter III:

$$\Phi_a\left(x^\mu, x^m\right) \to \widetilde{U}_{int}\left(x^\mu, x^m\right)\Phi_a\left(x^\mu, x^m\right), \tag{4.174}$$

$$\boldsymbol{F}_a\left(x^\mu, x^m\right) \to \widetilde{U}_{int}\left(x^\mu, x^m\right)\boldsymbol{F}_a\left(x^\mu, x^m\right). \tag{4.175}$$

After these internal rotations are introduced, the action becomes

$$\begin{aligned}
S_a &= \int d^Dx\left[\Phi_a^*g^{\mu\nu}\left(\partial_\mu\right)\left(\partial_\nu\right)\Phi_a + \boldsymbol{F}_a^*\boldsymbol{F}_a\right] \\
&= \int d^Dx[\phi_r^{int*}\phi_a^{r*}g^{\mu\nu}\left(\partial_\mu + imv_{\mu c}\sigma^c\right) \\
&\qquad \cdot\left(\partial_\nu + imv_{\nu d}\sigma^d\right)\phi_a^s\phi_s^{int} + F_r^{int*}F_a^{r*}F_a^sF_s^{int}] \\
&= \int d^4x[\phi_a^{r*}g^{\mu\nu}\langle r|\left(\partial_\mu + imv_{\mu c}\sigma^c\right)\sum_t|t\rangle \\
&\qquad \cdot\langle t|\left(\partial_\nu + imv_{\nu d}\sigma^d\right)|s\rangle\phi_a^s + F_a^{r*}\langle r|\sum_t|t\rangle\langle t|s\rangle F_a^s] \\
&= \int d^4x\left[\phi_a^{r*}g^{\mu\nu}\left(\delta_{rt}\partial_\mu - iA_\mu^it_i^{rt}\right)\left(\delta_{ts}\partial_\nu - iA_\nu^jt_j^{ts}\right)\phi_a^s + F_a^{r*}\delta_{rt}\delta_{ts}F_a^s\right] \\
&= \int d^4x\left[\phi_a^*g^{\mu\nu}\left(\partial_\mu - iA_\mu^it_i\right)\left(\partial_\nu - iA_\nu^jt_j\right)\phi_a + F_a^*F_a\right]
\end{aligned} \tag{4.176}$$

where we define as before

$$mv_{\mu c}\sigma^c = -A_\mu^i \sigma_i, \tag{4.177}$$

$$\sigma_i = mK_i^n v_{nc}\sigma^c, \tag{4.178}$$

$$t_i^{rs} = \langle r| \left(-iK_i\right) |s\rangle \, . \tag{4.179}$$

In this alternative approach we have again obtained the gauge interactions for scalar boson, as a low energy approximation.

H. Primitive Gaugino and Gravitino Fields

Let us now consider more general rotations which mix bosonic and fermionic degrees of freedom. First consider the global supersymmetry transformation

$$\Psi \to \Psi' = \mathcal{U}\,\Psi \tag{4.180}$$

or, with bosonic and fermionic fields shown separately,

$$\begin{pmatrix} \Psi_b \\ \Psi_f \end{pmatrix} \to \begin{pmatrix} \Psi'_b \\ \Psi'_f \end{pmatrix} = \begin{pmatrix} \mathcal{U}_{bb} & \mathcal{U}_{bf} \\ \mathcal{U}_{fb} & \mathcal{U}_{ff} \end{pmatrix} \begin{pmatrix} \Psi_b \\ \Psi_f \end{pmatrix}. \tag{4.181}$$

If

$$\mathcal{U}^\dagger \mathcal{U} = 1 \tag{4.182}$$

the action

$$S = \int d^D x \left[\frac{1}{2m} \partial^M \Psi^\dagger \partial_M \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} b \left(\Psi^\dagger \Psi\right)^2 \right] \tag{4.113}$$

is invariant under this transformation, so the theory has a primitive supersymmetry according to the definition given above. The elements of $\mathcal{U}_{bb}$ and $\mathcal{U}_{ff}$ are ordinary commuting variables, like the components of Ψ_b. The elements of $\mathcal{U}_{bf}$ and $\mathcal{U}_{fb}$ are anticommuting Grassmann variables, like the components of Ψ_f.

Now let us replace the picture of a rotating GUT-scale condensate by a more general picture in which all the fields of the vacuum contain a rotation described by a supermatrix $\mathcal{U}$ which varies as a function of the spacetime coordinates. With a possible redefinition of the fermion fields, we can choose $\mathcal{U}_{ff} = \mathcal{U}_{bb}$ and write

$$\Psi^{vac} = \mathcal{U}\, n_{vac}^{1/2}\, \Psi^0 \tag{4.183}$$

where Ψ^0 is constant and

$$\Psi^{vac} = \langle \Psi \rangle_{vac} = \begin{pmatrix} \langle \Psi_b \rangle_{vac} \\ \langle \Psi_f \rangle_{vac} \end{pmatrix} \tag{4.184}$$

$$\mathcal{U} = \begin{pmatrix} \mathcal{U}_{bb} & \mathcal{U}_{bf} \\ \mathcal{U}_{fb} & \mathcal{U}_{bb} \end{pmatrix} \tag{4.185}$$

$$\Psi^{0\dagger}\Psi^0 = 1\,. \tag{4.186}$$

The generalizations of our earlier equations in Chapter III with no mixing of bosons and fermions, are

$$mV^{\mu} = -i\mathcal{U}^{-1}\partial^{\mu}\mathcal{U} \tag{4.187}$$

$$\partial_m \mathcal{U} = \partial_m U = i\, U\, m\, v_m \tag{4.188}$$

$$V^M = V_{\alpha}^M \sigma^{\alpha} + V_c^M \sigma^c \quad , \quad V_M = V_{M\alpha}\sigma^{\alpha} + V_{Mc}\sigma^c \tag{4.189}$$

$$E_{\mu c} = \mathcal{A}_{\mu}^i K_i^n v_{nc} \tag{4.190}$$

$$E_{\mu c} = -V_{\mu c} \tag{4.191}$$

where the last two expressions in (4.188) implicitly multiply a 2×2 identity matrix, and it is assumed that the internal coordinate space contains no supersymmetric rotations.

The fact that $\mathcal{U}$ is unitary implies that $\partial_M \mathcal{U}^\dagger \mathcal{U} = -\mathcal{U}^\dagger \partial_M \mathcal{U}$ with $\mathcal{U}^\dagger = \mathcal{U}^{-1}$, or

$$mV_M = i\partial_M \mathcal{U}^\dagger \mathcal{U} \tag{4.192}$$

so that

$$V_M^\dagger = V_M \quad , V^{M\dagger} = V^M \quad . \tag{4.193}$$

We can then write, e.g.,

$$V^M = \begin{pmatrix} V^M_{bb} & V^M_{bf} \\ V^{M\dagger}_{bf} & V^M_{bb} \end{pmatrix} . \tag{4.194}$$

At this point, the logic in Chapter III can be repeated with

$$v_M \to V_M \quad , \quad v^M \to V^M \tag{4.195}$$

$$e^\mu_\alpha \to E^{\ \mu}_{\ \alpha} = \begin{pmatrix} e^\mu_\alpha & f^\mu_\alpha \\ f^{\mu\dagger}_\alpha & e^\mu_\alpha \end{pmatrix} \tag{4.196}$$

$$A^i_\mu \to \mathcal{A}^i_\mu = \begin{pmatrix} A^i_\mu & B^i_\mu \\ B^{i\dagger}_\mu & A^i_\mu \end{pmatrix} . \tag{4.197}$$

In particular, we obtain

$$\Psi\left(x^\mu, x^m\right) = \mathcal{U}\left(x^\mu, x^m\right) \Psi^r\left(x^\mu\right) \psi_r^{int}\left(x^m\right), \tag{4.198}$$

$$\partial_\mu \Psi = \mathcal{U}\left(x^\mu, x^m\right)\left(\partial_\mu + imV_{\mu\alpha}\sigma^\alpha + imV_{\mu c}\sigma^c\right) \Psi^r \psi_r^{int} \tag{4.199}$$

$$
\begin{aligned}
\int d^dx\, \Psi^\dagger\, \partial^\mu \partial_\mu\, \Psi = &\int d^dx\, \psi_r^{int\dagger} \Psi^{r\dagger} \eta^{\mu\nu} \left(\partial_\mu + imV_{\mu\alpha}\sigma^\alpha + imV_{\mu c}\sigma^c\right) \\
&\qquad \times \left(\partial_\nu + imV_{\nu\beta}\sigma^\beta + imV_{\nu d}\sigma^d\right) \Psi^s \psi_s^{int} \\
= &\, \Psi^{r\dagger}\, \eta^{\mu\nu} \langle r| \left(\partial_\mu + imV_{\mu\alpha}\sigma^\alpha + imV_{\mu c}\sigma^c\right) \\
&\qquad \times \sum_t |t\rangle\langle t|\, \left(\partial_\nu + imV_{\nu\beta}\sigma^\beta + imV_{\nu d}\sigma^d\right) |s\rangle\, \Psi^s \\
= &\, \Psi^{r\dagger}\, \eta^{\mu\nu} \left[\delta_{rt} \left(\partial_\mu + imV_{\mu\alpha}\sigma^\alpha\right) - i\mathcal{A}_\mu^i t_i^{rt}\right] \\
&\qquad \times \left[\delta_{ts} \left(\partial_\nu + imV_{\nu\beta}\sigma^\beta\right) - i\mathcal{A}_\nu^j t_j^{ts}\right]\, \Psi^s \\
= &\, \Psi_{ext}^\dagger\, \eta^{\mu\nu} \left[\left(\partial_\mu - i\mathcal{A}_\mu^i t_i\right) + imV_{\mu\alpha}\sigma^\alpha\right] \\
&\qquad \times \left[\left(\partial_\nu - i\mathcal{A}_\nu^j t_j\right) + imV_{\nu\beta}\sigma^\beta\right]\, \Psi_{ext}
\end{aligned}
\tag{4.200}
$$

$$
\begin{aligned}
S_L = &\int d^4x\, \Psi_{ext}^\dagger \times \\
&\left(\frac{1}{2m}\mathcal{D}^\mu \mathcal{D}_\mu + \frac{1}{2} iV_\alpha^\mu \sigma^\alpha \mathcal{D}_\mu + \frac{1}{2}\mathcal{D}_\mu iV_\alpha^\mu \sigma^\alpha - \frac{1}{2} mV_\alpha^\mu V_{\mu\alpha} + \mu_{ext}\right) \Psi_{ext}
\end{aligned}
\tag{4.201}
$$

where

$$\mathcal{D}_\mu = \partial_\mu - i\mathcal{A}_\mu^i t_i\,. \tag{4.202}$$

We also have the generalization

$$\Psi^{0\dagger} n_{vac}^{1/2} \left[\left(\frac{1}{2} mV^\mu V_\mu - \frac{1}{2m}\partial^\mu \partial_\mu - \mu_{ext}\right) - i\left(\frac{1}{2}\partial^\mu V_\mu + V^\mu \partial_\mu\right)\right] n_{vac}^{1/2} \Psi^0 = 0\,. \tag{4.203}$$

Adding this equation to its Hermitian conjugate gives a still more general Bernoulli equation

$$\frac{1}{2} m\Psi^{0\dagger}\, V^\mu V_\mu\, \Psi^0 + P_{ext} = \mu_{ext} \tag{4.204}$$

where

$$P_{ext} = -\frac{1}{2m} n_{vac}^{-1/2} \partial^\mu \partial_\mu n_{vac}^{1/2}. \tag{4.205}$$

As before, it is assumed that the basic texture of the vacuum field rotations is such that

$$V_0^k = V_a^0 = 0 \quad , \quad k, a = 1, 2, 3, \tag{4.206}$$

and that the nonzero gauge potentials are not coupled to Ψ^0 at energies well below the GUT scale,, so that (4.204) reduces to

$$\frac{1}{2} m V_\alpha^\mu V_{\alpha\mu} + P_{ext} = \mu_{ext}. \tag{4.207}$$

When $\partial_\mu n_{vac}^{1/2}$ and $\partial^\mu V_\mu$ are neglected, (4.201) then simplifies to

$$S_L = \int d^4x \, \Psi_{ext}^\dagger \left(\frac{1}{2m} \mathcal{D}^\mu \mathcal{D}_\mu + i E_\alpha^\mu \sigma^\alpha \mathcal{D}_\mu \right) \Psi_{ext} \, . \tag{4.208}$$

Since m is comparable to the Planck mass, it is reasonable to assume that the first term can be neglected, giving

$$S_L = \int d^4x \, \Psi_{ext}^\dagger \, E_\alpha^\mu \sigma^\alpha \mathcal{D}_\mu \, \Psi_{ext} \tag{4.209}$$

or, with e_α^μ again slowly varying,

$$S_L = \int d^4x \, e \, \overline{\Psi}_{ext}^\dagger \, E_\alpha^\mu \sigma^\alpha \mathcal{D}_\mu \, \overline{\Psi}_{ext} \tag{4.210}$$

$$\overline{\Psi}_{ext} = e^{-1/2} \, \Psi_{ext} \quad , \quad e = \det \left(e_{\alpha\mu} \right) \, . \tag{4.211}$$

According to (4.196) and (4.197), the bosonic fields play the same role as before. Namely, e_α^μ is the vierbein representing the gravitational field, and A_μ^i is the potential representing the gauge fields of a grand-unified theory – an $SO(10)$ theory [63],[64],[65] if the dimension of the internal space is 10. The fermionic fields can be interpreted in an equally simple way: namely, f_α^μ corresponds to a spin 2 gravitino and B_μ^i to spin 1 gauginos. Again, we have generalized the usual vocabulary, so that the superpartner of the graviton is defined to be the gravitino, and the superpartners of gauge bosons to be gauginos, even though these fermions would have quite unconventional properties in their initial primitive forms. However, just as in the case of the sfermions, these primitive superpartners must be transformed to the physical gauginos and gravitinos of standard SUSY. We will partially accomplish this in the present dissertation, and will give a more complete treatment elsewhere.

I. Spin 1 Gaugino Transformed to Spin 1/2 Gaugino

When the gauge and gaugino fields are introduced before the sfermion is transformed into a spin 0 scalar field, at low energy the action is given by

$$S = \int d^4x \left[\begin{pmatrix} \tilde{\psi}^{\dagger}_{\dot{\alpha}} & \psi^{\dagger}_{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu & g\bar{\sigma}^{\mu\dot{\alpha}\alpha}\tilde{A}_\mu \\ g\bar{\sigma}^{\mu\dot{\alpha}\alpha}\tilde{A}^{\dagger}_\mu & i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu \end{pmatrix} \begin{pmatrix} \tilde{\psi}_\alpha \\ \psi_\alpha \end{pmatrix} \right] . \tag{4.212}$$

When we define

$$\begin{pmatrix} i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu & g\bar{\sigma}^{\mu\dot{\alpha}\alpha}\tilde{A}_\mu \\ g\bar{\sigma}^{\mu\dot{\alpha}\alpha}\tilde{A}^{\dagger}_\mu & i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu \end{pmatrix} = \begin{pmatrix} A & C \\ D & B \end{pmatrix} , \tag{4.213}$$

the action is

$$\begin{aligned} S &= \int d^4x \left[\begin{pmatrix} \tilde{\psi}^{\dagger} & \psi^{\dagger} \end{pmatrix} \begin{pmatrix} A & C \\ D & B \end{pmatrix} \begin{pmatrix} \tilde{\psi}_\alpha \\ \psi_\alpha \end{pmatrix} \right] \\ &= \int d^4x \Bigg[\begin{pmatrix} \tilde{\psi}^{\dagger} & \psi^{\dagger} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ DA^{-1} & 1 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{pmatrix} \begin{pmatrix} 1 & A^{-1}C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\psi} \\ \psi \end{pmatrix} \Bigg] \end{aligned} \tag{4.214}$$

and then the functional integral is

$$
\begin{aligned}
Z &= \int D\tilde{\psi}^\dagger D\tilde{\psi} D\psi^\dagger D\psi \; e^{iS} \\
&= \int D\tilde{\psi}^\dagger D\tilde{\psi} D\psi^\dagger D\psi \exp\left[i \int d^4x \begin{pmatrix} \tilde{\psi}^\dagger & \psi^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ DA^{-1} & 1 \end{pmatrix} \right. \\
&\qquad\qquad \left. \cdot \begin{pmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{pmatrix} \begin{pmatrix} 1 & A^{-1}C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\psi} \\ \psi \end{pmatrix} \right] \\
&= \int D\tilde{\psi}'^\dagger D\tilde{\psi}' D\psi'^\dagger D\psi' \mathrm{sdet}\left(\begin{pmatrix} 1 & 0 \\ DA^{-1} & 1 \end{pmatrix}^{-1} \right) \mathrm{sdet}\left(\begin{pmatrix} 1 & A^{-1}C \\ 0 & 1 \end{pmatrix}^{-1} \right) \\
&\qquad\qquad \cdot \exp\left[i \int d^4x \begin{pmatrix} \tilde{\psi}'^\dagger & \psi'^\dagger \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{pmatrix} \begin{pmatrix} \tilde{\psi}' \\ \psi' \end{pmatrix} \right] \\
&= \int D\tilde{\psi}'^\dagger D\tilde{\psi}' D\psi'^\dagger D\psi' \\
&\qquad \cdot \exp\left[i \int d^4x \begin{pmatrix} \tilde{\psi}'^\dagger & \psi'^\dagger \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{pmatrix} \begin{pmatrix} \tilde{\psi}' \\ \psi' \end{pmatrix} \right]
\end{aligned} \tag{4.215}
$$

where we have used

$$\mathrm{sdet}\,(M_1 M_2) = \mathrm{sdet} M_1 \mathrm{sdet} M_2$$

$$\begin{aligned} \frac{M}{M} = 1 \;\rightarrow\; \mathrm{sdet}\left(\frac{M}{M}\right) &= 1 \\ \rightarrow\; \mathrm{sdet}\,(M)\,\mathrm{sdet}\left(\frac{1}{M}\right) &= 1 \\ \rightarrow\; \mathrm{sdet}\left(\frac{1}{M}\right) &= \frac{1}{\mathrm{sdet}\,(M)}, \end{aligned} \tag{4.216}$$

$$\mathrm{sdet}\begin{pmatrix} 1 & 0 \\ DA^{-1} & 1 \end{pmatrix} = \mathrm{sdet}\begin{pmatrix} 1 & A^{-1}C \\ 0 & 1 \end{pmatrix} = 1. \tag{4.217}$$

Since A and $B - DA^{-1}C$ are not diagonal matrices, we define unitary operators by

$$A_{\text{diag}} = U_{\tilde{\psi}} A U^{\dagger}_{\tilde{\psi}}, \tag{4.218}$$

$$\tilde{\psi}'_{\text{diag}} = U_{\tilde{\psi}} \tilde{\psi}', \tag{4.219}$$

$$\left(B - DA^{-1}C\right)_{\text{diag}} = U_{\psi}\left(B - DA^{-1}C\right) U^{\dagger}_{\psi} \tag{4.220}$$

$$\psi'_{\text{diag}} = U_{\psi} \psi', \tag{4.221}$$

and the action is rewritten as

$$\begin{aligned} Z = \int D\tilde{\psi}'^{\dagger}_{\text{diag}} D\tilde{\psi}'_{\text{diag}} D\psi'^{\dagger}_{\text{diag}} D\psi'_{\text{diag}} \exp\Bigg[i \int d^4x \begin{pmatrix} \tilde{\psi}'^{\dagger}_{\text{diag}} & \psi'^{\dagger}_{\text{diag}} \end{pmatrix} \\ \cdot \begin{pmatrix} A_{\text{diag}} & 0 \\ 0 & (B - DA^{-1}C)_{\text{diag}} \end{pmatrix} \begin{pmatrix} \tilde{\psi}'_{\text{diag}} \\ \psi'_{\text{diag}} \end{pmatrix} \Bigg] \end{aligned}$$

$$
\begin{aligned}
&= \prod_j \frac{\pi^2}{\det\left(iA_{\text{diag}}\right)} \det\left(i\left(B - DA^{-1}C\right)_{\text{diag}}\right) \\
&= \prod_j \frac{\pi^2}{\det\left(-i\partial_\mu\partial^\mu\right)\det\left(-i\right)} \det\left(i\left(B - DA^{-1}C\right)_{\text{diag}}\right) \\
&= \int D\phi'^{*}_{\text{diag}} D\phi'_{\text{diag}} DF'^{*}_{\text{diag}} DF'_{\text{diag}} D\psi'^{\dagger}_{\text{diag}} D\psi'_{\text{diag}} \\
&\qquad \cdot \exp[i \int d^4x[\begin{pmatrix} \phi'^{*}_{\text{diag}} & F'^{*}_{\text{diag}} & \psi'^{\dagger}_{\text{diag}} \end{pmatrix} \\
&\qquad\quad \cdot \begin{pmatrix} \partial_\mu\partial^\mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \left(B - DA^{-1}C\right)_{\text{diag}} \end{pmatrix} \begin{pmatrix} \phi'_{\text{diag}} \\ F'_{\text{diag}} \\ \psi'_{\text{diag}} \end{pmatrix} \\
&= \int D\phi'^{*} D\phi' DF'^{*} DF' D\psi'^{\dagger} D\psi' \\
&\qquad \cdot \exp\left[i \int d^4x[\begin{pmatrix} \phi'^{*} & F'^{*} & \psi'^{\dagger} \end{pmatrix}\right. \\
&\qquad\quad \cdot \left.\begin{pmatrix} \partial_\mu\partial^\mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i\bar{\sigma}^\mu\partial_\mu - g\bar{\sigma}^\mu \tilde{A}^{\dagger}_\mu \left(i\bar{\sigma}^\nu\partial_\nu\right)^{-1} g\bar{\sigma}^\xi \tilde{A}_\xi \end{pmatrix} \begin{pmatrix} \phi' \\ F' \\ \psi' \end{pmatrix}\right]
\end{aligned}
\tag{4.222}
$$

This is the fermion-sfermion decoupled formalism.

On the other hand, the action with a standard spin 1/2 gaugino (here the gauge field is not included for simplicity) is

$$S = \int d^4x \left[\begin{pmatrix} \phi^* & F^* & \psi^\dagger_{\dot\alpha} \end{pmatrix} \begin{pmatrix} \partial^\mu\partial_\mu & 0 & -\sqrt{2}g\lambda^\alpha \\ 0 & 1 & 0 \\ -\sqrt{2}g\lambda^{\dagger\dot\alpha} & 0 & i\bar\sigma^\mu\partial_\mu \end{pmatrix} \begin{pmatrix} \phi \\ F \\ \psi_\alpha \end{pmatrix} \right]$$

$$= \int d^4x \Bigg[\begin{pmatrix} \phi^* & F^* & \psi^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ DA^{-1} & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{pmatrix} \begin{pmatrix} 1 & A^{-1}C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ F \\ \psi \end{pmatrix} \Bigg], \qquad (4.223)$$

where

$$\begin{pmatrix} A_{2\times 2} & C_{2\times 1} \\ D_{1\times 2} & B_{1\times 1} \end{pmatrix} = \begin{pmatrix} \partial^\mu\partial_\mu & 0 & -\sqrt{2}g\lambda^\alpha \\ 0 & 1 & 0 \\ -\sqrt{2}g\lambda^{\dagger\dot\alpha} & 0 & i\bar\sigma^\mu\partial_\mu \end{pmatrix} \qquad (4.224)$$

Then the functional integration is given by

$$Z = \int D\phi^* D\phi DF^* DF D\psi^\dagger D\psi \exp\Bigg[i \int d^4x \begin{pmatrix} \phi^* & F^* & \psi^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ DA^{-1} & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{pmatrix} \begin{pmatrix} 1 & A^{-1}C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ F \\ \psi \end{pmatrix} \Bigg]. \qquad (4.225)$$

Then by redefining the fields as

$$\begin{pmatrix} \phi' \\ F' \\ \psi' \end{pmatrix} = \begin{pmatrix} 1 & A^{-1}C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ F \\ \psi \end{pmatrix}, \tag{4.226}$$

$$\begin{aligned} \begin{pmatrix} \phi'^* & F'^* & \psi'^\dagger \end{pmatrix} &= \begin{pmatrix} \phi^* & F^* & \psi^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C^\dagger \left(A^{-1}\right)^\dagger & 1 \end{pmatrix} \\ &= \begin{pmatrix} \phi^* & F^* & \psi^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D \left(A^{-1}\right)^\dagger & 1 \end{pmatrix} \end{aligned} \tag{4.227}$$

we obtain

$$\begin{aligned} Z = \int D\phi^* D\phi DF^* DF D\psi^\dagger D\psi \exp\Bigg[i \int d^4x \begin{pmatrix} \phi'^* & F'^* & \psi'^\dagger \end{pmatrix} \\ \cdot \begin{pmatrix} A & 0 \\ 0 & B - DA^{-1}C \end{pmatrix} \begin{pmatrix} \phi' \\ F' \\ \psi' \end{pmatrix} \Bigg] \\ = \int D\phi'^* D\phi' DF'^* DF' D\psi'^\dagger D\psi' \mathrm{sdet} \begin{pmatrix} 1 & 0 \\ DA^{-1} & 1 \end{pmatrix} \mathrm{sdet} \begin{pmatrix} 1 & A^{-1}C \\ 0 & 1 \end{pmatrix} \\ \cdot \exp\Bigg[i \int d^4x \begin{pmatrix} \phi'^* & F'^* & \psi'^\dagger \end{pmatrix} \begin{pmatrix} D^\mu D_\mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B - DA^{-1}C \end{pmatrix} \begin{pmatrix} \phi' \\ F' \\ \psi' \end{pmatrix} \Bigg] \end{aligned}$$

$$= \int D\phi'^* D\phi' DF'^* DF' D\psi'^\dagger D\psi' \exp[i \int d^4x \begin{pmatrix} \phi'^* & F'^* & \psi'^\dagger \end{pmatrix}$$

$$\cdot \begin{pmatrix} \partial^\mu\partial_\mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i\bar{\sigma}^\mu\partial_\mu - \begin{pmatrix} -\sqrt{2}g\lambda^\dagger & 0 \end{pmatrix} \begin{pmatrix} \partial^\mu\partial_\mu & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\sqrt{2}g\lambda \\ 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \phi' \\ F' \\ \psi' \end{pmatrix}$$

$$= \int D\phi'^* D\phi' DF'^* DF' D\psi'^\dagger D\psi' \exp\Bigg[i \int d^4x \begin{pmatrix} \phi'^* & F'^* & \psi'^\dagger \end{pmatrix}$$

$$\cdot \begin{pmatrix} \partial^\mu\partial_\mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i\bar{\sigma}^\mu\partial_\mu - 2g^2\lambda^\dagger \frac{1}{\partial^\mu\partial_\mu}\lambda \end{pmatrix} \begin{pmatrix} \phi' \\ F' \\ \psi' \end{pmatrix} \Bigg], \tag{4.228}$$

and this is the fermion-sfermion decoupled formalism of the usual SUSY.

Comparing the result from our primitive SUSY and standard SUSY, we see that the condition required for the two results to be identical is

$$ig^2\bar{\sigma}^{\mu\dot{a}b}\tilde{A}^\dagger_\mu \frac{\sigma^\kappa_{b\dot{b}}\partial_\kappa}{2\partial^\nu\partial_\nu}\bar{\sigma}^{\xi\dot{b}c}\tilde{A}_\xi = -2g^2\lambda^{\dagger\dot{a}}\frac{1}{\partial^\mu\partial_\mu}\lambda^c. \tag{4.229}$$

Then as an example we can take

$$\lambda^c = \eta_{\dot{b}}\bar{\sigma}^{\xi\dot{b}c}\tilde{A}_\xi \tag{4.230}$$

$$\lambda^{\dagger\dot{a}} = \tilde{A}^\dagger_\mu\bar{\sigma}^{\mu\dot{a}b}\eta_b \tag{4.231}$$

where we have defined

$$\eta_b\eta_{\dot{b}} = -\frac{i}{4}\sigma^\kappa_{b\dot{b}}\partial_\kappa. \tag{4.232}$$

We need to check the existence of η_b. The matrix $\eta_b\eta_{\dot{b}}$ is written as

$$\eta_b\eta_{\dot{b}} = \begin{pmatrix} \eta_1\eta_{\dot{1}} & \eta_1\eta_{\dot{2}} \\ \eta_2\eta_{\dot{1}} & \eta_2\eta_{\dot{2}} \end{pmatrix} \tag{4.233}$$

and the determinant is

$$\det \eta_b\eta_{\dot{b}} = 0. \tag{4.234}$$

Therefore, when the determinant of the right hand side of (4.232) is zero, η_b exists. This determinant is

$$\det\left(-\frac{i}{4}\sigma^{\kappa}_{b\dot{b}}\partial_\kappa\right) = -\frac{1}{16}\left(\partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3\right), \tag{4.235}$$

and only when we have

$$\partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3 = 0 \tag{4.236}$$

does η_b exist. We leave a more general treatment of gauginos and gravitinos for further work which will extend the results of this dissertation.

CHAPTER V

STATISTICAL ORIGIN OF THE BOSONIC ACTION

In this chapter we turn to a different issue, following a treatment in Ref. [47] which is included in this dissertation for completeness. Here we consider the origin of the phenomenological action (4.113). We will show that this action follows from a simple microscopic and statistical picture. Our starting point is a single fundamental system which consists of N_w identical "whits", with N_w variable. ("Whit", whose meanings include "particle" and "least possible amount", is an appropriate name for the irreducible objects that are postulated here.) Each whit can exist in any of M_w states, with the number of whits in the ith state represented by n_i. A *microstate* of the fundamental system is specified by the number of whits and the state of each whit. A *macrostate* is specified by only the occupancies n_i of the states.

As discussed below, D of the states are used to define D coordinates x^M in Euclidean spacetime, m_w of the states are used to define observable fields ϕ_k, and the remaining $(M_w - m_w - D)$ states may be regarded as corresponding to fields that are unobservable (at least at the energy scales considered here).

Let us begin by defining an initial set of coordinates X^M in terms of the occupancies n_M:

$$X^M = \pm n_M a_0 \tag{5.1}$$

where $M = 0, 1, ..., D-1$. It is convenient to include a fundamental length a_0 in this definition, so that we can later express the coordinates in conventional units. As one might expect, a_0 will eventually turn out to be comparable to the Planck length:

$$a_0 \sim \ell_P = (16\pi G)^{1/2} \tag{5.2}$$

since $a_0 = m^{-1} \sim m_P^{-1} = \ell_P$ according to (5.32).

With the definition (5.1), positive and negative coordinates correspond to the same occupancies. There are two relevant facts, however, which make this definition physically acceptable: First, two points whose coordinates differ by a minus sign are typically separated by cosmologically large distances. Second, and more importantly, the fields ϕ_k need not return to their original values when they are evolved, according to their equation of motion, from points with positive coordinates to points with negative coordinates. I. e., the classical field configurations described by the two sets of points can be regarded as distinct, and in this sense the points themselves are distinct. The different branches of the field configuration are analogous to the branches of a multivalued function like $z^{1/2}$, which are taken to correspond to distinct Riemann sheets.

At extremely small distances, spacetime is discrete in the present theory, with a finite interval between two adjacent points X^M and $X^M + \delta X^M$: $\delta X^M = a_0$. The X^M are divided into 4 external coordinates X^μ and $(D-4)$ internal coordinates X^m. In the internal space it is natural to have variations on a length scale that is comparable to ℓ_P. In external spacetime, on the other hand, we wish to consider fields which vary much more slowly, and it is convenient to average over a more physically meaningful length scale. Let us therefore consider a D-dimensional rectangular box centered on a point $\bar{X}$, with X^M ranging from $\bar{X}^M - a^M/2$ to $\bar{X}^M + a^M/2$. For the $(D-4)$ coordinates of internal space, a^m is taken to be the original fundamental length a_0. For the 4 coordinates of external spacetime, a^μ is taken to be an arbitrary length a, and we will find that the final form of the action is independent of this parameter.

In this coarse-grained picture, the density of whits in the ith state is

$$\rho_i\left(\bar{X}\right) = N_i/\Delta V \quad , \quad i = 1, 2, ..., M_w \tag{5.3}$$

where

$$N_i = \sum_X n_i(X) \quad , \quad \Delta V = \prod_M a^M = a^4 a_0^{D-4} \tag{5.4}$$

and the values of X are those lying within the box centered on $\bar{X}$. Let

$$\phi_k^2 = \rho_k \quad , \quad k = 1, 2, ..., m_w. \tag{5.5}$$

The initial bosonic fields ϕ_k are then real (but defined only up to a phase factor ± 1).

Let $\bar{S}(\bar{X})$ be the entropy of the single box at $\bar{X}$ for a given set of densities ρ_i, as defined by $\bar{S}(\bar{X}) = \log W(\bar{X})$ (in units with $k_B = \hbar = c = 1$). Here $W(\bar{X})$ is the total number of microstates in this box at fixed ρ_i or N_i: $W(\bar{X}) = \mathcal{N}(\bar{X})!/\Pi_i N_i(\bar{X})!$, with

$$\mathcal{N}(\bar{X}) = \sum_i N_i(\bar{X}). \tag{5.6}$$

The total number of available microstates for all points $\bar{X}$ is $W = \Pi_{\bar{X}} W(\bar{X})$, so the total entropy is

$$\bar{S} = \sum_{\bar{X}} \bar{S}(\bar{X}) \tag{5.7}$$

$$\bar{S}(\bar{X}) = \log\Gamma\left(\mathcal{N}(\bar{X}) + 1\right) - \sum_i \log\Gamma\left(N_i(\bar{X}) + 1\right). \tag{5.8}$$

It follows that

$$\frac{\partial \bar{S}}{\partial N_i(\bar{X})} = \psi\left(\mathcal{N}(\bar{X}) + 1\right) - \psi\left(N_i(\bar{X}) + 1\right) \tag{5.9}$$

$$\frac{\partial^2 \bar{S}}{\partial N_{i'}(\bar{X})\, \partial N_i(\bar{X})} = \psi^{(1)}\left(\mathcal{N}(\bar{X}) + 1\right) - \psi^{(1)}\left(N_i(\bar{X}) + 1\right)\delta_{i'i} \tag{5.10}$$

where $\psi(z) = d\log\Gamma(z)/dz$ and $\psi^{(n)}(z) = d^{n+1}\log\Gamma(z)/dz^{n+1}$ are the digamma and polygamma functions, with the asymptotic expansions [81]

$$\psi\ (z) = \log z - \frac{1}{2z} - \sum_{l=1}^{\infty} \frac{B_{2l}}{2l\, z^{2l}} \tag{5.11}$$

$$\psi^{(n)}\ (z) = (-1)^{n-1} \left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{l=1}^{\infty} B_{2l} \frac{(2l+n-1)!}{(2l)!\, z^{n+2l}} \right] \tag{5.12}$$

as $z \to \infty$. For $a \gg \ell_P$, we have $\mathcal{N}\left(\bar{X}\right) >>> \bar{n}_\mu = \left(\bar{X}^\mu / a_0\right)^2 >>> 1$, so it is an extremely good approximation to write

$$\frac{\partial \bar{S}}{\partial N_k\left(\bar{X}\right)} = \log \mathcal{N}\left(\bar{X}\right) - \psi\left(N_k\left(\bar{X}\right) + 1\right) \tag{5.13}$$

$$\frac{\partial^2 \bar{S}}{\partial N_{k'}\left(\bar{X}\right) \partial N_k\left(\bar{X}\right)} = -\psi^{(1)}\left(N_k\left(\bar{X}\right) + 1\right) \delta_{k'k}. \tag{5.14}$$

We could express $\bar{S}$ as a Taylor series expansion about the bare vacuum with $\phi_k\left(\bar{X}\right) = 0$ for all k and $\bar{X}$:

$$\bar{S} = S_{bare} + \sum_{\bar{X},k} \sum_{n} b_n\left(\bar{X}\right) N_k\left(\bar{X}\right)^n \tag{5.15}$$

$$b_1\left(\bar{X}\right) = \log \mathcal{N}_{bare}\left(\bar{X}\right) - \psi\ (1) \tag{5.16}$$

$$b_{n+1} = -\psi^{(n)}\ (1)\,/n! \quad , \quad n = 1, 2, ... \tag{5.17}$$

with

$$\psi\ (1) = -\gamma \quad , \quad \gamma = \text{ Euler's constant} \tag{5.18}$$

$$\psi^{(n)}\ (1) = (-1)^{n+1}\, n!\, \zeta\,(n+1) \tag{5.19}$$

where $\mathcal{N}_{bare}\left(\bar{X}\right)$ is the value of $\mathcal{N}\left(\bar{X}\right)$ when $N_k\left(\bar{X}\right) = 0$ for all the observable states k and $\zeta\,(z)$ is the zeta function. This is not physically appropriate, however, because bosonic fields exhibit extremely large zero-point fluctuations in the physical vacuum [82]. (These are analogous to the zero-point oscillations $\langle x^2 \rangle$ of a harmonic oscillator, but with a very large number of modes extending up to a Planck-scale cutoff.) In fact, it is consistent with both standard physics and the treatment of this paper to assume that

$$\left\langle \phi_k^2 \right\rangle_{vac} = \left\langle \rho_k \right\rangle_{vac} = \left\langle N_k \right\rangle_{vac} / \Delta V \sim \ell_P^{-D}. \tag{5.20}$$

Since there is no initial distinction between the states ϕ_k, it is reasonable to perform a Taylor series expansion about the same value N_{vac} for each k, where

$$N_{vac} \sim \ell_P^{-D} \Delta V \sim (a/\ell_P)^4 >>> 1 \tag{5.21}$$

if, e.g., $a^{-1} \sim 10^{10}$ TeV (with $\ell_P^{-1} = m_P \sim 10^{15}$ TeV). It is then an extremely good approximation to use the asymptotic formulas above and write

$$\bar{S} = S_{vac} + \sum_{\bar{X},k} a_1 \Delta N_k \left(\bar{X}\right) + \sum_{\bar{X},k} a_2 \left[\Delta N_k \left(\bar{X}\right)\right]^2 \tag{5.22}$$

$$\Delta N_k \left(\bar{X}\right) = N_k \left(\bar{X}\right) - N_{vac} \tag{5.23}$$

$$a_1 = \log \mathcal{N}_{vac} - \log N_{vac} \quad , \quad a_2 = -1/\left(2N_{vac}\right) \tag{5.24}$$

where $\mathcal{N}_{vac}\left(\bar{X}\right)$ is the value of $\mathcal{N}\left(\bar{X}\right)$ when $N_k\left(\bar{X}\right) = N_{vac}$ for all k, and the neglected terms are of order $\left[\Delta N_k\left(\bar{X}\right)/N_{vac}\right]^n \Delta N_k\left(\bar{X}\right)$, $n \geq 2$.

It is not conventional or convenient to deal with ΔN_k and $\left(\Delta N_k\right)^2$, so let us instead write $\bar{S}$ in terms of the fields ϕ_k and their derivatives $\partial\phi_k/\partial x^M$ via the following procedure: First, we can switch from the original points $\bar{X}$, which are defined to be the centers of the boxes, to a new set of points $\widetilde{X}$, which will be defined to be the corners of the boxes. It is easy to see that

$$\bar{S} = S_{vac} + \sum_{\widetilde{X},k} a_1 \left\langle \Delta N_k \left(\bar{X}\right)\right\rangle + \sum_{\widetilde{X},k} a_2 \left\langle \left[\Delta N_k \left(\bar{X}\right)\right]^2 \right\rangle \tag{5.25}$$

where $\langle \cdots \rangle$ in the present context indicates an average over the 2^D boxes labeled by $\bar{X}$ which have the common corner $\widetilde{X}$. Second, we can write $\Delta N_k = \Delta\rho_k \Delta V = (\langle\Delta\rho_k\rangle + \delta\rho_k)\,\Delta V$, with $\langle\delta\rho_k\rangle = 0$:

$$\bar{S} = S_{vac} + \sum_{\widetilde{X},k} a_1 \left\langle \langle\Delta\rho_k\rangle + \delta\rho_k \right\rangle \Delta V + \sum_{\widetilde{X},k} a_2 \left\langle \left(\langle\Delta\rho_k\rangle + \delta\rho_k\right)^2 \right\rangle (\Delta V)^2 \tag{5.26}$$

$$= S_{vac} + \sum_{\widetilde{X},k} a_1 \langle\Delta\rho_k\rangle \Delta V + \sum_{\widetilde{X},k} a_2 \left[\langle\Delta\rho_k\rangle^2 + \left\langle (\delta\rho_k)^2 \right\rangle\right] (\Delta V)^2 . \tag{5.27}$$

Each of the 2^D points $\bar{X}$ surrounding $\widetilde{X}$ is displaced by $\pm a/2$ along the x^μ axes and $\pm a_0/2$ along the x^m axes. The last term above can therefore be rewritten

$$\left\langle (\delta\rho_k)^2 \right\rangle = \sum_\mu \left(\frac{\partial\rho_k}{\partial X^\mu}\right)^2 \left(\frac{a}{2}\right)^2 + \sum_m \left(\frac{\partial\rho_k}{\partial X^m}\right)^2 \left(\frac{a_0}{2}\right)^2 \tag{5.28}$$

$$= \sum_\mu \rho_k \left(\frac{\partial\phi_k}{\partial X^\mu}\right)^2 a^2 + \sum_m \rho_k \left(\frac{\partial\phi_k}{\partial X^m}\right)^2 a_0^2 \tag{5.29}$$

where the neglected terms involve higher derivatives and higher powers of a and a_0. Since $\rho_k = \rho_{vac} + \Delta\rho_k$, with $\Delta\rho_k <<< \rho_{vac} = N_{vac}/\Delta V$ for normal fields, it is an extremely good approximation to replace ρ_k by ρ_{vac} in the above expression, and to neglect the term involving $a_2 (\Delta V)^2 (\Delta\rho_k)^2 = -(\Delta N_k)^2 / 2N_{vac}$, so that we have

$$\bar{S} = S'_{vac} + \sum_{\widetilde{X},k} \Delta V \left\{ \mu\bar{\phi}_k^2 - \frac{1}{2m} \left[\sum_\mu \left(\frac{\partial\bar{\phi}_k}{\partial X^\mu}\right)^2 \left(\frac{a}{a_0}\right)^2 + \sum_m \left(\frac{\partial\bar{\phi}_k}{\partial X^m}\right)^2 \right] \right\} \tag{5.30}$$

where

$$m = a_0^{-1} \quad , \quad \mu = m\left(\log\mathcal{N}_{vac} - \log N_{vac}\right) \quad , \quad \bar{\phi}_k = \phi_k/m \tag{5.31}$$

and $S'_{vac} = S_{vac} - \sum_{\widetilde{X},k} N_{vac}\left(\log\mathcal{N}_{vac} - \log N_{vac}\right)$. Recall that

$$m \sim m_P = \ell_P^{-1}. \tag{5.32}$$

The philosophy behind the above treatment is simple: We essentially wish to replace $\langle f^2 \rangle$ by $(\partial f/\partial x)^2$, and this can be accomplished because

$$\left\langle f^2 \right\rangle - \langle f \rangle^2 = \left\langle (\delta f)^2 \right\rangle \approx \left\langle (\partial f/\partial x)^2 \, (\delta x)^2 \right\rangle = (\partial f/\partial x)^2 \, (a/2)^2 \,. \tag{5.33}$$

The form of (5.30) also has a simple interpretation: The entropy $\bar{S}$ increases with the number of whits, but decreases when the whits are not uniformly distributed.

In the continuum limit,

$$\sum_{\widetilde{X}} \Delta V = \sum_{\widetilde{X}} a^4 a_0^{D-4} \rightarrow \int d^D X = \int_a^\infty d^4 X \int_{a_0}^\infty d^{D-4} X \tag{5.34}$$

(5.30) becomes

$$\begin{aligned} \bar{S} = S'_{vac} &+ \int_a^\infty d^4 X \int_{a_0}^\infty d^{D-4} X \sum_k \\ &\times \left\{ \mu \bar{\phi}_k^2 - \frac{1}{2m} \left[\sum_\mu \left(\frac{\partial \bar{\phi}_k}{\partial X^\mu} \right)^2 \left(\frac{a}{a_0} \right)^2 + \sum_m \left(\frac{\partial \bar{\phi}_k}{\partial X^m} \right)^2 \right] \right\} \\ = S'_{vac} &+ \int_{a_0}^\infty d^D x \, \sum_k \left[\mu \Phi_k^2 - \frac{1}{2m} \sum_M \left(\frac{\partial \Phi_k}{\partial x^M} \right)^2 \right] \end{aligned} \tag{5.35}$$

where

$$x^m = X^m \quad , \quad x^\mu = (a_0/a) \, X^\mu \quad , \quad \Phi_k = (a_0/a)^2 \, \bar{\phi}_k. \tag{5.36}$$

The lower limit on each integral is the cutoff imposed by the size of the rectangular boxes used in the coarse-graining above: a for X^μ, a_0 for X^m, and a_0 for any x^M. The continuum limit is an extremely good approximation for slowly varying fields in external spacetime, but only a moderately good approximation within the internal space, where the order parameter varies on a length scale comparable to ℓ_P. This implies that terms involving higher derivatives $\partial^n \widetilde{\phi}_k / \partial \left(x^m \right)^n$ can be significant in the internal space.

Notice that the final form (5.35) is independent of the arbitrary length a which was used for coarse-graining in external spacetime. The fields must be rescaled as a

is varied, but this is already a familiar feature in standard physics [83].

A physical configuration of all the fields $\phi_k(x)$ corresponds to a specification of all the densities $\rho_k(x)$. In the present picture, the probability of such a configuration is proportional to $W = e^{\bar{S}}$. In the Euclidean path integral, the probability is proportional to e^{-S_E}, where S_E is the Euclidean action. We conclude that

$$S_E = -\bar{S} + \text{constant}. \tag{5.37}$$

Choosing the constant to be zero, and employing the Einstein summation convention for all repeated indices, we obtain

$$S_E = -S'_{vac} + \int d^D x \left(\frac{1}{2m} \frac{\partial \Phi_k}{\partial x_M} \frac{\partial \Phi_k}{\partial x^M} - \mu \Phi_k \Phi_k \right). \tag{5.38}$$

The above result neglects interactions among the observable and unobservable fields, which will arise from the higher-order terms neglected above. Since a detailed treatment of these interactions would be quite complicated, we resort at this point to a phenomenological description: We assume that probability can flow out of and into each field, and that this effect can be modeled by a random optical potential $i\widetilde{V}$ which has a Gaussian distribution, with

$$\left\langle \widetilde{V} \right\rangle = 0 \quad , \quad \left\langle \widetilde{V}(x)\, \widetilde{V}(x') \right\rangle = b\, \delta(x - x') \tag{5.39}$$

where b is a constant.

If we also assume that the number of observable real fields Φ_k is even, we can group them in pairs to form complex fields $\Psi_{b,k}$. Then we finally have $S_E = S_0 + \bar{S}_E \left[\Psi_b, \Psi_b^\dagger \right]$ with

$$\bar{S}_E \left[\Psi_b, \Psi_b^\dagger \right] = \int d^D x \left(\frac{1}{2m} \partial^M \Psi_b^\dagger \partial_M \Psi_b - \mu\, \Psi_b^\dagger \Psi_b + i\widetilde{V}\, \Psi_b^\dagger \Psi_b \right) \tag{5.40}$$

where Ψ_b is the vector with components $\Psi_{b,k}$. This is the starting point for the discussion following Eq. (4.103) on page 125.

CHAPTER VI

CONCLUSION

Here we will summarize the logical development of the present theory, starting with the microscopic statistical picture at the end of this dissertation, and finishing with the Standard Model presented at the beginning.

As mentioned in the Introduction, our microscopic picture is motivated by the fact that a Euclidean path integral in quantum physics is equivalent to a partition function in statistical physics. This suggests that a fundamental description of Nature should start with some sort of statistical picture. The true picture is likely be richer than the one presented here, in the same sense that the description of the hydrogen atom in quantum electrodynamics is richer than the Bohr model, but it may nevertheless be related more closely to the ideas presented here than to those which are currently more fashionable.

The present theory is more ambitious than other attempts at a fundamental theory in that it aspires to explain the origins of

- Lorentz invariance
- gravity
- gauge fields and their symmetry
- supersymmetry
- fermionic fields
- bosonic fields
- quantum mechanics
- spacetime.

At the same time, it involves the familar concepts of grand unification, supersymmetry, higher dimensions, and topological defects.

The present theory begins by postulating a single fundamental system which consists of N_w identical but distinguishable "whits", with N_w variable. Each whit can exist in any of M_w states, with the number of whits in the ith state represented by n_i. A *microstate* of the fundamental system is specified by the number of whits and the state of each whit. A *macrostate* is specified by only the occupancies n_i of the states.

D of the states are used to define D coordinates x^M in Euclidean spacetime, with the value of x^M proportional to the occupancy n_M. Spacetime is then discrete, with a lattice spacing that is comparable to the Planck length ℓ_P. There is consequently an energy cutoff comparable to the Planck energy m_P (in units with $\hbar = c = 1$). m_w of the states are used to define real bosonic fields ϕ_k, with ϕ_k^2 proportional to the density of whits in the kth state. Later the real fields are combined in pairs to form complex bosonic fields.

We compute the entropy $\bar{S}$ to lowest order in the fields and their derivatives, and then define the Euclidean action S by $S = -\bar{S}$. The result (5.38) does not have a lower bound, so we must add the assumption that there is an unspecified perturbing environment which can be represented by the addition of a random imaginary potential to finally yield (5.40).

Fermionic fields and a primitive form of supersymmetry are obtained in Chapter IV, via a simplified version of the arguments used to introduce unphysical versions of SUSY in the context of disordered systems in condensed matter physics. Random fluctuations in the imaginary potential, due to an unseen perturbing environment, have been eliminated and replaced by the new fermionic variables, with the calculated value of any physical quantity F left unchanged.

The next step is a transformation from Euclidean to Lorentzian time, via an inverse Wick rotation. It is important to recognize that a single Lorentzian time is obtained from *all* Euclidean times. I.e., one must regard the physical reality at a single Lorentzian time as a superposition of contributions from all of the original Euclidean times. It is also important to recognize that classical trajectories and observers can be defined only after the transformation to Lorentzian time. The basic point is that a physical time coordinate should be defined in such a way that it can trace out classical paths through the totality of all states in the path integral (i.e., all possible field configurations over all points in spacetime). The Lorentzian time satisfies this requirement whereas the Euclidean time does not. A more detailed discussion of this somewhat philosophical point will be given elsewhere.

In Lorentzian spacetime, we now assume a simple cosmological model:

(1) In the early universe, the "$-\mu$" term in the fundamental action causes one of the bosonic fields to form a condensate near the Planck or GUT scale.

(2) As this condensate forms, a topological defect, which we call an instanton, is frozen into a $d = D - 4$ dimensional internal space. This instanton is assumed to have d-dimensional spherical symmetry. If $d = 10$, one obtains an $SO(10)$ grand unified gauge theory. Details of the internal space, and the origin of the gauge fields, are given in Chapter III.

(3) In 4-dimensional external spacetime, there is a general $U(2)$ rotation of the fields which is present from the beginning (just as a nonzero angular momentum is present from the beginning in the formation of a planetary system or galaxy). This $U(2)$ rotation is somewhat analogous to the rotation of the $U(1)$ order parameter $\psi_s = n_s^{1/2} e^{i\theta_s}$ in the complex plane for an ordinary superfluid which is flowing with a velocity $\vec{v}_s = \vec{\nabla}\theta_s$. If one shifts to the "frame of reference" associated with the "flow" of the 2-component fields, we showed in Chapter III that one obtains the usual action for a Weyl fermion at low energy (compared to the Planck scale). I.e., we obtained the action that is appropriate for a fundamental fermion field with the correct coupling to

the gravitational vierbein e_α^μ, which is defined in the present theory to be essentially a "superfluid velocity" component v_α^μ: $e_\alpha^\mu = -v_\alpha^\mu$.

(4) Initially the action for fundamental bosons has the same form as that for fermions. Through a transformation of the initial boson fields, however, we were able to rewrite the action so that one obtain exactly the standard action for normal scalar boson fields and auxiliary fields. It is remarkable that this transformation leaves both the action and the measure in the path integral unchanged, while leading to the fields and action required for conventional physics with supersymmetry.

(5) While focusing on the novel aspects of the present theory, we have also obtained some interesting results in the context of conventional SUSY. For example, we showed that invariance of both the action and the "volume element" in the functional integral under a sypersymmetry transformation is sufficient to guarantee closure of the SUSY algebra.

(6) At this point one has the fermions and sfermions, as well as the Higgs bosons and Higginos, of the Standard Model augmented by SUSY. In addition, we have an $SO(10)$ unified gauge theory with the correct couplings to these other fields. These results support the viability of the present approach. On the other hand, there are other aspects of conventional physics which we have not yet derived and must therefore postulate: The Einstein-Hilbert action of gravity, and the Maxwell-Yang-Mills action of the gauge fields, as well as the corresponding action terms for gauginos and gravitinos, are assumed to arise from a response of the vacuum to these fields that is analogous to the diamagnetic response of electrons in a metal. We must also assume that various interaction terms – for example, Yukawa couplings – arise from our original action via mechanisms that we have not yet explored. There is thus a considerable amount of work left to do in extending the theory.

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APPENDIX A

SOME NOTATION AND CONVENTIONS

Here we introduce some notation and conventions which are used in this dissertation.

A. Units

We use the convention

$$h = c = 1. \tag{A.1}$$

There are then dimensional relations given by

$$[\text{mass}] = [\text{energy}] = [\text{time}]^{-1} = [\text{length}]^{-1}. \tag{A.2}$$

B. Relativity and Tensors

We use the metric tensor

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.3}$$

where the Greek indices, μ, ν etc., denote 4-space $0, 1, 2, 3$ and the Roman indices, i, j etc., denote 3-space $1, 2, 3$. Vector indices are raised or lowered by the metric tensor as

$$x_\mu = \eta_{\mu\nu} x^\nu \quad \text{and} \quad x^\mu = \eta^{\mu\nu} x_\nu, \tag{A.4}$$

where $x^\mu = (x^0, \vec{x})$ and then $x_\mu = \eta_{\mu\nu} x^\nu = (-x^0, \vec{x})$. The " $\cdot$ " product of two 4 vectors is defined by

$$p \cdot x = \eta_{\mu\nu} p^\mu x^\nu = -p^0 x^0 + \vec{p}\vec{x}. \tag{A.5}$$

The derivative operator is defined by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right). \tag{A.6}$$

When the energy-momentum dispersion relation, $p^0 = |\vec{p}|$ for the massless field and $p^0 = \sqrt{|\vec{p}|^2 + m^2}$ for massive field with mass m is satisfied, we have

$$\begin{aligned} p^2 &= p^\mu p_\mu \\ &= -\left(p^0\right)^2 + |\vec{p}|^2 \\ &= 0 \qquad\qquad \text{for the massless case,} \qquad (A.7) \\ &= -m^2 \qquad\quad \text{for the massive case,} \qquad (A.8) \end{aligned}$$

where the repeated indices are assumed to be summed.

C. Quantum Mechanics

The energy and momentum operators are defined as

$$p^\mu = i\partial^\mu. \tag{A.9}$$

We define the Pauli sigma matrices as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.10}$$

With an identity matrix σ^0,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.11}$$

we define σ^μ and $\bar{\sigma}^\mu$ as

$$\sigma^\mu = \left(\sigma^0, \sigma^i\right), \qquad \bar{\sigma}^\mu = \left(\sigma^0, -\sigma^i\right). \tag{A.12}$$

σ^μ and $\bar{\sigma}^\mu$ are related to the metric tensor $\eta^{\mu\nu}$ as

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = -2\eta^{\mu\nu} \mathbf{1}_{2\times 2}. \tag{A.13}$$

D. Dirac Matrices Algebra

We choose the 4×4 Dirac matrices γ as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{A.14}$$

where σ^μ and $\bar{\sigma}^\mu$ are defined in (A.12), and then γ^5 is given by

$$\begin{aligned} \gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{A.15}$$

The anticommutation relation of γ^μ and γ^5 is

$$\begin{aligned} \{\gamma^\mu, \gamma^5\} &= \gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu \\ &= \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \\ &= 0. \end{aligned} \tag{A.16}$$

The anticommutation relation of the Dirac matrices is

$$
\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \\
&= \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \\
&= -2\eta^{\mu\nu}\mathbf{1}_{4\times 4},
\end{aligned}
\tag{A.17}
$$

where we have used (A.13) in the last line.

From the trace of (A.17),

$$
\begin{aligned}
tr\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) &= tr\left(\gamma^\mu\gamma^\nu\right) + tr\left(\gamma^\nu\gamma^\mu\right) \\
&= 2tr\left(\gamma^\mu\gamma^\nu\right) \\
&\overset{!}{=} tr\left(-2g^{\mu\nu}\mathbf{1}_{4\times 4}\right) \\
&= -8\eta^{\mu\nu},
\end{aligned}
$$

where in the 2nd line we have used $tr\left(\gamma^\mu\gamma^\nu\right) = tr\left(\gamma^\nu\gamma^\mu\right)$, or in general

$$
tr\left(\gamma^\mu\gamma^\nu \cdots \gamma^\rho\gamma^\sigma\right) = tr\left(\gamma^\sigma\gamma^\rho \cdots \gamma^\nu\gamma^\mu\right). \tag{A.18}
$$

In the 3rd line "$\overset{!}{=}$" means "required", and

$$
tr\left(\gamma^\mu\gamma^\nu\right) = -4\eta^{\mu\nu}. \tag{A.19}
$$

According to (A.14) and (A.15),

$$tr\left(\gamma^{\mu}\right) = tr\left(\gamma^{5}\right) = 0. \tag{A.20}$$

The product of any odd number of γ's can be always reduced to the sum of single γ's by using (A.17)−(A.19). (A.20) tells us that

$$tr\left(\text{any odd number of } \gamma\prime s\right) = 0. \tag{A.21}$$

The trace of $\gamma^{\mu}\gamma^{\nu}\gamma^{5}$ is also zero:

$$\begin{aligned}
tr\left(\gamma^{\mu}\gamma^{\nu}\gamma^{5}\right) &= tr\left(\left[-4g^{\mu\nu} - \gamma^{\nu}\gamma^{\mu}\right]\gamma^{5}\right) \\
&= -4g^{\mu\nu}\underbrace{tr\left(\gamma^{5}\right)}_{=0} - tr\left(\gamma^{5}\gamma^{\mu}\gamma^{\nu}\right) \\
&= -tr\left(\gamma^{5}\gamma^{\mu}\gamma^{\nu}\right) = -tr\left(\gamma^{\mu}\gamma^{\nu}\gamma^{5}\right),
\end{aligned}$$

$$\rightarrow tr\left(\gamma^{\mu}\gamma^{\nu}\gamma^{5}\right) = 0, \tag{A.22}$$

where we have used (A.17) in the 1st line and (A.16) in the last line. Finally, we obtain the trace of $\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}$:

$$\begin{aligned}
tr\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= tr\left(\left[-2\eta^{\mu\nu} - \gamma^{\nu}\gamma^{\mu}\right]\gamma^{\rho}\gamma^{\sigma}\right) \\
&= -2\eta^{\mu\nu}tr\left(\gamma^{\rho}\gamma^{\sigma}\right) - tr\left(\gamma^{\nu}\left[-2\eta^{\mu\rho} - \gamma^{\rho}\gamma^{\mu}\right]\gamma^{\sigma}\right) \\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} + 2\eta^{\mu\rho}tr\left(\gamma^{\nu}\gamma^{\sigma}\right) + tr\left(\gamma^{\nu}\gamma^{\rho}\left[-2\eta^{\mu\sigma} - \gamma^{\sigma}\gamma^{\mu}\right]\right) \\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} - 2\eta^{\mu\sigma}tr\left(\gamma^{\nu}\gamma^{\rho}\right) - tr\left(\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}\right) \\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + 8\eta^{\mu\sigma}\eta^{\nu\rho} - tr\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right),
\end{aligned}$$

where we have used $tr\left(\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}\right) = tr\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right)$, and therefore

$$tr\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) = 4\eta^{\mu\nu}\eta^{\rho\sigma} - 4\eta^{\mu\rho}\eta^{\nu\sigma} + 4\eta^{\mu\sigma}\eta^{\nu\rho}. \tag{A.23}$$

E. Delta Function and Fourier Transform

A simple definition of the delta function is given by

$$\delta(x) = \frac{d}{dx}\theta(x) \qquad \text{where } \theta(x) = 0 \text{ for } x < 0 \text{ and } 1 \text{ for } x > 0. \tag{A.24}$$

The delta function satisfies

$$\int d^n x \delta^{(n)}(x) = 1, \tag{A.25}$$

$$\int d^n x \delta^{(n)}(x - x_a) f(x) = f(x_a). \tag{A.26}$$

The Fourier transform used here is defined as

$$f(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ip\cdot x} f(p) \quad \text{and} \quad f(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\vec{x}} f(\vec{p}), \tag{A.27}$$

$$f(p) = \int d^4 x e^{-ip\cdot x} f(x) \quad \text{and} \quad f(\vec{p}) = \int d^3 p e^{-i\vec{p}\vec{x}} f(\vec{x}), \tag{A.28}$$

with

$$\int d^4 x e^{-i(p-p')\cdot x} = (2\pi)^4 \delta^{(4)}(p - p'). \tag{A.29}$$

F. Left and Right Handed Spinor Fields

The 4 component Dirac spinor Ψ is given by

$$\Psi = \begin{pmatrix} \eta_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix} \tag{A.30}$$

where η_α is a left hand 2 component spinor and $\chi^{\dagger\dot{\alpha}}$ is a right hand 2 component spinor. When Ψ in the Lagrangian density

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi$$

is replaced by (A.30), then $\mathcal{L}$ is rewritten as

$$\mathcal{L} = i\eta^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \eta_\alpha + i\chi^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \chi^{\dagger\dot{\alpha}}. \tag{A.31}$$

In this dissertation, we redefine η_α and $\chi^{\dagger\dot{\alpha}}$ as

$$\begin{aligned}\psi_{L\alpha} &= \eta_\alpha, \\ \psi_R^{\dot{\alpha}} &= \chi^{\dagger\dot{\alpha}},\end{aligned}$$

and the Lagrangian density has the following form instead of (A.31):

$$\mathcal{L} = i\psi^\dagger_{L\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_{L\alpha} + i\psi_R^{\dagger\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi_R^{\dot{\alpha}}.$$

When whether a field is right hand or left hand is not explicitly mentioned, it is assumed that

$$i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi$$

is the right hand field's Lagrangian density, and

$$i\psi'^\dagger\sigma^\mu\partial_\mu\psi'$$

is the left hand field's Lagrangian density, where the field with "†" is assumed to be always to the left of the sigma matrix.

APPENDIX B

COMPLEX AND REAL REPRESENTATIONS

When the generators of a representation D are given by T_a, the commutation relation is written as

$$[T_a, T_b] = i f_{abc} T_c. \tag{B.1}$$

The complex conjugate of the commutation relation is

$$\begin{aligned} [T_a^*, T_b^*] &= -i f_{abc} T_c^* \\ \rightarrow [-T_a^*, -T_b^*] &= i f_{abc} \left(-T_c^*\right), \end{aligned} \tag{B.2}$$

and $-T_a^*$ satisfies the same commutation relation as T_a. Since $H_i \subset T_a$, where H_i are the Cartan generators, $-H_i^*$ also satisfies (B.2). The Cartan generators are Hermitian and their complex conjugates have the same eigenvalues, or weights. I.e., when the weight of H_i is μ_i, the weight of $-H_i^*$ becomes $-\mu_i$. The complex conjugate of a representation D is denoted as $\bar{D}$. When $D = \bar{D}$, D is called a real representation, and when $D \neq \bar{D}$, D is called a complex representation.

When the representation D is real, both weights μ_i and $-\mu_i$ are required to be in the same representation. This means that there exists a trivial mapping $T_a \rightarrow -T_a^*$ given by

$$T_a = -O T_a^* O^{-1}, \tag{B.3}$$

and since the T_a are Hermitian

$$\begin{aligned} T_a^\dagger &= -O \left(T_a^\dagger\right)^* O^{-1} \\ &= -O T_a^T O^{-1} \end{aligned}$$

$$\underset{T}{\rightarrow} T_a^* = -\left(O^{-1}\right)^T T_a O^T, \tag{B.4}$$

where O is a matrix. Then by substituting (B.4) into (B.3), we have

$$T_a = O\left(O^{-1}\right)^T T_a O^T O^{-1},$$

$$\rightarrow O^T O^{-1} T_a = T_a O^T O^{-1}$$

$$\rightarrow \left[T_a, O^T O^{-1}\right] = 0. \tag{B.5}$$

To satisfy (B.5) for general generators T_a, $O^T O^{-1}$ is required to be proportional to the identity (Schur's lemma), and

$$O^T O^{-1} = cI, \tag{B.6}$$

$$\rightarrow O^T = cO, \tag{B.7}$$

where c is a constant. As double application of the transpose returns us to the original matrix, (B.7) becomes

$$O = cO^T$$

and by using this in (B.7) we have

$$O^T = c^2 O^T.$$

Therefore,

$$c = \pm 1,$$

and we finally obtain

$$O^T = \pm O, \tag{B.8}$$

where + holds for a real representation and − for a pseudo-real representation.

When there is a matrix M which transforms T_a to make it purely imaginary (and thus antisymmetric in order to be Hermitian), we have

$$T_a'^* = -T_a', \tag{B.9}$$

with

$$T_a' = M^{-1} T_a M.$$

Then the relation between T_a and T_a^* is derived to be

$$\begin{aligned} T_a' &= (T_a')^\dagger \\ &= (T_a'^*)^T \\ &= -T_a'^T \\ &= -M^T T_a^T \left(M^{-1}\right)^T \\ &\overset{!}{=} M^{-1} T_a M, \end{aligned} \tag{B.10}$$

so

$$\begin{aligned} T_a &= -M M^T T_a^T \left(M^{-1}\right)^T M^{-1} \\ &= -M M^T T_a^T \left(M M^T\right)^{-1} \\ &= -M M^T T_a^* \left(M M^T\right)^{-1} \quad \text{because } T_a = T_a^\dagger. \end{aligned} \tag{B.11}$$

Therefore,

$$O = M M^T,$$

$$\rightarrow O^T = +O,$$

and when T_a can be transformed into purely imaginary matrices T_a', the representation is real. The reverese of the argument is also true and when the representation is real there is a matrix M to transform T_a into a purely imaginary T_a', or when the representation is pseudo-real ($O^T = -O$), T_a cannot be transformed into pure imaginary matrices T'_a.

For a more detailed discussion, please see Ref. [62].

APPENDIX C

SPINOR ALGEBRA

Here we will review the two component spinor algebra. There are two kinds of two component Weyl spinors given by

$$\chi_\alpha \text{ and } \chi^\dagger_{\dot\alpha} \quad \text{where } \alpha = 1, 2 \text{ and } \dot\alpha = \dot 1, \dot 2,$$

which transform as the representations $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$, respectively, and these are related by Hermitian conjugation

$$\chi_\alpha = \left(\chi^\dagger_{\dot\alpha}\right)^\dagger \text{ and } \chi^\dagger_{\dot\alpha} = \left(\chi_\alpha\right)^\dagger, \tag{C.1}$$

$$\chi^\alpha = \left(\chi^{\dagger\dot\alpha}\right)^\dagger \text{ and } \chi^{\dagger\dot\alpha} = \left(\chi^\alpha\right)^\dagger. \tag{C.2}$$

Raising and lowering of an index are achieved by using the antisymmetric tensors

$$\epsilon^{\beta\alpha} = -\epsilon_{\beta\alpha} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{C.3}$$

defined by

$$\chi_\beta = \epsilon_{\beta\alpha}\chi^\alpha \text{ and } \chi^\beta = \epsilon^{\beta\alpha}\chi_\alpha, \tag{C.4}$$

and

$$\epsilon^{\dot\beta\dot\alpha} = -\epsilon_{\dot\beta\dot\alpha} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{C.5}$$

since

$$\chi_{\dot\beta} = \epsilon_{\dot\beta\dot\alpha}\chi^{\dot\alpha} \text{ and } \chi^{\dot\beta} = \epsilon^{\dot\beta\dot\alpha}\chi_{\dot\alpha}, \tag{C.6}$$

where the repeated indices are assumed to be summed. Since the ϵ are antisymmetric tensors, we obtain

$$\epsilon^{\beta\alpha}\epsilon_{\gamma\delta} = -\left(\delta^{\beta}_{\ \gamma}\delta^{\alpha}_{\ \delta} - \delta^{\beta}_{\ \delta}\delta^{\alpha}_{\ \gamma}\right), \tag{C.7}$$

$$\epsilon^{\dot\beta\dot\alpha}\epsilon_{\dot\gamma\dot\delta} = -\left(\delta^{\dot\beta}_{\ \dot\gamma}\delta^{\dot\alpha}_{\ \dot\delta} - \delta^{\dot\beta}_{\ \dot\delta}\delta^{\dot\alpha}_{\ \dot\gamma}\right), \tag{C.8}$$

and when $\alpha = \gamma$ we obtain

$$\begin{aligned}\epsilon^{\beta\alpha}\epsilon_{\alpha\delta} &= -\left(\delta^{\beta}_{\ \alpha}\delta^{\alpha}_{\ \delta} - \delta^{\beta}_{\ \delta}\delta^{\alpha}_{\ \alpha}\right)\\ &= -\left(\delta^{\beta}_{\ \delta} - 2\delta^{\beta}_{\ \delta}\right)\\ &= \delta^{\beta}_{\ \delta}.\end{aligned} \tag{C.9}$$

$$\epsilon^{\dot\beta\dot\alpha}\epsilon_{\dot\alpha\dot\delta} = \delta^{\dot\beta}_{\ \dot\delta}. \tag{C.10}$$

The Weyl spinors χ_α and $\chi^\dagger_{\dot\alpha}$ are transformed by the $SL\left(2, C\right)$ group, with $M_\beta^{\ \ \alpha}$ and $M^{\dagger\dot\alpha}_{\ \ \ \dot\beta}$ respectively, which are 2×2 complex matrices with determinant 1,

$$\chi'_\beta = M_\beta^{\ \ \alpha}\chi_\alpha, \tag{C.11}$$

and the Hermitian conjugate gives us

$$\chi'^\dagger_{\dot\beta} = \chi^\dagger_{\dot\alpha}M^{\dagger\dot\alpha}_{\ \ \ \dot\beta}. \tag{C.12}$$

The $SL\left(2, C\right)$ scalars are produced by

$$\eta^\alpha\chi_\alpha \equiv \eta\chi, \tag{C.13}$$

$$\chi^\dagger_{\dot\alpha}\eta^{\dagger\dot\alpha} = \left(\eta\chi\right)^\dagger \equiv \chi^\dagger\eta^\dagger, \tag{C.14}$$

where we follow the convention that the contraction of the undotted indices is from the upper left to the lower right and the contraction of the dotted indices is from the lower left to the upper right. Therefore, χ_α can be considered to be two component column, and χ^α to be two component row, with $\chi^{\dagger\dot\alpha}$ two component column and $\chi^\dagger_{\dot\alpha}$

two component row. From (C.11), raising the index by using $\epsilon^{\alpha\beta}$, we obtain

$$\begin{aligned}\chi'^{\alpha} &= \epsilon^{\alpha\beta}\chi'_{\beta} \\ &= \epsilon^{\alpha\beta}M_{\beta}{}^{\gamma}\chi_{\gamma} \\ &= \epsilon^{\alpha\beta}M_{\beta}{}^{\gamma}\epsilon_{\gamma\delta}\chi^{\delta}.\end{aligned}$$

Since $\chi^{\alpha}\chi_{\alpha}$ is an $SL\left(2,C\right)$ scalar,

$$\begin{aligned}\chi'^{\alpha}\chi'_{\alpha} &= \epsilon^{\alpha\beta}M_{\beta}{}^{\gamma}\epsilon_{\gamma\delta}\chi^{\delta}M_{\alpha}{}^{\xi}\chi_{\xi} \\ &= \chi^{\delta}\epsilon^{\alpha\beta}M_{\beta}{}^{\gamma}\epsilon_{\gamma\delta}M_{\alpha}{}^{\xi}\chi_{\xi} \\ &\overset{!}{=} \chi^{\alpha}\chi_{\alpha},\end{aligned}$$

and we obtain

$$\epsilon^{\alpha\beta}M_{\beta}{}^{\gamma}\epsilon_{\gamma\delta} \equiv \left(M^{-1}\right)_{\delta}{}^{\alpha}. \tag{C.15}$$

Similarly from (C.12),

$$\begin{aligned}\chi'^{\dagger}_{\dot\alpha}\chi'^{\dagger\dot\alpha} &= \chi^{\dagger}_{\dot\beta}M^{\dagger\dot\beta}{}_{\dot\alpha}\epsilon^{\dot\alpha\dot\xi}\chi^{\dagger}_{\dot\gamma}M^{\dagger\dot\gamma}{}_{\dot\xi} \\ &= \chi^{\dagger}_{\dot\beta}M^{\dagger\dot\beta}{}_{\dot\alpha}\epsilon^{\dot\alpha\dot\xi}\epsilon_{\dot\gamma\dot\delta}M^{\dagger\dot\gamma}{}_{\dot\xi}\chi^{\dagger\dot\delta} \\ &\overset{!}{=} \chi^{\dagger}_{\dot\alpha}\chi^{\dagger\dot\alpha},\end{aligned}$$

and we obtain

$$\epsilon^{\dot\alpha\dot\xi}\epsilon_{\dot\gamma\dot\delta}M^{\dagger\dot\gamma}{}_{\dot\xi} = \left(M^{\dagger-1}\right)^{\dot\alpha}{}_{\dot\delta}. \tag{C.16}$$

Next we review the properties of the sigma matrices σ^{μ} and $\bar\sigma^{\mu}$. We define the spinor indices on σ^{μ} and $\bar\sigma^{\mu}$ as

$$\sigma^{\mu}_{\alpha\dot\alpha} \text{ and } \bar\sigma^{\mu\dot\alpha\alpha}, \tag{C.17}$$

which means that σ^{μ} has lower indices, with the left side always undotted and the right side always dotted, and $\bar\sigma^{\mu}$ has upper indices, with the left side always dotted and the right side always undotted. We obtain a new spinor η_α from $\chi^{\dot\alpha}$ as

$$\eta_\alpha = V_\mu \sigma^\mu_{\alpha\dot\alpha} \chi^{\dagger\dot\alpha},$$

where V_μ is a vector. Then by raising the index, we obtain

$$\begin{aligned}\eta^\alpha &= \epsilon^{\alpha\beta}\eta_\beta \\ &= \epsilon^{\alpha\beta} V_\nu \sigma^\nu_{\alpha\dot\alpha}\chi^{\dagger\dot\alpha} \\ &= \epsilon^{\alpha\beta} V_\nu \sigma^\nu_{\beta\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\chi^\dagger_{\dot\beta} \\ &= -V_\nu \epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\sigma^\nu_{\beta\dot\alpha}\chi^\dagger_{\dot\beta}.\end{aligned}$$

By using η^α and η_α, we obtain an $SL\,(2, C)$ scalar,

$$\begin{aligned}\eta^\alpha\eta_\alpha &= -V_\nu \epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\sigma^\nu_{\beta\dot\alpha}\chi^\dagger_{\dot\beta} V_\mu \sigma^\mu_{\alpha\dot\alpha}\chi^{\dagger\dot\alpha} \\ &= -V_\nu V_\mu \chi^\dagger_{\dot\beta}\epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\sigma^\nu_{\beta\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\chi^{\dagger\dot\alpha},\end{aligned}$$

and the right hand side is required to be an $SL\,(2, C)$ scalar. This is satisfied when

$$\epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\sigma^\nu_{\beta\dot\alpha}\sigma^\mu_{\alpha\dot\alpha} \equiv \bar\sigma^{\nu\dot\beta\alpha}\sigma^\mu_{\alpha\dot\alpha}, \tag{C.18}$$

and

$$\begin{aligned}V_\nu V_\mu \epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\sigma^\nu_{\beta\dot\alpha}\sigma^\mu_{\alpha\dot\alpha} &= V_\nu V_\mu \bar\sigma^{\nu\dot\beta\alpha}\sigma^\mu_{\alpha\dot\alpha} \\ &= V_\nu V_\mu \frac{\bar\sigma^{\nu\dot\beta\alpha}\sigma^\mu_{\alpha\dot\alpha} + \bar\sigma^{\mu\dot\beta\alpha}\sigma^\nu_{\alpha\dot\alpha}}{2} \\ &= V_\nu V_\mu \frac{-2g^{\nu\mu}\delta^{\dot\beta}{}_{\dot\alpha}}{2} \\ &= -V^\mu V_\mu \delta^{\dot\beta}{}_{\dot\alpha},\end{aligned}$$

where we have used

$$\bar\sigma^{\nu\dot\beta\alpha}\sigma^\mu_{\alpha\dot\alpha} + \bar\sigma^{\mu\dot\beta\alpha}\sigma^\nu_{\alpha\dot\alpha} = -2g^{\nu\mu}\delta^{\dot\beta}{}_{\dot\alpha}, \tag{C.19}$$

and we finally obtain

$$\begin{aligned}\eta^\alpha\eta_\alpha &= -V_\nu V_\mu \chi^\dagger_{\dot\beta}\epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\sigma^\nu_{\beta\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\chi^{\dagger\dot\alpha}\\ &= V^\mu V_\mu \delta^{\dot\beta}{}_{\dot\alpha}\chi^\dagger_{\dot\beta}\chi^{\dagger\dot\alpha}\\ &= V^\mu V_\mu \chi^\dagger_{\dot\alpha}\chi^{\dagger\dot\alpha}.\end{aligned}$$

Therefore, from (C.18), we get

$$\epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\sigma^\nu_{\beta\dot\alpha} = \bar\sigma^{\nu\dot\beta\alpha}. \tag{C.20}$$

By applying $\epsilon_{\gamma\alpha}\epsilon_{\dot\gamma\dot\beta}$ on both sides of (C.20), we find

$$\begin{aligned}\epsilon_{\gamma\alpha}\epsilon_{\dot\gamma\dot\beta}\bar\sigma^{\nu\dot\beta\alpha} &= \epsilon_{\gamma\alpha}\epsilon_{\dot\gamma\dot\beta}\epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\sigma^\nu_{\beta\dot\alpha}\\ &= \delta_\gamma{}^\beta\delta_{\dot\gamma}{}^{\dot\alpha}\sigma^\nu_{\beta\dot\alpha}\\ &= \sigma^\nu_{\gamma\dot\gamma},\end{aligned} \tag{C.21}$$

where we have used (C.9). The Hermitian conjugate of (C.19) gives us

$$\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta} + \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta} = -2g^{\nu\mu}\delta_\alpha{}^\beta. \tag{C.22}$$

To obtain the identity $\bar\sigma^{\nu\dot\beta\delta}\bar\sigma_\nu^{\dot\delta\gamma} = -2\epsilon^{\dot\delta\dot\beta}\epsilon^{\gamma\delta}$, we start from (C.19),

$$\begin{aligned}&\bar\sigma^{\nu\dot\beta\alpha}\sigma^\mu_{\alpha\dot\alpha} + \bar\sigma^{\mu\dot\beta\alpha}\sigma^\nu_{\alpha\dot\alpha} = -2g^{\nu\mu}\delta^{\dot\beta}{}_{\dot\alpha},\\ &\to \epsilon_{\dot\alpha\dot\gamma}\epsilon_{\alpha\beta}\left[\bar\sigma^{\nu\dot\beta\alpha}\bar\sigma^{\mu\dot\gamma\beta} + \bar\sigma^{\mu\dot\beta\alpha}\bar\sigma^{\nu\dot\gamma\beta}\right] = -2g^{\nu\mu}\delta^{\dot\beta}{}_{\dot\alpha},\\ &\underset{\times g_{\mu\nu}}{\to} \epsilon_{\dot\alpha\dot\gamma}\epsilon_{\alpha\beta}\left[\bar\sigma^{\nu\dot\beta\alpha}\bar\sigma_\nu^{\dot\gamma\beta}\right] = -4\delta^{\dot\beta}{}_{\dot\alpha},\\ &\underset{\times\epsilon^{\dot\delta\dot\alpha}}{\to} \epsilon_{\alpha\beta}\left[\bar\sigma^{\nu\dot\beta\alpha}\bar\sigma_\nu^{\dot\delta\beta}\right] = -4\epsilon^{\dot\delta\dot\beta},\\ &\underset{\times\epsilon^{\gamma\delta}}{\to} \epsilon^{\gamma\delta}\epsilon_{\alpha\beta}\left[\bar\sigma^{\nu\dot\beta\alpha}\bar\sigma_\nu^{\dot\delta\beta}\right] = -4\epsilon^{\dot\delta\dot\beta}\epsilon^{\gamma\delta},\\ &\to -\left(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\gamma_\beta\delta^\delta_\alpha\right)\left[\bar\sigma^{\nu\dot\beta\alpha}\bar\sigma_\nu^{\dot\delta\beta}\right] = -4\epsilon^{\dot\delta\dot\beta}\epsilon^{\gamma\delta}\end{aligned}$$

$$\rightarrow -\bar{\sigma}^{\nu\dot{\beta}\gamma}\bar{\sigma}_{\nu}^{\dot{\delta}\delta} + \bar{\sigma}^{\nu\dot{\beta}\delta}\bar{\sigma}_{\nu}^{\dot{\delta}\gamma} = -4\epsilon^{\dot{\delta}\dot{\beta}}\epsilon^{\gamma\delta}.$$

The left hand side is antisymmetric under $\gamma \leftrightarrow \delta$, and the right hand side is also, so we obtain

$$\bar{\sigma}^{\nu\dot{\beta}\delta}\bar{\sigma}_{\nu}^{\dot{\delta}\gamma} = -2\epsilon^{\dot{\delta}\dot{\beta}}\epsilon^{\gamma\delta}. \tag{C.23}$$

Similarly, from (C.22) we obtain

$$\sigma^{\mu}_{\alpha\dot{\alpha}}\sigma_{\mu\gamma\dot{\beta}} = -2\epsilon_{\dot{\beta}\dot{\alpha}}\epsilon_{\gamma\alpha}. \tag{C.24}$$

Multiplying $\sigma^{\xi}_{\gamma\dot{\beta}}$ by (C.19) gives

$$\sigma^{\xi}_{\gamma\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}} + \sigma^{\xi}_{\gamma\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}} = -2g^{\nu\mu}\sigma^{\xi}_{\gamma\dot{\alpha}},$$

and with $\mu \leftrightarrow \xi$ (C.25) becomes

$$\sigma^{\mu}_{\gamma\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\alpha}\sigma^{\xi}_{\alpha\dot{\alpha}} + \sigma^{\mu}_{\gamma\dot{\beta}}\bar{\sigma}^{\xi\dot{\beta}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}} = -2g^{\nu\xi}\sigma^{\mu}_{\gamma\dot{\alpha}}.$$

By adding these two equations,

$$\sigma^{\xi}_{\gamma\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}} + \sigma^{\mu}_{\gamma\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\alpha}\sigma^{\xi}_{\alpha\dot{\alpha}} + \left(\sigma^{\xi}_{\gamma\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}} + \sigma^{\mu}_{\gamma\dot{\beta}}\bar{\sigma}^{\xi\dot{\beta}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\right) = -2g^{\nu\mu}\sigma^{\xi}_{\gamma\dot{\alpha}} - 2g^{\nu\xi}\sigma^{\mu}_{\gamma\dot{\alpha}}.$$

The expression inside the parenthesis is rewritten as

$$\sigma^{\xi}_{\gamma\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}} + \sigma^{\mu}_{\gamma\dot{\beta}}\bar{\sigma}^{\xi\dot{\beta}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}} = -2g^{\xi\mu}\delta^{\alpha}_{\gamma}\sigma^{\nu}_{\alpha\dot{\alpha}} = -2g^{\xi\mu}\sigma^{\nu}_{\gamma\dot{\alpha}},$$

and we obtain

$$\sigma^{\xi}_{\gamma\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}} + \sigma^{\mu}_{\gamma\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\alpha}\sigma^{\xi}_{\alpha\dot{\alpha}} = 2g^{\xi\mu}\sigma^{\nu}_{\gamma\dot{\alpha}} - 2g^{\nu\mu}\sigma^{\xi}_{\gamma\dot{\alpha}} - 2g^{\nu\xi}\sigma^{\mu}_{\gamma\dot{\alpha}}. \tag{C.25}$$

Similarly, multiplying $\bar{\sigma}^{\xi\dot{\gamma}\alpha}$ by (C.22) gives

$$\bar{\sigma}^{\xi\dot{\gamma}\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} + \bar{\sigma}^{\xi\dot{\gamma}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta} = -2g^{\nu\mu}\bar{\sigma}^{\xi\dot{\gamma}\beta},$$

and with $\xi \leftrightarrow \nu$ this equation becomes

$$\bar{\sigma}^{\nu\dot{\gamma}\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\xi\dot{\alpha}\beta} + \bar{\sigma}^{\nu\dot{\gamma}\alpha}\sigma^{\xi}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta} = -2g^{\nu\xi}\bar{\sigma}^{\mu\dot{\gamma}\beta}.$$

By adding these two equations, we find

$$\bar{\sigma}^{\xi\dot{\gamma}\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} + \bar{\sigma}^{\nu\dot{\gamma}\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\xi\dot{\alpha}\beta} + \left(\bar{\sigma}^{\nu\dot{\gamma}\alpha}\sigma^{\xi}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta} + \bar{\sigma}^{\xi\dot{\gamma}\alpha}\sigma^{\nu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\right) = -2g^{\nu\mu}\bar{\sigma}^{\xi\dot{\gamma}\beta} - 2g^{\nu\xi}\bar{\sigma}^{\mu\dot{\gamma}\beta}$$

$$\rightarrow \bar{\sigma}^{\xi\dot{\gamma}\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} + \bar{\sigma}^{\nu\dot{\gamma}\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\sigma}^{\xi\dot{\alpha}\beta} = 2g^{\nu\xi}\bar{\sigma}^{\mu\dot{\alpha}\beta} - 2g^{\nu\mu}\sigma^{\xi}_{\gamma\dot{\alpha}} - 2g^{\nu\xi}\sigma^{\mu}_{\gamma\dot{\alpha}}. \tag{C.26}$$

More results on spinor algebra can be found in Ref. [84].

VITA

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