Mathematical Analysis of a Vibrating Wire

by

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Abstract. The purpose of this paper is to study the vibration of a wire by incorporating resistance to bending into the model leading to the string equation. Included here is a derivation of a fourth order, partial differential equation of motion for the wire, the analytical solution of the equation, a record of my attempts to experimentally verify the solution, and conclusions as to its value.

1. Introduction. Mathematical modeling is used in most aspects of engineering. One approximates a physical process, assuming that some factors are negligible. The validity of these assumptions determines the accuracy of the model (i.e. whether or not it closely predicts the physical situation). Traditionally, when one models a vibrating string, one assumes that the resistance to bending is negligible. This phenomenon can be seen physically--when one holds a string by one end in the hand, it hangs vertically. At the other extreme, when one holds a beam, such as a pencil, it remains at the angle at which it is first held. As a result, one assumes here that the internal shear is paramount. The tension, which in a string is the main factor in determining motion, is in the beam negligible.

My project has been to study vibration in which both

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[1] As the model for my format, I used the following: Ablowitz, Mark J., "Nonlinear Evaluation Equations--Continuous and Discrete," Siam Review, Vol. 19, No. 4 (Oct., 1977), pp. 663-684.

factors, tension and resistance to bending are present. To facilitate discussion of this study, please keep my distinction between the following terms in mind: a vibrating string is an ideal case where there is no resistance to bending; a vibrating beam is influenced only by shear so that axial tension is negligible; a vibrating wire is affected by both tension and resistance to bending. For the purposes of this discussion, a wire will include all vibrating cords which are influenced by tension and shear, even those larger than are ordinarily termed wires.

At the outset of the project, the two questions which guided my research were (1) how far must a string or beam vary from the ideal for the equations describing their motion to result in significant error? and (2) how can one determine the frequency of vibration in the interim region? I found no record in the A&M library of previous work done on this particular problem. My approach then was to study the derivations of the string and beam equations and to try to develop my own model in a similar way. In order to learn sufficient mathematics to accomplish this goal, I attended two mathematics lectures during the year: Numerical Analysis 417 and Topics in Applied Mathematics 312. I also consulted books on the subjects of mathematical physics, vibration, and partial differential equations. [2, 3, 4, 5]

2. <u>The Wire Equation</u>. The first step was to develop an appropriate model to which I could apply Newton's second law in order to derive an equation describing the wire's motion. Two options were available--I could incorporate axial tension into the ideal beam model or include internal shear in the ideal string model. In most diagrams of ideal beams, as in Fig. 1, one end is allowed to move along the horizontal so that the tension resulting from the motion need not be included.

Fig. 1 Model for an ideal beam

My inclination as an engineering student was to draw a free body diagram of the vibrating wire. This is shown in Fig. 2.

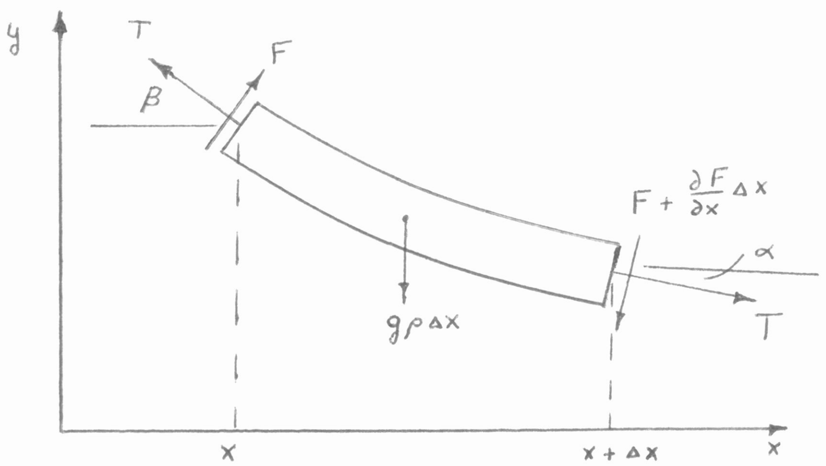

Fig. 2. Free-body diagram of forces acting on the wire

I made the following assumptions in this diagram:

1. Tension, T, and shear, F $>> g\rho\Delta x$
2. α, β are small (small vertical motion compared to length)
3. T = constant
4. $\frac{\partial x}{\partial t} = 0$
5. Density, ρ, is constant along length

From $\sum F_y = ma_y$

(1.) $F - (F + \frac{\partial F}{\partial x}\Delta x) + T\sin\beta - T\sin\alpha = ma_y$

For small β, $\sin\beta = \tan\beta = \left.\frac{\partial y}{\partial x}\right|_x$. Likewise, for small α, $\sin\alpha = \tan\alpha = \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}$. As a result,

(2.) $-\frac{\partial F}{\partial x}\Delta x + T\left[\left.\frac{\partial y}{\partial x}\right|_x - \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}\right] = ma_y$

Also, $ma_y = \rho\Delta x\frac{\partial^2 y}{\partial t^2}$

Substituting and dividing through by Δx,

(3.) $$-\frac{\partial F}{\partial x} + T\left[\frac{\left.\frac{\partial y}{\partial x}\right|_x - \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}}{\Delta x}\right] = \rho\frac{\partial^2 y}{\partial t^2}$$

For small Δx, $\left(\left.\frac{\partial y}{\partial x}\right|_x - \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}\right)/\Delta x$ approximates $\frac{\partial^2 y}{\partial x^2}$ by the definition of the derivative.

Now it remains to relate the shear force to the displacement. We can get an approximate expression relating the shear and the internal moment by considering Fig. 3.

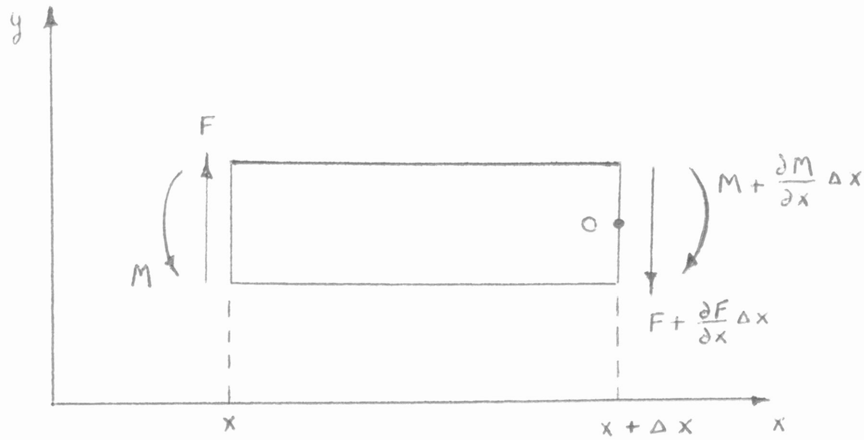

Fig 3
Free-body diagram for relating shear and moments in the wire

Setting $\sum M_o = 0$, we have

(4.) $F\Delta x + \frac{\partial M}{\partial x}\Delta x = 0$ or

(5.) $F = -\frac{\partial M}{\partial x}$.

From mechanics of materials, $M = -EI\frac{\partial^2 y}{\partial x^2}$ so that

$F = EI\frac{\partial^3 y}{\partial x^3}$ and $-\frac{\partial F}{\partial x} = -EI\frac{\partial^4 y}{\partial x^4}$.

From equation 3,

(6.) $-EI\frac{\partial^4 y}{\partial x^4} + T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2}$.

Finally,

(7.) $-a u_{xxxx} + b u_{xx} - u_{tt} = 0$

where $a = EI/\rho$ and $b = T/\rho$.

Equation 6 is a fourth order, linear, partial differential equation. It can be solved by the separation of variables technique which assumes a solution of the form $u(x,t) = X(x)T(t)$. Substituting this assumed solution into equation 6,

(8) $-aX''''T + bX''T = XT''$.

Dividing both sides by X and T,

$$(9.) \quad \frac{-aX'''' + bX''}{X} = \frac{T''}{T} .$$

The left side of this expression is solely a function of x ; the right side is solely a function of t . For the equation to hold for all x and t, both sides must be a constant. For instance, if one fixes t so that T''/T is constant, then $-aX'''' + bX''/X$ must be equal to that constant for all values of x.

Before proceeding further with the solution, it is first necessary to specify boundary and initial conditions for the problem. For the ideal case, the ends would be secured as in Fig. 4, so that the displacement is zero and the moment at each end is also zero. In terms of equation 6,

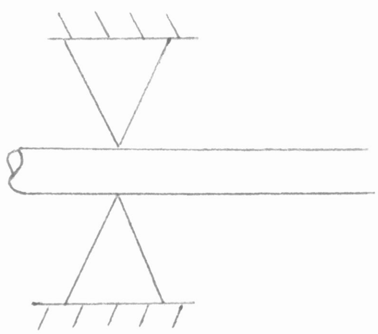

Fig. 4
Ideal Boundary Conditions

(10) $u(0,t) = 0$

(11) $u(L,t) = 0$

(12) $u_{xx}(0,t) = 0$

(13) $u_{xx}(L,t) = 0$

To uniquely specify the motion of the wire, one must give it either an initial displacement or an initial velocity. I chose to give the wire the initial displacement seen in Fig.5.

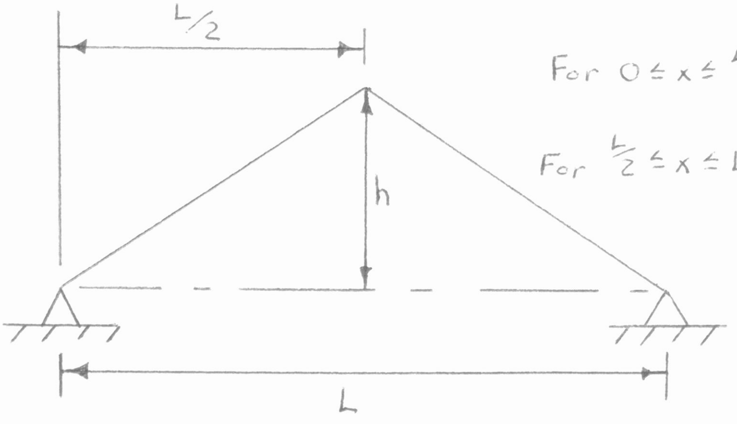

Fig. 5. Initial displacement of the wire

It is important to note that this initial condition is inconsistent with ordinary beam theory which allows no discontinuities in derivatives as at $L/2$ (plane sections remain planar). Although I considered no other initial displacement, one could replace this initial condition by a polynomial--for example, $x(L-x)$ -- or even better, smooth out the discontinuity by replacing a small length around it by a polynomial while leaving the rest of the wire straight. To find this polynomial, one has only to match the slope and position at each end of the small length with the slope and position of the lines.

With these constraints in mind, we now return to equation 8. Although both sides of this expression are constant, one must determine if the constant is positive, negative, or zero. By use of the boundary conditions, it can be shown that to avoid a trivial solution, this constant, λ^2, must be negative. Consequently,

(14.) $T'' + \lambda^2 T = 0$

(15.) $a X'''' - b X'' - \lambda^2 X = 0$.

The solution to equation 14 is well known,

(16.) $T = A\sin\lambda t + B\cos\lambda t$.

One begins to solve equation 15 by assuming a solution of the form Ce^{rt}, finding all possible values of r which satisfy the equation, and then forming the general solution by superposition. Substituting Ce^{rt} into equation 15, one gets,

(17.) $a r^4 - b r^2 - \lambda^2 = 0$,

By letting $u = r^2$, $a u^2 - b u - \lambda^2 = 0$ or

$$u = \frac{b \pm \sqrt{b^2 + 4a\lambda^2}}{2a}$$

. As a result,

$$r = \pm\sqrt{\frac{b \pm \sqrt{b^2 + 4a\lambda^2}}{2a}} \ .$$

Since $\sqrt{b^2 + 4a\lambda^2} > b$, there are two complex solutions and two real solutions for r .

Let $\phi = \sqrt{\dfrac{b + \sqrt{b^2 + 4a\lambda^2}}{2a}}$ and $J = \sqrt{\dfrac{\sqrt{b^2 + 4a\lambda^2} - b}{2a}}$

so that $r = \phi, -\phi, iJ,$ and $-iJ$.

Thus $X = A_1 e^{\phi x} + A_2 e^{-\phi x} + A_3 e^{i\delta x} + A_4 e^{-i\delta x}$.

Transforming these exponentials into trigonometric functions,

$$X = B_1 \cosh \phi x + B_2 \sinh \phi x + B_3 \cos \delta x + B_4 \sin \delta x .$$

By consideration of the boundary conditions, one concludes that B_1, B_2, and B_3 are zero, and that $B_4 \sin \delta L = 0$. To avoid a trivial solution, which would result if B_4 were zero $\delta = \frac{n\pi}{L}$.

Hence, $\frac{n\pi}{L} = \sqrt{\frac{\sqrt{b^2 + 4a\lambda^2} - b}{2a}}$. Solving for λ, one gets

$$(18.) \qquad \sqrt{\frac{\left[2a\left(\frac{n\pi}{L}\right)^2 + b\right]^2 - b^2}{4a}} .$$

Consequently, there is not simply a single solution, but an infinity of solutions corresponding to values of $n = 1, 2, 3 \cdots$.

Stretching the principle of superposition to include this infinity of solutions, the solution to the fourth order equation can be determined.

$$(19.) \qquad u(x,t) = \sum_{n=1}^{\infty} B_4 \sin \frac{n\pi x}{L} \left(A \sin \lambda_n t + B \cos \lambda_n t\right) .$$

Since the initial velocity is zero, then $u_t(x,0) = 0$. This condition is satisfied only when A is zero, so that the final solution becomes

$$(20.)\quad u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \cos \lambda_n t \; .$$

From the initial condition,

$$(21.)\quad u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} = f(x) \; .$$

Since $\left\{ \sin \frac{n\pi x}{L} \right\}$ forms an orthogonal and complete set for the fourth order differential equation $aX'''' - bX'' - \lambda^2 X = 0$, the theory of Fourier analysis allows one to conclude that $u(x,0)$ can be represented by the Fourier series in (21.) where the values of C_n are the Fourier coefficients of $f(x)$. Although this infinite series furnishes an analytical solution to the problem, to calculate values of $u(x,t)$ for specific values of x and t , one must use approximations obtained by partial sums of the series. This type of computation is easily accomplished by means of a computer. Fig. 6 gives the computer program for this calculation, for use on a Hewlett Packard 9830A with a plotter.

In equation 20, the values of λ_n give the different frequencies of vibration of the wire. For $n=1$, λ_n gives the fundamental frequency, and for larger values of n, the frequencies of harmonics result. From the solution to the equation for the ideal beam, one finds that $\lambda_n = \sqrt{a}\,\frac{n^2\pi^2}{L^2}$; for the ideal string, $\lambda_n = \sqrt{b}\,\frac{n\pi}{L}$. As the constants a and b approach zero in equation (18.), for λ_n derived from the wire equation, the value of λ_n approaches respectively λ_n (beam) and λ_n (string). Considering the limit as b

```
10 SCALE -1,4,-2,2
20 DISP "IF YOU WANT AXES DRAWN KEY IN 0 ";
30 INPUT P
40 IF P>0 THEN 70
50 XAXIS 0,0.1,-1,4
60 YAXIS 0,0.1,-2,2
70 DISP "WHAT IS A";
80 INPUT A1
90 DISP "WHAT IS B";
100 INPUT B1
110 E=PI/2
120 D=PI
130 DISP "WHAT IS T";
140 INPUT T
150 DISP "WHAT IS INITIAL HEIGHT";
160 INPUT H
170 DISP "PLOTTING INCREMENT";
180 INPUT I1
190 FOR X=0 TO D STEP I1
200 S=0
210 FOR K=1 TO 10
215 IF A1=0 THEN 225
220 G=(((K↑2)*2*A1+B1)↑2-B1↑2)/(4*A1))↑0.5
221 GOTO 230
225 G=K*SQR(B1)
230 L=G*D
240 M=G*E
250 C=(2*SIN(M)-SIN(L))*8*H/(L*(2*L-SIN(2*L)))
270 S=S+C*SIN(K*X)*COS(G*T)
280 NEXT K
290 PLOT X,S
300 NEXT X
310 PEN
320 DISP "NEW A,B ? YES=1,NO=0";
330 INPUT Q
340 IF Q=1 THEN 70
350 GOTO 10
360 STOP
```

T = time   H = initial height of string

K = n   G = λ_n

C = Fourier coefficient

S = u(x,t) after a certain number of summations

Fig. 6. Computer program for the solution to u(x,t)

approaches zero,

$$\lim_{b \to 0} \sqrt{\frac{\left[2a\left(\frac{n\pi}{L}\right)^2\right]^2}{4a}} = \sqrt{a\left(\frac{n\pi}{L}\right)^4} = \sqrt{a}\,\frac{n^2\pi^2}{L^2}$$

To find the limit as a approaches zero for λ_n, one makes use of L'Hopital's rule.

$$\left(\lim_{a \to 0}\right)^2 = \frac{\left[2a\left(\frac{n\pi}{L}\right)^2 + b\right]^2 - b^2}{4a} = \frac{\frac{\partial}{\partial a}\left(\left[2a\left(\frac{n\pi}{L}\right)^2 + b\right]^2 - b^2\right)}{\frac{\partial}{\partial a}(4a)} = \frac{2\left[2a\left(\frac{n\pi}{L}\right)^2 + b\right]2\left(\frac{n\pi}{L}\right)^2}{4}$$

Hence, $\lim_{a \to 0} \lambda_n = \sqrt{b}\,\frac{n\pi}{L}$.

The experimental testing of the analytical solution played a vital role in this project since it would verify the correctness of the assumptions made in deriving the model. As a result, I spent the better part of the spring semester on experimentation.

3. Experimentation: The first experiment, a crude one, was made simply to practice experimental procedure, and to determine which parameters were difficult to control. I was not attempting to directly prove the hypothesis and thus had no rigorous procedure in mind. To start with, wires of different composition and diameter were stretched between two clamps on a table, and the tension was varied by turning a screw on one of the clamps. Next, the frequency of vibration was measured by matching the tone of the vibrating wire with

the tone produced by a function generator attached to a small amplifier. The table served well as a sounding board since it was necessary for the tone generated by the wire to be amplified. Although this experiment does not on the surface appear significant, and indeed it produced no significant results, I did learn an important lesson from it: To prove my hypothesis, an experiment would require greater accuracy than I had anticipated, and much greater accuracy than that achieved from matching tones by ear.

By now, I was able to discern what would be required of the final experiment. In the first place, it was necessary to find both the tension and frequency simultaneously to see if they agreed with the model. In addition, the reason that the experiment needed to be so accurate was that I lacked the equipment to perform tests on large wire, such as cables. It would require very sturdy supports and too much tension. As a result, I was limited to using smaller wire (.050" diameter) in which the frequency variation from that predicted by the string equation was not as great.

Although the tension in the wire was easy to measure--one end was fixed while the other end, with a mass attached, was hung over a support--the frequency was more difficult to determine. In my first attempt to verify the hypothesis, I wanted to measure the variation in tension in the wire about some mean tension. This tension would vary at a rate equal to the frequency. To this end, I attached a force transducer in series with the wire. The output of the force transducer was recorded by a strip-chart recorder. This set-up is seen in Fig. 7.

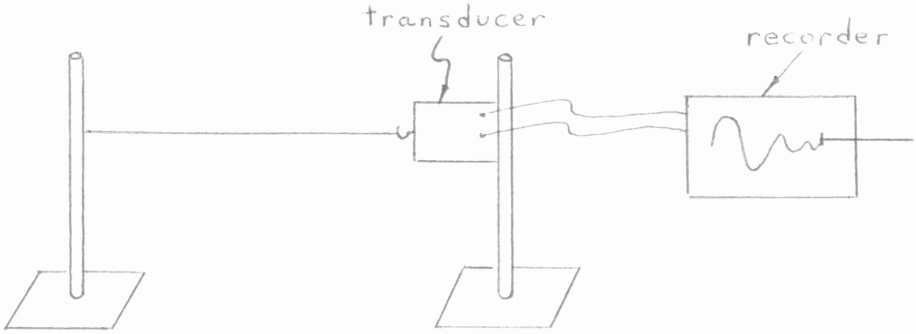

Fig. 7. Set-up for measuring tension variation in the wire

Of course, this method presupposed that the tension variation was significant, which conflicted with my initial assumption. At the beginning, it did look promising, but less promising, however, with each successive try. The frequency seemed independent of tension. I concluded that when plucking the wire, a vibration ensued in the supports for the wire. It was this vibration that caused the tension variation in the wire and which was measured by the transducer. The actual tension variation was too small to be detected.

My next experiment involved the use of a strobe light, which emits light intermittently at a known frequency. Thus, if the light flashes at the frequency of vibration of the wire, the wire will be illuminated at the same position in each cycle and will appear stopped. Since the amplitude of vibration was small, I planned to use a microscope to observe the wire. After repeated attempts, I still had not succeeded in in exactly stopping the wire, but instead found a range over which it was almost motionless, the wire stopping at a slightly different position each time. Hence, this method was not accurate enough to justify any conclusions about the model for the vibrating wire.

Finally, I looked toward electrical theory for a conclusive experiment. I considered letting the wire vibrate in a magnetic field and monitoring the change in current running through the wire. However, I would need an amplifier to measure the current variation, and I had little hope of getting an amplifier. Although an amplifier could be built from circuit components, a simpler method was suggested to me. It was to focus a beam of light upon the wire, and to detect the interruption of this beam by a photo-voltaic cell. This photo-voltaic cell emits, at most, 0.5 volts, and by observing its output, one could find the frequency of vibration of the wire. Fig. 8 gives a schematic of the experiment.

A laser provided the beam of light and no lens was needed for focusing. The response of the photo-voltaic cell was sent to one oscilloscope for monitoring and to a second scope, with a single trigger, to stop the signal so that the period and thus the frequency could be measured. In a completely dark room, the maximum voltage from the photo-cell, resulting from the laser, was 20 mv. Consequently, it was necessary to use the oscilloscopes instead of the more accurate frequency

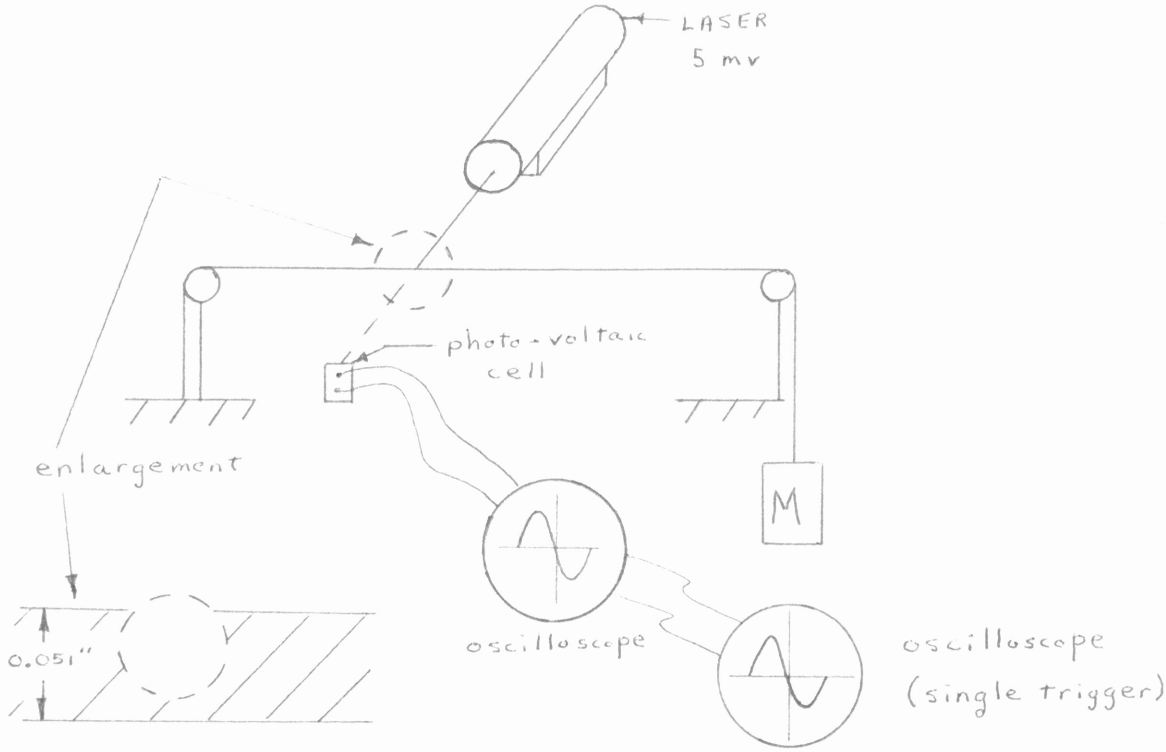

Fig. 8. Set-up for measuring frequency with a photo-voltaic cell

counter, since the counter requires at least 50 mv in order to trigger.

Although this effort was my final experiment of the semester, it also was not accurate enough to justify any unqualified conclusions as to the model for the vibrating wire. Table 1 shows the results of the above experiment for spring brass, 0.051 in. diameter, and compares the experimentally measured frequency with the frequency calculated from the wire model. I performed the same experiment on music wire and aluminum wire of the same diameter, and these results were similar although not as close to the calculated values. I expected the spring brass experiment to be more decisive however, since it had the greatest resistance to bending.

Table 1: Comparison of measured frequency with frequency calculated from wire and string equations

M (kg)	f (measured)	f(wire equation)	f(string eq.)
4	51.95	50.34	50.06
4.5	53.33	53.36	53.09
5	57.19	56.22	55.96
5.5	59.70	58.94	58.70
6	62.50	61.54	61.31
6.5	64.52	64.03	63.81
7	66.67	66.43	66.22
8	70.18	70.99	70.79
9	75.76	75.27	75.08
10	79.37	79.33	79.14
11	83.33	83.18	83.01
12	86.96	86.87	86.70
13	91.91	90.40	90.24

4. Conclusion: In spite of the fact that the measured frequency does not correspond with the value which the wire equation predicted, I still feel it is a valid predictor of frequency for a vibrating wire. For such a small deviation from the string equation which would be expected for spring brass of .051" diameter, the final experiment was not accurate enough. The definitive test of the model would be to use wire with much larger diameter in which the resistance to

bending is more influential.

The value of the wire equation is that it can predict frequencies over the entire spectrum, ranging from the ideal string to the ideal beam. As the resistance to bending becomes negligible, the frequencies from the wire and string equations correspond. Furthermore, as the characteristics of the wire approach those of a beam, the frequencies given by the wire and beam equations are the same. However, more work needs to be done to take into account the fact that when a large wire is fixed at the end, the assumption that the moment at each end is zero could perhaps result in error.

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Dr. Bruce J. Ver West--Physics

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