A Distributed Converging Overland Flow Model
1. Mathematical Solutions

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In models for overland flow based on kinematic wave theory the friction parameter is assumed to be constant. This paper studies a converging geometry and allows continuous spatial variability in the parameter. Parameter variability results in a completely distributed approach, reduces the need to use a complex network model to simulate watershed surface runoff, and saves much computational time and effort. This paper is the first in a series of three. It develops analytical solutions for a converging geometry with no infiltration and temporally constant lateral inflow. Part 2 discusses the effect of infiltration on the runoff process, and part 3 discusses application of the proposed model to natural agricultural watersheds.

INTRODUCTION

Since its formulation by *Lighthill and Whitham* [1955], its application to watershed modeling by *Henderson and Wooding* [1964] and *Wooding* [1965*a*, *b*, 1966], and the subsequent demonstration of its applicability to problems of hydrologic significance by *Woolhiser and Liggett* [1967], the kinematic wave theory has been utilized increasingly in numerous investigations of watershed runoff modeling [*Brakensiek*, 1967; *Woolhiser*, 1969; *Woolhiser et al.*, 1970; *Eagleson*, 1972; *Singh*, 1974, 1975*a*, *b*, *c*, *d*]. In these investigations the formulation has been as follows:

Continuity equation for plane section

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = q(x, t) \qquad (1)$$

Kinematic approximation to momentum equation

$$Q = uh = \alpha h^n \qquad (2)$$

where h is local depth of flow, u is local average velocity, $q(x, t)$ is rate of effective lateral inflow per unit area, Q is rate of outflow per unit width, α is the kinematic wave friction relationship parameter, and $n > 1$. For a specified T, $q(x, t) = 0$ when $t > T$.

The parameter α is constant. One approach that partly relaxes this assumption of parameter constancy is to employ a network model which considers the parameter to be different for different elements in the network geometry. Although a network model may be made so complex as to provide an almost perfect representation of the watershed system, it will be too complex and too time consuming to be of any operational value. In the past, four simplified geometric configurations have been hypothesized to represent the geometry of a natural watershed. Accordingly, kinematic wave models of watershed surface runoff can be classified into four groups: (1) converging overland flow model (Figure 1), (2) Wooding's runoff model (Figure 2), (3) composite section model (Figure 3), and (4) cascade model (Figure 4).

These models entail varying degrees of geometric abstractions and are either lumped or at most quasi-distributed, depending upon the characterization of the parameter α. The converging overland flow model [*Woolhiser*, 1969; *Singh*, 1974, 1975*a*, *b*] is a lumped parameter model. It has the highest degree of geometric abstraction. Wooding's model [*Wooding*, 1965*a*, *b*, 1966] has a lesser degree of abstraction. This is also a lumped parameter or at most quasi-distributed model if the parameter α is allowed to vary from one element to another in the network geometry. The composite section model [*Singh*, 1974], a combination of the two previous models, has an even lesser degree of abstraction. This model will be quasi-distributed if the parameter α is allowed to be different for different elements in the network geometry. The cascade model [*Brakensiek*, 1967; *Kibler and Woolhiser*, 1970; *Singh*, 1974] has the least order of abstraction and hence is more realistic. The cascade network geometry can be made so complex as to provide an almost perfect representation of the watershed geometry. Permitting the parameter α to vary from one element to another will make the cascade model a quasi-distributed model.

A consideration of watershed runoff dynamics suggests that the watershed surface roughness characteristics have more influence on the runoff generation process than the watershed geometry does. This contention was expressed in a recent study by *Singh* [1974] which concluded that regardless of its complexity the geometry of a natural watershed can be transformed into a simple converging section geometry which would preserve the hydrologic response to a large degree. This same view was expressed much earlier by *Woolhiser* [1969]. In the present study the roughness characteristics are represented by the parameter α. It then follows that the above geometric configurations are advanced primarily to represent the spatial distribution of the parameter α better. It is then argued that the necessity of a complex geometric configuration can be eliminated by simply allowing the parameter α to vary continuously in space. By so doing, the resulting model will be simpler in geometry (for example, a converging section) and completely distributed. It is interesting to note that this concept of parameter variability is not an artificial one but is consistent with runoff dynamics. This is the hypothesis that this series of papers attempts to develop and test by considering its application to natural agricultural watersheds. Before proceeding further, it must be made clear that we are not suggesting here that geometric details will have no influence on runoff.

DISTRIBUTED CONVERGING OVERLAND FLOW MODEL

The kinematic wave equations of continuity and momentum for a converging section are [*Singh*, 1974]

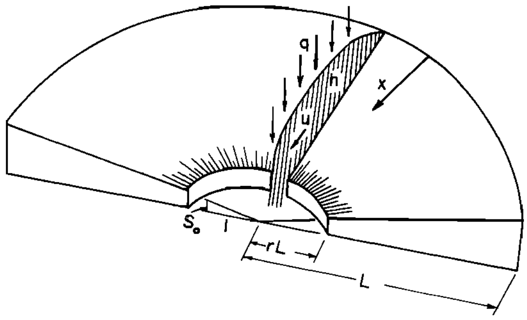

Fig. 1. Geometry of converging overland flow model.

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = q(x, t) + \frac{uh}{L - x} \tag{3}$$

$$Q = uh = \alpha(x, t)h^n \tag{4}$$

where L is the length of the converging section (Figure 1). For a specified rainfall duration T, $q(x, t) = 0$ when $t > T$. We assume that $n > 1$. Eliminating u in (3) and (4), we get

$$\frac{\partial h}{\partial t} + n\alpha(x, t)h^{n-1} \frac{\partial h}{\partial x} = q(x, t) + \frac{\alpha(x, t)h^n}{L - x} - h^n \frac{\partial \alpha(x, t)}{\partial x} \tag{5}$$

Equation (5) holds in $S = \{0 < x < L(1 - r), t > 0\}$. In the context of the watershed runoff problem the boundary conditions, representing an initially dry surface, are

$$h(x, 0) = 0 \qquad 0 \le x \le L(1 - r)$$
$$h(0, t) = 0 \qquad 0 \le t \le T \tag{6}$$

It is physically plausible that $h(0, t)$ should not be specified for $t > T$; that is, the solution of (5) in S below $t = T$ subject to (6) should extend into S above $t = T$. This will be seen to be true in the mathematical discussion below.

We note two special cases of (5). When $\alpha(x, t) = \alpha$, a constant, we get

$$\frac{\partial h}{\partial t} + n\alpha h^{n-1} \frac{\partial h}{\partial x} = q(x, t) + \frac{\alpha h^n}{L - x} \tag{7}$$

This case has been investigated by *Woolhiser* [1969], *Woolhiser et al.* [1970], and *Singh* [1974, 1975*a*, *b*, *c*, *d*]. When $\alpha(x, t) = \alpha(x)$, we get

$$\frac{\partial h}{\partial t} + n\alpha(x)h^{n-1} \frac{\partial h}{\partial x} = q(x, t) + \frac{\alpha(x)h^n}{L - x} - \alpha'(x)h^n \tag{8}$$

In this paper we study (8) in S subject to the boundary conditions (6).

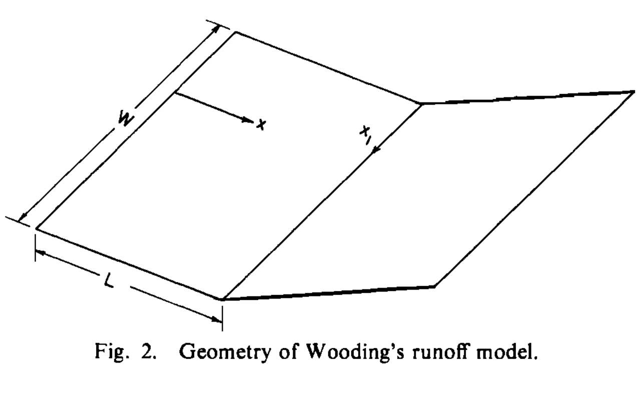

Fig. 2. Geometry of Wooding's runoff model.

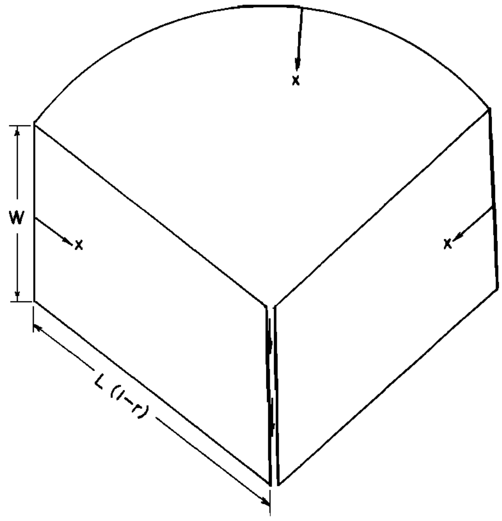

Fig. 3. Geometry of composite section model.

MATHEMATICAL SOLUTIONS

The method of characteristics can be used to solve (8) and (6). The characteristic equations are

$$dt/ds = 1 \qquad dx/ds = n\alpha(x)h^{n-1}$$

$$\frac{dh}{ds} = q(x, t) + \frac{\alpha(x)h^n}{L - x} - \alpha'(x)h^n$$

where s is a parameter. Through each point of (x, t, h) space there is a unique characteristic curve. The solution of (8) and (6) is the surface formed by all the characteristic curves through the segment $t = 0, 0 \le x \le L(1 - r)$ and the segment $x = 0, 0 \le t \le T$ (in Appendix A we show that this solution extends into all of S above $t = T$). Figure 5 shows the projections of these characteristic curves onto the (x, t) plane.

To obtain the surface formed by the characteristic curves, we take x as a parameter instead of s. Then

$$dt/dx = [n\alpha(x)h^{n-1}]^{-1} \tag{9}$$

$$\frac{dh}{dx} = \frac{q(x, t)}{n\alpha(x)h^{n-1}} + \frac{h}{n(L - x)} - \frac{\alpha'(x)h}{n\alpha(x)} \tag{10}$$

The initial conditions are $t(0) = t_0$, $h(0) = 0$ or $t(x_0) = 0$, $h(x_0) = 0$. We distinguish two cases, A and B.

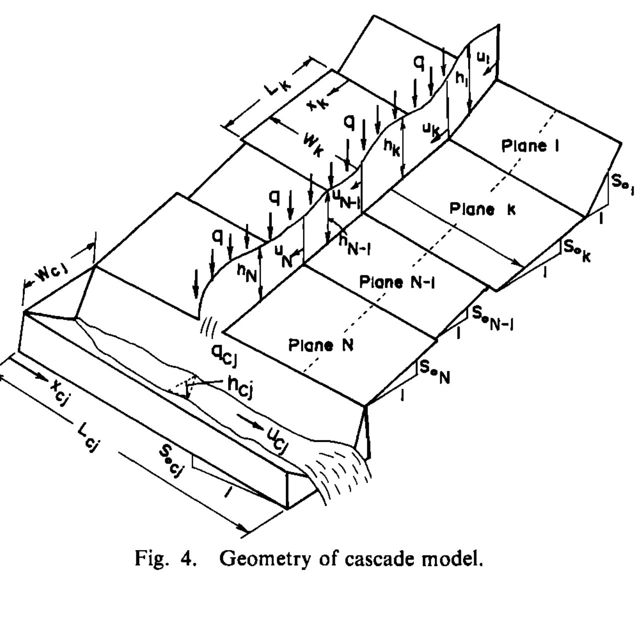

Fig. 4. Geometry of cascade model.

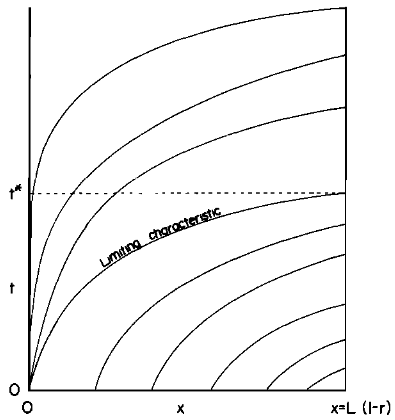

Fig. 5. Kinematic wave diagram.

Case A. The curve $t = t(x, 0)$ through the origin intersects $x = L(1 - r)$ before it intersects $t = T$. Let $t^* = t(L(1 - r), 0)$. As is shown in Figure 6, $t^* \leq T$. Case A represents an equilibrium situation wherein t^* is identical to the watershed equilibrium time. Here t^* is independent of T.

Case B. The curve $t = t(x, 0)$ through the origin intersects $t = T$ before it intersects $x = L(1 - r)$. As is shown in Figure 7, $t^* > T$. This case represents a partial equilibrium situation. Here t^* will depend on T and is not equal to t^* of case A.

The solutions to these two cases will completely characterize the surface runoff hydrograph. We examine the two cases in detail. In case A, S is divided into three parts, as is shown in Figure 6. First, we obtain the surface formed by the characteristics passing through the segment $0 \leq t_0 \leq T$, $x = 0$, i.e., the solution in domain D_2. The initial conditions are $t(0) = t_0$, $h(0) = 0$, and $0 \leq t_0 \leq T$. The solution surface is then expressed in terms of x and t_0: $t = t(x, t_0)$, $h = h(x, t_0)$, and $x = x$. We will assume that under appropriate conditions on $\alpha(x)$ and $q(x, t)$ the curves $t = t(x, t_0)$ do not, for distinct values of t_0, intersect in S. It will be seen in Appendix B that this is true for $q(x, t) = q$, a constant. The curve $t(x, t_0)$ is an increasing function of x for fixed t_0, since $h(x) > 0$ in S (from (11) below), and by our nonintersection assumption it is an increasing function of t_0. Thus we can solve for t_0 in $t = t(x, t_0)$, and we can therefore express h as a function of x and t.

The solution of (10) subject to $h(0) = 0$ is

$$h(x, t_0) = \left[\frac{1}{\alpha(x)(L - x)} \int_0^x (L - \xi) \cdot q[\xi, t(\xi, t_0)]\, d\xi\right]^{1/n} \tag{11}$$

Inserting (11) into (9) and integrating, we get

$$t(x, t_0) = t_0 + \int_0^x \frac{(L - \eta)^{(n-1)/n}}{n\alpha(\eta)^{1/n}} \cdot \left\{\int_0^\eta (L - \xi) q[\xi, t(\xi, t_0)]\, d\xi\right\}^{-(n-1)/n} d\eta \tag{12}$$

Equation (12) is an integral equation for $t(x, t_0)$. Inserting the solution of (12) into (11), we get $h(x, t_0)$.

There are three special cases of (11) and (12) that yield explicit solutions. The first is $q(x, t)$ independent of t. The second is, more particularly, $q(x, t) = q$, a constant. Then the solution is (here and henceforth $\beta = (q/2)^{1/n}$, and $\gamma = n^{-1}(2/q)^{(n-1)/n}$)

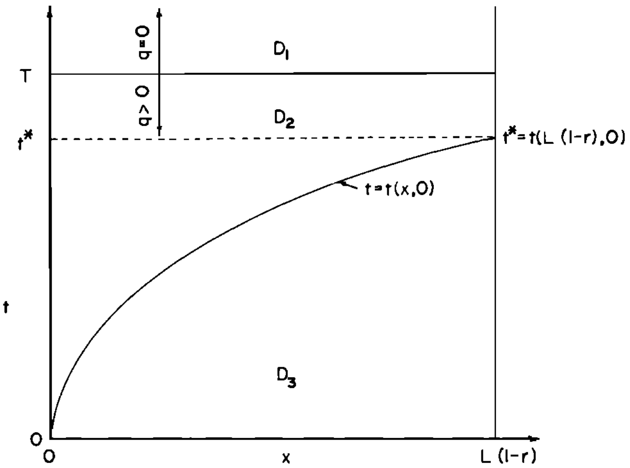

Fig. 6. Solution domain for equilibrium hydrograph.

$$h(x, t_0) = \beta\left[\frac{L^2 - (L - x)^2}{\alpha(x)(L - x)}\right]^{1/n} \tag{13}$$

$$t(x, t_0) = t_0 + \gamma \int_0^x \frac{1}{\alpha(\eta)^{1/n}} \left[\frac{L - \eta}{L^2 - (L - \eta)^2}\right]^{(n-1)/n} d\eta \tag{14}$$

The third special case is $q(x, t)$ and $\alpha(x)$ both constant. In this case we can express (14) in terms of the beta function and incomplete beta function by introducing the change of variable $\xi = [(L - \eta)/L]^2$. We get

$$t(x, t_0) = t_0 + (\gamma/2)\, (L/\alpha)^{1/n}\, [\beta(a, b) - \beta_\phi(a, b)]$$

where

$$a = 1 - 2n^{-1} \qquad b = n^{-1} \qquad \phi = [(L - x)/L]^2$$

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$$

and

$$\beta_\phi(a, b) = \int_0^\phi \xi^{a-1}(1 - \xi)^{b-1}\, d\xi$$

$$= \frac{\phi^a(1 - \phi)^b}{a}\left[1 + \sum_{j=0}^{\infty} \frac{\beta(a + 1, j + 1)}{\beta(a + b, j + 1)} \phi^{j+1}\right]$$

where $a > 0$, $b > 0$, and $0 < \Phi < 1$.

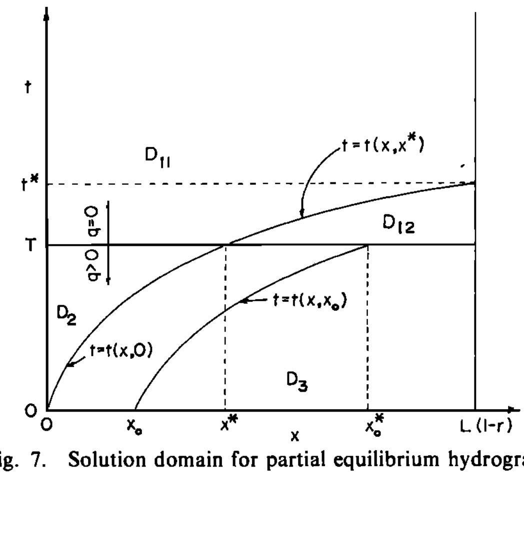

Fig. 7. Solution domain for partial equilibrium hydrograph.

When $q(x, t)$ is independent of t, $h(x, t)$ is also independent of t in D_2. This may be seen directly from (11).

To obtain the surface formed by the characteristics through the segment $0 \leq x < L(1 - r)$ on the x axis, i.e., the solution in D_3, we solve (9) and (10) subject to

$$t(x_0) = 0 \qquad h(x_0 = 0 \qquad 0 \leq x_0 < L(1 - r) \tag{15}$$

Then the solution surface is expressed in terms of x and x_0:

$$t = t(x, x_0) \qquad h = h(x, x_0) \qquad x = x$$

We assume again that the curves $t = t(x, x_0)$ do not intersect for distinct values of x_0. Thus $t(x, x_0)$ is, for fixed x_0, an increasing function of x and for fixed x a decreasing function of x_0. This nonintersection property will be proved in Appendix B under the conditions $q(x, t)$, a constant, and $(L - x)/\alpha(x)$, a decreasing function of x. The solution of (9), (10), and (15) is

$$h(x, x_0) = \left\{\frac{1}{\alpha(x)(L - x)} \int_{x_0}^{x} (L - \xi) \cdot q[\xi, t(\xi, x_0)]\, d\xi\right\}^{1/n} \tag{16}$$

$$t(x, x_0) = \int_{x_0}^{x} \frac{(L - \eta)^{(n-1)/n}}{n\alpha(\eta)^{1/n}} \cdot \left\{\int_{x_0}^{\eta} (L - \xi) q[\xi, t(\xi, x_0)]\, d\xi\right\}^{-(n-1)/n} d\eta \tag{17}$$

Equation (17) is an integral equation for $t(x, x_0)$. Inserting the solution of (17) into (16), we get $h(x, x_0)$. The solutions are explicit when $q(x, t)$ is independent of t. In particular, when $q(x, t) = q$,

$$h(x, x_0) = \beta\left[\frac{(L - x_0)^2 - (L - x)^2}{\alpha(x)(L - x)}\right]^{1/n} \tag{18}$$

$$t(x, x_0) = \gamma \int_{x_0}^{x} \frac{1}{\alpha(\eta)^{1/n}} \cdot \left[\frac{L - \eta}{(L - x_0)^2 - (L - \eta)^2}\right]^{(n-1)/n} d\eta \tag{19}$$

and also if $\alpha(x) = \alpha$,

$$h(x, x_0) = \beta\alpha^{-1/n}\left[\frac{(L - x_0)^2 - (L - x)^2}{L - x}\right]^{1/n} \tag{20}$$

$$t(x, x_0) = \gamma\alpha^{-1/n} \cdot \int_{x_0}^{x} \left[\frac{L - \eta}{(L - x_0)^2 - (L - \eta)^2}\right]^{(n-1)/n} d\eta \tag{21}$$

Introducing the change of variable $\xi = [(L - \eta)/(L - x_0)]^2$ into (21), we obtain an expression for $t(x, x_0)$ in terms of beta and incomplete beta functions:

$$t(x, x_0) = \frac{\gamma}{2}\left(\frac{L - x_0}{\alpha}\right)^{1/n} [\beta(a, b) - \beta_\phi(a, b)]$$

where

$$a = 1 - \frac{1}{2n} \qquad b = \frac{1}{n} \qquad \phi = \left(\frac{L - x}{L - x_0}\right)^2$$

It is clear that in D_3, $h(x, t)$ depends on both x and t even when one or both of the functions $q(x, t)$ and $\alpha(x)$ are constant.

To obtain the solution in D_1, we solve (9) and (10) with $q(x, t) = 0$, subject to

$$t(x_0^*) = T \qquad h(x_0^*) = h(x_0^*, T)$$

where $h(x_0^*, T)$ is obtained from (11) and (12). The solution in D_1 will be expressed in terms of x and x_0^*:

$$t = t(x, x_0^*) \qquad h(x, x_0^*) \qquad x = x$$

The solution is

$$h(x, x_0^*) = h(x_0^*, T)\left[\frac{\alpha(x_0^*)(L - x_0^*)}{\alpha(x)(L - x)}\right]^{1/n} \tag{22}$$

$$t(x, x_0^*) = T + \frac{1}{nh(x_0^*, T)^{n-1}[\alpha(x_0^*)(L - x_0^*)]^{(n-1)/n}} \cdot \int_{x_0^*}^{x} \frac{(L - \eta)^{(n-1)/n}}{\alpha(\eta)^{1/n}}\, d\eta \tag{23}$$

When $q(x, t) = q$, we obtain

$$h(x_0^*, T) = \beta\left[\frac{L - (L - x_0^*)^2}{\alpha(x_0^*)(L - x_0^*)}\right]^{1/n} \qquad h(x, x_0^*) = \beta\left[\frac{L^2 - (L - x_0^*)^2}{\alpha(x)(L - x)}\right]^{1/n} \tag{24}$$

$$t(x, x_0^*) = T + \frac{\gamma}{[L^2 - (L - x_0^*)^2]^{(n-1)/n}} \cdot \int_{x_0^*}^{x} \frac{(L - \eta)^{(n-1)/n}}{\alpha(\eta)^{1/n}}\, d\eta \tag{25}$$

and if in addition, $\alpha(x) = \alpha$,

$$h(x, x_0^*) = \frac{\beta}{\alpha^{1/n}}\left[\frac{L^2 - (L - x_0^*)^2}{L - x}\right]^{1/n} \tag{26}$$

$$t(x, x_0^*) = T + \frac{n\gamma}{(2n - 1)\alpha^{1/n}} \cdot \frac{(L - x_0^*)^{(2n-1)/n} - (L - x)^{(2n-1)/n}}{[L^2 - (L - x_0^*)^2]^{(n-1)/n}} \tag{27}$$

We note that in D_1, $h(x, t)$ depends on both x and t. The curves $t = t(x, x_0^*)$ fill out the entire domain D_1 as x_0^* ranges from 0 to $L(1 - r)$ (Appendix A).

We summarize case A for the general $q(x, t)$.

1. In domain D_3 the solution is given by (16) and (17). Here the parameter x_0 assumes values on the segment $0 \leq x < L(1 - r)$, $t = 0$.

2. In domain D_2 the solution is given by (11) and (12). Here the parameter t_0 assumes values on the segment $x = 0$, $0 \leq t \leq T$.

3. In domain D_1 the solution is given by (22) and (23). Here the parameter x_0^* assumes values on the segment $0 \leq x < L(1 - r)$, $t = T$.

We consider now, in case A, with $q(x, t) = q$, h as a function of t for fixed x. That is, we want to know the appearance of the curve cut out of the surface $h(x, t)$ by a plane perpendicular to the x axis. In domain D_3 we have

$$\frac{\partial h(x, t)}{\partial t} = \frac{\partial h(x, x_0)}{\partial x_0}\frac{\partial x_0}{\partial t} = \frac{h_{x_0}(x, x_0)}{t_{x_0}(x, x_0)} \tag{28}$$

From (18) we see that $h_{x_0}(x, x_0) < 0$. Since $t_{x_0}(x, x_0) < 0$, $h_t(x, t) > 0$ if $(x, t) \in D_3$. In domain D_2, $h(x, t)$ is independent of t. In domain D_1 we have, from (24), $h_{x_0^*}(x, x_0^*) > 0$; since $t_{x_0^*}(x, x_0^*) < 0$, $h_t(x, t) < 0$. From (25), $t \to \infty$ for fixed x is equivalent to $x_0^* \to 0$ for fixed x, and from (24), $h(x, t) \to 0$ as $t \to \infty$. In case A, $h(x, t)$ has, for fixed x, the graph shown in Figure 8.

We may obtain the approximate behavior of $h(x, t)$ for large t (and therefore small x_0^*) by setting $x_0^* = 0$ in the integral in (25) and then eliminating x_0^* between (24) and (25):

$$h(x, t) = \frac{\psi(x)}{(t - T)^{1/(n-1)}}$$

$$\psi(x) = \left[\int_0^x \alpha(\eta)^{-1/n}(L - \eta)^{(n-1)/n}\, d\eta\right]^{1/(n-1)} \cdot \{n^{1/(n-1)}[\alpha(x)(L - x)]^{1/n}\}^{-1} \tag{29}$$

We note that the decline to 0 as $t \to \infty$ is not exponential. Thus if $n = \frac{3}{2}$, $h(x, t)$ goes to 0 as t^{-2}.

We obtain in case A the equilibrium time t_e and the equilibrium depth h_e:

$$t_e = t^* = t[L(1 - r), 0] \qquad h_e = h\,[L(1 - r), 0]$$

where $t(x, 0)$ and $h(x, 0)$ are given by (11) and (12) with $t_0 = 0$. In particular, when $q(x, t) = q$, we get from (13) and (14)

$$h_e = \left[\frac{qL(1 - r^2)}{2r\sigma}\right]^{1/n} \qquad \sigma = \alpha[L(1 - r)]$$

$$t_e = \gamma \int_0^{L(1-r)} \frac{1}{\alpha(\eta)^{1/n}} \left[\frac{L - \eta}{L^2 - (L - \eta)^2}\right]^{(n-1)/n} d\eta$$

If in addition, $\alpha(x) = \alpha$,

$$h_e = \left[\frac{qL(1 - r^2)}{2r\alpha}\right]^{1/n}$$

$$t_e = \gamma\left(\frac{L}{\alpha}\right)^{1/n}\left\{\frac{\Gamma(1 - (2n)^{-1})\Gamma n^{-1}}{\Gamma(1 + (2n)^{-1})} - \frac{r^2 - n^{-1}(1 - r)^{1/n}}{1 - (2n)^{-1}} \cdot \left[1 + \sum_{j=0}^{\infty} \frac{\beta(2 - (2n)^{-1}, j + 1)}{\beta(1 + (2n)^{-1}, j + 1)} r^2(j + 1)\right]\right\}$$

If r is small,

$$t_e = \gamma\left(\frac{L}{\alpha}\right)^{1/n} \frac{\Gamma(1 - (2n)^{-1})\Gamma n^{-1}}{\Gamma(1 + (2n)^{-1})}$$

There is a simple criterion when $q(x, t) = q$ which distinguishes cases A and B. From (14) we obtain, by setting $t_0 = 0$ and the left side equal to T,

$$T = \gamma \int_0^x \frac{1}{\alpha(\eta)^{1/n}} \left[\frac{L - \eta}{(L^2 - (L - \eta)^2}\right]^{(n-1)/n} d\eta \tag{30}$$

Equation (30) has a root x^* between 0 and $L(1 - r)$ in case B and does not have a root in case A. Since the right side $F(x)$ of (30) is an increasing function of x, it is sufficient to determine the value of F at $x = L(1 - r)$: $F[L(1 - r)] \le T$, case A; $F[L(1 - r)] > T$, case B.

We discuss now case B. In this case, S is divided in four parts as shown in Figure 7. Let x^* be the solution of $T = t(x, 0)$. Let D_{11} be the domain above $t = T$ and above $t = t(x, x^*)$; the curve $t = t(x, x^*)$ is just the prolongation of $t = t(x, 0)$ beyond $t = T$. Let D_{12} be the domain bounded by $t = T$, $t = t(x, x^*)$, and $x = L(1 - r)$. The domain D_2 is bounded by $t = T$, $x = 0$, and $t = t(x, 0)$. The domain D_3 is bounded by $t = t(x, 0)$, $t = T$, $t = 0$, and $x = L(1 - r)$. Then in D_2 the solution is given by (11) and (12), in D_3 the solution is given by (16) and (17), and in D_{11} the solution is given by (22) and (23), where $0 < x_0^* \le x^*$. For the particular cases $q(x, t)$ constant and both $q(x, t)$ and $\alpha(x)$ constant the expressions in case A apply.

To obtain the solution in D_{12}, let x_0^* be the solution of $T = t(x, x_0)$. Thus x_0^* and x_0 are bounded by the equation $T = t(x_0^*, x_0)$. Let

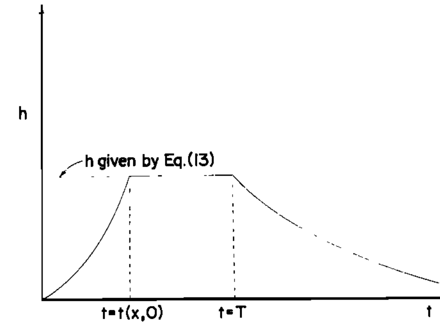

Fig. 8. Variation of the depth of flow with time for equilibrium case.

$$f(x_0, x_0^*) = \int_{x_0}^{x_0^*} (L - \xi)q[\xi, t(\xi, x_0)]\, d\xi$$

Then from (16) and (17),

$$h(x; x_0^*, x_0) = \left[\frac{f(x_0, x_0^*)}{\alpha(x)(L - x)}\right]^{1/n} \tag{31}$$

$$t(x; x_0^*, x_0) = T + \frac{1}{n} f(x_0, x_0^*)^{-(n-1)/n} \cdot \int_{x_0^*}^{x} \frac{(L - \eta)^{(n-1)/n}}{\alpha(\eta)^{1/n}} d\eta \tag{32}$$

When $q(x, t) = q$, we get

$$h(x; x_0^*, x_0) = \beta\left[\frac{(L - x_0)^2 - (L - x_0^*)^2}{\alpha(x)(L - x)}\right]^{1/n} \tag{33}$$

$$t(x; x_0^*, x_0) = T + \frac{\gamma}{[(L - x_0)^2 - (L - x_0^*)^2]^{(n-1)/n}} \cdot \int_{x_0^*}^{x} \frac{(L - \eta)^{(n-1)/n}}{\alpha(\eta)^{1/n}} d\eta \tag{34}$$

and the equation $T = t(x_0^*, x_0)$ becomes

$$T = \gamma \int_{x_0}^{x_0^*} \frac{1}{\alpha(\eta)^{1/n}} \cdot \left[\frac{L - \eta}{(L - x_0)^2 - (L - \eta)^2}\right]^{(n-1)/n} d\eta \tag{35}$$

If in addition, $\alpha(x) = \alpha$, then

$$h(x; x_0^*, x_0) = \beta\alpha^{-1/n}\left[\frac{(L - x_0)^2 - (L - x_0^*)^2}{L - x}\right]^{1/n} \tag{36}$$

$$t(x; x_0^*, x_0) = T + \frac{n\gamma}{(2n - 1)\alpha^{1/n}} \cdot \frac{(L - x_0^*)^{(2n-1)/n} - (L - x)^{(2n-1)/n}}{[(L - x_0)^2 - (L - x_0^*)^2]^{(n-1)/n}} \tag{37}$$

$$T = \gamma\alpha^{-1/n} \int_{x_0}^{x_0^*} \left[\frac{L - \eta}{(L - x_0)^2 - (L - \eta)^2}\right]^{(n-1)/n} d\eta \tag{38}$$

As in case A, we consider $h(x, t)$ as a function of t for fixed x when $q(x, t) = q$. When $0 < x \le x^*$, the discussion in case A applies, and Figure 9 exhibits the behavior. When $x^* < x \le L(1 - r)$, then $h(x, t)$ increases in D_3 and decreases in D_{11} by the same arguments as those used in case A. Therefore the maximum occurs in D_{12}. In Appendix C we show that if $\alpha(x) = \alpha$, then $h(x, t)$ increases in D_{12}, and so the maximum occurs on $t = t(x, x^*)$. Equation (29) also applies in case B.

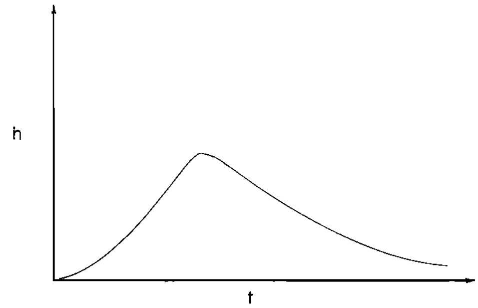

Fig. 9. Variation of the depth of flow with time for partial equilibrium case.

In case A when $q(x, t) = q$ and also in case B when $0 < x \leq x^*$ and $q(x, t) = q$,

$$h_{max}(x) = \beta\left[\frac{L^2 - (L - x)^2}{\alpha(x)(L - x)}\right]^{1/n}$$

$$t_{max}(x) = \{t : t(x, 0) \leq t \leq T\}$$

In case B when $x^* < x \leq L(1 - r)$, $q(x, t) = q$, and $\alpha(x) = \alpha$,

$$h_{max}(x) = \beta\alpha^{-1/n}\left[\frac{L^2 - (L - x^*)^2}{L - x}\right]^{1/n}$$

$$t_{max}(x) = T + \frac{n\gamma}{(2n - 1)\alpha^{1/n}} \cdot \frac{(L - x^*)^{(2n-1)/n} - (L - x)^{(2n-1)/n}}{[L^2 - (L - x^*)^2]^{(n-1)/n}}$$

We define t^* as the time of intersection of $t = t(x, 0)$ with $x = L(1 - r)$ in case A and as the time of intersection of $t = t(x, x^*)$ with $x = L(1 - r)$ in case B. Thus t^* is a function of T. Let $q(x, t) = q$, and define $T_0 = F[L(1 - r)]$; here $F(x)$ is the right side of (30). Then $t^*(T) = T_0$ when $T \geq T_0$, and when $T < T_0$, $t^*(T)$ is defined through x^* by $T = F(x^*)$ and

$$t^* = F(x^*) + \gamma\left(\frac{1}{L^2 - (L - x^*)^2}\right)^{(n-1)/n} \cdot \int_{x^*}^{L(1-r)} \frac{(L - \eta)^{(n-1)/n}}{\alpha(\eta)^{1/n}}\, d\eta \qquad (39)$$

Since

$$\frac{dt^*}{dT} = \frac{dt^*}{dx^*}\frac{dx^*}{dT}$$

and since $dx^*/dT > 0$, the sign of dt^*/dT is the same as the sign of dt^*/dx^*. It is easily checked that $dt^*/dx^* < 0$. Thus $t^*(T)$ is a decreasing function of T when $0 < T < T_0$. As $T \to 0$,

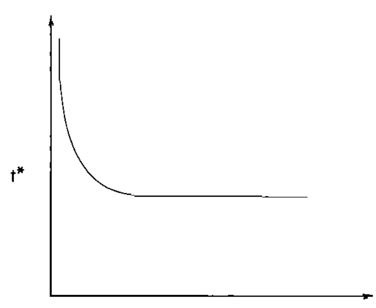

Fig. 10. Variation of the time when the characteristic issuing from the origin intersects the downstream boundary with rainfall duration.

$x^* \to 0$, and we see from (39) that $t^* \to \infty$, as is shown in Figure 10.

Concluding Remarks

The discussion above and in the appendices can be carried out on the assumption $q(x, t) = q(x)$ rather than $q(x) = q$ with only a slight increase in mathematical complexity. But the essential features of the solution $h(x, t)$ are not changed. For the converging overland flow model the assumption $q(x, t) = q(t)$ does not lead to explicit solutions even when t rather than x is selected as a parameter in the equations of the characteristics.

Appendix A

We will show that the curves $t = t(x, t_0)$, $0 \leq t_0 \leq T$, fill out all of S above $t = t(x, 0)$. For this purpose it is sufficient to prove that for fixed $x > 0$, $t(x, t_0) \to \infty$ as $t_0 \to T$. Together with our assumption that the curves $t = t(x, t_0)$ do not intersect in S for distinct values of t_0, this implies that $h(x, t)$ is defined throughout S.

We make the following assumptions on $q(x, t)$, $\alpha(x)$, and n:

$$n > 1 \qquad 0 < \alpha_1 \leq \alpha(x) \leq \alpha_2$$

$$0 < q(x, t) \leq q \quad \text{if} \quad t \leq T \qquad q(x, t) = 0 \quad \text{if} \quad t > T$$

From (11) we get

$$0 < h(x, t_0) < \left[\frac{1}{Lr\alpha_1}\int_0^x qL\, d\xi\right]^{1/n} = \left(\frac{qx}{r\alpha_1}\right)^{1/n} \qquad (A1)$$

and from (9),

$$\frac{dt}{dx} > \frac{1}{n\alpha_2}\left(\frac{qx}{r\alpha_1}\right)^{-(n-1)/n} = \frac{c_1}{n}x^{-1+(1/n)} \qquad (A2)$$

Integrating (A2) between 0 and x, we get

$$t(x, t_0) > t_0 + c_1x^{1/n} \qquad (A3)$$

Let $x^*(t_0)$ be the solution of $T = t(x^*, t_0)$. Then from (A3),

$$x^*(t_0) < [(T - t_0)/c_1]^n$$

We have, referring to (12),

$$\int_0^\eta (L - \xi)q[\xi, t(\xi, t_0)]\, d\xi < \int_0^{x^*(t_0)} qL\, d\xi = qLx^*(t_0) < qL\left(\frac{T - t_0}{c_1}\right)^n$$

and therefore

$$t(x, t_0) > t_0 + \int_0^x \frac{(Lr)^{(n-1)/n}}{n\alpha_2^{1/n}}\left[qL\left(\frac{T - t_0}{c_1}\right)^n\right]^{-(n-1)/n} d\eta = t_0 + \frac{c_2x}{(T - t_0)^{n-1}} \qquad (A4)$$

It follows from (A4) that $t(x, t_0) \to \infty$ as $t_0 \to T$ for fixed $x > 0$.

It follows from (A1) that

$$0 < h(x, t) < (qx/r\alpha_1)^{1/n} \qquad (A5)$$

Equation (A5) implies that $h(0, t) = 0$ for $t > T$.

Appendix B

When $q(x, t) = q$, the curves $t = t(x, t_0)$ do not, for distinct values of t_0, intersect in S. This follows from (14) and (25); the curves of (25) are the prolongations beyond $t = T$ of the curves of (14). Equation (14) implies that $t(x, t_0)$ is, for fixed x, an increasing function of t_0, and (25) implies that $t(x, x_0^*)$ is, for fixed x, a decreasing function of x_0^*.

To prove that the curves $t = t(x, x_0)$ do not intersect in D_3, we impose the condition that $(L - x)/\alpha(x)$ is a decreasing function of x; we retain the condition $q(x, t) = q$. Under these conditions we show that (19) is, for fixed x, a decreasing function of x_0. We write the integral in (19) as the difference of two integrals, one from x_0 to L and the other from x to L; we extend the definition of $\alpha(x)$ to $L(1 - r) < x \leq L$ by $\alpha(x) = \alpha[L(1 - r)]$. The integral from x to L is an increasing function of x_0, so its negative is a decreasing function of x_0. In the integral from x_0 to L we introduce the change of variable $\zeta = (L - \eta)/(L - x_0)$. Then it becomes

$$\int_0^1 \left\{\frac{L - x_0}{\alpha[L - \zeta(L - x_0)]}\right\}^{1/n} \left(\frac{\zeta}{1 - \zeta^2}\right)^{(n-1)/n} d\zeta \tag{B1}$$

Now if

$$x = L - \zeta(L - x_0) = (1 - \zeta)L + \zeta x_0$$

then

$$\frac{L - x_0}{\alpha[L - \zeta(L - x_0)]} = \frac{L - x}{\zeta\alpha(x)}$$

and it is clear that for fixed ζ the brace in (B1) is a decreasing function of x_0. Thus (B1) is a decreasing function of x_0, and therefore $t(x, x_0)$ is a decreasing function of x_0 in D_3.

To conclude the discussion, we need to prove that the curves $t = t(x; x_0^*, x_0)$ do not intersect in D_{12}. We will show that $t(x; x_0^*, x_0)$ is, for fixed x, a decreasing function of x_0. In (35) we introduce, as above, the variable ζ:

$$T = \alpha \int_{(L-x_0^*)/(L-x_0)}^1 \left\{\frac{L - x_0}{\alpha[L - \zeta(L - x_0)]}\right\}^{1/n} \cdot \left(\frac{\zeta}{1 - \zeta^2}\right)^{(n-1)/n} d\zeta \tag{B2}$$

The term in braces in (B2) is a decreasing function of x_0 for fixed ζ, so $(L - x_0^*)/(L - x_0)$ is a decreasing function of x_0. Introducing ζ in (34), we get

$$t(x; x_0^*, x_0) = T + \frac{\gamma}{\left[1 - \left(\dfrac{L - x_0^*}{L - x_0}\right)^2\right]^{(n-1)/n}} \cdot \int_{(L-x)/(L-x_0)}^{(L-x_0^*)/(L-x_0)} \left\{\frac{L - x_0}{\alpha[L - \zeta(L - x_0)]}\right\}^{1/n} \zeta^{(n-1)/n}\, d\zeta$$

Since

$$\left[1 - \left(\frac{L - x_0^*}{L - x_0}\right)^2\right]^{-1} \qquad \frac{L - x_0^*}{L - x_0} \qquad \frac{L - x_0}{\alpha[L - \zeta(L - x_0)]}$$

are decreasing functions of x_0 and $(L - x)/(L - x_0)$ is an increasing function of x_0, $t(x; x_0^*, x_0)$ is a decreasing function of x_0.

APPENDIX C

We investigate the behavior of $h(x, t)$ for fixed x in D_{12}. We assume first that $q(x, t) = q$ and $(L - x)/\alpha(x)$ is a decreasing function of x. It is sufficient, from (33), to consider

$$(L - x_0)^2 - (L - x_0^*)^2 \tag{C1}$$

as a function of t for fixed x. We may write (C1) as

$$G(x_0, z) = (L - x_0)^2(1 - z^2) \qquad z = (L - x_0^*)/(L - x_0) \qquad 0 < z < 1 \tag{C2}$$

where by (B2),

$$T = \gamma \int_z^1 \left\{\frac{L - x_0}{\alpha[L - \zeta(L - x_0)]}\right\}^{1/n} \cdot \left(\frac{\zeta}{1 - \zeta^2}\right)^{(n-1)/n} d\zeta \tag{C3}$$

We have

$$\frac{\partial G}{\partial t} = \frac{dG}{dz}\frac{dz}{dx_0}\frac{\partial x_0}{\partial t} \tag{C4}$$

where on the left of (C4), G is a function of x and t and on the right, a function of z. Since $dz/dx_0 < 0$ and $\partial x_0/\partial t < 0$, it is clear from (C4) that the sign of $\partial G/\partial t$, and therefore the sign of $\partial h/\partial t$, agrees with the sign of dG/dz.

We will now specialize the hypotheses on $\alpha(x)$ still further: $\alpha(x) = \alpha$ when $x^* < x \leq L(1 - r)$. Then (C3) becomes

$$\frac{T\alpha^{1/n}}{\gamma(L - x_0)^{1/n}} = \int_z^1 \left(\frac{\zeta}{1 - \zeta^2}\right)^{(n-1)/n} d\zeta \tag{C5}$$

Combining (C5) and (C2), we get

$$G(x_0, z) = g(z) = \alpha^2\left(\frac{T}{\gamma}\right)^{2n}(1 - z^2)\left\{\left[\int_z^1 \left(\frac{\zeta}{1 - \zeta^2}\right)^{(n-1)/n} d\zeta\right]^{2n}\right\}^{-1} \tag{C6}$$

Calculating $g'(z)$, we see that the sign of $g'(z)$ is determined by

$$n(1 - z^2)^{1/n} - z^{1/n}\int_z^1 \left(\frac{\zeta}{1 - \zeta^2}\right)^{(n-1)/n} d\zeta \tag{C7}$$

Since $\zeta/(1 + \zeta) \leq \frac{1}{2}$ when $0 \leq \zeta \leq 1$,

$$\int_z^1 \left(\frac{\zeta}{1 - \zeta^2}\right)^{(n-1)/n} d\zeta < (\tfrac{1}{2})^{(n-1)/n} n(1 - z)^{1/n}$$

and therefore (C7) is greater than

$$n\{(1 - z^2)^{1/n} - [(\tfrac{1}{2})^{n-1}(z - z^2)]^{1/n}\} \tag{C8}$$

Because of

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$$

the sign of (C8) is the same as that of

$$1 - z^2 - \frac{1}{2^{n-1}}(z - z^2) = (1 - z)\left[1 + \left(1 - \frac{1}{2^{n-1}}\right)z\right] \tag{C9}$$

Since (C9) is positive for $0 < z < 1$, we conclude that $g'(z) > 0$ for $0 < z < 1$ and finally that $\partial h/\partial t > 0$. Thus on the hypothesis $\alpha(x) = \alpha$ when $x^* < x \leq L(1 - r)$, $h(x, t)$ is, for fixed x, an increasing function of t in D_{12}.

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