THE $\overline{\partial}$-NEUMANN OPERATOR AND THE KOBAYASHI METRIC

A Dissertation

by

MIJOUNG KIM

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2003

Major Subject: Mathematics

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Approved as to style and content by:

Harold P. Boas
(Chair of Committee)

Emil J. Straube
(Member)

William B. Johnson
(Member)

Vivek Sarin
(Member)

Al Boggess
(Head of Department)

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Major Subject: Mathematics

ABSTRACT

The $\overline{\partial}$-Neumann Operator and the Kobayashi Metric. (August 2003)

Mijoung Kim, B.S., Duksung Women's University;

M.S., Seoul National University

Chair of Advisory Committee: Dr. Harold P. Boas

We study the $\overline{\partial}$-Neumann operator and the Kobayashi metric. We observe that under certain conditions, a higher-dimensional domain fibered over Ω can inherit noncompactness of the $\overline{\partial}$-Neumann operator from the base domain Ω. Thus we have a domain which has noncompact $\overline{\partial}$-Neumann operator but does not necessarily have the standard conditions which usually are satisfied with noncompact $\overline{\partial}$-Neumann operator. We define the property K which is related to the Kobayashi metric and gives information about holomorphic structure of fat subdomains. We find an equivalence between compactness of the $\overline{\partial}$-Neumann operator and the property K in any convex domain. We also find a local property of the Kobayashi metric [Theorem IV.1], in which the domain is not necessary pseudoconvex.

We find a more general condition than finite type for the local regularity of the $\overline{\partial}$-Neumann operator with the vector-field method. By this generalization, it is possible for an analytic disk to be on the part of boundary where we have local regularity of the $\overline{\partial}$-Neumann operator. By Theorem V.2, we show that an isolated infinite-type point in the boundary of the domain is not an obstruction for the local regularity of the $\overline{\partial}$-Neumann operator.

To my parents

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Most of all though I would like to express my gratitude from the bottom of my heart to my adviser, Professor Harold P. Boas, who through his deep insight, inexhaustible patience, and constant encouragement has been an inspiration to me during these years as his student.

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CHAPTER I

INTRODUCTION

This dissertation is intended to show our new results related to the $\overline{\partial}$-Neumann operator and the Kobayashi metric. The $\overline{\partial}$-Neumann operator and the Kobayashi metric are topics in the study of several complex variables. The prerequisite is familiarity with real analysis and one complex variable. Some knowledge in distribution theory and elliptic partial differential equations will also be helpful.

In Chapter II, as background for the $\overline{\partial}$-Neumann operator and the Kobayashi metric, not only do we give the definitions of the Bergman kernel, the Bergman projection, Carathéodory pseudodistance, Kobayashi pseudodistance, operators $\overline{\partial}$, $\overline{\partial}^*$, and the complex Laplacian $\Box$, but also we mention some recent progress.

Chapter III is devoted to noncompactness of the $\overline{\partial}$-Neumann operator. We geometrically analyze domains having noncompact $\overline{\partial}$-Neumann operator. Let Ω be a smooth, bounded, pseudoconvex Reinhardt domain in $\mathbb{C}^n$ with noncompact $\overline{\partial}$-Neumann operator. We observe that under certain conditions, a higher-dimensional domain fibered over Ω can inherit noncompactness of the $\overline{\partial}$-Neumann operator from the base domain Ω. Thus we have a domain which has noncompact $\overline{\partial}$-Neumann operator but does not necessarily have the standard conditions which usually are satisfied with noncompact $\overline{\partial}$-Neumann operator. Also we define the notion of a fat subdomain A of any domain Ω. (A is called a fat subdomain of Ω if the restriction operator $L^2(\Omega) \cap \mathcal{O}(\Omega) \to L^2(A)$ is not a compact operator.)

In Chapter IV we find a certain relation between the compactness of the $\overline{\partial}$-Neumann operator and the Kobayashi metric. We define the property K which gives

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information about holomorphic structure of fat subdomains. We find an equivalence between compactness of the $\overline{\partial}$-Neumann operator and the property K in any convex domain. We also find a local property of the Kobayashi metric [Theorem IV.1], in which the domain is not necessarily pseudoconvex.

In Chapter V, we find a more general condition than finite type for the local regularity of the $\overline{\partial}$-Neumann operator. In that condition we observe that it is possible for an analytic disk to be on the part of boundary where we have local regularity of the $\overline{\partial}$-Neumann operator. By our Theorem V.2, we show that an isolated infinite-type point in the boundary of the domain is not an obstruction for the local regularity of $\overline{\partial}$-Neumann operator. In this chapter, we introduce both the vector-field method for the regularity of $\overline{\partial}$-Neumann operator and the worm domain.

We are still working on better conditions for every theorem we prove. In particular, hopefully we expect to get better conditions for Theorem V.2 in Chapter V.

CHAPTER II

BACKGROUND FOR THE $\overline{\partial}$-NEUMANN OPERATOR AND THE KOBAYASHI METRIC

A. Bergman Kernel and Bergman Projection

A Bergman kernel is a function of a pair of complex variables with the reproducing kernel property defined for any domain Ω in which there exist nonzero analytic functions of class $L^2(\Omega)$ with respect to Lebesgue measure. Let $H(\Omega)$ be the Hilbert space of all square-integrable holomorphic functions on Ω. A function $B_\Omega : \Omega \times \Omega \to \mathbb{C}$ is called the Bergman kernel of $H(\Omega)$ if

(i) $B_\Omega(\cdot, y) \in H(\Omega),\ y \in \Omega,$

(ii) $f(y) = (f, B_\Omega(\cdot, y))_{H(\Omega)},\ f \in H(\Omega),\ y \in \Omega.$

The property (ii) is called the reproducing property [3]. To obtain explicit kernels a fundamental role is played by orthonormal bases for $H(\Omega)$ on Ω. The Bergman kernel can rarely be calculated explicitly; unless the domain Ω has a great deal of symmetry, in which case a useful orthonormal basis can be determined, there are few techniques for determining a Bergman kernel. The study of the Bergman kernel of the Hilbert space $H(\Omega)$ is deeply related to function theory, but also has various applications to the geometry of bounded domains. However, we restrict our attention to the subject related to our results. As an important property of the Bergman kernel, the transformation rule for the Bergman kernel (later we will use this rule in Chapter III) says that if $F : G \to D$ is a biholomorphic mapping between the domains $G, D \subset \mathbb{C}^n$ then we have

$$B_D(F(z), F(w)) \det F'(z) \overline{\det F'(w)} = B_G(z, w), \qquad z, w \in G. \tag{II.1}$$

The Bergman kernel is well understood on certain classes of domains. For example, when the domain is strongly pseudoconvex, the Bergman kernel function is differentiable up to the boundary [22]. But the case of arbitrary smooth, bounded, pseudoconvex domains remains open. Differentiability of the Bergman kernel is related to the properties of the Bergman projection.

The Bergman projection is the orthogonal projection from $L^2(\Omega)$ onto the space of all square-integrable holomorphic functions on Ω. Also it can be reformulated in terms of the reproducing kernel property. The continuity of the Bergman projection from the space $C^\infty(\overline{\Omega})$ into itself is a very interesting topic. An important application of the Bergman projection is that a biholomorphic mapping between two smooth bounded domains can be extended smoothly up to the boundaries of the domains when the Bergman projections on the two domains are continuous. When a domain is pseudoconvex, it plays an important role with the $\overline{\partial}$-Neumann operator in partial differential equations and differential geometry (when a domain is pseudoconvex, it has a bounded $\overline{\partial}$-Neumann operator in $L^2(\Omega)$) [11].

In this dissertation, we usually consider a pseudoconvex domain since we apply the property of $\overline{\partial}$-Neumann operator we observed to the Bergman projection. In Chapter V, we will say more about the Bergman Projection.

The Bergman kernel B_Ω leads to the following positive semidefinite Hermitian form

$$M_\Omega(z;X) := \sum_{i,j=1}^{n} \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log B_\Omega(z,z) X_i \overline{X_j},\ z \in \Omega,\ X \in \mathbb{C}^n.$$

The pseudometric

$$\mathbf{M}_\Omega := \sqrt{M_\Omega(z;X)},\ z \in \Omega,\ X \in \mathbb{C}^n,$$

induced by M_Ω is called the Bergman pseudometric on Ω. It is invariant under biholomorphic maps and induces the Bergman pseudodistance, which inherits the invariance property. Recently comparing invariant pseudodistances has been an interesting topic. In the next section we turn to holomorphically invariant metrics that arise directly from extremal problems.

B. Kobayashi Metric and Kobayashi Completeness

As one of the most beautiful results in classical complex analysis, the Riemann mapping theorem says that every simply connected plane domain except the whole complex plane is biholomorphically equivalent to the unit disk. On the other hand, the Euclidean ball and the bidisk in $\mathbb{C}^2$ are topologically equivalent simply connected domains, but they are not biholomorphic.

H. Poincaré observed that even inside the class of bounded simply connected domains there is no single model (up to biholomorphisms) as there is in the plane case. Therefore many mathematicians have thought it important to study objects that are invariant under biholomorphic mappings. For this purpose, the invariant pseudodistance is very popular.

Among invariant distances, the Carathéodory pseudodistance and the Kobayashi pseudodistance are most famous. Here we give the definitions of the Carathéodory pseudodistance and the Kobayashi pseudodistance. Let $\mathcal{O}(\Omega', \Omega'')$ be the space of holomorphic functions from Ω' to Ω''. For any domain Ω in $\mathbb{C}^n$, $n \geq 1$, put

$$c_\Omega(z', z'') := \sup\{p(f(z'), f(z'')) \mid f \in \mathcal{O}(\Omega, D)\},\ \ z', z'' \in \Omega,$$

where p is the Poincaré distance in the unit disk D in $\mathbb{C}$. We call c_Ω the Carathéodory pseudodistance for Ω. For $(p, v) \in \Omega \times \mathbb{C}^n$, the Kobayashi metric $K(p, v)$ is defined by

$$K(p, v) = \inf\left\{\frac{1}{|c|} : f(0) = p, f'(0) = cv, f \in \mathcal{O}(D, \Omega)\right\}.$$

Then we define $k_\Omega(z', z'') = \inf\{\int_0^1 K(r(t), r'(t))dt$: r is a piecewise C^1-curve in Ω from z' to $z''\}$. We call k_Ω the Kobayashi pseudodistance on Ω.

Both the Carathéodory pseudodistance and the Kobayashi pseudodistance are motivated by the extremal problem that arises from the Riemann mapping theorem. They endow virtually any domain with a pseudodistance. Moreover, these distances turn out to be invariant under biholomorphic mappings. Many important results have been obtained from studying the interaction of the two distances. Now we mention two famous results. One, observed by elementary tools, is that the Carathéodory and the Kobayashi pseudodistance coincide in the unit disk in $\mathbb{C}^1$. The other one, observed by Lempert, is that the distances (Carathéodory and Kobayashi) agree in convex domains in $\mathbb{C}^n$ [24].

In particular, one of our main results is related to properties of the Kobayashi metric and the Kobayashi pseudodistance. Therefore we focus on the Kobayashi metric and the Kobayashi pseudodistance.

We say that a domain $\Omega \subset \mathbb{C}^n$ is k-hyperbolic if the Kobayashi pseudodistance is a distance. Any bounded domain in $\mathbb{C}^n$ is k-hyperbolic. Therefore, the Kobayashi pseudodistance is a distance in any bounded domain in $\mathbb{C}^n$. (Later, we will consider only bounded domains in $\mathbb{C}^n$.)

Naturally, we consider the completeness of the topology induced by this pseudodistance. This is also a property that depends on the type of domain. For instance, convex domains are Kobayashi-complete. Since $c_\Omega \leq k_\Omega$, every Carathéodory-complete domain is Kobayashi-complete [21].

In Chapter IV, we discuss more about the Kobayashi metric and Kobayashi pseudodistance.

C. The $\overline{\partial}$-Neumann Problem

In this section, first we present the definitions of key words: the Cauchy-Riemann equations, the $\overline{\partial}$ operator, and the complex Laplacian, background for the $\overline{\partial}$-Neumann operator.

Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geq 1$. We call the following equations the Cauchy-Riemann equations:

$$\overline{\partial} u = f \text{ in } \Omega. \tag{II.2}$$

It is necessary that $\overline{\partial} f = 0$ in Ω in order for equation (II.2) to be solvable because $\overline{\partial}^2 = 0$. The precise definitions of the forms and $\overline{\partial}$ are as follows.

$L^2_{(p,q)}$ is the space of (p,q)-forms whose coefficients are in $L^2(\Omega)$. Any (p,q)-form $f \in L^2_{(p,q)}(\Omega)$ can be expressed as $f = \sum'_{I,J} f_{I,J}\, dz^I \wedge d\overline{z}^J$, where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multi-indices and $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\overline{z}^J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$. The notation $\sum'$ means summation over strictly increasing multi-indices.

The operator $\overline{\partial}$ is defined by

$$\overline{\partial} f = \sum_{I,J}{}' f_{I,J} \sum_{k=1}^{n} \frac{\partial f_{I,J}}{\partial \overline{z}_k}\, d\overline{z}_k \wedge dz^I \wedge d\overline{z}^J. \tag{II.3}$$

An element $f \in L^2_{(p,q-1)}$ is in the domain of $\overline{\partial}$ if $\overline{\partial} f$, defined in the distribution sense, belongs to $L^2_{(p,q)}(\Omega)$.

The Hilbert space adjoint of $\overline{\partial}$ is denoted by $\overline{\partial}^*$. An element $f \in L^2_{(p,q-1)}$ belongs to $\mathrm{Dom}(\overline{\partial}^*)$ if there exists a $g \in L^2_{(p,q-1)}(\Omega)$ such that for every $\psi \in \mathrm{Dom}(\overline{\partial}) \cap L^2_{(p,q-1)}(\Omega)$, we have

$$(f, \overline{\partial}\psi) = (g, \psi), \tag{II.4}$$

in which case $\overline{\partial}^* f$ is defined to be equal to g. The operators $\overline{\partial}_{(p,q)}$ and $\overline{\partial}^*_{(p,q)}$ are linear, closed, densely defined operators on $L^2_{(p,q)}$.

Now for fixed $0 \leq p \leq n$, $1 \leq q \leq n$, we define the Laplacian of the $\overline{\partial}$ complex as follows.

Definition II.1. *Let* $\square_{(p,q)} = \overline{\partial}_{(p,q-1)}\overline{\partial}^*_{(p,q)} + \overline{\partial}^*_{(p,q+1)}\overline{\partial}_{(p,q)}$ *be the operator from* $L^2_{(p,q)}(\Omega)$ *to* $L^2_{(p,q)}(\Omega)$ *such that* $\mathrm{Dom}(\square_{(p,q)}) = \{f \in L^2_{(p,q)}(\Omega) \mid f \in \mathrm{Dom}(\overline{\partial}_{(p,q)}) \cap \mathrm{Dom}(\overline{\partial}^*_{(p,q)})$*;* $\overline{\partial}_{(p,q)} f \in \mathrm{Dom}(\overline{\partial}^*_{(p,q+1)})$ *and* $\overline{\partial}^*_{(p,q)} f \in \mathrm{Dom}(\overline{\partial}_{(p,q-1)})\}$.

$\square_{(p,q)}$ is a linear, closed, densely defined self-adjoint operator in $L^2_{(p,q)}(\Omega)$.

1. L^2 Existence Theorem

The following L^2 existence theorem holds for the $\overline{\partial}$-Neumann operator.

Theorem II.1 (Hörmander [11]). *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n$*,* $n \geq 2$*. For each* $0 \leq p \leq n$*,* $1 \leq q \leq n$*, there exists a bounded operator* $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ *such that the following hold.*

1. $R(N_{(p,q)}) \subset \mathrm{Dom}(\square_{(p,q)})$, $N_{(p,q)}\square_{(p,q)} = \square_{(p,q)}N_{(p,q)} = I$ *on* $\mathrm{Dom}(\square_{(p,q)})$.
2. *For any* $f \in L^2_{(p,q)}(\Omega)$, $\quad f = \bar{\partial}\bar{\partial}^* N_{(p,q)} f \oplus \bar{\partial}^*\bar{\partial} N_{(p,q)} f$.
3. $\bar{\partial} N_{(p,q)} = N_{(p,q+1)}\bar{\partial}$ *on* $\mathrm{Dom}(\bar{\partial})$, $1 \leq q \leq n-1$.
4. $\bar{\partial}^* N_{(p,q)} = N_{(p,q-1)}\bar{\partial}^*$ *on* $\mathrm{Dom}(\bar{\partial}^*)$, $2 \leq q \leq n$.
5. *Let* δ *be the diameter of* Ω*. The following estimates hold for any* $f \in L^2_{(p,q)}(\Omega)$*:*

$$\|N_{(p,q)} f\| \leq \frac{e\delta^2}{q},$$
$$\|\overline{\partial} N_{(p,q)} f\| \leq \sqrt{\frac{e\delta^2}{q}}\, \|f\|,$$
$$\|\overline{\partial}^* N_{(p,q)} f\| \leq \sqrt{\frac{e\delta^2}{q}}\, \|f\|,$$

where $\|\cdot\|$ *is the norm in* L^2.

Corollary II.1. *Let Ω and $N_{(p,q)}$ be the same as Theorem II.1, where $0 \leq p \leq n$, $1 \leq q \leq n$. For any $\alpha \in L^2_{(p,q)}(\Omega)$ such that $\overline{\partial}\alpha = 0$, the $(p, q-1)$-form*

$$u = \overline{\partial}^* N_{(p,q)}\alpha \tag{II.5}$$

satisfies the equation $\overline{\partial}u = \alpha$ and the estimate

$$\|u\|^2 \leq \frac{e\delta^2}{q}\|\alpha\|^2. \tag{II.6}$$

The solution u is called the canonical solution to the equation (II.2), and it is the unique solution which is orthogonal to $\ker(\overline{\partial})$.

The existence of the $\overline{\partial}$-Neumann operator for $q = 0$, $N_{(p,0)}$, is also important. Let $\Box_{(p,0)} = \overline{\partial}^*\overline{\partial}$ on $L^2_{(p,0)}(\Omega)$. We define

$$\mathcal{H}_{(p,0)}(\Omega) = \{f \in L^2_{(p,0)}(\Omega) \mid \overline{\partial}f = 0\}. \tag{II.7}$$

This space $\mathcal{H}_{(p,0)}(\Omega)$ is a closed subspace of $L^2_{(p,0)}$ since $\overline{\partial}$ is a closed operator. Let $H_{(p,0)}(\Omega)$ denote the projection from $L^2_{(p,q)}(\Omega)$ onto the set $\mathcal{H}_{(p,0)}(\Omega)$. We have the following theorem [11].

Theorem II.2. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. There exists an operator $N_{(p,0)} : L^2_{(p,0)} \to L^2_{(p,0)}$ such that*

1. $R(N_{(p,0)}) \subset \mathrm{Dom}(\Box_{(p,0)})$, $N_{(p,0)}\Box_{(p,0)} = \Box_{(p,0)}N_{(p,0)} = I - H_{(p,0)}$,
2. *for every* $f \in L^2_{(p,0)}(\Omega)$, $f = \overline{\partial}^*\overline{\partial}N_{(p,0)}f \oplus H_{(p,0)}(\Omega)f$, *and*
3. $\overline{\partial}N_{(p,0)} = N_{(p,1)}\overline{\partial}$ *on* $\mathrm{Dom}(\overline{\partial})$, $\overline{\partial}^* N_{(p,1)} = N_{(p,1)}\overline{\partial}^*$ *on* $\mathrm{Dom}(\overline{\partial}^*)$.

The Bergman projection P is the orthogonal projection from $L^2_{(0,0)}(\Omega)$ onto $\mathcal{H}_{(0,0)}(\Omega)$. By Theorem II.2, P can be reformulated by

$$P = I - \overline{\partial}^* N_{(0,1)}\overline{\partial}. \tag{II.8}$$

2. Globally Regular Operators

We say that an operator is globally regular when it maps the space $C^\infty(\overline{\Omega})$ into itself (this is the space of functions that extend smoothly across the boundary). If for every nonnegative integer s the operator acts continuously from the Sobolev space $H^s(\Omega)$ to itself (this is the space of functions or forms with square-integrable derivatives through order s), we say that the operator is exactly regular.

Boas and Straube proved equivalence of regularity for the Bergman projection and the $\overline{\partial}$-Neumann operator when the domain is a bounded smooth pseudoconvex domain [6].

The regularity of the $\overline{\partial}$-Neumann operator has been studied extensively when the domain has smooth boundary. Many classes of domains have been found to have regularity of the $\overline{\partial}$-Neumann operator: for example strongly pseudoconvex domains, finite-type domains, and domains admitting plurisubharmonic defining functions [11].

On the other hand, D. Barrett [1] showed the failure of exact regularity of the $\overline{\partial}$-Neumann operator on the worm domains constructed by K. Diederich and J. Fornæss [14]. Later M. Christ showed the failure of global regularity in the worm domains [12].

The worm domains are pseudoconvex domains with C^∞-boundaries which do not have plurisubharmonic defining functions on bD [5]. Before D. Barrett's observation about the worm domain, it had been thought that smoothness of a pseudoconvex domain would be sufficient for the regularity of the $\overline{\partial}$-Neumann operator. Up to now, necessary and sufficient conditions for global regularity of the $\overline{\partial}$-Neumann operator are not known.

Recently, the regularity of the $\overline{\partial}$-Neumann operator on domains which do not have a smooth boundary has been studied. For example, Michel and Shaw proved that the $\overline{\partial}$-Neumann operator is bounded on Sobolev (1/2)-spaces on a bounded pseudoconvex domain Ω in $\mathbb{C}^n$ with a plurisubharmonic Lipschitz defining function [27]. But they have this result only for Sobolev (1/2)-spaces. Therefore there still remains a big gap for the regularity of the $\overline{\partial}$-Neumann operator. In the strictly pseudoconvex case, Engliš observed that the singular support of Nf is contained in the singular support of the strictly plurisubharmonic Lipschitz defining function [16]. This is a direct result from Catlin's pseudolocal estimate for the $\overline{\partial}$-Neumann operator. In Chapter V, we will discuss a more general condition for pseudolocal estimates for the $\overline{\partial}$-Neumann operator.

3. Compactness in the $\overline{\partial}$-Neumann Problem

Kohn and Nirenberg [23] found that global regularity for the canonical solution does hold when a certain estimate, known as a compactness estimate, holds for the domain Ω. A compactness estimate is said to hold for the $\overline{\partial}$-Neumann problem on Ω if for every $\epsilon > 0$, there is a function $\zeta_\epsilon \in C_0^\infty(\Omega)$ such that

$$\|f\|^2 \leq \epsilon Q(f,f) + \|\zeta_\epsilon f\|_{-1}^2,\ f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*).$$

Here $Q(f,f)$ refers to the form $(\overline{\partial} f, \overline{\partial} f) + (\overline{\partial}^* f, \overline{\partial}^* f)$, and $\|\cdot\|_{-1}^2$ refers to the Sobolev norm of order -1 for forms on $\mathbb{C}^n$. A sufficient condition for the compactness of the $\overline{\partial}$-Neumann operator called property (P) was found by Catlin. The precise definition follows below.

The boundary of a domain Ω satisfies property (P) if for every positive number M there is a plurisubharmonic function $\lambda \in C^\infty(\overline{\Omega})$ with $0 \leq \lambda \leq 1$, such that for all $z \in b\Omega$,

$$\sum_{i,j=1}^{n} \frac{\partial^2 \lambda}{\partial z_i \partial \overline{z}_j}(z) t_i \overline{t}_j \geq M|t|^2,\ t \in \mathbb{C}^n.$$

By the virtue of a consequence of the analysis of finite-type points in [10], it is not hard to see that if a domain is strictly pseudoconvex except at finitely many points, then the boundary of the domain satisfies property (P). In fact, if the infinite-type points of the boundary have two-dimensional Hausdorff measure zero, then the domain satisfies property (P) [4].

Here we restrict our attention to some results related to our main results [Chapters III, IV]. We consider on which domains a disk in the boundary is a sufficient condition for failure of compactness. The following proposition was observed by David Catlin.

Theorem II.3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with Lipschitz boundary. If the boundary of Ω contains an analytic disk, then the $\overline{\partial}$-Neumann operator N_1 on Ω is not compact [19].*

It is not known whether Theorem II.3 holds in higher dimensions. However, Fu and Straube proved that a necessary and sufficient condition for the compactness of the $\overline{\partial}$-Neumann operator N_1 in convex domains is the absence of an analytic disk on the boundary [18].

Let Ω be a domain (an open connected set) in $\mathbb{C}^n$. We say that Ω is a Reinhardt domain if whenever $z = (z_1, \ldots, z_n) \in \Omega$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$, $(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) \in \Omega$. Pseudoconvex Reinhardt domains are locally convexifiable at most of their boundary points. Therefore, it is not surprising that the observation about the case of convex domains also give results on the class of Reinhardt domains [19].

Theorem II.4 (Fu, Straube). *Let Ω be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. If the boundary of Ω does not contain an analytic disk of dimension 1, then $\overline{\partial}$-Neumann operator N_1 on $(0,1)$-forms is compact.*

This theorem implies that when a pseudoconvex Reinhardt domain has a noncompact $\overline{\partial}$-Neumann operator N_1, there is an analytic disk on the boundary. It gives an idea about the proof of Theorem III.1 in the next chapter.

CHAPTER III

INHERITANCE OF NONCOMPACTNESS OF THE $\overline{\partial}$-NEUMANN PROBLEM

A. Introduction

For many years, it has been an open question whether every smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$ with an analytic disk in the boundary necessarily has a noncompact $\overline{\partial}$-Neumann operator N. Some partial results are known. For instance, the answer is affirmative both for domains in $\mathbb{C}^2$ [19] and for convex domains in $\mathbb{C}^n$ [18]. It remains open, for example, whether an analytic disk in the boundary of a complete Reinhardt domain in $\mathbb{C}^3$ necessarily obstructs compactness of N. On the other hand, it is known that in the case of Reinhardt domains, disks in the boundary are the only possible obstructions to compactness of N [19].

In this chapter I show that noncompactness of the $\overline{\partial}$-Neumann operator on a smooth, bounded, pseudoconvex Reinhardt domain Ω in $\mathbb{C}^2$ implies an analogous noncompactness for higher-dimensional domains fibered over Ω under a suitable size restriction on the fibers. The main result is as follows.

Theorem III.1. *Suppose that Ω is a smooth, bounded, pseudoconvex Reinhardt domain in $\mathbb{C}^2$ whose $\overline{\partial}$-Neumann operator N is noncompact (on the space of square-integrable $(0,1)$-forms). A sufficient condition for noncompactness of the $\overline{\partial}$-Neumann operator N of a smooth, bounded, pseudoconvex domain G in $\mathbb{C}^n$ $(n \geq 3)$ fibered over Ω is that there exist a constant C such that all points (z, w) of G (where $z \in \Omega$, $w \in \mathbb{C}^{n-2}$) satisfy the restriction $\|w\| < Cd(z, b\Omega)^{1/2}$.*

In the next section, I will introduce the notion of "fat subdomain" and some lemmas that enter into the proof of the theorem.

B. Fat Subdomains

Let G be a domain in $\mathbb{C}^n$, and let A be a subdomain of G. If there is a sequence $\{f_j\}$ of holomorphic functions in the unit ball of $L^2(G)$ such that no subsequence of $\{f_j\}$ converges in $L^2(A)$, then A is said to be a *fat subdomain* of G. In other words, A is a fat subdomain of G if the restriction operator $L^2(G) \cap \mathcal{O}(G) \to L^2(A)$ is not a compact operator.

For example, if there is a point p in the boundary of G and a neighborhood U of p in $\mathbb{C}^n$ such that $A \cap U = G \cap U$, then A is a fat subdomain of G. This condition is not necessary, however. For instance, let E be the unit disk $\{z \mid |z| < 1\}$ in $\mathbb{C}$ and let A be $\{(x, y) \in E \mid 0 < x < 1 \text{ and } 0 < y < (1 - x)^p\}$, where $p > 0$. Then A is a fat subdomain of E if (and only if) $p \leq 1$. One can easily check this by taking the sequence of holomorphic functions $\{f_j\}$ to be the sequence of normalized Bergman kernel functions $\{K_E(z, p_j)/\sqrt{K_E(p_j, p_j)}\,\}$, where the sequence $\{p_j\}$ approaches the point $(1, 0)$.

I recall the definition of the Bergman kernel function. Let $H(D)$ denote the space of square-integrable holomorphic functions on a domain D in $\mathbb{C}^n$. By the Riesz representation theorem, for each fixed point w in D there is a unique element of $H(D)$, denoted by $K_D(\cdot, w)$, such that

$$f(w) = (f, K_D(\cdot, w)) = \int_D f(z)\overline{K_D(z, w)}\, dV_z$$

for all $f \in H(D)$. This function $K_D(z, w)$ is called the Bergman kernel function for D.

The following lemma is contained in [18].

Lemma III.1. *If Ω is a bounded convex domain in $\mathbb{C}^n$, and $0 < R \leq 1$, then*

1. *for any points $p_0 \in b\Omega$ and $p_1 \in \Omega$ there exist positive constants C and δ_0 such*

that the Bergman kernel function K_Ω satisfies the inequality

$$K_\Omega(p_\delta, p_\delta) > C K_\Omega(p_{\delta/R}, p_{\delta/R})$$

for any $\delta \in (0, \delta_0)$, where $p_\delta := p_0 + \delta(p_1 - p_0)/\|p_1 - p_0\|$;

2. *if $0 \in b\Omega$ then the scaled domain $R\Omega$ is a fat subdomain of Ω.*

Part (1) of Lemma III.1 is identical with [18, Lemma 4.1, part (1)], and I omit the proof. The proof of the second part of Lemma III.1 is contained in [18, proof of the implication (1) $\Rightarrow$ (2) in Theorem 1.1]. I recall the proof for the convenience of the reader.

Let p_1 be an arbitrary point in $R\Omega$, and set $p_j = p_1/j$ for $j \in \mathbb{N}$. Let $f_j(\cdot) = K_\Omega(\cdot, p_j)/\sqrt{K_\Omega(p_j, p_j)}$. Then $\|f_j\|_\Omega = 1$. The reproducing property of $K_{R\Omega}(p_j, \cdot)$ applied to the function $K_\Omega(\cdot, p_j)$ implies, via the Cauchy-Schwarz inequality, that

$$K_\Omega(p_j, p_j) \leq \|K_\Omega(\cdot, p_j)\|_{R\Omega} \sqrt{K_{R\Omega}(p_j, p_j)}\,.$$

Consequently,

$$\|f_j\|^2_{R\Omega} = \frac{\|K_\Omega(\cdot, p_j)\|^2_{R\Omega}}{K_\Omega(p_j, p_j)} \geq \frac{K_\Omega(p_j, p_j)}{K_{R\Omega}(p_j, p_j)}.$$

By the transformation rule for the Bergman kernel function,

$$K_{R\Omega}(p_j, p_j) = R^{-2n} K_\Omega(p_j/R, p_j/R).$$

Therefore by part (1) of the lemma, the right-hand side of the preceding inequality is bounded below by a positive constant independent of j. Thus the sequence $\{f_j\}$ is bounded away from 0 in the norm of $L^2(R\Omega)$. On the other hand, the sequence $\{f_j\}$ tends to 0 pointwise. Consequently, the sequence $\{f_j\}$ has no subsequence converging in $L^2(R\Omega)$.

I now recall briefly the $\overline{\partial}$-Neumann operator from Chapter II. When a domain Ω

is bounded and pseudoconvex, the (unbounded) self-adjoint, surjective operator $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ has a bounded inverse operator acting on $(0,q)$-forms. This operator $N = N_q$ is called the $\overline{\partial}$-Neumann operator. I refer the reader to [17], [20] and the recent survey [9] and book [11] for background on the $\overline{\partial}$-Neumann problem. In this paper, I consider only N_1. The compactness condition can be reformulated in the following way [19].

Lemma III.2. *Let Ω be a bounded pseudoconvex domain, $1 \leq q \leq n$. Then the following are equivalent.*

(1) The $\overline{\partial}$-Neumann operator N_q is compact from $L^2_{(0,q)}(\Omega)$ to itself.

(2) The canonical solution operators $\bar{\partial}^* N_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q-1)}(\Omega)$ and $\bar{\partial}^* N_{q+1} : L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega)$ are compact.

C. Proof of Theorem III.1

I will prove a little more than is stated in the theorem: the smoothness of the boundary is needed only in a neighborhood of the base domain Ω.

Let $z = (z_1, z_2)$ denote the coordinate in the space $\mathbb{C}^2$, and let w denote the coordinate in the space $\mathbb{C}^{n-2}$. I divide the proof of the theorem into the following three steps.

Step 1: A local model There is a point p in the boundary of Ω and a neighborhood U of p in $\mathbb{C}^2$ such that $G \cap (U \times \mathbb{C}^{n-2})$ is biholomorphic to a domain G' that has in its boundary the affine disk $\{(z_1, 0, 0) \mid |z_1| < 1\}$. Moreover $\{z_2 \mid (z_1, z_2, w) \in G'\}$ is contained in the disk $\{z_2 \mid |z_2 - 1| < 1\}$.

Step 2: Geometry of the model The local model domain is nested between two product domains of the form

$$\{z_1 \in \mathbb{C} \mid |z_1| < r\} \times \{(z_2, w) \in \mathbb{C}^{n-1} \mid |z_2 - r|^2 + \|w\|^2 < r^2\}$$

for two suitable values of r.

Step 3: Analysis From the geometry of the local model, a standard argument leads to noncompactness of the $\overline{\partial}$-Neumann operator of the original domain G.

Proof of Step 1. By hypothesis, Ω has a noncompact $\overline{\partial}$-Neumann operator N. But since Ω is also a bounded pseudoconvex Reinhardt domain, Ω has an analytic disk in its boundary by [19, Theorem 5.2]. Since the boundary of Ω is smooth, this analytic disk cannot be entirely contained in the part of the boundary of Ω where either $z_1 = 0$ or $z_2 = 0$. Let p be a point of the analytic disk where both coordinates are nonzero. Near such a point p, a pseudoconvex Reinhardt domain is locally convexifiable (see, for example, [31]).

Let ϕ be a biholomorphic map defined on a neighborhood U of p in $\mathbb{C}^2$ such that $\Omega' := \phi(U \cap \Omega)$ is convex. Then Ω' also has an analytic disk in its boundary. Because Ω' is convex, we may assume by [18, §2] that the analytic disk in its boundary is an affine analytic disk. After an affine coordinate change, we may assume that $\phi(p) = (0, 0)$, the affine disk lies in the z_1 coordinate plane, and a supporting hyperplane for Ω' at $(0, 0)$ is given by $\{x_2 = 0\}$. After a linear fractional transformation in the z_2 coordinate, we may assume further that Ω' is contained in the set $\{(z_1, z_2) \mid |z_2 - 1| < 1\}$.

We extend ϕ to $U \times \mathbb{C}^{n-2}$ by making ϕ the identity in the remaining variables. Set $G' = \phi(G \cap (U \times \mathbb{C}^{n-2}))$. □

Proof of Step 2. Consider slicing G' with a complex hyperplane on which the value of z_1 is constant. Since the $(n-1)$-dimensional slice is contained in $\{(z_2, w) \mid$

$|z_2 - 1| < 1\}$, and the origin is a smooth boundary point, the slice contains the ball $\{(z_2, w) \mid |z_2 - r_1|^2 + \|w\|^2 < r_1^2\}$ for a sufficiently small radius r_1.

The biholomorphic map ϕ distorts distance by a bounded amount, so the hypothesis of the theorem about the size restriction on the fibers of G carries over to G'. This restriction implies that the slice is contained in a ball $\{(z_2, w) \mid |z_2 - r_2|^2 + \|w\|^2 < r_2^2\}$ for a sufficiently large radius r_2.

The radii r_1 and r_2 depend continuously on the value of z_1, so we may choose values of the radii that are independent of z_1 for z_1 in a neighborhood of the origin. $\square$

Proof of Step 3. After possibly shrinking U and r_1, we may assume that G' contains the set $\{z_1 \in \mathbb{C} \mid |z_1| < 3r_1\} \times \{(z_2, w) \in \mathbb{C}^{n-1} \mid |z_2 - 3r_1|^2 + \|w\|^2 < (3r_1)^2\}$. By Lemma III.1.(2), the ball $\{(z_2, w) \mid |z_2 - r_1|^2 + \|w\|^2 < r_1^2\}$ is a fat subdomain of the ball $\{(z_2, w) \mid |z_2 - r_2|^2 + \|w\|^2 < r_2^2\}$ in $\mathbb{C}^{n-1}$. It follows easily that the product domain

$$A := \left\{z_1 \in \mathbb{C} \mid |z_1| < \tfrac{r_1}{2}\right\} \times \left\{(z_2, w) \in \mathbb{C}^{n-1} \mid |z_2 - \tfrac{r_1}{2}|^2 + \|w\|^2 < (\tfrac{r_1}{2})^2\right\}$$

is a fat subdomain of the product domain

$$\{z_1 \in \mathbb{C} \mid |z_1| < r_2\} \times \{(z_2, w) \in \mathbb{C}^{n-1} \mid |z_2 - r_2|^2 + \|w\|^2 < r_2^2\}.$$

Consequently, A is a fat subdomain of G'.

Let $\{f_j\}$ be a sequence of holomorphic functions in the unit ball of $L^2(G')$ having no subsequence that converges in $L^2(A)$. Let χ be a smooth cutoff function of one real variable that is identically equal to 0 for $t \geq 2r_1$ and identically equal to 1 for $t \leq r_1$. Let α_j denote the pullback to G under ϕ of the $(0,1)$-form $\overline{\partial}(f_j(z_1, z_2, w)\chi(|z_1|)\chi(\|(z_2, w)\|))$.

The form α_j is $\overline{\partial}$-closed, and the function $g_j := \overline{\partial}^* N\alpha_j$ represents the canonical solution on G to the equation $\overline{\partial} u = \alpha_j$. Seeking a contradiction, let us suppose that the $\overline{\partial}$-Neumann operator N for G (and hence the operator $\overline{\partial}^* N$) is compact. Then after passing to a subsequence, we may assume that the sequence $\{g_j\}$ converges in $L^2(G)$.

On G', define a sequence of functions $\{h_j\}$ via

$$h_j(z_1, z_2, w) = g_j \circ \phi^{-1}(z_1, z_2, w) - f_j(z_1, z_2, w)\chi(|z_1|)\chi(\|(z_2, w)\|).$$

The functions h_j are holomorphic on G', and $h_j = g_j \circ \phi^{-1}$ when $|z_1| > 2r_1$. Since the $g_j \circ \phi^{-1}$ are converging in $L^2(G')$, the mean-value property of holomorphic functions implies that the h_j are converging in $L^2(A)$. But $f_j(z_1, z_2, w)\chi(|z_1|)\chi(\|(z_2, w)\|) = f_j$ on A, so it follows that the f_j are converging in $L^2(A)$. This contradiction shows that the $\overline{\partial}$-Neumann operator N for G cannot be compact. $\square$

Corollary III.1. *Let $\phi : [0, 1] \to [0, 1]$ be a concave C^∞ function such that $\phi \equiv 1$ on $[0, \frac{1}{4}]$ and $\phi(r) = 1 - r^2$ on $[\frac{3}{4}, 1]$. Let $D = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_2|^2 \leq \phi(|z_1|)\}$. If G is a smooth pseudoconvex domain with base domain D, and if there exists a constant C such that $\|w\| < Cd(z, \partial D)^{1/2}$ for all points $(z, w) \in G$, then G has a noncompact $\overline{\partial}$-Neumann operator N.*

That the base domain D in the corollary has a noncompact $\overline{\partial}$-Neumann operator was first observed by Ligocka [25]. Corollary III.1 provides some simple concrete examples of higher-dimensional domains that have noncompact $\overline{\partial}$-Neumann operator although they are neither product domains nor convex domains with analytic disks in the boundary.

CHAPTER IV

THE $\overline{\partial}$-NEUMANN OPERATOR AND THE KOBAYASHI METRIC

A. Introduction

In this chapter we study a condition on the Kobayashi metric near a boundary point p of a pseudoconvex domain in $\mathbb{C}^n$ that is related to compactness of the $\overline{\partial}$-Neumann operator. We call this condition property K.

If Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ that is Kobayashi complete near p, and if Ω has a compact $\overline{\partial}$-Neumann operator, then property K is necessarily satisfied. The precise definition of property K is given in the next section.

Product domains are examples of domains that have noncompact $\overline{\partial}$-Neumann operator. If a domain Ω does not have property K near a boundary point p, then Ω can be well approximated near p by a product domain, in a sense made precise below in Lemma IV.1.

In particular, we have the following result about the Kobayashi metric and product domains. Let D denote the open unit disk in $\mathbb{C}^1$. When Ω is a domain in $\mathbb{C}^n$ and v is a vector, $d_v(z)$ denotes the radius of the largest affine disk in Ω with center z and direction v, that is, $d_v(z) = \sup\{r \mid z + rDv \subset \Omega\}$.

Theorem IV.1. *If Ω is a bounded domain in $\mathbb{C}^n$ that is Kobayashi complete near a boundary point p and there is some $\epsilon < 1$ and some $v \in \mathbb{C}^n$ such that for all z close enough to p, the Kobayashi metric K satisfies*

$$K(z, v) \lesssim \frac{1}{d_v(z)^\epsilon}$$

(when $\epsilon = 1$ the above equation is always true), then Ω is locally a product space near p.

We also give an example in which property K is not satisfied (and hence the $\overline{\partial}$-Neumann operator is not compact).

In the next section, I define terminology. Section C contains the main results. There are some applications in Section D.

B. Terminology and Definition of Property K

Definition IV.1 (Fat subdomain with mass at p). *Let Ω be a domain in $\mathbb{C}^n$, and let A be a fat subdomain of Ω. If $A \cap U$ is still a fat subdomain of Ω for every open neighborhood U of p in $\mathbb{C}^n$, then we say that A has mass at p.*

A domain Ω is called Kobayashi complete if any Cauchy sequence $\{z_j\}_{j\in\mathbb{N}}$ with respect to the Kobayashi pseudodistance converges to a point $z_0 \in \Omega$, i.e., $\{k_\Omega(z_j, z_0)\}$ converges to 0.

Some examples of Kobayashi complete domains are strongly pseudoconvex domains, convex domains, and bounded pseudoconvex Reinhardt domains containing 0. It is an open problem whether every bounded balanced domain with C^∞ Minkowski function is Kobayashi complete. Up to now, it is also an open problem whether every bounded pseudoconvex domain with C^∞-smooth boundary is Kobayashi complete [21].

Definition IV.2 (Property K). *We say that Ω has property K near $p \in b\Omega$ if the following property is satisfied:*

For every fat subdomain A having mass at p of $b\Omega$, $\forall v \in \mathbb{C}^n$, $\forall \epsilon < 1$, there is a sequence $\{q_n\}$ approaching p in $\Omega \cap (A + vD)$ such that

$$K(q_n, v) \gtrsim \frac{1}{d_v(q_n)^\epsilon}.$$

The following theorem gives examples of domains that have property K and examples of domains that do not have this property. We will give the proof of this theorem in Section D.

Theorem IV.2. *If Ω is a bounded convex domain in $\mathbb{C}^n$, then the following are equivalent.*

(1) Ω has property K at every p in $b\Omega$.

(2) The $\overline{\partial}$-Neumann operator N_1 is compact.

(3) There is no affine complex disk in the boundary of Ω.

Although the property K and compactness of N_1 are equivalent in convex domains, this equivalence does not hold in general. Here is an example of a pseudoconvex Reinhardt domain in $\mathbb{C}^2$ with compact $\overline{\partial}$-Neumann operator N_1. Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1, \, 0 < |z_1| < 1\}$. This domain is not Kobayashi complete and also does not have the property K near $(0, 0)$ in $b\Omega$ [19, p. 150].

C. Main Proof

Theorem IV.3. *If Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ that is Kobayashi complete near p and has smooth boundary near p in $b\Omega$, and Ω does not satisfy property K, then Ω has a noncompact $\overline{\partial}$-Neumann operator.*

Before we prove Theorem IV.3, we need Lemmas IV.1 and IV.2.

Lemma IV.1. *Suppose that Ω is a bounded domain in $\mathbb{C}^n$ that is Kobayashi complete near p, and Ω does not have property K near p in $b\Omega$. Then there is a fat subdomain A having mass at p such that after linearly changing the coordinate system there is a coordinate system $(z_1, \ldots, z_n)$ on some open neighborhood U_0 of p in $\mathbb{C}^n$ such that*

$\exists C_0$ for which $(1/4)C_0 D \times \pi(\{A \cap U_0 \mid |z_1| < (1/2)C_0\}) \subset \Omega$, where $\pi : \Omega \to \mathbb{C}^{n-1}$ is the natural projection $\pi(z) = (z_2, \ldots, z_n)$.

Proof of Lemma IV.1. By the hypotheses, there are a fat subdomain A having mass at p, a nonzero vector $v \in \mathbb{C}^n$, a number $\epsilon < 1$, and a constant C_ϵ such that

$$K(z, v) \leq \frac{C_\epsilon}{d_v(z)^\epsilon} \tag{IV.1}$$

when $z \in (A + Dv) \cap (\Omega \cap U)$, where U is a small enough open neighborhood of p in $\mathbb{C}^n$. (When $\epsilon = 1$, the above equation is always true.) Here I need to prove the following two steps.

Step 1 There is an open $U_0 \subset\subset U$ containing p such that for $\forall z \in U_0 \cap A$, we have $C_0 Dv + z \subset \Omega$.

Step 2 After linearly changing the coordinate system, v is a unit vector in the z_1-direction, and $(1/4)C_0 D \times \pi(\{A \cap U_0 \mid |z_1| < (1/2)C_0\}) \subset \Omega$.

Proof of Step 1. Fix an open set $U_0 \subset\subset U$ in $\mathbb{C}^n$. We may assume that $\Omega \cap U$ is Kobayashi complete. Choose C_0 such that $C_0 = \frac{1}{\|v\|} \min(d(U_0, \Omega \cap U^c), 1)$. Suppose that $\exists q \in A \cap U_0$ such that $C_0 Dv + q \not\subset \Omega$. Let $|z_0|$ be the minimum value in $\{|z| \mid zv + q \in b\Omega\}$; obviously $|z_0| < C_0$. We define a curve r from q to $z_0 v + q$ in $\mathbb{C}^n$ via $r(s) = sz_0 v + q$, $r(0) = q$, $r(1) = z_0 v + q$, $r'(s) = z_0 v$. Since $\|sz_0 v\| < d(U_0, \Omega \cap U^c)$, it follows that $r(s) \in (A + C_0 Dv) \cap (\Omega \cap U)$ if $s < 1$. We choose s_j approaching 1. By the property of the Kobayashi pseudodistance, the inequality in (IV.1), and the inequality $d_v(r(s)) \geq (1 - s)(|z_0|)$, we have

$$\begin{aligned} k_\Omega(q, r(s_j)) &\leq \int_0^{s_j} K(r(s), r'(s))ds \leq |z_0| \int_0^{s_j} K(r(s), v)ds \\ &\leq |z_0| \int_0^{s_j} \frac{C_\epsilon}{d_v(r(s))^\epsilon} ds \leq \frac{|z_0|}{|z_0|^\epsilon} \int_0^{s_j} \frac{C_\epsilon}{(1-s)^\epsilon} ds \leq M. \end{aligned}$$

But $r(s_j)$ approaches $z_0v + q \in b\Omega$ as j goes to infinity, which contradicts the Kobayashi completeness of Ω near p. Therefore $C_0Dv + z \subset \Omega$, for all $z \in U_0 \cap A$. $\square$

Proof of Step 2. We choose coordinates $(z_1, \ldots, z_n)$ such that v is a unit vector in the z_1-direction. For a fixed $z = (z_1, \ldots, z_n) \in U_0 \cap A$, by Step 1, if $|w| < C_0$, then $wv + z \in \Omega$, that is, $wv + z = (w + z_1, z_2, \ldots, z_n) \in \Omega$. If we choose any $(z_1, z') \in (1/4)C_0D \times \pi(\{A \cap U_0 \mid |z_1| < (1/2)C_0\})$, then there is a point $(z_1^0, z') \in A \cap U_0$, $|z_1^0| < (1/2)C_0$. This implies that $|z_1 - z_1^0| \leq C_0$. By the argument just above, $(z_1, z') = (z_1 - z_1^0)v + (z_1^0, z') \in \Omega$. So we get $(1/4)C_0D \times \pi(\{A \cap U_0 \mid |z_1| < (1/2)C_0\}) \subset \Omega$. $\square$

This completes the proof of Lemma IV.1. $\square$

Lemma IV.2. *If Ω has a fat subdomain A having mass at p in $b\Omega$ which is a product space and has smooth boundary near p, then Ω has noncompact $\overline{\partial}$-Neumann operator.*

Proof of Lemma IV.2. We give a sketch of the proof of Lemma IV.2, which is a standard argument [19]. We may assume that there is a coordinate system $(z_1, \ldots, z_n)$ on a neighborhood U of p such that $p = 0$ and $A = C_0D \times W$, where W is an open set in $\mathbb{C}^{n-1}$. Since $(C_0/2)D \times W$ is still a fat subdomain having mass at p, we may assume that there is a holomorphic sequence $\{f_j\}_{j=1}^{\infty}$ which lies in the unit ball of $L^2(\Omega)$ and which has no subsequence that converges in $L^2((C_0/2)D \times W)$. Denote by $\chi(t)$ a smooth cut-off function that is identically equally to 1 for $0 \leq t \leq C_0/2$ and identically equal to 0 for $t \geq 2C_0/3$. Let $z' = (z_2, \ldots, z_n)$. Let α_j be $\overline{\partial}(\chi(|z_1|)f_j((z_1, z')))$, which is $\overline{\partial}$ closed on Ω. Let $g_j = \overline{\partial}^* N\alpha_j$. Suppose that N is a compact operator. By Lemma III.2, we may assume that $\overline{\partial}^* N$ is also a compact operator. After passing to a subsequence, we may assume that $\{g_j\}_{j=1}^{\infty}$ converges in $L^2(\Omega)$. Let $h_j((z_1, z')) = \chi(|z_1|)f_j((z_1, z')) - g_j((z_1, z'))$. Then h_j is holomorphic. If $|z_1| \geq 2C_0/3$

on A, then $h_j = g_j$. Using the mean-value property of holomorphic functions, we see that h_j converges in $L^2(A)$, hence χf_j converges on $L^2((C_0/2)D \times W)$. This is a contradiction. $\square$

Proof of Theorem IV.3. Lemma IV.1 implies that Ω has a fat subdomain with mass at p that is a product domain. By Lemma IV.2, Ω has a noncompact $\overline{\partial}$-Neumann operator. $\square$

D. Applications

Theorem IV.1 and Theorem IV.4 are the applications of Lemma IV.1. Now we give the proof of Theorem IV.1.

Proof of Theorem IV.1. By the hypotheses, there are a sufficiently small open neighborhood U of p in $\mathbb{C}^n$ and a constant C_ϵ such that for $z \in U \cap \Omega$,

$$K(z, v) \leq \frac{C_\epsilon}{d_v(z)^\epsilon}.$$

We may assume that $U \cap \Omega$ is Kobayashi complete. The hypothesis of Theorem IV.1 gives inequality (IV.1) in the proof of Lemma IV.1 with the set A replaced by the set $U \cap \Omega$. We may assume that v is a unit vector in z_1-direction.

Now we can follow directly the argument of Lemma IV.1. There are C_0 and an open neighborhood U_0 of p in $\mathbb{C}^n$ such that $(1/4)C_0 D \times \pi(\{\Omega \cap U_0 \mid |z_1| < (1/2)C_0\}) \subset \Omega$. Hence if we suppose that $|z_1| \leq (1/4)C_0$ for all $z = (z_1, \ldots, z_n) \in U_0$, then we get $\Omega \cap U_0 = U_0 \cap ((1/4)C_0 D \times \pi(\{\Omega \cap U_0 \mid |z_1| < (1/2)C_0\}))$. Near $p = 0$, Ω is locally a product space. $\square$

Theorem IV.4. *Suppose that Ω is a bounded domain that is Kobayashi complete near p in $\partial\Omega$. The following are equivalent.*

(1) For z close enough to p, for some $\epsilon < 1$ there exists C_ϵ such that

$$K(z,v) \leq \frac{C_\epsilon}{d_v(z)^\epsilon}.$$

(2) For z close enough to p, for all $\epsilon \leq 1$ there exists C_ϵ such that

$$K(z,v) \leq \frac{C_\epsilon}{d_v(z)^\epsilon}.$$

(3) There is a small open neighborhood U of p in $\mathbb{C}^n$ and a constant M such that

$$K(z,v) \leq M \qquad \text{for all } z \in U \cap \Omega.$$

(4) The domain Ω is locally a product space as follows. Let $p = 0$. After linearly changing the coordinate system then there are C_0, an open neighborhood U of 0 in $\mathbb{C}^n$, and an open set W in $\mathbb{C}^{n-1}$ such that $U \cap (C_0 D \times W) = \Omega \cap U$.

Proof of Theorem IV.4. By theorem IV.1, (1) implies (4). Suppose that (4) is true, then there is a constant C and open neighborhood U_0 of 0 such that if $z = (z_1, z') \in U_0 \cap ((C/2)D \times W)$, then $d_v(z) \geq C/2$. So we get $K(z,v) \leq 1/d_v(z) \leq 2/C$. Thus (3) is satisfied.

Now we show that (3) implies (2). The domain Ω is bounded, so there is a constant M_0 such that $d_v((z_1, z')) \leq M_0$, $1 \leq M_0^\epsilon(\frac{1}{d_v(z)^\epsilon})$ for all ϵ. This implies that $K(z,v) \leq MM_0^\epsilon(\frac{1}{d_v(z)^\epsilon})$. Let $C_\epsilon = MM_0^\epsilon$. So we get $K(z,v) \leq \frac{C_\epsilon}{d_v(z)^\epsilon}$. Obviously (2) implies (1). □

We wondered whether the property (3) in Theorem IV.4 may be invariant under biholomorphism. But Theorem IV.4 gives a negative answer because product structure is not preserved under biholomorphism. Here is an example. The domain $\Omega = \{(z,w) \mid z \in D, |w| < R(z)\}$ can be mapped onto the unit dicylinder by some biholomorphism if $-\ln R(z)$ is harmonic in D [29]. This example is enough to show that the product structure is not invariant under biholomorphism.

Now we introduce the localization principle of the Bergman kernel in the case of a bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$ (see [15], [28]). Let Ω be a bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, $z^0 \in b\Omega$. Then for any sufficiently small neighborhood U of z^0, for $z \in U' \cap \Omega$, where U' is a smaller neighborhood $U' \subset\subset U$, we have:

$$\frac{1}{c} B_{U\cap\Omega}(z, z) \leq B_\Omega(z, z) \leq B_{U\cap\Omega}(z, z).$$

Corollary IV.1. *If Ω is a smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$ that is Kobayashi complete near p, and there are a neighborhood U, a nonzero vector v, and $M > 0$ such that for $q \in \Omega \cap U$,*

$$K(q, v) \leq M,$$

then after linearly changing the coordinate system such that $p = 0$ and v is a unit vector in the z_1-direction, we have the following inequality for $z = (z_1, z') \in \Omega \cap U_1$, $z' \in \mathbb{C}^{n-1}$:

$$B_{C_0 D}(z_1, z_1) \times B_{\pi(U_0)}(z', z') \lesssim B_\Omega(z, z) \lesssim B_{C_0 D}(z_1, z_1) \times B_{\pi(U_0)}(z', z'),$$

where U_0 and $U_1 \subset U_0$ are subneighborhoods of U containing p and C_0 is the constant from Theorem IV.4.

Geometrically, $\pi(U_0)$ is the intersection of the hypersurface supported by v at p and $\Omega \cap U_0$. The point of the Corollary is that the rate of blow-up of the Bergman kernel $B_\Omega(z, z)$ as $z \to p$ is comparable to the rate of blow-up of the Bergman kernel $B_{\pi(U_0)}$ of a lower-dimensional domain. Theorem IV.4 and the argument above imply this Corollary.

Now we give Proof of Theorem IV.2.

Proof of Theorem IV.2. The proof of (2) $\leftrightarrow$ (3) is in [18, Theorem 1.1]. Now we show that (2) implies (1). Suppose that Ω does not have property K near some point p in $b\Omega$. Convexity of Ω implies Kobayashi completeness. By Theorem IV.3 (when the domain is convex, smoothness is not necessary), Ω has a noncompact $\overline{\partial}$-Neumann operator.

To prove that (1) implies (3), we assume that there is an affine complex disk on the boundary of Ω. After linearly changing the coordinate system, we may assume that $0 \in \partial\Omega$. Let $A = \{z_1 \in \mathbb{C}^1 \mid |z_1| < 1\} \times \Omega_2 \subset \Omega$, where $\Omega_2 = (1/2)\{z' \in \mathbb{C}^{n-1} \mid (0, z') \in \Omega\}$. By Lemma III.1 (2), the convexity of Ω implies that A is a fat subdomain having mass at 0 of Ω. The complete proof of existence of the fat subdomain which is a product space after linearly changing the coordinate system is contained in [18]. Denote by $v = (1, 0, \ldots, 0)$. For all $z \in A \cap \{(z_1, z') \mid |z_1| < (1/4)\}$, $d_v(z)$ is uniformly lower and upper bounded. There exist M_1 and M_2 such that $M_1 < 1/d_v(z) < M_2$. This implies that $K(z, v) \leq 1/d_v(z) \leq M_2(M_1^{-\epsilon})/d_v(z)^{\epsilon}$. Thus Ω does not have property K. $\square$

CHAPTER V

LOCAL REGULARITY OF THE $\overline{\partial}$-NEUMANN OPERATOR

A. Introduction

Kohn proved that if Ω is a strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, then the $\overline{\partial}$-Neumann operator is regular. Catlin, one of Kohn's students, proved [10] by almost classical machinery of Kohn and Nirenberg [23] that if p is of finite type in the sense of D'Angelo [13] and p is a smooth boundary point, then a pseudolocal estimate for some ϵ holds near p. Here we give the precise definition of the pseudolocal estimate. One says that a pseudolocal estimate of order $\epsilon > 0$ holds in U if for each $k \geq 0$ and any functions $\xi_1, \xi_2 \in C_0^\infty(U)$ with $\xi_1 = 1$ on the support of ξ_2 there exists a constant $C > 0$ such that

$$\|\xi_2 N\alpha\|_{(k+2)\epsilon} \leq C(\|\xi_1\alpha\|_{k\epsilon} + \|\alpha\|_0) \tag{V.1}$$

for all $\alpha \in L^2_{(p,q)}$, $q > 0$. Here $\|\cdot\|_s$ stands for the ordinary Sobolev s-norm on Ω. Further, we say that N is locally regular in U if whenever α is C^∞-smooth up to the boundary of Ω, the restriction to U of $N\alpha$ is also C^∞-smooth up to the boundary. As a direct application of Catlin's pseudolocal estimate near finite-type points, we have the following theorem.

Theorem V.1. *If Ω has a smooth boundary near p, and p is of finite type in the sense of D'Angelo [13], then there is an open neighborhood U on which a pseudolocal estimate holds for some ϵ, and N is locally regular in U [16].*

In this chapter, we study whether suitable assumptions (more general than finite type) on $b\Omega \cap U$ imply local regularity of N on U. A natural situation to consider is the case when Ω admits a local plurisubharmonic defining function. The counterexample of the worm domain shows that an additional hypothesis is needed to obtain a positive result. Each boundary point of the worm domain has a neighborhood U in which there is a local plurisubharmonic defining function. By Christ's theorem the $\overline{\partial}$-Neumann problem is not globally regular in the worm domain [12], so local regularity must fail on at least one of these patches U. Hence, existence of a local plurisubharmonic defining function on $\Omega \cap U$ may not be a sufficient condition for the regularity of N on U. We found by analysis of the proof in [7] a sufficient additional condition.

Theorem V.2. *We assume that Ω is a smooth, bounded, pseudoconvex domain, and $\overline{U}$ is polynomially convex in $\mathbb{C}^n$. If there are open sets $V_0 \subset V_1 \subset U$ such that*

(1) there is a plurisubharmonic function ρ on V_1 such that $\{z \mid \rho(z) < 0\} = \Omega \cap V_1$ and $\nabla\rho$ vanishes nowhere on $V_1 \cap b\Omega$,

(2) for $p \in b\Omega \cap (\overline{U} \setminus (V_0))$, p is of finite type,

then N is locally regular in U, and for each $s \geq 0$ there exists a constant C_s such that $\|Nf\|_{W^s(\Omega\cap U)} \leq C_s\|f\|_{W^s(\Omega)}$ for $f \in W^s_{p,q}(\Omega)$, $0 \leq p \leq n$, $1 \leq q \leq n$.

We will prove Theorem V.2 in the next section. In section C, we give several corollaries as applications of Theorem V.2.

B. Main Proof

1. The Vector Field Method

The vector field method is a well-known method for proving the regularity of the $\overline{\partial}$-Neumann operator. For the convenience of the reader, we briefly introduce this method.

Let Ω be a smooth bounded pseudoconvex domain, and let r be a smooth defining function for Ω. Set

$$L_n = \frac{4}{|\nabla r|^2} \sum_{i,j}^{n} \frac{\partial r}{\partial \bar{z}_j} \frac{\partial}{\partial z_j}.$$

For $1 \leq j < k \leq n$, set

$$L_{(jk)} = \frac{\partial r}{\partial z_j} \frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k} \frac{\partial}{\partial z_j}.$$

Denote by $X_n = (|\nabla r|/\sqrt{2})L_n$ the globally defined type $(1,0)$ vector field which is transverse to the boundary everywhere. The field X_n vanishes nowhere on the boundary. Thus, near every boundary point $p \in \partial\Omega$, we may choose tangential type (1,0) vector fields $X_1, \ldots, X_{n-1}$ so that $X_1, \ldots, X_{n-1}$ together with X_n form an orthonormal basis of the space of type (1,0) vector fields in some open neighborhood of p.

Following [11], we define the Condition (T).

Definition V.1 (Condition (T)**).** *For any given* $\varepsilon > 0$ *there exists a smooth real vector field* $\mathrm{T} = \mathrm{T}_\varepsilon$*, depending on* ε*, defined in some open neighborhood of* $\overline{\Omega}$ *and tangent to the boundary with the following properties:*

(1) On the boundary, T *can be expressed as*

$$\mathrm{T} = \Theta_\epsilon(z)(L_n - \overline{L}_n) \quad \mathrm{mod}\ (\mathrm{T}^{(1,0)}(b\Omega) \oplus \mathrm{T}^{(0,1)}(b\Omega))$$

for some smooth function $\Theta_\varepsilon(z)$ *with* $|\Theta_\varepsilon(z)| \geq \delta > 0$ *for all* $z \in b\Omega$*, where* δ *is a positive constant independent of* ε*.*

(2) If S *is any one of the vector fields* L_n*,* $\overline{L}_n$*,* $L_{(jk)}$*,* $\overline{L}_{(jk)}$*,* $1 \leq j < k \leq n$*, then*

$$[\mathrm{T}, S]|_{b\Omega} = A_s(z)L_n \quad \mathrm{mod}\ (\mathrm{T}^{(1,0)}(b\Omega) \oplus \mathrm{T}^{(0,1)}(b\Omega), \overline{L}_n)$$

for some smooth function $A_s(z)$ *with* $\sup_{b\Omega} |A_s(z)| < \varepsilon$*.*

Remark. Near a boundary point p, we have, say, $\partial r/\partial z_n(p) \neq 0$. Thus, for each

$j = 1, \ldots, n-1$, we may write

$$X_j = \sum_{k=1}^{n-1} c_{jk} L_{kn}$$

for some smooth functions c_{jk}. It follows that if condition (T) holds on Ω, then property (2) of condition (T) is still valid with S taken to be X_j or $\overline{X}_j$ for $j = 1, \ldots, n-1$, where the X_j's are defined as above in some small open neighborhood of p. Here we use the same explanation about condition (T) and $X_n, \ldots, X_1$ as that in [11, 6.2]. Now we state the well known theorem.

Theorem V.3. *Let Ω be a smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with a smooth defining function r. Suppose that Condition* T *holds on Ω. Then the $\overline{\partial}$-Neumann operator N maps $W^s_{(p,q)}(\Omega)$, $0 \leq p \leq n$, $1 \leq q \leq n$, boundedly into itself for each nonnegative real number s [11].*

Boas and Straube proved that when the domain Ω has a plurisubharmonic defining function, the condition T holds on Ω.

Theorem V.4. *Let $\Omega \in \mathbb{C}^n$, $n \geq 2$, be a smooth, bounded, pseudoconvex domain admitting a plurisubharmonic defining function $r(z)$. Then the $\overline{\partial}$-Neumann operator N is exactly regular on $W_{(p,q)}(\Omega)$ for $1 \leq q \leq n$ and all real $s \geq 0$ [7].*

By analyzing the proof of Theorems V.3 and V.4, we can observe that when a smooth, bounded, pseudoconvex domain has a plurisubharmonic defining function on some neighborhood U, then the condition T holds on $U \cap \Omega$. That point is needed in the proof of Theorem V.2.

2. Proof of Theorem V.2

Before we prove Theorem V.2, we need Lemma V.1.

Lemma V.1. *Suppose that Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$. If there is an open set U such that the part of $b\Omega$ in $\overline{U}$ is smooth, and for $p \in b\Omega \cap \overline{U}$, p is of finite type, then N is locally regular on U, and for each $s \geq 0$ there exists a constant C_s such that for $f \in W^s_{p,q}(\Omega)$, $0 \leq p \leq n$, $1 \leq q \leq n$,*

$$\|Nf\|_{W^s(\Omega\cap U)} \leq C_s \|f\|_{W^s(\Omega)}. \tag{V.2}$$

Proof of Lemma V.1. For $p \in b\Omega \cap \overline{U}$, by Theorem V.1, there is an open neighborhood V_p of p on which there is $C_{p,s}$ such that for $f \in C^\infty(\Omega \cap \overline{V}_p)$, for $s \geq 0$,

$$\|Nf\|_{W^s(\Omega\cap V_p)} \leq C_{p,s} \|f\|_{W^s(\Omega)}.$$

Since $\overline{\Omega} \cap \overline{U}$ is compact, we may assume that the union of a finite number of V_{p_i} cover $b\Omega \cap \overline{U}$. Here we use $\|f\|_s^2$ to control the interior term. So we can choose C_s such that

$$\|Nf\|_{W^s(\Omega\cap U)} \leq C_s \|f\|_{W^s(\Omega)}.$$

This completely proves Lemma V.1. $\square$

Proof of Theorem V.2. We explain about the strategy of the proof. We construct special approximating smooth subdomains Ω_ϵ, and then for some $V_2, V_0 \subset V_2 \subset U$, we show $\|N_\epsilon f\|_{W^s(\Omega\cap V_2)} \lesssim \|f\|_{W^s(\Omega)}$ for $N_\epsilon f \in C^\infty(\overline{\Omega} \cap V_2)$. By passing to the limit as $\epsilon \to 0$, we will get $\|Nf\|_{W^s(\Omega\cap V_0)} \lesssim \|f\|_{W^s(\Omega)}$ and also $\|Nf\|_{W^s(\Omega\cap(U\setminus V_0))} \lesssim \|f\|_{W^s(\Omega)}$ by Lemma V.1.

Fix V_2', $V_0 \subset\subset V_2' \subset U$. Now we construct a smooth pseudoconvex domain Ω_ϵ for each ϵ such that $b\Omega_\epsilon \cap V_2' = b\Omega \cap V_2'$, and for $q \in b\Omega_\epsilon \setminus V_0$, q is finite type.

We may assume that $\overline{V_2'}$ is a compact polynomially convex domain since $\overline{U}$ is. By [32, Lemma 4], there is a nonnegative plurisubharmonic function ψ such that $\psi^{-1}(0) = \overline{V_2'}$. Then we use an idea similar to [2].

Let r be a C^∞ defining function for Ω. We may assume that for some positive η, $-(-r(z))^\eta$ is a strictly plurisubharmonic function on Ω, and we may assume that η is the reciprocal of a positive integer (we want $\psi^{1/\eta}$ to be C^∞) [30].

Define $\rho_\epsilon(z) = -(-r(z))^\eta + \epsilon\psi(z)$. The function ρ_ϵ is strictly plurisubharmonic on Ω. The domain Ω_ϵ will be given by $\Omega_\epsilon = \{z \in \Omega \mid \rho_\epsilon(z) < 0\}$. That implies $b\Omega_\epsilon \cap V_2' = b\Omega \cap V_2'$. We need to show why Ω_ϵ is smooth. First, we consider points in $b\Omega_\epsilon \cap b\Omega$. Now $\Omega_\epsilon = \{z \mid R(z) < 0\}$ where $R(z) = r(z) + (\epsilon\psi(z))^{1/\eta}$. Since R is a C^∞ defining function for $b\Omega_\epsilon$ which agrees to infinite order with r on $b\Omega_\epsilon \cap b\Omega$, where $\nabla r \neq 0$, it follows that $\nabla R \neq 0$. On $b\Omega_\epsilon \setminus b\Omega$, $\nabla\rho_\epsilon(z) \neq 0$ for sufficiently small ϵ, since $\|\nabla r\|$ is below bounded on $b\Omega$.

D'Angelo's Theorem [13] and hypothesis (2) of the theorem yield that the boundary points of Ω_ϵ in $b\Omega \cap (V_2' \setminus V_0)$ are finite type. So we can conclude that the boundary points of $b\Omega_\epsilon \setminus V_0$ are of finite type. By the hypotheses (1), (2), and [8], we can extend the vector field T to Ω_ϵ. Theorem V.4 yields that Ω_ϵ has regularity of the $\overline{\partial}$-Neumann operator.

Claim 1. *There is a constant C_s such that for $N_\epsilon f \in C^\infty(\overline{\Omega} \cap V_2)$, where $V_0 \subset\subset V_2 \subset\subset V_2'$,*

$$\|N_\epsilon f\|_{W^s(\Omega\cap V_2)} \leq C_s \|f\|_{W^s(\Omega)} \tag{V.3}$$

(C_s is independent of ϵ and we may assume that $\Omega_\epsilon \cap V_2 = \Omega \cap V_2$).

Proof of Claim 1. Denote $u = N_\epsilon f$. We will estimate $\|u\|_{W^s(\Omega\cap V_2)}$. The initial step $s = 0$ is obvious, since N_ϵ is a bounded operator in $L^2(\Omega_\epsilon)$. This boundedness is independent of ϵ because the constant depends only on the diameter of the domain Ω_ϵ by Hörmander's theorem (Theorem II.1).

Here we prove the case $s = 1$ in detail. The general case follows in a similar way by induction on s. By the hypothesis of Theorem V.2, there is a smooth plurisubharmonic function ρ on V_1 such that $\{z \mid \rho < 0\} \cap (V_1 \cap V_2) = \Omega_\epsilon \cap (V_1 \cap V_2) = \Omega \cap (V_1 \cap V_2)$.

Near $b\Omega_\epsilon \cap (V_1 \cap V_2)$, the condition (T) holds. For $\varepsilon > 0$, we can define a vector field T_ε on $\Omega_\epsilon \cap (V_1 \cap V_2)$ which satisfies conditions (1) and (2) in condition (T). By multiplying with a smooth cut-off function χ that is identically one on V_0 and zero outside of V_2, we can extend T_ε to Ω_ϵ. For convenience, we write $\mathrm{T}_\varepsilon = \mathrm{T}$.

We choose boundary coordinate charts $\{U_\alpha\}_{\alpha=1}^m$ such that $\{U_\alpha\}_{\alpha=1}^m$ and $U_0 = \Omega_\epsilon \cap V_2'$ form an open cover of $\overline{\Omega_\epsilon \cap V_2}$. Let $\{\zeta_\alpha\}_{\alpha=0}^m$ be a fixed partition of unity subordinate to $\{U_\alpha\}_{\alpha=1}^m$. On each U_α, $1 \leq \alpha \leq m$, let $w_{\alpha k}$, $k = 1, \ldots, n$, be an orthonormal frame of $(1,0)$-forms dual to $X_{\alpha k}$, $k = 1, \ldots, n$. We note that $w_{\alpha n} = w_n$, $\alpha = 1, \ldots, m$, is a globally defined type $(1,0)$-form dual to $X_n = (|\nabla r|/\sqrt{2})L_n$. Similarly, $X_{\alpha n} = X_n$, $\alpha = 1, \ldots, m$, is also a globally defined type $(1,0)$ vector field.

The form u can be locally expressed on U_α as $\sum' u^\alpha_{(I,J)} w^{\alpha,I} \wedge \overline{w}^{\alpha,J}$, where $w^{\alpha,I} = w_{\alpha i_1} \wedge \cdots \wedge w_{\alpha i_p}$ and $\overline{w}^{\alpha,J} = \overline{w}_{\alpha j_1} \wedge \cdots \wedge \overline{w}_{\alpha j_q}$.

Set

$$\|\overline{X}u\|^2_{\Omega_\epsilon \cap V_0} = \sum_{\alpha=1}^m \sideset{}{'}\sum_{I,J} \sum_{k=1}^n \|\overline{X}_{\alpha k}(\zeta_\alpha u^\alpha_{I,J})\|^2_{\Omega_\epsilon \cap V_0},$$
$$\|X'u\|^2_{\Omega_\epsilon \cap V_0} = \sum_{\alpha=1}^m \sideset{}{'}\sum_{I,J} \sum_{k=1}^{n-1} \|X_{\alpha k}(\zeta_\alpha u^\alpha_{I,J})\|^2_{\Omega_\epsilon \cap V_0},$$
$$\mathrm{T}u = \mathrm{T}(\zeta_0 u) + \sum_{\alpha=1}^m \sideset{}{'}\sum_{I,J} \mathrm{T}(\zeta_\alpha u^\alpha_{I,J}) w^{\alpha,I} \wedge \overline{w}^{\alpha,J}.$$

By the standard calculus (we recommend the reader to refer to [11, p. 130]) we get

$$\|\overline{X}u\|^2_{\Omega_\epsilon \cap V_0} + \|X'u\|^2_{\Omega_\epsilon \cap V_0} \leq C_1(\|f\|^2 + \|u\|^2 + (sc)\|\mathrm{T}u\|^2). \tag{V.4}$$

The constant C_1 is independent of ϵ, and (sc) can be as small as we wish. If we can control $\|\mathrm{T}u\|_{\Omega_\epsilon \cap V_0}$, then $\|u\|_{W^1(\Omega_\epsilon \cap V_0)}$ can be estimated. In view of (V.4) and the hypothesis (1) on the vector field T, if we control $\|\mathrm{T}u\|_{\Omega_\epsilon \cap V_0}$, then we control all derivatives of u, that is, we can estimate $\|u\|_{W^1(\Omega_\epsilon\cap V_0)}$. Therefore, we call $X_{\alpha 1}, \ldots, X_{\alpha n-1}, \overline{X}_{\alpha 1}, \ldots, \overline{X}_{\alpha n-1}$ and $\overline{X}_{\alpha n}$, $1 \le \alpha \le m$, *good directions.*

Now we estimate $\|\mathrm{T}u\|_{L^2(\Omega_\epsilon\cap V_0)}$. First by the basic L^2 estimate, we get

$$\|\mathrm{T}u\|^2 \le C_2(\|\overline{\partial}\mathrm{T}u\|^2 + \|\overline{\partial}_\epsilon^*\mathrm{T}u\|^2). \tag{V.5}$$

The constant C_2 is independent of ϵ, since $\mathrm{supp}(\mathrm{T}u) \subset V_2$ where the boundaries of Ω and Ω_ϵ agree. Hence the operators $\overline{\partial}^*$ and $\overline{\partial}_\epsilon^*$ agree on the form $\mathrm{T}u$. We estimate the right-hand side as follows.

$$\begin{aligned}\|\overline{\partial}\mathrm{T}u\|^2 &= (\overline{\partial}\mathrm{T}u, \overline{\partial}\mathrm{T}u)\\ &= (\overline{\partial}^*\overline{\partial}u, -\mathrm{T}^2u) + (\overline{\partial}u, [[\overline{\partial}, \mathrm{T}], \mathrm{T}]u) + (-\overline{\partial}\mathrm{T}u, [\overline{\partial}, \mathrm{T}]u)\\ &\quad + \|[\overline{\partial}, \mathrm{T}]u\|^2 + ([\overline{\partial}, \mathrm{T}]u, \overline{\partial}\mathrm{T}u) + \mathrm{O}((\|\overline{\partial}\mathrm{T}u\| + \|u\|_{W^1(\Omega_\epsilon\cap V_2)})\|\overline{\partial}u\|).\end{aligned}$$

Note that

$$\mathrm{Re}\{(-\overline{\partial}\mathrm{T}u, [\overline{\partial}, \mathrm{T}]u) + ([\overline{\partial}, \mathrm{T}]u, \overline{\partial}\mathrm{T}u)\} = 0.$$

With a similar estimate for $\|\overline{\partial}_\epsilon^*\mathrm{T}u\|^2$, we obtain

$$\begin{aligned}&\|\overline{\partial}\mathrm{T}u\|^2 + \|\overline{\partial}_\epsilon^*\mathrm{T}u\|^2\\ &\quad \le C_3(\|f\|_1^2 + \|[\overline{\partial}, \mathrm{T}]u\|^2 + \|[\overline{\partial}_\epsilon^*, \mathrm{T}]u\|^2) + (sc)\|\mathrm{T}u\|^2 + (sc)\|u\|^2_{W^1(\Omega_\epsilon\cap V_2)}.\end{aligned} \tag{V.6}$$

To estimate $\|[\overline{\partial}, \mathrm{T}]u\|^2_{\Omega_\epsilon\cap V_2}$ and $\|[\overline{\partial}_\epsilon^*, \mathrm{T}]u\|^2_{\Omega_\epsilon\cap V_2}$, we use the hypothesis on T on $\Omega_\epsilon \cap V_0$ and (2) in the hypothesis of Theorem V.2. We need to separately estimate $\|[\overline{\partial}, \mathrm{T}]u\|^2_{\Omega_\epsilon\cap V_2}$ as $\|[\overline{\partial}, \mathrm{T}]u\|^2_{\Omega_\epsilon\cap V_0}$ and $\|[\overline{\partial}, \mathrm{T}]u\|^2_{\Omega_\epsilon\cap(V_2\setminus V_0)}$. The second part can be controlled by $\|f\|_1$ by Lemma V.1 since every point $p \in \partial(\Omega_\epsilon) \cap \overline{(V_2 \setminus V_0)}$ is of finite type. The domain Ω_ϵ has the same boundary as Ω on $\overline{(V_2 \setminus V_0)}$, so we get

$$\|[\overline{\partial}, \mathrm{T}]u\|_{\Omega_\epsilon\cap(V_2\setminus V_0)} + \|[\overline{\partial}_\epsilon^*, \mathrm{T}]u\|_{\Omega_\epsilon\cap(V_2\setminus V_0)} \le C_4\|u\|_{\Omega_\epsilon\cap(V_2\setminus V_0)} \le C_5\|f\|_{W^1(\Omega)}. \tag{V.7}$$

Here we need to emphasize why C_4 is independent of ϵ. In the proof of Lemma V.1, we use a pseudolocal estimate near a finite-type point p. The constant C in (V.1) depends only on the boundary near p and the diameter of the domain. But $b\Omega_\epsilon$ has the same boundary on V_2', and there is an uniform bound on the diameter of Ω_ϵ.

So we need to estimate only the first part. On each boundary coordinate chart the commutator between T and $X_1, \ldots, X_{n-1}, \overline{X}_1, \ldots, \overline{X}_{n-1}$ can be controlled by the hypothesis on T. Thus we need to consider the commutator of T and $\overline{X}_n$ (or X_n) which occurs, when commuting T and $\overline{\partial}$ (or $\overline{\partial}^*$), only for those multi-indices (I, J) with $n \notin J$ (or $n \in J$). Such terms can be handled as follows:

$$
\begin{aligned}
[\overline{X}_n, \mathrm{T}](\zeta_\alpha u^\alpha_{I,J}) &= [(|\nabla r|/\sqrt{2})\overline{L}_n, \mathrm{T}](\zeta_\alpha u^\alpha_{I,J}) \\
&= (|\nabla r|/\sqrt{2})[\overline{L}_n, \mathrm{T}](\zeta_\alpha u^\alpha_{I,J}) - (\mathrm{T}(|\nabla r|)/|\nabla r|)\overline{X}_n(\zeta_\alpha u^\alpha_{I,J})
\end{aligned}
$$

for $\alpha = 1, \ldots, m$. Using the basic estimate, we obtain

$$
\sum_{\alpha=1}^{m} \sum_{n\in J}{}' \|\mathrm{T}(|\nabla r|)/|\nabla r|)\overline{X}_n(\zeta_\alpha u^\alpha_{I,J})\|^2_{\Omega_\epsilon\cap V_0} \le C_6(\|\overline{\partial} u\|^2 + \|\overline{\partial}^*_\epsilon u\|^2) \le C_7\|f\|^2_\Omega.
$$

The remaining commutator terms can be estimated by the hypothesis on T on $\Omega_\epsilon \cap V_0$. Since C_7 depends only on the diameter of Ω_ϵ, we can choose C_7 independently of ϵ. Now

$$
\begin{aligned}
&\|[\overline{\partial}, \mathrm{T}]u\|^2_{\Omega_\epsilon\cap V_0} \\
&\quad \le C_4'\Bigg(\sum_{\alpha=1}^{m} \sum_{I,J}{}' \left\| \frac{A}{\Theta_\varepsilon}\mathrm{T}(\zeta_\alpha u^\alpha_{I,J})\right\|^2_{\Omega_\epsilon\cap V_0} + \|\overline{X}u\|^2_{\Omega_\epsilon\cap V_0} + \|X'u\|^2_{\Omega_\epsilon\cap V_0} + \|f\|^2_{W^1(\Omega)}\Bigg) \\
&\quad \le \left(\frac{\varepsilon}{\delta}\right)^2 \|\mathrm{T}u\|^2 + C_5'(\|\overline{X}u\|^2_{\Omega_\epsilon\cap V_0} + \|X'u\|^2_{\Omega_\epsilon\cap V_0} + \|f\|^2_1). \quad \text{(V.8)}
\end{aligned}
$$

For $\|[\overline{\partial}^*_\epsilon, \mathrm{T}]u\|_{\Omega_\epsilon\cap V_0}$ we commute T with X_n if $n \in J$. Hence

$$
[X_n, \mathrm{T}](\zeta_\alpha u^\alpha_{I,J}) = (|\nabla r|/\sqrt{2})[L_n, \mathrm{T}](\zeta_\alpha u^\alpha_{I,J}) - (\mathrm{T}(|\nabla r|)/|\nabla r|)X_n(\zeta_\alpha u^\alpha_{I,J})
$$

for $\alpha = 1, \ldots, m$. Observe that $\pm X_n(\zeta_\alpha u^\alpha_{I,J})$ appears in the coefficient of $w^{\alpha,I} \wedge \overline{w}^{\alpha,H}$ with $\{n\} \cup H = J$ in $\overline{\partial}^* u$.

Meanwhile, all the other terms in the coefficient of $w^{\alpha,I} \wedge \overline{w}^{\alpha,H}$ are differentiated by $X_1, \ldots, X_{n-1}$ only. Thus we have

$$\sum_{\alpha=1}^{m} \sum_{n\in J}{}' \|(\mathrm{T}(|\nabla r|)/|\nabla r|)X_n(\zeta_\alpha u^\alpha u_{I,J})\|^2_{\Omega_\epsilon\cap V_0} \leq C_7'(\|\overline{\partial}^*_\epsilon u\|^2 + \|X' u\|^2_{\Omega_\epsilon\cap V_0} + \|f\|^2).$$

The first term can be controlled as before. Here we use $\|f\|_1^2$ to control the interior term:

$$\begin{aligned} &\|[\overline{\partial}^*, \mathrm{T}]u\|^2_{\Omega_\epsilon\cap V_0} + \|[\overline{\partial}, \mathrm{T}]u\|^2_{\Omega_\epsilon\cap V_0} \\ &\qquad \leq C_8(\|f\|^2 + \|X'u\|^2_{\Omega_\epsilon\cap V_0} + \|\overline{X}u\|^2_{\Omega_\epsilon\cap V_0} + \|f\|_1^2) + r\|\mathrm{T}u\|^2, \end{aligned} \tag{V.9}$$

where $r > 0$ is a constant that can be as small as we wish. Combining equations (V.7) and (V.9), we get

$$\begin{aligned} &\|[\overline{\partial}^*_\epsilon, \mathrm{T}]u\|^2 + \|[\overline{\partial}, \mathrm{T}]u\|^2 \\ &\qquad \leq C_8(\|f\|^2_{W^1(\Omega_\epsilon)} + \|X'u\|^2_{\Omega_\epsilon\cap V_0} + \|\overline{X}u\|^2_{\Omega_\epsilon\cap V_0}) + r\|\mathrm{T}u\|^2. \end{aligned} \tag{V.10}$$

By (V.5), (V.6), and (V.10),

$$\|\mathrm{T}u\|^2 \leq C_8'(\|f\|^2_{W^1(\Omega_\epsilon)} + (sc)\|u\|^2_{W^1(\Omega_\epsilon\cap V_2)}). \tag{V.11}$$

Inequalities (V.4), (V.7), and (V.11) imply

$$\begin{aligned} &\|u\|_{W^1(\Omega_\epsilon\cap V_2)} \leq (\|u\|_{W^1(\Omega_\epsilon\cap V_0)} + \|u\|_{W^1(\Omega_\epsilon\cap V_2\setminus V_0)}) \\ &\quad \leq C_9(\|u\|_{W^1(\Omega_\epsilon\cap V_2\setminus V_0)} + \|X'u\|_{\Omega_\epsilon\cap V_0} + \|\overline{X}u\|_{\Omega_\epsilon\cap V_0} + \|\mathrm{T}u\|) + (sc)\|u\|_{W^1(\Omega_\epsilon\cap V_2)}. \end{aligned}$$

Finally we get

$$\|N_\epsilon f\|_{W^1(\Omega\cap V_2)} \leq C_1\|f\|_{W^1(\Omega)}.$$

By using induction, and a similar procedure to the first case, independently of ϵ, we get the equation (V.3) in Claim 1. This completely proves Claim 1. $\square$

By Claim 1, and the regularity of the $\bar{\partial}$-Neumann operator of Ω_ϵ, if $f \in C^\infty(\overline{\Omega}) \subset C^\infty(\overline{\Omega}_\epsilon)$, then $N_\epsilon f \in W^s(\overline{\Omega} \cap V_2)$, and we have

$$\|N_\epsilon f\|_{W^s(\Omega\cap V_0)} \leq \|N_\epsilon f\|_{W^s(\Omega\cap V_2)} \leq C_s \|f\|_{W^s(\Omega)}. \tag{V.12}$$

Now $N_\epsilon f$ converges weakly to some function B in $W^s(\Omega \cap V_0)$, and $N_\epsilon f$ converges to Nf in $L^2(\Omega \cap V_0)$ [26]. We can conclude that $Nf = B$ and Nf is in $W^s(\Omega \cap V_0)$. The constant C_s in (V.12) is independent of ϵ, so we have

$$\|Nf\|_{W^s(\Omega\cap V_0)} \leq C_s \|f\|_{W^s(\Omega)}.$$

After replacing U in Lemma V.1 with $U \setminus V_0$, we have completely proved Theorem V.2. $\square$

Remark. In the situation of Theorem V.2, it is possible for an analytic disk to be on the part of boundary where we have local regularity of the $\bar{\partial}$-Neumann operator.

C. Application

By Corollary V.1, we show that an isolated infinite-type point in the boundary of the domain is not an obstruction for local regularity of the $\bar{\partial}$-Neumann operator.

Corollary V.1. *Suppose that Ω is a smooth, bounded, pseudoconvex domain, and p is a boundary point of Ω. If there is an open neighborhood U of p such that for $q \in b\Omega \cap \overline{U}$, $q \neq p$, q is finite type, then N is locally regular on U, and for $s > 0$, there is C_s such that for $f \in C^\infty(\overline{\Omega})$ we have*

$$\|Nf\|_{W^s(\Omega\cap U)} \leq C_s \|f\|_{W^s(\Omega)}.$$

Proof of Corollary V.1. There is a small number $\sigma > 0$ such that $B_\sigma(p) = \{z \mid d(z,p) < \sigma\} \subset\subset U$. Now $\overline{B_\sigma(p)}$ is a compact, polynomially convex set. By the same argument as in the proof of Theorem V.2, we can construct a smooth bounded pseudoconvex domain Ω_ϵ such that $b\Omega_\epsilon \cap B_\sigma(p) = b\Omega \cap B_\sigma(p)$.

We define a vector field $\mathrm{T} = L_n - \overline{L}_n$ where $L_n = \frac{4}{|\nabla r|^2} \sum_{j=1}^n \frac{\partial r}{\partial \bar{z}_j} \frac{\partial}{\partial z_j}$ on U. By multiplying with a smooth cut-off function χ that is identically one around p and zero outside of U, we can define a smooth vector field T on Ω. There is an open neighborhood V_ε of p contained in $B_\sigma(p)$ such that T satisfies (1) and (2) in Condition T on $b\Omega \cap V_\varepsilon$.

We replace V_0 in the equations (V.7) and (V.9) with V_ε. The corollary follows by the same argument as in the proof of Theorem V.2. $\square$

Corollary V.2. *Let Ω, p, and U be the same as in Theorem V.2. Then for $f \in C^\infty(\overline{\Omega})$ we have $Pf \in C^\infty(\overline{\Omega} \cap U)$, where P is the Bergman projection.*

Corollary V.2 is a direct result from Theorem V.2 and the identity $P = I - \overline{\partial}^* N_{(0,1)} \overline{\partial}$.

CHAPTER VI

SUMMARY

We have shown in Chapter III that under certain conditions, a higher-dimensional domain fibered over Ω can inherit noncompactness of the $\overline{\partial}$-Neumann operator from the base domain Ω. We tried to generalize the inheritance of noncompactness of the $\overline{\partial}$-Neumann operator from the base domain, but more work needs to be done for the generalization.

In Chapter IV we showed by using the property K that a certain local property of a domain is a sufficient condition for noncompactness of the $\overline{\partial}$-Neumann operator. We wondered whether the condition in Theorem IV.1 may be invariant under biholomorphisms, but Theorem IV.4 gives a negative answer because product structure is not preserved under biholomorphisms.

Using the vector-field method, we found a more general condition than finite type for the local regularity of the $\overline{\partial}$-Neumann operator. By this generalization, it is possible for an analytic disk to be on the part of the boundary where we have the local regularity of the $\overline{\partial}$-Neumann operator. In Theorem V.2, we showed that an isolated infinite-type point in the boundary of the domain is not an obstruction for local regularity of the $\overline{\partial}$-Neumann operator.

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VITA

Mijoung Kim was born on July 2, 1967, to parents Hyunuk Kim and Sangyun Jo, in Seoul, Korea. She completed her Bachelor of Science degree in mathematics at Duksung Women's University, in Korea, 1991, and completed her Master of Science degree in mathematics at Seoul National University, in Korea, 1993. From 1998 to 2003 she was a graduate research assistant in the mathematics department at Texas A&M University. She currently works as post-doctoral researcher at Pohang University of Science and Technology. Her permanent mailing address is 90 Bungy Sungbukdong 1 Ga Sungbukgu Seoul, Korea.

The typist for this thesis was Mijoung Kim.