12. FINITE ELEMENTS MODELLING OF MECHANICAL SYSTEMS

Reference, J.M. Reedy, "Introduction to the Finite Element Method," John Wiley.

The finite element method is a piecewise application of the variational methods.

Here we will provide a fundamental introduction to the method and in agreement with our previous studies of mechanical systems and the assumed modes method.

So far what we have learned on mechanical systems (MDOF) is of fundamental nature and independent of the method we choose to solve the problem.

The beauty of the FEM method lies on its simplicity since its formulation is independent of the actual response of the system. That is, little knowledge about what one is expected to get as an answer is required.

Let's review the Assumed Modes Method:

In any mechanical system, we have shown that the Hamiltonian:

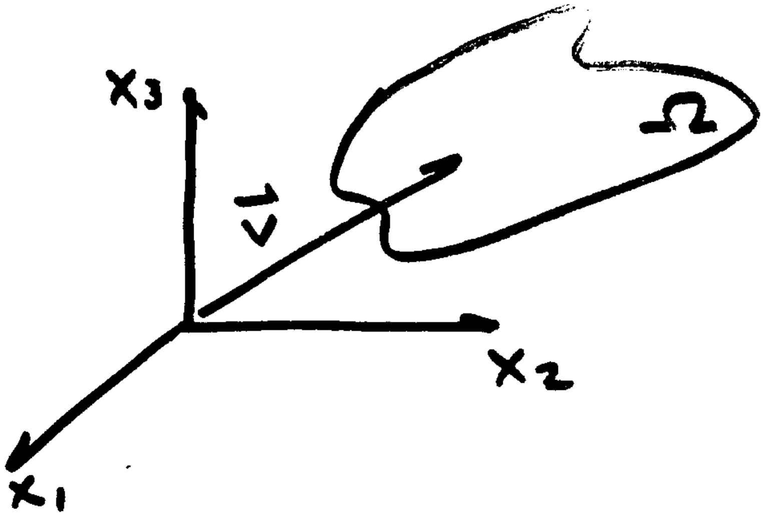
$$\delta \int_{t_1}^{t_2} \left(T - V + W_{ext} \right) dt = 0 \qquad (1)$$

is the fundamental principle of mechanics from which all laws can be derived, i.e. Newton's Laws and/or Lagrangian Mechanics.

We know that in general the kinetic energy and the strain energy of the system are general functions of the displacements and its time derivatives, i.e.

$$\begin{aligned} T &= T\left(\vec{V}, \dot{\vec{V}}, \text{material properties}\right) \\ V &= V\left(\vec{V}, \dot{\vec{V}}, \text{material properties}\right) \end{aligned} \qquad (2)$$

where $(\bullet) = d/dt$ and $\vec{V} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$ are the displacements of a material point in the material domain of interest.



$$\Omega = \Omega(x_i)$$

$$\vec{v} = \vec{v}(x_i, t)$$

In the <u>assumed modes</u> method an <u>approximation</u> to a continuous system was created by letting the displacement function be expressed in the form:

$$\vec{v} = \sum_{i=1}^{N} \psi_i(x_j)\, v_i(t) \qquad (3)$$

where <u>each</u> $\psi_i(\Omega)$ describes a deflected shape of the entire system.

As shown in past classes, substitution of (3) into (1) leads to an N-DOF mathematical model of the mechanical system in the form:

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{Q} \qquad (4)$$

when $\mathbf{M} = \mathbf{M}^T$ and $\mathbf{K} = \mathbf{K}^T$ are the N x N mass and stiffness matrices
Q is the vector of generalized forces.

The coefficients of **M** and K are determined from relations of the shape functions and its derivatives.

$$(\,'\,) = d/dx$$

<u>For example</u>:
for a <u>bar subjected to axial motion</u> we had:

$$m_{ij} = m_{ji} = \int_0^L \rho A\, \psi_i \psi_j\, dx \; ; \quad k_{ij} = k_{ji} = \int_0^L EA\, \psi_i' \psi_j'\, dx \qquad (5.a)$$

while, for a <u>beam subjected to transverse</u> deformations we had:

$$m_{ij} = m_{ji} = \int_0^L \rho A\, \psi_i \psi_j\, dx \; ; \quad k_{ij} = k_{ji} = \int_0^L EI\, \psi_i'' \psi_j''\, dx \qquad (5.b)$$

etc.

The set of $\{\psi_i\}_{i=1}^{N}$ was required to satisfy the following relationships:

$$\left.\begin{array}{l}\psi_i \text{ must be linearly independent} \\ \psi_i \text{ must satisfy the essential B.C.'s}\end{array}\right\} \text{i.e., be an admissible set.}$$

and be sufficiently differentiable.

However, there is a number of problems associated with these requirements:

a) a complex system geometry requires of complex shape function which may be difficult to choose for the unexperienced user.

b) the ψ_i are usually defined over the entire domain and thus lead to highly coupled system of equations.

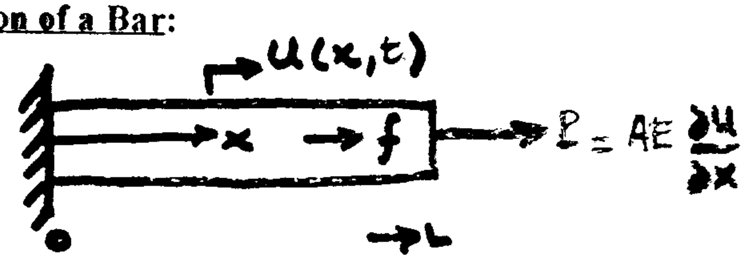
c) the ψ_i functions may be related to a particular problem, and consequently, are difficult to be generalized to other problems.

The FEM overcomes these difficulties and provides a sound basis for the analysis of mechanical systems.

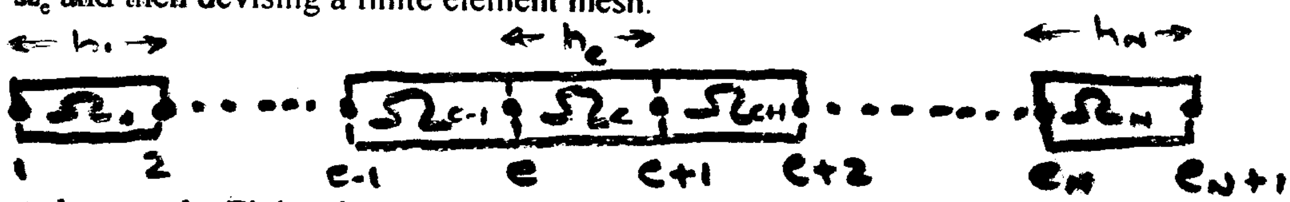
The FEM can be envisioned in the present context as an application of the assumed modes method wherein the shape functions $\{\psi_i\}$ represent deflection over just a portion (finite element) of the structure, with the elements being assembled to form the structural system.

Element matrices for Axial Deformation of a Bar:

Consider as shown in the figure, bar subjected to axial deformation.



The first step in the FEM is to discretize the domain Ω into a series of finite elements Ω_e and then devising a finite element mesh.

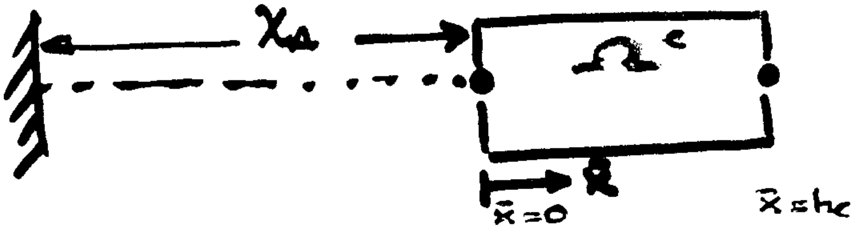


● denotes the Finite element Nodes or joints, i.e., the intersection points.

Do not mistake these nodes with vibration nodes!!

Derivation of finite element equation:

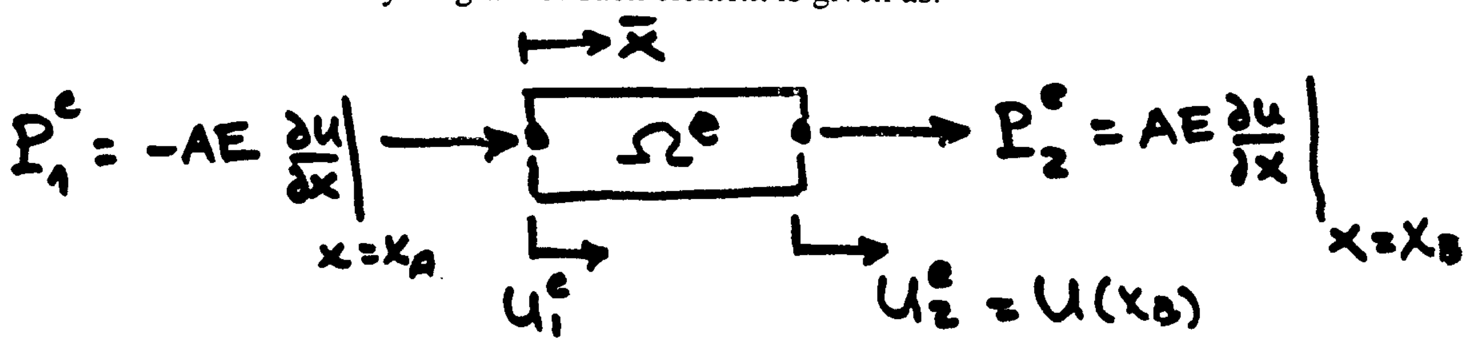
A typical element $\Omega^e = (x_A, x_B)$ is isolated from the mesh.



$$\bar{x} = x - x_A$$

$$h_e = x_B - x_A$$

and the free body diagram of such element is given as:



Where $u_1^{(e)}$ and $u_2^{(e)}$ are the nodal displacements at the tip of the element and $P_1^{(e)}$ and $P_2^{(e)}$ are the nodal axial forces coming from the reaction with the neighboring elements.

The kinetic and potential (strain) energy for the element are given by:

$$T^e = \frac{1}{2}\int_0^{h_e} \rho A \left(\frac{\partial u}{\partial t}\right)^2 d\bar{x} \qquad (6.a)$$

$$V^e = \frac{1}{2}\int_0^{h_e} EA \left(\frac{\partial u}{\partial x}\right)^2 d\bar{x} \qquad (6.b)$$

and the virtual work of external forces over the element is given by:

$$\delta W_{ext} = \int_0^{h_e} f(x)\,\delta u^e\, d\bar{x} + \delta u(x_A)\, P_2^e + \delta u(x_B)\, P_1^e \qquad (7)$$

The Hamiltonian

$$\delta \int_\Omega \{T - V + W_{ext}\}\ dt = 0$$

holds over the entire system i.e., for $x \in [0, L]$, and in particular it is also valid over the element $\Omega^e = (x_A, x_B)$, i.e.

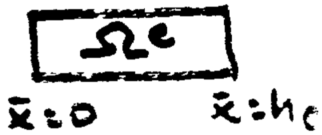
$$\delta \int_\Omega \{T^e - V^e + W^e_{ext}\}\, dt = 0 \qquad (8)$$

Let over the element Ω^e, the displacement **u** be given by

$$u^e(\bar{x},t) = \sum_{i=1}^{2} \psi_i^e(\bar{x})\, u_i^e(t) \qquad (9)$$

where $u_1^e = u_{(XA)}$ and $u_2^e = u_{(XB)}$ are the nodal displacements and $\psi_1^{(e)}$, $\psi_2^{(e)}$ are shape (or approximation) functions that must be admissible for the problem. (Note that here we presume to know $u_1^{(e)}$ and $u_2^{(e)}$).

This choice then leads us to select from (9)



$$u^e = \psi_1^e u_1^e + \psi_2^e u_2^e$$

at $x = x_A$; $\bar{x} = 0$; $u^e(0) = u_1^e = \psi_1^e(0)\, u_1^e + \psi_2^e(0)\, u_2^e$

Then, $\psi_1^e(0) = 1$, $\psi_2^e(0) = 0$ (10)

at $x = x_B$; $\bar{x} = h_e$; $u^e(h_e) = u_2^e = \psi_1^e(h_e)\, u_1^e + \psi_2^e(h_e)\, u_2^e$

Then, $\psi_1^e(h_e) = 0$, $\psi_2^e(h_e) = 1$

(Note) n > 2 in (9) is also possible, although we will not consider the case of higher order elements.

Substitution of (9) into (6) gives:

$$T^e = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} m_{ij}^e\, \dot{u}_i^e\, \dot{u}_j^e \qquad (11.a)$$

$$V = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} k_{ij}^e\, u_i^e\, u_j \qquad (11.b)$$

where $$m_{ij} = m_{ji} = \int_0^{h_e} (\rho A)^e \psi_i^e \psi_j^e \, d\bar{x} \qquad (12.a)$$

$$i,j = 1,2$$

$$K_{ij} = K_{ji} = \int_0^{h_e} (EA) \frac{d\psi_i^e}{d\bar{x}} \frac{d\psi_j^e}{d\bar{x}} \, d\bar{x} \qquad (12.b)$$

are the elements of the mass and stiffness matrices.

Note that from (12.b), the shape functions need to be at least once differentiable <u>over</u> the element Ω^e, i.e.,

$$\psi_i^e \in C^1(\Omega^e) \qquad (13)$$

Substitution of (9) into (7) leads to:

$$\delta W_{ext} = [\int_0^{h_e} f \, \psi_i^e \, d\bar{x}] \, \delta u_i^e + \delta u_1^e \psi_1^e(0) \, P_1^e + \delta u_2^e \psi_2^e(h_e) \, P_2^e$$

$$= \underbrace{[\int_0^{h_e} f \, \psi_i^e \, d\bar{x}]}_{F_i^e} \delta u_i^e + \delta u_1^e \, P_1^e + \delta u_2^e \, P_2^e \qquad (14)$$

where

$$P_1^e = -EA \left.\frac{\partial u}{\partial x}\right|_{\bar{x}=0} \qquad (15)$$

$$P_2^e = +EA \left.\frac{\partial u}{\partial x}\right|_{\bar{x}=h_e}$$

are the nodal forces connecting to the neighboring elements.

Substitution of (11) and (14) into the Hamiltonian eqn. (8) leads to the element system of equations:

$$\sum_{j=1}^{2} m_{ij}^e \ddot{u}_j^e + \sum_{j=1}^{2} K_{ij}^e u_j^e = F_i^e + P_i^e \qquad (16.a)$$

$$M^e \ddot{u}^e + K^e u^e = F^e + P^e \qquad (16.b)$$

where: $$m_{ij}^e = m_{ji}^e = \int_0^{h_e} (\rho A)^e \psi_i^e \psi_j^e \, d\bar{x}$$ (17)

$$k_{ij}^e = k_{ji}^e = \int_0^{h_e} (EA)^e \psi_i'^e \psi_j'^e \, d\bar{x}$$

$F_i^e = \int_0^{h_e} f_x(\bar{x}) \psi_i^e \, d\bar{x}$ components of distributed force vector

$P_1^e \; ; \; P_2^e$ nodal forces.

(16) can be also be written as:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^e \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^e \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\text{primary unknowns}} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}^e + \underbrace{\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^e}_{\text{secondary unknowns.}}$$

At this point, we need to construct the shape functions $\{\psi_i\}_{i=1}^2$.

These need to satisfy:

$\psi_1(0) = 1 \; ; \; \psi_1(h_e) = 0$

$\psi_2(0) = 0 \; ; \; \psi_2(h_e) = 1$ such that they satisfy the geometric or essential conditions.

and $\psi_i \in C^1(\Omega^e)$

We choose

$$\psi_1 = a_1 \bar{x} + b_1 \quad ; \quad \psi_2 = a_2 \bar{x} + b_2 \qquad (18.a)$$

Note that ψ_1 and ψ_2 are linear combinations of the linearly independent complete set $\{1, \bar{x}\}$.

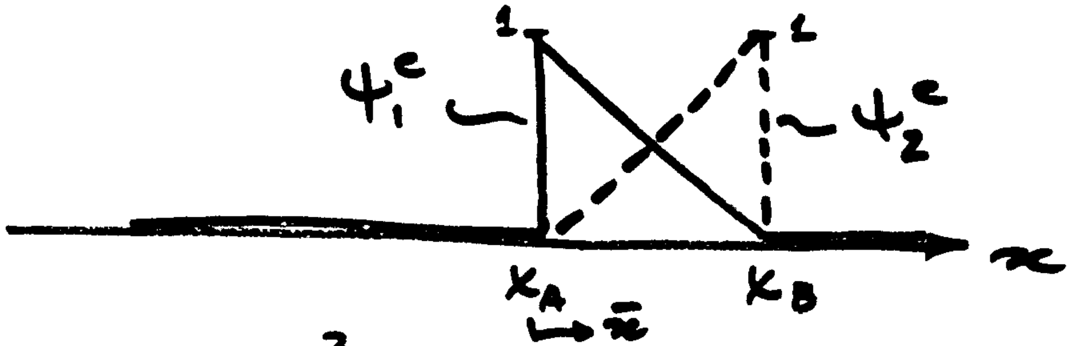
at $\bar{x} = 0$: $\psi_1 = 1 = a_1 0 + b_1 \rightarrow b_1 = 1$

$\psi_2 = 0 = a_2 0 + b_2 \rightarrow b_2 = 0$ (18.b)

at $\bar{x} = h_e$: $\psi_1 = 0 = a_1 h_e + 1 \rightarrow a_1 = -1/h_e$

$\psi_2 = 1 = a_2 h_e + 0 \rightarrow a_2 = 1/h_e$

$$\psi_1^e = 1 - \frac{\bar{x}}{h_e} \quad ; \quad \psi_2^e = \frac{\bar{x}}{h_e} \qquad (18.c)$$



Note that $$\sum_{i=1}^{2} \psi_i^e(\bar{x}) = 1 = \psi_1^e + \psi_2^e$$

are a partition of unity (This means the shape functions ψ_i^e will be able to model rigid body motions).

Note that the set $\{\psi_i^e\}$ is different from zero only on Ω^e and elsewhere is zero. This quality is called a local support and it is extremely important for compact or banded forms in the global matrices (M and K).

The derivation of the shape (or approximation or interpolation) functions ψ_i^e does not depend on the problem. They do depend on the type of element (geometry, number of nodes or joints and number of primary unknowns).

Substitution of the shape functions (18) into the mass and stiffness matrices gives:

$$K^e = \frac{AE}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad F^e = \frac{f_e h_e}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (19)$$

$$M^e = \frac{\rho A h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

here we have considered that the bar geometric and material properties (A,ρ,E) are uniform within the element.

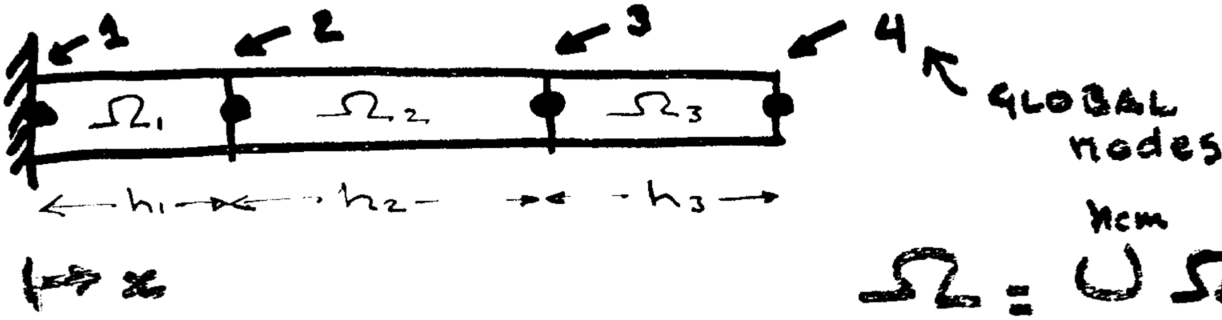
The system of equation for element Ω^e are then:

$$\frac{\rho A h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix}^e + \frac{AE}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^e = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}^e + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^e \quad (20)$$

Note that **[K^e]** is <u>singular</u>.

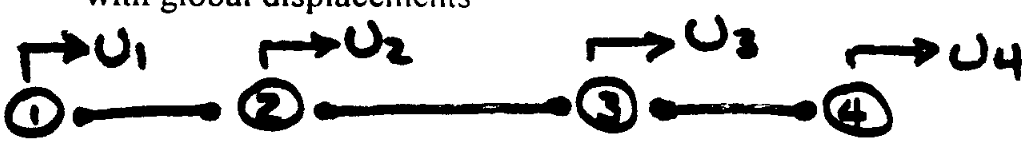
We perform (repeatedly) the same procedure for each element (subdivision) in the domain of interest, and then we must perform the interconnection or assembly of these elements.

Equation (20) has been derived for an arbitrary typical element, and thus, it holds for any element from the FEM mesh. For the sake of discussion, suppose that the domain of the problem $\Omega = (\emptyset, L)$ is divided into 3 elements of possibly unequal lengths, i.e.

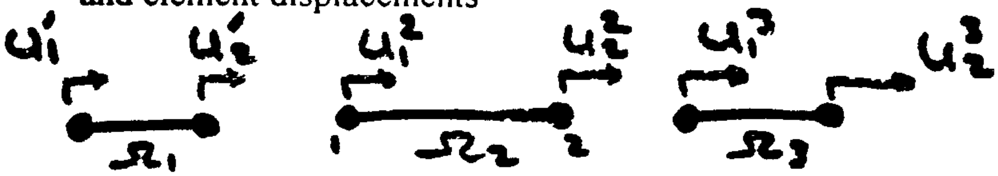


$$\Omega = \bigcup_{e=1}^{nem} \Omega^e$$

with global displacements



and element displacements



The elements are connected at global nodes (2) and (3) and the displacement **u** needs to be continuous (i.e., no cracks, fractures, etc.) at these locations, then we must have element (local) $\qquad\qquad$ global

that $$u_2^1 = u_1^2 = U_2$$
$$u_2^2 = u_1^3 = U_3 \qquad (21)$$

and also $$u_1^1 = U_1 .$$
$$u_2^3 = U_4$$

These are called the interelement continuity conditions. (Also derivatives d/dx)

The correspondence between local nodes and the global nodes can be expressed in the form of an array called the **BOOLEAN** or **CONNECTIVITY** ARRAY or MATRIX:

$b_{ij} \equiv$ the global node number corresponding to the j-th node of element I

$I = 1,2 \ldots N$ number elements on mesh.
$j = 1,2 \ldots N_n$ number of nodes per element.

In this case, we have that $B = \{b_{ij}\}$ is given by:

$$B = \begin{array}{cl} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} & \begin{array}{l} \text{first element} \\ \text{second element} \\ \text{third element} \end{array} \\ \begin{array}{cc} \text{local node \#1} & \text{local node \#2} \end{array} & \end{array}$$

Repetition of a number in B indicates that the coefficients of $[K^e]$ and $[M^e]$ associated with the number add up.

In a computer implementation of the FEM scheme, the connectivity array is used extensively for automatic assembly of the global system of equations.

For the 3 elements in question, the element eqns. (20) are written in global coordinate or nodes as:

<u>1st element</u>

$$(22.1):\quad \left(\frac{\rho A h}{6}\right)_1 \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{bmatrix} + \left(\frac{AE}{h}\right)_1 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} F_1^1 + P_1^1 \\ F_2^1 + P_2^1 \\ 0 \\ 0 \end{bmatrix}$$

<u>2nd element</u>

$$(22.2):\quad \left(\frac{\rho A h}{6}\right)_2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{bmatrix} + \left(\frac{AE}{h}\right)_2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ F_1^2 + P_1^2 \\ F_2^2 + P_2^2 \\ 0 \end{bmatrix}$$

<u>3rd element</u>:

$$(22.3):\quad \left(\frac{\rho A h}{6}\right)_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{bmatrix} + \left(\frac{AE}{h}\right)_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_1^3 + P_1^3 \\ F_2^3 + P_2^3 \end{bmatrix}$$

Equation (22) indicates contributions of each element to the <u>overall</u> problem. The equation of the global system are obtained by superposition (addition) of eqns. (22), i.e.

$$\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{bmatrix} + \qquad (23.1)$$

$$\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} F_1^1 + P_1^1 \\ F_2^1 + P_2^1 + F_1^2 + P_1^2 \\ \\ \\ \end{bmatrix}$$

$$M_g \ddot{U} + K_g U = F_g + P_g \qquad (23.2)$$

$$M_g = \bigcup_e m^e$$

$$K_g = \bigcup_e k^e \qquad (24)$$

$$F_g = \bigcup_e F^e$$ → global vector of distributed forces

$$P_g = \bigcup_e P^e$$ → global vector of <u>nodal</u> forces.

Note that the system of eqns. (23.1) is <u>tridiagonal</u> and thus, its solution can be attained very easily.

Imposition of Boundary Conditions

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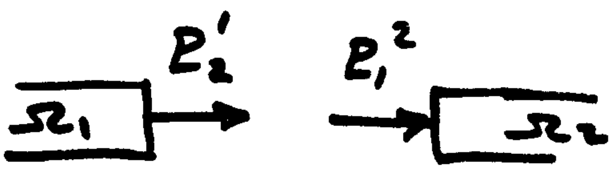
2) <u>Internal Nodal Forces</u>

In general, due to continuity (action = reaction) we must have in eqn.. (23.1):

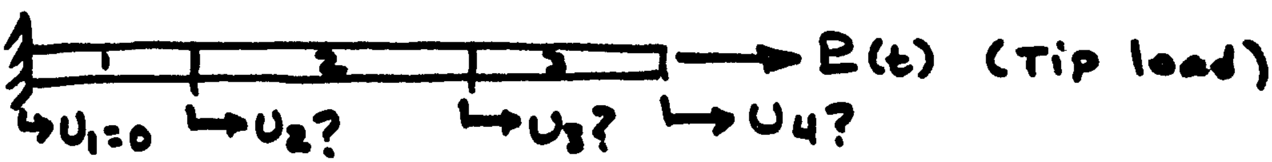
$$P_2^1 + P_1^2 = 0 \qquad (24.a)$$



unless nodal point forces are specified, i.e. then

$$P_2^1 + P_1^2 = \text{specified value of nodal force (concentrated)} \qquad (24.b)$$

for the problem under consideration we have that:



and no distributed force, so then $F_g = 0$ and (23.2) comes to be:

$$M_G \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{bmatrix} + K_G \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} P_1^1 \\ 0 \\ 0 \\ P_2^2 = P(t) \end{bmatrix} \qquad (25)$$

Note that the essential B.C.'s specified is $U_1 = \varnothing$ while the reaction force P_1^1 is unknown. At this moment (25) can not be resolved since K_G is singular.

The global finite element equation can be partitioned conveniently in the following form:

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{U}_a \\ \ddot{U}_d \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} U_a \\ U_d \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad (26)$$

Where the vector $\{U_a\}$ contains the <u>active</u> degrees of freedom while the $\{U_d\}$ vector contains the <u>specified</u> (constant) displacements. Note that $\{\ddot{U}_d\} = \varnothing$ and <u>also</u>

$K_{ab} = K_{ba}$; the partitioned global matrices are symmetric

$M_{ab} = M_{ba}$; F_1 is the global vector of <u>known applied</u> external forces.

F_2 is the global vector of <u>unknown</u> reaction forces.

(26) can be written as:

$$M_{11} \ddot{U}_a + K_{11} U_a + K_{12} U_d = F_1 \qquad (a)$$
$$M_{21} \ddot{U}_a + K_{21} U_a + K_{22} U_d = F_2 \qquad (b) \qquad (27)$$

or

$$M_{11} \ddot{U}_a + K_{11} U_a = F_1 - K_{12} U_d \qquad (28.a)$$

Once (28.a) is solved and the <u>active</u> displacements are known, i.e., <u>$\{U_a(t)\}$</u> is found, from (27.b) we can calculate the unknown internal forces as:

$$F_2 = M_{21} \ddot{U}_a + K_{21} U_a + K_{22} U_d \qquad (28.b)$$

For the example problem discussed we know that the essential B.C. is $U_1 = \emptyset$ while $P_1^{(1)}$ is unknown and P(t) is specified.

Then from eqns. (23.1) we have to solve the 3 DOF systems:

$$\begin{bmatrix} m_{22} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{bmatrix} + \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P(t) \end{bmatrix} \quad (29.a)$$

$$M_{11} \ddot{U}_a + K_{11} U_a = F_1 \quad (29.b)$$

where

$$U_a = [U_2 \ U_3 \ U_4]^T ; \quad F_1 = [0, 0, P]^T$$

$$U_d = [U_1 = 0]^T \qquad F_2 = [P_1^1]^T \quad (30)$$

and once (29) is solved:

from first eqn..: $M_{11}^1 \ddot{U}_1 + M_{12}^1 \ddot{U}_2 + K_{11}^1 U_1 + K_{12}^1 U_2 = P_1^1$

is obtained. $\rightarrow$ $P_1^1 = M_{12}^1 \ddot{U}_2 + K_{12}^1 U_2$ (31)

In the example problem, the satisfaction of the essential constant $U_1 = \emptyset$ removes the singularity of the stiffness matrix (i.e. removes the rigid body **mode**).

For the example case, considering elements of equal length, i.e. $h_e = L/3$, the eqns. (29) are written as:

$$\frac{\rho A L}{18} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{bmatrix} + \frac{3AE}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P(t) \end{bmatrix} \quad (32)$$

Now eqn.. (32) can be solved in the usual way, i.e. we can write (32) as:

$$M \ddot{U} + K U = P(t) \quad (33)$$

+<u>Initial Conditions</u>.

Solve the homogeneous form of (33)

$$\left[\mathbf{K} - \omega_i^2 \mathbf{M}\right] {}^i\phi = 0$$

Note:
here N denotes the active degrees of freedom in the system!

obtain modal matrix $\Phi = \left[{}^1\phi \ldots\ldots {}^N\phi \right]$

Transform (33) to modal coordinates with:

$$U(t) = \Phi\, \eta(t)$$

as:

$$\mathcal{M} \ddot{\eta} + \mathcal{K} \eta = \mathcal{P}$$

where

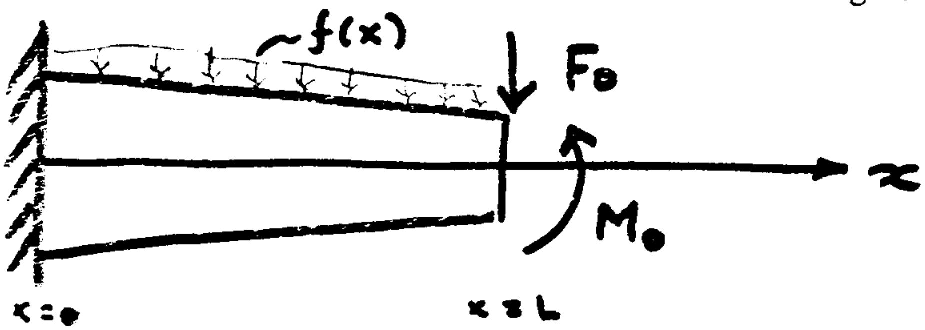
$$\left.\begin{aligned} \mathcal{M} &= \Phi^T M \Phi = \\ \mathcal{K} &= \Phi^T K \Phi = \end{aligned}\right\} \text{diagonal}$$

$$\mathcal{P} = \Phi^T$$

and obtain the solution = $\eta(t)$
and back to physical coordinates.

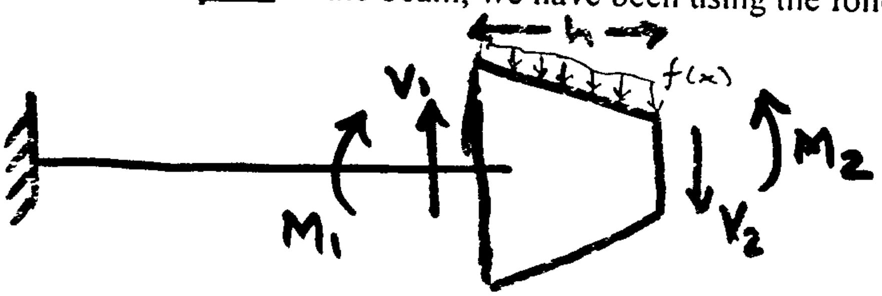
Finite Element Modeling for Bending of Elastic Beams

Consider the example cantilever beam shown in the figure



material properties ρ, E
geometry A, I

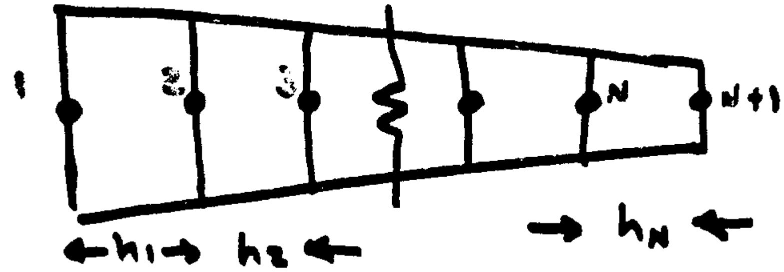
and for a piece of the beam, we have been using the following sign convention



$$M = EI \frac{\partial^2 v}{\partial x^2}$$

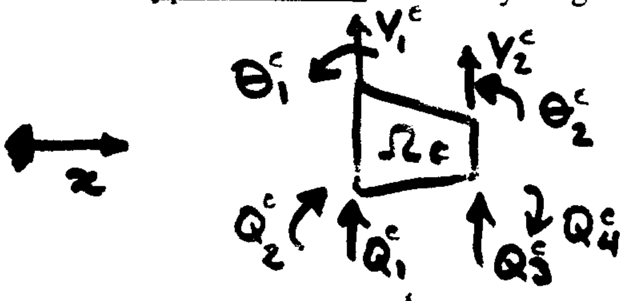
$$V = \frac{\partial M}{\partial x}$$

In the finite element method, we discretize the beam into a series of finite elements Ω^e



$$\Omega = \bigcup_e \Omega_e$$

and a typical element free body diagram is given as:



→ primary variable

→ secondary variable

where $\quad \overset{\leftarrow h_e \rightarrow}{x_e \qquad x_{e+1}}$

$$\theta = \partial v / \partial x \quad (36.a)$$

$$Q_1 = \frac{\partial}{\partial x}\left[EI\frac{\partial^2 v}{\partial x^2}\right]_{x_e}$$

$$Q_3 = -\frac{\partial}{\partial x}\left[EI\frac{\partial^2 v}{\partial x^2}\right]_{x_{e+1}}$$

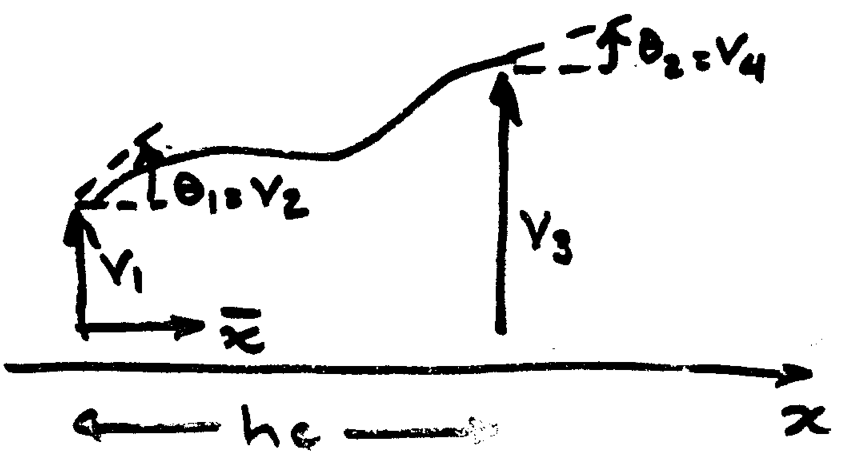
Shear force at ends of beam. (36.b)

$$Q_2 = \left[EI\frac{\partial^2 v}{\partial x^2}\right]_{x_e}$$

$$Q_4 = -\left[EI\frac{\partial^2 v}{\partial x^2}\right]_{x_{e+1}}$$

Bending moments at ends of beam. (36.c)

$$x_{e+1} = x_e + h_e$$



$$V_2 = \theta_1 , V_4 = \theta_2$$

$$V_1 = v(x_e, t)$$

$$V_3 = v(x_{e+1}, t)$$

$$V_2 = \theta(x_e, t)$$

$$V_4 = \theta(x_{e+1}, t)$$

$$\bar{x} = x - x_e$$

We assume that the displacement $v^e(x,t)$ and rotation $\theta^e(x,t)$ on the finite element Ω^e is given by the <u>approximation</u>.

$$v^e(x,t) = \sum_{i=1}^{4} \psi_i^e(\bar{x})\, v_i^e(t)$$

$$\theta^e(x,t) = \sum_{i=1}^{4} \frac{d\psi_i^e}{d\bar{x}}\, v_i^e(t) \qquad (37)$$

From our discussion for the assumed modes method, we know that the **elements of the** element mass and stiffness matrices are given by:

$$m_{ij}^e = m_{ji}^e = \int_0^{h_e} \rho A\, \psi_i^e \psi_j^e \, d\bar{x} \qquad (38.a)$$

$$i,j = 1,\ldots 4$$

$$k_{ij}^e = k_{ji}^e = \int_0^{h_e} EI\, \frac{d^2\psi_i^e}{d\bar{x}^2}\, \frac{d^2\psi_j^e}{d\bar{x}^2}\, d\bar{x} \qquad (38.b)$$

Note here that
$$\{\psi_i^e\}_{i=1}^4 \in C^2(\Omega^e)$$

As before, we require that

since

$$\psi_1(0) = 1\ ;\quad \psi_2 = \psi_3 = \psi_4 = 0 \ \text{at}\ \bar{x} = 0 \rightarrow v(x_e) = V_1$$

$$\psi_3(h_e) = 1;\quad \psi_1 = \psi_2 = \psi_4 = 0 \ \text{at}\ \bar{x} = h_e \rightarrow v(x_{e+1}) = V_3$$

$$\psi_2'(0) = 1;\quad \psi_1' = \psi_3' = \psi_4' = 0 \ \text{at}\ \bar{x} = 0 \rightarrow \theta(x_e) = V_2 \qquad (39)$$

$$\psi_4'(h_e) = 0;\quad \psi_1' = \psi_2' = \psi_3' = 0 \ \text{at}\ \bar{x} = h_e \rightarrow \theta(x_{e+1}) = V_4$$

for each ψ_i^e we have to satisfy four conditions and the ψ_i^e need to be at least twice differentiable. Equation (39) guarantees that they will be linearly independent and satisfying the geometric (essential) constraints of the problem. The lowest order polynomial with four constants and C^2 is of the form:

$$v(\bar{x},t) = c_1 + c_2 \frac{\bar{x}}{h_e} + c_3\left(\frac{\bar{x}}{h_e}\right)^2 + c_4\left(\frac{\bar{x}}{h_e}\right)^3 \qquad (40)$$

$$\theta = \frac{\partial v}{\partial x} = \frac{c_2}{h_e} + \frac{2c_3\bar{x}}{h_e^2} + \frac{3c_4\bar{x}^2}{h_e^3} \qquad (41)$$

so at the extremes (boundary) of the element:

$$\text{at } \bar{x}=0 \quad V_1 = C_1$$
$$V_2 = \theta_1 = C_2/h_e$$
$$\text{at } \bar{x}=h_e \quad V_3 = C_1 + C_2 + C_3 + C_4$$
$$V_4 = \theta_2 = \frac{C_2}{h_e} + \frac{2C_3}{h_e} + \frac{3C_4}{h_e}$$

or

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/h_e & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1/h_e & 2/h_e & 3/h_e \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \qquad (42)$$

inverting (42) and writing the $\{C_i\}$'s in terms of the $\{V_i\}$'s, <u>we obtain</u>

$$v^e(\bar{x},t) = \sum_{i=1}^{4} \psi_i^e \cdot \nu_i(t) \qquad (37)$$

where

$$\psi_1^e = 1 - 3(\bar{x}/h_e)^2 + 2(\bar{x}/h_e)^3$$
$$\psi_2^e = \bar{x}\,(1 - \bar{x}/h_e)^2 \qquad (43)$$
$$\psi_3^e = 3(\bar{x}/h_e)^2 - 2(\bar{x}/h_e)^3$$
$$\psi_4^e = \bar{x}\,(\bar{x}/h_e)(\bar{x}/h_e - 1)$$

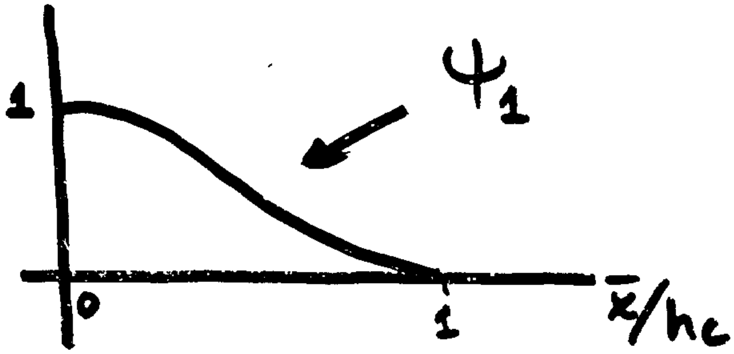
and

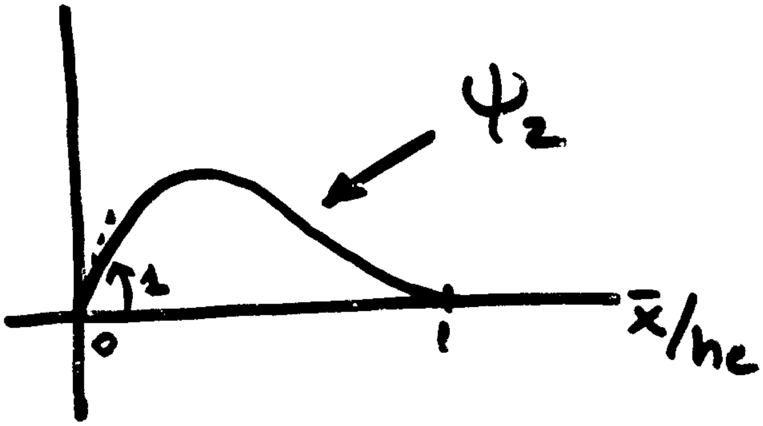
$$\psi_1' = -6\bar{x}/h_e^2 + 6\bar{x}^2/h_e^3$$
$$\psi_2' = 1 - 4\bar{x}/h_e + 3\bar{x}^2/h_e^2 \qquad (44)$$
$$\psi_3' = 6\bar{x}/h_e^2 - 6\bar{x}^2/h_e^3 = -\psi_1'$$
$$\psi_4' = 3\bar{x}^2/h_e^2 - 2\bar{x}/h_e$$

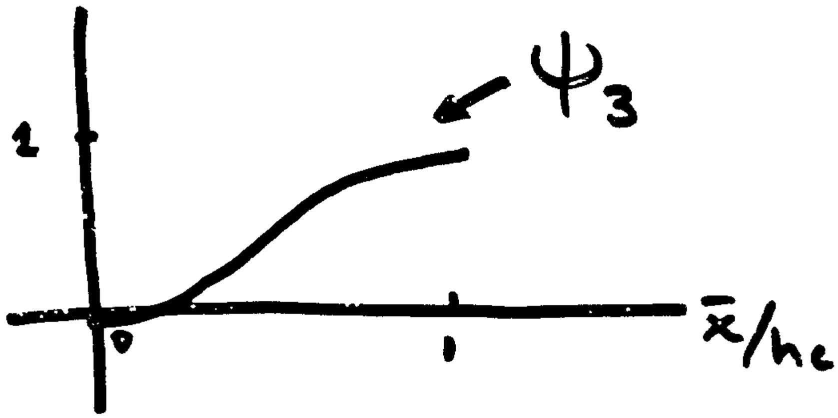
and

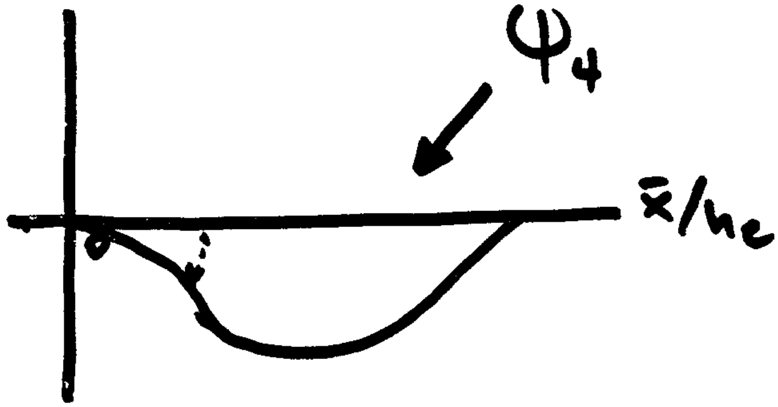
$$\psi_1'' = -6/h_e^2 + 12\bar{x}/h_e^3 = -\psi_3''$$

$$\psi_2'' = -4/h_e + 6\bar{x}/h_e^2 \; ; \quad \psi_4'' = 6\bar{x}/h_e^2 - 2/h_e$$


ψ_1
1
0
1
$\bar{x}/h_c$


ψ_2
1
0
1
$\bar{x}/h_c$


ψ_3
1
0
1
$\bar{x}/h_c$


ψ_4
0
$\bar{x}/h_c$

Note that the shape functions (38) satisfy identically (39) and are linearly independent.

Substitution of (43) and (45) into (38.a) and (38.b), respectively, renders the following <u>element</u> mass and stiffness matrices.

$$K^e = \frac{EI}{h_e^3}\begin{bmatrix} 12 & 6h_e & -12 & 6h_e \\ & 4h_e^2 & -6h_e & 2h_e^2 \\ \text{sym} & & 12 & -6h_e \\ & & & 4h_e^2 \end{bmatrix} \qquad (46)$$

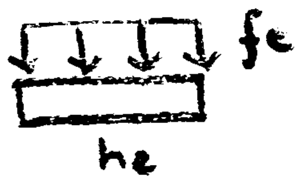
$$M^e = \frac{\rho A h_e}{420}\begin{bmatrix} 156 & 22h_e & 54 & -13h_e \\ & 4h_e^2 & 13h_e & -3h_e^2 \\ \text{SYM} & & 156 & -22h_e \\ & & & 4h_e^2 \end{bmatrix} \qquad (47)$$

and the vector of generalized forces $\{F_i^e\}$ is given by:

$$F_i^e = \int_0^{h_e} f(\bar{x},t)\,\psi_i^e(\bar{x})\,d\bar{x} \qquad (48.a)$$

assuming a constant distributed force $f(x,t)$ over the element we obtain:

$$F^e = \left\{ \frac{f_e h_e}{2}, \frac{f_e h_e^2}{12}, \frac{f_e h_e}{2}, -\frac{f_e h_e^2}{12} \right\}^T \qquad (48.b)$$



and the constraint nodal forces are obtained from

$$\delta W = Q_1^e \delta v_1 + Q_3^e \delta v_3 - Q_2^e \delta v_2 - Q_4^e \delta v_4$$

or

$$Q^e = \{Q_1, -Q_2, Q_3, -Q_4\}^T \qquad (49)$$

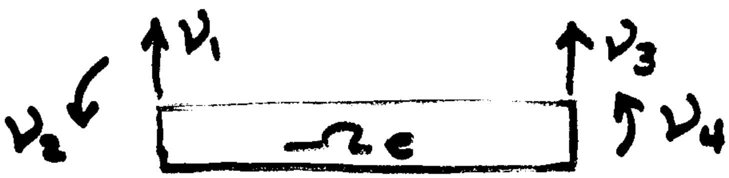
Note that the <u>distributed force</u> is equal or equivalent to the nodal shear forces $\left(\frac{f_e h_e}{2}\right)$

at the ends.

Then, the system of equations for the element Ω^e are given as:

$$\mathbf{M}^e \ddot{\mathbf{\nu}}^e + \mathbf{K}^e \mathbf{\nu}^e = \mathbf{F}^e + \mathbf{Q}^e \qquad (50)$$

where $\mathbf{\nu} = [\nu_1, \nu_2, \nu_3, \nu_4]^T$



Assembly of element matrices to produce the global system of equations is easily done as exemplified before. Here, we need to be careful to keep continuity of displacements at nodal (joints) and also the continuity of constraint forces at the joints.

The global system of equations will then be given as:

$$\mathbf{M}_G \ddot{\mathbf{\nu}}_G + \mathbf{K}_G \mathbf{\nu}_G = \mathbf{F}_G + \mathbf{Q}_G \qquad (51)$$

where the sub-index G means global, and

$$\mathbf{M}_G = \bigcup \mathbf{M}^e \qquad \text{global FEM matrix}$$

$$\mathbf{K}_G = \bigcup \mathbf{K}^e \qquad \text{global FEM stiffness matrix} \qquad (52)$$

$$\mathbf{F}_G = \bigcup \mathbf{F}^e \qquad \text{global FEM distributed lateral force vector}$$

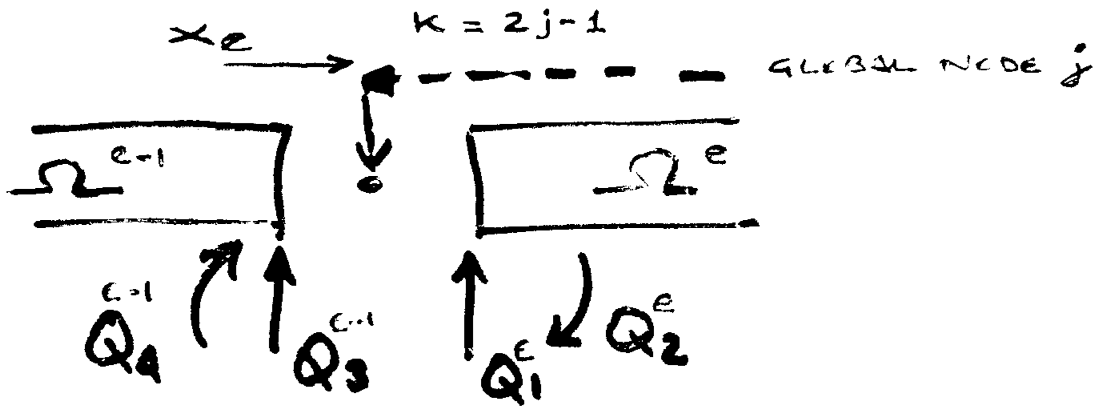
and $$\mathbf{Q}_G = \bigcup_{e=1}^{Nem} \mathbf{Q}^e \qquad \text{global FEM vector of constraint force}$$

$$V_G = [\, v_1,\ \underset{\theta_1}{\underset{\downarrow}{v_2}},\ v_3,\ \underset{\theta_2}{\underset{\downarrow}{v_4}} \ldots\ldots \,]^T$$

is the vector of generalized nodal displacements

The global FEM vector of constraint force Q_G will generally have zero-components at internal nodes. The elements of this vector are of the form:

$$Q_{G_K} = Q_3^{e-1} + Q_1^{e} \quad ; \quad Q_{G_{K+1}} = Q_4^{e-1} + Q_2^{e} \qquad (53)$$



Equations (53) constitute a statement of equilibrium of forces at the nodal interface (boundary) of the element.

Recall from equations (36.b) and (36.c) that

$$Q_3^{e-1} + Q_1^{e} = -\frac{\partial}{\partial x}\left[EI\frac{\partial^2 v}{\partial x^2}\right]\Bigg|_{x=x_e} + \frac{\partial}{\partial x}\left[EI\frac{\partial^2 v}{\partial x^2}\right]\Bigg|_{x=x_e} = 0$$

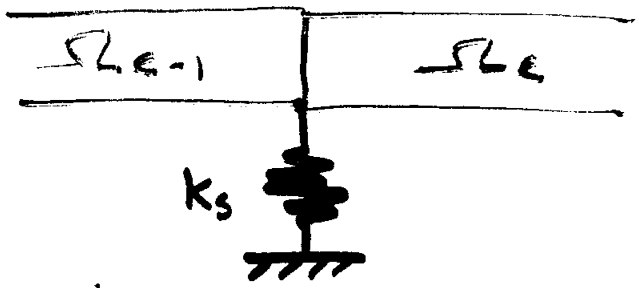
$$Q_4^{e-1} + Q_2^{e} = -\left[EI\frac{\partial^2 v}{\partial x^2}\right]\Bigg|_{x=x_e} + \left[EI\frac{\partial^2 v}{\partial x^2}\right]\Bigg|_{x=x_e} = 0$$

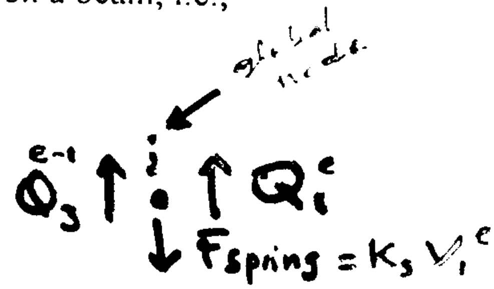
if no <u>external nodal</u> forces or moments are applied.

Thus, then $\quad Q_{G_K} = Q_{G_{K+1}} = 0$

However, if an external nodal shear force or moment is applied at the joint, then the components of the force vector Q_G will be non-zero.

Consider, for example, the case of an intermediate spring on a beam; i.e.,





we need then to have

$$Q_3^{e-1} + Q_1^e + F_{spring} = 0$$

$\underline{k = 2j - 1}$

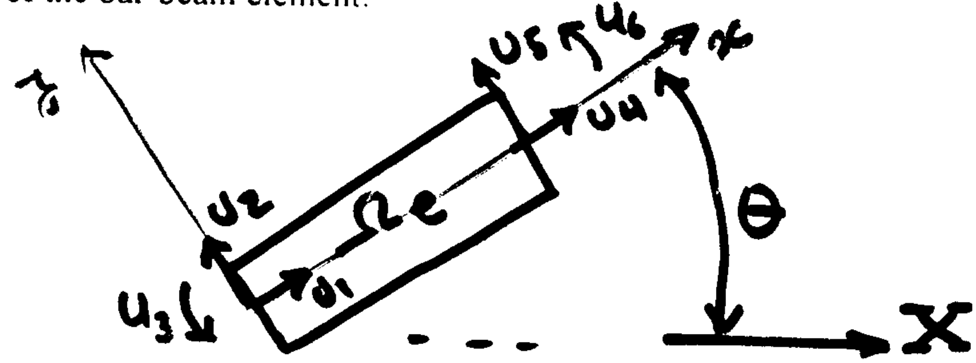
so, $Q_3^{e-1} + Q_1^e = -F_{spring} = -k_s v_1^e$ <u>but</u> $V_K = v_1^e = v_3^{e-1}$

↑global displacement

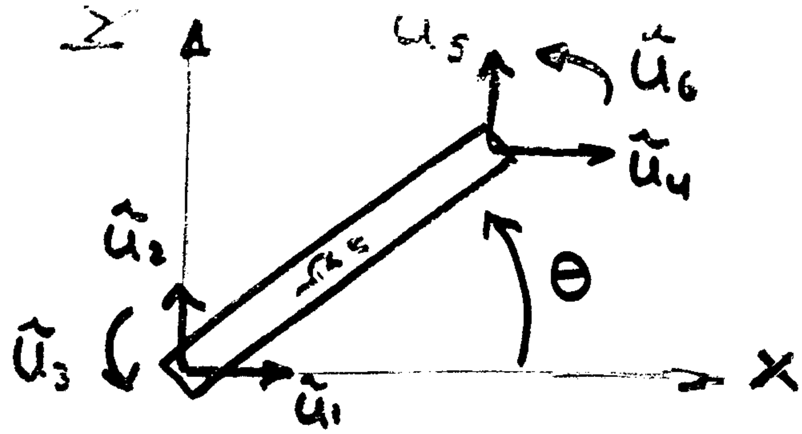
Note that this restoring force depends on the element displacement, and, therefore, it is unknown. At this point, you need to modify the global stiffness matrix and add the contribution of the <u>support stiffness k_s</u>.

<u>Transformation of Element Matrices:</u>

A plane frame element has three displacement coordinates at each end (two orthogonal displacements and one rotation). As shown in the figure, this element has a local coordinate system (x,y) where the x-coordinate is aligned with the major axis (length) of the bar-beam element.



The x-axis is tilted θ degrees relative to a <u>global</u> (inertial) coordinate system (X, Y) to which all elements in the structure will be related. In the (X,Y) coordinate system, the displacements are given as:



The transformation between the $\tilde{U}_i$ displacements in (X,Y) to the U_i displacements in the local coordinate system is given by the transformation equation

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_4 \\ \hat{u}_5 \\ \hat{u}_6 \end{bmatrix} \qquad \boxed{u^e = T_e \hat{u}^e} \quad (1)$$

θ is characteristic for each element, i.e. $\theta = \theta^e$

The <u>element</u> equations in the (x,y) coordinate system are given as

$$M^e \ddot{u}^e + K^e u^e = F^e + Q^e \qquad (2)$$

where $M^e = M^T$ element mass matrix

$K^e = K^T$ element stiffness matrix

(3) F^e element distributed forces (load) vector

Q^e element vector of nodal constraint force (loads)

Substitution of (1) into (2) and premultiplication by T_e^T gives the following equation:

$$T^T M T \hat{u} + T^T K T \hat{u} = T^T F + T^T Q \qquad (4)$$

and letting

$$\hat{M}^e = T_e^T M^e T_e$$

$$\hat{K}^e = T_e^T K^e T_e$$

mass and stiffness matrices relative to the (X,Y) coordinate system

(5)

$$\hat{F}^e = T_e^T F^e$$

$$\hat{Q}^e = T_e^T Q^e$$

force vectors relative to (X,Y) coordinate system

we write equation (4) as:

$$\hat{M}^e \ddot{\hat{u}}^e + \hat{K}^e \hat{u}^e = \hat{F}^e + \hat{Q}^e \qquad (6)$$

Note that $\hat{M}^e$ and $\hat{K}^e$ are still symmetric matrices, i.e.

$$\hat{M}^T = (T^T M T)^T = T^T (T^T M)^T$$

$$= T^T M^T T \underset{\substack{\uparrow \\ \text{since } M^T = M}}{=} T^T M T = \hat{M}$$

The assembly of the element matrices proceeds in the usual way to obtain the global system of equations:

$$\hat{M}_G \ddot{\hat{U}}_G + \hat{K}_G U_G = \hat{F}_G + \hat{Q}_G \qquad (7)$$

where

$$\hat{M}_G = \bigcup_e \hat{M}^e \qquad \hat{K}_G = \bigcup_e \hat{K}^e \qquad (8)$$

etc....

Constraints. Reduction of Degrees of Freedom:

So far, we have assumed that all generalized displacements are independent of each other. This assumption has lead to the system of equations:

$$M\ddot{U} + KU = F + Q \qquad (9)$$

where I have omitted the (G) subindex and the superindex (^) for simplicity.

Frequently, there arises a need for specifying relationships among system displacement coordinates. This is equivalent to specify a number $\hat{N}$ of displacements which are L.I. and the rest $\hat{N}+1$, $\hat{N}+2, \ldots$ depend on the $\hat{N}$ displacements. The discussion is now related to constraint equation of the form:

$$f_i(u_{\hat{N}+1}, u_{\hat{N}+2}, \ldots u_N) - g_i(u_1, u_2, \ldots u_{\hat{N}}) = 0 \qquad i = \hat{N}+1, \ldots N \qquad (10)$$

and where $\hat{N} < N$

equation (10) can be written in matrix form as:

$$RU = \begin{bmatrix} R_{da} & R_{dd} \end{bmatrix} \begin{bmatrix} U_a \\ U_d \end{bmatrix} \qquad (11)$$

where U_d is the vector of N_d dependent coordinates, and
U_a is the vector of $\hat{N} = N_a$ independent or ACTIVE coordinates

and such that $\hat{N} + N_d = N_a + N_d = N$

from (11) we can find the dependent coordinates from:

$$R_{da}U_a + R_{dd}U_d = 0 \qquad (12)$$

$$U_d = -R_{dd}^{-1} R_{da} U_a = T_{da} U_a$$

where T_{da} is the N_d x N_a matrix transformation between active to dependent degrees of freedom. Now, the total global displacement vector can be written as:

$$U = [U_{1a}, U_{2a}, \dots U_{Na}, U_{1d}, U_{2d}, \dots U_{Nd}]^T$$

$$U = \begin{bmatrix} U_a \\ U_d \end{bmatrix} = \begin{bmatrix} \underset{Na \times Na}{I_{aa}} \\ \underset{Nd \times Na}{T_{da}} \end{bmatrix} U_a = \underset{N \times Na}{T}\ U_a \qquad (13)$$

where $\mathbf{I}_{aa}$ is the Na x Na unitary matrix.

Substitution of (13) into (9) and premultiplying by T^T gives:

$$\underbrace{T^T M T}_{M_a}\ \ddot{U}_a + \underbrace{T^T K T}_{K_a}\ U_a = T^T F + T^T Q \qquad (14)$$

$$M_a \ddot{U}_a + K_a U_a = F_a + Q_a$$

where the active mass and stiffness matrices are reduced to be square (N_a x N_a) and the system of equation (14) only accounts for the active degrees of freedom.