Handout 8

Modal Analysis of MDOF Systems with Proportional Damping

The governing equations of motion for a *n*-DOF linear mechanical system with viscous damping are:

$$\mathbf{M}\ddot{\mathbf{U}}+\mathbf{D}\dot{\mathbf{U}}+\mathbf{K}\mathbf{U}_{(t)}=\mathbf{F}_{(t)} \tag{1}$$

where $\mathbf{U},\dot{\mathbf{U}},$ and $\ddot{\mathbf{U}}$ are the vectors of generalized displacement, velocity and acceleration, respectively; and $\mathbf{F}_{(t)}$ is the vector of generalized (external forces) acting on the system.
$\mathbf{M},\mathbf{D},\mathbf{K}$ represent the matrices of inertia, viscous damping and stiffness coefficients, respectively[1].

The solution of Eq. (1) is uniquely determined once initial conditions are specified. That is,

$$\text{at } t=0 \;\rightarrow \mathbf{U}_{(0)}=\mathbf{U}_o,\;\; \dot{\mathbf{U}}_{(0)}=\dot{\mathbf{U}}_o \tag{2}$$

Consider the case in which the damping matrix **D** is of the form

$$\mathbf{D}=\alpha\,\mathbf{M}+\beta\,\mathbf{K} \tag{3}$$

where α, β are constants[2], usually empirical. This type of damping is known as PROPORTIONAL, i.e proportional to either the mass **M** of the system, or the stiffness **K** of the system, or both.

[1] The matrices are square with *n*-rows = *n* columns, while the vectors are n-rows.
[2] These constants have physical units, α is given in [1/sec] and β in [sec]

Proportional damping is rather unique, since only one or two parameters, α, β, appear to fully describe the complexity of damping, irrespective of the system number of DOFs, *n*. This is clearly not realistic. Hence, proportional damping is not a rule but rather the exception.

Nonetheless the approximation of proportional damping is useful since, most times damping is quite an elusive phenomenon, i.e. difficult to model (predict) and hard to measure but for a few DOFs.

Next, consider one already has found the natural frequencies and natural modes (eigenvectors) for the UNDAMPED case, i.e. given $\mathbf{M}\ddot{\mathbf{U}}+\mathbf{K}\,\mathbf{U}=\mathbf{0}$,

$$\left\{\omega_i, \boldsymbol{\varphi}_{(i)}\right\}_{i=1,2\ldots n} \text{ satisfying } \left[-\mathbf{M}\,\omega_i^2+\mathbf{K}\right]\boldsymbol{\varphi}_{(i)}=\mathbf{0},\ {}_{i=1,\ldots n}. \qquad (4)$$

$$\text{with properties } \boldsymbol{\Phi}^T\mathbf{M}\,\boldsymbol{\Phi}=\left[M\right];\quad \boldsymbol{\Phi}^T\mathbf{K}\,\boldsymbol{\Phi}=\left[K\right] \qquad (5)$$

As in the undamped modal analysis, consider the **modal transformation**

$$\mathbf{U}_{(t)}=\boldsymbol{\Phi}\,\mathbf{q}_{(t)} \qquad (6)$$

And with $\dot{\mathbf{U}}_{(t)}=\boldsymbol{\Phi}\,\dot{\mathbf{q}}_{(t)}$; $\ddot{\mathbf{U}}_{(t)}=\boldsymbol{\Phi}\,\ddot{\mathbf{q}}_{(t)}$, then EOM (1) becomes:

$$\mathbf{M}\boldsymbol{\Phi}\ddot{\mathbf{q}}+\mathbf{D}\boldsymbol{\Phi}\dot{\mathbf{q}}+\mathbf{K}\boldsymbol{\Phi}\mathbf{q}=\mathbf{F}_{(t)} \qquad (7)$$

which offers no advantage in the analysis. However, premultiply the equation above by $\boldsymbol{\Phi}^T$ to obtain

$$\left(\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi}\right)\ddot{\mathbf{q}}+\left(\boldsymbol{\Phi}^T\mathbf{D}\boldsymbol{\Phi}\right)\dot{\mathbf{q}}+\left(\boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi}\right)\mathbf{q}=\boldsymbol{\Phi}^T\mathbf{F}_{(t)} \qquad (8)$$

And using the modal properties, Eq. (5), and

$$\mathbf{\Phi}^T \mathbf{D} \mathbf{\Phi} = \mathbf{\Phi}^T \left(\alpha \mathbf{M} + \beta \mathbf{K} \right) \mathbf{\Phi} = \alpha \, \mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} + \beta \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi}$$

$$\mathbf{\Phi}^T \mathbf{D} \mathbf{\Phi} = \alpha \left[M \right] + \beta \left[K \right] \rightarrow \left[D \right] \tag{9}$$

i.e. a diagonal matrix known as **proportional modal damping**. Then Eq. (7) becomes

$$\left[M \right] \ddot{\mathbf{q}} + \left[D \right] \dot{\mathbf{q}} + \left[K \right] \mathbf{q} = \mathbf{Q} = \mathbf{\Phi}^T \mathbf{F}_{(t)} \tag{10}$$

Thus, the equations of motion are **uncoupled in modal space**, since [M], [D], and [K] are diagonal matrices. Eq. (10) is just a set of n-uncoupled ODEs. That is,

$$\begin{aligned} &M_1 \ddot{q}_1 + D_1 \dot{q}_1 + K_1 q_1 = Q_1 \\ &M_2 \ddot{q}_2 + D_2 \dot{q}_2 + K_2 q_2 = Q_2 \\ &\ldots\ldots \\ &M_n \ddot{q}_n + D_n \dot{q}_n + K_n q_n = Q_n \end{aligned} \tag{11}$$

Or $$M_j \ddot{q}_j + D_j \dot{q}_j + K_j q_j = Q_j \quad , \quad {}_{j=1,2\ldots n} \tag{12}$$

Where $\omega_{n_j} = \sqrt{K_j / M_j}$ and $D_j = \alpha M_j + \beta K_j$. Modal damping ratios are also easily defined as

$$\zeta_j = \frac{D_j}{2\sqrt{K_j M_j}} = \frac{\alpha M_j + \beta K_j}{2\sqrt{K_j M_j}} \quad ; j=1,2,\ldots.n \tag{13}$$

For **damping proportional to mass only**, $\beta = 0$, and

$$\zeta_j = \frac{\alpha M_j}{2\sqrt{K_j M_j}} = \frac{\alpha}{2\,\omega_{n_j}} \tag{13a}$$

i.e., the *j*-modal damping ratio decreases as the natural frequency increases.

For **damping proportional to stiffness only**, $\alpha = 0$, **(structural damping)** and

$$\zeta_j = \frac{\beta K_j}{2\sqrt{K_j M_j}} = \frac{\beta\, \omega_{n_j}}{2} \tag{13b}$$

i.e., the *j*-modal damping ratio increases as the natural frequency increases. In other words, higher modes are more increasingly more damped than lower modes.

The response for each modal coordinate satisfying the modal Eqn. $M_j\, \ddot{q}_j + D_j\, \dot{q}_j + K_j\, q_j = Q_j, \quad {}_{j=1,2\ldots n}$ proceeds in the same way as for a single DOF system (See Handout 2).

First, find initial values in modal space $\left\{ q_{o_j}, \dot{q}_{o_j} \right\}$. These follow from either

$$\mathbf{q}_o = \mathbf{\Phi}^{-1} \mathbf{U}_o \;\; ; \;\; \dot{\mathbf{q}}_o = \mathbf{\Phi}^{-1}\, \dot{\mathbf{U}}_o \tag{14}$$

or

$$\mathbf{q}_o = \left[M\right]^{-1} \mathbf{\Phi}^T \mathbf{M}\, \mathbf{U}_o\,, \qquad \dot{\mathbf{q}}_o = \left[M\right]^{-1} \mathbf{\Phi}^T \mathbf{M}\, \dot{\mathbf{U}}_o \tag{15a}$$

$$q_{o_k} = \frac{1}{M_k} \boldsymbol{\varphi}_{(k)}^T \left(\mathbf{M}\, \mathbf{U}_o \right), \; \dot{q}_{o_k} = \frac{1}{M_k} \boldsymbol{\varphi}_{(k)}^T \left(\mathbf{M}\, \dot{\mathbf{U}}_o \right) \tag{15b}$$

k=1,….n

Free response in modal coordinates

Without modal forces, $Q=0$, the modal EOM is

$$M_j \ddot{q}_{H_j} + D_j \dot{q}_{H_j} + K_j q_{H_j} = 0 = Q_j \qquad (16)$$

with solution, for **an elastic underdamped mode** $\zeta_j < 1$

$$q_{H_j} = e^{-\zeta_j \omega_{d_j} t}\left(C_j \cos\left(\omega_{d_j} t\right) + S_j \sin\left(\omega_{d_j} t\right)\right) \quad \text{if } \omega_{n_j} \neq 0 \ (17a)$$

where $\omega_{d_j} = \omega_{n_j}\sqrt{1-\zeta_j^2}$, $\omega_{n_j} = \sqrt{K_j/M_j}$ and

$$C_j = q_{o_j};\ S_j = \frac{\dot{q}_{o_j} + \zeta_j \omega_{n_j} q_{o_j}}{\omega_{d_j}} \qquad (17b)$$

See Handout (2a) for modal responses corresponding to overdamped and critically damped SDOF system.

Forced response in modal coordinates

For step-loads, $Q_{S\,j}$, the modal equations are

$$M_j \ddot{q}_j + D_j \dot{q}_j + K_j q_j = Q_{S\,j} \qquad (18)$$

with solution, for **an elastic underdamped mode** $\zeta_j < 1$

$$q_j = e^{-\zeta_j \omega_{d_j} t}\left(C_j \cos\left(\omega_{d_j} t\right) + S_j \sin\left(\omega_{d_j} t\right)\right) + q_{S_j} \quad \omega_{n_j} \neq 0 \ (19a)$$

where $\omega_{d_j} = \omega_{n_j}\sqrt{1-\zeta_j^2}$, $\omega_{n_j} = \sqrt{K_j / M_j}$ and

$$q_{S_j} = \frac{Q_{S_j}}{K_j};\ C_j = \left(q_{o_j} - q_{S_j}\right);\ S_j = \frac{\dot{q}_{o_j} + \zeta_j \omega_{n_j} C_j}{\omega_{d_j}} \tag{19b}$$

See Handout (2a) for physical responses corresponding to overdamped and critically damped SDOF system.

For periodic-loads,

Consider the case of force excitation with frequency $\Omega \neq \omega_{n_j}$ and acting for very long times. The EOMs in physical space are

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{D}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F_P}\cos(\Omega t)$$

The modal equations are

$$M_j\,\ddot{q}_j + D_j\,\dot{q}_j + K_j\,q_j = Q_{P_j}\cos(\Omega t) \tag{20}$$

with solutions

for **an elastic mode,** $\omega_{n_j} \neq 0$

$$\begin{aligned} q_j &= q_{transient} + q_{ss(t)} = \\ & e^{-\zeta_j \omega_{n_j} t}\left(C_j \cos\left(\omega_{d_j} t\right) + S_j \sin\left(\omega_{d_j} t\right)\right) + \\ & C_{c_j}\cos(\Omega t) + C_{s_j}\sin(\Omega t) \end{aligned} \tag{21}$$

The **steady state or periodic response** is of importance, since the transient response will disappear because of damping dissipative effects. Hence, the *j*-mode response is:

$$q_{PS_j} = \left(\frac{Q_{P_j}}{K_j} \right) A_j \cos\left(\Omega t - \psi_j \right) \tag{22}$$

Let $f_j = \frac{\Omega}{\omega_{n_j}}$ be a j_{th}-*mode* excitation frequency ratio. Then, define

$$A_j = \frac{1}{\sqrt{\left(1 - f_j^2\right)^2 + \left(2\zeta_j f_j\right)^2}} \quad \text{and} \quad \tan\left(\psi_j\right) = \frac{2\zeta_j f_j}{\left(1 - f_j^2\right)} \tag{23}$$

Recall that φ_j is a **phase angle** and A_j is an **amplitude ratio** for the j_{th}-mode.

Note that depending on the magnitude of the excitation frequency Ω, the frequency ratio for a particular mode, say *k*, determines the regime of operation, i.e. below, above or around the natural frequency.

Using the **mode displacement method**, the response in physical coordinates is

$$\mathbf{U} \approx \sum_{j=1}^{m} \left(\boldsymbol{\varphi}_j \frac{Q_{P_j}}{K_j} A_j \cos\left(\Omega t - \psi_j \right) \right) \tag{24}$$

And recall that $K_j = \omega_{n_j}^2 M_j = \boldsymbol{\varphi}_{(j)}^T \mathbf{K} \boldsymbol{\varphi}_{(j)}$ and $Q_{P_j} = \boldsymbol{\varphi}_{(j)}^T \mathbf{F_P}$.

A **mode acceleration method** can also be easily developed to give

$$\mathbf{U} \approx \mathbf{U_{SP}} \cos\left(\Omega t\right) - \sum_{j=1}^{m} \frac{2\zeta_j}{\omega_j} \boldsymbol{\varphi}_j \dot{q}_{PS_j} - \sum_{j=1}^{m} \frac{\boldsymbol{\varphi}_j}{\omega_j^2} \ddot{q}_{PS_j} \tag{25}$$

where $\mathbf{U_{SP}} = \mathbf{K}^{-1}\mathbf{F_p}$. (please demonstrate Eq. (25) above). Note that the mode acceleration method cannot be applied modes if there are any rigid body modes.

Frequency response functions for damped MDOF systems.

The **steady state or periodic modal response** for *j*-mode is:

$$q_{PS_j} = \left(\frac{Q_{P_j}}{K_j}\right) A_j \cos\left(\Omega t - \psi_j\right) \qquad (22)$$

Or, taking the real part of the following complex number expression

$$q_{PS_j} = \left(\frac{Q_{P_j}}{K_j}\right) H_j\, e^{i\Omega t} \qquad (26)$$

where

$$H_j = \frac{1}{\left(1 - f_j^2\right) + i\left(2\zeta_j\, f_j\right)} \qquad (27)$$

with $i = \sqrt{-1}$ is the imaginary unit, and where $f_j = \dfrac{\Omega}{\omega_{n_j}}$ is the j_{th}-*mode* excitation frequency ratio. Then, recall from Eqs. (23)

$$A_j = \left|H_j\right| \frac{1}{\sqrt{\left(1 - f_j^2\right)^2 + \left(2\zeta_j\, f_j\right)^2}} \quad \text{and } \psi_j = \arg\left(H_j\right) \qquad (28)$$

Using the **modal transformation,** the periodic response $\mathbf{U_P}$ in physical coordinates is

$$\mathbf{U_P} = \sum_{j=1}^{n} \left(\boldsymbol{\varphi}_j \frac{Q_{P_j}}{K_j} A_j \cos\left(\Omega t - \psi_j\right) \right) \qquad (24)$$

or take the real part of the equation below

$$\mathbf{U_P} = \mathbf{\Phi}\,\mathbf{q} = \sum_{j=1}^{n}\left(\boldsymbol{\varphi}_j\, q_j\right) = \sum_{j=1}^{n}\left(\boldsymbol{\varphi}_j \frac{\boldsymbol{\varphi}_j^T \mathbf{F_P}}{K_j} H_j\, e^{i\Omega t}\right)$$

$$= \left\{\sum_{j=1}^{n}\left(\boldsymbol{\varphi}_j \boldsymbol{\varphi}_j^T \frac{H_j}{K_j}\mathbf{F_P}\right)\right\} e^{i\Omega t} \tag{29}$$

Now, the product $\boldsymbol{\varphi}_j\, \boldsymbol{\varphi}_j^T = \mathbf{matrix}(n \times n)$. That is, define the elements of the complex – frequency response matrix **H** as

$$H_{p,q} = \left(\frac{\boldsymbol{\varphi}_{j_p} \boldsymbol{\varphi}_{j_q}^T}{K_j}\left(\frac{1}{\left(1 - f_j^2\right) + i\left(2\zeta_j\, f_j\right)}\right)\right) \tag{30}$$

p,q =1,2…. *n*. The response in physical coordinates thus becomes:

$$\mathbf{U_P} = \mathbf{H}\,\mathbf{F_P}\, e^{i\Omega t} \tag{31}$$

Or in component form,

$$U_{P_j} = \left(\sum_{r=1}^{n} H_{j,r}\, F_{P_r}\right) e^{i\Omega t}\ ;\quad {}_{j=1,2..n} \tag{32}$$

The components of the frequency response matrix **H** are determined numerically or experimentally. In any case, the components of **H** depend on the excitation frequency (Ω). Determining the elements of **H** seems laborious and (perhaps) its physical meaning remains elusive.

Direct Method to Find Frequency Responses in MDOF Systems

Nowadays, with fast computing power at our fingertips, the young engineer prefers to pursue a more direct approach, one known as **brute force or direct aproach**. Recall that the equation of motion is

$$\mathbf{M}\,\ddot{\mathbf{U}}+\mathbf{D}\,\dot{\mathbf{U}}+\mathbf{K}\,\mathbf{U}=\mathbf{F_P}\cos(\Omega t)$$

Or

$$\mathbf{M}\,\ddot{\mathbf{U}}+\mathbf{D}\,\dot{\mathbf{U}}+\mathbf{K}\,\mathbf{U}=\mathrm{Re}\left(\mathbf{F_P}\;e^{i\Omega t}\right) \quad (33)$$

Assume a periodic solution of the form $\mathbf{U}=\mathbf{V_P}\;e^{i\Omega t}$ (34)

where $\mathbf{V_P}$ is a vector in the complex domain. Substitution of Eq. (34) into eq. (33) gives

$$\left[\mathbf{K}+i\,\Omega\mathbf{D}-\Omega^2\,\mathbf{M}\right]\mathbf{V_P}=\mathbf{F_P} \quad (35)$$

Define at each excitation frequency the complex impedance (dynamic stiffness) matrix as:

$$\mathbf{K_{D(\Omega)}}=\left[\mathbf{K}+i\,\Omega\mathbf{D}-\Omega^2\,\mathbf{M}\right] \quad (36)$$

And find the **vector** of physical responses (amplitude and phase) as

$$\mathbf{V_P}=\left[\mathbf{K_{D(\Omega)}}\right]^{-1}\mathbf{F_P} \quad (37)$$

Since $\mathbf{V_P}=\mathbf{V_{P_{\mathit{real}}}}+i\,\mathbf{V_{P_{\mathit{imaginary}}}}$, the physical response for each DOF follows as:

$$
\begin{aligned}
&U_r = V_{P_r}\cos\left(\Omega t - \gamma_r\right); \quad {\scriptstyle r=1,2...n} \\
&V_{P_r} = \sqrt{V_{P_r-real}^2 + V_{P_r-imaginary}^2}; \;\; \tan\left(\gamma_r\right) = -\left(V_{P_r-imaginary} \big/ V_{P_r-real} \right)
\end{aligned} \tag{38}
$$

The direct method requires calculating the inverse of the dynamic stiffness matrix at each excitation frequency. The computational effort to perform this task could be excessive but for systems with a few DOFs (*n* small).

STEP FORCED RESPONSE of 2-DOF mechanical system with proportional damping

ORIGIN := 1

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The equations of motion are:

$$M \cdot \frac{d^2}{dt^2} X + D \cdot \frac{d}{dt} X + K \cdot X = F_o \qquad (1)$$

where **M,D, K** are matrices of inertia, damping and stiffness coefficients; and **X, V**=d**X**/dt, d^2**X**/dt^2 are the vectors of physical displacement, velocity and acceleration, respectively.
The FORCED undamped response to the initial conditions, at t=0, **Xo,Vo=**d**X/**dt, follows:

For proportional damping, **D** = α **M** + β **K,** so the undamped mode analysis can be used.
α & β are physical constants usually determined from measurements of modal damping.

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The equations of motion are:

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \cdot \frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \cdot \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_{1o} \\ F_{2o} \end{pmatrix} \qquad (2)$$

1. Set elements of inertia, stiffness & damping matrices

DATA FOR problem

$$M := \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix} \cdot kg \qquad K := \begin{pmatrix} 2 \cdot 10^6 & -1 \cdot 10^6 \\ -1 \cdot 10^6 & 2 \cdot 10^6 \end{pmatrix} \cdot \frac{N}{m} \qquad n := 2$$ # of DOF

Note M and K are symmetric matrices

example $\alpha := 0.0 \cdot \frac{1}{s}$ $\beta := .001 \cdot s$

$$D := \alpha \cdot M + \beta \cdot K$$

$$D = \begin{pmatrix} 2 \times 10^3 & -1 \times 10^3 \\ -1 \times 10^3 & 2 \times 10^3 \end{pmatrix} N \cdot \frac{s}{m}$$

initial conditions $X_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot m$ $V_o := \begin{pmatrix} 0.0 \\ 0 \end{pmatrix} \cdot \frac{m}{sec}$

Applied force vector: $F_o := \begin{pmatrix} 10000 \\ -5000 \end{pmatrix} \cdot N$

analysis

2. Find eigenvalues (undamped natural frequencies) and eigenvectors

Set determinant of system of eqns = 0

$$\Delta = \left[\left(K_{11} - M_{11}\cdot\omega^2\right)\cdot\left(K_{22} - M_{22}\cdot\omega^2\right) - \left(K_{12} - M_{12}\cdot\omega^2\right)\cdot\left(K_{21} - M_{21}\cdot\omega^2\right)\right] = 0 \qquad (2a)$$

$$\Delta = a\cdot\omega^4 + b\cdot\omega^2 + c = \left(a\cdot\lambda^2 + b\cdot\lambda + c\right) = 0 \quad \text{with} \quad \lambda = \omega^2 \qquad (2b)$$

where the coefficients are:

$$a := M_{1,1}\cdot M_{2,2} - M_{1,2}\cdot M_{2,1}$$

$$b := K_{1,2}\cdot M_{2,1} - K_{1,1}\cdot M_{2,2} - K_{2,2}\cdot M_{1,1} + K_{2,1}\cdot M_{1,2} \qquad (2c)$$

$$c := K_{1,1}\cdot K_{2,2} - K_{1,2}\cdot K_{2,1}$$

The roots of equation (2b) are:

$$\lambda_1 := \frac{\left[-b - \left(b^2 - 4\cdot a\cdot c\right)^{.5}\right]}{2\cdot a} \qquad \lambda_2 := \frac{\left[-b + \left(b^2 - 4\cdot a\cdot c\right)^{.5}\right]}{2\cdot a} \qquad (3)$$

also known as eigenvalues. The **natural frequencies** follow as:

$$j := 1 .. n \qquad \omega_j := \left(\lambda_j\right)^{.5} \qquad f := \frac{\omega}{2\cdot\pi} \qquad \omega = \begin{pmatrix} 112.6 \\ 217.53 \end{pmatrix}\frac{rad}{sec} \qquad (4)$$

$$f = \begin{pmatrix} 17.92 \\ 34.62 \end{pmatrix} Hz$$

Note that: $\Delta\left(\omega_1\right) = \Delta\left(\omega_2\right) = 0$

For each eigenvalue, the eigenvectors (**natural modes)** are

$$j := 1 .. n$$

$$a_j := \begin{bmatrix} 1 \\ \dfrac{K_{1,1} - M_{1,1}\cdot\lambda_j}{-\left(K_{1,2} - M_{1,2}\cdot\lambda_j\right)} \end{bmatrix}$$

Set arbitrarily first element of vector = 1

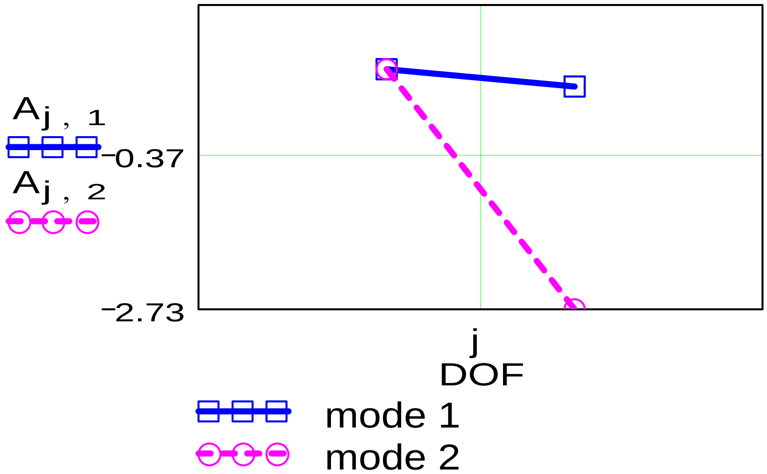
$$a_1 = \begin{pmatrix} 1 \\ 0.73 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 1 \\ -2.73 \end{pmatrix} \qquad (5)$$

MODAL matrix $\quad A^{\langle j\rangle} := a_j$

A is the matrix of eigenvectors (undamped modal matrix): each column corresponds to an eigenvector

$$A = \begin{pmatrix} 1 & 1 \\ 0.73 & -2.73 \end{pmatrix}$$

Plot the mode shapes:



3. Modal transformation of physical equations to (natural) modal coordinates

Using transformation: $X = A \cdot q$ (6)

EOMs (1) become uncoupled in modal space:

$$M_m \cdot \frac{d^2}{dt^2} q + D_m \cdot \frac{d}{dt} q + K_m \cdot q = Q_m \quad (7)$$

with modal force vector: $Q_m = A^T \cdot F_o$ (8)

and initial conditions (modal displacement=q and modal velocity dq/dt=s)

$$q_o = M_m^{-1} \cdot \left(A^T \cdot M \cdot X_o\right) \qquad s_o = M_m^{-1} \cdot \left(A^T \cdot M \cdot V_o\right) \quad (9)$$

The natural modes satisfy the orthogonality properties

$M_m := A^T \cdot M \cdot A$ $\quad M_m = \begin{pmatrix} 126.79 & -2.24 \times 10^{-14} \\ -1.58 \times 10^{-14} & 473.21 \end{pmatrix} kg$

$K_m := A^T \cdot K \cdot A$ $\quad K_m = \begin{pmatrix} 1.61 \times 10^{6} & 3.18 \times 10^{-10} \\ 3.51 \times 10^{-10} & 2.24 \times 10^{7} \end{pmatrix} \frac{N}{m}$ $\qquad \omega = \begin{pmatrix} 112.6 \\ 217.53 \end{pmatrix} s^{-1}$

$D_m := A^T \cdot D \cdot A$ $\quad D_m = \begin{pmatrix} 1.61 \times 10^{3} & 3.06 \times 10^{-13} \\ -1.8 \times 10^{-13} & 2.24 \times 10^{4} \end{pmatrix} s \frac{N}{m}$

or better $\qquad D_m := \alpha \cdot M_m + \beta \cdot K_m$

Define the **modal damping ratios and damped natural freqs:** $k := 1..n$

$$\zeta_k := \frac{Dm_{k,k}}{2 \cdot Mm_{k,k} \cdot \omega_k} \qquad \omega d_k := \omega_k \cdot \left[1 - (\zeta_k)^2\right]^{.5} \qquad (11)$$

$$\zeta = \begin{pmatrix} 0.06 \\ 0.11 \end{pmatrix}$$

Underdamped modes

UNDERDAMPED CASE

$$\omega = \begin{pmatrix} 112.6 \\ 217.53 \end{pmatrix} s^{-1} \qquad \omega d = \begin{pmatrix} 112.42 \\ 216.24 \end{pmatrix} s^{-1}$$

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4. Find Modal and Physical Response for given initial condition and Constant Force vector

4.a Find initial conditions in modal coordinates (displacement = q, velocity = s)

Set inverse of modal mass matrix $A_{inv} := Mm^{-1} \cdot (A^T \cdot M)$

$$q_o := A_{inv} \cdot X_o \qquad s_o := A_{inv} \cdot V_o$$

$$q_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m \qquad s_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m\,s^{-1}$$

4.b Find Modal forces:

$$Q_m := A^T \cdot F_o \qquad Q_m = \begin{pmatrix} 6.34 \times 10^3 \\ 2.37 \times 10^4 \end{pmatrix} N$$

4.c Build Modal responses: **two elastic modes - underdamped**

$j := 1..2$

$$qs_j := \frac{Qm_j}{Km_{j,j}}$$: **static displacement in modal space**

$$Ac_j := (qo_j - qs_j)$$: coefficients of cos & sin functions.

$$As_j := \frac{(so_j - \zeta_j \cdot \omega_j \cdot Ac_j)}{\omega d_j}$$

$$q_1(t) := e^{-\zeta_1 \cdot \omega_1 \cdot t} \cdot \left(A_{c_1} \cdot \cos\left(\omega_{d_1} \cdot t\right) + A_{s_1} \cdot \sin\left(\omega_{d_1} \cdot t\right)\right) + q_{s_1}$$

$$q_2(t) := e^{-\zeta_2 \cdot \omega_2 \cdot t} \cdot \left(A_{c_2} \cdot \cos\left(\omega_{d_2} \cdot t\right) + A_{s_2} \cdot \sin\left(\omega_{d_2} \cdot t\right)\right) + q_{s_2}$$

4.d Build Physical responses:

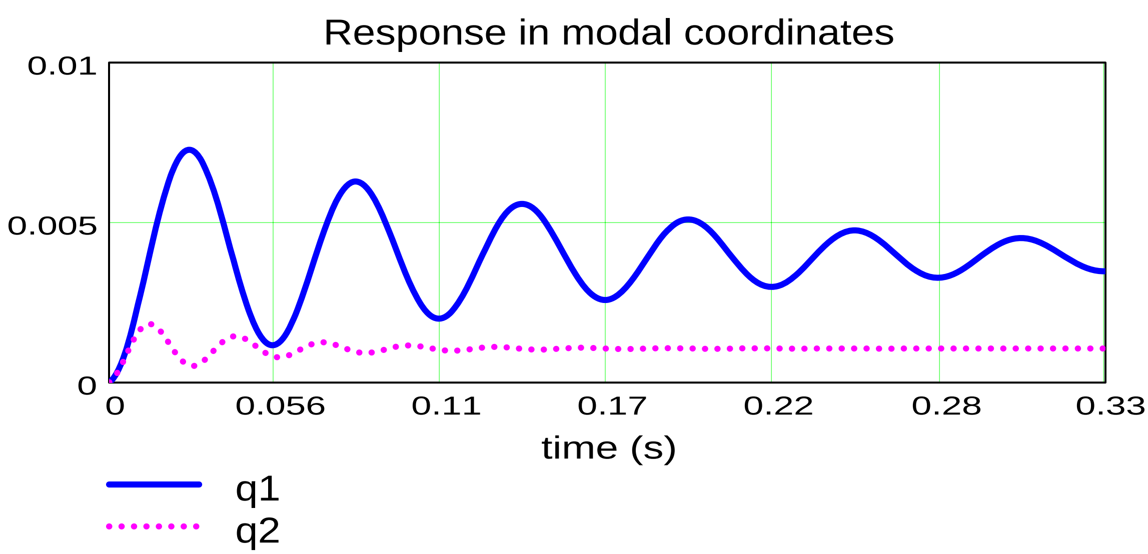
$$X(t) := a_1 \cdot q_1(t) + a_2 \cdot q_2(t)$$

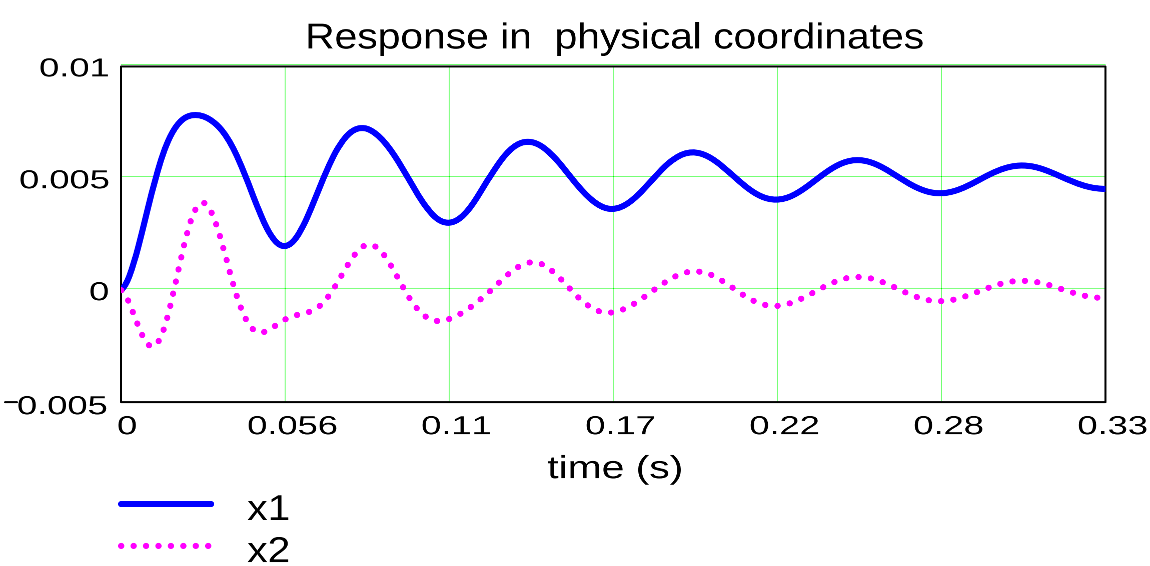
for plots:

$$T_{plot} := \frac{6}{f_1}$$

4.e Graphs of Modal and Physical responses:

analysis





5. Interpret response: analyze results, provide recommendations

Note the paramount effect of damping in attenuating the system response.

Recall for this example: $\zeta = \begin{pmatrix} 0.06 \\ 0.11 \end{pmatrix}$ $\omega_d = \begin{pmatrix} 112.42 \\ 216.24 \end{pmatrix} s^{-1}$ $\omega = \begin{pmatrix} 112.6 \\ 217.53 \end{pmatrix} s^{-1}$

S-S displacement

$$K^{-1} \cdot F_o = \begin{pmatrix} 5 \times 10^{-3} \\ 0 \end{pmatrix} m$$

compare to modal derived values:

$$A \cdot q_s = \begin{pmatrix} 5 \times 10^{-3} \\ 0 \end{pmatrix} m$$

STEP FORCED RESPONSE of 2-DOF mechanical system with proportional damping

ORIGIN := 1

Dr. Luis San Andres (c) MEEN 363, 617 February 2008

The equations of motion are:

$$M \cdot \frac{d^2}{dt^2} X + D \cdot \frac{d}{dt} X + K \cdot X = F_o \qquad (1)$$

where **M,D, K** are matrices of inertia, damping and stiffness coefficients; and **X, V**=d**X**/dt, d^2**X**/dt^2 are the vectors of physical displacement, velocity and acceleration, respectively.
The FORCED undamped response to the initial conditions, at t=0, **Xo,Vo=**d**X/**dt, follows:

For proportional damping, **D** = α **M** + β **K,** so the undamped mode analysis can be used.
α & β are physical constants usually determined from measurements of modal damping.

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The equations of motion are:

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \cdot \frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \cdot \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_{1o} \\ F_{2o} \end{pmatrix} \qquad (2)$$

1. Set elements of inertia, stiffness & damping matrices

DATA FOR problem

$$M := \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix} \cdot kg \qquad K := \begin{pmatrix} 1 \cdot 10^6 & -1 \cdot 10^6 \\ -1 \cdot 10^6 & 1 \cdot 10^6 \end{pmatrix} \cdot \frac{N}{m} \qquad n := 2$$

# of DOF

Note M and K are symmetric matrices

with a RIGID BODY MODE

example $\alpha := 0.0 \cdot \frac{1}{s}$ $\beta := .001 \cdot s$

$$D := \alpha \cdot M + \beta \cdot K$$

$$D = \begin{pmatrix} 1 \times 10^3 & -1 \times 10^3 \\ -1 \times 10^3 & 1 \times 10^3 \end{pmatrix} N \cdot \frac{s}{m}$$

initial conditions $X_o := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot m$ $V_o := \begin{pmatrix} 0.0 \\ 0 \end{pmatrix} \cdot \frac{m}{sec}$

Applied force vector: $F_o := \begin{pmatrix} 1000 \\ -980 \end{pmatrix} \cdot N$

2. Find eigenvalues (undamped natural frequencies) and eigenvectors

Set determinant of system of eqns = 0

$$\Delta = \left[\left(K_{11} - M_{11}\cdot\omega^2\right)\cdot\left(K_{22} - M_{22}\cdot\omega^2\right) - \left(K_{12} - M_{12}\cdot\omega^2\right)\cdot\left(K_{21} - M_{21}\cdot\omega^2\right)\right] = 0 \qquad (2a)$$

$$\Delta = a\cdot\omega^4 + b\cdot\omega^2 + c = \left(a\cdot\lambda^2 + b\cdot\lambda + c\right) = 0 \quad \text{with} \quad \lambda = \omega^2 \qquad (2b)$$

where the coefficients are:

$$a := M_{1,1}\cdot M_{2,2} - M_{1,2}\cdot M_{2,1}$$

$$b := K_{1,2}\cdot M_{2,1} - K_{1,1}\cdot M_{2,2} - K_{2,2}\cdot M_{1,1} + K_{2,1}\cdot M_{1,2} \qquad (2c)$$

$$c := K_{1,1}\cdot K_{2,2} - K_{1,2}\cdot K_{2,1}$$

The roots of equation (2b) are:

$$\lambda_1 := \frac{\left[-b - \left(b^2 - 4\cdot a\cdot c\right)^{.5}\right]}{2\cdot a} \qquad \lambda_2 := \frac{\left[-b + \left(b^2 - 4\cdot a\cdot c\right)^{.5}\right]}{2\cdot a} \qquad (3)$$

also known as eigenvalues. The **natural frequencies** follow as:

$$j := 1 .. n \qquad \omega_j := \left(\lambda_j\right)^{.5} \qquad f := \frac{\omega}{2\cdot\pi}$$

$$\omega = \begin{pmatrix} 0 \\ 173.21 \end{pmatrix}\frac{rad}{sec} \qquad (4)$$

$$f = \begin{pmatrix} 0 \\ 27.57 \end{pmatrix} Hz$$

Note that: $\Delta\left(\omega_1\right) = \Delta\left(\omega_2\right) = 0$

For each eigenvalue, the eigenvectors (**natural modes)** are

$$j := 1 .. n$$

$$a_j := \begin{bmatrix} 1 \\ \dfrac{K_{1,1} - M_{1,1}\cdot\lambda_j}{-\left(K_{1,2} - M_{1,2}\cdot\lambda_j\right)} \end{bmatrix}$$ Set arbitrarily first element of vector = 1

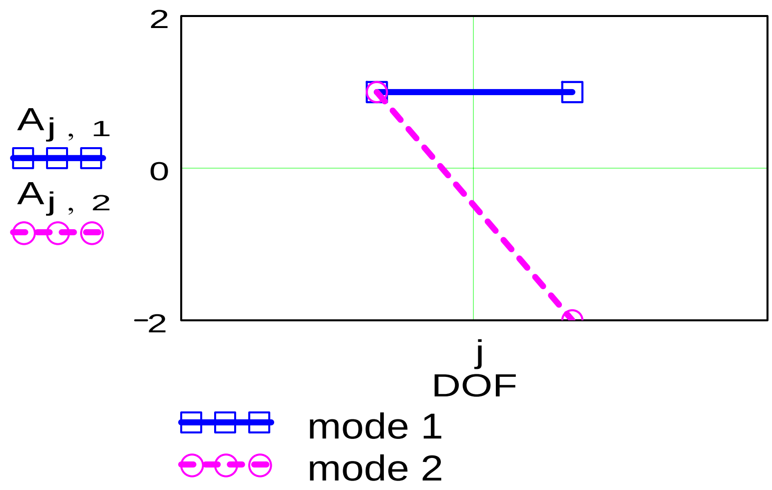
MODAL matrix $A^{\langle j\rangle} := a_j$

$$a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad (5)$$

A is the matrix of eigenvectors (undamped modal matrix): each column corresponds to an eigenvector

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

Plot the mode shapes:



3. Modal transformation of physical equations to (natural) modal coordinates

Using transformation: $X = A \cdot q$ (6)

EOMs (1) become uncoupled in modal space:

$$M_m \cdot \frac{d^2}{dt^2} q + D_m \cdot \frac{d}{dt} q + K_m \cdot q = Q_m \quad (7)$$

with modal force vector: $Q_m = A^T \cdot F_o$ (8)

and initial conditions (modal displacement=q and modal velocity dq/dt=s)

$$q_o = M_m^{-1} \cdot \left(A^T \cdot M \cdot X_o\right) \qquad s_o = M_m^{-1} \cdot \left(A^T \cdot M \cdot V_o\right) \quad (9)$$

The natural modes satisfy the orthogonality properties

$M_m := A^T \cdot M \cdot A$ $\quad M_m = \begin{pmatrix} 150 & 0 \\ 0 & 300 \end{pmatrix} kg$

$K_m := A^T \cdot K \cdot A$ $\quad K_m = \begin{pmatrix} 0 & 0 \\ 0 & 9 \times 10^6 \end{pmatrix} \frac{N}{m}$

$\omega = \begin{pmatrix} 0 \\ 173.21 \end{pmatrix} s^{-1}$

$D_m := A^T \cdot D \cdot A$ $\quad D_m = \begin{pmatrix} 0 & 0 \\ 0 & 9 \times 10^3 \end{pmatrix} s \frac{N}{m}$

or better $\quad D_m := \alpha \cdot M_m + \beta \cdot K_m$

Define the **modal damping ratios and damped natural freqs:** $k := 1 .. n$

$$\zeta_k := \frac{Dm_{k,k}}{2 \cdot Mm_{k,k} \cdot \omega_k} \qquad \omega d_k := \omega_k \cdot \left[1 - (\zeta_k)^2\right]^{.5} \qquad (11)$$

$$\zeta = \begin{pmatrix} 0 \\ 0.09 \end{pmatrix}$$

ONE RIGID BODY mode with null modal damping

UNDERDAMPED CASE

$$\omega = \begin{pmatrix} 0 \\ 173.21 \end{pmatrix} s^{-1} \qquad \omega d = \begin{pmatrix} 0 \\ 172.55 \end{pmatrix} s^{-1}$$

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4. Find Modal and Physical Response for given initial condition and Constant Force vector

4.a Find initial conditions in modal coordinates (displacement = q, velocity = s)

Set inverse of modal mass matrix $A_{inv} := Mm^{-1} \cdot \left(A^T \cdot M\right)$

$$q_o := A_{inv} \cdot X_o \qquad s_o := A_{inv} \cdot V_o$$

$$q_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m \qquad s_o = \begin{pmatrix} 0 \\ 0 \end{pmatrix} m\,s^{-1}$$

4.b Find Modal forces:

$$Q_m := A^T \cdot F_o \qquad Q_m = \begin{pmatrix} 20 \\ 2.96 \times 10^3 \end{pmatrix} N$$

4.c Build Modal responses:

rigid body mode - NO DAMPING

$$q_1(t) := q_{o_1} + s_{o_1} \cdot t + \frac{Q_{m_1}}{Mm_{1,1}} \cdot \frac{t^2}{2}$$

elastic mode - UNDERDAMPED

$j := 2$

$$q_{s_j} := \frac{Q_{m_j}}{Km_{j,j}}$$: **static displacement in modal space**

$$A_{c_j} := \left(q_{o_j} - q_{s_j}\right)$$: coefficients of cos & sin functions.

$$A_{s_j} := \frac{\left(s_{o_j} - \zeta_j \cdot \omega_j \cdot A_{c_j}\right)}{\omega_{d_j}}$$

$$q_2(t) := e^{-\zeta_2 \cdot \omega_2 \cdot t} \cdot \left(A_{c_2} \cdot \cos\left(\omega_{d_2} \cdot t\right) + A_{s_2} \cdot \sin\left(\omega_{d_2} \cdot t\right)\right) + q_{s_2}$$

for plots:

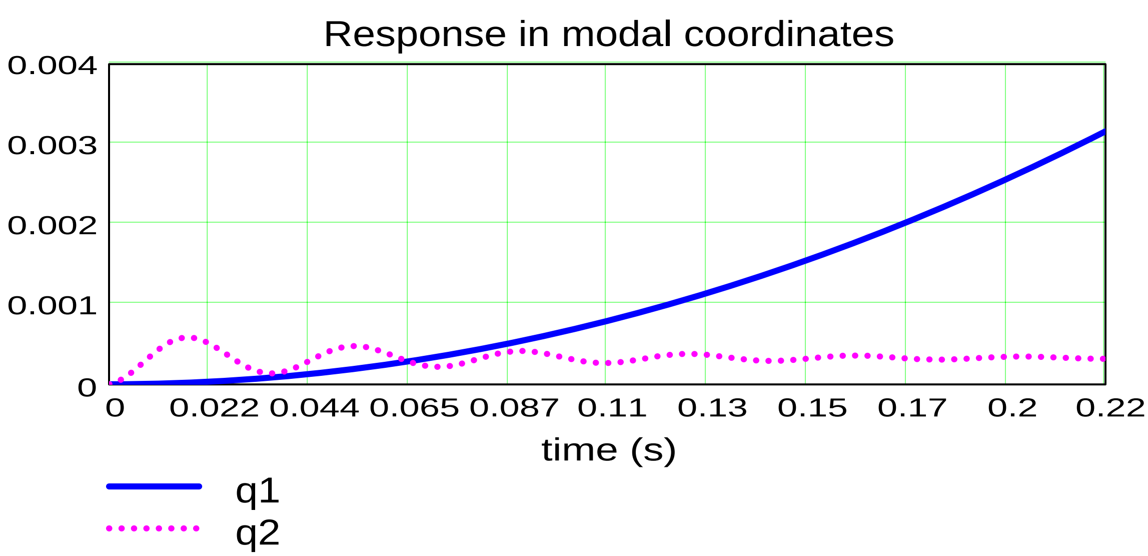
4.d Build Physical responses:

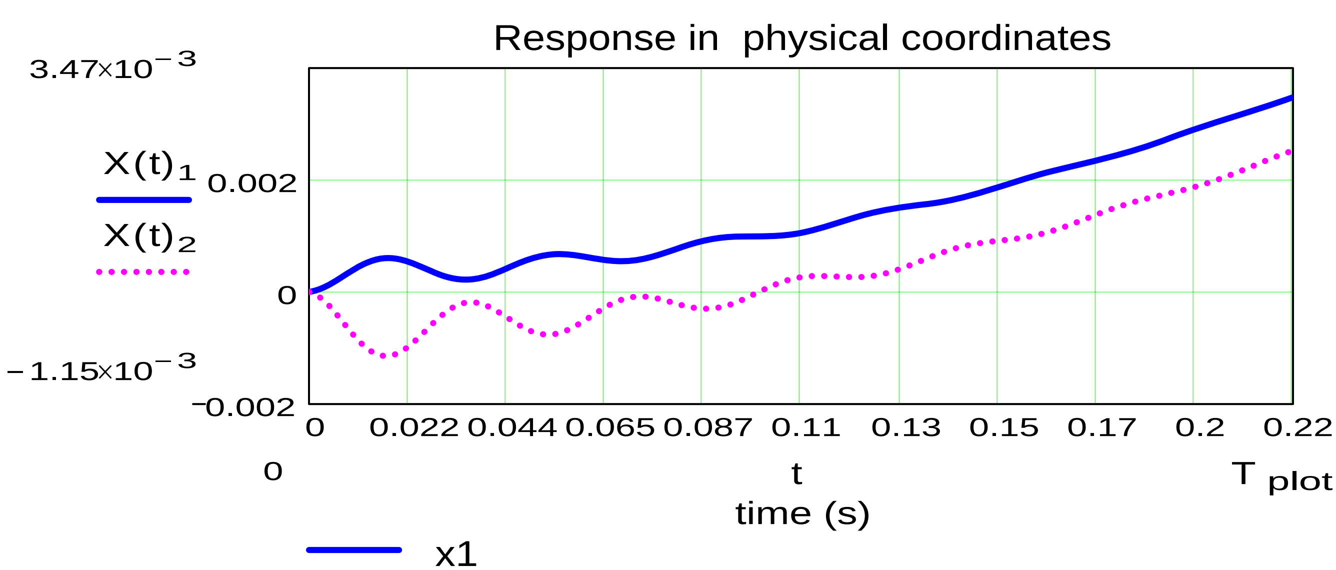
$$X(t) := a_1 \cdot q_1(t) + a_2 \cdot q_2(t)$$

$$T_{plot} := \frac{6}{f_2}$$

4.e Graphs of Modal and Physical responses:

analysis





x2

5. Interpret response: analyze results, provide recommendations

Note the paramount effect of damping in attenuating the system response.

Recall for this example: $\zeta = \begin{pmatrix} 0 \\ 0.09 \end{pmatrix}$ $\omega_d = \begin{pmatrix} 0 \\ 172.55 \end{pmatrix} s^{-1}$ $\omega = \begin{pmatrix} 0 \\ 173.21 \end{pmatrix} s^{-1}$

i.e., Modal damping ratios of 9% for elastic mode.