Derivation of Mode Acceleration Method for MDOF systems (proportional damping or light damping)

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Problem Statement

Determine the system response of a MDOF system with proportional damping using the **Mode Acceleration method**.

Solution

The differential equation governing the motion of a *n*-DOF linear system is:

$$[M]\ddot{X} + [C]\dot{X} + [K]X = P(t) \tag{1}$$

where [*M*], [*K*], [*C*] are the (*n*x*n*) matrices of (constant) mass, stiffness and damping coefficients. P(t) is a vector of *n*-external forces, time dependent, and *X(t)* is the vector of system displacements (physical responses). The physical damping is of proportional type, i.e. [*C*] = *a* [*M*] + *b* [*K*]

The system described by (1) has a set of natural frequencies $(\omega_i)_{i=i,..n}$ and associated modal (eigen) vectors $({}^i\phi)_{i=i,..n}$. Each pair $(\omega_i\ {}^i\phi)$ satisfies the fundamental relationship

$$[K]\ {}^i\phi = \omega_i^2\ [M]\ {}^i\phi, \qquad i=1,2,\ldots n \tag{2}$$

The physical response *X(t)* or solution to (1) can be found using modal analysis, i.e.

$$X(t) = [\Phi]\eta(t) = \sum_{i=1}^{n} {}^i\phi\ \eta_i \tag{3}$$

where $[\Phi] = \{ {}^1\phi\ \ {}^2\phi \ldots.. {}^n\phi \}$ is the modal matrix. Each of the components of the modal response vector $\eta(t)$ is obtained from solution of the (uncoupled) equations:

$$M_{m\,i}\,\ddot{\eta}_i + C_{m\,i}\,\dot{\eta}_i + K_{m\,i}\,\eta_i = Q_i \qquad i=1,2,\ldots n \tag{4}$$

where $Q = [\Phi]^T P$, and $(K_m, M_m, C_m)_i$ are the *i*-th **modal** stiffness, mass and damping coefficients obtained from:

$$[M_m] = [\Phi]^T[M][\Phi];\ [K_m] = [\Phi]^T[K][\Phi];[C_m] = [\Phi]^T[C][\Phi]; \tag{5}$$

In (3), using a number of modes *m* less than the *n*-DOF is known as **the mode displacement method, i.e.**

$$X(t) \cong \sum_{i=1}^{m} {}^i\phi\ \eta_i \ ; \quad m<n \tag{6}$$

The **mode acceleration method** aims to find exactly the system static response should *P* be a vector of constant generalized forces. In this case, the mode displacement method does poorly when just a few modes, $m<<n$, are used

To derive the appropriate equations, pre-multiply (1) by $[K]^{-1}$, i.e. the *flexibility matrix* (obviously this operation precludes any rigid body motion), to obtain:

$$[K]^{-1}[M]\ddot{X} + [K]^{-1}[C]\dot{X} + X = [K]^{-1}P(t)$$

and

$$X = [K]^{-1}P(t) - [K]^{-1}[M]\ddot{X} - [K]^{-1}[C]\dot{X} \qquad (7)$$

from (6) it follows that $\dot{X} \cong \sum_{i=1}^{m} \phi_i \dot{\eta}_i(t)$ and $\ddot{X} \cong \sum_{i=1}^{m} \phi_i \ddot{\eta}_i(t)$. Replacing these relationships into (7) gives:

$$X(t) \cong [K]^{-1} P(t) - \sum_{i=1}^{m} [K]^{-1} [M]\, {}^i\phi\, \ddot{\eta}_i(t) - \sum_{i=1}^{m} [K]^{-1} [C]\, {}^i\phi\, \dot{\eta}_i(t); \quad {}_{m<n} \qquad (8)$$

Let's work with the terms: $[K]^{-1}[M]\, {}^i\phi$ *and* $[K]^{-1}[C]\, {}^i\phi$. Since each pair $(\omega_i, {}^i\phi)$ satisfies the fundamental relationship

$$[K]\, {}^i\phi = \omega_i^2\, [M]\, {}^i\phi \qquad (2)$$

then

$$[K]^{-1}\, [M]\, {}^i\phi = (1/\omega_i^2)\, {}^i\phi \qquad (9.a)$$

and similarly,

$$[K]^{-1}\, [C]\, {}^i\phi = (2\xi_i / \omega_i)\, {}^i\phi \qquad (9.b)$$

where ξ_i is the *i*-th **modal** damping ratio defined as

$$\xi_i = \frac{C_{mi}}{C_{cr_i}} \quad ; \; [C_m] = [\Phi]^T [C][\Phi],\; C_{cr_i} = 2\left(K_{m_i}\, M_{m_i}\right)^{1/2} \qquad (10)$$

Note that in the equation above, $(K_m, M_m)_i$ are the *i*-th **moda**l stiffness and mass coefficients satisfying $\left(\frac{K_{m_i}}{M_{m_i}}\right)^{1/2} = \omega_i$

Replacing (9) into (8) gives the physical response of the system as:

$$X(t) \cong [K]^{-1} P(t) - \sum_{i=1}^{m} \frac{{}^i\phi}{\omega_i^2} \ddot{\eta}_i(t) - \sum_{i=1}^{m} \frac{2\xi_i\, {}^i\phi}{\omega_i} \dot{\eta}_i(t); \quad {}_{m<n} \qquad (10)$$

which is known as the **mode acceleration response method**. The first term in the response $[K]^{-1}P(t)$ corresponds to a "pseudostatic" static displacement due to *P(t)*.

Note that for $P = P_s$ *(constant)*, $X = X_s = [K]^{-1}P_s$ since all $\eta_i = 0$. This simple check certifies the accuracy of the mode acceleration method even when using few modes ($m<n$).

Reference:

MEEN 617 Handout #8 Modal Analysis of MDOF Systems with Proportional Damping, L. SanAndrés, 2008.