Important design issues and engineering applications of SDOF system Frequency response Functions

The following descriptions show typical questions related to the design and dynamic performance of a second-order mechanical system operating under the action of an external force of periodic nature, i.e. $F(t)=F_o \cos(\Omega t)$ *or* $F(t)=F_o \sin(\Omega t)$

The system EOM is: $M\,\ddot{X} + D\,\dot{X} + K\,X = F_o \cos\left(\Omega t\right)$

Recall that the system response is governed by its parameters, i.e. stiffness ***(K)***, mass ***(M)*** and viscous damping ***(D)*** coefficients. These parameters determine the fundamental natural frequency, $\omega_n = \sqrt{K/M}$, and viscous damping ratio, $\zeta = D/D_c$, with $D_c = 2\sqrt{KM}$

In all design cases below, let $r=(\Omega/\omega_n)$ as the frequency ratio, i.e. a parameter that sets the importance of excitation frequency with respect to the system natural frequency and determines the system performance.

PROBLEM TYPE 1

Consider a system excited by a periodic force of magnitude $\boldsymbol{F_o}$ with external frequency $\boldsymbol{\Omega}$.

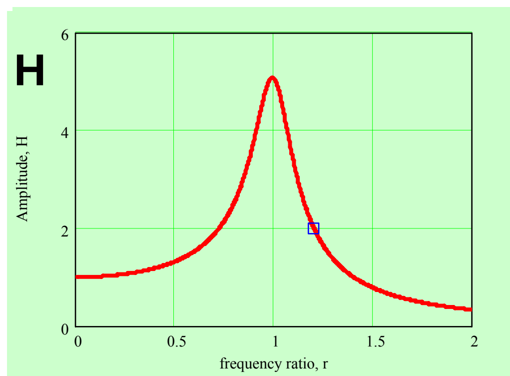
a) Determine the damping ratio ζ needed such that the amplitude of motion does not ever exceed (say) twice the displacement ($X_s=F_o/K$) for operation at a frequency (say) 20% above the natural frequency of the system.

b) With the result of (a), determine the amplitude of motion for operation with an excitation frequency coinciding with the system natural frequency. Is this response the maximum ever expected? Explain.

Recall that system periodic response is

$$X(t) = X_s H(r)\cos(\Omega t + \psi)$$

Solution. From the amplitude of FRF



$$\left|\frac{X}{X_s}\right| = H(r) = \frac{1}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}}$$

Set $r=r_a = 1.2$ and $|X/X_s|=H_a=2$, and find the damping ratio ζ from algebraic equation:

$$H_a^2\left(\left(1-r_a^2\right)^2 + (2\zeta r_a)^2\right) = 1$$

$$(2\zeta r_a)^2 = \frac{1}{H_a^2} - \left(1-r_a^2\right)^2 \Rightarrow \zeta = \frac{1}{2r_a}\left[\frac{1}{H_a^2} - \left(1-r_a^2\right)^2\right]^{1/2} = 0.099$$

Next, calculate the viscous damping coefficient $D=\zeta D_c$

For excitation at the natural frequency, i.e. at resonance, then $r=1$, $|X/X_s|=1/(2\zeta) = Q$. *Thus* $|X|=Q\,X_s$

The maximum amplitude of motion does not necessarily occur at $r=1$. In actuality, the value of frequency ratio (r_*) which maximizes the response, i.e. when $\left(\dfrac{\partial \left| \dfrac{X}{X_s} \right|}{\partial r} \right) = 0$, is

- after some algebraic manipulation -

$$r_* = \sqrt{\left(1-2\zeta^2\right)}; \text{ and } \left| \frac{X}{X_s} \right|_{\max} = \frac{1}{2\zeta} \frac{1}{\sqrt{\left(1-2\zeta^2\right)}}$$

Note that for small values of damping $\left| \dfrac{X}{X_s} \right|_{\max} \approx \dfrac{1}{2\zeta}$

PROBLEM TYPE 2

Consider a system exited by an imbalance (u), giving an amplitude of force excitation equal to $F_o = M\,u\,\mathbf{\Omega}^2$. Recall that $u = m\,e/M$, where m is the imbalance mass and e is its radial location

$$M\,\ddot{X} + D\,\dot{X} + K\,X = M\,u\,\Omega^2 \cos(\Omega t)$$

Recall that system periodic response is

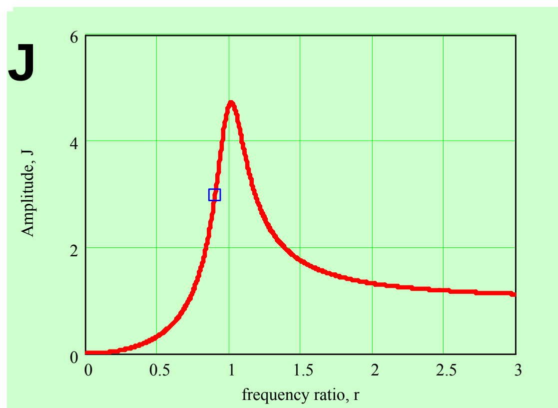
$$X(t) = u\,H(r)\cos(\Omega t + \psi)$$

a) What is the value of damping ζ necessary so that the system response never exceeds (say) three times the imbalance $\boldsymbol{u}$ for operation at a frequency (say) 10% below the natural frequency of the system.

b) With the result of (a), determine the amplitude of motion for operation with an excitation frequency coinciding with the system natural frequency.

Solution From the fundamental FRF amplitude ratio

$$\left|\frac{X}{u}\right| = J(r) = \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + (2\zeta\,r)^2}}$$



Set $r = 0.9$ and $|X/u| = J_a = 3$ and, and calculate the damping ratio ζ from algebraic equation.

$$\Rightarrow \zeta = \frac{1}{2r_a}\left[\frac{r_a^4}{J_a^2} - \left(1 - r_a^2\right)^2\right]^{1/2} = 0.107$$

Next, calculate the viscous damping coefficient, $D = \zeta D_c$

$r=1$, $|X/u| = 1/(2\zeta) = Q$. *Thus* $|X| = Q\, u$

The maximum amplitude of motion does not necessarily occur at $r=1$. The value of frequency ratio (r_*) which maximizes the response, i.e.

$\left(\partial \left|\frac{X}{u}\right| \Big/ \partial r\right) = 0$ is $r_* = \dfrac{1}{\sqrt{\left(1 - 2\zeta^2\right)}}$; and $\left|\dfrac{X}{u}\right|_{\max} = \dfrac{1}{2\zeta}\dfrac{1}{\sqrt{\left(1 - 2\zeta^2\right)}}$

Note that for small values of damping $\left|\dfrac{X}{u}\right|_{\max} \approx \dfrac{1}{2\zeta}$

PROBLEM TYPE 3

Consider a system excited by a periodic force of magnitude F_o and frequency Ω. Assume that the spring and dashpot connect to ground.

a) Determine the damping ratio needed such that the **transmitted force** to ground does not ever exceed (say) two times the input force for operation at a frequency (say) = 75% of natural frequency

$$F_{transmitted} = K\,X + D\,\dot{X}$$

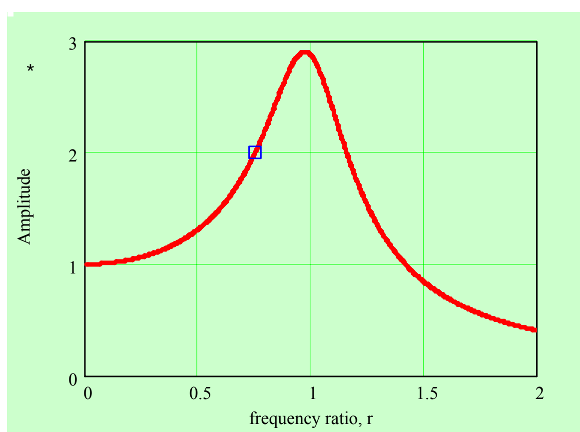
b) With the result of (a), determine the transmitted force to ground if the excitation frequency coincides with the system natural frequency. Is this the maximum transmissibility ever?

c) Provide a value of frequency such that the transmitted force is less than the applied force, irrespective of the damping in the system.

Solution: From the fundamental FRF amplitude

$$\left|\frac{F_{transmitted}}{F_o}\right| = A_{T(r)} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}$$

Set $A_T=2$ and $r=0.75$, and find the damping ratio ζ.



$$\Rightarrow \zeta = \frac{1}{2r_a}\left[\frac{1-A_T^2\left(1-r_a^2\right)^2}{A_T^2-1}\right]^{1/2}$$

$=0.186$

Next, calculate the viscous damping coefficient $D=\zeta D_c$

At resonance, $r=1$, $A_T = \frac{\left[1+(2\zeta)^2\right]^{.5}}{2\zeta}$. Then calculate the magnitude of the transmitted force.

Again, the maximum transmissibility occurs at a frequency f_* which satisfies $\left(\partial A_T / \partial r\right) = 0$. Perform the derivation and find a closed form solution.

Recall that operation at frequencies $r \geq \sqrt{2}$, i.e. for $\Omega \geq 1.414\ \omega_n$, (41 % above the natural frequency) determines transmitted forces less than the applied force (i.e. an **effective structural isolation** is achieved).

PROBLEM TYPE 4

Consider a system excited by a periodic force of magnitude $F_o=(2M\,g)$ (for example) and frequency Ω.

a) Determine the damping ratio ζ needed such that the maximum acceleration in the system does not exceed (say) **5 g**'s for operation at a frequency (say) 25% above the natural frequency of the system.
b) With the result of (a), determine the system acceleration for operation with an excitation frequency coinciding with the system natural frequency. Explain your result

Recall the periodic response is $X(t)=X_s H(r)\cos(\Omega t+\psi)$, then

$$\ddot{X}(t) = -\Omega^2 X_s H(r)\cos(\Omega t+\psi) = -\Omega^2 X(t)$$

Solution: From the amplitude of FRF

$$\left|\frac{\ddot{X}}{F_o / K}\right| = \frac{\omega_n^2 r^2}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}} \rightarrow \left|\frac{\ddot{X}}{F_o / M}\right| = \frac{r^2}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}}$$

Follow a similar procedure as in other problems above.

OTHER PROBLEMS

Think of similar problems and questions related to system dynamic performance.

In particular, you may also "cook up" similar questions related to the dynamic response of **first-order systems** (mechanical, thermal, electrical, etc). $M\dot{V} + DV = F_o \cos\left(\Omega t\right)$

Luis San Andrés - MEEN 363/617 instructor