DYNAMIC RESPONSE OF A SDOF SYSTEM TO ARBITRARY PERIODIC LOADS

Fourier Series

Forces acting on structures are frequently periodic or can be approximated closely by superposition of periodic loads. As illustrated, the function $F(t)$ is periodic but not harmonic.

Any periodic function, however, can be represented by a convergent series of harmonic functions whose frequencies are integer multiples of a certain fundamental frequency $\Omega$.

The integer multiples are called harmonics. The series of harmonic functions is known as a FOURIER SERIES, and written as

$$ F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega t) \tag{1} $$

with $F(t + T) = F(t)$ and where $T = \frac{2\pi}{\Omega}$ is the fundamental period. $a_n$, $b_n$ are the coefficients of the $n$th harmonic, and calculated from
\[ a_n = \frac{2}{T} \int_{t}^{t+T} F(t) \cos(n\Omega t) \, dt, \quad n = 0, 1, 2, \ldots \infty \]  

(2)

\[ b_n = \frac{2}{T} \int_{t}^{t+T} F(t) \sin(n\Omega t) \, dt, \quad n = 1, 2, \ldots \infty \]

each representing a measure of the participation of the harmonic content of \( \cos(n\Omega t) \) and \( \sin(n\Omega t) \), respectively. All the \( a_0, b_n, c_n \) have the units of a generalized force.

Note that \( \frac{1}{2} a_0 \) is the time averaged value of the function \( F(t) \).

In practice \( F(t) \) may be approximated by a relatively small number of terms. Some useful simplifications arise

If \( F(t) \) is an EVEN function, i.e., \( F(t) = F(-t) \) then, \( b_n = 0 \)

for all \( n \)

If \( F(t) \) is an ODD function, i.e., \( F(t) = -F(-t) \) then, \( a_n = 0 \)

for all \( n \)

The Fourier series representation, Eq. (1), can also be written as

\[
F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\Omega t - \beta_n)
\]

(3)

where \( c_n = \left(a_n^2 + b_n^2\right)^{1/2} \) and \( \beta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \), \( n = 1, 2, \ldots \infty \)

are the magnitude and phase angle respectively of each harmonic content.
PERIODIC FORCED RESPONSE OF AN UNDAMPED SDOF

In an undamped SDOF system, the steady state response (w/o the transient solution) produced by each sine and cosine term in the harmonic loading series is given by as

\[ X_s(t) = \frac{b_m}{K} \frac{\sin(m\Omega t)}{1 - f_m^2} \quad (4a) \]

\[ X_c(t) = \frac{a_m}{K} \frac{\cos(m\Omega t)}{1 - f_m^2} \quad m=1,2,\ldots \quad (4b) \]

where \( f_m = \frac{m\Omega}{\omega_n}, \quad \omega_n = \sqrt{K/M} \).

For the constant force \( a_0 \), the s-s response is simply

\[ X_0(t) = \frac{a_0}{K} \quad (4c) \]

Using the principle of superposition, then the total periodic response is expressed as the sum of the individual component as follows,

\[ X(t) = \frac{1}{K} \left( a_0 + \sum_{m=1}^{\infty} \frac{1}{1 - f_m^2} \left[ a_m \cos(m\Omega t) + b_m \sin(m\Omega t) \right] \right) \quad (5) \]

Note that for the undamped case if \( m\Omega = \omega_n \), i.e. there is a harmonic frequency equal to the natural frequency of the
system, then the system response will become UNBOUNDED (system failure).

PERIODIC FORCED RESPONSE OF A DAMPED SDOF

In a damped SDOF system, the **steady-state** response produced by each sine and cosine term in the harmonic load series is

\[
X_{c_m}(t) = \frac{a_m}{K} \left[ \frac{(1-f_m^2) \cos(m \Omega t) + (2 \zeta f_m) \sin(m \Omega t)}{(1-f_m^2)^2 + (2 \zeta f_m)^2} \right] \quad (6a)
\]

\[
X_{s_m}(t) = \frac{b_m}{K} \left[ \frac{-2 \zeta f_m \cos(m \Omega t) + (1-f_m^2) \sin(m \Omega t)}{(1-f_m^2)^2 + (2 \zeta f_m)^2} \right] \quad (6b)
\]

and for the constant term \( a_0 \), the s-s response is

\[
X_0 = \frac{a_0}{2K} \quad (6c)
\]

Superposition gives the total system response as

\[
X(t) = \frac{a_0}{2K} + \frac{1}{K} \sum_{m=1}^{\infty} \left[ \frac{a_m (1-f_m^2) - 2 \zeta f_m b_m}{(1-f_m^2)^2 + (2 \zeta f_m)^2} \cos(m \Omega t) \right] + \frac{1}{K} \sum_{m=1}^{\infty} \left[ \frac{b_m (1-f_m^2) + 2 \zeta f_m a_m}{(1-f_m^2)^2 + (2 \zeta f_m)^2} \sin(m \Omega t) \right] \quad (7)
\]
or,

\[
X(t) = \frac{a_0}{2K} + \frac{1}{K} \sum_{m=1}^{\infty} \left[ \frac{c_m}{(1 - f_m^2)^2 + (2 \zeta f_m \gamma)^2} \cos(m \Omega t - \gamma_m) \right]
\]

(8)

where,

\[ f_m = \frac{m \Omega}{\omega_n}; \quad \omega_n = \sqrt{K/M} \]

and

\[ c_m = \sqrt{a_m^2 + b_m^2} \]

\[ \gamma_m = \tan^{-1} \left( \frac{b_m (1 - f_m^2) + 2 \zeta f_m a_m}{a_m (1 - f_m^2) - 2 \zeta f_m b_m} \right), \quad m = 1, 2, \ldots, \infty \]
RESPONSE OF A SDOF SYSTEM TO NON-PERIODIC (ARBITRARY) FORCE EXCITATIONS

The steady-state response of a SDOF system to a periodic excitation of fundamental period $T$ is also periodic with a fundamental period $T=2\pi/\Omega$.

Consider now an arbitrary external force (transient, non-periodic, etc.) Clearly, in this case there is no steady-state response and the entire system response may be regarded as transient.

Various approaches can be used to obtain the system dynamic response, including direct numerical integration. A natural extension would be to use Fourier Integral transforms obtained from the Fourier series in the limit as the period $T \to \infty$. This topic is discussed at length in your textbook.

Another way to find the dynamic response to arbitrary load excitations is to regard the acting force function as a superposition of impulses of very short duration.

First, introduce the concept of unit impulse or Direct Delta Function as shown in the Figure. The mathematical definition of a unit impulse is

$$\delta(t-a) = 0 \text{ for } t \neq a \quad (9)$$

and such that

$$\int_{-\infty}^{+\infty} \delta(t-a) \, dt = 1$$
The time over which the function $\delta$ is different from zero is infinitesimally small, $\varepsilon \rightarrow 0$. Note that the physical units of $\delta$ are 1/sec.

An impulsive force of arbitrary magnitude $F$ applied at time $t=a$ is written conveniently as

$$F(t) = \hat{F} \delta(t-a)$$  \hspace{1cm} (10)

where $\hat{F}$ has the units of impulse, i.e. N-sec or lb-sec.

Now, consider a damped SDOF mechanical system and find the response to the impulsive force applied at time $a=0$, i.e.

$$M \dddot{X} + D \dot{X} + K X(t) = \hat{F} \delta(t-a)$$  \hspace{1cm} (11)

Recall that the force $F(t)$ acts over a very short time, $\varepsilon \rightarrow 0$. Now, integrate Eq. (11) over time in the interval $\Delta t = \varepsilon \rightarrow 0$,

$$\int_{0}^{\varepsilon} \left( M \dddot{X} + D \dot{X} + K X \right) dt = \int_{0}^{\varepsilon} \hat{F} \delta(t) dt = \hat{F} \int_{0}^{\varepsilon} \delta(t) dt = \hat{F}$$  \hspace{1cm} (12)

Then,

$$\lim_{\varepsilon \rightarrow 0} \int_{0}^{\varepsilon} M \dddot{X} \ dt = M \dddot{X}\bigg|_{0}^{\varepsilon} = M \left[ \dddot{X}(\varepsilon) - \dddot{X}(0) \right] = M \left[ \dddot{X}(0+) - \dddot{X}(0) \right]$$

$$\lim_{\varepsilon \rightarrow 0} \int_{0}^{\varepsilon} D \dot{X} \ dt = D \lim_{\varepsilon \rightarrow 0} \dot{X}\bigg|_{0}^{\varepsilon} = D \lim_{\varepsilon \rightarrow 0} \left[ X(0+) - X(0) \right] = 0$$
The notation $\dot{X}(0+)$ is interpreted as a change in velocity that occurs shortly after the time $\Delta t = \epsilon \to 0$ elapses. Note that there is no instantaneous change in displacement $X(0+) \approx X(0)$ because $\Delta t$ is too short for displacements to happen (no displacement jump!). Thus,

$$\dot{X}(0+) = \frac{\hat{F}}{M}$$  \hspace{1cm} (13)$$

when $\dot{X}(0) = 0$ (initial null velocity).

As a physical interpretation, the impulsive force produces an instantaneous change in velocity. Hence, one can regard the effect of the impulse applied at $t=0$ as the equivalent of an initial velocity equal to $(\hat{F}/M)$.

Recall that the free vibration response of an underdamped $((\zeta < 1) < 1)$ SDOF system to an initial velocity is,

$$X(t) = e^{-\zeta \omega_n t} \frac{1}{\omega_d M} \frac{\hat{F}}{\sin(\omega_d t)} \quad t > 0, \quad \zeta < 1$$  \hspace{1cm} (14)$$

where $\omega_n = \sqrt{\frac{K}{M}}$ and $\omega_d = \omega_n \sqrt{1 - \zeta^2}$.
The **unit impulse response** \( h(t) \) is simply obtained by letting \( \hat{F} = 1 \), so that

\[
h(t) = e^{-\zeta \omega_n t} \frac{1}{M \omega_d} \sin(\omega_d t); \quad t > 0
\]  

(15)

Let’s find the system response to an arbitrary force excitation function \( F(t) \). **Interpret** \( F(t) \) **as a train of (short time)** impulsive impulses of varying amplitude.

As shown in the Figure, at arbitrary time \( t = \tau \) and corresponding to the time increment \( \Delta \tau \), there is an impulse of magnitude \( F_\tau \Delta \tau \), and expressed as \( F_\tau \Delta \tau \delta_{(\tau-\tau)} \),

The response to a unit load impulse at \( t = \tau \) is \( h_{(\tau-\tau)} \). Then the contribution of \( F_\tau \Delta \tau \delta_{(\tau-\tau)} \) to the total response at time \( t \) is

\[
\Delta X_{(t,\tau)} = F_\tau \Delta \tau h_{(\tau-\tau)}
\]  

(16)

Thus, the total response is

\[
X_{(t)} = \sum F_\tau h_{(\tau-\tau)} \Delta \tau
\]  

(17)

As \( \Delta \tau \to 0 \), the summation becomes an integral, i.e.

\[
X_{(t)} = \int_{0}^{t} F_{\tau} h_{(\tau-\tau)} d\tau
\]  

(18)
This is known as the **Convolution or Duhamel's integral** and expresses the system response as the superposition of individual responses to impulse loads.

Replacing $h(t-\tau)$ from Eq. (15) into Eq. (18), gives the system response as

$$X(t) = \frac{1}{M \omega_d} \int_0^t F(\tau) e^{-\zeta \omega_d (t-\tau)} \sin(\omega_d [t-\tau]) \, d\tau$$

(19)

for all $t > 0$, with $\omega_n = \sqrt{K/M}$, $\omega_d = \omega_n \sqrt{1-\zeta^2}$

For **non-zero initial conditions in displacement and velocity**, the **superposition principle** allows to express the total response of the underdamped ($\zeta < 1$) SDOF to an arbitrary excitation force $F(t)$, i.e.

$$X(t) = e^{-\zeta \omega_n t} \left( X_0 \cos(\omega_d t) + \left( \frac{\dot{X}_0 + \zeta \omega_n X_0}{\omega_d} \right) \sin(\omega_d t) \right) +$$

$$\frac{1}{M \omega_d} \int_0^t F(\tau) e^{-\zeta \omega_n (t-\tau)} \sin[\omega_d (t - \tau)] \, d\tau$$

(20)

Note that for a SDOF system without any viscous damping, $\zeta = 0$, Eq. (20) simplifies to

$$X(t) = X_0 \cos(\omega_n t) + \frac{\dot{X}_0}{\omega_n} \sin(\omega_n t) + \frac{1}{M \omega_n} \int_0^t F(\tau) \sin[\omega(t - \tau)] \, d\tau$$

(21)
**Homework exercises:**
Determine ANALITICALLY the time response of an undamped ($\zeta=0$) SDOF system to the forcing functions depicted below.

![Diagram](image)

**Answers:**

\[
X(t) = \frac{Q_1}{K} \left[ 1 - \cos(\omega_n t) \right] \quad 0 \leq t \leq t_1
\]

\[
X(t) = \frac{Q_1}{K} \left[ \cos(\omega_n (t-t_1)) - \cos(\omega_n t) \right] - \frac{Q_2}{K} \left[ 1 - \cos(\omega_n (t_1-t_1)) \right] \quad t_1 \leq t \leq t_2
\]

a) \[
X(t) = \frac{Q_1}{K} \left[ \cos(\omega_n (t-t_1)) - \cos(\omega_n t) \right] - \frac{Q_2}{K} \cos(\omega_n (t-t_2)) \quad t_2 \leq t
\]
b) 

\[ X(t) = \frac{Q_1}{K} \left( 1 - \cos (\omega_n t) - \frac{\left\{ t - 1/\omega_n \sin (\omega_n t) \right\}}{\omega_n t_1} \right) 0 \leq t \leq t_1 \]

\[ X(t) = \frac{Q_1}{K} \left( -\cos (\omega_n t) + \frac{\left\{ \sin (\omega_n t) - \sin (\omega_n (t - t_1)) \right\}}{\omega_n t_1} \right) t_1 \leq t \]

where \( \omega_n = \sqrt{K/M} \)