APPENDIX C. DERIVATION OF EQUATIONS OF MOTION FOR MULTIPLE DEGREE OF FREEDOM SYSTEM

Consider a linear mechanical system with n-independent degrees of freedom. Let $\{x_i(t)\}_{i=1,..n}$ be the independent coordinates describing the motion of the system <u>about an equilibrium position</u>, and with $\{F_{i(t)}\}_{i=1,..n}$ as the set of external forces applied at each degree of freedom. The kinetic and potential energy of the system can be written in the following form,

$$T = \tfrac{1}{2}[\dot{x}]^T [M][\dot{x}], \;\; V = \tfrac{1}{2}[x]^T [K][x] \qquad (1)$$

where $[x] = \{x_1\, x_2\, x_3 \,.......x_n\}^T$, and $[\dot{x}] = \{\dot{x}_1\, \dot{x}_2\, \dot{x}_3 \,.......\dot{x}_n\}^T$ are the vectors of displacements and velocities; $[M]=\{m_{i,j}\}_{i,j=1,n}$ and $[K]=\{k_{i,j}\}_{i,j=1,n}$ are the *(nxn)* matrices of generalized inertia and stiffness coefficients, respectively. The elements of these matrices are constant coefficients.

Note that energies are scalar functions, i.e. $T=T^T$. Thus, taking the transpose of the potential energy we can easily show that the stiffness matrix must be symmetric, i.e.

$$\begin{aligned} &V = \tfrac{1}{2}[x]^T [K][x] = V^T = \tfrac{1}{2}\left([K][x]\right)^T \left([x]^T\right)^T = \tfrac{1}{2}[x]^T [K]^T [x], \\ &V - V^T = 0 = \tfrac{1}{2}[x]^T \left\{[K]-[K]^T\right\}[x], \quad \longrightarrow [K]^T = [K] \end{aligned} \qquad (2)$$

and it also follows that $[M]^T = [M]$. We have used in equation (2) above the following fundamental matrix property $\boldsymbol{(A^T B)^T = B^T A}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are general matrices.

The **viscous power dissipation** and **viscous dissipated energy** are given by the equations

$$P_v = [\dot{x}]^T [D][\dot{x}], \qquad E_v = \int_0^t P_v\, dt \qquad (3)$$

where $[D]=\{d_{i,j}\}_{i,j=1,n}$ is a matrix of constant damping coefficients. The work performed by external forces is given by,

$$W = \int d[x]^T [F] \qquad (4)$$

The **principle of conservation of mechanical** energy establishes that for any instant of time,

$$T + V + E_v = W + T_0 + V_0 \tag{5}$$

where $T_0 = \tfrac{1}{2}\left[\dot{x}_0\right]^T \left[M\right]\left[\dot{x}_0\right],\;\; V_0 = \tfrac{1}{2}\left[x_0\right]^T \left[K\right]\left[x_0\right]$ (6)

are the initial values of the system kinetic and potential energies, respectively.

Now, take the time derivative of equation (5) to obtain

$$\frac{d}{dt}\left(T + V + E_v - W\right) = 0 \tag{7}$$

Now,

$$\frac{d}{dt}(T) = \tfrac{1}{2}[\ddot{x}]^T[M][\dot{x}] + \tfrac{1}{2}[\dot{x}]^T[M][\ddot{x}] = \tfrac{1}{2}\left\{\left([\ddot{x}]^T[M][\dot{x}]\right)^T + [\dot{x}]^T[M][\ddot{x}]\right\} =$$

$$= \tfrac{1}{2}\left\{[\dot{x}]^T[M]^T[\ddot{x}] + [\dot{x}]^T[M][\ddot{x}]\right\} = \tfrac{1}{2}\left\{[\dot{x}]^T[M][\ddot{x}] + [\dot{x}]^T[M][\ddot{x}]\right\} = [\dot{x}]^T[M][\ddot{x}]$$

$$\frac{d}{dt}(V) = \tfrac{1}{2}[\dot{x}]^T[K][x] + \tfrac{1}{2}[x]^T[K][\dot{x}] = \tfrac{1}{2}\left\{[\dot{x}]^T[K][x] + \left([x]^T[K][\dot{x}]\right)^T\right\}$$

$$= \tfrac{1}{2}\left\{[\dot{x}]^T[K][x] + [\dot{x}]^T[K]^T[x]\right\} = \tfrac{1}{2}\left\{[\dot{x}]^T[K][x] + [\dot{x}]^T[K][x]\right\} = [\dot{x}]^T[K][x] \tag{8}$$

Since $[K]=[K]^T$ and $[M]=[M]^T$ and with $[\ddot{x}] = \left\{\ddot{x}_1\, \ddot{x}_2\, \ddot{x}_3 \,.......\ddot{x}_n\right\}^T$ as a vector of accelerations. Also,

$$\frac{d}{dt}E_v = \frac{d}{dt}\int_0^t P_v\, dt = P_v = \left[\dot{x}\right]^T\left[D\right]\left[\dot{x}\right] \tag{9}$$

And

$$\frac{d}{dt}W = \frac{d}{dt}\int d[x]^T[F] = \frac{d[x]^T}{dt}\left[F(t)\right] = [\dot{x}]^T\left[F(t)\right] \tag{10}$$

Substitution of equations (8), (9) and (10) into equation (7) gives

$$[\dot{x}]^T[M][\ddot{x}] + [\dot{x}]^T[K][x] + [\dot{x}]^T[D][\dot{x}] - [\dot{x}]^T[F] = 0$$

$$[\dot{x}]^T\left\{[M][\ddot{x}]+[K][x]+[D][\dot{x}]-[F]\right\}=0,$$

and since $[\dot{x}]^T \neq [0]$, then

Thus, the n-equations of motion for the n-dof system are given by

$$[M][\ddot{x}]+[D][\dot{x}]+[K][x]=[F(t)] \qquad (11).$$

The difficulty in using this approach is to devise a simple method to establish the elements of the matrices *[M], [K], [D]*. The use of the Lagrangian equations of motion is particularly useful in this case.

Derivation of equations of motion using Lagrange's approach[1]

Consider a mechanical system with n-independent degrees of freedom, and where $\left\{x_i, \dot{x}_i\right\}_{i=1,....n}$ are the generalized coordinates and velocities for each degree of freedom in the system. The Work performed on the system by external generalized forces is given by

$$W=\int\left(F_1\,dx_1+F_2\,dx_2+F_3\,dx_4+....F_n\,dx_n\right)=\int\sum_{i=1}^{n}F_i\,dx_i \qquad (12)$$

Here we use the term generalized to denote that the product of a generalized displacement, say x_i , and the generalized effort, F_i , produce units of work [N.m]. For example if $x_2=\theta$ denotes an angular coordinate, then the effort F_2 corresponds to a moment or torque.

Let the **total kinetic energy** and **potential energy** of the n-dof mechanical system be given by the generic expressions

$$\begin{aligned} T &= f\left\{\dot{x}_1\,\dot{x}_2\,\dot{x}_3.......\dot{x}_n\,,x_1,x_2\,,.....,x_n\,,t\right\} \\ V &= g\left\{x_1,x_2\,,.....,x_n\,,t\right\} \end{aligned} \qquad (13)$$

The kinetic energy above is a function of the generalized displacements, velocities and time, while the potential energy in a conservative system is only a function of the generalized displacements and time.

[1] Sources Meirovitch, L., Analytical Methods in Vibrations, pp. 30-50, and San Andrés, L., Vibrations Class Notes, 1996.

The viscous dissipated power is a general function of the velocities, i.e.

$$P_v = P_v \left\{ \dot{x}_1 \, \dot{x}_2 \, \dot{x}_3 \, ... \dot{x}_n \right\} \tag{14}$$

The n-equations of motion for the system are derived using the Lagrangian approach, i.e.

$$\frac{\partial}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} + \frac{1}{2}\frac{\partial P_v}{\partial \dot{x}_i} = F_i \qquad i=1,2,....n \tag{15}$$

Once you have performed the derivatives above for each coordinate, $i=1,\ldots n$, the resulting equations are of the form:

$$\begin{aligned}
&m_{11}\ddot{x}_1 + ... + m_{1n}\ddot{x}_n + d_{11}\dot{x}_1 + ... + d_{1n}\dot{x}_n + k_{11}x_1 + ... + k_{1n}x_n = F_1 \\
&m_{21}\ddot{x}_1 + ... + m_{2n}\ddot{x}_n + d_{21}\dot{x}_1 + ... + d_{2n}\dot{x}_n + k_{21}x_1 + ... + k_{2n}x_n = F_2 \\
&.... \\
&m_{n1}\ddot{x}_1 + ... + m_{nn}\ddot{x}_n + d_{n1}\dot{x}_1 + ... + d_{nn}\dot{x}_n + k_{n1}x_1 + ... + k_{nn}x_n = F_n
\end{aligned} \tag{16}$$

or written in matrix form as

$$[M][\ddot{x}] + [D][\dot{x}] + [K][x] = [F] \tag*{(17)=(11)}$$