APPENDIX C. DERIVATION OF EQUATIONS OF MOTION FOR MULTIPLE DEGREE OF FREEDOM SYSTEM

Consider a linear mechanical system with n-independent degrees of freedom. Let \( \{x_i(t)\}_{i=1,n} \) be the independent coordinates describing the motion of the system about an equilibrium position, and with \( \{F_{i(t)}\}_{i=1,n} \) as the set of external forces applied at each degree of freedom. The kinetic and potential energy of the system can be written in the following form,

\[
T = \frac{1}{2} \left[ \dot{x} \right]^T [M] [x], \quad V = \frac{1}{2} \left[ x \right]^T [K] [x] \tag{1}
\]

where \( \left[ x \right] = [x_1 \ x_2 \ x_3 \ \ldots \ x_n]^T \), and \( \left[ \dot{x} \right] = [\dot{x}_1 \ \dot{x}_2 \ \dot{x}_3 \ \ldots \ \dot{x}_n]^T \) are the vectors of displacements and velocities; \( [M] = \{m_{i,j}\}_{i,j=1,n} \) and \( [K] = \{k_{i,j}\}_{i,j=1,n} \) are the \((nxn)\) matrices of generalized inertia and stiffness coefficients, respectively. The elements of these matrices are constant coefficients.

Note that energies are scalar functions, i.e. \( T = T^T \). Thus, taking the transpose of the potential energy we can easily show that the stiffness matrix must be symmetric, i.e.

\[
\frac{1}{2}[x]^T[K][x] = V = \frac{1}{2} \left( [K][x] \right)^T \left[ x \right]^T = \frac{1}{2} \left[ x \right]^T [K]^T [x], \quad V - V^T = 0 = \frac{1}{2}[x]^T \left\{ [K] - [K]^T \right\} [x], \quad \rightarrow [K]^T = [K] \tag{2}
\]

and it also follows that \( [M]^T = [M] \). We have used in equation (2) above the following fundamental matrix property \((A^T B)^T = B^T A\), where \( A \) and \( B \) are general matrices.

The **viscous power dissipation** and **viscous dissipated energy** are given by the equations

\[
P_v = \left[ \dot{x} \right]^T [D] [\dot{x}], \quad E_v = \int_0^T P_v \, dt \tag{3}
\]

where \( [D] = \{d_{i,j}\}_{i,j=1,n} \) is a matrix of constant damping coefficients. The work performed by external forces is given by,

\[
W = \int d[x]^T [F] \tag{4}
\]
The principle of conservation of mechanical energy establishes that for any instant of time,

\[ T + V + E_v = W + T_0 + V_0 \]  

(5)

where \( T_0 = \frac{1}{2} [\dot{x}_0]^T [M][\dot{x}_0] \), \( V_0 = \frac{1}{2} [x_0]^T [K][x_0] \)

(6)

are the initial values of the system kinetic and potential energies, respectively.

Now, take the time derivative of equation (5) to obtain

\[ \frac{d}{dt}(T + V + E_v - W) = 0 \]  

(7)

Now,

\[ \frac{d}{dt}(T) = \frac{1}{2} [\ddot{x}]^T [M][\ddot{x}] + \frac{1}{2} [\dot{x}]^T [M][\dot{x}] = \frac{1}{2} \left( [\ddot{x}]^T [M][\ddot{x}] + [\dot{x}]^T [M][\dot{x}] \right) = [\dot{x}]^T [M][\dot{x}] \]  

(8)

\[ \frac{d}{dt}(V) = \frac{1}{2} [x]^T [K][x] + \frac{1}{2} [\dot{x}]^T [K][\dot{x}] = \frac{1}{2} \left( [x]^T [K][x] + [\dot{x}]^T [K][\dot{x}] \right) = [\dot{x}]^T [K][x] \]  

\[ \frac{d}{dt}(E_v) = \frac{d}{dt} \int_0^t P_v dt = P_v = [\dot{x}]^T [D][\dot{x}] \]  

(9)

And

\[ \frac{d}{dt} W = \frac{d}{dt} \int d[x]^T [F] = \frac{d}{dt} [x]^T [F(t)] = [\dot{x}]^T [F(t)] \]  

(10)

Substitution of equations (8), (9) and (10) into equation (7) gives

\[ [\dot{x}]^T [M][\dot{x}] + [\dot{x}]^T [K][x] + [\dot{x}]^T [D][\dot{x}] - [\dot{x}]^T [F] = 0 \]
\[
[\ddot{x}] = \left( [M][\ddot{x}] + [K][x] + [D][\dot{x}] - [F] \right) = 0,
\]

and since \([\ddot{x}] \neq [0]\), then

Thus, the n-equations of motion for the n-dof system are given by

\[
[M][\ddot{x}] + [D][\dot{x}] + [K][x] = [F(t)]
\]

(11).

The difficulty in using this approach is to devise a simple method to establish the elements of the matrices \([M], [K], [D]\). The use of the Lagrangian equations of motion is particularly useful in this case.

### Derivation of equations of motion using Lagrange’s approach

Consider a mechanical system with n-independent degrees of freedom, and where \(\{x_i, \dot{x}_i\}_{i=1,\ldots,n}\) are the generalized coordinates and velocities for each degree of freedom in the system. The Work performed on the system by external generalized forces is given by

\[
W = \int \left( F_1 \, dx_1 + F_2 \, dx_2 + F_3 \, dx_3 + \ldots + F_n \, dx_n \right) = \sum_{i=1}^{n} F_i \, dx_i
\]

(12)

Here we use the term generalized to denote that the product of a generalized displacement, say \(x_i\), and the generalized effort, \(F_i\), produce units of work \([N.m]\). For example if \(x_2 = \theta\) denotes an angular coordinate, then the effort \(F_2\) corresponds to a moment or torque.

Let the **total kinetic energy** and **potential energy** of the n-dof mechanical system be given by the generic expressions

\[
T = f \left\{ \dot{x}_1 \, \dot{x}_2 \, \ldots \, \dot{x}_n , x_1 , x_2 , \ldots , x_n , t \right\}
\]

\[
V = g \left\{ x_1 , x_2 , \ldots , x_n , t \right\}
\]

(13)

The kinetic energy above is a function of the generalized displacements, velocities and time, while the potential energy in a conservative system is only a function of the generalized displacements and time.

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The viscous dissipated power is a general function of the velocities, i.e.

$$P_v = P_v \left\{ \dot{x}_1, \dot{x}_2, \dot{x}_3, \ldots, \dot{x}_n \right\}$$  \hspace{1cm} (14)

The n-equations of motion for the system are derived using the Lagrangian approach, i.e.

$$\frac{\partial}{\partial t} \left( \frac{\partial T}{\partial \dot{x}_i} \right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} + \frac{1}{2} \frac{\partial P_v}{\partial \dot{x}_i} = F_i \quad i=1,2,\ldots,n$$ \hspace{1cm} (15)

Once you have performed the derivatives above for each coordinate, $i=1,\ldots,n$, the resulting equations are of the form:

\begin{align*}
    m_{11} \ddot{x}_1 + \ldots + m_{1n} \ddot{x}_n + d_{11} \dot{x}_1 + \ldots + d_{1n} \dot{x}_n + k_{11} x_1 + \ldots + k_{1n} x_n &= F_1 \\
    m_{21} \ddot{x}_1 + \ldots + m_{2n} \ddot{x}_n + d_{21} \dot{x}_1 + \ldots + d_{2n} \dot{x}_n + k_{21} x_1 + \ldots + k_{2n} x_n &= F_2 \\
    &\ldots \\
    m_{n1} \ddot{x}_1 + \ldots + m_{nn} \ddot{x}_n + d_{n1} \dot{x}_1 + \ldots + d_{nn} \dot{x}_n + k_{n1} x_1 + \ldots + k_{nn} x_n &= F_n
\end{align*} \hspace{1cm} (16)

or written in matrix form as

$$[M][\ddot{x}] + [D][\dot{x}] + [K][x] = [F]$$ \hspace{1cm} (17)=(11)