NOTES 5
DYNAMICS OF A RIGID ROTOR-FLUID FILM BEARING SYSTEM

Lecture 5 restates the analysis for static equilibrium in a journal bearing. Next, it considers the dynamics of the simplest rigid rotor bearing system supported on journal bearings. For small amplitude journal motions about an equilibrium position, the analysis proceeds to linearize the fluid film forces and introduces the concepts of bearing force coefficients, namely, 4 stiffnesses, 4 damping and 4 inertia coefficients. Formulas for the direct and cross-coupled stiffness and damping coefficients of a short length journal bearing are derived. The analysis focuses on the stability of the rigid rotor-bearing system to determine the threshold rotor speed at which the system loses its equivalent damping and develops ever growing motions at a whirl frequency that coincides with the rotor-bearing system natural frequency. The low and high magnitudes of the Sommerfeld number show whereas the system operates stably or not. The ½ whirl frequency ratio reveals a typical stability limit of lubricated journal bearings. The effect of rotor flexibility on further reducing the threshold speed of instability is noted since the rotor-bearing natural frequency is also lowered. An appendix provides a physical explanation of the follower force, induced by the cross-coupled stiffnesses, that drives the rotor bearing system into swirl. Remedies to avoid or delay the instability are given. Actual examples of instabilities and measurements in the author’s laboratory make evident the harmful, potentially catastrophic, whirl instability. A list of industrial or commercial bearing configurations with noted advantages and disadvantages complements the lecture.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>C</td>
<td>Radial clearance [m]</td>
</tr>
<tr>
<td>Cij</td>
<td>Bearing damping force coefficients, , i,j=X,Y [N.s/m]</td>
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<tr>
<td>D=2R</td>
<td>Bearing diameter</td>
</tr>
<tr>
<td>e</td>
<td>Journal eccentricity [m]</td>
</tr>
<tr>
<td>e, eϕ</td>
<td>Journal center radial and tangential velocities [m/s]</td>
</tr>
<tr>
<td>F</td>
<td>Fluid film force acting on journal surface [N].</td>
</tr>
<tr>
<td>Fo</td>
<td>½ Static load [N]</td>
</tr>
<tr>
<td>h</td>
<td>Film thickness, H=h/c</td>
</tr>
<tr>
<td>Kij</td>
<td>Bearing stiffness force coefficients, , i,j=X,Y [N/m]</td>
</tr>
<tr>
<td>Ke</td>
<td>Bearing equivalent stiffness [N/m]</td>
</tr>
<tr>
<td>Krot</td>
<td>Elastic rotor stiffness (one side) [N/m]</td>
</tr>
<tr>
<td>L</td>
<td>Bearing axial length</td>
</tr>
<tr>
<td>M</td>
<td>½ Mass of rigid rotor [kg]</td>
</tr>
<tr>
<td>Mij</td>
<td>Bearing added mass force coefficients, , i,j=X,Y [N.s²/m]</td>
</tr>
<tr>
<td>mc=Ps²</td>
<td>Dimensionless critical mass</td>
</tr>
<tr>
<td>P</td>
<td>Hydrodynamic pressure [Pa]</td>
</tr>
<tr>
<td>R</td>
<td>Bearing radius [m]</td>
</tr>
<tr>
<td>r, t</td>
<td>Moving coordinate system</td>
</tr>
<tr>
<td>S</td>
<td>Sommerfeld number</td>
</tr>
<tr>
<td>t</td>
<td>Time (s)</td>
</tr>
<tr>
<td>u</td>
<td>Mass imbalance [kg]</td>
</tr>
</tbody>
</table>
$X, Y$ Inertial coordinates system

$\Delta X, \Delta Y$ Small amplitude displacements in $(X,Y)$ coordinate system

$\Delta e, e_o, \Delta \phi$ Small amplitude displacements in $(r,t)$ coordinate system

$\varepsilon$ $e/C$. Journal eccentricity ratio

$\delta$ $u/C$. imbalance parameter

$\rho$ Fluid density [kg/m$^3$]

$\sigma$ $\frac{\mu \Omega LR}{4F_o} \left( \frac{L}{C} \right)^2$. Modified Sommerfeld number

$\mu$ Absolute viscosity [N.s/m$^2$]

$\Gamma$ $F_o / K_{rot}$. Static (elastic) sag at rotor midspan [m]

$\phi$ Journal attitude angle

$\Theta, \theta$ Circumferential coordinates

$\omega$ Characteristic whirl frequency [rad/s]

$\omega_n$ $(K_{eq}/M)^{1/2}$. Rotor-bearing system natural frequency [rad/s]

$\Omega$ Journal rotational speed [rad/s]

$\Omega_s$ Threshold speed of instability [rad/s]

**Subscripts**

$a$ Ambient value

$o$ Static or equilibrium condition

$s, fs$ Threshold of instability for rigid and flexible rotor

$XX, XY, YX, YY$ Indices of force coefficients in fixed $(X,Y)$ coordinate system

$rr, rt, tr, tt$ Indices of force coefficients in moving $(r,t)$ coordinate system
Equations of motion of a rigid rotor supported on plain journal bearings

Consider, as shown in Fig. 5.1, a symmetric rigid rotor of mass $2M$ that carries a static load $(2F_o=W)$ along the $X$ axis. Two identical plain journal bearings support the rotor. The equations of motion of the rotating system at constant rotational speed $\Omega$ are given by:

$$
M \ddot{X} = F_X + M\Omega^2 \sin(\Omega t) + F_o,
M \ddot{Y} = F_Y + M\Omega^2 \cos(\Omega t)
$$

where $u$ is the magnitude of the imbalance vector, $X(t)$ and $Y(t)$ are the coordinates of the rotor mass center, and $(F_X, F_Y)$ are the fluid film bearing reaction forces.

Since the rotor is rigid, the center of mass displacements are identical to those of the journal bearing centers, i.e.

$$
X(t) = e_X(t), \quad Y(t) = e_Y(t)
$$

The rotor-bearing static equilibrium is defined by

$$
F_{Xo} = -F_o, \quad F_{Yo} = 0, \quad \Rightarrow e_{Yo} = e_{Yo}, \quad \phi_o
$$

where $(e_o, \phi_o)$ are the static equilibrium journal eccentricity and attitude angle, respectively. The static fluid film reaction force components are such that:
Recall that $F = \sqrt{F_r^2 + F_t^2} = \sqrt{F_{r_0}^2 + F_{t_0}^2}$

At equilibrium, the region of positive fluid film pressure extends from $\theta_1 = 0$ to $\theta_2 = \pi$. In a short length journal bearing, the radial and tangential components of the static fluid film force $F_o$ are

\[
F_{r_0} = -\frac{\mu R L^3 \Omega}{C^2} \frac{\varepsilon^2}{(1-\varepsilon^2)^2}; \quad F_{t_0} = +\frac{\mu R L^3 \Omega}{C^2} \frac{\pi \cdot \varepsilon}{4(1-\varepsilon^2)^{3/2}}
\]

where $R = D/2$, $L$ and $C$ are the journal radius, axial length and radial clearance, respectively. $\varepsilon = e/C$ is the journal center eccentricity ratio, $\varepsilon < 1.0$; $\mu$ is the lubricant absolute viscosity, and $\Omega = (\text{rpm} \pi/30)$ is the rotor speed in rad/s. Figure 2 depicts the force components, radial and tangential, growing rapidly (non-linearly) with the journal eccentricity $e/C$.

Note that the short length bearing forces are proportional to the lubricant viscosity and rotor surface speed ($\Omega R$), the bearing length ($L^3$), and inversely proportional to the radial clearance ($C^2$). Most importantly, the bearing forces grow rapidly (non-linearly) with the journal eccentricity ($\varepsilon = e/C$).

Each bearing reaction force balances a fraction of the applied static load $F_o = \frac{1}{2} W$ for a symmetric rotor bearing system. Thus,

\[
F_o = \left( F_{r_0}^2 + F_{t_0}^2 \right)^{1/2} = \mu\Omega R \left( \frac{L}{C} \right)^2 \frac{\varepsilon}{4} \frac{\sqrt{16\varepsilon^2 + \pi^2 (1-\varepsilon^2)}}{(1-\varepsilon^2)^2}
\]
The equilibrium attitude angle ($\phi_o$) between the static load direction and the eccentricity vector is
\[
\tan(\phi_o) = -\frac{F_o}{F_r} = \frac{\pi \sqrt{1 - \varepsilon^2}}{4 \cdot \varepsilon}
\]  (5.7)

Note that as $\varepsilon \to 0$, $\phi_o \to \frac{\pi}{2}$ (journal eccentricity is perpendicular to the static load direction), whereas $\varepsilon \to 1$, $\phi_o \to 0$ (journal eccentricity parallel or aligned to load direction).

The bearing design parameter is the modified Sommerfeld number ($\sigma$)
\[
\frac{\mu \Omega LR}{4F_o} \left(\frac{L}{C}\right)^2 = \sigma = \frac{\left(1 - \varepsilon^2\right)^2}{\varepsilon \sqrt{16 \varepsilon^2 + \pi^2 (1 - \varepsilon^2)}}
\]  (5.8)

For a rated operating condition, $\sigma$ is known since the bearing geometry, speed, fluid type (viscosity) and load are known. Then Eqn. (5.8) gives a relationship to determine (iteratively) the equilibrium eccentricity ratio, $\varepsilon = e/c$, that generates the film reaction force balancing the applied static load $F_o$. Recall that,

**Large Sommerfeld ($\sigma$) numbers** (small load $W$, high speed $\Omega$, large lubricant viscosity $\mu$) determine small operating journal eccentricities or nearly centered operation, i.e. $\varepsilon \to 0.0$ and attitude angles approaching 90°; and

**Small Sommerfeld ($\sigma$) numbers** (large load $W$, low speed $\Omega$, light lubricant viscosity $\mu$) determine large operating eccentricities, i.e. $\varepsilon \to 1.0$ and attitude angle approaching 0°

Figures 4.6-8 in Lecture 4 depict the Sommerfeld number and attitude angle versus the journal eccentricity and the locus of the journal center within the clearance circle. The same figures are reproduced here in a smaller format.
Consider, as represented in Figure 5.3, small amplitude journal motions about the equilibrium position. These motions are defined as

\[ e_x = e_{x_o} + \Delta e_x(t), \quad e_y = e_{y_o} + \Delta e_y(t) \]  \hspace{1cm} (5.9.a)

or

\[ X = X_o + \Delta X(t), \quad Y = Y_o + \Delta Y(t) \] \hspace{1cm} (5.9.b)

or conversely,

\[ e(t) = e_o + \Delta e(t), \quad \phi(t) = \phi_o + \Delta \phi(t) \] \hspace{1cm} (5.9.c)

with

\[ \frac{dX}{dt} = \dot{e}_x = \Delta \dot{e}_x, \quad \frac{dY}{dt} = \dot{e}_y = \Delta \dot{e}_y \]

\[ \frac{d^2X}{dt^2} = \ddot{e}_x = \Delta \ddot{e}_x, \quad \frac{d^2Y}{dt^2} = \ddot{e}_y = \Delta \ddot{e}_y \] \hspace{1cm} (5.10)

The journal dynamic displacements in the \((r, t)\) coordinate system are related to those in the \((X,Y)\) fixed system by the linear transformation

\[ \begin{bmatrix} \Delta e_x \\ \Delta e_y \end{bmatrix} = \begin{bmatrix} \cos \phi_o & -\sin \phi_o \\ \sin \phi_o & \cos \phi_o \end{bmatrix} \begin{bmatrix} \Delta e(t) \\ \Delta \phi(t) \end{bmatrix} \] \hspace{1cm} (5.11)

Similar relationships hold for the journal center velocities and accelerations.

Figure 5.3. Small amplitude journal motions about a static equilibrium position.
Note that the small amplitude motions assumption means $\Delta e_x, \Delta e_y \ll C$, i.e., the journal dynamic displacements are much smaller than the bearing clearance.

The fluid film forces are general functions of the journal center displacements and velocities, i.e.

$$F_\alpha = F_\alpha [e_x(t), e_y(t), \dot{e}_x(t), \dot{e}_y(t)], \quad \alpha = X, Y$$

(5.12)

The assumption of small amplitude motions about an equilibrium position allows expressing the bearing reaction forces as a Taylor Series expansion around the static journal position $(e_{X_0}, e_{Y_0})$, i.e.

$$F_X = F_{X_0} + \frac{\partial F_X}{\partial X} \Delta X + \frac{\partial F_X}{\partial Y} \Delta Y + \frac{\partial F_X}{\partial \dot{X}} \Delta \dot{X} + \frac{\partial F_X}{\partial \dot{Y}} \Delta \dot{Y}$$

$$F_Y = F_{Y_0} + \frac{\partial F_Y}{\partial X} \Delta X + \frac{\partial F_Y}{\partial Y} \Delta Y + \frac{\partial F_Y}{\partial \dot{X}} \Delta \dot{X} + \frac{\partial F_Y}{\partial \dot{Y}} \Delta \dot{Y}$$

(5.13)

**Definition of dynamic force coefficients in fluid film bearings**

Fluid film bearing stiffness $(K_{ij})_{ij=X,Y}$, damping $(C_{ij})_{ij=X,Y}$ and inertia force coefficients are defined as

$$K_{ij} = -\frac{\partial F_i}{\partial X_j}; \quad C_{ij} = -\frac{\partial F_i}{\partial \dot{X}_j}; \quad M_{ij} = -\frac{\partial F_i}{\partial X_j}; \quad i,j=X,Y$$

(5.14)

For example, $K_{XY} = -\frac{\partial F_X}{\partial Y}$ corresponds to a stiffness produced by a fluid force in the $X$ direction due to a journal static displacement in the $Y$ direction. By definition, this coefficient is evaluated at the equilibrium position with other journal center displacements and velocities equal to zero. The negative sign in the definition assures that a positive magnitude stiffness coefficient corresponds to a restorative force.

The coefficients $(K_{XX}, K_{YY})$ are known as the **direct stiffness** terms, while the coefficients $(K_{XY}, K_{YX})$ are referred as **cross-coupled**. Figure 5.4 provides an idealized representation of the bearing force coefficients as mechanical parameters.

Fluid inertia or added mass coefficients $M_{ij} = -\frac{\partial F_i}{\partial \dot{X}_j}; \quad i,j=X,Y$ where $\{\dot{X}, \dot{Y}\}$ are journal center accelerations. Fluid inertia coefficients are of particular importance in superlaminar and turbulent flow bearings and seals handling liquids (large density). The inertia force coefficients or *apparent* masses have a sound physical interpretation and are always present in a fluid film bearing. Inertia coefficients are of large magnitude especially in dense liquids. However, the effect of inertia forces on the dynamic response of rotor-bearing systems is only of importance at
large excitation frequencies (This fact also holds for most mechanical systems subjected to fast transient motions).

\[ K_i = -\frac{\Delta F_i}{\Delta X_j} \]
\[ C_i = -\frac{\Delta F_i}{d(\Delta X_j)/dt} \]

**Figure 5.4. The “physical” representation of dynamic force coefficients in fluid film bearings**

Note that the defined force coefficients allow the representation of the dynamic fluid film bearing (or seal) forces in terms of fundamental mechanical parameters \{K, C, M\}. However, this does not mean that these coefficients must be accordance with customary knowledge. For example, the “viscous” damping coefficients may be negative, i.e. non-dissipative, or the stiffness coefficients non restorative or non conservative.

Fluid film force coefficients in the radial and tangential directions \((r, t)\) are also defined. Thus, the radial and tangential fluid film forces are expressed as (stiffness and damping for simplicity)

\[
F_r = F_{ro} + \frac{\partial F_r}{\partial e} \Delta e + \frac{\partial F_r}{e_o \partial \phi} e_o \Delta \phi + \frac{\partial F_r}{\partial \dot{e}} \Delta \dot{e} + \frac{\partial F_r}{e_o \partial \dot{\phi}} e_o \Delta \dot{\phi} + \frac{\partial F_r}{\partial \ddot{e}} \Delta \ddot{e} + \frac{\partial F_r}{e_o \partial \ddot{\phi}} e_o \Delta \ddot{\phi} \\
= F_{ro} - K_{r\phi} \Delta e - K_{r\phi} e_o \Delta \phi - C_{r\phi} \Delta \dot{e} - C_{r\phi} e_o \Delta \dot{\phi} \tag{5.15a}
\]

\[
F_t = F_{to} + \frac{\partial F_t}{\partial e} \Delta e + \frac{\partial F_t}{e_o \partial \phi} e_o \Delta \phi + \frac{\partial F_t}{\partial \dot{e}} \Delta \dot{e} + \frac{\partial F_t}{e_o \partial \dot{\phi}} e_o \Delta \dot{\phi} + \frac{\partial F_t}{\partial \ddot{e}} \Delta \ddot{e} + \frac{\partial F_t}{e_o \partial \ddot{\phi}} e_o \Delta \ddot{\phi} \\
= F_{to} - K_{t\phi} \Delta e - K_{t\phi} e_o \Delta \phi - C_{t\phi} \Delta \dot{e} - C_{t\phi} e_o \Delta \dot{\phi} \tag{5.15b}
\]

Note that \{\Delta \dot{e}, e_o \Delta \dot{\phi}\} are the journal center radial and tangential (small) velocities in the \((r, t)\) coordinate system, respectively.
The relationship between the force coefficients in both coordinate systems is easily determined from equation (5.11) as:

\[
\begin{bmatrix}
K_{XX} & K_{XY} \\
K_{XY} & K_{YY}
\end{bmatrix} = \begin{bmatrix}
\cos \phi_o & -\sin \phi_o \\
\sin \phi_o & \cos \phi_o
\end{bmatrix} \begin{bmatrix}
K_{rr} & K_{rt} \\
K_{tr} & K_{tt}
\end{bmatrix} \begin{bmatrix}
\cos \phi_o & \sin \phi_o \\
-\sin \phi_o & \cos \phi_o
\end{bmatrix}
\]  

(5.16)

\[
\begin{bmatrix}
C_{XX} & C_{XY} \\
C_{XY} & C_{YY}
\end{bmatrix} = \begin{bmatrix}
\cos \phi_o & -\sin \phi_o \\
\sin \phi_o & \cos \phi_o
\end{bmatrix} \begin{bmatrix}
C_{rr} & C_{rt} \\
C_{tr} & C_{tt}
\end{bmatrix} \begin{bmatrix}
\cos \phi_o & \sin \phi_o \\
-\sin \phi_o & \cos \phi_o
\end{bmatrix}
\]

Substitution of the force coefficient definitions (5.14) into equation (5.13) gives the following

\[
\begin{bmatrix}
F_X(t) \\
F_Y(t)
\end{bmatrix} = \begin{bmatrix}
F_{Xo} \\
F_{Yo}
\end{bmatrix} - \begin{bmatrix}
K_{XX} & K_{XY} \\
K_{XY} & K_{YY}
\end{bmatrix} \begin{bmatrix}
\Delta X \\
\Delta Y
\end{bmatrix} - \begin{bmatrix}
C_{XX} & C_{XY} \\
C_{XY} & C_{YY}
\end{bmatrix} \begin{bmatrix}
\Delta \dot{X} \\
\Delta \dot{Y}
\end{bmatrix}
\]  

(5.17)

And, the governing equations of motion for the rigid-rotor-bearing system, Eqn. (5.1) become

\[
\begin{bmatrix}
M & O \\
O & M
\end{bmatrix} \begin{bmatrix}
\Delta \ddot{X} \\
\Delta \ddot{Y}
\end{bmatrix} + \begin{bmatrix}
C_{XX} & C_{XY} \\
C_{XY} & C_{YY}
\end{bmatrix} \begin{bmatrix}
\Delta \dot{X} \\
\Delta \dot{Y}
\end{bmatrix} + \begin{bmatrix}
K_{XX} & K_{XY} \\
K_{XY} & K_{YY}
\end{bmatrix} \begin{bmatrix}
\Delta X \\
\Delta Y
\end{bmatrix} = M u \Omega^2 \begin{bmatrix}
sin \Omega t \\
\cos \Omega t
\end{bmatrix}
\]  

(5.18)

where \(F_{Xo} = F_o = \frac{1}{2} W\) and \(F_{Yo} = 0\). These differential equations are linear and represent the rotor-bearing system dynamics for small amplitude motions about the equilibrium position.

Fluid inertia effects are altogether neglected in the traditional stability analysis of rotor-lubricated bearing systems.

**Force coefficients for the short length journal bearing**

The general definition of fluid film bearing dynamic force coefficients is above. The analytical derivation of these coefficients for the short length journal bearing follows.

The film thickness for an aligned cylindrical journal bearing is

\[
h = C + e(t) \cos(\theta_o); \quad \theta = \Theta - \phi
\]  

(5.19)

For small amplitude motions about the equilibrium position, \(e(t) = e_o + \Delta e(t); \phi(t) = \phi_o + \Delta \phi(t)\), where \(\Delta e \) and \(\Delta \phi \) are small radial and angular displacement quantities, respectively.

Eqn. (5.19) is rewritten with \(\theta = \Theta - \phi_o\) as

\[
h = C + (e_o + \Delta e) \{\cos \theta \cos \Delta \phi + \sin \theta \sin \Delta \phi\},
\]
and, for small amplitude motions note that \( \cos(\Delta \phi) \approx 1, \sin(\Delta \phi) \approx \Delta \phi \). Then neglecting second order terms,

\[
h = C + e_o \cos \theta + \Delta e \cos \theta + e_o \Delta \phi \sin \theta = h_o + h_t
\]

(5.20)

where,

\[
h_o = C + e_o \cos \theta; \quad h_t = \Delta e \cos \theta + e_o \Delta \phi \sin \theta
\]

(5.21)

are the zeroth-order and first-order or perturbed film thicknesses, respectively.

Recall that the Reynolds equation for the short length journal bearing model is\(^1\):

\[
\frac{\partial}{\partial z} \left( \frac{h^3}{12 \mu} \frac{\partial P}{\partial z} \right) = \frac{\partial h}{\partial t} + \frac{\Omega}{2} \frac{\partial h}{\partial \theta} \quad (5.22)
\]

\[
\frac{\partial h}{\partial t} = \frac{\partial h_t}{\partial t} = \Delta \dot{e} \cos \theta + e_o \Delta \dot{\phi} \sin \theta
\]

and,

\[
\frac{\partial h}{\partial \theta} = \frac{\partial h_o}{\partial \theta} - \Delta e \sin \theta + e_o \Delta \phi \cos \theta; \quad \frac{\partial h_t}{\partial \theta} = -e_o \sin \theta \quad (5.23)
\]

Substitution of (5.23) into (5.22) gives:

\[
\frac{\partial}{\partial z} \left( \frac{h^3}{12 \mu} \frac{\partial P}{\partial z} \right) = \left\{ \Delta \dot{e} + \frac{\Omega}{2} e_o \Delta \phi \right\} \cos \theta + \left\{ e_o \left[ \Delta \dot{\phi} - \frac{\Omega}{2} - \Delta e \frac{\Omega}{2} \right] \right\} \sin \theta \quad (5.24)
\]

Integration of the pressure field on the journal surface gives the radial and tangential components of the fluid film force, i.e.,

\[
\begin{bmatrix} F_r \cr F_t \end{bmatrix} = 2 \int_{\theta = 0}^{\theta = \pi} \int_{\delta = 0}^{L/2} P(\theta, z, t) \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} R d\theta dz \quad (5.26)
\]

\(^1\) This equation is valid for \((L/D) < 0.50\) and incompressible, isoviscous lubricants. No thermal effects are accounted for in this simple form of the classical Reynolds equation.
where the positive (uncavitated) pressure region lies between $\theta_1 = 0$ and $\theta_2 = \pi$ when $P_a$ is set as zero (nil). Note that it is assumed the perturbed pressure field, due to small amplitude journal motions about the equilibrium position $(e_o, \phi_o)$, does not affect the extent of the steady state lubricant cavitation region, i.e. from 0 to $\pi$. This assumption is clearly void if the motions are large in character. By the way, the concept of linear force coefficients is also inadequate when motion amplitudes are large.

Substitution of Eqn. (5.25) into Eqn. (5.26) and integration in the axial direction renders

$$
\left( \frac{F_r}{F_i} \right) = \frac{\mu R L^3}{C^3} \int_0^{\pi} \left\{ \Delta \dot{e} + \frac{\Omega}{2} e_0 \dot{\phi} \right\} \frac{\cos \theta}{H^3} + \left\{ e_0 \left[ \Delta \dot{\phi} - \frac{\Omega}{2} \right] - \Delta e \frac{\Omega}{2} \right\} \frac{\sin \theta}{H^3} \left[ \frac{\cos \theta}{\sin \theta} \right] d\theta
$$

(5.27)

However, the cubic term in the denominator ($H^3$) also depends on the perturbed journal center displacements. A first-order Taylor series expansion of this terms gives for $h/C =$

$$
h^{-3} = h_0^{-3} - 3h_0^{-4} h_1
$$

(5.28)

where $h_0 = C + e_o \cos \theta$; $h_1 = \Delta e \cos \theta + e_o \Delta \phi \sin \theta$. Substitution of Eqn. (5.28) into (5.27) and neglecting second-order terms, i.e. products of small quantities such as $\Delta e \cdot \Delta \phi$, etc., gives after some considerable algebraic manipulation

$$
\begin{bmatrix}
F_r \\
F_i
\end{bmatrix} = \begin{bmatrix}
F_{r0} \\
F_{i0}
\end{bmatrix} - \frac{\mu R L^3}{C^3} \begin{bmatrix}
J_3^{11} & J_3^{12} \\
J_3^{20} & J_3^{21}
\end{bmatrix} \begin{bmatrix}
\Delta \dot{e} \\
e \Delta \dot{\phi}
\end{bmatrix} - \frac{\mu R L^3}{C^3} \frac{\Omega}{2} \begin{bmatrix}
-J_3^{11} + 3 \varepsilon J_4^{12} & J_3^{02} + 3 \varepsilon J_4^{21} \\
-J_3^{20} + 3 \varepsilon J_4^{21} & J_3^{11} + 3 \varepsilon J_4^{30}
\end{bmatrix} \begin{bmatrix}
\Delta e \\
e \Delta \phi
\end{bmatrix}
$$

(5.29)

where $J_i^{kj} = \int_{\theta_1=0}^{\theta_2=\pi} \left( \frac{\sin \theta}{H_0^i} \right)^k \left( \frac{\cos \theta}{H_0^i} \right)^j d\theta$ are definite integrals and $H_0 = (1 + \varepsilon \cos \theta)$.

The bearing stiffness and damping force coefficients are, from Eqn. (5.29), specified as
\[ K_{rr} = \mu RL^3 \frac{\Omega}{C^3} \frac{1}{2} \left\{ - J_3^{11} + 3 \varepsilon J_4^{12} \right\} = \frac{\mu RL^3 \Omega}{C^3} \frac{2 \varepsilon (1 + \varepsilon^2)}{(1 - \varepsilon^2)^3} = - \frac{\partial F_r}{\partial \varepsilon} \]

\[ K_{rt} = \mu RL^3 \frac{\Omega}{C^3} \frac{1}{2} \left\{ J_3^{02} + 3 \varepsilon J_4^{21} \right\} = \frac{\mu RL^3 \Omega}{C^3} \frac{\pi}{4 (1 - \varepsilon^2)^{3/2}} = - \frac{\partial F_r}{\partial \varphi} \]

\[ K_{tr} = \mu RL^3 \frac{\Omega}{C^3} \frac{1}{2} \left\{ J_3^{30} + 3 \varepsilon J_4^{31} \right\} = \frac{\mu RL^3 \Omega}{C^3} \frac{\pi (1 + 2 \varepsilon^2)}{4 (1 - \varepsilon^2)^{5/2}} = - \frac{\partial F_t}{\partial e} \]

\[ C_{rr} = \mu RL^3 \frac{J_3^{02}}{C^3} = \frac{\mu RL^3 \pi (1 + 2 \varepsilon^2)}{C^3 2 (1 - \varepsilon^2)^{3/2}} = - \frac{\partial F_r}{\partial \varepsilon} \]

\[ C_{rt} = \mu RL^3 \frac{J_3^{30}}{C^3} = \frac{\mu RL^3 \pi (1 + 2 \varepsilon^2)}{C^3 2 (1 - \varepsilon^2)^{3/2}} = - \frac{\partial F_t}{\partial \varphi} \]

\[ C_{tr} = \mu RL^3 \frac{J_3^{31}}{C^3} = \frac{\mu RL^3 (-2 \varepsilon)}{C^3 (1 - \varepsilon^2)^2} = C_{rr} = - \frac{\partial F_r}{\partial \varphi} = - \frac{\partial F_t}{\partial e} \]

The \((\Delta \varepsilon, e_0, \Delta \phi)\) correspond to the journal center radial and tangential velocities in the \((r, t)\) coordinate system, respectively. Note that the stiffness coefficients \((K_{ij})_{ij=r,t}\) are proportional to the rotational speed \((\Omega)\) and fluid viscosity \((\mu)\). The damping coefficients \((C_{ij})_{ij=r,t}\) are not a direct function of the angular speed but depend only on the fluid viscosity and the journal equilibrium position. Without journal rotation there cannot be a fluid film bearing stiffness.

**Dimensionless Force Coefficients**

The literature presents the force coefficients in dimensionless form according to

\[ k_{ij} = K_{ij} \frac{C}{F_0} \quad c_{ij} = C_{ij} \frac{C \Omega}{F_0} \quad \text{for} \quad i,j = X,Y \]

(5.32)

where \(F_0\) is the static load applied on each bearing (in the \(X\) direction). [Note that the total load \(W=2F_0\) is shared by the two bearings in a symmetric rotor mount].

Recall that \(F_0 = \frac{\mu \Omega (L / C)^2 LR}{4 \sigma}\), where \((\sigma)\) is the modified Sommerfeld Number defined as \(\text{(See Notes 4)}\)
\[
\mu \Omega LR \left( \frac{L}{C} \right)^2 = \sigma = \frac{(1-\varepsilon^2)^2}{\varepsilon \sqrt{16\varepsilon^2 + \pi^2(1-\varepsilon^2)}}
\] (4.8)

Using the following definitions:
\[
f_{ro} = -\frac{F_{ro}}{F_o} = \cos \phi_o = \frac{4\sigma \varepsilon^2}{(1-\varepsilon^2)^2}; \quad f_{io} = \frac{F_{io}}{F_o} = \sin \phi_o = \frac{\pi \sigma \varepsilon}{(1-\varepsilon^2)^{\frac{1}{2}}}
\] (5.33)

the dimensionless force coefficients in the \((r, t)\) coordinate system become,
\[
k_{rr} = f_{ro} \frac{2(1+\varepsilon^2)}{\varepsilon(1-\varepsilon^2)}; \quad c_{rr} = f_{io} \frac{2(1+2\varepsilon^2)}{\varepsilon(1-\varepsilon^2)}
\]
\[
k_{rt} = f_{ro} \frac{1}{\varepsilon}; \quad c_{rt} = c_{tr} = -f_{ro} \frac{2}{\varepsilon}
\]
\[
k_{tt} = -f_{io} \frac{(1+2\varepsilon^2)}{\varepsilon(1-\varepsilon^2)};
\]
\[
k_{tr} = f_{ro} \frac{1}{\varepsilon}; \quad c_{tr} = f_{io} \frac{2}{\varepsilon}
\] (5.34)

Force coefficients in the \((X,Y)\) coordinate system are easily obtained using the matrix transformation Eqn. (5.16). After a lengthy algebraic procedure,
\[
k_{xx} = K_{xx} \frac{C}{F_o} = \frac{f_{ro}}{\varepsilon(1-\varepsilon^2)} \left\{ f_{ro}^2 + 1 + 2\varepsilon^2 \right\}
\]
\[
k_{yy} = K_{yy} \frac{C}{F_o} = \frac{f_{ro}}{\varepsilon(1-\varepsilon^2)} \left\{ f_{io}^2 + 1 + \varepsilon^2 \right\}
\]
\[
k_{yx} = K_{yx} \frac{C}{F_o} = \frac{f_{io}}{\varepsilon(1-\varepsilon^2)} \left\{ f_{ro}^2 - 1 + \varepsilon^2 \right\}
\]
\[
k_{xy} = K_{xy} \frac{C}{F_o} = \frac{f_{io}}{\varepsilon(1-\varepsilon^2)} \left\{ f_{ro}^2 + 1 + 2\varepsilon^2 \right\}
\]
\[
k_{xy} = K_{xy} \frac{C}{F_o} = \frac{f_{ro}}{\varepsilon(1-\varepsilon^2)} \left\{ f_{ro}^2 - 1 + \varepsilon^2 \right\} = c_{yx}
\] (5.35)

recall that the \(X\)-direction is along the static load \(F_o\).

Figures 5.5 and 5.6 depict the dimensionless force coefficients, **stiffness and damping**, as functions of both the journal eccentricity and the modified Sommerfeld number \((\sigma)\), respectively. Both representations are necessary since sometimes the journal eccentricity is known a priori.
while most often, the design parameter, i.e. the Sommerfeld number, is known in advance. In
general, the physical magnitude of the stiffness and damping coefficients increases rapidly
(nonlinearly) with the journal eccentricity (load too!).

Note that the dimensionless force coefficients do not represent the actual physical trends. For
example, at $e_o=0$, $K_{XX}=K_{YY}=0$, but the dimensionless values $k_{XX}=k_{YY}=0$ in the figures show a
definite value. This peculiar result follows from the definition of dimensionless force coefficients
using the applied load ($F_o$). Thus, as $e_o\to 0$, the bearing load $F_o$ is also nil.
Figure 5.5. Short length journal bearing (dimensionless) stiffness and damping force coefficients vs. journal eccentricity (ε)
Figure 5.6. Short length journal bearing (dimensionless) stiffness and damping force coefficients vs. modified Sommerfeld number ($\sigma$)

$$\sigma = \frac{\mu \Omega LR}{4W} \left( \frac{L}{c} \right)^2$$
**Dynamic force Coefficients for journal centered operation, i.e. static load=0**

As the journal center approaches the bearing center, $e_0 \to 0$, and from the formulas,

$$K_{rr} = K_n = C_{rr} = C_n = 0$$  \hspace{1cm} (5.36)

$$\bar{k} = K_{rr} = -K_n = \frac{\mu RL^3 \Omega \pi}{C^3} = \frac{\Omega}{2} \bar{c}; \quad \bar{c} = C_n = C_{rr} = \frac{\mu RL^3 \pi}{C^3}$$

At $e \to 0$, $\phi_0 = 90^\circ$, so the force coefficients in the $(X,Y)$ system are given as:

$$\begin{align*}
[K_{XX} & K_{XY}] \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & +\bar{k} \\ -\bar{k} & 0 \end{bmatrix} \\
[C_{XX} & C_{XY}] \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \bar{c} \\ 0 & \bar{c} \end{bmatrix}
\end{align*}$$

Hence

$$K_{XY} = -K_{YX} = \bar{k} = \frac{\mu \Omega RL^3 \pi}{C^3} = \frac{\Omega}{2} \bar{c}; \quad C_{XX} = C_{YY} = \bar{c} = \frac{\mu RL^3 \pi}{C^3}$$  \hspace{1cm} (5.37)

Thus, at the centered journal position the bearing offers no direct (support) stiffness but only cross-coupled support. A small static load applied on the bearing will cause a journal displacement in a direction orthogonal (perpendicular) to the load. This phenomenon is found in nearly all fluid film bearings of rigid geometry.
Stability analysis of rigid rotor supported on plain journal bearings

For small amplitude journal motions about the equilibrium position \((e_0, \phi_0)\), the equations of motion of a rigid rotor supported on (linear) fluid bearings are:

\[
\begin{bmatrix}
M & O \\
O & M
\end{bmatrix}
\begin{bmatrix}
\Delta \ddot{X} \\
\Delta \ddot{Y}
\end{bmatrix} +
\begin{bmatrix}
C_{XX} & C_{XY} \\
C_{YX} & C_{YY}
\end{bmatrix}
\begin{bmatrix}
\Delta \dot{X} \\
\Delta \dot{Y}
\end{bmatrix} +
\begin{bmatrix}
K_{XX} & K_{XY} \\
K_{YX} & K_{YY}
\end{bmatrix}
\begin{bmatrix}
\Delta X \\
\Delta Y
\end{bmatrix} = M u \Omega^2 \left( \sin \Omega t \right) \cos \Omega t
\]

(5.38)

Introduce the dimensionless variables:

\[
\Delta x = \frac{\Delta X}{C}, \quad \Delta y = \frac{\Delta Y}{C}, \quad \tau = \Omega t, \quad \delta = \frac{u}{C}
\]

(5.39)

where \(C\) is the bearing radial clearance and \(\Omega\) is the journal or rotor speed (regarded as invariant). Substitution of Eqn. (5.39) into (5.38) gives:

\[
p^2 \left[ \frac{\Delta x''}{\Delta y''} + \frac{c_{XX} c_{YY}}{c_{YX} c_{XX}} \frac{\Delta x'}{\Delta y'} + \frac{k_{XX} k_{YY}}{k_{Xy} k_{YY}} \frac{\Delta x}{\Delta y} \right] = p^2 \delta \left[ \frac{\sin(\tau)}{\cos(\tau)} \right]
\]

(5.40)

where \(p = \frac{d}{d\tau} \), \(p^2 = \frac{CM^2 \Omega^2}{F_o}\) is a dimensionless mass, and \(k_{ij} = k_{ij} (C/F_o)\), \(c_{ij} = C_{ij} (C \Omega / F_o)\) are the dimensionless dynamic force coefficients.

It is of interest to study if the rotor-bearing system is **stable** for small amplitude journal center motions (perturbations) about the equilibrium position. To this end, set the imbalance parameter \(\delta = 0\) in the equations above to obtain,
If the rotor-bearing system is to become unstable, this will occur at a threshold speed of rotation \((\Omega_s)\) and the rotor will perform (undamped\(^2\)) orbital motions at a whirl frequency \((\omega_s)\). These motions, satisfying equation (5.42), are of the form:

\[
x = A e^{i\omega t} = A e^{i\bar{\omega} t} \quad y = B e^{i\omega t} = B e^{i\bar{\omega} t}, \quad j = \sqrt{-1}
\]

where \(\bar{\omega} = \omega_s / \Omega_s\) is known as the whirl frequency ratio, i.e. the ratio between the rotor whirl or precessional frequency and the rotor onset speed of instability.

Substitution of Eqn. (5.42) into (5.41) leads to:

\[
\left[ \begin{array}{cc}
- p_s^2 \bar{\omega}_s^2 + k_{xx} + j \bar{\omega}_s c_{xx} & k_{xy} + j \bar{\omega}_s c_{xy} \\
- k_{yx} + j \bar{\omega}_s c_{yx} & - p_s^2 \bar{\omega}_s^2 + k_{yy} + j \bar{\omega}_s c_{yy}
\end{array} \right] \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

The determinant of the system of equations must be zero for a non-trivial solution of the homogenous system of equations, i.e.

\[
\Delta = \left( - p_s^2 \bar{\omega}_s^2 + k_{xx} + j \bar{\omega}_s c_{xx} \right) \left( - p_s^2 \bar{\omega}_s^2 + k_{yy} + j \bar{\omega}_s c_{yy} \right) - \left( k_{yx} + j \bar{\omega}_s c_{yx} \right) \left( k_{xy} + j \bar{\omega}_s c_{xy} \right) = 0
\]

After a rather lengthy algebraic manipulation, the real and imaginary parts of \(\Delta\) above render,

\[
p_s^2 \bar{\omega}_s^2 = k_{eq} = \frac{k_{xx} c_{yy} + k_{yy} c_{xx} - c_{xx} k_{xy} - c_{yy} k_{yx}}{c_{xx} + c_{yy}} = \frac{C M \omega_s^2}{F_o}
\]

and

\[
\bar{\omega}_s^2 = \left( \frac{k_{eq} - k_{xx} \left( k_{eq} - k_{yy} \right) - k_{xy} k_{yx}}{c_{xx} c_{yy} - c_{xy} c_{yx}} \right) = \left( \frac{\omega_s}{\Omega_s} \right)^2
\]

For a given value of journal eccentricity \((\varepsilon_o)\), i.e. a given Sommerfeld number \((\sigma)\), one evaluates Eqn. (5.45) to obtain the dimensionless equivalent stiffness \(k_{eq}\), and then (5.46) to obtain the whirl frequency ratio \(\bar{\omega}_s\). This substitution then yields \(p_s^2 = k_{eq} / \bar{\omega}_s^2\) (system critical mass) which in turn renders the onset speed of instability \(\Omega_s\).

\(^2\) Recall that in a second order mechanical system an equivalent damping ratio \(>0\) causes the damping or attenuation of motions induced by small perturbations. A damping equal to zero produces sustained periodic motions without decay or growth and indicates the threshold between stability and instability (amplitude growing motions).
Figures 5.7 and 5.8 depict the whirl frequency ratio $\bar{\omega} = \omega_s/\Omega_s$ and the dimensionless threshold speed of instability ($p_s$) versus both the journal eccentricity and Sommerfeld number, respectively. Note that for near centered journal operation, i.e. large Sommerfeld numbers, the whirl frequency is 0.50, i.e. half-synchronous whirl.

Other important information is also obtained. If one assumes that the current (operating) rotational speed $\Omega$ is the onset speed of instability, then from the relations above, the magnitude of $\frac{1}{2}$ system mass ($M$) is obtained, and which would make the rotor-bearing system become unstable. This mass is known as the **critical mass**, $M_c$, and corresponds to the limit mass which the system can carry dynamically. If the total mass is equal or larger than twice $M_c$, then the system will be unstable at the rated speed $\Omega$.

The whirl frequency ratio, $\bar{\omega} = \omega_s/\Omega_s$, is the ratio between the rotor whirl frequency and the *onset speed of instability*. Note that this ratio, as given by Eqn. (5.46), depends only on the fluid film bearing characteristics and the equilibrium eccentricity, and it is independent of the rotor characteristics (rotor mass and flexibility).

The parameter $k_{eq}$ is a journal bearing (dimensionless) **equivalent stiffness** and depicted in Figures 5.5 and 5.6. From the definitions of threshold speed and whirl ratio, $p_s^2 = M \Omega_s^2 (C/F_o)$ and $\bar{\omega}_s = \omega_s/\Omega_s$, then

$$M \omega_s^2 = k_{eq} \left( \frac{F_o}{C} \right) = K_{eq}$$

Thus, the whirl or precessional frequency is given by

$$\omega_s = \sqrt{\frac{K_{eq}}{M}} = \omega_n$$  \hspace{1cm} (5.47)

i.e., the whirl frequency equals the **natural frequency** of the rigid rotor supported on journal bearings.

For operation close to the concentric position, $\varepsilon_0 \to 0$, i.e. large Sommerfeld numbers (no load condition), the force coefficients are, see Eqn. (5.37),

$$k_{xx} = k_{yy} = 0; \quad c_{xx} = c_{yy}; \quad k_{xy} = -k_{yx}; \quad c_{xy} = c_{yx} = 0$$

$$k_{eq} = (k_{xx}c_{xx} + c_{xy}k_{xy})/c_{xx} = 0$$

and

$$\frac{\omega_s}{\Omega_s} = \frac{k_{xy}}{c_{xx}} = 0.50 \quad \text{as} \quad \varepsilon \to 0$$  \hspace{1cm} (5.48)

---

3 Recall that each bearing carries half the static load, and also half the dynamic or inertia load ($2M_c \Omega^2$).
The 0.5 magnitude for whirl frequency ratio \((WFR)\) (or 50% whirl as is called in industry) is a characteristic of hydrodynamic plain journal bearings. It shows us that at the onset of instability the rotor whirls at its natural frequency, which equals to 50% of the rotor speed. Furthermore, under no externally applied loads, \(F_o=0\), as in vertically turbomachinery, the bearing possesses no support stiffness, i.e. \(K_{eq}=0\) and the system natural frequency \(\omega_n\) is zero, i.e. the rotor-bearing system whirls at all speeds.

Note that if \(k_{XY} = 0\), i.e. the fluid film bearing does not show cross-coupled effects, then the \(WFR = 0\), i.e. no whirl occurs and the system is ALWAYS stable. (Asymmetrical) cross-coupled stiffnesses are thus responsible for the instabilities so commonly observed in rotors mounted on journal bearings.

If the whirl frequency ratio is 0.50, then the maximum rotational speed that the rotor-bearing system can attain is just,

\[
\Omega_{\text{max}} = \frac{\omega_n}{0.50} = 2\omega_n = 2\omega_n
\]

i.e., twice or two times the natural frequency (or observed rigid rotor critical speed).

Figures 5.7, 5.8, and 5.9 show, respectively, the whirl frequency ratio, the dimensionless critical mass parameter \((p_s)\), and the dimensionless critical mass \((p_s)^2\) versus the Sommerfeld number and operating journal eccentricity. The results show that a rigid-rotor supported on plain journal bearings is always STABLE for operation with journal eccentricity ratios \(\varepsilon > 0.75\) (small Sommerfeld numbers) for all \(L/D\) ratios. Note that the critical mass and the whirl ratio are relatively insensitive for operation with eccentricities \(\varepsilon_0 < 0.50\).

Keep in mind that increasing the rotational speed of the rotor-bearing system determines larger Sommerfeld numbers, and consequently, operation at smaller journal eccentricities for the same applied static load. Thus, operation at ever increasing speeds will eventually lead to a rotor dynamically unstable system as the results show.

Effects of Rotor Flexibility

A similar analysis can be performed considering rotor flexibility. This analysis is more laborious though straightforward. The analysis shows that the whirl frequency ratio is not affected by the rotor flexibility. However, the onset speed of instability decreases dramatically!

The relationship for the threshold speed of instability of a flexible rotor is:

\[
p_{sf}^2 = \frac{p_s^2}{1 + k_{eq} \left(\Gamma \frac{C}{C}\right)}
\]

where the sub index \(f\) denotes the flexible rotor, \(K_{rot}\) is the rotor stiffness on each side of the center disk, and \(\Gamma = F_o/K_{rot}\) is the rotor static sag or elastic deformation at midspan.
The elastic shaft and bearing are mounted in series, i.e. the bearing and shaft flexibilities add (reciprocal of stiffnesses), and thus the equivalent system stiffness is lower than that of the bearings alone, and therefore the system natural frequency is lower.

Figure 5.10 shows the threshold speed of instability ($p_{sf}$) for a flexible rotor mounted on plain short length journal bearings. Note that the more flexible the rotor is, the lower the threshold speed of instability. If the fluid film bearings are designed too stiff (small Sommerfeld numbers), then the natural frequency of the rotor-bearing system is just $(K_{rot}/M)^{0.5}$, irrespective of the bearing configuration.

**Postcript**

See the Appendix to these notes for further understanding on the nature of the cross-coupled coefficients driving the whirl motion.

The MATHCAD programs attached include the algebraic formulas for evaluation of the bearing force coefficients in actual applications.
Figure 5.7. Whirl frequency ratio vs. modified Sommerfeld number ($\sigma$) and journal eccentricity ($\varepsilon$)
Figure 5.8. Dimensionless threshold speed of instability ($\rho_s$) vs. modified Sommerfeld number ($\sigma$) and journal eccentricity ($e/c$)
Figure 5.9. Dimensionless critical mass \( \left(m^* = \rho_s^2 \right) \) vs. modified Sommerfeld number \( (\sigma) \) and journal eccentricity \( (e/c) \).
Figure 5.10. Dimensionless threshold speed of instability ($\rho_s$) for flexible rotor vs. modified Sommerfeld number ($\sigma$). Static sag ($\Gamma/c$) varies.
References consulted

Stability and Imbalance Response of a Jeffcott-Rotor Supported on Short Length Journal Bearings

(c) Dr. Luis San Andres  UT/2000, TAMU/2006  Extended with eigen analysis: 10/00 TAMU

DATA for rotor

\[ W_T := 600 \times 4.448 \cdot N \]
\[ W := \frac{W_T}{2} \]
\[ W = 1.334 \times 10^3 \cdot N \]
\[ M := \frac{W}{g} \]
\[ M = 136.071 \cdot kg \]

BEARING GEOMETRY, OIL viscosity and Operating conditions:

\[ D := 0.15 \cdot m \]
\[ L := 2 \cdot 0.025 \cdot m \]
\[ c := 0.060 \cdot mm \]

\[ \mu_R := \frac{2.5 \times 10^{-6} \cdot 6894.757 \cdot N\cdot s}{m^2} \]

fluid viscosity at reference etemperature:

\[ T_R := 50 \]
\[ \alpha := 0.030 \]

Oil viscosity-temperature coefficient (1/deg C)

\[ T := 60 \text{ degC} \]

Top shaft speed for analysis

\[ \text{RPM}_{\text{max}} := 12000 \]

\[ k_{\text{shaft}} := \frac{8}{N \cdot m} \]

1/2 shaft stiffness

\[ \alpha := 0.2 \cdot c \]

Amplitude of imbalance on rotor disk

\[ \delta := \frac{W}{k_{\text{shaft}}} \]
\[ \Gamma := \frac{\delta}{c} \]
\[ \Gamma = 0.222 \]
\[ 1N = 1 \text{ kg m s}^{-2} \]

\[ \delta = 1.334 \times 10^{-5} \cdot m \]

\[ \text{if } \Gamma > 1 \text{ then rotor is quite flexible} \]

\[ \mu := \mu_R \cdot e^{-\alpha \cdot (T-T_R)} \]
\[ \mu = 0.013 \frac{N\cdot s}{m^2} \]

Operating viscosity
Journal eccentricity ratio and attitude angle for STATIC equilibrium

Short bearing solution for an isoviscous-incompressible lubricant

Power loss (kW) / bearing

Side flow rate / bearing (LPM)

Bearing flow rate (LPM)
Stiffness and damping force coefficients vs shaft speed

Equivalent bearing stiffness for rigid rotor

Whirl frequency ratio
Threshold speed of instability (rpm)

Critical Rotor Mass for rigid rotor

Recall rotor mass: \(2 \cdot M = 272.142 \text{ kg}\)

System goes unstable when ratio < 1

Eigenvalues rig rotor

damping ratio for system,

Search for intersection of real part with Y=0 axis to determine threshold speed of instability.
natural frequency - from eigenvalue analysis and from \((Keq/M)\) formula
**Effect of shaft flexibility on threshold speed of instability:**

Static Deflection \( (\Gamma = \delta/c) \) and Stiffness of *Flexible* shaft  \( \Gamma = 0.222 \)

\[
\Omega_{tf} := \frac{\Omega_n}{1 + k_{eq} \cdot \Gamma^2}
\]

**NEARLY RIGID ROTOR**

\( k_{shaft} = 1 \times 10^8 \, \text{N/m} \)
\( \delta = 1.334 \times 10^{-5} \, \text{m} \)

The threshold speed of instability is lower for the flexible rotor than for the rigid rotor model.

**Damping ratio of flex rotor system**

\[\delta = 1.334 \times 10^{-5} \, \text{m} \]
Synchronous imbalance response of flexible rotor

The equations of motion for both rotor and journal bearings are given below. The coordinates of rotor and disk motion have their origin at the static equilibrium position. No damping at rotor midspan, no mass lumped at the bearings.

\[
\begin{align*}
 m \left( \frac{d^2 X}{dt^2} \right) + k_{\text{shaft}} (X - x) &= m \cdot a \cdot \omega \cdot \cos(\omega \cdot t) \\
 m \left( \frac{d^2 Y}{dt^2} \right) + k_{\text{shaft}} (Y - y) &= m \cdot a \cdot \omega \cdot \sin(\omega \cdot t)
\end{align*}
\]

\[
\left( C_{1,J} \right) \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} + \left( K_{1,J} \right) \begin{pmatrix} x \\ y \end{pmatrix} = -k \begin{pmatrix} x - X \\ y - Y \end{pmatrix}
\]

\[
a = 1.2 \times 10^{-5} \text{ m}
\]

imbalance displacement

\[
\frac{a}{c} = 0.2
\]

The rotor disk \((X,Y)\) and journal center displacements \((x,y)\) are synchronous with the imbalance excitation, i.e.

\[
\begin{align*}
 X &= X_c \cos(\omega \cdot t) + X_s \sin(\omega \cdot t) \\
 x &= x_c \cos(\omega \cdot t) + x_s \sin(\omega \cdot t)
\end{align*}
\]

\[
\begin{align*}
 Y &= Y_c \cos(\omega \cdot t) + Y_s \sin(\omega \cdot t) \\
 y &= y_c \cos(\omega \cdot t) + y_s \sin(\omega \cdot t)
\end{align*}
\]
Exercise: Calculate the major and minor axes of the ellipses describing the (X,Y) motions. See Appendix A of Childs' Rotordynamics Book.

Notes: You could update this program to account for
a) bearing mass MB, a fraction of total rotor mass,
b) introduce damping at the rotor midspan, Cs.
Effect of shaft flexibility on threshold speed of instability:

Static Deflection ($\Gamma=\delta/c$) and Stiffness of Flexible shaft

$\Gamma = 2.224$

Threshold speed of instability

$$\Omega_{t_f} := \frac{\Omega_n}{1 + \frac{1}{k_{eq} \cdot \Gamma}}$$

Natural Frequency (rpm) of Flexible Shaft

$$\omega_n := WFR_n \cdot \Omega_{t_f}$$

The threshold speed of instability is lower for the flexible rotor than for the rigid rotor model.

Eigenvalues: flexible rotor

Damping ratio of flex rotor system

$$k_{shaft} = 1 \times 10^7 \frac{N}{m}$$

$$\delta = 1.334 \times 10^{-4} m$$
Synchronous imbalance response of flexible rotor

The equations of motion for both rotor and journal bearings are given below. The coordinates of rotor and disk motion have their origin at the static equilibrium position. No damping at rotor midspan, no mass lumped at the bearings.

\[
m \left( \frac{d^2 X}{dt^2} \right) + k_{\text{shaft}}(X - x) = m \cdot a \cdot \omega^2 \cdot \cos(\omega \cdot t)
\]
\[
m \left( \frac{d^2 Y}{dt^2} \right) + k_{\text{shaft}}(Y - y) = m \cdot a \cdot \omega^2 \cdot \sin(\omega \cdot t)
\]

**ROTOR**

**MASSLESS BEARINGS**

\[
\begin{bmatrix}
  (C_{1,J}) & (K_{I,J}) \\
  (K_{1,J}) & (C_{I,J})
\end{bmatrix}
\begin{bmatrix}
  \frac{dx}{dt} \\
  \frac{dy}{dt}
\end{bmatrix} + \begin{bmatrix}
  x \\
  y
\end{bmatrix} = -k \begin{bmatrix}
  x - X \\
  y - Y
\end{bmatrix}
\]

\[
a = 1.2 \times 10^{-5} \text{ m}
\]

\[
\frac{a}{c} = 0.2
\]

The rotor disk \((X,Y)\) and journal center displacements \((x,y)\) are synchronous with the imbalance excitation, i.e.

\[
X = X_c \cos(\omega \cdot t) + X_s \sin(\omega \cdot t)
\]

\[
x = x_c \cos(\omega \cdot t) + x_s \sin(\omega \cdot t)
\]

\[
Y = Y_c \cos(\omega \cdot t) + Y_s \sin(\omega \cdot t)
\]

\[
y = y_c \cos(\omega \cdot t) + y_s \sin(\omega \cdot t)
\]
Amplitudes of motion at rotor midspan

Amplitudes of motion at bearings

Exercise: Calculate the major and minor axes of the ellipses describing the (X,Y) motions. See Appendix A of Childs' Rotordynamics Book.

Notes: You could update this program to account for
a) bearing mass MB, a fraction of total rotor mass,
b) introduce damping at the rotor midspan, Cs.
OTHER TYPES OF LUBRICATED JOURNAL BEARINGS

Compressors, turbines, pumps, electric motors, electric generators and other rotating machines are commonly supported on fluid film bearings. In the past, the vast majority of these bearings were plain journal bearings. As machines have achieved higher speeds, rotor dynamic instability problems such as oil whirl have brought the need for other types of bearing configurations. Cutting axial grooves in the bearing to provide a different oil flow pattern across the lubricated surface generates some of these geometries. Other bearing types have various patterns of variable clearance (preload and offset) to create a pad film thickness that has strongly converging and diverging regions, thus generating a direct stiffness for operation even at the journal centered position. Various other geometries have evolved as well, such as the tilting pad bearings which allow each pad to pivot, and thus to take its own equilibrium position. This usually results in a strongly converging film region for each loaded pad and the near absence of cross-coupled stiffness coefficients.

TYPES OF HYDRODYNAMIC BEARINGS:

The Tables below list in a condensed form some of the advantages and disadvantages of various practical bearing configurations.
# Fixed Pad Non-Pre Loaded Journal Bearings

<table>
<thead>
<tr>
<th>Bearing Type</th>
<th>Advantages</th>
<th>Disadvantages</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plain Journal</td>
<td>1. Easy to make</td>
<td>1. Most prone to subsynchronous whirl</td>
<td>Round bearings are nearly always “crushed” to make elliptical bearings</td>
</tr>
<tr>
<td></td>
<td>2. Low Cost</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Partial Arc</td>
<td>1. Easy to make</td>
<td>1. Poor vibration resistance</td>
<td>Bearing used only in rather old machines</td>
</tr>
<tr>
<td></td>
<td>2. Low Cost</td>
<td>2. Oil supply not easily contained</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3. Low horsepower loss</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Axial Groove</td>
<td>1. Easy to make</td>
<td>1. Subject to oil whirl</td>
<td>Round bearings are nearly always “crushed” to make elliptical or multi-lobe</td>
</tr>
<tr>
<td></td>
<td>2. Low Cost</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Floating Ring</td>
<td>1. Relatively easy to make</td>
<td>2. Subject to oil (two whirl frequencies from inner and outer films (50% shaft speed, 50% [shaft + ring] speeds)</td>
<td>Used primarily on high speed turbochargers for diesel engines and P/C vehicles</td>
</tr>
<tr>
<td></td>
<td>2. Low Cost</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elliptical</td>
<td>1. Easy to make</td>
<td>1. Subject to oil whirl at high speeds</td>
<td>Probably most widely used bearing at low or moderate rotor speeds</td>
</tr>
<tr>
<td></td>
<td>2. Low Cost</td>
<td>2. Load direction must be known</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3. Good damping at critical speeds</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Offset Half (With Horizontal Split)</td>
<td>1. Excellent suppression of whirl at high speeds</td>
<td>1. Fair suppression of whirl at moderate speeds</td>
<td>High horizontal stiffness and low vertical stiffness - may become popular - used outside U.S.</td>
</tr>
<tr>
<td></td>
<td>2. Low Cost</td>
<td>2. Load direction must be known</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3. Easy to make</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Three and Four Lobe</td>
<td>1. Good suppression of whirl</td>
<td>1. Some types can be expensive to make properly</td>
<td>Currently used by some manufacturers as a standard bearing design</td>
</tr>
<tr>
<td></td>
<td>2. Overall good performance</td>
<td>2. Subject to whirl at high speeds</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3. Moderate cost</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
# FIXED PAD JOURNAL BEARINGS WITH STEPS, DAMS OR POCKETS

<table>
<thead>
<tr>
<th>Bearing Type</th>
<th>Advantages</th>
<th>Disadvantages</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pressure Dam (Single Dam)</strong></td>
<td>1. Good suppression of whirl</td>
<td>1. Goes unstable with little warning</td>
<td>Very popular in the petrochemical industry. Easy to convert elliptical over to pressure dam</td>
</tr>
<tr>
<td></td>
<td>2. Low cost</td>
<td>2. Dam may be subject to wear or build up over time</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3. Good damping at critical speeds</td>
<td>3. Load direction must be known</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4. Easy to make</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Multi-Dam Axial Groove or Multiple-Lobe</strong></td>
<td>1. Dams are relatively easy to place in existing bearings</td>
<td>1. Complex bearing requiring detailed analysis</td>
<td>Used as standard design by some manufacturers</td>
</tr>
<tr>
<td></td>
<td>2. Good suppression of whirl</td>
<td>2. May not suppress whirl due to nonbearing causes</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3. Relatively low cost</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4. Good overall performance</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Hydrostatic</strong></td>
<td>1. Good suppression of oil whirl</td>
<td>1. Poor damping at critical speeds</td>
<td>Generally high stiffness properties used for high precision rotors</td>
</tr>
<tr>
<td></td>
<td>2. Wide range of design parameters</td>
<td>2. Requires careful design</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3. Moderate cost</td>
<td>3. Requires high pressure lubricant supply</td>
<td></td>
</tr>
</tbody>
</table>

# NON-FIXED PAD JOURNAL BEARINGS

<table>
<thead>
<tr>
<th>Bearing Type</th>
<th>Advantages</th>
<th>Disadvantages</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tilting Pad Journal bearing</strong></td>
<td>1. Will not cause subsynchronous whirl (no cross coupling)</td>
<td>1. High Cost</td>
<td>Widely used bearing to stabilize machines with subsynchronous non-bearing related excitations</td>
</tr>
<tr>
<td><strong>Flexure pivot, tilting pad bearing</strong></td>
<td>1. Tolerance to misalignment.</td>
<td>2. Requires careful design</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2. Oil-free</td>
<td>3. Poor damping at critical speeds</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4. Hard to determine actual clearances</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>5. High horsepower loss</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>6. Load direction must be known</td>
<td></td>
</tr>
<tr>
<td><strong>Foil bearing</strong></td>
<td>1. High cost</td>
<td>1. High cost</td>
<td>Used mainly for low load support on high speed machinery (APU units).</td>
</tr>
<tr>
<td></td>
<td>2. Dynamic performance not well known for heavily loaded machinery.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3. Prone to subsynchronous whirl</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Plain journal bearing
Partial arc journal bearing
Journal bearing with axial grooves
Elliptical journal bearing
Two lobe bearing with offset
Three lobe bearing w/wo offset
Four lobe bearing w/wo offset
Floating ring journal bearing
Tilting pad journal bearing

Typical configurations of cylindrical journal bearings (1)
Pressure dam journal bearing

Typical configurations of cylindrical journal bearings (2)
Notes 5. APPENDIX B. TYPES OF JOURNAL BEARINGS. Dr. Luis San Andrés © 2009
References consulted