ROBUSTNESS OF COMPRESSED SENSING
IN SENSOR NETWORKS

A Senior Honors Thesis

by

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ABSTRACT

Robustness of Compressed Sensing in Sensor Networks (April 2008)

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Compressed sensing is a new theory that is based on the fact that many natural images can be sparsely represented in an orthonormal wavelet basis. This theory holds valuable implications for wireless sensor networks because power and bandwidth are limited resources. Applying the theory of compressed sensing to the sensor network data recovery problem, we describe a measurement scheme by which sensor network data can be compressively sampled and reconstructed. Then we analyze the robustness of this scheme to channel noise and fading coefficient estimation error. We demonstrate empirically that compressed sensing can produce significant gains for sensor network data recovery in both ideal and noisy environments.

ACKNOWLEDGEMENTS

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I INTRODUCTION[1]

Compressed sensing (CS) is a new method of signal and image measurement and reconstruction that takes advantage of the fact that many signals are sparse under some basis (typically a wavelet basis)[1]. Compressive measurement is accomplished by taking a small number of projections of the image onto a pseudo random basis and reconstructing the wavelet coefficients of the image from these projections.

Sensor network communication is one area that has not yet experienced the benefits that CS might produce. This is in part because of the novelty of this theory. The limiting characteristics of any wireless network are power, bandwidth, and signal distortion. For applications involving distributed measurements of some physical phenomenon (e.g. temperature, vibration) that may be wirelessly transmitted, CS holds promising improvements to these limits. Compressive measurement of a sensor network is accomplished by prompting each sensor to communicate its value simultaneously to a central base station in a phase coherent fashion [8]. Each sensor value is multiplied by a random number that changes each measurement and is known by the sensor and the base station. This compressive measurement scheme is analogous to the compressive measurement of an image where each sensing element in the network represents one pixel in the image. There has recently been some work discussing the power distortion latency relationship for such a scheme termed Compressive Wireless Sensing (CWS) in [7,8].

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[1] This thesis follows the style and format of *IEEE Transactions.*

In this thesis we apply linear programming to the problem of reconstruction of wireless sensor network data. We analyze the number of compressive measurements that are necessary to accurately reconstruct measured data and empirically determine the robustness of CS in the presence of noise and fading coefficient estimation error. The performance of linear programming for image reconstruction is compared with reconstruction results from other common CS reconstruction algorithms including both the LASSO (Least Absolute Shrinkage and Selection Operator) and LARS (Least Angle Regression) algorithms and the Orthogonal Matching Pursuit (OMP) algorithm. The novel part of our investigation of compressed signal reconstruction is that we examine the effects of two types of error that can occur in compressive measurement of sensor networks. First is the thermal noise at the base station or receiver. The second is estimation error that can arise from inaccurate estimation of the fading coefficient associated with the wireless channel between each sensor and the receiver.

In Chapter II we provide a background discussion on the method of compressed sensing and the principles of signal reconstruction from compressed measurements. We provide as a standard by which our measurements can be compared, the theoretical number of measurements that are required to accurately reconstruct a sparse signal. We also discuss the rate at which computational complexities of these algorithms increase with respect to data field size and the number of compressive measurements taken. In Chapter III we discuss our sensor network system model and the assumptions that will be used to while running empirical tests on the effectiveness of compressed sensing algorithms in the presence of noise. In Chapter IV we test the capabilities of a few popular reconstruction algorithms and compare them to linear programming. Determination of the minimum number of samples required for accurate data reconstruction, and the effects of channel noise and fading coefficient estimation error are discussed. In Chapter V we discuss the principal conclusions of this thesis and identify important areas for future work.

II COMPRESSED SENSING BACKGROUND

Considering Sparsity

Compressed sensing seeks to take advantage of the fact that many natural or manmade images are sparse under some wavelet basis for sampling and reconstruction. Consider an N element image which can be described as an $N\times 1$ vector, X. Any discrete signal in $\mathbb{R}^N$ can be represented in terms of an N element basis of column vectors, $\Psi_{\mathrm{N\times N}} = [\psi_1|\psi_2| \ldots |\psi_N]$. For our purposes we will assume that the basis is an orthonormal wavelet basis. The wavelet transform of X in this basis is given by

$$S = {\Psi_{\mathrm{N\times N}}}^{-1} X \quad \Rightarrow \quad X = \Psi_{\mathrm{N\times N}} S. \qquad (2.1)$$

Where the $N\times 1$ wavelet vector, S, represents the ordered wavelet coefficients of X. A signal is said to be K-sparse if there exists a wavelet basis, $\Psi_{\mathrm{N\times N}}$, in which X can be represented by only K non-zero elements (S has only K non-zero coefficients). Ideally K is much smaller than N. A typical compression algorithm would simply compute the K non-zero coefficients of S and store their amplitudes and locations within the wavelet basis.

This method of sampling a signal and then compressing it suffers from a few inherent inefficiencies. First, the entire N length signal must be measured which can be inefficient if N is very large. This is especially significant for images where the number of pixels scales quadratically with the length or width of the image. Second, the compression algorithm must compute all of the transform coefficients even though many of them are small and can be discarded. Third, the positions of each of the transform coefficients must be known and will therefore require storage. The most relevant inefficiency for the problem considered in this thesis is the fact that the entire *N* length signal must be measured. We propose the use of compressed sensing primarily as a tool to decrease the number of measurements required to accurately determine the sensor readings in a wireless sensor network.

Compressive Measurement

CS presents an alternate way to think about the problem of data acquisition, compression and transmission and in doing so presents an alternate method of data measurement for many applications. The remarkable characteristic of CS is that a *K* sparse signal can be encoded by multiplying it by a random matrix, $\Theta_{\mathrm{M\times N}}$, where *M* is much smaller than *N* but is larger than *K*. The result of this encoding method is the compressive measurement vector, *Y*, which is defined by

$$Y = \Theta_{\mathrm{M\times N}} X. \tag{2.2}$$

Here the length *N* image vector *X* has been encoded as an *M* length vector, but this does not necessarily point to any method for image reconstruction given the underdetermined nature of the system. This is where CS takes advantage of the sparsity of the wavelet coefficient vector *S*. Substituting the wavelet representation of *X* into equation 2.2 we have

$$Y = \Theta_{\mathrm{M\times N}}\, \Psi_{\mathrm{N\times N}} \mathrm{S}. \tag{2.3}$$

Let $\Phi_{M\times N} = \Theta_{M\times N}\,\Psi_{N\times N}$, then we have a single pseudo random matrix in which *Y* gives an underdetermined representation of the sparse vector *S*. This is helpful for mathematical simplification.

It is important to discuss how it is possible to reconstruct *S* from *Y* and to ensure that the probability of exact reconstruct can be made close to unity for this measurement scheme. This is a difficult problem because the locations of the *K* non-zero wavelet coefficients of *X* are unknown. The measurement vector *Y* is just a linear combination of the columns of $\Phi_{M\times N}$ which correspond to the non-zero coefficient in *S*. If the locations of the non-zero entries of *S* were known, finding a solution would simply be a matter of inverting the matrix corresponding to the ordered set of these entries. Here, reconstruction is possible so long as $M \geq K$. A necessary and sufficient condition to show that the $M \times K$ system has a numerically stable inverse is that for any *V* with the same non-zero entries as *S* we have

$$1 - \epsilon \leq \frac{\|\Theta_{M\times N} V\|_2}{\|V\|_2} \leq 1 + \epsilon \qquad (2.4)$$

for some $\epsilon > 0$ [1]. This means that the length of the vector *V* with non-zero components in these *K* coordinates is not affected by $\Theta_{M\times N}$.

It is quite interesting that the vector *S* can be reconstructed from *Y* with high probability even when the locations of the non-zero components are not known. A few reconstruction algorithms capable of solving this problem are discussed below. Another intuitive explanation for why such reconstruction is possible with high probability is that the measurement matrix $\Phi_{M\times N}$ is incoherent with the wavelet matrix $\Psi_{N\times N}$. That is none of the vectors in $\Phi_{\mathrm{M}\times\mathrm{N}}$ can be sparsely represented in $\Psi_{\mathrm{N}\times\mathrm{N}}$ and vice versa. Compressed sensing assures this quality by generating the matrix $\Theta_{\mathrm{M}\times\mathrm{N}}$ at random.

Signal Reconstruction and Constrained Optimization

The natural reconstruction problem where S is our target solution for an underdetermined set of linear constraints leads to the inference that the reconstruction problem at hand can be solved as a constrained optimization problem. The issues that must now be considered are determining the exact cost function and whether efficient solutions exist to the resulting optimization problems. The classic answer to this question is that we want to minimize the root mean square error between S and our reconstructed solution $\hat{S}$. This defines $\hat{S}$ as

$$\hat{S} = min \; \|S'\|_2 \;\; such\ that\ Y = \Phi_{\mathrm{M}\times\mathrm{N}} S'. \tag{2.4}$$

The orthogonality principle tells us that the optimal solution to an underdetermined system of equations is the solution that is orthogonal to all of the constraints. This solution, however, does not take advantage of the sparsity of an image in the wavelet basis.

A logical response to the goal of generating a minimal solution to this problem that is also sparse is to find the minimum ℓ_0 norm solution that satisfies the given constraints. This defines $\hat{S}$ as

$$\hat{S} = min \; \|S'\|_0 \;\; such\ that\ Y = \Phi_{\mathrm{M}\times\mathrm{N}} S'. \tag{2.5}$$

This would work in the ideal case and some CS recovery algorithms are implemented with this idea because it can be shown that a minimum ℓ_0 norm solution can be found with high probability with only $M \geq K + 1$ random measurements [2]. Unfortunately, this solution is forbiddingly complex and requires a combinatorial enumeration of each of the $\binom{N}{K}$ sparse subspaces of $\mathbb{R}^n$.

Compressed Sensing takes advantage of sparsity and counters the complexity and instability of these solutions by solving for the minimum ℓ_1 norm solution to this problem. This defines $\hat{S}$ as

$$\hat{S} = min\ \|S'\|_1 \ \ such\ that\ Y = \Phi_{\mathrm{M \times N}} S'. \quad (2.6)$$

The result is a solution that is not necessarily orthogonal to each of the constraints but it is easier to ensure sparsity of the solution because an ℓ_1 norm solution is in some sense sharper than an ℓ_2 norm solution. In many cases the minimum ℓ_1 norm solution is a good approximation of the minimum ℓ_0 norm solution. This can be geometrically understood by thinking about the way a minimum ℓ_p norm solution is constructed. For ℓ_2 norm recovery, the minimum solution is obtained by increasing the root mean square length of the target vector until it touches the optimization constraints. For ℓ_1 norm recovery the absolute value of each element of the solution vector is increased until the solution vector touches the optimization constraints. This is a simplification of the constrained optimization problem; however, it presents the basic concepts necessary to understand why minimum ℓ_1 norm solutions are useful for sparse signal reconstruction. An illustration of this concept is shown in Figure 2.1 for a constrained minimization problem in $\mathbb{R}^2$, i.e., when $N = 2, K = 1$, $M = 1$. Then, the optimization problem is to find a vector S in $\mathbb{R}^2$ with exactly one non-zero component such that we have one observation Y of the form $Y = \Phi_{M\times N}S$. In Figure 2.1, the ℓ_1 norm solution is compared to the ℓ_2 norm solution pictorially.

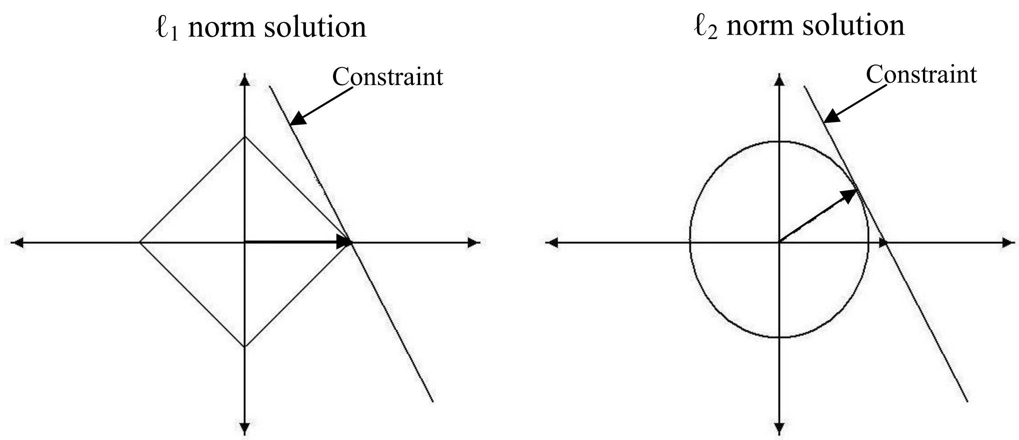

Fig. 2.1 - ℓ_1 vs. ℓ_2 norm solutions to constrained optimization problems in $\mathbb{R}^2$

It can be seen from the figure that the minimum ℓ_2 norm solution is a solution with both components of the vector being non-zero, i.e., the solution is not sparse. The minimum ℓ_1 norm solution results in only one of the two components being non-zero, i.e., the solution is sparse. The reason constraints are handled for the ℓ_1 and ℓ_2 norms as shown in Figure 2.1 can be understood by the algebraic definition of a line equation and the Pythagorean Theorem respectively. The important result from this discussion is that an ℓ_1 norm solution for a given constraint or set of constraints is much more likely to be sparse than ℓ_2 norm solutions.

Reconstruction Algorithms

Because the ℓ_1 norm has been found to accurately reconstruct sparse signal representations there has been much work to generate efficient algorithms for ℓ_1 norm reconstruction. One of the simplest solutions to this problem is a linear program called *basis pursuit* [3] which relies on conventional linear programming techniques. The computational complexity of this algorithm is polynomial in N while the number of measurements generally required for adequate reconstruction is given by $M = cK$ for $c > 1$. The constant, c, refers to an oversampling factor whose value is inversely dependent on sparsity.

Other minimum ℓ_1 norm recovery algorithms such as the LARS (Least Angle Regression) algorithm and one of its derivatives the LASSO (Least Absolute Shrinkage and Selection Operator) algorithm have been investigated in [4,6]. These greedy algorithms take advantage of geometric correlation between the targeted minimum ℓ_1 norm reconstruction and the coefficients most correlated with the measured response. These algorithms take advantage of quadratic programming concepts because they produce a weighted minimum ℓ_1 and ℓ_2 norm solution. The LASSO algorithm specifically solves

$$min\left\{\frac{1}{2M}\|Y - S'\|_2^2 + \lambda\|S'\|_1\right\} \; such\ that\ Y = \Phi_{\mathrm{M\times N}}S', \qquad (2.7)$$

for any parameter λ. The value of λ typically corresponds to the maximum correlation between the measurements Y and the observation matrix $\Phi_{\mathrm{M\times N}}$. The LARS algorithm is a derivative of the LASSO which solves a similar problem but approaches a solution more quickly.

The LARS algorithm specifically works by starting with all reconstruction coefficients equal to zero and finding the coefficient most correlated by the randomized measurements. The algorithm increases its value in that direction until another coefficient becomes equally correlated with the random measurements. At this point the target reconstruction steps in a direction equiangular between the original coefficient and the newly correlated coefficient. Changing the step size for this algorithm produces a trade-off between the error gained by adding new coefficients too infrequently and the number of iterations required to produce an accurate solution. It has been shown in [6] that thresholds θ_l and θ_u exist with the properties that for any $v > 0$ the number of samples required for accurate signal reconstruction with high probability can be given by

$$M > 2K(\theta_l + v)\log(N - K) + k + 1. \qquad (2.8)$$

The probability of accurate signal reconstruction converges to 1 as N increases. The limit θ_u similarly defines the number of samples under which the probability of accurate signal reconstruction converges to 0 as N increases.

A compressed sensing solver that seeks to compute the minimum ℓ_0 norm solution is the Orthogonal Matching Pursuit (OMP) algorithm [5]. This algorithm attempts to determine which columns of the pseudo random matrix $\Phi_{\mathrm{M\times N}}$ are most correlated to the measurement matrix Y. The column with the largest correlation is likely the largest coefficient of S. During an iteration, the column of $\Phi_{\mathrm{M\times N}}$ with the largest correlation to Y is found and its contribution to Y is subtracted. The resulting coefficient of S is determined and the process repeats until Y disappears or has a value smaller than some threshold of acceptable error. This algorithm should only need to iterate K times to successfully reconstruct S. It has been shown that with $M \geq cK \ln\left(\frac{N}{K}\right)$ it is possible to reconstruct every K sparse with a probability exceeding $1 - e^{-NM}$. There has been some critique that OMP cannot produce accurate results except in the simplest (noiseless) circumstances [5]. This leads to reason that the effectiveness of the algorithm may degrade swiftly in the presence of noise. This will be investigated in this thesis in Chapter IV.

We present an additional theoretical limit algorithm that will be used to determine a lower bound on the error present in signal reconstruction. Optimal solutions will be constructed by assuming a genie tells us which coefficients in the sparse vector S are non-zero. These are used to create a new matrix $\Phi_{M\times N}{}'$ which constructed from the columns corresponding to the locations of these non-zero coefficients. The values of the coefficients can be determined by orthogonally projecting our measurement vector Y onto this matrix. This is mathematically expressed as

$$C = (\Phi_{M\times N}{}'^{T}\Phi_{M\times N}{}')^{-1}\Phi_{M\times N}{}'^{\,T}Y. \qquad (4.1)$$

Here, C is a vector of the non-zero coefficients of S which can be used to reconstruct the simulated sensor network data. These coefficients are placed in there appropriate locations to determine S. This provides a theoretical limit for the capabilities of CS reconstruction algorithms because it characterizes the information that is available after compressive sampling and produces a solution that is orthogonal to our constraints in the wavelet domain. The results obtained from this algorithm are not necessarily indicative of the capabilities of an actual reconstruction algorithm because they are based on prior knowledge of the locations of non-zero coefficients. One of the strengths of the theory of compressed sensing is that accurate solutions may be obtained without knowing these locations.

It is important to note that the both the number of samples required to effectively reconstruct S and the computational complexity (approximated by the number of iterations required by an algorithm) is dominated by the sparseness factor *K*. This means that CS is a measurement scheme for which the complexity of measurement decreases with the density of information present in a signal field. The implications of this idea will alter the way many problems in engineering and the sciences are approached.

III COMPRESSIVE MEASUREMENT OF SENSOR NETWORKS

Compressive Wireless Sensing

Recent research has introduced an implementation of CS for sensor networks called Compressive Wireless Sensing (CWS) in which a central base station retrieves wireless sensor network data from a randomly distributed grid of transducers [7,8]. Much of this research has focused on the power - distortion - latency relationship for a projection of distributed sensor network data onto an underdetermined basis. The goal of this research was to discuss a theoretical model for the compressive sampling of wireless sensor network data. The effects of sensor measurement error, electromagnetic interference present at the base station, and channel phase estimation error were discussed as far as they pertain to accurate determination of the values of Y in [8].

Our research considers a similar distributed grid of sensors (transducers) that measure some physical data (e.g. temperature, pressure) and wirelessly transmit these measurements to a central base station simultaneously and phase coherently. A typical sensor network is shown in Figure 3.1. Here, each black dot represents a sensor with a wireless transmitter communicating sensor readings simultaneously to a base station. The extension to a randomly distributed grid is not difficult; however, we consider an ordered grid to simplify sensor addressing.

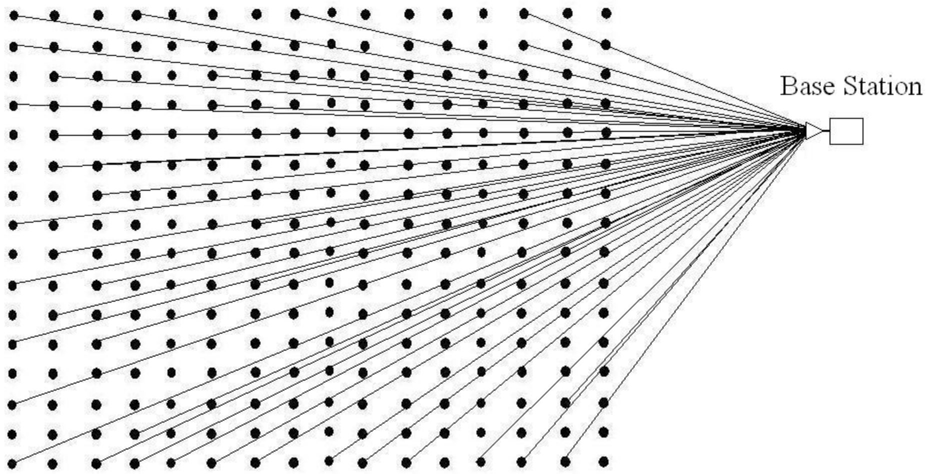

Figure 3.1 – Distributed Sensor Network Communication

Modeling and Assumptions

The novelty of the concept of CWS arises from the fact that projections of sensor network data can be naturally added at the base station because the sensors transmit their data phase coherently. Disregarding the effects of noise, the p_{th} observation of the sensor network data which corresponds to the p_{th} element of Y is given by

$$y_p = \sum_i \theta_{i,p} h_i x_i. \tag{3.1}$$

Here, $h_i \in \mathbb{R}$ is the phase dependent gain associated with the i_{th} sensor in the grid which is multiplied by the corresponding element in X. It is assumed that the value of h_i is known for each sensor and does not change during the M observations (it is independent of p). The value $\theta_{i,p}$ is an element of a Rademacher random matrix that enables random sampling of sensor network data. Equation 3.1 can be represented in matrix form as

$$Y = \Theta_{M\times N} H_{N\times N}\, \Psi_{N\times N} S, \tag{3.2}$$

where $H_{i,j} = h_i \; \forall \; j$. This assumes that an entire compressive reading of network data is completed before fading changes. The random matrix, $\Theta_{\mathrm{M\times N}}$, for CWS is constructed as a matrix of Rademacher random variables. This is useful because it allows each of the sensors to locally determine their own vector of Rademacher random variables using their address as a seed value. The base station is then able to construct $\Theta_{\mathrm{M\times N}}$ from the seed values given by the appropriate addresses [8].

An important focus of this experiment is to examine the capabilities of current compressed sensing reconstruction methods for noisy measurements. It is important to discuss how this noise is modeled in our simulated network. There are two different types of error that are analyzed. The first type is error due to noise present at the base station receiver. This is given by

$$Y = \Theta_{\mathrm{M\times N}} \mathrm{H}_{\mathrm{N\times N}} \, \Psi_{\mathrm{N\times N}} \mathrm{S} + Z. \tag{3.3}$$

Here Z represents an $\mathrm{M} \times 1$ vector of i.i.d. zero mean Gaussian random variables. The second type of error present in our analysis arises from fading coefficient estimation error that is introduced when the channel gain from each sensor to the base station is incorrectly estimated. The assumption here is that there is a zero mean i.i.d. Gaussian random error, w_i, associated with our knowledge of each h_i. An individual element, y_p, of Y can then be described by

$$y_p = \sum_i h_i \theta_{i,p} x_i + z_p = \sum_i h_i \theta_{i,p} \, \psi_{i,p} s_i + z_p. \tag{3.4}$$

Here, h_i and h_i' represent the actual phase gain and estimated phase gain associated with the i_{th} channel respectively given by

$$h_i = h'_i + w_i. \tag{3.5}$$

In vector form we have

$$Y = \Theta_{\mathrm{M\times N}} \mathrm{H}_{\mathrm{N\times N}}\, \Psi_{\mathrm{N\times N}} \mathrm{S} + Z. \tag{3.6}$$

IV RESULTS AND DISCUSSIONS

Compressive Wireless Sensing Simulation

In this chapter we provide some simulation results for compressive measurement of wireless sensor network data using the sensor network model discussed in Chapter III. Simulated sensor network data was generated by sparsely populating a matrix with normal random variables and computing the two dimensional inverse discrete wavelet transform (IDWT) of these generated coefficients. An example of this wireless sensor network data is shown in Figure 4.1.

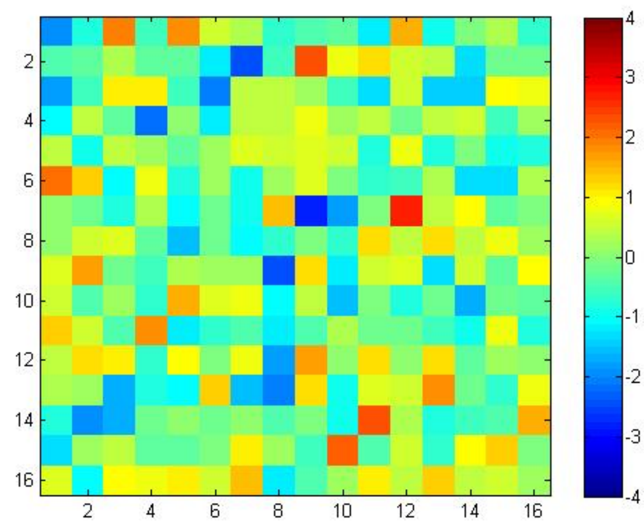

Figure 4.1 – Sample Wireless Sensor Network Data

This data was then ordered as the $N \times 1$ vector, X, by stacking each of the columns in the matrix. For our measurements we choose N as 256. The sparsifying matrix $\Psi_{\mathrm{N\times N}}$ was then generated by computing and vectorizing the two dimensional DWT of each standard basis element in $\mathbb{R}^{16\times 16}$. Reconstruction of each data set was accomplished using linear programming techniques and implementations of the LASSO, LARS, and OMP algorithms provided by Stanford University's Sparselab toolkit [9].

Minimum Sample Size

It is first necessary to determine how many measurements are required to accurately reconstruct sensor network data in a noiseless environment. This is important because it allows us to approximate the theoretical advantages of this measurement scheme. It also provides a benchmark for tests involving channel noise and fading coefficient estimation error. Figure 4.2 represents the root mean square error measured by this test averaged over 200 data sets with an average sparsity of five percent.

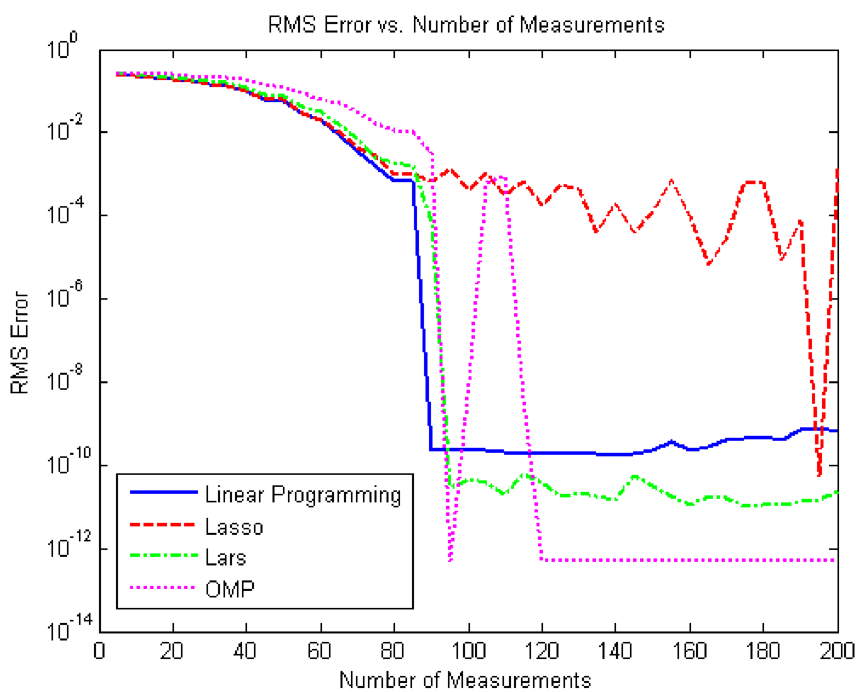

Figure 4.2 – RMS Error vs. Number of Measurements

The results of this test are interesting to discuss, especially for measurement sizes larger than ninety. There are significant inconsistencies in the rate of decay of RMS error for the LASSO and OMP algorithm reconstructions. There also appears to be a limit at which further increases in sample size do not appear to improve signal reconstruction quality for each algorithm. A histogram of the RMS error of measurements from Figure 4.2 is useful to understand how the spikes in error for certain sample sizes are characterized. This histogram shown in Figure 4.3 depicts the range of the RMS reconstruction errors in a stacked format that were averaged to plot Figure 4.2.

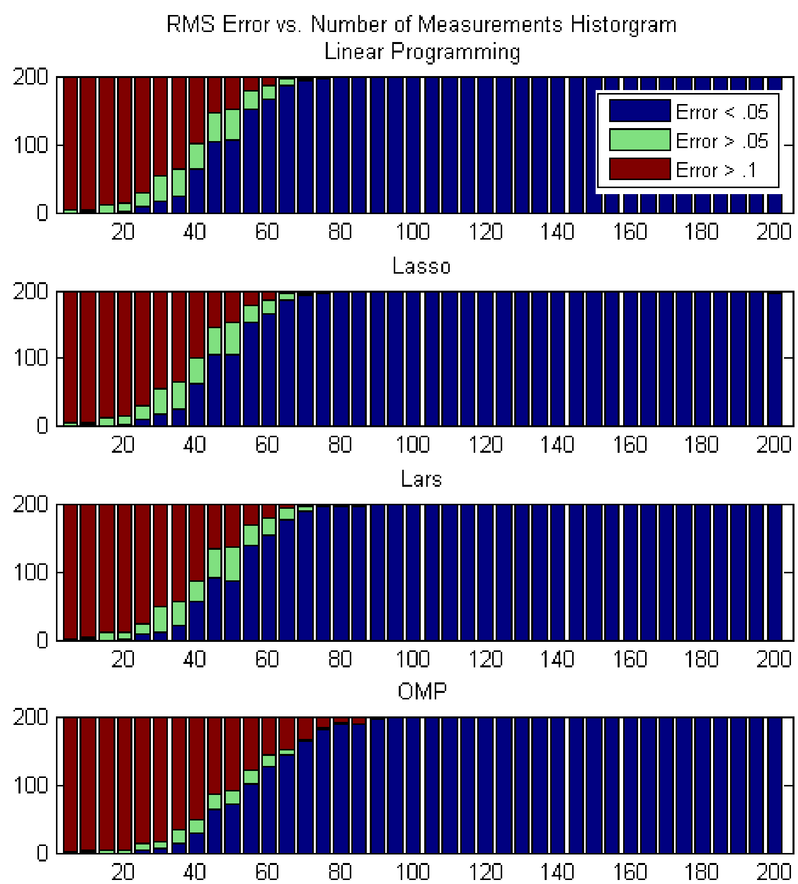

Figure 4.3 – RMS Error vs. Number of Measurements Histogram

It may be difficult to see how the RMS errors for larger measurement sizes are characterized from the view of the histogram in Figure 4.3. It can be concluded, however that the spikes in error in Figure 4.2 are not characteristic of consistent errors at specific sample sizes. The inconsistencies in the LASSO, LARS, and OMP algorithms appear to arise from reconstruction failures for individual data sets. Figure 4.4 provides a clearer view of this assertion by displaying the top of the stacked histogram in Figure 4.3 much more closely. The histogram in Figure 4.4 uses the same legend as Figure 4.3.

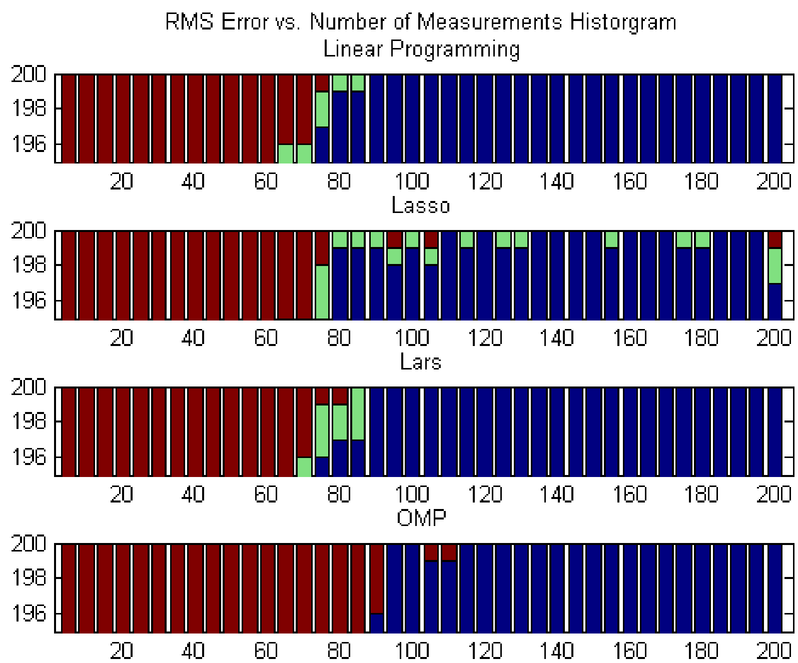

Figure 4.4 – RMS Error vs. Number of Measurements Histogram With Shifted Y-Axis

Notice that even when the number of compressive measurements is very high (ex: 200) the LASSO algorithm produces a very high error for a few sensor network data sets. This causes the average RMS error to be very high. A more detailed analysis of these failures is left for future work.

From Figure 4.2 and Figure 4.3, it can be seen that only seventy samples are required for accurate data reconstruction for each algorithm in a noiseless environment. This result emphasizes the gains which may be made by compressive measurement and the current deficiencies of measurement schemes which do not take advantage of signal sparsity. Traditional sampling schemes require the value of each sensor to be measured individually which is especially restrictive in environments where data is available for short durations. Tests determining the effects of noise on signal reconstruction will be investigated with a minimum of seventy samples.

Effects of Channel Noise

It is important to characterize the effects of channel noise on sensor network data reconstruction. As discussed Chapter III, the sensor network measurement vector, *Y*, with additive channel noise, *Z*, is given by

$$Y = \Theta_{M\times N} H_{N\times N}\, \Psi_{N\times N} S + Z. \tag{4.1}$$

The RMS error measured for reconstructions is plotted against signal to noise ratios from 2.5dB to 50 dB for each reconstruction algorithm in Figure 4.5. This test considers data which is five percent sparse and averaged over 100 measurements.

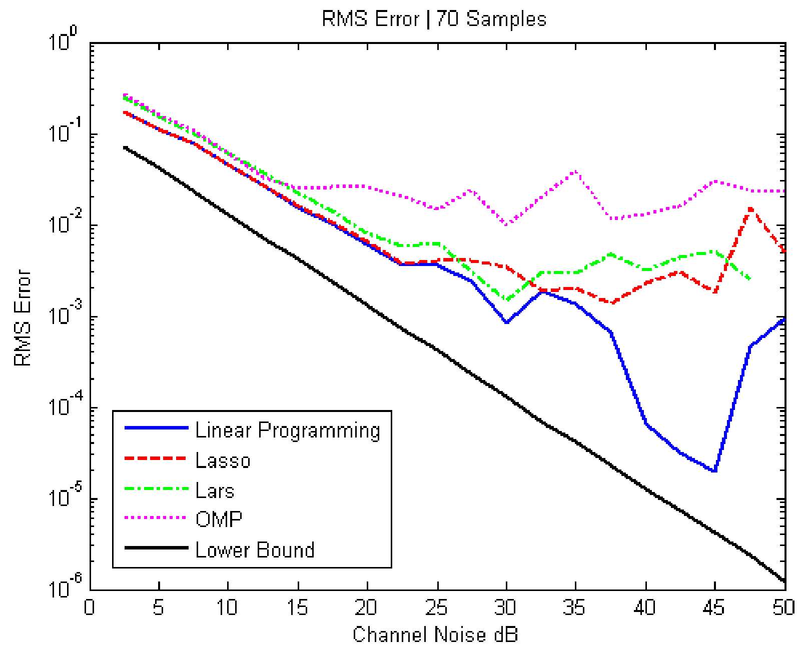

Figure 4.5 – RMS Error vs. Channel Noise for 70 Compressive Measurements

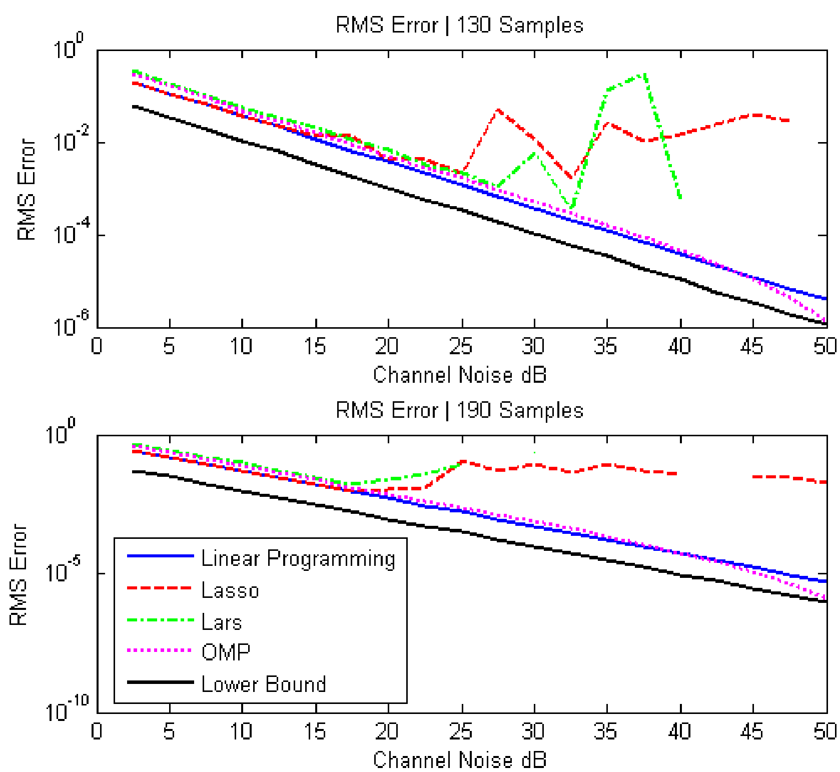

Figure 4.6 – RMS Error vs. Channel Noise for 130 and 190 Compressive Measurements

The lower bound in Figures 4.5, 4.6, 4.8 and 4.9 represents reconstruction from the theoretical limit algorithm discussed in Chapter II. The errors in signal reconstruction above this limit must then correspond to the fact that coefficients generated from sparse (minimum ℓ_0 or ℓ_1 norm) solutions may not correspond to orthogonal solutions and that ignorance of non-zero coefficient location may produce non-zero coefficients with incorrect positions within S.

One interesting result from Figure 4.5 is that the reconstruction error of each of the reconstruction algorithms tested improves at a rate equal to the theoretical limit for large channel noise variance. Each algorithm appears to have a limit after which a decrease in noise variance does not improve signal reconstruction consistently. We show that these limits arise in part from because of the small number of compressive measurements as this test was repeated for larger compressive measurement sizes in Figure 4.6.

Notice in Figures 4.6 that the rate of decay of RMS error with SNR is the same for the linear programming, the OMP algorithm, and the theoretical limit based receiver. This shows that the receiver optimally trades-off RMS error for SNR. Perhaps the most puzzling result from this test is that the LARS and LASSO algorithms maintained a limit after which decreasing noise variance did not produce improvement in signal reconstruction quality with larger numbers of compressive measurements. It is also surprising that increasing the number of compressive samples and decreasing the measurement noise actually serves to degrade reconstruction quality and stability for these algorithms. The capabilities for these algorithms are not discussed in detail in this thesis, however, this failure of the LARS and LASSO algorithms is likely the result of the small number of sensors in the simulated network and the particular software that was used to implement these algorithms. In Chapter II, it was discussed that the probability for accurate signal reconstruction converges to one as *N* increases. It may be assumed that 256 is too small a network size for consistent network data reconstructions for these algorithms. The SNRs in Figure 4.6 where the RMS error is not plotted corresponds to datasets that contain NaN results causing a test failure. Because this failure can result from the given algorithm returning a single NaN value, a histogram of the RMS error is included for several sample sizes over 2000 data reconstructions with an SNR of 30dB in Figure 4.7. The limits of the y-axis of the histogram in Figure 4.7 are selected for best view of high error and NaN results. The values on the x-axis represent the minimum RMS error allowed for each section of the histogram.

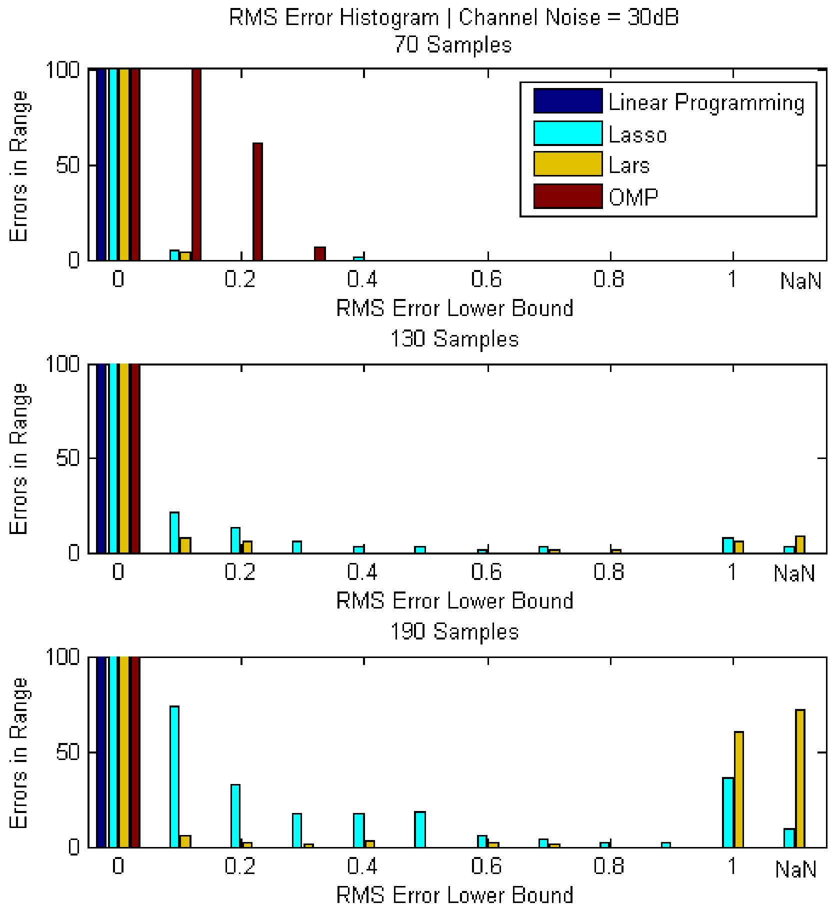

Figure 4.7 – RMS Error Histogram for Channel Noise of 30dB

There is a large number of NaN errors measured for the LARS and less notably the LASSO algorithms. It is interesting that the Linear Programming and OMP algorithms do not return any NaN errors. The larger errors apparent for the OMP algorithm at smaller sample sizes likely occur because of the difficulty of computing minimum ℓ_0 norm solutions with a small number of measurements especially in the presence of noise. Even small noise levels can disrupt ℓ_0 norm reconstruction because of the difficulty of choosing a threshold for non-zero coefficients.

Effects of Fading Estimation Error

Fading estimation error can have interesting effects on signal reconstruction error. This is partially because it is possible for small SNRs to cause a sign error on the reconstructed value of an individual sensor if the channel gain for that particular sensor is small. The measurement vector, *Y*, when this noise is present is given by

$$Y = \Theta_{M\times N} H_{N\times N}\, \Psi_{N\times N} S. \tag{4.2}$$

Where $H_{N\times N}$ is the real fading coefficient matrix and each row of $H_{N\times N}$ is the same vector. The real fading coefficient of the i_{th} sensor is given by

$$h_i = h_i{}' + w_i. \tag{4.3}$$

Where w_i corresponds to the error in the estimation of the i_{th} fading coefficient. This is discussed in detail in chapter III.

The tests used to study the effects of channel noise on sensor network reconstruction were repeated for fading estimation error. The RMS error measured for reconstructions with fading coefficient estimation error variances from -2.5dB to -50 dB for each reconstruction algorithm is shown in Figure 4.8. This test considers data which is five percent sparse and averaged over 100 measurements.

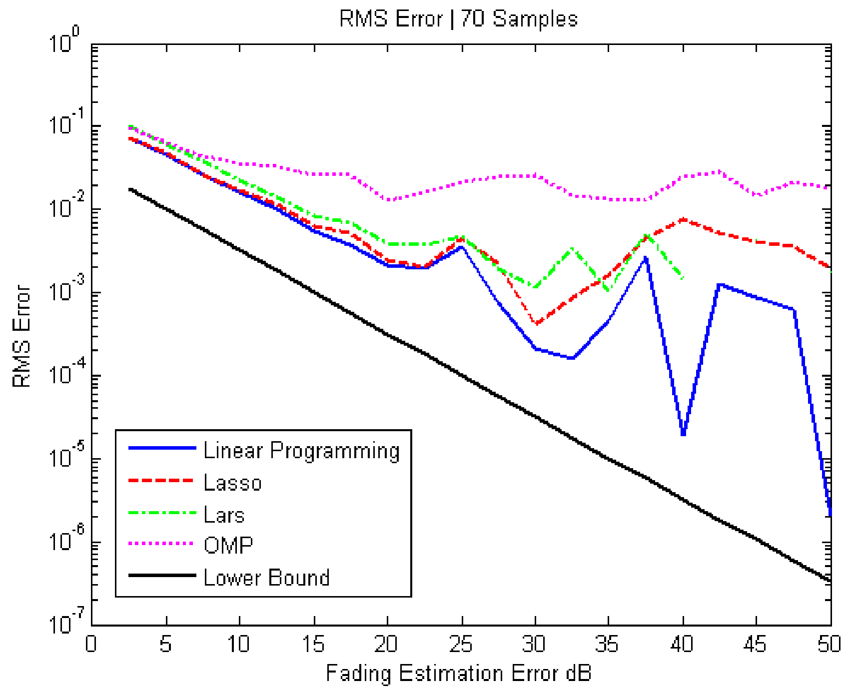

Figure 4.8 – RMS Error vs. Fading Estimation Error for 70 Compressive Measurements

The results for this test are similar to those for the channel noise test with seventy measurements. An important result from both Figure 4.5 and 4.8 is that Linear Programming reconstructions result in smaller error measurements than each of the other algorithms. It is interesting that the small sample size of this test dominates the decrease in RMS error much more quickly in Figure 4.8 than in Figure 4.5. Close inspection of these figures reveals that the fading estimation error simply does not degrade data reconstruction quality as much as channel noise. This is reflected in the theoretical limit reconstruction algorithm as well. This test was repeated with larger sample sizes in Figure 4.9.

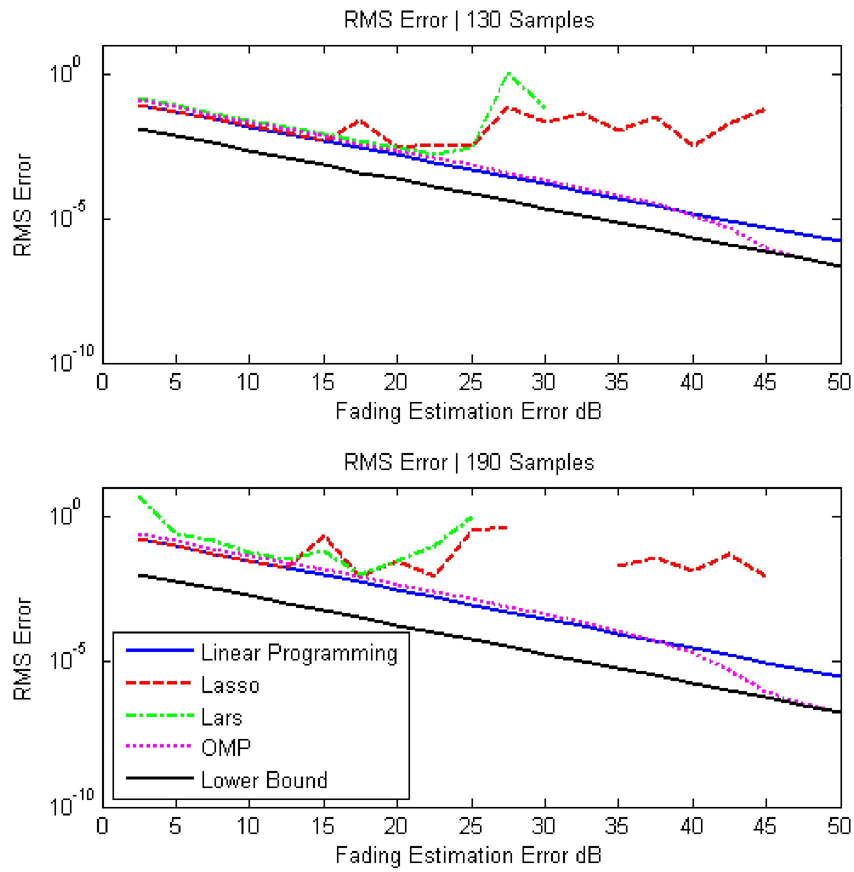

Figure 4.9 - RMS Error vs. Fading Estimation Error for

130 and 190 Compressive Measurements

The test results shown in Figure 4.9 are consistent with the results of Figure 4.6 in that the RMS errors of the Linear Programming and OMP data reconstructions decay at the same rate as the theoretical limit algorithm. OMP reconstructions improve more swiftly and approach the theoretical limit for small noise variances. This is likely the result of the hard threshold for minimum signal reconstruction error present in the algorithm [9]. Non-zero coefficients in the wavelet domain with a value smaller than this threshold will not be present in the solution because they correspond to a change in the solution that is smaller than the minimum acceptable error. The LARS and LASSO algorithms continue to produce inconsistent results for large sample sizes and small error variances. A histogram of the RMS error is plotted for fading estimation error equal to 30 dB in Figure 4.10.

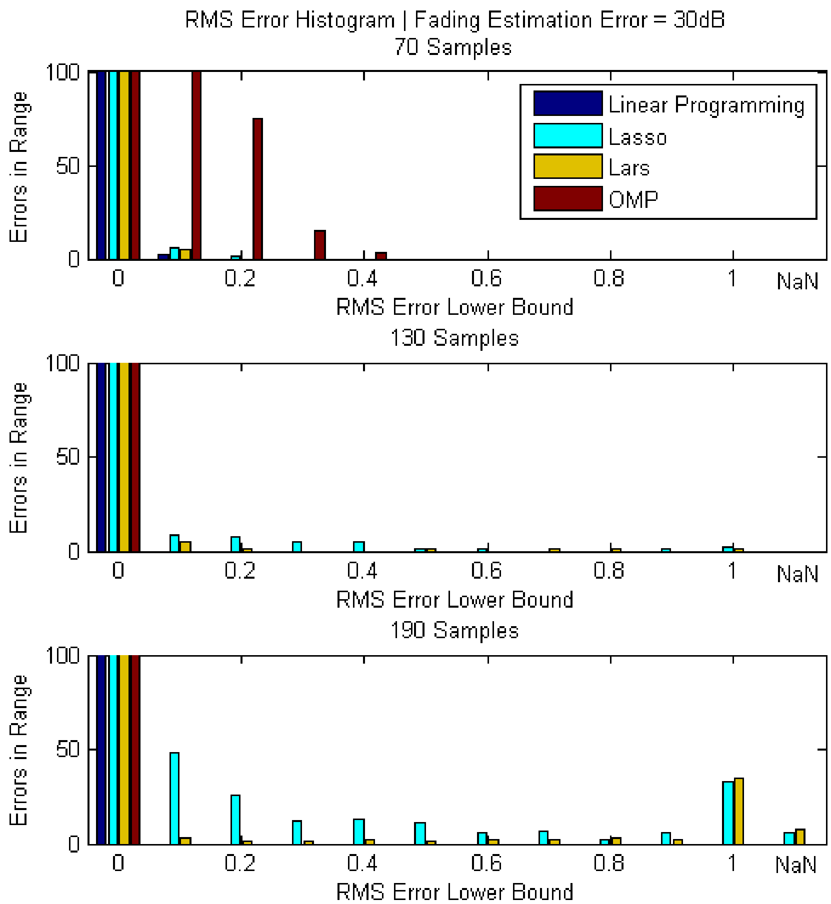

Figure 4.10 – RMS Error Histogram for Fading Estimation Error of 30dB

Comparing Figure 4.10 with Figure 4.7 further demonstrates that fading estimation error does not degrade data reconstruction quality as much as much as channel noise except for a small decrease in performance of the OMP algorithm for compressive measurement sizes.

V CONCLUSIONS AND FURTHER RESEARCH

There are a few specific conclusions which may be drawn from the results of our study. First is that the method of compressed sensing can significantly improve the quality of sensor network data measurement over the traditional scheme individually measuring each sensor reading. This can be seen from Figure 4.3 where it becomes clear that only roughly 70 samples are required for reconstruction of sensor network data. Second is that noise from the channel as well as estimation errors do not adversely affect the performance of linear programming based reconstruction algorithms. Surprisingly, as seen from Figure 4.6, the rate of decay of reconstruction error with SNR is nearly optimal for linear programming based reconstruction algorithms. Thus compressed sensing can be a robust and efficient alternative to conventional methods of data collection and reconstruction in sensor networks. The particular implementations of the LASSO and LARS algorithms that were obtained from the Stanford sparselab tool kit failed occasionally causing a large increase in the average RMS error which produces somewhat abnormal results. A more detailed investigation of these failures is very important and should be considered in future work.

One of the disadvantages of the current technique is that it is not very power efficient because each of the sensors needs to transmit M times. Future research will discuss methods by which the number of sensor transmissions may be decreased. One method which will be studied is whether the sensing matrix $\Phi_{M\times N}$ can be made sparse. It will be interesting to try to determine a way to estimate the RMS error that will occur from different reconstruction methods prior to applying the algorithm to reconstruction.

This work may shed light onto the ways that current algorithms may be improved. This research has only investigated fading estimation error for slow fading. It will be interesting to determine how fast fading affects compressive receiver performance.

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EDUCATION

Texas A&M University
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Major: Electrical Engineering
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RESEARCH

Texas A&M University Undergraduate Research Fellows Program Class of 2008

- Thesis: *Robustness of Compressed Sensing in Sensor Networks*
- Collaborated with faculty advisor in researching capabilities of compressive measurement scheme for efficient data measurement and reconstruction in sensor networks

WORK HISTORY

Baker Oil Tools — Houston, TX
Electrical Engineering Summer Internship — 2007

- Designed and programmed simulation tool for marketing and training purposes to familiarize users with system software and expected well response
- Designed algorithm for a position sensor to empirically determine position of sliding sleeve for the hydraulically adjusted valve

Baker Atlas — Aldine, TX
Electrical Engineering Summer Internship — 2006

- Administered environmental testing and failure analysis on oil service and measurement equipment

HONORS AND AWARDS

- Received Graduate Merit Fellowship Award
- Graduation with Fellows Honors
- Dean's List Awards
- Awarded Baker Hughes Texas A&M University Endowment Scholarship
- College of Engineering Scholarship (2005-2008)
- The Bolton Award (2005-2006)
- The Robert M. Kennedy Award (2007-2008)
- ETA KAPPA NU membership
- Eagle Scout Award