

SINGULAR SUBFACTORS OF II_1 FACTORS

A Dissertation

by

ALAN DANIEL WIGGINS

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2007

Major Subject: Mathematics

SINGULAR SUBFACTORS OF II_1 FACTORS

A Dissertation

by

ALAN DANIEL WIGGINS

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Roger Rance Smith
Committee Members,	Ken Dykema
	David Kerr
	P. Fred Dahm
Head of Department,	Al Boggess

May 2007

Major Subject: Mathematics

ABSTRACT

Singular Subfactors of II_1 Factors. (May 2007)

Alan Daniel Wiggins, B.A., Indiana University

Chair of Advisory Committee: Dr. Roger Rance Smith

We examine the notion of α -strong singularity for subfactors N of a II_1 factor M , which is a metric quantity that relates the distance of a unitary to a subalgebra with the distance between that subalgebra and its unitary conjugate. Using work of Popa, Sinclair, and Smith, we show that there exists an absolute constant $0 < c < 1$ such that all singular subfactors are c -strongly singular. Under the hypothesis that $N' \cap \langle M, e_N \rangle$ is 2-dimensional, we prove that finite index subfactors are α -strongly singular with a constant that tends to 1 as the Jones Index tends to infinity and infinite index subfactors are 1-strongly singular. We provide examples of subfactors satisfying these conditions using group theoretic constructions. Specifically, if P is a II_1 factor and G is a countable discrete group acting on P by outer automorphisms, we characterize the elements x of $P \rtimes G$ such that $x(P \rtimes H)x^* \subseteq P \rtimes H$ for some subgroup H of G . We establish that proper finite index singular subfactors do not have the weak asymptotic homomorphism property, in contrast to the case for masas. In the infinite index setting, we discuss the role of the semigroup of one-sided normalizers with regards to the question of whether all infinite index singular subfactors have the weak asymptotic homomorphism property. Finally, we provide a characterization of singularity for finite index subfactors in terms of the traces of projections in $N' \cap \langle M, e_N \rangle$ and use this result to show that fixed point subfactors of outer \mathbb{Z}_p for p prime are regular. The characterization extends to infinite index subfactors by replacing "singular" with "contains its one-sided normalizers."

To my Uncles Corky and Danilo.

ACKNOWLEDGMENTS

From Texas A&M, I would first like to thank my advisor, Roger Smith, who offered advice and encouragement almost from the moment I arrived. Without your direction and willingness to hear me talk, sometimes about total lies, I would not have been able to get to this point. I also thank Stuart White for engaging me in discussions about problems I ordinarily wouldn't have thought about and for helping me navigate the treacherous minefield that is the job application process. Try to keep your feet off of other people's chairs at Glasgow. Thanks to Thomas Schlumprecht for making my transition here so easy and to Ken Dykema, Gilles Pisier, and Bill Johnson for putting up with my stupid questions. Being in the math department at A&M made me realize that I had found my community, and I will miss being here. Special thanks to Monique Stewart, the true power in the department, without whom I would still be wondering when to fill out my degree plan.

From the University of Texas, I would like to thank my advisor Haskell Rosenthal for introducing me to functional analysis and for continuing to meet with me and listen to my ideas even when not in the best of health. Thanks also to Vrej Zarikian for convincing me to transfer to A&M and to Ted Odell for honestly describing what to expect as a researcher. I thank all my friends from UT, especially Kate, Adam, Judy, Cecilia, Kurt, Chris, Christine, Kelly, Rafael, and Mike. "I was so afraid that you would all be math nerds!" Yeah, me, too! What the hell happened?

I give thanks and love to my parents for always supporting me, even when I was a sullen teen trying to deny their existence, and to my brother, Tom, who enters graduate school just as I am leaving it. You've been more important to me than I could say.

Finally, I have to thank my wife Jennifer, who has endured eight years worth of

brutal Texas summers and mathematical conversations of varying degrees of sobriety in order for me to get a silly robe and hood. I love you!

TABLE OF CONTENTS

	Page
1. INTRODUCTION	1
2. PRELIMINARIES	4
2.1. Background and Notation	4
2.2. Group von Neumann Algebras, Crossed Products, and Ex- amples of II_1 Factors	6
2.2.1. Group von Neumann Algebras	6
2.2.2. Crossed Products	7
2.3. The Standard Representation	9
2.4. The Pull-Down Homomorphism	12
2.5. Averaging Over Subalgebras	13
3. SINGULARITY AND STRONG SINGULARITY FOR MASAS . . .	14
4. HILBERT MODULES AND THE JONES INDEX	17
4.1. Definitions and Properties	17
4.2. Galois Theory for Subfactors	22
4.3. Hilbert Bimodules and $N' \cap \langle M, e_N \rangle$	23
4.4. Pimsner-Popa Bases	30
5. STRONG SINGULARITY ESTIMATES FOR SINGULAR SUB- FACTORS	32
6. EXAMPLES OF SINGULAR SUBFACTORS	40
6.1. Normalizers in Crossed Products	40
6.2. Singular Subfactors	47
6.3. The WHP and Finite Index Subfactors	49
6.4. Infinite Index Singular Subfactors and One-Sided Normalizers	52
7. A CHARACTERIZATION OF SINGULARITY FOR SUBFACTORS	57
8. CONCLUSION	63
REFERENCES	64

VITA	68
----------------	----

1. INTRODUCTION

The idea of distinguishing subalgebras of a II_1 factor M by its normalizing unitaries in M dates back to Jacques Dixmier [7] in the context of maximal abelian $*$ -subalgebras (masas) A of M . Though Dixmier's approach for masas has been quite successful and spawned a variety of techniques and results, there has been relatively little work done on normalizers of subalgebras of II_1 factors.

Sinclair and Smith [30], [28] recently developed the notion of α -strong singularity for a subalgebra of a II_1 factor where $0 < \alpha \leq 1$. One could hope that singular subalgebras are all α -strongly singular for some α or even $\alpha = 1$, the best possible constant. The latter was proved for masas in [31].

At the other end of the spectrum from masas, the modern study of subfactors of a II_1 factor was initiated in the early 1980's through work of Vaughan Jones [13] defining the Jones Index. Given the considerable attention focused on subfactors since this time and the many useful techniques developed, subfactors are prime candidates for examining the question of whether singularity implies α -strong singularity. We became convinced that if $\alpha = 1$ were not possible, one should be able to obtain a constant dependent upon the Jones Index. Under additional hypotheses, our main result is that this is indeed so. We now detail the contents of this work.

In section 2 we establish notation that will hold for the whole of this work. We detail the construction of group von Neumann algebras and crossed products and define basic concepts applicable to all subalgebras of II_1 factors which shall be employed in later sections. In section 3, we define singularity, α -strong singularity, and the weak asymptotic homomorphism property (WAHP) for subalgebras of a II_1 factor and discuss the equivalence between these concepts for maximal abelian $*$ -subalgebras.

This dissertation follows the style of the Journal of Functional Analysis.

The next section, section 4, develops the theory of Hilbert modules and the Jones Index. We include a useful picture for left Hilbert modules due to Sorin Popa and remark upon the behavior of the Jones Index in crossed product factors. Next, we discuss Hilbert bimodules and their relation to relative commutants, including a proof of the description of irreducible submodules for crossed product factors in Proposition 4.3.3. We end the section by touching on Pimsner-Popa bases.

Section 5 contains the main results of this work. Theorem 5.0.3 shows that when the higher relative commutant $N' \cap \langle M, e_N \rangle$ is 2-dimensional, proper finite index singular subfactors of M are α -strongly singular in M where $\alpha = \sqrt{\frac{[M:N]-2}{[M:N]-1}}$. As this constant tends to one as the index tends to infinity, this suggests that infinite index singular subfactors are strongly singular, a variant on the proof for the finite index case yields strong singularity for an infinite index inclusion $N \subseteq M$ when $N' \cap \langle M, e_N \rangle$ is 2-dimensional. Using results from [26], we obtain an absolute constant $c = \frac{1}{25}$ for which all singular subfactors are c -strongly singular. The constant obtained is less than the values produced in Theorem 5.0.3 and so is not optimal in at least those cases.

A method for constructing singular subfactors is discussed in section 6. Subsection 6.1 gives Theorem 6.1.2, which characterizes elements x of $P \rtimes G$ such that $x(P \rtimes H)x^* \subseteq P \rtimes H$ for some subgroup H of G . As corollaries, we obtain explicit descriptions of both the normalizing algebra of $P \rtimes H$ in $P \rtimes G$ and of the normalizing unitaries themselves. In subsection 6.2, we give examples of singular subfactors with index greater than or equal to 4 satisfying the hypotheses of Theorem 5.0.3. This relies upon Proposition 4.3.3.

Though the weak asymptotic homomorphism property implies strong singularity for any subalgebra of M , using a Pimsner-Popa basis we show in subsection 6.3 that no proper singular finite index subfactor of M possesses this property. Thus, the

WAHP and singularity cannot be equivalent in general. In [28], the authors exhibited infinite index subfactors with the WAHP. We consider in subsection 6.4 the question of whether all infinite index singular subfactors have the WAHP. We end with a discussion of existence questions for singular and strongly singular subfactors.

The penultimate section 7 demonstrates the equivalence between singularity for a finite index subfactor N of M and a lower bound on the trace of projections subordinate to e_N^\perp in $N' \cap \langle M, e_N \rangle$ in Theorem 7.0.1. We use this result to show that fixed point subfactors of outer actions of finite groups on a II_1 factor never have $N' \cap \langle M, e_N \rangle$ two-dimensional. In particular, the only fixed point subfactors without nontrivial intermediate subfactors, those of \mathbb{Z}_p actions, are regular. When N is infinite index, we obtain a version of Theorem 7.0.1 by replacing normalizers with one-sided normalizers. We conclude with directions of future interest.

2. PRELIMINARIES

2.1. Background and Notation

We will assume familiarity with the basics of von Neumann algebra theory, as can be found in Chapter 5 of [14]. The reference for this section is [15] unless otherwise noted. SOT and WOT will stand for strong-operator topology and weak-operator topology, respectively, and "normal" means WOT continuous on bounded subsets. The term "separable" for a von Neumann algebra means that the algebra admits a representation on separable Hilbert space \mathcal{H} . Herein, we shall only concern ourselves with separable von Neumann algebras. $B(\mathcal{H})$ will denote the bounded linear operators over \mathcal{H} and all von Neumann algebras in $B(\mathcal{H})$ shall be assumed to be *unital*; that is, they contain the identity operator I on \mathcal{H} . Two projections p and q in a von Neumann algebra S are *Murray-von Neumann equivalent* (or merely equivalent) in S if there exists an operator $v \in S$ such that $vv^* = p$ and $v^*v = q$. We call v a partial isometry. The notation S' stands for all the operators in $B(\mathcal{H})$ that commute with S .

Every von Neumann algebra is a direct integral (a generalization of direct sum) of *factors*, von Neumann algebras whose multiplicative center consists of scalar multiples of the identity operator on $B(\mathcal{H})$. Factors occur in three general types. Type *I* factors are those with minimal projections and are isomorphic (as von Neumann algebras) to $B(\mathcal{H})$ for some \mathcal{H} . The equivalence class of a projection in $B(\mathcal{H})$ is completely determined by the dimension of its range space. In a type *III* factor, any two nonzero projections are equivalent. Type *III*'s are further classified into (sub)types III_λ for $0 \leq \lambda \leq 1$. The best-understood type *III*'s are the hyperfinites, the closure in the weak topology of an ascending union of matrix algebras. Their classification was accomplished by Alain Connes [5], Uffe Haagerup [10], and Wolfgang Krieger [17]

with contributions by many others. In short, there is up to isomorphism only one hyperfinite factor of type III_λ for $0 < \lambda \leq 1$, but uncountably many nonisomorphic hyperfinite type III_0 's.

A von Neumann algebra S is called *semifinite* if it has "no" type III 's in its direct integral decomposition. The quotations indicate "up to a set of measure zero" in a sense we shall be deliberately imprecise about. Any semifinite von Neumann algebra admits a normal, faithful, semifinite tracial weight ρ . A weight is a map initially defined on the positive cone S^+ of S mapping into $[0, \infty]$ that respects multiplication by nonnegative reals and is linear. The adjectives are defined as follows:

- *normal*: There exist a sequence of normal functionals ρ_i on S such that $\rho(x) = \sum_{i=1}^{\infty} \rho_i(x)$ for all $x \in S^+$;
- *faithful*: If $x \in S$, then $\rho(x^*x) = 0$ if and only if $x = 0$;
- *semifinite*: $\{x \in S^+ : \rho(x) < \infty\}$ is WOT dense in S^+ ;
- *tracial*: For all $x \in S$, $\rho(xx^*) = \rho(x^*x)$.

Note that the existence of such a weight on a type III factor is impossible. However, given such a ρ , its definition can be extended to a functional on the linear span of $\{x \in S^+ : \rho(x) < \infty\}$. For all elements x, y in this set, we have

$$\rho(xy) = \rho(yx). \quad (2.1)$$

When ρ is actually a functional on all of S , we shall take (2.1) as the definition of tracial.

Herein, we shall be concerned with factors of type II , those which have no minimal projections yet are not pathologically symmetrical like type III 's. They are divided into subtypes II_1 and II_∞ . Throughout, M will denote a separable II_1 factor.

All II_1 factors M possess a unique normal, faithful tracial state τ (or τ_M when there is ambiguity). A von Neumann algebra S is called *finite* if it admits a normal, faithful tracial state τ , though τ will not be unique unless S is a factor. At least in the separable case, every II_∞ factor is the tensor product of a type II_1 factor with $B(\mathcal{H})$ where \mathcal{H} is infinite dimensional ($M \otimes M_n(\mathbb{C})$ is again a II_1 factor).

For a von Neumann subalgebra B of M , we denote by $\mathcal{U}(B)$ the group of unitaries in N , $\mathcal{N}_M(B)$ the group of all unitaries in M such that $uBu^* = B$, and $\mathcal{P}(B)$ the set of all projections in B .

2.2. Group von Neumann Algebras, Crossed Products, and Examples of II_1 Factors

2.2.1. Group von Neumann Algebras

Let G be a countable discrete group. We define

$$\ell_2(G) = \{f : G \rightarrow \mathbb{C} : \sum_{g \in G} |f(g)|^2 < \infty\}.$$

The set of functions $\{\delta_g\}_{g \in G}$ such that $\delta_g(k) = 1$ if $k = g$ and zero otherwise is an orthonormal basis for $\ell_2(G)$. We can then construct a homomorphism $\lambda : G \rightarrow \mathcal{U}(B(\ell_2(G)))$ by

$$\lambda(g)(\delta_k) = \delta_{gk}$$

for all $g, k \in G$. Taking the double commutant of the set $\{\lambda(g)\}_{g \in G}$ gives a von Neumann algebra which we denote by $L(G)$. We obtain a normal, faithful, tracial state by the vector functional

$$\tau_{L(G)}(x) = \langle x\delta_{1_G}, \delta_{1_G} \rangle$$

where 1_G is the identity of G . If, in addition, the conjugacy class of every nonidentity element in G is infinite (ICC for Infinite Conjugacy Class), then $L(G)$ is a II_1 factor.

- Examples 2.2.1.** 1) If G is ICC and amenable (for example, $G = S_\infty$, the group of all permutations on a countable set which fix all but finitely many elements), then $L(G)$ is the hyperfinite II_1 factor due to Connes' proof that injectivity implies hyperfiniteness [6]. Uniqueness of the hyperfinite II_1 factor is due to Murray and von Neumann [19].
- 2) If $G = \mathbb{F}_n$, the free group on $n \geq 2$ generators, then $L(\mathbb{F}_n)$ is not hyperfinite, and deciding whether $L(\mathbb{F}_n)$ is isomorphic to $L(\mathbb{F}_m)$ for $m \neq n$ is one of the major open problems in the theory of von Neumann algebras.
- 3) If G has Kazhdan's property T and is ICC ($G = \text{SL}_n(\mathbb{Z})$ with $n > 2$, for example), then $L(G)$ is not isomorphic to any of the factors previously described.

We can also make a homomorphism $\rho : G \rightarrow \mathcal{U}(B(\ell_2(G)))$ by

$$\rho(g)(\delta_k) = \delta_{kg^{-1}}$$

for all $g, k \in G$. If we denote by $R(G)$ the double commutant of the set $\{\rho(g)\}_{g \in G}$, then $R(G)$ is a II_1 factor if G is ICC. Moreover, $L(G)' = R(G)$ and $R(G)$ is anti-isomorphic to $L(G)$ by extending the map $\lambda(g) \mapsto \rho(g^{-1})$.

2.2.2. Crossed Products

Let S be a von Neumann algebra represented on \mathcal{H} and let G be a countable discrete group acting by automorphisms $g \mapsto \alpha_g$ on S . We can then define a von Neumann algebra on the Hilbert space $\mathcal{H} \otimes_2 \ell_2(G)$ which we will denote by $S \rtimes_\alpha G$. We shall usually drop the α and write $S \rtimes G$. Explicitly, we identify $\mathcal{H} \otimes_2 \ell_2(G)$ with the Hilbert space of all square-summable functions f mapping from G to \mathcal{H} . Define maps $\pi_\alpha : S \rightarrow B(\mathcal{H} \otimes_2 \ell_2(G))$ and $u : G \rightarrow B(\mathcal{H} \otimes_2 \ell_2(G))$ by

$$(\pi_\alpha(x)f)(g) = \alpha_{g^{-1}}(x)f(g)$$

and

$$(u(h)f)(g) = f(h^{-1}g)$$

for all $f \in \mathcal{H} \otimes_2 \ell_2(G)$, $x \in S$, and $g, h \in G$. Then π_α is a normal $*$ -isomorphism and $u(g)$ is a unitary for all $g \in G$. We will usually denote $u(g)$ by u_g and drop the π_α in $\pi_\alpha(x)$. The crossed product of S by G , $S \rtimes G$, is $(\pi_\alpha(S) \cup u(G))''$. One can check that this is independent of the particular representation of S and that there is always a normal, faithful conditional expectation \mathbb{E}_S from $S \rtimes G$ to S . The u_g 's satisfy

$$u_g x u_g^* = \alpha_g(x)$$

for all $x \in S$, $g \in G$.

Elements of the crossed product $S \rtimes G$ may be treated as formal sums

$$x = \sum_{g \in G} x_g u_g$$

where the coefficients $x_g = \mathbb{E}_S(x u_g^*)$ uniquely determine x . The convergence, however, is not generally in strong operator topology if G is infinite [18]. If ψ is a normal linear functional on S , the function Ψ defined for $x \in S \rtimes G$ by

$$\Psi(x) = \psi(x_{1_G})$$

is a normal linear functional on $S \rtimes G$. If ψ is a state, then so is Ψ , and if ψ is tracial then Ψ is as well. We will use these observations to construct II_1 factors.

Examples 2.2.2. 1) Suppose $S = L^\infty(X, \mathcal{A}, \mu)$ for some probability space X . S has a normal, faithful, tracial state given by integration against μ . Let G be a countable discrete group acting on S by automorphisms $g \mapsto \theta_g$ such that the action is

- (a) *Free*: If $g \in G$, $g \neq 1_G$, then $\mu\{x \in X : \theta_g(x) = x\}$ is measure-zero;

(b) *Ergodic*: If $Y \in \mathcal{A}$ and $\mu(\theta_g(Y) \setminus Y) = 0$ for all $g \in G$, then either $\mu(Y) = 0$ or $\mu(X \setminus Y) = 0$;

(c) *Measure-preserving*: If $Y \subseteq X$, then $\theta_g(Y) \in \mathcal{A}$ if and only if $Y \in \mathcal{A}$. Moreover, if $Y \in \mathcal{A}$, $\mu(\theta_g(Y)) = 0$ if and only if $\mu(Y) = 0$.

Then the induced action α_g on $L^\infty(X, \mathcal{A}, \mu)$ is an automorphism of $L^\infty(X, \mathcal{A}, \mu)$ that is

(a)' *Free*: If there exists a $g \in G$ and $f \in L^\infty(X, \mathcal{A}, \mu)$ such that

$$\alpha_g(h)f = fh$$

for all $h \in L^\infty(X, \mathcal{A}, \mu)$, then $g = 1_G$ or $f = 0$;

(b)' *Ergodic*: If there exists an $f \in L^\infty(X, \mathcal{A}, \mu)$ such that $\alpha_g(f) = f$ for all $g \in G$, then f is a scalar multiple of the identity function on X .

These definitions generalize to actions of groups on arbitrary von Neumann algebras. Freeness ensures that no element in $S \rtimes G$ outside of S may commute with S (so that if $S = L^\infty(X, \mathcal{A}, \mu)$, S is maximal abelian in $S \rtimes G$), and the ergodicity condition shows that $S \rtimes G$ is a factor. The discussion preceding this example shows that $S \rtimes G$ is of type II_1 .

2) Suppose $S = M$ is a II_1 factor. Let G be a countable discrete group acting by outer automorphisms on M (except, of course, for the identity of G). Then the action is automatically free and ergodic, so $M \rtimes G$ is a II_1 factor.

2.3. The Standard Representation

We can use the tracial state τ to define an inner product on M by

$$\langle x, y \rangle = \tau(xy^*)$$

for all x, y in M . Completing M with respect to this inner product yields a Hilbert space which shall be denoted by $L^2(M, \tau)$ or usually, simply $L^2(M)$. We will denote $x \in M$ considered as an element of $L^2(M)$ by \hat{x} . Define a homomorphism $\pi_\tau : M \rightarrow B(L^2(M))$ first on the dense subspace $M\hat{I}$ by

$$\pi_\tau(x)\hat{y} = \hat{xy}.$$

Then π_τ is a normal $*$ -isomorphism onto its image and so $\pi_\tau(M) = \pi_\tau(M)''$. We will henceforth identify M with $\pi_\tau(M)$ and regard $M \subseteq B(L^2(M))$ unless otherwise noted. The vector \hat{I} is both cyclic and separating for M .

There is an isometric antilinear involution J defined first on $M\hat{I}$ by

$$J(x\hat{I}) = x^*\hat{I}. \tag{2.2}$$

and then by extension to $L^2(M)$. The von Neumann algebra JMJ is equal to M' , which parallels the situation when $M = L(G)$ for an ICC discrete group G . In fact, $\ell_2(G)$ is nothing more than $L^2(L(G))$ and J is the map $\delta_g \mapsto \delta_{g^{-1}}$. We call the representation of M on $L^2(M)$ the *standard representation*.

For any subalgebra B of M , the Hilbert space $L^2(B)$ is isometric to the subspace of $L^2(M)$ defined as the 2-norm closure of B in $L^2(M)$. We shall denote this space by $L^2(B)$ as well. We can then define the orthogonal projection of $L^2(M)$ onto $L^2(B)$, denoted by e_B . For all x in M ,

$$e_B x(\hat{I}) = e_B(x\hat{I}) = \mathbb{E}_B(x)\hat{I}. \tag{2.3}$$

where \mathbb{E}_B is the unique normal, faithful, trace-preserving conditional expectation onto B . \mathbb{E}_B can either be defined starting first with e_B and showing that $e_B(M\hat{I}) = B\hat{I}$ [12] or by using a Radon-Nikodym type theorem for factors [15]. If $x \in M$ satisfies

$xe_B = e_Bx$, then

$$x\hat{I} = xe_B\hat{I} = e_Bx\hat{I} = \mathbb{E}_B(x)\hat{I}$$

by equation (2.3). Since \hat{I} is separating for M , we have $x = \mathbb{E}_B(x)$. Then x must be in B and $\{e_B\}' \cap M \supseteq B$. As the conditional expectation is bimodular, $\{e_B\}' \cap M \subseteq B$, and so we have equality.

The von Neumann algebra generated by M and e_B , denoted by $\langle M, e_B \rangle$, is equal to $JB'J$ since J and e_B commute. The algebra $\langle M, e_B \rangle$ is then finite if and only if B' is. However, it is always semifinite and possesses a special tracial weight.

We denote by Tr (or $\text{Tr}_{\langle M, e_B \rangle}$ when ambiguous) the normal, faithful, semifinite tracial weight on $\langle M, e_B \rangle$ such that

$$\text{Tr}(xe_By) = \tau(xy) \tag{2.4}$$

for all $x, y \in M$. To prove rigorously that such a weight exists requires some work, which we shall shirk and instead refer the interested reader to the forthcoming book by Allan Sinclair and Roger Smith [29]. With this particular choice, $\text{Tr}(e_B) = 1$.

Consider now a finite sum in $\langle M, e_B \rangle$ of the form

$$x_0 + \sum_{i=1}^n x_i e_B y_i \tag{2.5}$$

where $x_i, y_i \in M$ for $0 \leq i \leq n$. The set of all such sums is a unital self-adjoint algebra in $\langle M, e_B \rangle$ containing M and e_B , hence it is weakly dense in $\langle M, e_B \rangle$. As we shall note later, more is true when B is a II_1 factor.

2.4. The Pull-Down Homomorphism

Suppose that there exists a partial isometry $v \in \langle M, e_B \rangle$ with $q = vv^* \leq e_B$ and $p = v^*v \in B' \cap \langle M, e_B \rangle$. Then we may define a homomorphism $\phi : B \rightarrow B$ by

$$e_B v x v^* e_B = \phi(x) e_B \quad (2.6)$$

for all $x \in B$. A priori, there is no reason why $\phi(I) = I$. Since $e_B \langle M, e_B \rangle e_B = B e_B$, the range of ϕ is contained in B and so ϕ is well-defined. The only properties of ϕ that are not straightforward are multiplicativity and WOT-continuity. If $x, y \in B$, then since $p \in B'$,

$$\begin{aligned} \phi(x)\phi(y)e_B &= \phi(x)e_B v y v^* e_B = e_B v x v^* e_N v y v^* e_N \\ &= e_B v x p y v^* e_B = e_B v p x y v^* e_B \\ &= e_B v (xy) v^* e_B \\ &= \phi(xy) e_B \end{aligned}$$

and so ϕ is multiplicative. Now suppose $x_i \rightarrow x$ WOT in B . Then for all $\xi, \zeta \in L^2(M)$,

$$\begin{aligned} \langle (\phi(x_i) - \phi(x)) e_B \xi, \zeta \rangle &= \langle e_B v x_i v^* e_B \xi, \zeta \rangle - \langle e_B v x v^* e_B \xi, \zeta \rangle \\ &= \langle (x_i - x) v^* e_B \eta, v^* e_B \zeta \rangle \rightarrow 0, \end{aligned}$$

so $\lim_{i \rightarrow \infty} \phi(x_i) e_B = \phi(x) e_B$ and ϕ is WOT continuous.

For all $x \in B$, we have that

$$\begin{aligned} \phi(y)v &= \phi(y) v v^* v = \phi(y) e_B v \\ &= e_B v y v^* e_B v = e_B v y p \\ &= e_N v p y = e_N v y \\ &= v y. \end{aligned} \quad (2.7)$$

This property will be particularly useful in later sections when B is a subfactor.

2.5. Averaging Over Subalgebras

Let B_0, B be two subalgebras of M and define C to be the weakly closed convex hull of the set $\{we_Bw^* : w \in B_0\}$ where $w \in \mathcal{U}(B_0)$. Then we have the following proposition, as detailed in [26] :

Proposition 2.5.1. *C admits a unique element of minimal 2-norm denoted by h such that*

- 1) $h \in (B_0)' \cap \langle M, e_B \rangle$;
- 2) $\text{Tr}(e_B h) = \text{Tr}(h^2)$;
- 3) $\text{Tr}(h) = 1$;
- 4) $1 - \text{Tr}(e_B h) \leq \|\mathbb{E}_B - \mathbb{E}_{B_0}\|_{\infty,2}^2$.

In 4),

$$\|T\|_{\infty,2} = \sup_{\substack{x \in M \\ \|x\| \leq 1}} \|Tx\|_2.$$

While existence of h follows from basic Hilbert space geometry, identifying h is difficult due to the unpredictability of $(B_0)' \cap \langle M, e_B \rangle$. In Theorem 5.0.3, we provide an example where h can be explicitly determined.

3. SINGULARITY AND STRONG SINGULARITY FOR MASAS

In this section, A will be a maximal abelian *-subalgebra (masa) in M . A is said to be

- i) *Regular* if $\mathcal{N}_M(A)'' = M$;
- ii) *Semi-Regular* if $\mathcal{N}_M(A)''$ is a II_1 factor;
- iii) *Singular* if $\mathcal{N}_M(A)'' = A$.

These definitions may be extended, somewhat naively, to subalgebras of M . Dixmier provided examples of all three types of masas in the hyperfinite II_1 factor R . While the crossed product of an $L^\infty(X, \mathcal{A}, \mu)$ space by the free, measure preserving, ergodic action of a countable discrete group G on X yields natural examples of regular masas in II_1 factors, determining whether a given masa is singular is in general quite difficult. It should be noted, however, that when $G = \mathbb{Z}$ and the action is strongly or weakly mixing, the von Neumann algebra generated by $u(G)$ in the crossed product is a singular masa [28].

This difficulty led Allan Sinclair and Roger Smith to define the notion of α -strong singularity in [30] as an analytical quantity which would imply the algebraic condition of singularity. Introduced for masas in a II_1 factor M , the definition was extended to subalgebras B in [28] by Sinclair, Smith, and Guyan Robertson. A von Neumann subalgebra B of M is α -strongly singular if there is a constant $0 < \alpha \leq 1$ such that for all unitaries u in M ,

$$\alpha \|u - \mathbb{E}_B(u)\|_2 \leq \|\mathbb{E}_B - \mathbb{E}_{uBu^*}\|_{\infty, 2}. \quad (3.1)$$

If $\alpha = 1$, then B is said to be strongly singular. It is easy that α -strong singularity for any α implies singularity, since if u normalizes B , the right hand side of the inequality

in equation (3.1) is zero. We must then have $\|u - \mathbb{E}_B(u)\|_2 = 0$, which implies that $u = \mathbb{E}_B(u)$ and so $u \in B$. Examples of strongly singular masas were provided in both [30] and [28] and by Robertson in [27].

In [30], the asymptotic homomorphism property for a masa A in M was defined as the existence of a unitary $u \in A$ such that

$$\lim_{|n| \rightarrow \infty} \|\mathbb{E}_A(xu^n y) - \mathbb{E}_A(x)u^n \mathbb{E}_A(y)\|_2 = 0 \quad (3.2)$$

for all x and y in M . The asymptotic homomorphism property was used to show that various singular masas coming from groups, such as the generator masas in $L(\mathbb{F}_n)$, are strongly singular. However, the full force of this property is not necessary to determine that singular masas are strongly singular. In [31], Sinclair, Smith, Stuart White and I showed that singularity and strong singularity are equivalent to a formally stronger property which first appeared in [28] and has come to be known as the weak asymptotic homomorphism property, or WAHP, in M .

A subalgebra B is said to have the WAHP in M if for every $\varepsilon > 0$ and for all $x_1, \dots, x_n, y_1, \dots, y_m$ in M , there exists a unitary u in B with

$$\|\mathbb{E}_B(x_i u y_j) - \mathbb{E}_B(x_i)u \mathbb{E}_B(y_j)\|_2 < \varepsilon \quad (3.3)$$

for every $1 \leq i \leq n, 1 \leq j \leq m$. Using the equivalence between the WAHP and singularity, it was shown in [31] that the tensor product of singular masas in II_1 factors is again a singular masa in the tensor product factor. This result has been extended by Ionut Chifan in [3] to show that for masas A_1 and A_2 in II_1 factors M_1 and M_2 , respectively, $\mathcal{N}_{M_1 \overline{\otimes} M_2}(A_1 \overline{\otimes} A_2)'' = \mathcal{N}_{M_1}(A_1)'' \overline{\otimes} \mathcal{N}_{M_2}(A_2)''$.

The key ingredient in the proof that singular masas have the WAHP is Popa's Intertwining Theorem [21]. Since this theorem applies to subalgebras of M , one might hope that it may be employed to show that all singular subalgebras of M have the

WAHP in M . We will show in subsection 6.3 that this is not the case for proper finite index singular subfactors.

4. HILBERT MODULES AND THE JONES INDEX

4.1. Definitions and Properties

All theorems in this section are due to Vaughan Jones unless otherwise stated. For the remainder of this work, N will denote a *subfactor* of M ; that is, N is a subalgebra of M which is a II_1 factor. A *left* (resp, right) *Hilbert M -module* is merely a Hilbert space \mathcal{H} complete with a $*$ -isomorphism of M (resp., M^{op}) into $B(\mathcal{H})$. A Hilbert $N - M$ -bimodule (or correspondence, using Alain Connes' terminology) is both a left Hilbert N -module and a right Hilbert M -module.

Examples 4.1.1. • $L^2(M)$ is both an $M - M$ and $N - N$ Hilbert bimodule. Hence it is also an $N - M$ and an $M - N$ bimodule.

- $L^2(N)$ is an $N - N$ bimodule, but is only a left or right M -module if $M = N$.
- If p is any projection in M , then the closure of $M\hat{p}$ in $L^2(M)$ is a left Hilbert M -module, abusively denoted by $L^2(M)p$.
- The ℓ_2 -direct sum of Hilbert modules is again a Hilbert module.

Two left Hilbert M -modules \mathcal{H}_1 and \mathcal{H}_2 (with representations π_1 and π_2 of M) are said to be isomorphic if there is a unitary $u : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$(u\pi_1(x))h = (\pi_2(x)u)h$$

for all x in M and all h in \mathcal{H}_1 . The obvious generalization holds for right Hilbert M -modules.

If \mathcal{H} is any left Hilbert M -module with a cyclic vector, then \mathcal{H} is module-isomorphic to $L^2(M)p$ for some $p \in \mathcal{P}(M)$. Using this result, we can show that if \mathcal{H} is any (separable) left Hilbert N -module, then \mathcal{H} is module-isomorphic to a

direct sum

$$\mathcal{H} \cong \bigoplus_{i=1}^{\infty} L^2(M)p_i.$$

The module isomorphism class is completely determined by $\sum_{i=1}^{\infty} \tau(p_i)$. We call this extended real number the (left-)module dimension of \mathcal{H} over M , and denote it by $\dim_M(\mathcal{H})$.

We now develop a picture, taken from notes by Sorin Popa, that allows one to easily compute many properties of the Jones Index.

Example 4.1.2. Let $0 < \alpha < \infty$ and let \mathcal{H} be a Hilbert N -module with dimension α . Let k be the smallest integer greater than or equal to α and consider the von Neumann algebra

$$M_k(N) \cong N \otimes M_k(\mathbb{C}).$$

$M_k(N)$ is a II_1 factor with tracial state given by $\tau_k((x_{i,j})_{i,j=1}^k) = \frac{1}{k} \sum_{i=1}^k \tau(x_{i,i})$. Let p_α in N be a projection with $\tau(p_\alpha) = k - \alpha$. Let q_α be the projection in $M_k(N)$ defined by

$$q_\alpha = \begin{pmatrix} I & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & I & 0 \\ 0 & 0 & \dots & 0 & p_\alpha \end{pmatrix}$$

if α is not an integer and the identity of $M_k(N)$ otherwise.

We will define N_α to be the compression of $M_k(N)$ by q_α . Then N_α is a II_1 factor, so we may represent N_α on $L^2(N_\alpha)$. Let e be the projection in N_α with I in the $(1, 1)$ position and zeroes elsewhere and consider the compression of N_α by e on the space

$eL^2(N_\alpha)$. This compression is isomorphic to N , and if we look at $eN_\alpha\hat{q}_\alpha$, we obtain

$$\begin{aligned} & \begin{pmatrix} I & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & \dots & x_{1,k-1} & x_{1k}p_\alpha \\ \vdots & \ddots & \ddots & \vdots \\ x_{k-1,1} & \dots & x_{k-1,k-1} & x_{k-1,k}p_\alpha \\ p_\alpha x_{k,1} & \dots & p_\alpha x_{k,k-1} & p_\alpha x_{k,k}p_\alpha \end{pmatrix} \hat{q}_\alpha \\ &= \begin{pmatrix} x_{11} & \dots & x_{1,k-1} & x_{1,k}p_\alpha \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \hat{q}_\alpha \end{aligned}$$

The left Hilbert N -module $eL^2(N_\alpha)$ is isomorphic to $\mathcal{H} = \bigoplus_{i=1}^{k-1} L^2(N) \oplus L^2(N)p_\alpha$ by the map $u : eL^2(N_\alpha) \rightarrow \mathcal{H}$ defined on $eN_\alpha\hat{q}_\alpha$ by

$$u \left(\begin{pmatrix} x_{11} & \dots & x_{1,k-1} & x_{1,k}p_\alpha \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right) = \frac{1}{\sqrt{\tau_k(q_\alpha)}} (x_{11}\hat{I}, x_{12}\hat{I}, \dots, x_{1,k-1}\hat{I}, x_{1k}p_\alpha\hat{I})$$

and extended to $eL^2(N_\alpha)$. The compression $eN_\alpha e$ acting on $eL^2(N_\alpha)$ is then equal to uNu^* . Since N_α was represented in standard form on $L^2(N_\alpha)$, if J_α is the associated isometric involution, then

$$N'_\alpha = J_\alpha N_\alpha J_\alpha.$$

Therefore N' is isomorphic to $J_\alpha N_\alpha J_\alpha$ by u and so N' is a II_1 factor.

One of the many ways in which Example 4.1.2 is useful is that it allows one to deal easily with projections.

Theorem 4.1.3. *Let \mathcal{H} be a left Hilbert N -module. Then*

i) If $p \in \mathcal{P}(N')$, $p\mathcal{H}$ is a left Hilbert N -module. If $\dim_N(\mathcal{H}) < \infty$, then $\dim_N(p\mathcal{H}) = \tau_{N'}(p) \dim_N(\mathcal{H})$;

ii) If $p \in \mathcal{P}(N)$, $p\mathcal{H}$ is a left Hilbert pNp module and $\dim_{pNp}(p\mathcal{H}) = \frac{1}{\tau_N(p)} \dim_N(\mathcal{H})$.

As $L^2(M)$ is a left Hilbert N -module, it is module isomorphic to a sum $\bigoplus_{i=1}^{\infty} L^2(M)p_i$.

We now define the index.

Definition 4.1.4. (*Jones index*) The Jones Index of N in M , denoted by $[M : N]$, is defined as $\sum_{i=1}^{\infty} \tau(p_i)$ where $L^2(M) \cong \bigoplus_{i=1}^{\infty} L^2(N)p_i$.

Though it is clear that the index can never be less than one, the possible values greater than one are less transparent. A by-now classical yet still remarkable result of Jones is that the index cannot take arbitrary values in $[1, \infty]$.

Theorem 4.1.5. (Jones) $[M : N]$ takes values in the set $\{4 \cos^2(\frac{\pi}{n}) | n \geq 3\} \cup [4, \infty]$. Moreover, every such index value is attained by a subfactor inside the hyperfinite II_1 factor.

In the discrete $\{4 \cos^2(\frac{\pi}{n})\}$ range, all subfactors N have $N' \cap M = \mathbb{C}I$ or are *irreducible*. It is one of the major open problems of subfactor theory to decide whether one can construct an irreducible subfactor of the hyperfinite II_1 factor for each index value in the continuous $[4, \infty]$ range.

If $\alpha = [M : N] < \infty$, example 4.1.2 shows that $L^2(M)$ is isomorphic to $eL^2(N_\alpha)$ as left Hilbert N -modules. On $L^2(M)$, we have already determined that $N' = J\langle M, e_N \rangle J$. On $eL^2(N_\alpha)$, the commutant of $eN_\alpha e \cong N$ is eN'_α , which is isomorphic to N' . Regarding e_N on \mathcal{H} as the projection onto the first coordinate, we have that $u^*e_N u = e$ where u is the map in example 4.1.2. The normalized trace of this projection in eN_α is exactly $\frac{1}{\tau_\alpha(q_\alpha)} = \frac{1}{[M:N]}$ and therefore the normalized trace of e_N in $\langle M, e_N \rangle$ is $\frac{1}{[M:N]}$. Hence, if $[M : N] < \infty$ then $\langle M, e_N \rangle$ is a II_1 factor.

Conversely, if $\langle M, e_N \rangle$ is a II_1 factor, then using the normalized trace on $\langle M, e_N \rangle$ and the fact that $JN'J = \langle M, e_N \rangle$, we may construct an isomorphism between $\langle M, e_N \rangle$ and N_α for some $1 \leq \alpha < \infty$. This isomorphism induces a module isomorphism at the level of Hilbert spaces, and therefore we find that $[M : N] < \infty$. We record this fact and other properties of the index.

Theorem 4.1.6. *Let $N \subseteq M$ be an inclusion of subfactors. Then*

- i) $[M : N] < \infty$ if and only if N' is finite;*
- ii) $[M : N] = 1$ if and only if $M = N$;*
- iii) If $P \subseteq N$ is a II_1 factor, then $[M : P] = [M : N] \cdot [N : P]$;*
- iv) If N' is finite, then $[\langle M, e_N \rangle : M] = [M : N]$;*
- v) If N' is finite, $[M : N] = [N' : M']$;*
- vi) If $p \in \mathcal{P}(N' \cap M)$, then $[pMp : pN] = \tau_M(p)\tau_{N'}(p)[M : N]$.*

We can use *vi)* of Theorem 4.1.6 to prove

Theorem 4.1.7. *If $[M : N] < \infty$, then any family of orthogonal projections in $N' \cap M$ has cardinality at most $\sqrt{[M : N]}$. In particular,*

$$\dim(N' \cap M) \leq [M : N].$$

If $[M : N] < \infty$, then by part *iv)* of Theorem 4.1.6, we may repeat the Jones construction on $L^2(\langle M, e_N \rangle)$ and produce a projection e_M from $L^2(\langle M, e_N \rangle)$ to $L^2(M)$ implementing a normal, faithful, trace-preserving conditional expectation \mathbb{E}_M from $\langle M, e_N \rangle$ to M . We can then obtain the following corollary:

Corollary 4.1.8. $[M : N] \notin (1, 2)$

Proof. If $M \neq N$, e_N and e_N^\perp are in $\langle M, e_N \rangle$. By *iii*) of Theorem 4.1.7 and Theorem 4.1.6,

$$4 < [\langle M, e_N \rangle : N] = [\langle M, e_N \rangle : M] \cdot [M : N] = [M : N]^2. \quad \square$$

Now for $x_i, y_i \in M$, $0 \leq i \leq n$.

$$\left(x_0 + \sum_{i=1}^n x_i e_N y_i \right) e_N = \left(x_0 + \sum_{i=1}^n x_i \mathbb{E}_N(y_i) \right) e_N \in M e_N. \quad (4.1)$$

Under the assumption that N has finite index in M , one can calculate that $\mathbb{E}_M(e_N) = \frac{1}{[M:N]}$. Combining this with equation (4.1), we can establish

$$x e_N = [M : N] \mathbb{E}_M(x e_N) e_N \quad (4.2)$$

for all $x \in \langle M, e_N \rangle$. We have then shown that for every $x \in \langle M, e_N \rangle$ for $N \subseteq M$ a finite-index inclusion, there exists an element $x_0 \in M$ with the property that $x e_N = x_0 e_N$. This element is necessarily unique. Equation (4.2) shows that all sums of the form $\sum_{i=1}^n x_i e_N y_i$ comprise a 2-sided ideal in $\langle M, e_N \rangle$ for a finite index inclusion and so is equal to $\langle M, e_N \rangle$.

4.2. Galois Theory for Subfactors

One of the satisfying features of the Jones Index is how it relates to the index for groups. Let G be a countable discrete group with an outer action on the II_1 factor P and let H be any subgroup of G . Then from subsection 2.2, $P \rtimes G$ and $P \rtimes H$ are II_1 factors. We have

Theorem 4.2.1. $[P \rtimes G : P \rtimes H] = [G : H]$.

We can further develop the analogy between subgroups and subfactors to obtain a form of Galois Theory for subfactors. The first result characterizes the intermediate

subfactors between P and $P \rtimes G$ and is due, in its ultimate form, to Hisashi Choda [4].

Theorem 4.2.2. *There exists a one-to-one correspondence between the class of all subgroups K of G and the class of all intermediate subalgebras B containing P of $P \rtimes G$.*

Since every algebra of the form $P \rtimes K$ is again a factor, every $B \subseteq M$ containing P must be a factor. In particular, every subfactor containing $P \rtimes H$ must be of the form $P \rtimes K$ for $H \leq K \leq G$. The second result establishes the fixed point version of Theorem 4.2.2 in the case of finite groups and is also due to Choda [4]. For a subgroup K of G , define

$$P^K = \{x \in P : \alpha_k(x) = x \ \forall k \in K\}.$$

Theorem 4.2.3. *If $|G| < \infty$ then there exists a Galois correspondence between the class of all subgroups of G and the class of all subalgebras B of P containing P^G , i.e., all subalgebras of P containing P^G are of the form P^K for $K \leq G$.*

Note that by ergodicity of the action, $P^{\{1_G\}} = P$.

4.3. Hilbert Bimodules and $N' \cap \langle M, e_N \rangle$

We discuss properties of Hilbert bimodules further. Much of this material can be found in [12].

Using *iii)* and *iv)* from Theorem 4.1.6 and Theorem 4.1.7, the dimension of $N' \cap \langle M, e_N \rangle$ is less than $[M : N]^2$. Regardless of the type decomposition of the von Neumann algebra $N' \cap \langle M, e_N \rangle$, a simple calculation shows that e_N is minimal in

$N' \cap \langle M, e_N \rangle$. We have

$$\begin{aligned}
 e_N(N' \cap \langle M, e_N \rangle)e_N &= e_N N' e_N \cap e_N \langle M, e_N \rangle e_N \\
 &= e_N N e_N \cap N e_N \\
 &= e_N(N' \cap N)e_N \\
 &= e_N \mathbb{C}.
 \end{aligned}$$

Proposition 4.3.1. *The equivalence classes of minimal central projections in $N' \cap \langle M, e_N \rangle$ are in one-to-one correspondence with isomorphism classes of irreducible sub-bimodules of $L^2(M)$.*

Proof. (\Rightarrow) Suppose p is any projection in $N' \cap \langle M, e_N \rangle$. Then consider the subspace $pL^2(M)$. For all $x \in N$, $y \in M$,

$$x(py\hat{I}) = p(xy\hat{I}) \in pL^2(M)$$

so $pL^2(M)$ is a left N -module, and since $JN'J = \langle M, e_N \rangle$,

$$(JxJ)py\hat{I} = p(JxJy\hat{I}) = p(yx^*\hat{I}) \in pL^2(M)$$

and so $pL^2(M)$ is a right N -module. Given a projection $q \in \mathcal{Z}(N' \cap \langle M, e_N \rangle)$, let p be a minimal projection under q . Then $pL^2(M)$ is an irreducible $N - N$ bimodule, and the equivalence class of p determines an equivalence class for $pL^2(M)$.

(\Leftarrow) Suppose $K \subseteq L^2(M)$ is an irreducible $N - N$ bimodule. Let $p : L^2(M) \rightarrow K$ be the orthogonal projection onto K . Then since N is unital, $NK = K$ and $JNJK = K$, p is in both N' and $(JN'J)' = \langle M, e_N \rangle$, and so $p \in N' \cap \langle M, e_N \rangle$. Since K is irreducible, p must be minimal by (\Rightarrow) and the equivalence class of K determines the equivalence class of p . \square

Proposition 4.3.1 is part of a much larger picture for a finite index inclusion.

By Theorem 4.1.7, $N' \cap \langle \langle M, e_N \rangle, e_M \rangle$ is finite dimensional, as is $M' \cap \langle \langle M, e_N \rangle, e_M \rangle$. Iterating the Jones construction, we obtain II_1 factors M_i with $M_1 := \langle M, e_N \rangle$, $M_2 := \langle \langle M, e_N \rangle, e_M \rangle, \dots$ and $M_0 := M$. The (necessarily finite dimensional) algebras of the form $N' \cap M$, $N' \cap M_2, \dots$ captures the information of the $N - M$ bimodules in the Hilbert space $L^2(M_k)$. Similarly, the algebras $N' \cap M_{2k+1}$, $M' \cap M_{2k+1}$, and $M' \cap M_{2k}$ represent, respectively, the $N - N$ bimodules in $L^2(M_k)$, the $M - N$ bimodules in $L^2(M_k)$ and the $M - M$ bimodules in $L^2(M_k)$. We obtain a double sequence of inclusions of finite dimensional algebras

$$\begin{array}{ccccccc} N' \cap M & \subseteq & N' \cap \langle M, e_N \rangle & \subseteq & N' \cap \langle \langle M, e_N \rangle, e_M \rangle & \subseteq & \dots \\ \cup & & \cup & & \cup & & \\ \mathbb{C}I & \subseteq & M' \cap \langle M, e_N \rangle & \subseteq & M' \cap \langle \langle M, e_N \rangle, e_M \rangle & \subseteq & \dots \end{array}$$

called the standard invariant, which captures essentially all the bimodule information. Popa showed in [25] that when $[M : N] < 4$ and M is hyperfinite, the weak closure of the towers are isomorphic to the original inclusion, so that the subfactors may be completely reconstructed from finite dimensional information. He also proved that this holds in general for a larger class of *amenable* subfactors, but in [2] it was shown that there exist infinitely many nonisomorphic index 6 subfactors of the hyperfinite II_1 factor with the same standard invariant. In this work, we shall only have occasion to use $N' \cap \langle M, e_N \rangle$.

We may characterize intermediate subfactors between N and M using projections from $N' \cap \langle M, e_N \rangle$ in the case where N has trivial relative commutant. If $N \subseteq N_1 \subseteq M$, then $\{e_{N_1}\}' \cap M = N_1 \supseteq N$. Since e_{N_1} commutes with J , we have $e_{N_1} \in N' \cap \langle M, e_N \rangle$. A more in depth study of when projections in $N' \cap \langle M, e_N \rangle$ correspond to intermediate subfactors can be found in [1].

Proposition 4.3.2. *If N is irreducible and $p \in \mathcal{P}(N' \cap \langle M, e_N \rangle)$, then $\{p\}' \cap M$ is*

an intermediate subfactor N_1 .

Proof. Let $p \in \mathcal{P}(N' \cap \langle M, e_N \rangle)$. Then from the proof of Proposition 4.3.1, $pL^2(M)$ is an $N - N$ bimodule. Define

$$N_1 = \{x \in M : xk, x^*k \in pL^2(M) \ \forall k \in pL^2(M)\}.$$

Then N_1 is a von Neumann algebra containing N and is a II_1 factor since N is irreducible.

We claim that $N_1 = \{p\}' \cap M$. Take $x \in \{p\}' \cap M$. Then for all $y \in M$,

$$x(py\hat{I}) = p(xy\hat{I}) \in pL^2(M).$$

Since p is self-adjoint, $x^* \in \{p\}' \cap M$ and the same calculation then applies to x^* . Hence $x \in N_1$, so $\{p\}' \cap M \subseteq N_1$. Now since $I \in N_1$, $N_1K = K$ and therefore $p \in N_1'$. This implies that $N_1 \subseteq \{p\}'$, and since $N_1 \subseteq M$, we get that $N_1 \subseteq \{p\}' \cap M$. \square

Owing to proposition 4.3.1, we may define two $N - N$ sub-bimodules of $L^2(M)$ to be equivalent if there exists a partial isometry in $N' \cap \langle M, e_N \rangle$ between their range projections. The following proposition is well-known to subfactor theorists and can be extracted from [16]. We include a proof for the sake of completeness.

Proposition 4.3.3. *If $N = P \rtimes H$ and $M = P \rtimes G$, then an $N - N$ sub-bimodule of $L^2(M)$ is irreducible if and only if it has the form*

$$\overline{\left\{ \sum_k x_k u_k \hat{I} : x_k \in P, k \in HgH \right\}}^{\|\cdot\|_2}$$

for some $g \in G$.

Proof. First, recall from the discussion at the beginning of this subsection that for any subfactor N of a II_1 factor M , e_N is a minimal projection in $N' \cap \langle M, e_N \rangle$. Also,

if x is any element in $N' \cap \langle M, e_N \rangle$, then uxu^* is in $N' \cap \langle M, e_N \rangle$ for all unitaries u in $\mathcal{N}_M(N)$. This last observation applies in particular to $N = P$, $M = P \rtimes G$, and the unitaries u_g for $g \in G$.

We first demonstrate the proof in the case where $H = 1_G$, the identity of G . The double cosets are then the elements of G . The general result follows from this case.

Case 1: H is the identity element 1_G of G , i.e. $N = P$

We claim that all inequivalent irreducible $N - N$ sub-bimodules of $L^2(P \rtimes G)$ are given by

$$L^2(N)u_g := \overline{\{xu_g\hat{I} : x \in N, g \in G\}}^{\|\cdot\|_2}.$$

Define projections P_g in $N' \cap \langle M, e_N \rangle$ by

$$P_g = u_g e_N u_g^*$$

for all $g \in G$. Then the range of P_g is $L^2(N)u_g$ and since $\mathbb{E}_N(u_g) = 0$, $P_g P_h = 0$ unless $h = g$. Now suppose K is a nontrivial $N - N$ submodule of $L^2(N)u_g$. Then the projection P_K from $L^2(M)$ onto K is in $N' \cap \langle M, e_N \rangle$ by the proof of Proposition 4.3.1. Therefore the projection $u_g^* P_K u_g \in N' \cap \langle M, e_N \rangle$ and is subordinate to e_N . By minimality of e_N , $u_g^* P_K u_g = e_N$, and so $K = L^2(N)u_g$. This shows that $L^2(N)u_g$ is an irreducible $N - N$ bimodule.

To see that $L^2(N)u_g$ and $L^2(N)u_h$ are inequivalent as $N - N$ bimodules if $h \neq g$, suppose v is a partial isometry in $N' \cap \langle M, e_N \rangle$ with $v^*v = P_g$, $vv^* = P_h$. If we define a partial isometry in $N' \cap \langle M, e_N \rangle$ by

$$w = u_g^* v u_g$$

then $w^*w = e_N$ and $w^*w = P_{g^{-1}h}$. Hence we may assume that $g = 1_G$ and $P_g = e_N$.

Then

$$v = vv^*vv^*v = P_hve_N = u_h e_N u_h^* v e_N.$$

Since v is in $\langle M, e_N \rangle$,

$$e_N u_h^* v e_N \in e_N \langle M, e_N \rangle e_N = N e_N$$

and so $v = u_h y e_N$ for some y in N . Then

$$v\hat{I} = u_h y \hat{I} = \alpha_h(y) u_h \hat{I} \in N u_h \hat{I}.$$

Now since $v \in N' \cap \langle M, e_N \rangle = N' \cap JN'J$, for all $z \in N$ we obtain

$$\begin{aligned} z\alpha_h(y)u_h\hat{I} &= zv\hat{I} = vz\hat{I} \\ &= vJz^*J\hat{I} = Jz^*Jv\hat{I} \\ &= Jz^*J\alpha_h(y)u_h\hat{I} = \alpha_h(y)u_hz\hat{I} \\ &= \alpha_h(y)\alpha_h(z)u_h\hat{I}. \end{aligned}$$

Since the vector \hat{I} is separating for M , we get that

$$z\alpha_h(y)u_h = \alpha_h(y)\alpha_h(z)u_h.$$

As u_h is a unitary, we obtain that $z\alpha_h(y) = \alpha_h(y)\alpha_h(z)$ for all z in N . The outer action of G implies we must have either $\alpha_h(y) = 0$ or $h = 1_G$ and $\alpha_h(y)$ is a nonzero scalar multiple of the identity. But if $\alpha_h(y) = 0$, then $y = 0$ and for all z in N ,

$$vz\hat{I} = zv\hat{I} = zu_hy\hat{I} = 0$$

which would imply v is the zero map. As $v^*v = e_N$, we must have that $h = 1_G$. Then $\alpha_h(y)$ and hence y is a nonzero scalar multiple of the identity. This shows that $v\hat{I}$ is in $L^2(N)$ and so the irreducible $N - N$ sub-bimodules $\{L^2(N)u_g\}_{g \in G}$ of $L^2(M)$ are

mutually inequivalent and form a complete decomposition

$$L^2(M) = \bigoplus_{g \in G} L^2(N)u_g$$

as an ℓ_2 direct sum.

Case 2: $H \neq 1_G$.

Let K be an $N - N$ bimodule in $L^2(M)$. As P is contained in $P \rtimes H = N$, K is then a $P - P$ bimodule, and so there is a set $F \subseteq G$ with

$$K = \bigoplus_{g \in F} L^2(N)u_g$$

as an ℓ_2 direct sum. Therefore K contains an element of the form $u_g \hat{I}$ for some g in G . Since K is an $N - N$ bimodule, it will then contain all elements of the form

$$\sum_{k \in HgH} x_k u_k \hat{I}$$

with x_k in P and the sum converging in 2-norm. For a fixed g , this is an $N - N$ bimodule, and so it follows that if K is irreducible, this is all of K .

Let us then set

$$K_g := \overline{\left\{ \sum_{k \in HgH} x_k u_k \hat{I} \mid x_k \in P \right\}}^{\|\cdot\|_2}$$

for a given $g \in G$ of H . We have shown that each K_g is an irreducible $N - N$ bimodule, and we now demonstrate that K_g and $K_{g'}$ are inequivalent $N - N$ bimodules if $HgH \neq Hg'H$.

Since $P \subseteq N$, $N' \cap \langle M, e_N \rangle$ is contained in $P' \cap \langle M, e_p \rangle$ and so is commutative. Hence, there can be no partial isometry in $N' \cap \langle M, e_N \rangle$ between the projections onto K_g and $K_{g'}$ if $HgH \neq Hg'H$. Therefore, the K_g 's are a complete listing of the inequivalent, irreducible $N - N$ bimodules. \square

4.4. Pimsner-Popa Bases

One advantage of working with crossed product factors is that when $M = P \rtimes G$, $N = P \rtimes H$, and $[G : H] < \infty$, a left module basis for M over N is given by a complete set of left coset representatives for G over H . If $[G : H] = \infty$, it is still true that a set of left coset representatives can be used as an infinite basis, but the convergence is not in WOT [18]. The convergence can be regarded as taking place in $L^2(M)$, however.

The analog for general finite index inclusions was developed in [20] and extended in [12]. A *Pimsner-Popa basis* for a finite index inclusion of subfactors $N \subseteq M$ is a collection of elements $\lambda_1, \dots, \lambda_k$ in M with k any integer greater than or equal to $[M : N]$ such that

$$\text{i) } \sum_{j=1}^k \lambda_j e_N \lambda_j^* = 1$$

$$\text{ii) } \sum_{j=1}^k \lambda_j e_N (\lambda_j^* \xi) = \xi$$

for every $x \in M$, $\xi \in L^2(M)$.

An important aspect of these bases is that if $\lambda_1, \dots, \lambda_k$ is a basis for M over N and $\gamma_1, \dots, \gamma_n$ is a basis for N over P , then $\{\gamma_j \lambda_i : 1 \leq i \leq k, 1 \leq j \leq n\}$ is a basis for M over P .

As the crossed product example suggests, the situation for Pimsner-Popa bases is more complicated when $[M : N] = \infty$. Although the crossed product allows us to still choose a basis from M , in the general infinite index setting this may not be possible. As Popa observes in [25], we can always find a sequence of unbounded operators affiliated to M satisfying the defining conditions of a basis (the actual statement in [25] is much more general) with convergence in 2-norm.

We can think of the basis as "coming from $L^2(M)$ " in the following sense: For a

vector $\xi \in L^2(M)$, define a linear operator η_ξ on $M\hat{I}$ by

$$\eta_\xi(x\hat{I}) = Jx^*J\xi$$

for all $x \in M$. The map η_ξ is bounded if and only if $\xi = y\hat{I}$ for some $y \in M$ [12].

Now suppose T is an unbounded operator on $L^2(M)$ affiliated to M and set $\xi = T\hat{I}$.

Then since $JMJ = M'$,

$$\eta_\xi x\hat{I} = Jx^*J\xi = Jx^*JT\hat{I} = TJx^*J\hat{I} = Tx\hat{I}.$$

Therefore we may identify an unbounded Pimsner-Popa basis $\{\lambda_i\}_{i=1}^\infty$ with the collection of vectors $\{\lambda_i\hat{I}\}$ in $L^2(M)$.

5. STRONG SINGULARITY ESTIMATES FOR SINGULAR SUBFACTORS

We begin by producing an absolute constant α for which all singular subfactors of M are strongly singular. We employ the following result, which is Theorem 5.4 in [26].

Theorem 5.0.1. (Popa, Smith, & Sinclair) *Suppose $\delta > 0$ and N, N_0 are two subfactors of M with $\|\mathbb{E}_N - \mathbb{E}_{N_0}\|_{\infty,2} \leq \delta$. Then there exist projections $q_0 \in N_0$, $q \in N$, $q'_0 \in N'_0 \cap M$, $q' \in N' \cap M$, $p_0 = q_0 q'_0$, $p = q q'$, and a partial isometry v in M such that $vp_0 N_0 p_0 v^* = p N p$, $vv^* = p$, $v^*v = p_0$, and*

$$\|1 - v\|_2 \leq 13\delta, \quad \tau(p) = \tau(p_0) \geq 1 - 67\delta^2. \quad (5.1)$$

Theorem 5.0.1 is a consequence of a more general result for subalgebras, also in [26].

Theorem 5.0.2. *Let N be a singular subfactor in M . Then N is $\frac{1}{25}$ strongly singular in M .*

Proof. Suppose N is singular in M . Let p, p_0, q, q_0 and v be as in Theorem 5.0.1 with $N_0 = u N u^*$ for some unitary u in M and some δ to be specified later. Then

$$N' \cap M = N'_0 \cap M = \mathbb{C}I$$

so q and q_0 are scalar multiples of I . Therefore, $p \in N$ and $p_0 \in N_0$.

Consider the partial isometry $v_1 = vu$. Then

$$v_1 N v_1^* = vu N u^* v^* = p N p \subseteq N \quad (5.2)$$

and

$$v_1^* N v_1 = u^* v^* N v u = u^* p_0 N_0 p_0 u \subseteq u^* u N u^* u = N. \quad (5.3)$$

Also, $v_1^* v_1 = p$ and $v_1 v_1^* = u^* p_0 u$. Our goal is to produce $u_1 \in \mathcal{U}(N)$ and $f \in \mathcal{P}(N)$

so that $\|v_1 - u_1 f\|_2$ is controlled by $\|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}$. We will then employ equation (5.1) to obtain the strong singularity estimate.

Suppose initially that $\tau(p) = \frac{n-1}{n} \geq \frac{1}{2}$ for some positive integer n . Let $e_{11}, e_{22}, \dots, e_{nn} = p^\perp$ be mutually orthogonal projections in N with $\tau(e_{ii}) = \frac{1}{n}$ for all i and $\sum_{i=1}^n e_{ii} = I$. Let e_{1j} be partial isometries in N with $e_{1j}^* e_{1j} = e_{jj}$ and $e_{1j} e_{1j}^* = e_{11}$ and set $e_{j1} = e_{1j}^*$. Extend to a system of matrix units in N by

$$e_{ij} = e_{i1} e_{1j}.$$

Let u_{ij} be a system of matrix units in N constructed in the same manner using the projections $u_{ii} = v_1 e_{ii} v_1^*$ for $1 \leq i \leq n-1$ and $u_{nn} = (u^* p_0 u)^\perp$. We define w_1 in M with $w_1 N w_1^* = N$ by

$$w_1 = v_1 + u_{n1} v_1 e_{1n}.$$

As

$$\begin{aligned} w_1^* w_1 &= (v_1^* + e_{n1} v_1^* u_{1n})(v_1 + u_{n1} v_1 e_{1n}) \\ &= p + e_{n1} v_1^* u_{11} v_1 e_{1n} + e_{n1} v_1^* u_{1n} v_1 + v_1^* u_{n1} v_1 e_{1n} \\ &= p + e_{n1} e_{11} e_{1n} + e_{n1} v_1^* u_{1n} u_{nn} (u^* p_0 u) v_1 + v_1^* (u^* p_0 u) u_{nn} u_{n1} v_1 e_{1n} \\ &= p + p^\perp \\ &= I, \end{aligned}$$

w_1 is a unitary since M is a II_1 factor. We have assumed that N is singular, so w_1 must then be in N . As p is in N , the partial isometry $w_1 p$ is in N . But

$$\begin{aligned} w_1 p &= v_1 p + u_{n1} v_1 e_{1n} p = v_1 + u_{n1} v_1 e_{1n} e_{nn} p \\ &= v_1 + u_{n1} v_1 e_{1n} p^\perp p \\ &= v_1 \end{aligned}$$

and so v_1 is in N .

Now suppose that $\tau(p) \geq 1 - \varepsilon > \frac{1}{2}$. Then there exists an integer $n \geq 2$ with

$$\frac{n-1}{n} \leq 1 - \varepsilon < \frac{n}{n+1}.$$

Choose $f \in N$, $f \leq p$ with $\tau(f) = \frac{n-1}{n}$ and let $v_2 = v_1 f$. Then v_2 satisfies equations (5.2) and (5.3), so there exists a unitary w_2 in N with $v_2 = w_2 f$. Therefore,

$$\|v_1 - w_2 f\|_2^2 = \|v_1 - v_2\|_2^2 = \|v_1 - v_1 f\|_2^2 \leq \|1 - f\|_2^2 = \frac{1}{n}. \quad (5.4)$$

But $1 - \varepsilon < \frac{n}{n+1}$, hence $\frac{1}{\varepsilon} < n + 1$, and so

$$\frac{1 - \varepsilon}{\varepsilon} < \frac{n}{n+1} \cdot \frac{1}{\varepsilon} < n.$$

Using the assumption that $\varepsilon < \frac{1}{2}$

$$\frac{1}{n} < \frac{\varepsilon}{1 - \varepsilon} < 2\varepsilon.$$

Combining this with equation (5.4), we obtain $\|v_1 - w_2 f\|_2^2 < 2\varepsilon$.

Then, with $\varepsilon = 67\delta^2$, there exists a projection $f \in N$ and a unitary u_1 in $\mathcal{N}_M(N) = N$ with

$$\|v_1 - u_1 f\|_2^2 \leq 134\delta^2$$

and so $\|v_1 - u_1 f\|_2 \leq \sqrt{134}\delta$. Hence, if we set $\delta = \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}$, then using equation (5.1),

$$\begin{aligned} \|u - \mathbb{E}_N(u)\|_2 &\leq \|u - u_1 f\|_2 \leq \|u - v_1\|_2 + \|v_1 - u_1 f\|_2 \\ &\leq \|u - v_1\|_2 + \sqrt{134}\delta = \|u - vu\|_2 + \sqrt{134}\delta \\ &= \|1 - v\|_2 + \sqrt{134}\delta \leq 13\delta + \sqrt{134}\delta \\ &= (13 + \sqrt{134})\|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2} \leq 25\|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}. \end{aligned}$$

This establishes that N is $\frac{1}{25}$ -strongly singular in M if $67\delta^2 < \frac{1}{2}$, that is, if

$$\delta = \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2} < \frac{1}{\sqrt{134}} < \frac{1}{11}.$$

However, if $\delta \geq \frac{1}{11}$, then

$$\frac{1}{25} \|u - \mathbb{E}_N(u)\|_2 \leq \frac{1}{25} < \delta$$

and so the theorem follows. \square

Under the assumption that $N' \cap \langle M, e_N \rangle$ is 2-dimensional, we can improve the constant in Theorem 5.0.2. Note that if $[M : N] > 2$ and $N' \cap \langle M, e_N \rangle$ is 2-dimensional, then N is automatically singular in M , as any u in $\mathcal{N}_M(N) \setminus \mathcal{U}(N)$ yields the projection ue_Nu^* in $N' \cap \langle M, e_N \rangle$. This projection is not e_N since $\{e_N\}' \cap M = N$ and it is also not e_N^\perp since

$$\mathrm{Tr}(e_N^\perp) > \mathrm{Tr}(e_N) = \mathrm{Tr}(ue_Nu^*).$$

By Goldman's Theorem ([9] or [12]), all index 2 subfactors are regular.

Theorem 5.0.3. *Let $N \subseteq M$ be a singular subfactor with $N' \cap \langle M, e_N \rangle$ two-dimensional.*

If $[M : N] < \infty$, then N is $\sqrt{\frac{[M : N] - 2}{[M : N] - 1}}$ -strongly singular in M . If $[M : N] = \infty$, then N is strongly singular in M .

Proof. Let $N \subseteq M$ be a singular inclusion of subfactors and suppose $N' \cap \langle M, e_N \rangle$ is 2-dimensional. Let h be the element of minimal 2-norm defined by averaging e_N over uNu^* as in subsection 2.5. Recall that $h \in (uNu^*)' \cap \langle M, e_N \rangle$. This algebra has basis ue_Nu^* and $ue_N^\perp u^*$, so that $h = \alpha(ue_Nu^*) + \beta(ue_N^\perp u^*)$ for some scalars α and β . By 3) in Proposition 2.5.1,

$$1 = \mathrm{Tr}(h) = \alpha + \lambda\beta,$$

where $\lambda = [M : N] - 1$. If $[M : N] = \infty$, then $\mathrm{Tr}(e_N^\perp) = \infty$, which implies that $\beta = 0$,

and therefore $\alpha = 1$. If $[M : N]$ is finite, then

$$\alpha = 1 - \lambda\beta, \quad (5.5)$$

and expanding $\text{Tr}(e_N h)$ yields

$$\begin{aligned} \text{Tr}(e_N h) &= \alpha \text{Tr}(e_N u e_N u^*) + \beta \text{Tr}(e_N - e_N u e_N u^*) \\ &= \alpha \text{Tr}(e_N \mathbb{E}_N(u) \mathbb{E}_N(u^*)) + \beta (\text{Tr}(e_N) - \text{Tr}(e_N \mathbb{E}_N(u) \mathbb{E}_N(u^*))) \\ &= \alpha \tau(\mathbb{E}_N(u) \mathbb{E}_N(u^*)) + \beta (1 - \tau(\mathbb{E}_N(u) \mathbb{E}_N(u^*))) \\ &= \alpha \|\mathbb{E}_N(u)\|_2^2 + \beta \|u - \mathbb{E}_N(u)\|_2^2, \end{aligned}$$

since $1 = \|u\|_2^2 = \|\mathbb{E}_N(u)\|_2^2 + \|u - \mathbb{E}_N(u)\|_2^2$. Setting $k = \|u - \mathbb{E}_N(u)\|_2^2$ and substituting the formula for α from equation (5.5) gives

$$\text{Tr}(e_N h) = (1 - \lambda\beta)(1 - k) + \beta k.$$

Using 2) from Proposition 2.5.1, we have

$$\begin{aligned} (1 - \lambda\beta)(1 - k) + \beta k &= \text{Tr}(e_N h) = \text{Tr}(h^2) \\ &= (1 - \lambda\beta)^2 + \lambda\beta^2, \end{aligned}$$

and so $(\lambda^2 + \lambda)\beta^2 - (\lambda + k + \lambda k)\beta + k = 0$. We may then solve for β in terms of λ and k , obtaining the roots $\beta = \frac{k}{\lambda}$ and $\beta = \frac{1}{1 + \lambda}$.

Suppose that $\beta = \frac{1}{1 + \lambda}$. Then

$$\alpha = 1 - \frac{\lambda}{1 + \lambda} = \frac{1}{1 + \lambda} = \beta,$$

and so

$$h = \beta(u e_N u^*) + \beta(u e_N^\perp u^*) = \beta I = \frac{1}{1 + \lambda} I.$$

Since h is an element of the weakly closed convex hull of $\{u e_N u^* : u \in \mathcal{U}(u N u^*)\}$,

there exist natural numbers $\{n_j\}_{j=1}^\infty$, positive reals $\{\gamma_i^{(j)}\}_{i=1}^{n_j}$ with $\sum_{i=1}^{n_j} \gamma_i^{(j)} = 1$ and unitaries $\{w_i^{(j)}\}_{i=1}^{n_j}$ in N with

$$\lim_{j \rightarrow \infty} \sum_{i=1}^{n_j} \gamma_i^{(j)} u w_i^{(j)} u^* e_N u (w_i^{(j)})^* u^* = \frac{1}{1+\lambda} I$$

in WOT.

Then also

$$\lim_{j \rightarrow \infty} \sum_{i=1}^{n_j} \gamma_i^{(j)} w_i^{(j)} u^* e_N u (w_i^{(j)})^* = \frac{1}{1+\lambda} I$$

in WOT and

$$\lim_{j \rightarrow \infty} e_N \left(\sum_{i=1}^{n_j} \gamma_i^{(j)} w_i^{(j)} u^* e_N u (w_i^{(j)})^* \right) = \frac{e_N}{1+\lambda}$$

in WOT. Taking the trace of both sides yields

$$\mathrm{Tr} \left(\lim_{j \rightarrow \infty} e_N \left(\sum_{i=1}^{n_j} \gamma_i^{(j)} w_i^{(j)} u^* e_N u (w_i^{(j)})^* \right) \right) = \mathrm{Tr} \left(\frac{e_N}{1+\lambda} \right) = \frac{1}{1+\lambda}.$$

However, for any n_j , $1 \leq j < \infty$,

$$\begin{aligned} \mathrm{Tr} \left(e_N \left(\sum_{i=1}^{n_j} \gamma_i^{(j)} w_i^{(j)} u^* e_N u (w_i^{(j)})^* \right) \right) &= \mathrm{Tr} \left(e_N \left(\sum_{i=1}^{n_j} \gamma_i^{(j)} w_i^{(j)} \mathbb{E}_N(u^*) \mathbb{E}_N(u) (w_i^{(j)})^* \right) \right) \\ &= \tau \left(\sum_{i=1}^{n_j} \gamma_i^{(j)} w_i^{(j)} \mathbb{E}_N(u^*) \mathbb{E}_N(u) (w_i^{(j)})^* \right) \\ &= \|\mathbb{E}_N(u)\|_2^2, \end{aligned}$$

and so $\frac{1}{1+\lambda} = \|\mathbb{E}_N(u)\|_2^2$. We obtain that

$$k = 1 - \|\mathbb{E}_N(u)\|_2^2 = 1 - \frac{1}{1+\lambda} = \frac{\lambda}{1+\lambda}.$$

Then $\frac{k}{\lambda} = \frac{1}{1+\lambda}$, and so the only instance where $\beta = \frac{1}{1+\lambda}$ is when $k = \frac{\lambda}{1+\lambda}$, and there the two roots are identical.

We may then take $\beta = \frac{k}{\lambda}$ and so $\alpha = 1 - \lambda\beta = 1 - k$ when $[M : N]$ is finite.

Hence

$$h = (1 - k)(ue_N u^*) + \frac{k}{\lambda}(ue_N^\perp u^*). \quad (5.6)$$

By 2) and 4) of Proposition 2.5.1,

$$\begin{aligned} \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}^2 &\geq 1 - \text{Tr}(e_N h) \\ &= 1 - \text{Tr}(h^2) \\ &= 1 - \left((1 - k)^2 + \frac{k^2}{\lambda} \right) \\ &= k \left(2 - \left(1 + \frac{1}{\lambda} \right) k \right) \end{aligned}$$

and therefore

$$\|u - \mathbb{E}_N(u)\|_2^2 = k \leq \frac{1}{2 - (1 + \frac{1}{\lambda})k} \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}^2. \quad (5.7)$$

As $k \leq 1$,

$$2 - \left(1 + \frac{1}{\lambda} \right) k \geq 2 - \left(1 + \frac{1}{\lambda} \right) = 1 - \frac{1}{\lambda},$$

and it follows that

$$\begin{aligned} \|u - \mathbb{E}_N(u)\|_2^2 &\leq \frac{1}{1 - \frac{1}{\lambda}} \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}^2 \\ &= \frac{\lambda}{\lambda - 1} \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}^2 \\ &= \frac{[M : N] - 1}{[M : N] - 2} \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}^2. \end{aligned}$$

Hence N is $\sqrt{\frac{[M : N] - 2}{[M : N] - 1}}$ -strongly singular in M .

If $[M : N] = \infty$, then as previously noted, $\alpha = 1$ and so $h = ue_N u^*$. Therefore,

$$\|u - \mathbb{E}_N(u)\|_2^2 = 1 - \text{Tr}(e_N h) \leq \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}^2$$

so that N is strongly singular in M and the proof is complete. \square

We end this section by showing that in the situation of Theorem 5.0.3, when unitaries are close enough to a finite index subfactor in 2-norm they satisfy the equation for strong singularity.

Corollary 5.0.4. *Under the hypotheses of Theorem 5.0.3, if $[M : N] < \infty$ and*

$$\|u - \mathbb{E}_N(u)\|_2 \leq \sqrt{\frac{[M : N] - 1}{[M : N]}},$$

then

$$\|u - \mathbb{E}_N(u)\|_2 \leq \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}.$$

Proof. Recall $k = \|u - \mathbb{E}_N(u)\|_2^2$ and $\lambda = [M : N] - 1$. If $k \leq \frac{\lambda}{\lambda+1}$, then

$$\begin{aligned} 2 - \left(1 + \frac{1}{\lambda}\right)k &= 2 - \left(\frac{\lambda+1}{\lambda}\right)k \\ &\geq 2 - \left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda}{\lambda+1}\right) = 1 \end{aligned}$$

Using equation (5.7),

$$\|u - \mathbb{E}_N(u)\|_2^2 = k \leq k \left(2 - \left(1 + \frac{1}{\lambda}\right)k\right) \leq \|\mathbb{E}_N - \mathbb{E}_{uNu^*}\|_{\infty,2}^2. \quad \square$$

6. EXAMPLES OF SINGULAR SUBFACTORS

M is always a singular subfactor of itself, but is not particularly exciting from this standpoint. If $[M : N] = 4 \cos^2(\frac{\pi}{n})$ for $n \neq 3, 4$ or 6 , then N is automatically singular in M , as follows:

Set $P = \mathcal{N}_M(N)$. Then if $P \neq N$,

$$4 > [M : N] = [M : P] \cdot [P : N] > 2 \cdot [M : P]$$

and so $[M : P] < 2$. By Corollary 4.1.8, $[M : P] = 1$ and so $P = M$. Therefore N is either singular or regular. However, by results of Jones [11], a regular subfactor must have integer index. Therefore N is singular.

Our goal is to produce singular subfactors for larger values of the index. We shall proceed by first analyzing the structure of crossed products of II_1 factors.

6.1. Normalizers in Crossed Products

Throughout this subsection, we will suppose that P is a II_1 factor represented on \mathcal{H} and G is a countable discrete group with an outer action α on P . We will show, as a consequence of Theorem 6.1.2, that the only unitary normalizers of the crossed product of $P \rtimes H$ for H a subgroup of G are contained in the algebra $P \rtimes \mathcal{N}_G(H)$, where

$$\mathcal{N}_G(H) = \{g \in G : gHg^{-1} = H\}.$$

This result will appear in [34] as Theorem 4.1 and will be the tool we use to exhibit singular subfactors of integer index. We begin with a lemma (also to appear in [34]) that determines the von Neumann algebra generated by a certain diagonal subalgebra of $B(\mathcal{H} \otimes_2 \ell_2(G))$.

Lemma 6.1.1. *In $B(\mathcal{H} \otimes_2 \ell^2(G))$, let*

$$\mathbf{A} = \{(b_{g,h})_{g,h \in G} : b_{g,h} = \delta_{g,h} z \alpha_g(b)\}$$

where $b \in P$ and $z \in P' \subseteq B(\mathcal{H})$. Then

$$\mathbf{A}'' = \{(c_{g,h})_{g,h \in G} : c_{g,h} = \delta_{g,h} z_g\}$$

where $z_g \in B(\mathcal{H})$, $\sup_{g \in G} \|z_g\| < \infty$.

Proof. Suppose $A = (a_{g,h})_{g,h \in G} \in \mathbf{A}'$ and $B = (b_{g,h})_{g,h \in G} \in \mathbf{A}$. Then for a fixed g and h ,

$$\begin{aligned} (AB)_{g,h} &= \sum_{k \in G} a_{g,k} b_{k,h} \\ &= a_{g,h} b_{h,h} \\ &= a_{g,h} z \alpha_h(b). \end{aligned}$$

Interchanging B and A gets that $(BA)_{g,h} = z \alpha_g(b) a_{g,h}$. Since $AB = BA$,

$$a_{g,h} z \alpha_h(b) = z \alpha_g(b) a_{g,h} \tag{6.1}$$

for all $g, h \in G$. Setting $b = I$ in equation (6.1) yields that $a_{g,h} z = z a_{g,h}$, so $a_{g,h} \in P$ for all $g, h \in G$.

Setting $z = I$ in equation (6.1) gets $a_{g,h} \alpha_h(b) = \alpha_g(b) a_{g,h}$, and if $h = g$, then

$$a_{g,g} \alpha_g(b) = \alpha_g(b) a_{g,g}.$$

Hence $a_{g,g} \in P'$, and so $a_{g,g} \in P \cap P' = \mathbb{C}I$ for all $g \in G$.

If $g \neq h$, then by setting $c = \alpha_h(b)$ in equation (6.1),

$$a_{g,h} c = \alpha_g \circ \alpha_{h^{-1}}(c) a_{g,h}.$$

This implies that $\alpha_g \circ \alpha_{h^{-1}} = \alpha_{gh^{-1}}$ is inner unless $a_{g,h} = 0$ for all $g, h \in G, g \neq h$. Therefore,

$$\mathbf{A}' = \{(a_{g,h})_{g,h \in G} : a_{g,g} \in \mathbb{C}I, a_{g,h} = \delta_{g,h}a_{g,g}\}$$

and $\sup_{g \in G} \|a_{g,g}\| < \infty$, from which the result follows. \square

With the aid of Lemma 6.1.1, we can prove the main result of this subsection.

Theorem 6.1.2. *If H is a subgroup of G , then $x \in P \rtimes G$, $x(P \rtimes H)x^* \subseteq P \rtimes H$ implies $x \in P \rtimes \mathcal{N}_G(H)$. Furthermore, this occurs if and only if $x = x_0 u_g$ for some $x_0 \in P \rtimes H$ and $g \in \mathcal{N}_G(H)$.*

Proof. Let $x \in P \rtimes G$. Then

$$x = \sum_{g \in G} a_g u_g$$

where u_g is the unitary representation of a group element in $P \rtimes G$, $a_g := \mathbb{E}_P(xu_{g^{-1}})$ and the sum converges in 2-norm. Let α_g denote the automorphism on P associated to g .

If $g \notin \mathcal{N}_G(H)$, then there exists $h \in H$ with $s := ghg^{-1} \notin H$. We then obtain a bijective map $\phi : G \rightarrow G$ such that for all $g \in G$, $gh\phi(g) = s$.

Since $x(P \rtimes H)x^* \subseteq P \rtimes H$, if $b \in P$, then $x(bh)x^*$ is in $P \rtimes H$. This means that the Fourier coefficient associated to all k not in H has to be zero. In particular,

$$\sum_{k \in G} a_k \alpha_g(b) \alpha_s(a_{\phi(k)}^*) = 0 \quad (6.2)$$

since this is the coefficient for s . The convergence here can be regarded as SOT [18].

Now order the elements of G as g_1, g_2, \dots . The row operator on $\mathcal{H} \otimes_2 \ell^2(G)$ defined

by

$$\begin{pmatrix} a_{g_1} & a_{g_2} & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

is bounded, as follows:

It suffices to show that

$$\begin{pmatrix} a_{g_1}^* & 0 & \dots \\ a_{g_2}^* & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

is bounded.

Pick $\xi \in \mathcal{H}$, $\|\xi\| = 1$. Then

$$\begin{aligned} \left\| \begin{pmatrix} a_{g_1}^* & 0 & \dots \\ a_{g_2}^* & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \xi \\ 0 \\ \vdots \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} a_{g_1}^* \xi \\ a_{g_2}^* \xi \\ \vdots \end{pmatrix} \right\|^2 \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \langle a_{g_i}^* \xi, a_{g_i}^* \xi \rangle \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \langle a_{g_i} a_{g_i}^* \xi, \xi \rangle \\ &= \langle (\lim_{n \rightarrow \infty} \sum_{i=1}^n a_{g_i} a_{g_i}^*) \xi, \xi \rangle \end{aligned}$$

Now from [18], it follows that in SOT,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n a_{g_i} a_{g_i}^* = \mathbb{E}_P(xx^*)$$

and so

$$\begin{aligned}
\left\| \begin{pmatrix} a_{g_1}^* & 0 & \dots \\ a_{g_2}^* & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \xi \\ 0 \\ \vdots \end{pmatrix} \right\|^2 &= \langle (\lim_{n \rightarrow \infty} \sum_{i=1}^n a_{g_i} a_{g_i}^*) \xi, \xi \rangle \\
&= \langle \mathbb{E}_P(xx^*) \xi, \xi \rangle \\
&\leq \|x\|^2 \|\xi\|^2
\end{aligned}$$

which demonstrates the result.

This also implies that

$$\begin{pmatrix} \alpha_s(a_{\phi(g_1)}^*) & 0 & \dots \\ \alpha_s(a_{\phi(g_2)}^*) & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

defines a bounded linear operator on $H \otimes_2 \ell^2(G)$. Therefore, the product

$$\begin{pmatrix} a_{g_1} & a_{g_2} & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \alpha_{g_1}(b) & 0 & \dots \\ 0 & \alpha_{g_2}(b) & \vdots \\ \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \alpha_s(a_{\phi(g_1)}^*) & 0 & \dots \\ \alpha_s(a_{\phi(g_2)}^*) & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = 0$$

since this matrix multiplication yields automatic zeroes except in the $(1_G, 1_G)$ position. In this position, the entry is $\sum_{k \in G} a_k \alpha_g(b) \alpha_s(a_{\phi(k)}^*) = 0$ from equation (6.2).

If we multiply by arbitrary diagonal operators

$$\begin{pmatrix} z & 0 & \dots \\ 0 & z & \vdots \\ \vdots & \ddots & \ddots \end{pmatrix}$$

where $z \in P'$, we obtain

$$\begin{pmatrix} a_{g_1} & a_{g_2} & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} z\alpha_{g_1}(b) & 0 & \dots \\ 0 & z\alpha_{g_2}(b) & \vdots \\ \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \alpha_s(a_{\phi(g_1)}^*) & 0 & \dots \\ \alpha_s(a_{\phi(g_2)}^*) & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = 0$$

Then by taking WOT limits of the middle terms,

$$\begin{pmatrix} a_{g_1} & a_{g_2} & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 & 0 & \dots \\ 0 & x_2 & \vdots \\ \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \alpha_s(a_{\phi(g_1)}^*) & 0 & \dots \\ \alpha_s(a_{\phi(g_2)}^*) & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = 0$$

where the x_i 's are in $B(\mathcal{H})$ and $\sup_i \|x_i\| < \infty$ by Lemma 6.1.1. Setting $x_i = 0$ save for the entry corresponding to g (recall g was a fixed group element not in $\mathcal{N}_G(H)$) gives

$$a_g x \alpha_s(a_{g^{-1}})^* = 0$$

since $\phi(g) = g^{-1}$. Then choosing x to be a nonzero partial isometry with domain space contained in the range of $\alpha_s(a_{g^{-1}})$ and mapping into the cokernel of a_g forces a contradiction unless $a_g = 0$.

So in the fourier expansion of x , all coefficients associated to $g \notin \mathcal{N}_G(H)$ are zero, hence x is in $P \rtimes \mathcal{N}_G(H)$ as claimed. We have then established the first assertion of the theorem.

To prove the last assertion, let $x = x_0 u_g$ for some $g \in G$, $x_0 \in P \rtimes H$. Then for all $h \in H$, $ghg^{-1} = h_0 \in H$. Then for all $b \in P$,

$$x(bu_h)x^* = x_0 \alpha_g(b) u_g u_h u_g^{-1} x_0^* = x_0 \alpha_g(b) u_{h_0} x_0^* \in P \rtimes H$$

and so $x(P \rtimes H)x^* \subseteq P \rtimes H$.

Now suppose $x(P \rtimes H)x^* \subseteq P \rtimes H$. Then $x \in P \rtimes \mathcal{N}_G(H)$ by the first part of

the theorem, so $x = \sum_{g \in \mathcal{N}_G(H)} x_g u_g$. For all $h \in H$, $b \in P$

$$x(bu_h)x^* = \sum_{g,k \in \mathcal{N}_G(H)} x_g \alpha_g(b) u_{ghk^{-1}} x_k^* \in P \rtimes H.$$

If $ghk^{-1} \in H$, then $ghk^{-1} = h'$ for some $h' \in H$, and so $gh = h'k$. However, since $k \in \mathcal{N}_G(H)$ and $H \triangleleft \mathcal{N}_G(H)$, $kH = Hk$, and so there exists an $h'' \in H$ with

$$gh = h'k = kh''.$$

Since left cosets are either disjoint or equal, $gH = kH$.

If $ghk^{-1} \notin H$, then setting $s = ghk^{-1}$ and mimicking the proof of the first part of the theorem shows that either x_g or x_k must be zero. Therefore, the only allowable nonzero coefficients of x are those associated to the group elements from a single left coset of H . That is, we must have $x = x_0 g$ for some $g \in \mathcal{N}_G(H)$ and $x_0 \in P$. \square

By taking x to be a normalizing unitary of N , we obtain

Corollary 6.1.3. $\{\mathcal{N}_{P \rtimes G}(P \rtimes H)\}'' = P \rtimes \mathcal{N}_G(H)$.

Corollary 6.1.4. *If x and y are nonzero elements of $P \rtimes G$ that satisfy*

$$x(P \rtimes H)y \subseteq P \rtimes H,$$

then x and y are in $P \rtimes \mathcal{N}_G(H)$.

Proof. The goal is to show that $x(P \rtimes H)x^* \subseteq P \rtimes H$ and then appeal to Theorem 6.1.2. The same result will hold for y . If $x(P \rtimes H)y \subseteq P \rtimes H$, then

$$(x(P \rtimes H)y)(y^*(P \rtimes H)x^*) \subseteq P \rtimes H.$$

If $u \in \mathcal{U}(P \rtimes H)$,

$$\begin{aligned} (x(P \rtimes H)uy)(y^*u^*(P \rtimes H)x^*) &= (x(P \rtimes H)u^*uy)(y^*u^*u(P \rtimes H)x^*) \\ &= (x(P \rtimes H)y)(y^*(P \rtimes H)x^*) \\ &\subseteq P \rtimes H \end{aligned}$$

since $(P \rtimes H)u = P \rtimes H$. This implies that if $\{u_i\}_{i=1}^n$ in $P \rtimes H$ are unitaries and $\{\lambda_i\}_{i=1}^n$ are scalars with $\sum_{i=1}^n \lambda_i = 1$, we have that

$$x(P \rtimes H)\left(\sum_{i=1}^n \lambda_i u_i y y^* u_i^*\right)(P \rtimes H)x^* \subseteq P \rtimes H.$$

Then if t is the element of minimal 2-norm in the weak closure of the convex hull of $\{u y y^* u^* : u \in \mathcal{U}(P \rtimes H)\}$, it follows that

$$x(P \rtimes H)t(P \rtimes H)x^* \subseteq P \rtimes H$$

But since t is in $P \rtimes H \cap (P \rtimes H)'$, t is a nonzero scalar since

$$\tau(t) = \tau(y y^*) > 0$$

hence

$$x(P \rtimes H)x^* \subseteq P \rtimes H$$

and so x is in $M \rtimes \mathcal{N}_G(H)$ by Theorem 6.1.2. □

6.2. Singular Subfactors

Using Corollary 6.1.4, we can obtain examples of singular subfactors for higher index values using crossed products. All we need do is exhibit countable discrete groups G with proper subgroups H such that $\mathcal{N}_G(H) = H$ and $[G : H] < \infty$, as any such G admits a proper outer action on the hyperfinite II_1 factor. We now give examples of

such pairs of groups.

Example 6.2.1. Let $G = S_n$, the permutation group on $n > 2$ elements. Suppose $H \cong S_{n-1}$ is the subgroup of G consisting of all permutations which fix a single element. For simplicity, let us assume H fixes the 1st element. Let K be any intermediate subgroup properly containing H and suppose K contains the transposition $(1j)$ for $j \neq 1$. If $i \neq 1$, then $(ij) \in H$, and $(ij) \circ (1j) \circ (ij) = (1i) \in K$. Therefore K will contain all transpositions and so is equal to S_n .

Inductively assume that if K contains an element not in H that is a product of fewer than $m > 1$ transpositions, then $K = G$. Suppose that $\gamma \in K$ is the product of m transpositions. If $\gamma = \gamma_1 \circ \gamma_2$ where γ_1 and γ_2 are disjoint, then only one, say γ_1 , has

$$\gamma_1(1) \neq 1.$$

Therefore $\gamma_2 \in H$, and so $\gamma_1 = \gamma \circ \gamma_2^{-1} \in H$. Since γ_1 is not in H and is the product of fewer than m transpositions, by the inductive hypothesis, $K = G$.

If $\gamma \in K$ is not a product of disjoint permutations, then γ is a cycle. We can then write $\gamma = (1j) \circ \gamma_1$ where $\gamma_1 \in H$. Hence $(1j) = \gamma \circ \gamma_1^{-1} \in K$ and so by the initial inductive step, $K = G$. Then by induction, the only subgroup of G properly containing H is G itself.

We now know that $\mathcal{N}_G(H)$ is either H or G . However,

$$(1n) \circ (2n) \circ (1n) = (12) \notin H$$

and so $\mathcal{N}_G(H) = H$.

Example 6.2.1 may be generalized to the case where $G = S_\infty$ to produce an infinite index singular subfactor. Theorem 4.2.2 for subfactors shows that for any pair $N = P \rtimes H$ and $M = P \rtimes G$, the intermediate subfactors of N in M are all of the form

$P \rtimes K$ for some intermediate subgroups $H \leq K \leq G$. The calculations in Example 6.2.1 show that for those choices of G and H , there are no intermediate subfactors. This implies that there is no immediate obstruction preventing $N' \cap \langle M, e_N \rangle$ from being 2-dimensional. In fact, we have

Theorem 6.2.2. *The group-subgroup pairs in Example 6.2.1 yield factors which satisfy the hypotheses of Theorem 5.0.3.*

Also included is the case $G = S_\infty$. Using Proposition 4.3.3, all we need do is determine the double cosets of H in G .

Proposition 6.2.3. *With G and H as in Example 6.2.1, there are two double cosets of H in G .*

Proof. Take H to be the subgroup of G that fixes the 1st element. Let σ be the transposition $(1n)$ and let $\gamma \in G \setminus H$. Then $\gamma(1) \neq 1$. Take $\pi \in H$ with $\pi(n) = \gamma^{-1}(1)$. We then have

$$(\gamma \circ \pi \circ \sigma)(1) = (\gamma \circ \pi)(n) = \gamma(\pi(n)) = \gamma(\gamma^{-1}(1)) = 1,$$

and so $\theta = \gamma \circ \pi \circ \sigma \in H$. Then $\gamma = \theta \circ \sigma \circ \pi^{-1} \in H\sigma H$, and this shows that H has exactly 2 double cosets in G . \square

The proof of Proposition 6.2.3 extends verbatim to the case where $G = S_\infty$. Therefore there are only two irreducible bimodules, and hence only two nontrivial projections in $N' \cap \langle M, e_N \rangle$.

6.3. The WHP and Finite Index Subfactors

Theorem 6.3.1 below shows that any proper finite index subfactor N of M does not have the WHP in M , defined by equation (3.3).

Theorem 6.3.1. *If $N \subseteq M$ is a II_1 factor with $1 < [M : N] < \infty$, then N does not have the WAHP in M .*

Proof. It will be advantageous to use a Pimsner-Popa basis obtained by first choosing k to be the least integer greater than or equal to $[M : N]$. We then select a collection of orthogonal projections $\{p_j\}_{j=1}^k$ in $\langle M, e_N \rangle$ with $p_1 = e_N$, $\sum_{j=1}^k p_j = 1$ and $\text{Tr}(p_i) \leq 1$ with equality except possibly for $j = k$.

Let v_1, v_2, \dots, v_k be partial isometries in $\langle M, e_N \rangle$ such that $e_N = v_1$, $v_j v_j^* = p_j$ and $v_j^* v_j \leq e_N$. The desired basis is given by the unique elements $\lambda_j \in M$ from equation (4.2) with the property that

$$\lambda_j e_N = v_j e_N.$$

Observe that $\lambda_1 = 1$. Since for $i \neq j$,

$$\begin{aligned} \mathbb{E}_N (\lambda_i^* \lambda_j) e_N &= e_N \lambda_i^* \lambda_j e_N = e_N v_i^* v_j e_N \\ &= e_N v_i^* p_i p_j v_j e_N = 0, \end{aligned}$$

we have that $\mathbb{E}_N (\lambda_i^* \lambda_j) = 0$ for $i \neq j$. In particular, $\mathbb{E}_N (\lambda_j) = 0$ for all $1 < j \leq k$. It is worth noting that this is the original construction in [20].

Now suppose $1 < [M : N] < \infty$ and $\lambda_1, \dots, \lambda_k$ are chosen as indicated. We will show that the WAHP fails for the sets $\{x_i = \lambda_i\}$ and $\{y_j = \lambda_j^*\}$, $1 < i, j \leq k$. Let u be any unitary in N . Then since

$$\tau(\mathbb{E}_N(x)) = \tau(x) = \text{Tr}(e_N x)$$

for all x in M ,

$$\begin{aligned}
\sum_{i,j=2}^k \|\mathbb{E}_N(\lambda_i^* u \lambda_j)\|_2^2 &= \sum_{i,j=2}^k \tau(\mathbb{E}_N(\lambda_j^* u^* \lambda_i) \mathbb{E}_N(\lambda_i^* u \lambda_j)) \\
&= \sum_{i,j=2}^k \tau(\lambda_j^* u^* \lambda_i \mathbb{E}_N(\lambda_i^* u \lambda_j)) \\
&= \sum_{i,j=2}^k \text{Tr}(e_N \lambda_j^* u^* \lambda_i \mathbb{E}_N(\lambda_i^* u \lambda_j)) \\
&= \sum_{i,j=2}^k \text{Tr}(e_N \lambda_j^* u^* \lambda_i e_N \lambda_i^* u \lambda_j e_N).
\end{aligned}$$

Using this equality, the fact that u commutes with e_N , and $\sum_{j=1}^k \lambda_j e_N \lambda_j^* = 1$, we get

$$\begin{aligned}
\sum_{i,j=2}^k \|\mathbb{E}_N(\lambda_i^* u \lambda_j)\|_2^2 &= \sum_{i,j=2}^k \text{Tr}(e_N \lambda_j^* u^* \lambda_i e_N \lambda_i^* u \lambda_j e_N) \\
&= \sum_{i,j=2}^k \text{Tr}(u^* \lambda_i e_N \lambda_i^* u \lambda_j e_N \lambda_j^*) = \text{Tr}(u^* (1 - e_N) u (1 - e_N)) \\
&= \text{Tr}((1 - e_N) u^* u) = [M : N] - 1 > 0.
\end{aligned}$$

This implies that for any given unitary u in N , there are indices $1 < i, j \leq k$ with

$$\|\mathbb{E}_N(\lambda_i^* u \lambda_j)\|_2 \geq \frac{\sqrt{[M : N] - 1}}{k - 1},$$

and so the WAHP fails to hold. □

Combining the previous theorem with the results from subsection 6.2, we immediately arrive at the desired conclusion.

Corollary 6.3.2. *There exist II_1 factors M with singular subfactors that do not have the WAHP in M .*

6.4. Infinite Index Singular Subfactors and One-Sided Normalizers

As we saw in the previous subsection, no nontrivial finite index subfactor of M can have the WAHP. Examples of infinite index singular subfactors with the WAHP were given in [28] and the subfactors of free group factors examined in [30] can easily be shown to have the WAHP. We consider the case $M = L(S_\infty)$ and $N = L(H)$ for H as in subsection 6.2. Theorem 5.0.3 shows that $L(H)$ is strongly singular in $L(S_\infty)$.

Lemma 6.4.1. *If $G = S_\infty$ and H is a subgroup of G that fixes a single element, then for all $\sigma_1, \dots, \sigma_n \in G$, $\gamma_1, \dots, \gamma_m \notin H$, there exists an element $\pi \in H$ with*

$$\sigma_i \circ \pi \circ \gamma_j \notin H \quad (6.3)$$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Hence, if $M = L(S_\infty)$ and $N = L(H)$,

$$\mathbb{E}_N(\sigma_i \circ \pi \circ \gamma_j) = 0. \quad (6.4)$$

Proof. Again suppose H fixes the 1st element. By the definition of S_∞ , there exists $s \in \mathbb{N}$ with $\gamma_j(r) = \sigma_i(r) = r$ for all $r > s$ and for all $1 \leq i \leq n$, $1 \leq j \leq m$. As $\gamma_1, \dots, \gamma_m \notin H$, $\gamma_j(1) \neq 1$ for all $1 \leq j \leq m$. Let t_1, \dots, t_k be the distinct images of 1 under γ_j for some j . Let π_l be the transposition $t_l \mapsto s + t_l$ and define

$$\pi = \prod_{l=1}^k \pi_l.$$

Then since $t_l \neq 1$ for any $1 \leq l \leq k$, we have that $\pi \in H$. For a given j , there exists $1 \leq l \leq k$ with $\gamma_j(1) = t_{l_j}$. Therefore,

$$\sigma_i \circ \pi \circ \gamma_j(1) = \sigma_i \circ \pi(t_{l_j}) = \sigma_i(s + t_{l_j}) = s + t_{l_j} \neq 1$$

for any $1 \leq i \leq n$, and so $\sigma_i \circ \pi \circ \gamma_j \notin H$. □

As noted in the proof of Lemma 4.1 in [28] when H is abelian, equation 6.3 is

equivalent to the condition: if $x_1, \dots, x_n \in G$, then

$$H \subseteq \bigcup_{i,j=1}^n x_i H x_j$$

implies that $x_i \in H$.

From Lemma 6.4.1, an approximation argument yields that $L(H)$ has the WAHP in $L(S_\infty)$ and the corresponding result for crossed products with these groups. Both cases have similar proofs, so we only detail the case of group von Neumann algebras.

Theorem 6.4.2. *If $G = S_\infty$ and H is a subgroup fixing a single element, $L(H)$ has the WAHP in $L(G)$. If P is a II_1 factor admitting an outer action of G , then $P \rtimes H$ has the WAHP in $P \rtimes G$.*

Proof. Let $x_1, \dots, x_n, y_1, \dots, y_m \in L(G)$ with $\mathbb{E}_{L(H)}(x_i) = \mathbb{E}_{L(H)}(y_j) = 0$ for all $1 \leq i \leq n, 1 \leq j \leq m$. Also assume $\|x_i\| = \|y_j\| = 1$ (this causes no loss in generality). Then for every $\varepsilon > 0$, there exists $x'_i, y'_j \in \mathbb{C}([G])$, $\mathbb{E}_{L(H)}(x'_i) = \mathbb{E}_{L(H)}(y'_j) = 0$ and

$$\|x_i - x'_i\|_2, \|y_j - y'_j\|_2 < \frac{\varepsilon}{2}.$$

Since $x'_i, y'_j \in \mathbb{C}([G])$, $x'_i = \sum_{l=1}^{r_i} \alpha_l^{(i)} u_{\sigma_l^{(i)}}$, $y'_j = \sum_{l=1}^{c_j} \beta_l^{(j)} u_{\gamma_l^{(j)}}$ for some $\gamma_l^{(j)}, \sigma_l^{(i)} \in G$ and $r_i, c_j \in \mathbb{N}$. Then none of the permutations in $\left(\bigcup_{i,l} \sigma_l^{(i)}\right) \cup \left(\bigcup_{j,l} \gamma_l^{(j)}\right)$ are in H . Construct $\pi \in H$ as in Lemma 6.4.1 for the sets $\left\{\bigcup_{i=1}^n \{\cup_{l=1}^{r_i} \sigma_l^{(i)}\}\right\}$ and

$\left\{ \bigcup_{j=1}^m \{ \cup_{l=1}^{c_j} \gamma_l^{(j)} \} \right\}$. Then

$$\begin{aligned}
& \| \mathbb{E}_{L(H)} (x_i u_\pi y_j) \|_2 \\
& \leq \| \mathbb{E}_{L(H)} (x_i u_\pi y_j) - \mathbb{E}_{L(H)} (x_i u_\pi y'_j) \| \\
& \quad + \| \mathbb{E}_{L(H)} (x_i u_\pi y'_j) - \mathbb{E}_{L(H)} (x'_i u_\pi y'_j) \| + \| \mathbb{E}_{L(H)} (x'_i u_\pi y'_j) \|_2 \\
& \leq \| x_i u_\pi y_j - x_i u_\pi y'_j \|_2 + \| x_i u_\pi y'_j - x'_i u_\pi y'_j \|_2 + \| \mathbb{E}_{L(H)} (x'_i u_\pi y'_j) \|_2 \\
& \leq \| y_j - y'_j \|_2 + \| x_i - x'_i \|_2 + \| \mathbb{E}_{L(H)} (x'_i u_\pi y'_j) \|_2 \\
& < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \| \mathbb{E}_{L(H)} (x'_i u_\pi y'_j) \|_2 \\
& = \varepsilon
\end{aligned}$$

as Theorem 6.4.1 shows that $\| \mathbb{E}_{L(H)} (x'_i u_\pi y'_j) \|_2 = 0$. Therefore $L(H)$ has the WAHP in $L(G)$. \square

Given the many examples of infinite index subfactors with the WAHP, one might be led to conjecture that every infinite index singular subfactor possesses this property. The following example, first brought to our attention by Ken Dykema, shows that a bit more care is necessary.

Example 6.4.3. Consider $L(\mathbb{Z}) \cong L_\infty([0, 1])$. Let $P \cong L(\mathbb{F}_\infty)$ be the free product of infinitely many copies of $L(\mathbb{Z})$, indexed by \mathbb{Z} . Let

$$M = P \rtimes \mathbb{Z}$$

where the (outer) action of \mathbb{Z} is given by shifting the free product. Define

$$N = \ast_{i=1}^\infty L(\mathbb{Z}).$$

Then N is an infinite index subfactor of P and hence of M . If u is the unitary that implements the action of \mathbb{Z} in M , then $\mathbb{E}_P(u) = \mathbb{E}_P(u^*) = 0$. Therefore $\mathbb{E}_N(u) = 0$.

However, $uNu^* \subsetneq N$, and so for any unitary $w \in N$, uwu^* is a unitary in N . Hence, for all $w \in \mathcal{U}(M)$,

$$\|\mathbb{E}_N(uwu^*)\|_2 = \|uwu^*\|_2 = 1$$

and so N does not have the WAHP in M .

The phenomenon $uBu^* \subsetneq B$ does not occur for masas since uAu^* is also a masa in M . Similarly, if N is a finite index subfactor of M and $uNu^* \subseteq N$, then

$$[M : N] = [uMu^* : uNu^*] = [M : uNu^*] = [M : N][N : uNu^*].$$

Since $[M : N] < \infty$, we obtain $[N : uNu^*] = 1$ and hence $N = uNu^*$. Owing to Smith [32], the factor N considered in Example 6.4.3 is singular.

We may amend the situation for infinite index subfactors by considering

$$\{u \in \mathcal{U}(M) : uNu^* \subset N\},$$

the semigroup of one-sided normalizers of N in M . Results in the subsequent section support consideration of this object. As previously noted, if N is finite index, this is a group and coincides with $\mathcal{N}_M(N)$.

Question 6.4.4. *If N is infinite index in M and contains its semigroup of one-sided normalizers, must N have the WAHP in M ?*

We end this section by briefly discussing existence questions. Recently, Stefaan Vaes has proved that there exists a factor M such that every finite index irreducible subfactor N is equal to M [33]. Since singular subfactors are in particular irreducible, this shows that there exist factors with no proper finite index singular subfactors. On the other hand, Popa has shown in [24] that there always exist singular masas in separable II_1 factors. The proper analog of the question for masas, then, is to ask

whether there always exist infinite index hyperfinite singular or α -strongly singular subfactors of any separable II_1 factor.

An example in the hyperfinite II_1 factor of an infinite index subfactor with the WAHP was provided in [28]. In [23], Popa remarks that any irreducible maximal hyperfinite subfactor of a II_1 factor is singular. Popa has also shown that every separable II_1 factor has a semi-regular masa that is contained in some (necessarily irreducible) hyperfinite subfactor [22]. We may then use Zorn's Lemma to obtain irreducible maximal hyperfinite subfactors for any separable II_1 factor. Therefore, any separable II_1 factor has an infinite index hyperfinite singular subfactor. This does not a priori answer the question of whether there exist infinite index α -strongly singular hyperfinite subfactors or hyperfinite subfactors with the WAHP in any II_1 factor. Maximal hyperfinite subfactors in any II_1 factor were first exhibited in [8].

7. A CHARACTERIZATION OF SINGULARITY FOR SUBFACTORS

We begin with a reformulation of singularity in terms of the traces of projections in $N' \cap \langle M, e_N \rangle$ when $[M : N] < \infty$.

Theorem 7.0.1. *Suppose $N \subseteq M$ is a finite index subfactor inclusion. Then N is singular in M if and only if every nonzero projection $q \in N' \cap \langle M, e_N \rangle$ subordinate to e_N^\perp has $\text{Tr}(q) > 1$.*

Proof. (\Leftarrow) Suppose every nonzero projection $q \leq e_N^\perp$ in $N' \cap \langle M, e_N \rangle$ has trace greater than one. Let $u \in \mathcal{N}_M(N)$ and assume $u \in N'$. Then $\mathbb{E}_N(u) = \tau(u)$ and

$$\begin{aligned} 1 &= \text{Tr}(ue_Nu^*) = \text{Tr}(e_Nue_Nu^*e_N) + \text{Tr}(e_N^\perp ue_Nu^*e_N^\perp) \\ &= \|\mathbb{E}_N(u)\|_2^2 + \text{Tr}((e_N^\perp ue_Nu^*)(ue_Nu^*e_N^\perp)) \\ &= |\tau(u)|^2 + \|e_N^\perp ue_Nu^*\|_{2,\text{Tr}}^2. \end{aligned} \tag{7.1}$$

By the Kaplansky formula, $p = (e_N \vee ue_Nu^*) - e_N$ is equivalent to $p_0 = ue_Nu^* - (e_N \wedge ue_Nu^*)$ in $N' \cap \langle M, e_N \rangle$. The projection p is the range projection of $e_N^\perp ue_Nu^*$ and so is subordinate to e_N^\perp . Similarly, p_0 is the range projection of $ue_Nu^*e_N^\perp$ and so subordinate to ue_Nu^* . Since $\text{Tr}(p) = \text{Tr}(p_0) \leq \text{Tr}(ue_Nu^*) = 1$, we conclude that $p_0 = p = 0$. This implies that $e_N^\perp ue_Nu^* = 0$ and so

$$ue_Nu^* = e_Nue_Nu^*e_N + e_Nue_Nu^*e_N^\perp + e_N^\perp ue_Nu^*e_N + e_N^\perp ue_Nu^*e_N^\perp = e_Nue_Nu^*e_N.$$

From equation (7.1), $|\tau(u)| = 1$. Then

$$e_Nue_Nu^*e_N = \mathbb{E}_N(u) \mathbb{E}_N(u^*) e_N = |\tau(u)|^2 e_N = e_N$$

and so we have that $e_N = ue_Nu^*$. As $\{e_N\}' \cap M = N$, $u \in N$. As u was also assumed to be in N' , and therefore u is a scalar.

We have shown that if $u \in \mathcal{N}_M(N)$ commutes with N then u is a scalar and

so in N . Now suppose $u \in \mathcal{N}_M(N)$ and does not commute with N . If there exists $w \in \mathcal{U}(N)$ with $wxw^* = uxu^*$ for all $x \in N$, then $(w^*u)x = x(w^*u)$. Hence $w^*u \in N'$ and normalizes N , so $\mathbb{E}_N(w^*u) = \tau(w^*u)$ has absolute value one as previously demonstrated. But $w \in N$ so

$$\tau(w^*u) = \mathbb{E}_N(w^*u) = w^*\mathbb{E}_N(u).$$

Since $|\tau(w^*u)| = 1$, we then have

$$\|u\|_2 = 1 = |\tau(w^*u)| = \|w^*\mathbb{E}_N(u)\|_2 = \|\mathbb{E}_N(u)\|_2.$$

From this we obtain $\|u - \mathbb{E}_N(u)\|_2 = 0$ and so $u \in N$.

If $u \in \mathcal{N}_M(N)$, $u \notin N'$, and there does not exist a $w \in N$ with $uNu^* = wNw^*$, then the automorphism $\phi : N \rightarrow N$, $\phi(x) = uxu^*$ is outer. Then $\phi(x)u = ux$ for all $x \in N$, so by taking conditional expectations,

$$\phi(x)\mathbb{E}_N(u) = \mathbb{E}_N(u)x$$

for all $x \in N$. Since outer automorphisms are free, this implies that $\mathbb{E}_N(u) = 0$ and so $u \notin N$.

We have now established that if $u \in \mathcal{N}_M(N)$ and $u \notin N'$, then either $u \notin N$ and $\mathbb{E}_N(u) = 0$ or $u \in N$. If $\mathbb{E}_N(u) = 0$, then ue_Nu^* is orthogonal to e_N since

$$e_Nue_Nu^* = \mathbb{E}_N(u)e_Nu^* = 0.$$

This implies that $ue_Nu^* \leq e_N^\perp$, and since $\text{Tr}(ue_Nu^*) = 1$, $\mathbb{E}_N(u)$ cannot be 0. Hence u must be in N and so N is singular.

(\Rightarrow) Suppose N is singular and $q_0 \in N' \cap \langle M, e_N \rangle$ is any projection with $\text{Tr}(q_0) \leq 1$ and $q_0 \leq e_N^\perp$. Then there exists a partial isometry $v \in \langle M, e_N \rangle$ with $vv^* = q \leq e_N$ and $v^*v = q_0$. Let ϕ denote the pull-down homomorphism from N to N defined in

subsection 2.4 by $e_N v y v^* e_N = \phi(y) e_N$ for all $y \in N$. Then from equation (2.7), we have

$$\phi(y)v = vy. \quad (7.2)$$

Consider the adjoint of equation (7.2). Then

$$y v^* e_N = v^* \phi(y) e_N = v^* e_N \phi(y) \quad (7.3)$$

as e_N commutes with all y in N . By equation (4.2), there is a unique element z in M with

$$z e_N = v^* e_N = v^*, \quad (7.4)$$

and hence, $z\phi(y) = yz$ for all y in N by applying equation (7.3) to \hat{I} . This then implies that zz^* commutes with N . Since $N' \cap M$ is trivial, z is then a scalar multiple of a unitary by the polar decomposition of z^* , and if v is nonzero, z is nonzero. If $v \neq 0$, then with u the unitary in the polar decomposition of z , $u\phi(y) = yu$ for all y in N , and so u normalizes N .

As N is singular, this forces u and hence z to be in N . But then e_N commutes with z , and so

$$\begin{aligned} z\hat{I} &= z e_N \hat{I} = e_N z \hat{I} \\ &= e_N z e_N \hat{I} = e_N v^* \hat{I} \\ &= e_N v^* v v^* \hat{I} = e_N q_0 v^* \hat{I} \\ &= 0 \end{aligned}$$

since q_0 is a subprojection of e_N^\perp . Since \hat{I} is separating for M , $z = 0$. By equation (7.4), $v^* = z e_N$, so v^* and hence v must be zero. Therefore there exists no nonzero subprojection of e_N^\perp in $N' \cap \langle M, e_N \rangle$ equivalent to a subprojection of e_N . Since $\langle M, e_N \rangle$

is a factor, this implies that every nonzero subprojection f of e_N^\perp in $N' \cap \langle M, e_N \rangle$ has

$$\mathrm{Tr}(f) > 1$$

where Tr is the non-normalized trace on $\langle M, e_N \rangle$. □

Using Theorem 7.0.1, we can establish what is perhaps a well-known corollary showing that fixed point algebras of finite groups never fall under the hypotheses of theorem 5.0.3.

Corollary 7.0.2. *Let p be prime and suppose \mathbb{Z}_p acts on the II_1 factor M by outer automorphisms. Then the fixed point algebra $N = M^{\mathbb{Z}_p}$ is regular.*

Proof. If $p = 2$ then the result is trivial by Goldman's theorem [9]. Let M be a II_1 factor admitting a proper, outer \mathbb{Z}_p action α on M where p is any prime larger than 2. The action is implemented in $B(L^2(M))$ by a unitary u through defining

$$u(x\hat{I}) = \alpha(x)\hat{I}.$$

By Theorem 4.2.3, N is either regular or singular since any intermediate subfactor must be the fixed point algebra of a subgroup of \mathbb{Z}_p .

First, observe that $N' \cap \langle M, e_N \rangle$ is abelian as follows: Any element $x \in \langle M, e_N \rangle \cong M \rtimes \mathbb{Z}_p$ (see Proposition A.4.1 in [12] for this isomorphism) may be written as $\sum_{j=0}^{p-1} x_j u^j$ with x_j in M . If x also commutes with N , then x_j commutes with N for each j . Since $N' \cap M = \mathbb{C}$, we get that x is in the von Neumann algebra generated by u . This algebra is abelian and contained in $N' \cap \langle M, e_N \rangle$, so $N' \cap \langle M, e_N \rangle$ is abelian and equal to the von Neumann algebra generated by u , i.e. the group algebra of \mathbb{Z}_p .

Let q_0, q_1, \dots, q_{p-1} be the spectral projections for u , which are a basis for $N' \cap \langle M, e_N \rangle$. As e_N is minimal in $N' \cap \langle M, e_N \rangle$, $e_N = q_m$ for some $0 \leq m \leq p-1$, and

$m = 0$ since for all x in N ,

$$ux\hat{I} = \alpha(x)\hat{I} = x\hat{I} = e_Nx\hat{I}.$$

As

$$\sum_{j=0}^{p-1} \text{Tr}(q_j) = \text{Tr}\left(\sum_{j=0}^{p-1} q_j\right) = \text{Tr}(I) = [M : N],$$

there must exist a $j \neq 1$ with $\text{Tr}(q_j) \leq 1$. But this contradicts Theorem 7.0.1 as q_j is necessarily a subprojection of e_N^\perp . We conclude that N cannot be singular in M and so must be regular in M . \square

If $G \neq \mathbb{Z}_p$ is any other finite group with $|G| > 3$, then G admits a proper, nontrivial subgroup H . Then the projection onto $L^2(M^H)$ is in $M^G \cap \langle M, e_{M_G} \rangle$. This shows that $M^G \cap \langle M, e_{M_G} \rangle$ cannot be 2-dimensional and thus cannot satisfy the hypotheses of Theorem 5.0.3 (In fact, $N' \cap \langle M, e_N \rangle \cong \mathbb{C}([G])$).

Observe that if M , N , and u are as in Example 6.4.3, then $u^*e_Nu \in N' \cap \langle M, e_N \rangle$. Since $\mathbb{E}_N(u) = 0$, we must have $u^*e_Nu < e_N^\perp$, and $\text{Tr}(u^*e_Nu) = 1$. Adding weight to the notion that one-sided normalizers replace normalizers in the infinite index setting, we obtain

Theorem 7.0.3. *If $[M : N] = \infty$, then N contains its semigroup of one-sided normalizers if and only if every nonzero projection in $q \in N' \cap \langle M, e_N \rangle$ subordinate to e_N^\perp has $\text{Tr}(q) > 1$.*

Proof. (\Leftarrow) is identical to the finite index case. For the other direction, suppose N contains its one-sided normalizers and q, q_0 , and v are as in Theorem 7.0.1. We may then define ϕ as before, and we still have equation (7.2),

$$\phi(x)v = vx$$

for all x in N . However, we can no longer use equation (4.2) so cavalierly. In its

absence, we employ Lemmas (9.4.2), (9.4.38) and (9.4.39) (at the time of this writing) in the forthcoming book by Sinclair and Smith [29]. Assuming $v \neq 0$, the first lemma allows us to find a nonzero partial isometry $w \in M$ with $xw = w\phi(x)$. As in the proof of Theorem 7.0.1, we can find a unitary $u \in M$ with $\phi(x)u = ux$, and since N contains its one-sided normalizers, $u \in N$. Therefore, $uxu^* = \phi(x)$ and it follows from the definition of ϕ that

$$uxu^*e_N = \phi(x)e_N = vxv^*e_N.$$

Setting $x = I$ gives us that $e_N = vv^*$ and so

$$\|v^*(x\hat{I})\|_2 = \|x\|_2 \quad (7.5)$$

for all $x \in N$.

We then use the latter two lemmas to obtain a sequence $v_n \in M$ with

$$xv_n = v_n\phi(x) \quad (7.6)$$

for all $x \in N$ and

$$\lim_{n \rightarrow \infty} \|v_n\hat{I} - v^*\hat{I}\|_2 = 0. \quad (7.7)$$

If there exists a subsequence $\{n_i\}_{i=1}^\infty$ with $v_{n_i} = 0$ for all i , then $v^*\hat{I} = 0$, which contradicts equation (7.5). We may then assume $v_n \neq 0$ for all n . As $e_N^\perp v^* = v^*$, by multiplying v_n on the left by e_N^\perp we may assume further that $\mathbb{E}_N(v_n) = 0$. Again using the proof of Theorem 7.0.1, each v_n is a nonzero scalar multiple of a unitary $u_n \in M$ with $xu_n = u_n\phi(x)$. Since N contains its one-sided normalizers, u_n and hence v_n are in N . But then $\mathbb{E}_N(v_n) = 0$ implies $v_n = 0$ for all n which is again a contradiction. Since this stems from assuming v nonzero, we are forced to conclude that $v = 0$. Therefore, there are no nonzero subprojections q_0 of e_N^\perp with $\text{Tr}(q_0) \leq 1$. \square

8. CONCLUSION

The developments in Sections 6 and 7 both form the basis of and point the way to the continued study of normalizers in II_1 factors. It seems possible that when $N = P \rtimes S_2$ and $M = P \rtimes S_3$, brute calculation could decide whether N is strongly singular in M . As yet, we have been unable to do so. Using an analog of Theorem 7.0.1, Roger Smith, Stuart White, and I have been working towards a fuller understanding of the normalizing algebra of subfactors in a II_1 factor. A combination of these methods and those for masas could yield results for normalizing algebras of subalgebras of II_1 factors.

REFERENCES

- [1] D. Bisch, A note on intermediate subfactors, *Pacific J. Math.* 163 (2) (1994) 201–216.
- [2] D. Bisch, R. Nicoara, S. Popa, Continuous families of hyperfinite subfactors with the same standard invariant, preprint, arXiv:math.OA/0604460, To appear *Internat. J. Math.* (2006).
- [3] I. Chifan, On the normalizing algebra of a MASA in a II_1 factor, preprint, arXiv:math.OA/0606225 (2006).
- [4] H. Choda, A Galois correspondence in a von Neumann algebra, *Tôhoku Math. J.* (2) 30 (4) (1978) 491–504.
- [5] A. Connes, Une classification des facteurs de type III, *C. R. Acad. Sci. Paris Sér. A-B* 275 (1972) A523–A525.
- [6] A. Connes, Classification of injective factors. Cases II_1 , II_∞ , III_λ , $\lambda \neq 1$, *Ann. of Math.* (2) 104 (1) (1976) 73–115.
- [7] J. Dixmier, Sous-anneaux abéliens maximaux dans les facteurs de type fini, *Ann. of Math.* (2) 59 (1954) 279–286.
- [8] B. Fuglede, R. V. Kadison, On a conjecture of Murray and von Neumann, *Proc. Nat. Acad. Sci. U. S. A.* 37 (1951) 420–425.
- [9] M. Goldman, On subfactors of factors of type II_1 , *Mich. Math. J.* 7 (1960) 167–172.
- [10] U. Haagerup, Connes’ bicentralizer problem and uniqueness of the injective factor of type III_1 , *Acta Math.* 158 (1-2) (1987) 95–148.

- [11] V. Jones, Sur la conjugaison de sous-facteurs de facteurs de type III₁, C. R. Acad. Sci. Paris Sér. A-B 284 (11) (1977) A597–A598.
- [12] V. Jones, V. S. Sunder, Introduction to subfactors, vol. 234 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1997.
- [13] V. F. R. Jones, Index for subfactors, Invent. Math. 72 (1) (1983) 1–25.
- [14] R. V. Kadison, J. R. Ringrose, Fundamentals of the theory of operator algebras. Vol. I, vol. 15 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1997, Elementary theory, Reprint of the 1983 original.
- [15] R. V. Kadison, J. R. Ringrose, Fundamentals of the theory of operator algebras. Vol. II, vol. 16 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1997, Advanced theory, Corrected reprint of the 1986 original.
- [16] H. Kosaki, S. Yamagami, Irreducible bimodules associated with crossed product algebras, Internat. J. Math. 3 (5) (1992) 661–676.
- [17] W. Krieger, On ergodic flows and the isomorphism of factors, Math. Ann. 223 (1) (1976) 19–70.
- [18] R. Mercer, Convergence of Fourier series in discrete crossed products of von Neumann algebras, Proc. Amer. Math. Soc. 94 (2) (1985) 254–258.
- [19] F. J. Murray, J. Von Neumann, On rings of operators, Ann. of Math. (2) 37 (1) (1936) 116–229.
- [20] M. Pimsner, S. Popa, Entropy and index for subfactors, Ann. Sci. École Norm. Sup. (4) 19 (1) (1986) 57–106.

- [21] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w -rigid groups part I, preprint.
- [22] S. Popa, On a problem of R. V. Kadison on maximal abelian $*$ -subalgebras in factors, *Invent. Math.* 65 (2) (1981/82) 269–281.
- [23] S. Popa, Maximal injective subalgebras in factors associated with free groups, *Adv. in Math.* 50 (1) (1983) 27–48.
- [24] S. Popa, Singular maximal abelian $*$ -subalgebras in continuous von Neumann algebras, *J. Funct. Anal.* 50 (2) (1983) 151–166.
- [25] S. Popa, Classification of subfactors and their endomorphisms, vol. 86 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1995.
- [26] S. Popa, A. M. Sinclair, R. R. Smith, Perturbations of subalgebras of type II_1 factors, *J. Funct. Anal.* 213 (2) (2004) 346–379.
- [27] G. Robertson, Singular MASAS of von Neumann algebras: examples from the geometry of spaces of nonpositive curvature, in: *Travaux mathématiques. Fasc. XIV*, *Trav. Math.*, XIV, Univ. Luxemb., Luxembourg, 2003, pp. 153–165.
- [28] G. Robertson, A. M. Sinclair, R. R. Smith, Strong singularity for subalgebras of finite factors, *Internat. J. Math.* 14 (3) (2003) 235–258.
- [29] A. M. Sinclair, R. R. Smith, Finite von Neumann algebras and masas, unpublished manuscript, Department of Mathematics, Texas A&M University, 2007.
- [30] A. M. Sinclair, R. R. Smith, Strongly singular masas in type II_1 factors, *Geom. Funct. Anal.* 12 (1) (2002) 199–216.

- [31] A. M. Sinclair, R. R. Smith, S. A. White, A. Wiggins, Strong singularity of singular masas, arXiv:math.OA/0601594, To appear Illinois J. Math.
- [32] R. R. Smith, Personal communication, Department of Mathematics, Texas A&M University (2007).
- [33] S. Vaes, Factors of type II_1 without non-trivial finite index subfactors, preprint, arXiv:math.OA/0610231 (2006).
- [34] S. White, A. Wiggins, Semi-regular masas of transfinite length, preprint, arXiv:math.OA/0611615, To appear Internat. J. Math. (2006).

VITA

Name: Alan Daniel Wiggins

Address: c/o Department of Mathematics, 1326 Stevenson Center,
Vanderbilt University, Nashville, TN, 37240

Education: B.A., Mathematics and Philosophy, Indiana University, 1999
Ph.D., Mathematics, Texas A&M University, 2007