ROBUSTNESS ANALYSIS OF LINEAR ESTIMATORS USING DIFFERENTIAL GEOMETRY TECHNIQUES

A Thesis

by

RAJESHWARY TAYADE

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 2003

Major Subject: Major Electrical Engineering

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Approved as to style and content by:

D.R. Halverson
(Chair of Committee)

Krishna Narayanan
(Member)

Raymond Carroll
(Member)

Ali Abur
(Member)

Chanan Singh
(Head of Department)

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ABSTRACT

Robustness Analysis of Linear Estimators
Using Differential Geometry Techniques. (May 2003)

Rajeshwary Tayade, M.S., Texas A&M University

Chair of Advisory Committee: Dr. D.R. Halverson

Robustness of a system has been defined in various ways and a lot of work has been done to model the system robustness , but quantifying or measuring robustness has always been very difficult. In this research we consider a simple system of a linear estimator and then attempt to model the system performance and robustness in a geometrical manner which admits an analysis using the differential geometric concepts of slope and curvature. We try to compare two different types of curvatures, namely the curvature along the maximum slope of a surface and the square-root of the absolute value of sectional curvature of a surface, and observe the values to see if both of them can alternately be used in the process of understanding or measuring system robustness. In this process we have worked on two different examples and taken readings for many points to find if there is any consistency in the two curvatures.

To My Parents

ACKNOWLEDGMENTS

I would like to express my sincere appreciation to Dr.Halverson, whose guidance and support has made this research successful. I would also like to thank my family and friends for their encouragement and support

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INTRODUCTION

The theory of estimation was originally developed within the broad area of statistics and today it is applied in many disciplines of science and engineering such as Control Systems, Digital Signal Processing and Telecommunications. Several examples can be enumerated from varied fields such as estimation of the noise variance of a communication channel, mathematical modelling of a human operator, determination of a satellite orbit and many others. Estimation is needed in general when the dynamical behavior of a system needs to be characterized by statistical distributions and multidimensional variables and is called stochastic estimation. Stochastic estimation consists of assigning a value (possibly random) called the 'Estimate' to some unknown system state or system parameter, based on noise-corrupted observations involving some function of the state or parameter [1]. Thus any function that assigns a value to each observation is an estimator.

A. Estimation Theory Background

Estimators can be broadly classified into three categories based on the assumptions we make at design time. These are 1. Parametric estimators. 2. Non-Parametric estimators. 3. Robust Estimators.

1. Parametric Estimation

In this model, it is assumed that the joint and marginal distributions of the observed signal and the parameter to be estimated are known. Example is a Bayesian estimator,

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which requires the knowledge of an a priori probability density function that gives the statistical properties of the parameters or states to be estimated.

2. Non-Parametric Estimation

Few assumptions are made about the statistical properties of the quantities to be estimated. Here we do not know the exact distribution of the unknown, but we may know the generalized family of the distribution. For example, we may know that the distribution is from a family of symmetric distributions and hence could be Gaussian, Laplace, etc. This type of estimator is appropriate when considerable lack of knowledge is present regarding the underlying random process [2]. However, the generality of its application can often result in undistinguished performance.

3. Robust Estimation

A robust estimator has not been formally defined uniquely for a universal context, but is important in practice since it lies somewhere between Parametric and Non-Parametric estimation. In this case, we do not know the exact distribution, but we have an idea of the nominal, that is, we know the value that is most likely to occur. For example if we are studying ocean signals, the noise process consists of many random constituents and its impossible to know their distributions, but we can use the central limit theorem and approximate the distribution to be Gaussian, which is then our nominal. Note that this is an approximation because the appropriate Central Limit theorem requires infinite terms to be present in the summation, while in reality there might be only a finite number. Robust estimators have the advantage of being able to exploit what knowledge is available through the employment of a nominal, while allowing a controlled degree of uncertainty in the knowledge.

Estimation can also be classified as Parameter estimation and Estimation of

Random Variables. In parameter estimation, we try to determine the value of one or more parameters such as the mean or variance while in the later we want to find the value that a random variable will assume. In this research we have focussed on estimation of random variables to develop our results.

B. Estimator Performance

Every estimator is associated with a corresponding cost-function that in turn is a function of the error between the estimate and the true value of the unknown [1]. The objective of the estimator is to reduce this cost. A list of commonly used criteria or cost functions used for estimator design is given below:

1. Minimization of Average Cost: For example, the estimate can be designed to minimize the average square error or the average absolute error.
2. Minimization of Maximum Absolute Error
3. Selection of the mode of the *a posteriori* probability density function.
4. Selection of the Median of the *a posteriori* probability density function.
5. Conditional Maximum Likelihood, i.e. given that the conditional probability function $p(y|x)$ is known, find x such that $p(y_0|x)$ is maximum for a particular observation y_0.

Thus we can have estimators that minimize the average cost, or minimizes the maximum cost.

C. Linear Estimator of a Random Variable

A Linear estimator is one in which the estimator of a random variable is a linear function of the observed random variables. Linear estimators are generally preferred in many applications because of their simple design. One of the most commonly used linear filter is the Kalman filter. In this research we concentrate on the most simple linear estimator that is used to estimate a random variable Y using information obtained by observing a realization of the random variable X,

$$\hat{Y} = KX \tag{1.1}$$

where X and Y are zero mean, real random variables and K is some constant. The error criteria or cost function is the Mean Square Error,

$$\begin{aligned} MSE &= E(Y-\hat{Y})^2 \\ &= E(Y^2) - 2KE(XY) + K^2E(X^2) \end{aligned} \tag{1.2}$$

Optimal estimator is obtained by choosing a K that minimizes the MSE. Thus differentiating 1.2 w.r.t. K (or by using the Projection Theorem), we obtain,

$$K = \frac{E(XY)}{E(X^2)} \tag{1.3}$$

These are nothing but the variance and covariance of the random variables X and Y. Thus the design of the optimal estimator depends on the covariance matrix [2],

$$C = \begin{pmatrix} a^2 & c \\ c & b^2 \end{pmatrix} \tag{1.4}$$

where $a^2 = E(X^2);\ b^2 = E(Y^2);\ c = E(XY)$

D. Definition of Performance Surface and Robustness

The estimator defined above can be viewed as a parametric estimator since for designing an optimal estimator, we need to know the exact covariance matrix values of the random variables X and Y. This is however a highly optimistic design and in most cases it is difficult to know the exact values. In such a design situation we need to use a robust estimator as described in the previous section. Thus we have to decide on a nominal value of the covariance matrix that can be used. The actual values of the elements of the covariance matrix may be different from the nominal and hence the estimator may no longer remain optimal. We need to know how much the system performance changes as the values of (a, b, c) change.

Here we define robustness as the sensitivity of the system performance to the change in design parameters (covariance matrix in this case). Loosely, the system (linear estimator) is robust if the performance does not change much with the values of the covariance matrix elements. Performance of the system is inversely proportional to the cost function, and since the MSE directly gives us the performance we use equation 1.2 and obtain the performance function $P(a, b, c)$. In general if the parameters a, b and c are assumed to be independently varying variables, P will be a solid embedded in 3-dimensional space. This would admit the application of the techniques of [3], which apply to variations from a nominal in affine space. But in many situations, it is possible that we have knowledge of some functional relation between the elements of the covariance matrix instead of simple affine space. We would like to make use of this extra information in the analysis of the system performance and robustness. For example we may not know the exact values of the variances, but we may know the value of the correlation coefficient $\lambda = \frac{E(XY)}{\sqrt{E(X^2)E(Y^2)}}$ This gives us the relation

$$c = \lambda ab \tag{1.5}$$

This parameter surface thus formed is shown in Figure 1.

The information given by the parameter equation can be used along with the

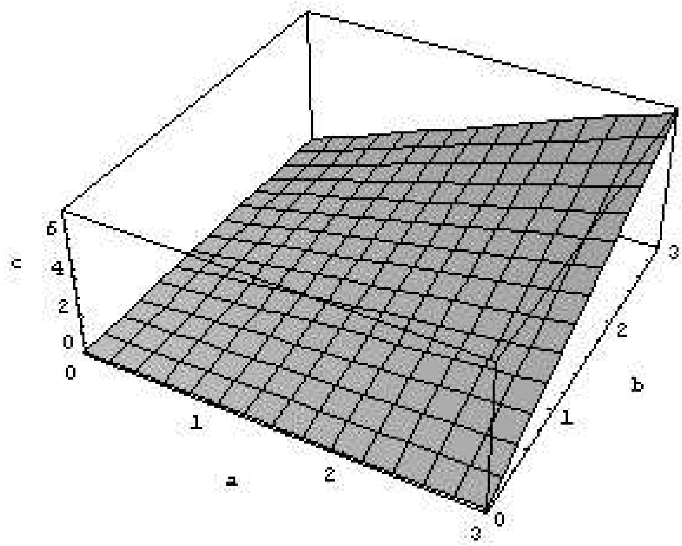

Fig. 1. Parameter Surface

performance function and we can reduce P to a function of only two variables

$$P(a, b) = a^2 - 2Kc(a, b) + K^2a^2 \tag{1.6}$$

Thus instead of having a 3-dimensional solid, we have a 2-dimensional surface embedded in 3-dimensional space. The height of the surface above the $a - b$ plane at any point gives the MSE for that point. Note that here we are writing c as a function of a and b explicitly, and our results depend on this assignment of independent and dependent variables. The choice seems reasonable since we are more likely to know something about a and b because they involve first order statistics. The less understood c is then allowed to depend on a and b. But this assignment is not unique.

In the above relation, the value K is fixed at design time based on the nominal values of the parameters a, b and c and then we plot the performance surface as the parameters deflect away from the nominal. This means that if the performance surface is almost flat, then the MSE does not vary much with a and b, and thus the system is robust. On the other hand if we have a performance surface that looks like an inverted mountain, then the points near the peak are highly sensitive to the parameters a and b, and a small change in the parameters will cause a drastic change in performance, making the system highly un-robust. The simplest way to check the performance change at any point is to measure the slope of the tangent to the performance surface at that point. Robust estimation and detection has been addressed to in [3]-[5]. Also, similar system modelling using differential geometry concepts has been done in [6] and slope was the primary tool used to quantify system robustness. The concept of employing manifold curvature to admit measurement of system robustness was introduced in [7].

E. Motivation to Analyze Curvature

Consider the parametric representation of a space curve (i.e. a curve that does not lie in a plane) given by

$$x_1 = x_1(t)\ ; x_2 = x_2(t);\ x_3 = x_3(t) \tag{1.7}$$

Where (x_1, x_2, x_3) is any point on the curve and each point is a real function of the variable t. If the functions are analytic in some interval T, then they have continuous derivatives of all orders in the interval T. Let x represent any point defined by the triplet (x_1, x_2, x_3), then the value of the function at $t + \delta$ can be determined using the Taylor series expansion. When we consider slope, we use only the first two terms of the series and we can get a more closer approximation by using the second derivative also.

In this research we try to analyze this second order derivative which directly involves the curvature at that point. A simple example that motivates this study is shown in Figure 2. At the peak point, the slope is almost zero, but the curvature is

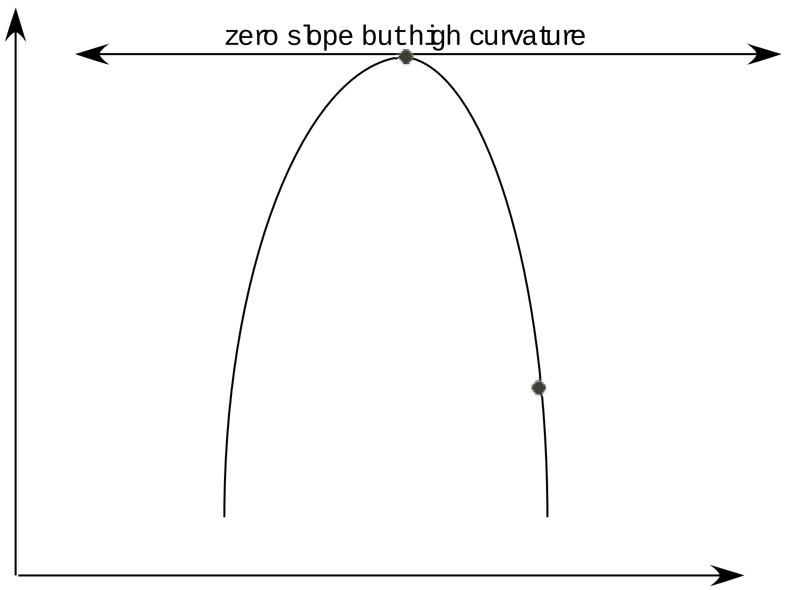

Fig. 2. Zero Slope, High Curvature

not, and hence if we consider just the slope in the local region, the system may look robust which is misleading. Another example, shown in Figure 3 is when the slope is high, while the actual change in system performance is not so high. In this case, the slope of the tangent is almost close to -1, which might imply a substantial change in the performance, but the actual performance does not vary much.

Regarding slope, any variation of a or b in a particular direction may change the performance; as can be approximated by the directional derivative of the performance surface at that point. In order to consider the worst case, we take the maximum directional derivative by searching all directions. Note that this approximation is valid if the step is infinitesimal, but we need to take into account the higher order derivatives if the step size is larger, thus motivating the use of curvature.

In the past various areas in mathematics such as vector algebra, matrix algebra, quadratic forms, probability theory and others have been used as tools for modelling

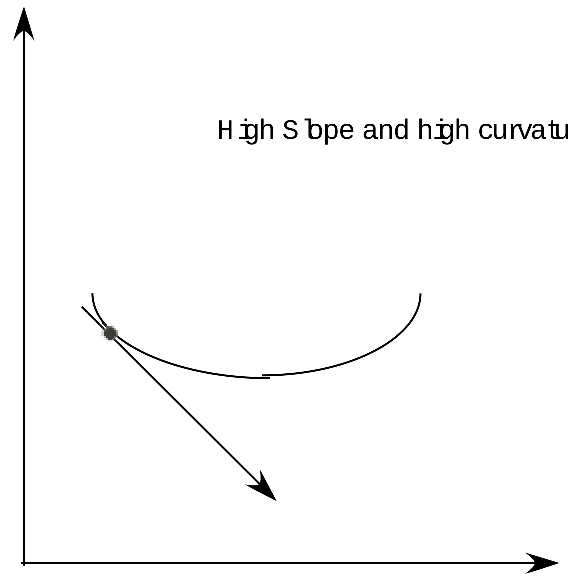

Fig. 3. High Slope, Low Curvature

estimators and better understanding of the theory. In our research we employ differential geometry techniques for the analysis of our system.

When we think of surface curvature, the first definition that comes up is the Gaussian curvature. In our case however, curvature is always used only in conjunction with slope in a particular direction, and since here we consider the maximum slope, the curvature along the maximum slope direction also seems to be a natural choice. In fact, because of its relation to the maximum slope, this curvature may be preferred over the Gaussian for worst case analysis. But we can also think of examples where the maximum slope and the curvature along maximum slope might not be the worst case. For example, consider a surface shaped as an egg as shown in Figure 4. If the egg is titled a little so that the maximum slope is along the major axis, the curvature along maximum slope being very small, our performance change in the direction of maximum slope will be seen as very small. But if we happen to step in the direction of the minor axis, which has higher curvature, the actual change in performance will

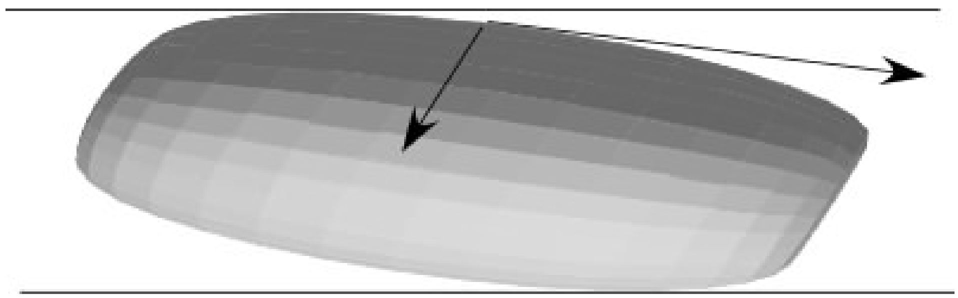

Fig. 4. Surface with Higher Gaussian Curvature than CAMS

be much higher. Thus the curvature along maximum slope may not always give the true worst case. On the other hand, Gaussian curvature, which is the product of the minimum and maximum curvatures at any point, will be much more relevant to the worst case in such cases.

Our objective is thus to observe both the types of curvatures and find if they are consistent, i.e., check if they have values that are close enough so that either of the curvatures could be used interchangeably as a measure of performance change. Our approach will be to take two different examples and calculate both these curvatures for various points. In Chapter II, we give a brief background of differential geometry and define the various terms that we employ in our calculations. In Chapter III, we derive and/or present the expressions used for calculating maximum slope, curvature along maximum slope (CAMS) and the Gaussian curvature. Finally, in Chapter IV we describe the examples that we used, present the observations and results and conclusions.

CHAPTER II

DIFFERENTIAL GEOMETRY BACKGROUND

A curve in space can be defined either as the intersection of two surfaces or by using parametric equations [8]. In this section we use the parametric equations to define various terms but later when we describe the examples, the curves we use are obtained by intersection of two surfaces. The parametric equations of a space curve is given by

$$x_1 = x_1(t);\ x_2 = x_2(t);\ x_3 = x_3(t) \tag{2.1}$$

Here (x_1, x_2, x_3) can be thought as the three axes of a three dimensional space. Thus every point on the curve will be defined by the triplet $(x_1(t), x_2(t), x_3(t))$, where $(x_1(t), x_2(t), x_3(t))$ are three functions of the real variable t. We also assume that these functions are single valued and that the curve be analytic for a particular interval T. By analytic we mean that in this interval, the function will be continuous and also will have continuous derivatives of all orders. Such a curve is called a regular curve.

A curve described above can be imagined as a trace of a particle moving in space. The position of the particle at any time t is given by 2.1. The velocity of the particle at any time t is given by the tangent vector of the curve for that value of t. Before we make further definitions we need to express the curve in terms of the arc length s. The length of the arc as the particle moves from t_0 to t is given by

$$s = \int_{t_0}^{t} |x'(t)| \mathrm{d}t \tag{2.2}$$

Here we have an expression for s as a function of $t : s = f(t)$. To show that s is a regular parameter, we note that this function is also analytic. Differentiating equation 2.2 we have, $\mathrm{d}s = |x'(t)|\mathrm{d}t$; and we know that since $x'(t) \neq 0; \mathrm{d}s/\mathrm{d}t$ does not vanish. If we take the inverse function $t = \phi(s)$, we can conclude that this function is also analytic and its first derivative never vanishes.Thus we can obtain the curve in terms of the arc length s, which will be a regular parameter for the curve. Let us denote the derivative of x with respect to s as x'_s and with respect to t as x'.

$$x'_s = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{x'}{|x'|} \tag{2.3}$$

A. Tangent Vector

According to [8], "The tangent vector at a point P of a curve is the limiting position of the secant joining P to a neighboring point P', when P' approaches P along the curve." Figure 5 shows the tangent vector of a surface and the equation is given by

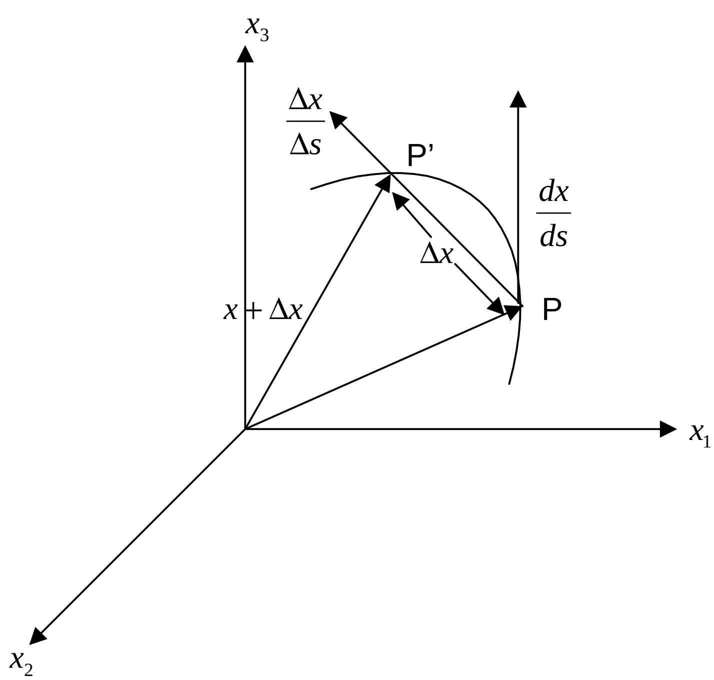

Fig. 5. Tangent Vector for a Curve in Space

$$lim_{\triangle s}\frac{\triangle x}{\triangle s} = \frac{\mathrm{d}x}{\mathrm{d}s} = x'_s \tag{2.4}$$

In terms of parameter t the tangent vector is given by $\frac{x'}{|x'|}$ from 2.3

1. Osculating and Normal Plane

"There is a unique plane through a regular point P of a curve, not a straight line, which has contact of at least the second order with the curve", as stated in [8] It can be also seen (see [9]) as "the plane passing through three consecutive points of the curve, which means the limiting position of a plane through three nearby points of the curve when two of these points approach the third." The *osculating plane* is obtained as the plane passing through P and perpendicular to the vector $x' \times x''$ and the equation of the plane is $(X - x).(x' \times x'')$. For a plane curve, the osculating plane at any point of the curve is the plane of the curve. Another important definition is the *Normal plane* which is the plane through the point P, and perpendicular to the tangent line at P. All the lines in this plane that go through P are normal to P.

2. The Moving Trihedral

For a regular point P on a curve, the principal normal is the normal lying in the osculating plane and *binormal* is the normal perpendicular to the osculating plane. Thus the *principal normal* is perpendicular to the tangent vector at that point and lies in the same plane as the osculating plane, and the *binormal* is perpendicular to the plane containing these two vectors. The tangent, *principal normal* and the *binormal* at any point are thus mutually perpendicular and together they are called the *trihedral* at point P.

The moving trihedron is shown in Figure 6. Let us denote the Trihedron by (α, β, γ) and the equations of these three vectors are given by

$$TangentVector: \ \alpha = \frac{x'}{|x'|} \tag{2.5}$$

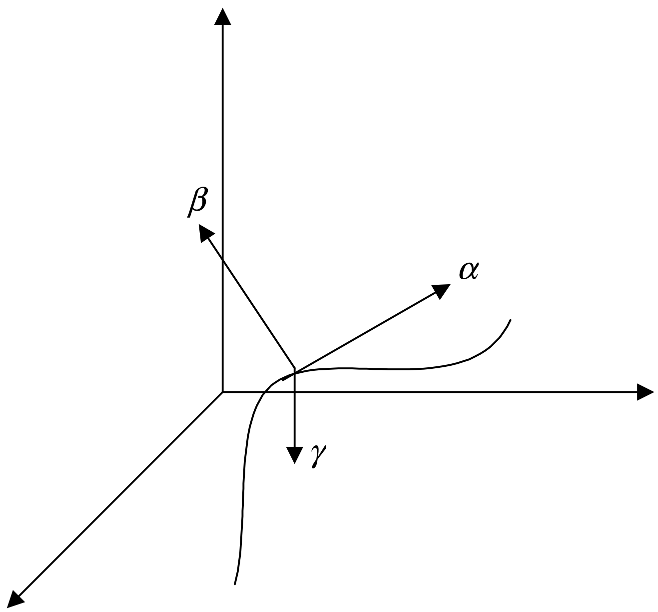

Fig. 6. The Moving Trihedron

$$Principal\ normal:\ \beta = \frac{(x' \times x'') \times x'}{|x'||x'.x''|} \tag{2.6}$$

$$Binormal:\ \gamma = \frac{x' \times x''}{x'.x''} \tag{2.7}$$

In terms of parameter s, the equations of the unit vectors α, β and γ are given by

$$\alpha = x'_s; \tag{2.8}$$

$$\beta = \frac{x''_s}{|x''_s|}; \tag{2.9}$$

$$\gamma = \frac{x'_s \times x''_s}{x''_s} \tag{2.10}$$

B. Curvature

As stated in [8], "There is a unique circle, which has contact of at least the second order with a given curve, not a straight line, at a regular point. It is the circle in the osculating plane whose radius is R and whose center lies on the positive half of the principal normal, in the point $x + R\beta$. This circle is known as the osculating circle, or the circle of curvature, of the curve at that point." If we go back to the description of a curve as the trace of a particle moving in space, then curvature can be thought of as the rate of change of speed of the particle or its acceleration as it moves around in space. Thus curvature is defined as the rate of change of the tangent vector as we proceed along the curve. The vector $K = \frac{\mathrm{d}\alpha}{\mathrm{d}s}$ is called the curvature vector and its length $|K|$ is the value of the curvature at that point [10]. Thus the expression for curvature is

$$K = |x''_s| \tag{2.11}$$

Curvature is also defined as the inverse of the radius of curvature R of the osculating circle. In terms of the parameter t, the equation of curvature is obtained as

$$K = \frac{1}{R} = \frac{[(x'x'').(x'x'')]^{1/2}}{(x'.x')^{3/2}} \tag{2.12}$$

CHAPTER III

DERIVATION OF PARAMETERS USED FOR ROBUSTNESS ANALYSIS

In this section we define and derive the expressions for the parameters that we are employing in our analysis namely maximum slope , Curvature Along Maximum Slope (CAMS) and Sectional Curvature.

A. Maximum Slope

Directional derivative of a function $f(u_1, u_2)$ in the direction of a unit vector (v_1, v_2) is given by

$$D_v = (\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}).(v_1, v_2) \tag{3.1}$$
$$= v_1\frac{\partial f}{\partial u_1} + v_2\frac{\partial f}{\partial u_2} \tag{3.2}$$

To maximize D_v w.r.t (v_1, v_2) Consider the Lagrange Multiplier:

$$J = v_1\frac{\partial f}{\partial u_1} + v_2\frac{\partial f}{\partial u_2} - \lambda(v_1^2 + v_2^2) \tag{3.3}$$
$$\frac{\partial J}{\partial v_1} = \frac{\partial f}{\partial u_1} - 2\lambda v_1 = 0 \tag{3.4}$$
$$\frac{\partial J}{\partial v_2} = \frac{\partial f}{\partial u_2} - 2\lambda v_2 = 0 \tag{3.5}$$

This gives

$$v_1 = \frac{-\partial f/\partial u_1}{2\lambda} \tag{3.6}$$
$$v_2 = \frac{-\partial f/\partial u_2}{2\lambda} \tag{3.7}$$

this is the direction vector that will have the maximum slope. Thus

$$D_{v_{max}} = -\frac{(\partial f/\partial u_1)^2}{2\lambda} - \frac{(\partial f/\partial u_2)^2}{2\lambda} \tag{3.8}$$

Also since we need to consider a unit vector, $v_1^2 + v_2^2 = 1$

hence,

$$\frac{(\partial f/\partial u_1)^2 + (\partial f/\partial u_2)^2}{4\lambda^2} = 1 \tag{3.9}$$

$$2\lambda = \pm[(\partial f/\partial u_1)^2 + (\partial f/\partial u_2)^2]^{1/2} \tag{3.10}$$

The maximum directional derivative is given by:

$$D_{v_{max}} = \frac{(\partial f/\partial u_1)^2 + (\partial f/\partial u_2)^2}{\sqrt{(\partial f/\partial u_1)^2 + (\partial f/\partial u_2)^2}} \tag{3.11}$$

$$= \sqrt{(\partial f/\partial u_1)^2 + (\partial f/\partial u_2)^2} \tag{3.12}$$

B. Curvature Along Maximum Slope

Consider a surface given by $u_3 = f(u_1, u_2)$.

From 3.6 and 3.7, we know that the direction of maximum slope is given by

$$(v_1, v_2) = -\frac{1}{\lambda}(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}) \tag{3.13}$$

We take a plane through the nominal $(\tilde{u_1}, \tilde{u_2}, f(\tilde{u_1}, \tilde{u_2}))$, perpendicular to the (u_1, u_2) plane and cutting the surface in a curve along the maximum slope, as shown in Figure 7. Equation of this plane is

$$u_1 = \tilde{u_1} + t\frac{\partial f}{\partial u_1} \quad u_2 = \tilde{u_2} + t\frac{\partial f}{\partial u_2} \quad u_3 = arbitrary \tag{3.14}$$

where t varies over the reals, and the equation of the curve obtained by cutting the surface with 3.14 is given by

$$\alpha(t) = (u_1(t), u_2(t), f(u_1(t), u_2(t))) \tag{3.15}$$

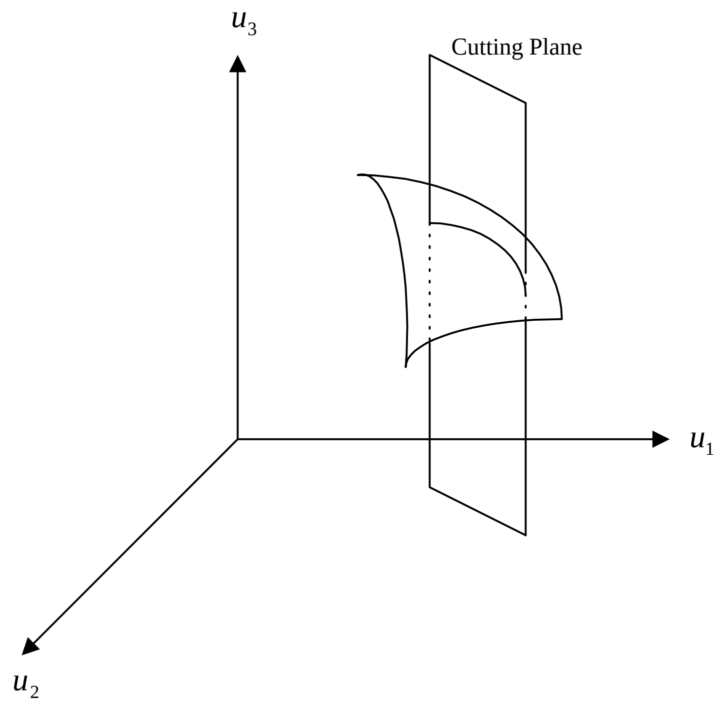

Fig. 7. Curvature Along Maximum Slope

where

$$u_1(t) = \tilde{u_1} + t\frac{\partial f}{\partial u_1}|_{\tilde{u_1},\tilde{u_2}} \quad u_2(t) = \tilde{u_2} + t\frac{\partial f}{\partial u_2}|_{\tilde{u_1},\tilde{u_2}} \tag{3.16}$$

Therefore curvature at the nominal in the direction of maximum slope is

$$K = [(\alpha'\alpha')(\alpha''\alpha'') - (\alpha'\alpha'')^2]^{1/2}/(\alpha'\alpha')^{3/2}|_{t=0} \tag{3.17}$$

Figure 8 shows an example, in which the surface is given by

$$f(u_1, u_2) = (u_1 - 1)^2 + 2(u_2 - 1)^2 \;\; where \;\; u_1 = u_1(t) \;\; and \;\; u_2 = u_2(t) \tag{3.18}$$

The surface is a simple ellipsoid with, for a nominal of $\tilde{u_1} = 0$, $\tilde{u_2} = 1$, the maximum slope $= 2$, and the direction of maximum slope is given by a vector tangent to the surface at the nominal given by $v_1 = 1$ and $v_2 = 0$. The vertical cutting plane in the direction of maximum slope and the curve obtained in that direction are shown in

Figure 9 and Figure10 respectively.

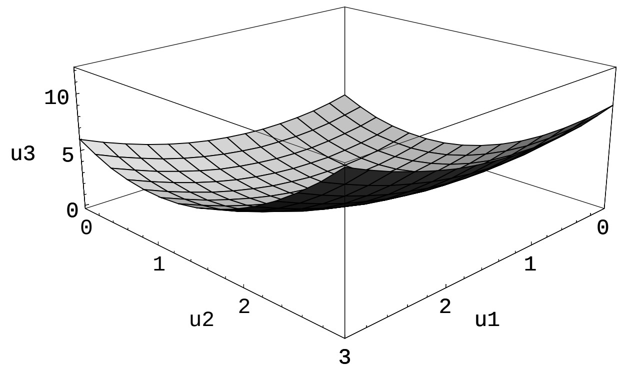

Fig. 8. Ellipsoid Surface

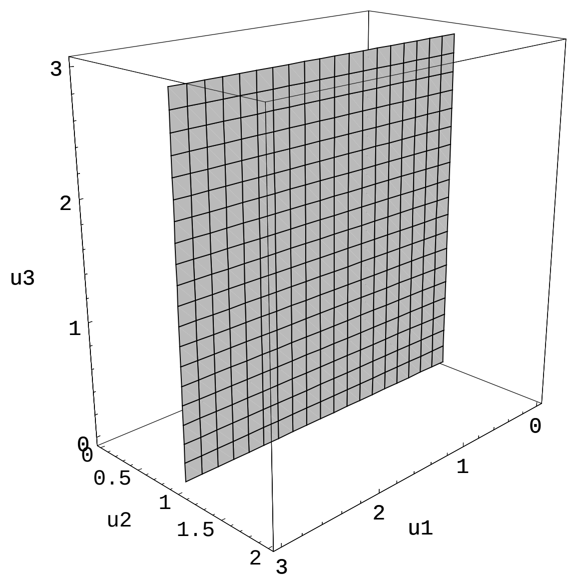

Fig. 9. Cutting Plane in the Direction of Maximum Slope

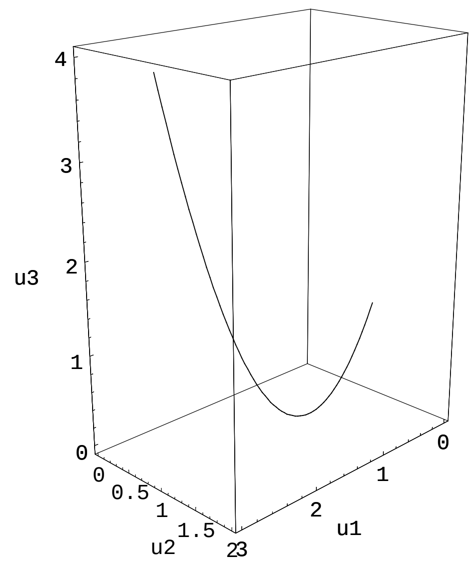

Fig. 10. Curve Obtained in the Direction of Maximum Slope

C. Gaussian Curvature

German mathematician Gauss has developed the theory of surfaces and curves on the surfaces with a totally different approach. The theory described earlier involves a major contribution of Monge, who interpreted a surface as the envelope of a solid body, and hence the properties of a surface in his description depend on its relation to the surrounding Euclidean space [9]. Gauss, on the other hand saw a surface as a thin sheet which exists on its own and need not be attached to any three-dimensional body. Such a sheet that is independent of the enclosing space is called a manifold. Thus a manifold is a two-dimensional entity and in space it does not have any orientation. According to [9], "A two-dimensional being living on this surface would be unaware of the space of which the surface may be a part of"; for example, an ant crawling on a large sphere would not know that it is on a spherical surface, but it can find the path of the shortest distance between any two points measured along the surface of the sphere, or the angle between any two directions on the surface. These properties are said to be *intrinsic* to the surface. The concept of Gaussian curvature was developed from this theory and is called the *bending invariant*. By this we mean that the value of the Gaussian curvature remains unchanged under all deformations that do not involve stretching, straining or tearing. An idea of such bending is obtained by deforming the above mentioned sheet without changing its elastic properties. This essentially means that the measurements we make should be independent of the co-ordinate system we are assuming the surface to be in. For example, "a curve drawn on this sheet conserves its length even if its shape is changed as a result of the bending" [9]. All these properties of Gaussian curvature makes it a very attractive parameter to measure. One important reason for considering Gaussian curvature is that as we move to higher dimension surfaces, i.e., when there are more than two random variables involved, measuring Curvature Along Maximum Slope will become increasingly difficult, while there are pre-derived formulae available for measuring Gaussian curvature.

To understand the concept of Gaussian curvature intuitively, we can imagine a surface that is embedded in Euclidean space. For such a surface, there exists a normal at any point. We cut the surface by a plane normal to point P, for example, and get a curve that will have some value of curvature at P. Figure 11 shows an example of a normal cutting plane for a surface. The cutting plane is then rotated about P to get a set of curvatures. The Gaussian curvature at P is then the product of the minimum and maximum curvatures at P. Thus Gaussian curvature can be thought as curvature of a surface and not of a curve.

Here we present the formula that we used to calculate Gaussian curvature along with the explanation of the different terms used. The Gaussian curvature at any point P on a surface M defined by $\vec{X}(u_1, u_2) = (X_1(u_1, u_2), X_2(u_1, u_2), X_3(u_1, u_2))$ is

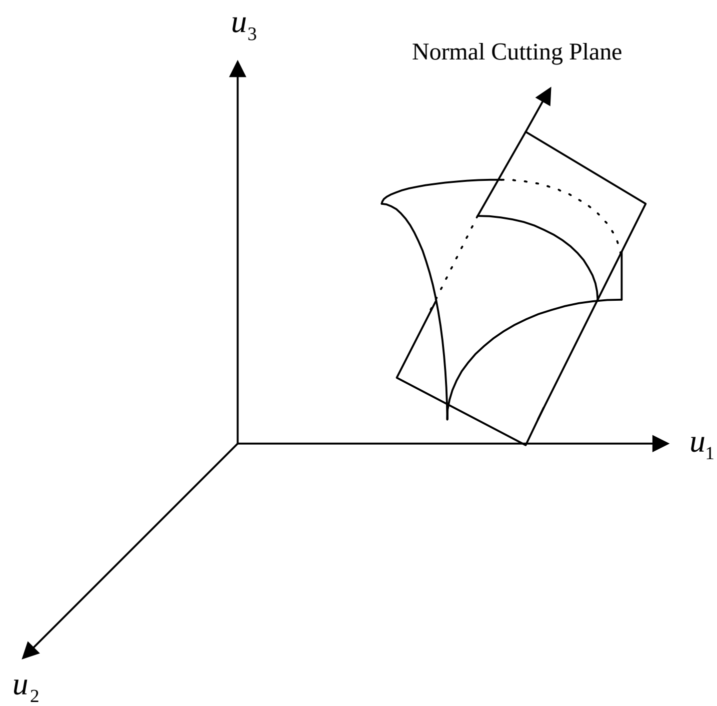

Fig. 11. Curve Obtained by Taking Normal Sections

given by

$$K = \frac{L}{g},\ where\ L = det(L_{ij})\ and\ g = det(g_{ij}), \tag{3.19}$$

where we now define these two terms used in the expression for Gaussian Curvature. A curve $\alpha(t)$ on the surface can be written as $\vec{X}(u_1(t), u_2(t))$. Also, the unit normal vector to M is then given by

$$\vec{u} = \frac{\vec{X_1} \times \vec{X_2}}{\|\vec{X_1} \times \vec{X_2}\|} \tag{3.20}$$

where $\vec{X_i} = \frac{\partial \vec{X}}{\partial u_i}$. Also define the continuous second partial as $X_{ij} = \frac{\partial^2 \vec{X}}{\partial u_1 \partial u_2}$. We define L_{ij} as the projection of $\vec{X_{ij}}$ in the direction of $\vec{u}$

$$L_{ij} = \langle \vec{X_{ij}}, \vec{u} \rangle \tag{3.21}$$

$$= \langle \vec{X_{ij}}, \frac{\vec{X_1}, \vec{X_2}}{\|\vec{X_1} \times \vec{X_2}\|} \rangle \tag{3.22}$$

This is called the second fundamental form of Gauss. The matrix g_{ij} is defined as

$$g_{ij} = \langle \vec{X}_i, \vec{X}_j \rangle \tag{3.23}$$

and this is called the first fundamental form of Gauss.

D. Sectional Curvature

Gaussian curvature described above is defined for two-dimensional surfaces; for surfaces of higher dimension, we have sectional curvature, which is nothing but the Gaussian curvature of a two-dimensional section of the original multiple dimension surface. The practical computation for sectional curvature of an n-dimensional surface would be quite involved, and in our research, since we have used only two dimensional surfaces, we present the basic expressions that were used to calculate Gaussian curvature using the generalized formula. Let u_1, u_2, u_3 define the coordinate axes of a three dimensional space. In general we have the surface $u_3 = f(u_1, u_2)$. Define $x_1 = u_1$, $x_2 = u_2$ and $u_3 = f(x_1, x_2)$ The tangent vectors are then obtained as

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial u_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial}{\partial u_3}\frac{\partial u_3}{\partial x_1} \tag{3.24}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial}{\partial u_2}\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial u_3}\frac{\partial u_3}{\partial x_2}, \tag{3.25}$$

where the $\frac{\partial}{\partial u_i}$ are the tangent vectors for R^3 The elements of the g matrix defined above are then obtained as

$$g_{11} = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right\rangle \; g_{12} = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle \tag{3.26}$$

$$g_{21} = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right\rangle \; g_{22} = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2} \right\rangle \tag{3.27}$$

And these may be calculated from 3.24 and 3.25 and via the ordinary inner product on R^3.

Let us denote the inverse of matrix g_{ij} as g^{ij}. We have the formula for the sectional curvature of a two-dimensional surface as

$$R_{sec} = -\frac{R_{1212}}{g_{12}^2 - g_{11}g_{22}} \tag{3.28}$$

Here

$$R_{ijkl} = \sum_{p=1}^{2} g_{ip} R^p_{jkl} \tag{3.29}$$

and R_{jkh} are the curvature tensors defined as

$$R^i_{jkh} = \frac{\partial \Gamma^i_{jk}}{\partial x_k} - \frac{\partial \Gamma^i_{jk}}{\partial x_h} + \sum_{p=1}^{2} (\Gamma^i_{pk}\Gamma^p_{jk} - \Gamma^i_{ph}\Gamma^p_{jk}) \tag{3.30}$$

Note that R^i_{jhk} is skew in h,k which implies $R^1_{jhh} = 0 \ \forall h$ and $R^i_{jkh} = -R^i_{jhk}$. The term Γ^h_{ji} used in the above expression is called the Christofel Symbol and is obtained using the formula given below:

$$\Gamma^h_{ji} = \frac{1}{2} g_{h1} \left[\frac{\partial g_{j1}}{\partial x_i} + \frac{\partial g_{i1}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_1} \right] + \frac{1}{2} g_{h2} \left[\frac{\partial g_{j2}}{\partial x_i} + \frac{\partial g_{i2}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_2} \right] \tag{3.31}$$

We remark that the expression for sectional curvature 3.28 can be easily generalized for higher dimensions, as with 3.29-3.31.

CHAPTER IV

RESULTS AND CONCLUSIONS

Here we present two examples involving different parameter relations for which the aforementioned curvatures were calculated. The following procedure is used for the analysis of the surface slope and curvatures with application to the linear estimation of a random variable Y in terms of X:

1. We first decide on a nominal covariance matrix for (X, Y) and then design the linear estimator, $\hat{Y} = KX$, by determining K for the given nominal according to the mean square error (MSE) criterion. We also assume that we have some additional information about the parameters (a, b, c) which compose the covariance matrix (see 1.4) expressed by the parameter surface. Using this relation, then the performance surface can be plotted using the performance function, defined by MSE.

2. The actual system parameters are likely to vary from the nominal by a small amount. Thus we decide on some step size (variation from nominal), and then take a step in the direction of a or b accordingly. The values of maximum slope, Curvature Along Maximum Slope (CAMS), and Gaussian curvature are then calculated for the new system state (point on the performance surface). As mentioned before, the Gaussian curvature is defined for a surface and not for a curve and hence it can be thought as a two-dimensional value. To compare it to a curvature of a curve (CAMS) relevant to our application, we use the square-root of the absolute value of the Gaussian curvature obtained. Also note that since Gaussian curvature is the product of the minimum and maximum curvatures at any point, the square-root will give us the geometric mean of the

extreme curvatures at that point.

3. The choice of step size is important here, since we might not always know of the allowed variation. The maximum allowed deflection from the nominal defines the area 'local' to the nominal. We can define our own 'local' region using various methods, for example, one of the methods is to see the variation in the CAMS as we take steps away from the nominal in all directions. Since for a local region, the CAMS should not vary much, if the ratio of the minimum CAMS to maximum CAMS for a given nominal is observed to be less than 1.5 (for example), then that amount of step size defines the span of the local region. This concept reflects the intended interpretation of 'local' implying restrained variations. In our experiment however, we do not use this method, instead we take three fixed steps of 0.05 each, so that the farthest distance moved from any nominal would be 1.7 units. The exact distribution of the points around the nominal is explained in the further sections.

4. For all the different nominals, the two types of curvatures that we measure are then compared. The procedure is repeated for various nominals so that the union of the local regions covers an important subset of the parameter surface.

A. Example 1

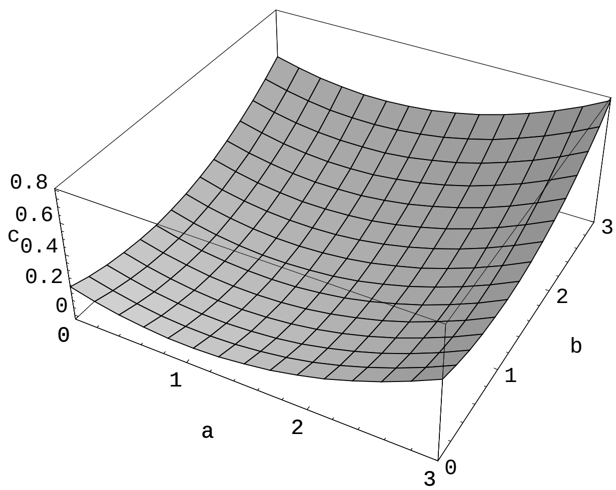

Fig. 12. Parameter Surface Example 1

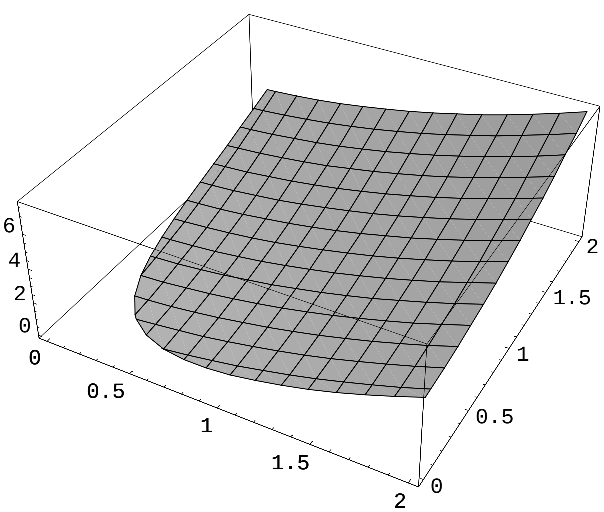

Fig. 13. Performance Surface Example 1

In the first example, the parameters (a, b, c) are related by the relation:

$$c(a, b) = 0.1(a-1)^2 + 0.1(b-1)^2 - 0.01 \tag{4.1}$$

Note here that we choose c to be a function of a and b for convenience but in practice there could be any other relation between the three parameters and our analysis will be independent of it. The parameter surface is shown in Figure 12. If we choose an arbitrary nominal as $(0.4, 1.4)$, then using equation 1.3 we obtain $K = 0.2625$. The performance function $P(a, b)$ is given by 1.6, which plays the role of $f(u_1, u_2)$ in Chapter III. The performance surface can then be plotted and is shown in Figure 13. Let us refer to this surface as surface1.

B. Example 2

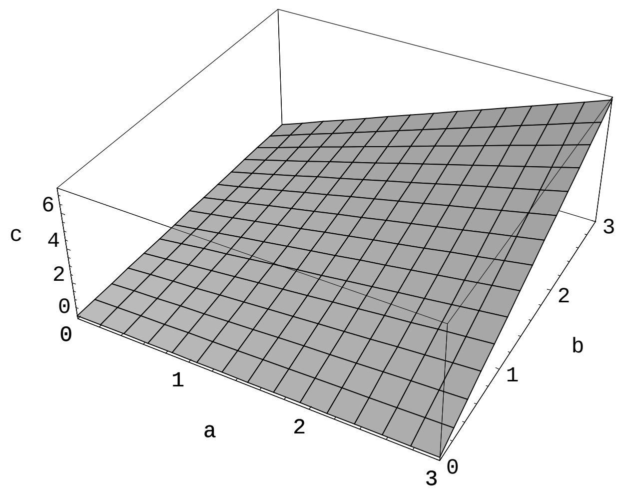

Fig. 14. Parameter Surface Example 2

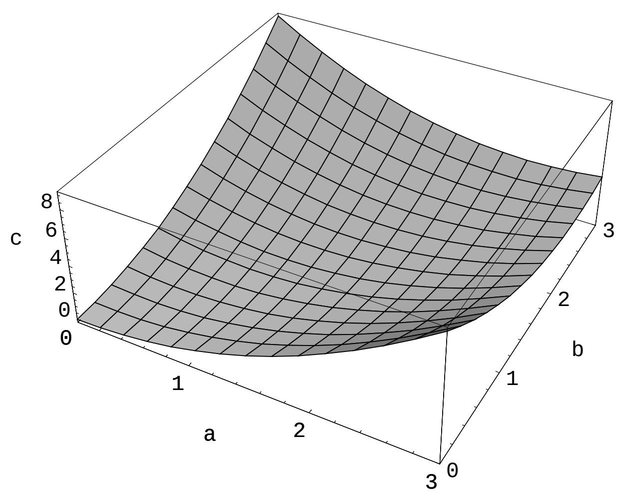

Fig. 15. Performance Surface Example 2

In the second example we assume that a, b, c are related via a known constant, the correlation coefficient λ. The parameter equation is then given by $c = \lambda ab$. The parameter surface for $\lambda = 0.8$ is plotted in Figure 14 and the corresponding performance surface for $K = 1$ is shown in Figure 15. This surface will be referred as surface 2.

C. Results

In this section, we present the readings for different curvature values that were obtained from theses two examples. For each surface, we take five different nominals and the readings are shown in tables below.

1. Deciding on the Nominals

We are going to consider two different examples of parameter surface, and in order to reach some conclusions it is important that we try multiple values for the nominal in each example. Note again that for each nominal value, we get a different performance surface because K is changed according to 1.3 based on the nominal; thus, for each case we have a unique surface. The procedure used for selecting a nominal is that we choose the first value of (a, b) arbitrarily and then check for some constraints to see if it can be an allowable nominal. Some of the constraints that we used are: a) The slopes of the chosen nominal should be low enough so that effect of curvature is not negligible and b) the value of the MSE should be acceptable, (i.e., within a range of practical interest) depending on the application using the estimator. After choosing the nominal, we need to decide on the variation from the nominal, keeping in mind that this analysis is for a local region. As mentioned in the earlier section, the concept of 'local' may vary, since the acceptable variation of the actual parameters from the nominal depends on the particular application. In our analysis, we do not assume any particular criteria for acceptable variation, and simply move away from the nominal in concentric squares that will require a variation of 0.05 units in the parameter values a or b in rectangular coordinates. If we take three steps in any direction, we form three such squares, and eight points on each square, corresponding to eight different directions. There are four points each at a distance of 0.05, 0.07, 0.1, 0.14, 0.15 and 0.21 from the nominal.

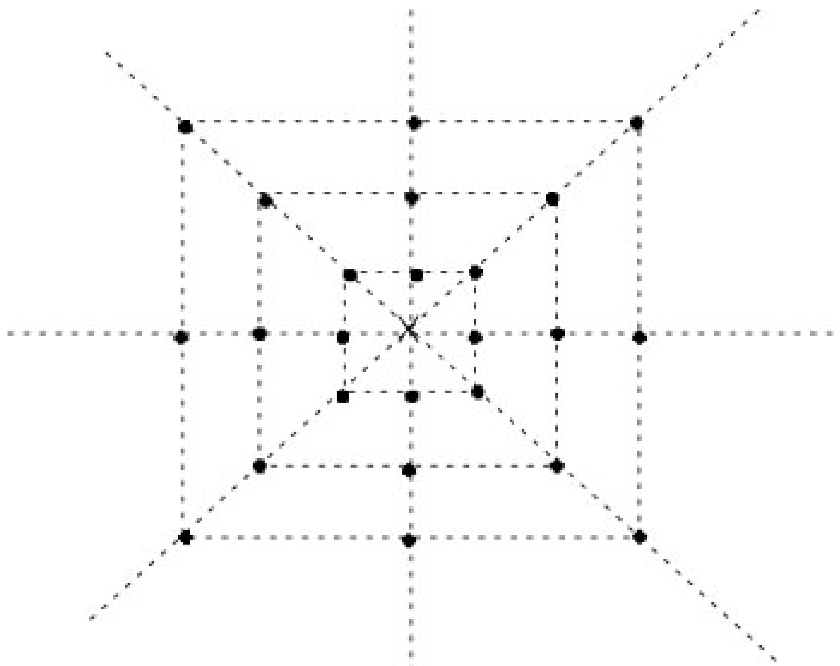

Fig. 16. Choice of Points

Graphical representation of the distribution of points around the nominal is shown in Figure 16. In the following section, we present the values at the points for the maximum slope, the curvature along maximum slope and the square-root of the Gaussian curvature for various values of nominals. Every table is accompanied by a graph showing the variation of CAMS and square-root of Gaussian curvature with distance. To plot these graphs, we group together all the points at the same distance and then take the average of the curvature values at these points. This method is fine for the Gaussian curvature, but might not be the best one for CAMS, since the direction of CAMS could be different for every point in the group. However we can make a assumption that surface does not have much changes within this small distance and hence the direction of maximum slope does not vary a lot. For each nominal there is thus an associated square centered at the nominal and containing points used to evaluate the quantities of interest. We then arrange the squares approximately uniformly over the parameter surface by choosing their nominal center points appropriately. We remark that our choices in the following examples are not unique but do provide a useful sampling of points. The increment of 0.05 and corresponding three squares were found to provide a small enough region to be considered local for our examples while simultaneously large enough so that the number of nominals required to cover the parameter surface of interest was not inordinately large. Tables I through V give the curvature and slope values for different nominals for the first example surface. The variation of the two curvature values w.r.t to distance away from the nominal are shown in Figures 17-21. Tables VI through X and Figures 23 - 26 correspond to the second example surface.

Table I. Curvature and Slope for Surface 1, Nominal 0.4,0.3

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal: 0.4, 0.3 Step = 0.05			
$(0.35, 0.3)$	0.781491	0.791636	0.419543
$(0.35, 0.35)$	0.866879	0.713846	0.385828
$(0.4, 0.25)$	0.7025	0.84906	0.452472
$(0.4, 0.3)$	0.786023	0.778792	0.417701
$(0.4, 0.35)$	0.870968	0.703891	0.384269
$(0.45, 0.25)$	0.707763	0.832313	0.450235
$(0.45, 0.3)$	0.79073	0.765725	0.415793
$(0.45, 0.35)$	0.875218	0.693721	0.382654
Nominal: 0.4, 0.3 Step = 0.1			
$(0.3, 0.3)$	0.777136	0.804225	0.421319
$(0.3, 0.4)$	0.949669	0.645383	0.355319
$(0.4, 0.3)$	0.786023	0.778792	0.417701
$(0.4, 0.4)$	0.956955	0.630134	0.352742
$(0.5, 0.2)$	0.633062	0.862469	0.482428
$(0.5, 0.3)$	0.795609	0.752468	0.413823
$(0.5, 0.4)$	0.964844	0.614114	0.349972
Nominal: 0.4, 0.3 Step = 0.15			
$(0.25, 0.45)$	1.03393	0.578249	0.326616
$(0.4, 0.3)$	0.786023	0.778792	0.417701
$(0.4, 0.45)$	1.04373	0.560725	0.323433
$(0.55, 0.3)$	0.800657	0.739052	0.411791
$(0.55, 0.45)$	1.05479	0.54171	0.319878

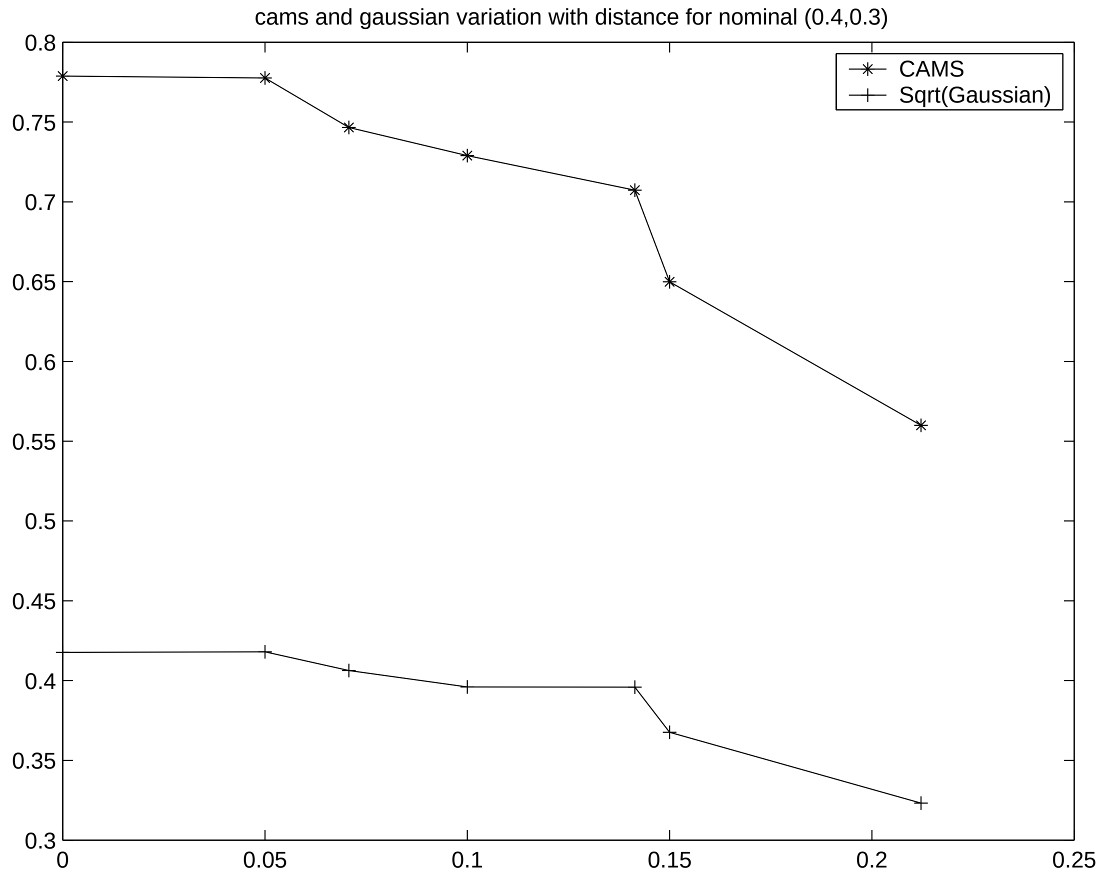

Fig. 17. Surface 1 Curvatures and Slopes for Nominal 0.4,0.3

Table II. Curvature and Slope for Surface 1, Nominal 0.5, 0.2

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal: 0.5, 0.2; Step = 0.05			
$(0.45, 0.2)$	0.525858	1.18439	0.290333
$(0.45, 0.25)$	0.615786	1.08228	0.268721
$(0.5, 0.2)$	0.526981	1.17793	0.290064
$(0.5, 0.25)$	0.616745	1.07767	0.268491
$(0.55, 0.15)$	0.440231	1.23845	0.310451
$(0.55, 0.2)$	0.528126	1.17138	0.28979
$(0.55, 0.25)$	0.617724	1.07299	0.268256
Nominal: 0.5, 0.2; Step = 0.1			
$(0.4, 0.3)$	0.705872	0.973863	0.247366
$(0.5, 0.2)$	0.526981	1.17793	0.290064
$(0.5, 0.3)$	0.707525	0.967239	0.246981
$(0.6, 0.2)$	0.529295	1.16475	0.28951
$(0.6, 0.3)$	0.709251	0.960385	0.246579
Nominal: 0.5, 0.2; Step = 0.15			
$(0.4, 0.3)$	0.796804	0.864912	0.226692
$(0.5, 0.2)$	0.526981	1.17793	0.290064
$(0.5, 0.3)$	0.798975	0.857679	0.226212
$(0.6, 0.2)$	0.530486	1.15804	0.289225
$(0.6, 0.3)$	0.801292	0.850042	0.225702

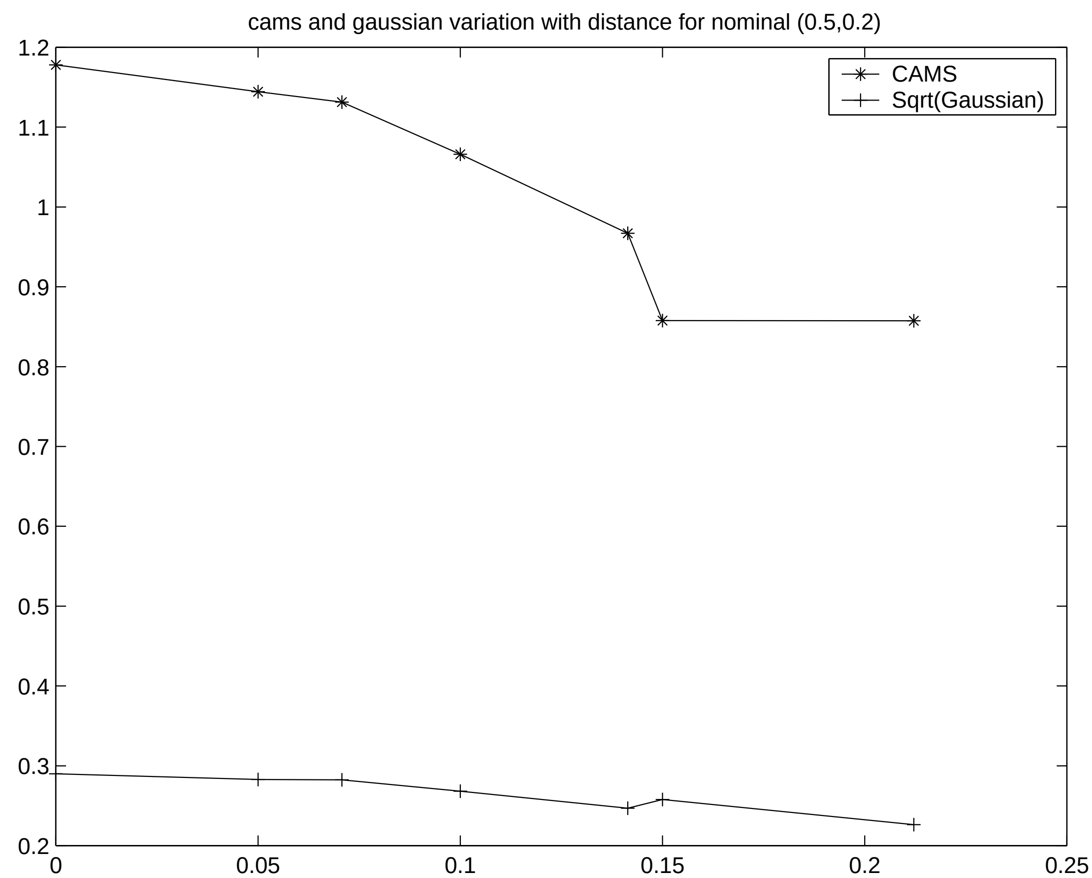

Fig. 18. Surface 1 Curvatures and Slopes for Nominal 0.5,0.2

Table III. Curvature and Slope for Surface 1, Nominal 0.3, 0.4

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal 0.3,0.4; Step = 0.05			
$(0.25, 0.4)$	1.16476	0.416271	0.226692
$(0.25, 0.45)$	1.23705	0.378701	0.290064
$(0.3, 0.35)$	1.12373	0.42958	0.226212
$(0.3, 0.4)$	1.19269	0.393892	0.289225
$(0.3, 0.45)$	1.26337	0.359769	0.225702
$(0.35, 0.35)$	1.15506	0.40392	0.226692
$(0.35, 0.4)$	1.22225	0.371886	0.290064
$(0.35, 0.45)$	1.29132	0.341006	0.226212
Nominal 0.3,0.4; Step = 0.1			
$(0.2, 0.5)$	1.28745	0.359482	0.499099
$(0.3, 0.3)$	1.05686	0.466059	0.62655
$(0.3, 0.4)$	1.19269	0.393892	0.547521
$(0.3, 0.5)$	1.33552	0.327699	0.476493
$(0.4, 0.3)$	1.12486	0.407964	0.585514
$(0.4, 0.4)$	1.25334	0.350448	0.515924
$(0.4, 0.5)$	1.38996	0.29601	0.452381
Nominal 0.3,0.4; Step = 0.15			
$(0.3, 0.4)$	1.19269	0.393892	0.547521
$(0.3, 0.55)$	1.4089	0.297949	0.444345
$(0.45, 0.25)$	1.10268	0.405711	0.598569
$(0.45, 0.4)$	1.28585	0.329731	0.499875
$(0.45, 0.55)$	1.48859	0.257752	0.412441

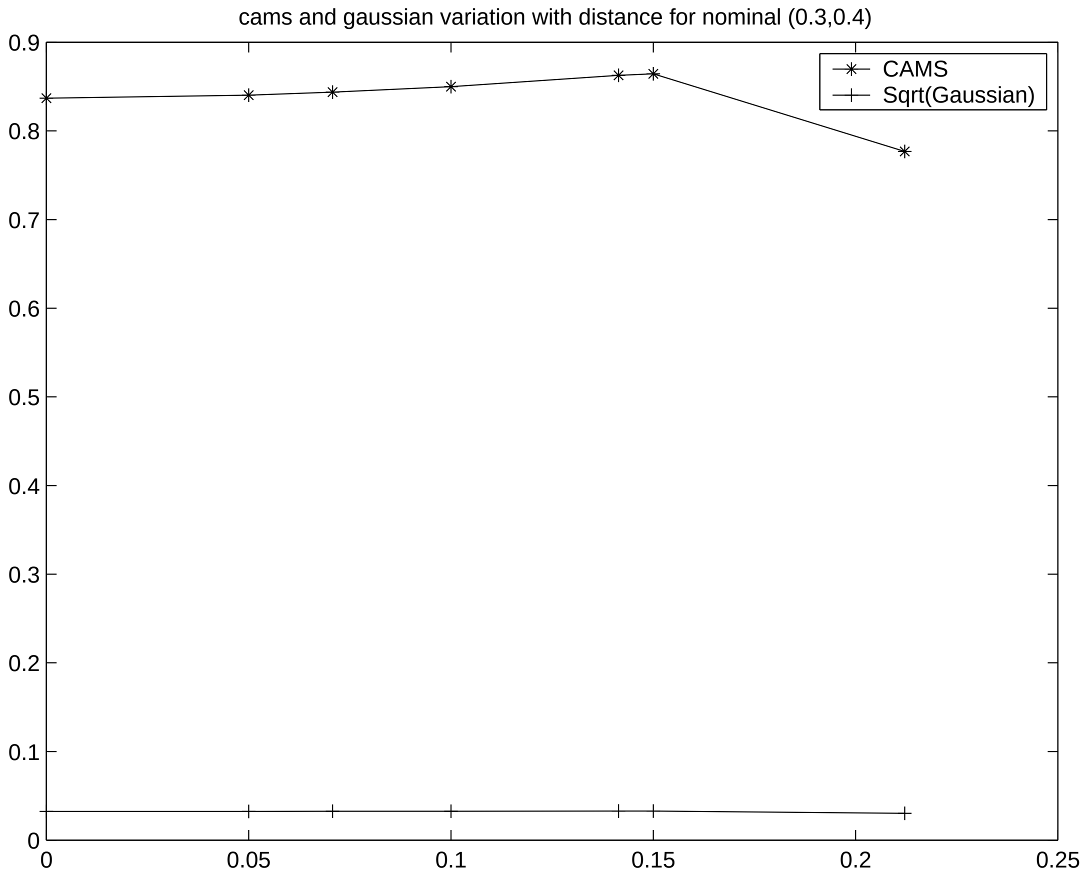

Fig. 19. Surface 1 Curvatures and Slopes for Nominal 0.3,0.4

Table IV. Curvature and Slope for Surface 1, Nominal 0.5, 0.4

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal 0.5,0.4;Step = 0.05			
$(0.45, 0.35)$	0.757528	0.960148	0.0355522
$(0.45, 0.4)$	0.852943	0.837038	0.0323898
$(0.45, 0.45)$	0.94846	0.727229	0.0294559
$(0.5, 0.35)$	0.757537	0.960114	0.0355519
$(0.5, 0.4)$	0.852951	0.837013	0.0323896
$(0.5, 0.45)$	0.948468	0.727211	0.0294557
$(0.55, 0.35)$	0.757545	0.960079	0.0355516
$(0.55, 0.4)$	0.852959	0.836988	0.0323893
$(0.55, 0.45)$	0.948475	0.727192	0.0294555
Nominal 0.5,0.4;Step = 0.1			
$(0.4, 0.3)$	0.662248	1.0947	0.0388954
$(0.4, 0.4)$	0.852935	0.837063	0.0323901
$(0.4, 0.5)$	1.04405	0.63097	0.0267718
$(0.5, 0.3)$	0.662268	1.0946	0.0388947
$(0.5, 0.4)$	0.852951	0.837013	0.0323896
$(0.5, 0.5)$	1.04406	0.630942	0.0267714
$(0.6, 0.3)$	0.662288	1.09451	0.0388939
$(0.6, 0.4)$	0.852967	0.836962	0.0323891
$(0.6, 0.5)$	1.04407	0.630914	0.0267711
Nominal 0.5,0.4;Step = 0.15			
$(0.35, 0.4)$	0.852927	0.837089	0.0323903

Table IV. Continued.

Points	Maximum Slope	CAMS	Sqrt Sectional
$(0.35, 0.55)$	1.13969	0.547522	0.0243395
$(0.5, 0.25)$	0.567219	1.23584	0.0423335
$(0.5, 0.4)$	0.852951	0.837013	0.0323896
$(0.5, 0.55)$	1.1397	0.547491	0.024339
$(0.65, 0.25)$	0.567255	1.23563	0.0423322
$(0.65, 0.4)$	0.852975	0.836937	0.0323888
$(0.65, 0.55)$	1.13972	0.54746	0.0243387

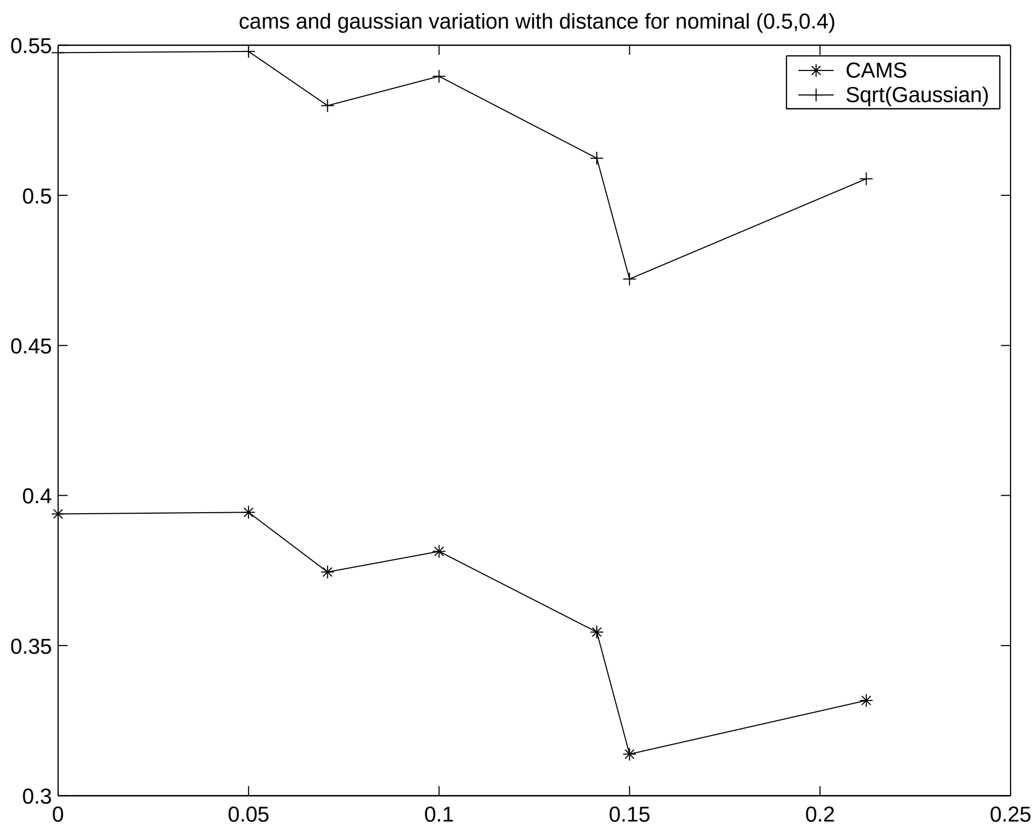

Fig. 20. Surface 1 Curvatures and Slopes for Nominal 0.5,0.4

Table V. Curvature and Slope for Surface 1, Nominal 0.2,0.6

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal: 0.2, 0.6; Step = 0.05			
$(0.15, 0.6)$	2.11704	0.265606	0.484446
$(0.15, 0.65)$	2.16298	0.24538	0.467671
$(0.2, 0.55)$	2.27782	0.248991	0.429131
$(0.2, 0.6)$	2.31876	0.232545	0.416467
$(0.2, 0.65)$	2.36077	0.217072	0.404009
$(0.25, 0.55)$	2.49607	0.210851	0.367291
$(0.25, 0.6)$	2.53349	0.198818	0.357974
$(0.25, 0.65)$	2.572	0.187324	0.348731
Nominal: 0.2, 0.6; Step = 0.1			
$(0.2, 0.5)$	2.23802	0.266417	0.441966
$(0.2, 0.6)$	2.31876	0.232545	0.416467
$(0.2, 0.7)$	2.40381	0.202553	0.391786
$(0.3, 0.5)$	2.69068	0.185462	0.322298
$(0.3, 0.6)$	2.7582	0.167789	0.308522
$(0.3, 0.7)$	2.83008	0.151246	0.294766
Nominal: 0.2, 0.6; Step = 0.15			
$(0.2, 0.45)$	2.19942	0.284815	0.454934
$(0.2, 0.6)$	2.31876	0.232545	0.416467
$(0.2, 0.75)$	2.44783	0.188962	0.379821

Table V.Continued.

Points	Maximum Slope	CAMS	Sqrt Sectional
$(0.35, 0.45)$	2.89909	0.159998	0.282374
$(0.35, 0.6)$	2.99064	0.14079	0.267064
$(0.35, 0.75)$	3.09178	0.122824	0.251503

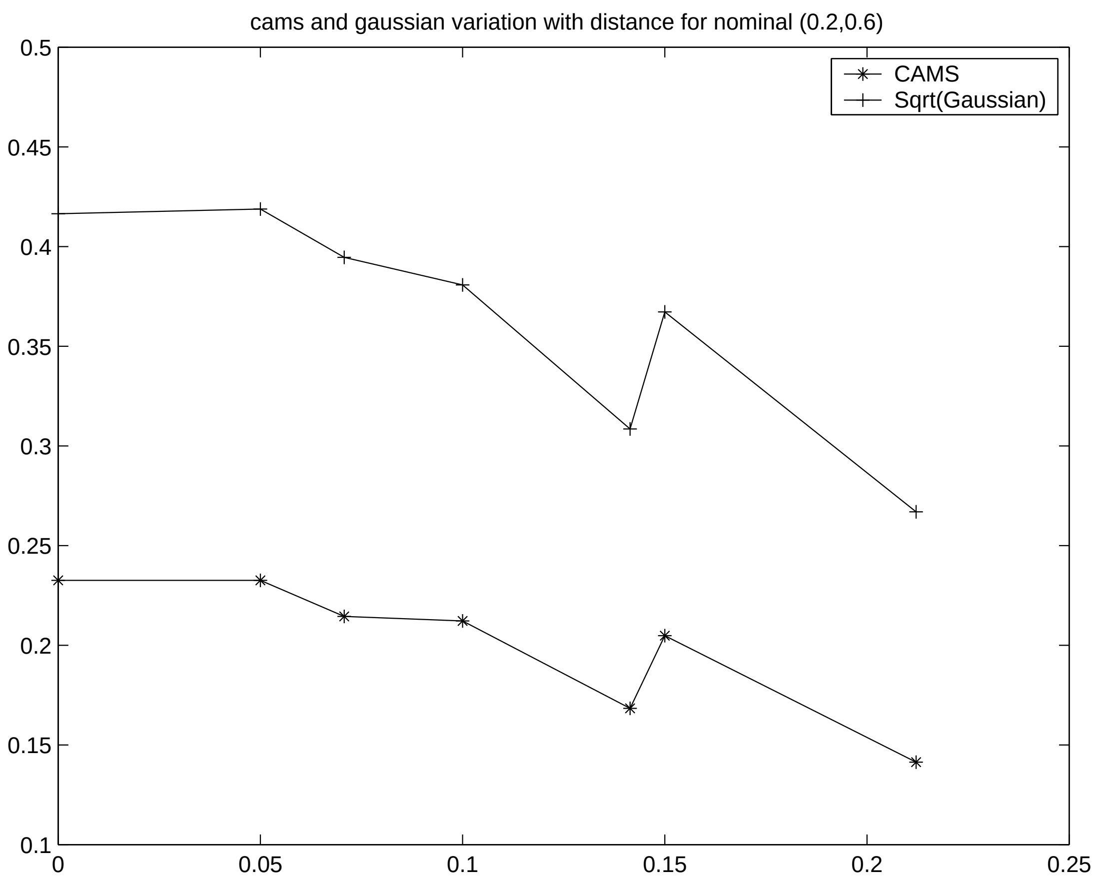

Fig. 21. Surface 1 Curvatures and Slope for Nominal 0.2,0.6

Table VI. Curvature and Slope for Surface 2, Nominal 0.2,0.3

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal 0.2,0.3; Step = 0.05			
$(0.15, 0.25)$	0.482009	1.46301	0.477044
$(0.15, 0.3)$	0.58207	1.2932	0.439106
$(0.15, 0.35)$	0.682165	1.12998	0.401186
$(0.2, 0.25)$	0.476038	1.46977	0.479269
$(0.2, 0.3)$	0.576	1.30132	0.441424
$(0.2, 0.35)$	0.676027	1.13833	0.403482
$(0.25, 0.25)$	0.470239	1.4751	0.481423
$(0.25, 0.3)$	0.570071	1.30856	0.443688
$(0.25, 0.35)$	0.670007	1.14611	0.405738
Nominal 0.2,0.3; Step = 0.1			
$(0.1, 0.2)$	0.388046	1.62318	0.51094
$(0.1, 0.3)$	0.588275	1.28426	0.436737
$(0.1, 0.4)$	0.788571	0.971456	0.362475
$(0.2, 0.2)$	0.376191	1.63206	0.514995
$(0.2, 0.3)$	0.576	1.30132	0.441424
$(0.2, 0.4)$	0.776093	0.987681	0.366891
$(0.3, 0.2)$	0.365234	1.63106	0.518687
$(0.3, 0.3)$	0.564287	1.31486	0.445895
$(0.3, 0.4)$	0.764024	1.00248	0.371198
Nominal 0.2,0.3; Step = 0.15			
$(0.1, 0.2)$	0.294138	1.77091	0.541066

Table VI. Continued.

Points	Maximum Slope	CAMS	Sqrt Sectional
$(0.1, 0.3)$	0.594613	1.27455	0.434318
$(0.1, 0.4)$	0.895132	0.830386	0.32637
$(0.2, 0.2)$	0.276586	1.76979	0.546101
$(0.2, 0.3)$	0.576	1.30132	0.441424
$(0.2, 0.4)$	0.876185	0.852746	0.332566
$(0.3, 0.2)$	0.261895	1.72514	0.550144
$(0.3, 0.3)$	0.558653	1.32017	0.448046
$(0.3, 0.4)$	0.858047	0.873018	0.338591

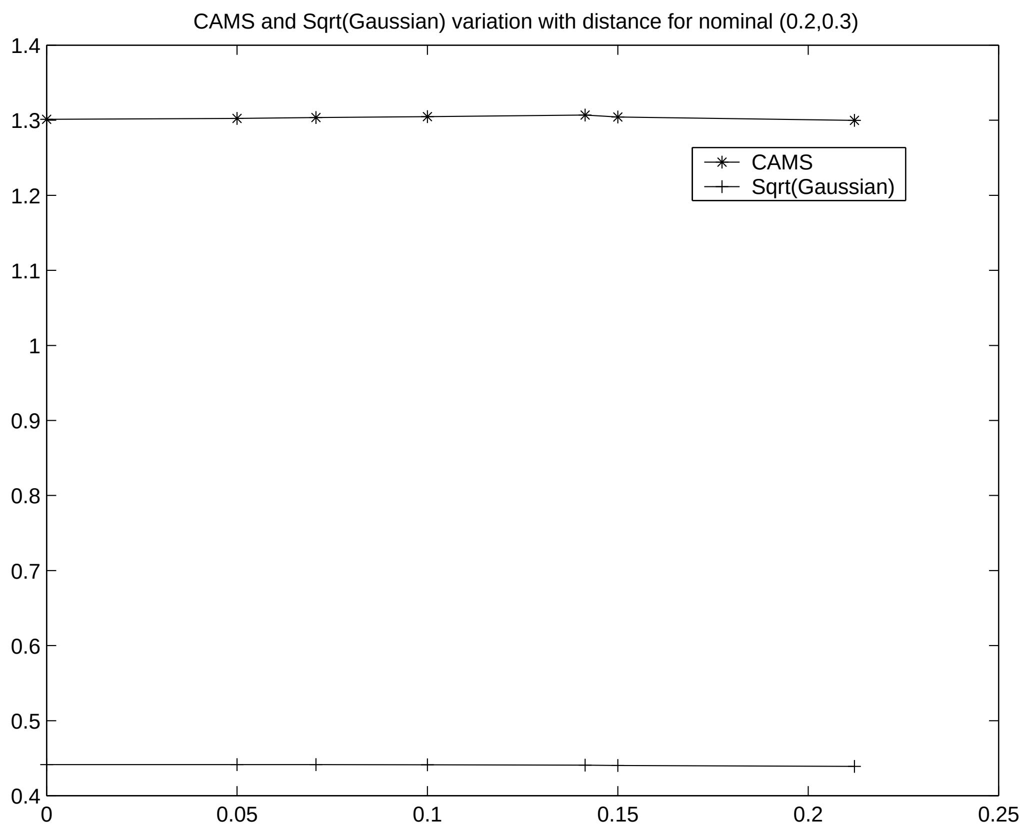

Fig. 22. Surface 2 Curvatures and Slopes for Nominal 0.2,0.3

Table VII. Curvature and Slope for Surface 2, Nominal 0.4,0.5

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal 0.4,0.5; Step = 0.05			
$(0.35, 0.45)$	0.865001	0.865357	0.280226
$(0.35, 0.5)$	0.96502	0.745644	0.253667
$(0.35, 0.55)$	1.06506	0.642013	0.22953
$(0.4, 0.45)$	0.860015	0.871124	0.281611
$(0.4, 0.5)$	0.96	0.750819	0.254943
$(0.4, 0.55)$	1.06001	0.64656	0.230689
$(0.45, 0.45)$	0.855074	0.876769	0.28299
$(0.45, 0.5)$	0.95502	0.755918	0.256214
$(0.45, 0.55)$	1.055	0.651059	0.231847
Nominal 0.4,0.5; Step = 0.1			
$(0.3, 0.4)$	0.770004	0.995134	0.30755
$(0.3, 0.5)$	0.970081	0.740397	0.252387
$(0.3, 0.6)$	1.17022	0.54923	0.20676
$(0.4, 0.4)$	0.760066	1.00775	0.310514
$(0.4, 0.5)$	0.96	0.750819	0.254943
$(0.4, 0.6)$	1.16004	0.557141	0.208849
$(0.5, 0.4)$	0.750337	1.01957	0.313433
$(0.5, 0.5)$	0.950082	0.760934	0.257481
$(0.5, 0.6)$	1.15	0.564935	0.210935
Nominal 0.4,0.5; Step = 0.15			
$(0.25, 0.35)$	0.67501	1.1394	0.336552
$(0.25, 0.5)$	0.97518	0.735082	0.251104

Table VII. Continued.

Points	Maximum Slope	CAMS	Sqrt Sectional
$(0.25, 0.65)$	1.27545	0.470718	0.186502
$(0.4, 0.35)$	0.66017	1.15926	0.341196
$(0.4, 0.5)$	0.96	0.750819	0.254943
$(0.4, 0.65)$	1.26009	0.480935	0.189309
$(0.55, 0.35)$	0.645882	1.17627	0.345689
$(0.55, 0.5)$	0.945186	0.765864	0.258743
$(0.55, 0.65)$	1.24501	0.490993	0.192114

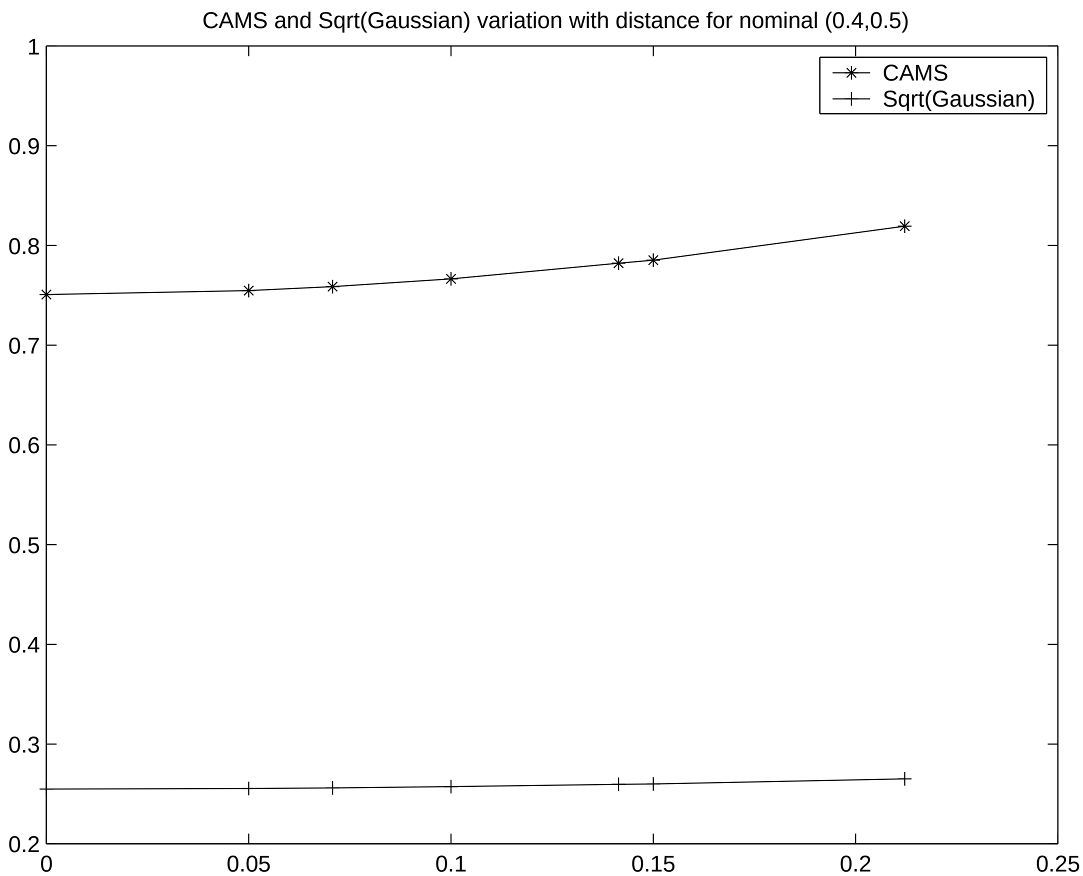

Fig. 23. Surface 2 Curvatures and Slopes for Nominal 0.2,0.3

Table VIII. Curvature and Slope for Surface 2, Nominal 0.6, 0.2

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal 0.6, 0.2, step: 0.05			
$(0.55, 0.15)$	0.285335	1.77828	0.120804
$(0.55, 0.2)$	0.385334	1.62501	0.11375
$(0.55, 0.25)$	0.485337	1.45638	0.105734
$(0.6, 0.15)$	0.284003	1.78006	0.120889
$(0.6, 0.2)$	0.384	1.62714	0.113851
$(0.6, 0.25)$	0.484002	1.45865	0.105845
$(0.65, 0.15)$	0.282672	1.78181	0.120973
$(0.65, 0.2)$	0.382667	1.62927	0.113953
$(0.65, 0.25)$	0.482667	1.46091	0.105955
Nominal 0.6, 0.2, step: 0.1			
$(0.5, 0.1)$	0.186675	1.89918	0.12624
$(0.5, 0.2)$	0.386668	1.62287	0.113648
$(0.5, 0.3)$	0.586677	1.28349	0.0971882
$(0.6, 0.1)$	0.184019	1.90141	0.12636
$(0.6, 0.2)$	0.384	1.62714	0.113851
$(0.6, 0.3)$	0.584006	1.28795	0.0974148
$(0.7, 0.1)$	0.181368	1.9035	0.126479
$(0.7, 0.2)$	0.381334	1.63138	0.114054
$(0.7, 0.3)$	0.581336	1.29241	0.0976414

Table VIII. Continued.

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal 0.6, 0.2, step: 0.15			
$(0.45, 0.05)$	0.0880404	1.97357	0.129635
$(0.45, 0.2)$	0.388002	1.62072	0.113546
$(0.45, 0.35)$	0.688021	1.11847	0.088667
$(0.6, 0.05)$	0.0840952	1.97201	0.129722
$(0.6, 0.2)$	0.384	1.62714	0.113851
$(0.6, 0.35)$	0.684012	1.12474	0.0889992
$(0.75, 0.05)$	0.0801776	1.96864	0.129805
$(0.75, 0.2)$	0.380002	1.63349	0.114155
$(0.75, 0.35)$	0.680005	1.13101	0.0893318

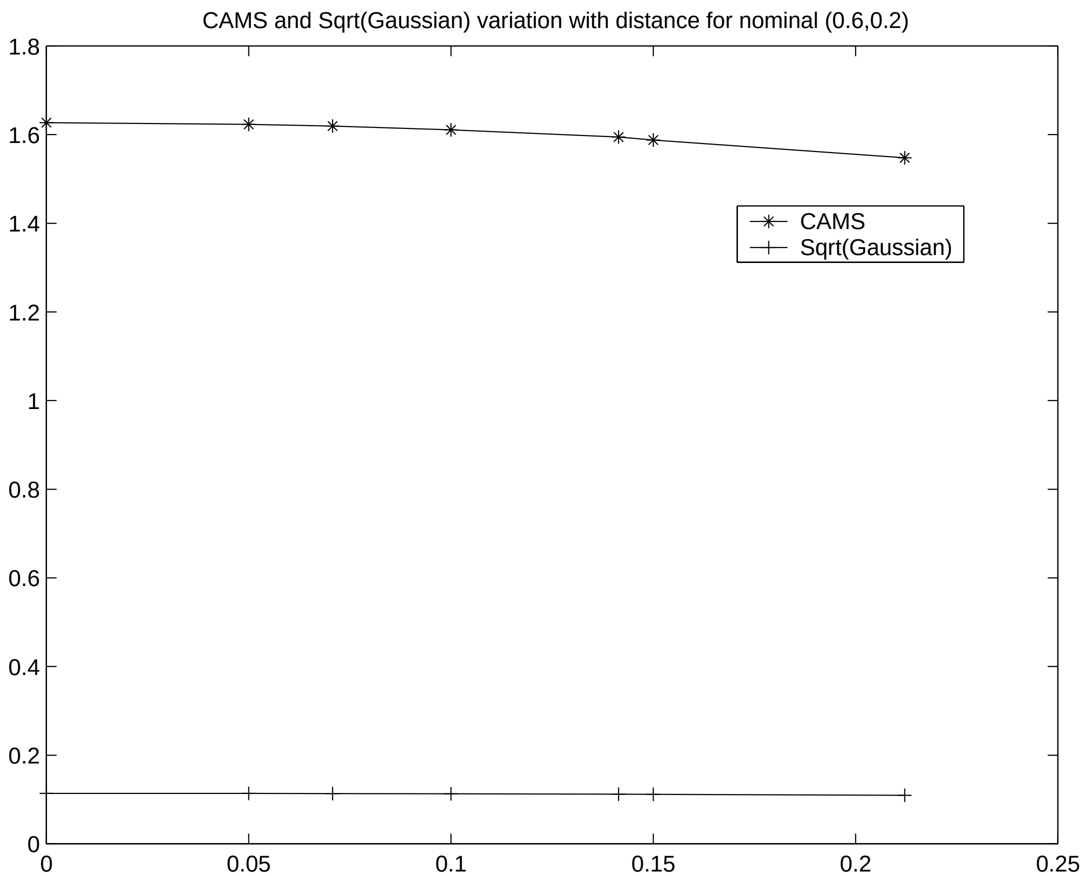

Fig. 24. Surface 2 Curvatures and Slopes for Nominal 0.6,0.2

Table IX. Curvature and Slope for Surface 2, Nominal 0.8,0.4

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal 0.8, 0.4, step: 0.05			
$(0.75, 0.35)$	0.670001	1.14669	0.135247
$(0.75, 0.4)$	0.770001	0.994875	0.12302
$(0.75, 0.45)$	0.870005	0.858936	0.111536
$(0.8, 0.35)$	0.668003	1.14979	0.135497
$(0.8, 0.4)$	0.768	0.997715	0.123258
$(0.8, 0.45)$	0.868002	0.861461	0.111758
$(0.85, 0.35)$	0.666007	1.15289	0.135747
$(0.85, 0.4)$	0.766001	1.00055	0.123497
$(0.85, 0.45)$	0.866001	0.863989	0.111979
Nominal 0.8, 0.4, step: 0.1			
$(0.7, 0.3)$	0.572003	1.30787	0.14765
$(0.7, 0.4)$	0.772003	0.992037	0.122782
$(0.7, 0.5)$	0.972019	0.737567	0.10076
$(0.8, 0.3)$	0.568014	1.31439	0.148158
$(0.8, 0.4)$	0.768	0.997715	0.123258
$(0.8, 0.5)$	0.968008	0.74197	0.101164
$(0.9, 0.3)$	0.564032	1.32086	0.148664
$(0.9, 0.4)$	0.764003	1.0034	0.123735
$(0.9, 0.5)$	0.964002	0.746388	0.10157
Nominal 0.8, 0.4, step: 0.15			
$(0.65, 0.25)$	0.474009	1.47525	0.160008

Table IX.Continued.

Points	Maximum Slope	CAMS	Sqrt Sectional
$(0.65, 0.4)$	0.774006	0.989199	0.122545
$(0.65, 0.55)$	1.07404	0.63301	0.0909933
$(0.8, 0.25)$	0.468038	1.48491	0.160746
$(0.8, 0.4)$	0.768	0.997715	0.123258
$(0.8, 0.55)$	1.06802	0.638693	0.0915415
$(0.95, 0.25)$	0.462088	1.49436	0.161479
$(0.95, 0.4)$	0.762006	1.00624	0.123974
$(0.95, 0.55)$	1.062	0.644411	0.0920924

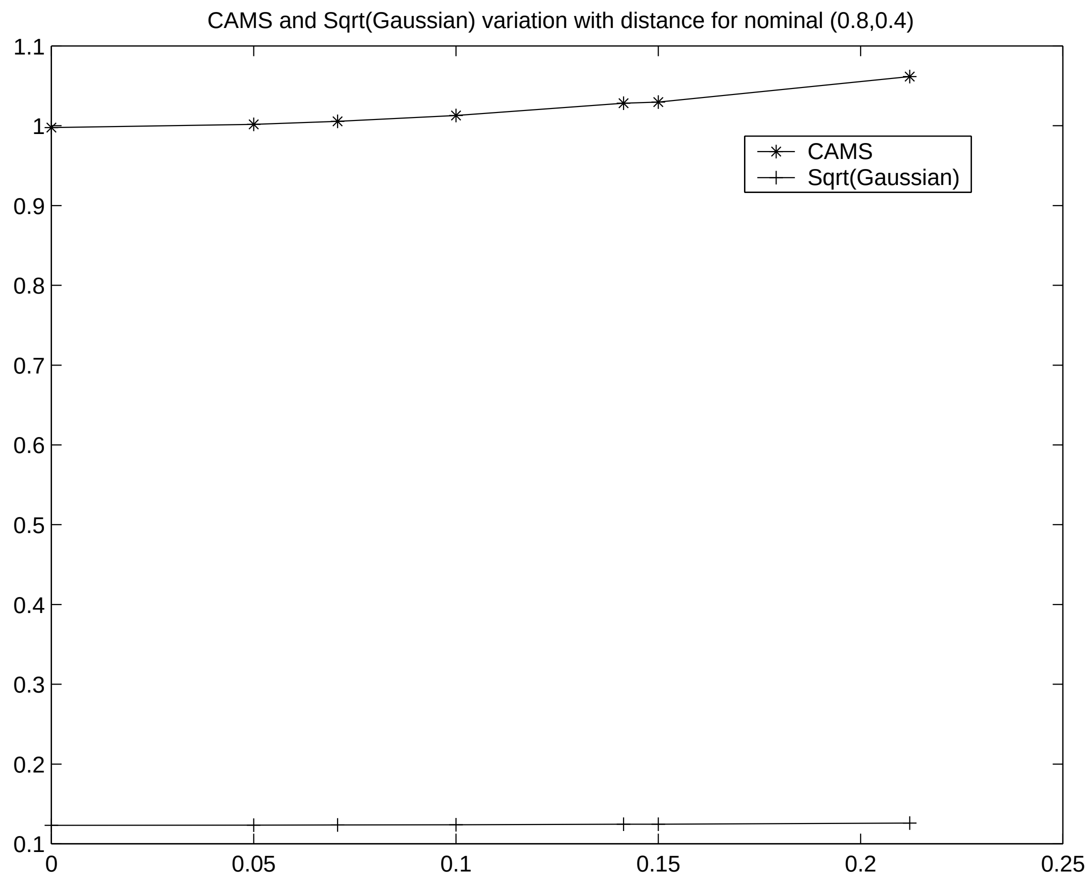

Fig. 25. Surface 2 Curvatures and Slopes for Nominal 0.8,0.4

Table X. Curvature and Slope for Surface 2, Nominal 0.5,0.3

Points	Maximum Slope	CAMS	Sqrt Sectional
Nominal 0.5, 0.3, step: 0.05			
$(0.45, 0.25)$	0.478401	1.46801	0.191356
$(0.45, 0.3)$	0.578402	1.29741	0.176203
$(0.45, 0.35)$	0.678411	1.1337	0.161036
$(0.5, 0.25)$	0.476006	1.47187	0.191712
$(0.5, 0.3)$	0.576	1.30132	0.176569
$(0.5, 0.35)$	0.676004	1.13741	0.161396
$(0.55, 0.25)$	0.473616	1.47569	0.192068
$(0.55, 0.3)$	0.573602	1.3052	0.176936
$(0.55, 0.35)$	0.673601	1.14112	0.161756
Nominal 0.5, 0.3, step: 0.1			
$(0.4, 0.2)$	0.380805	1.63192	0.20537
$(0.4, 0.3)$	0.580807	1.29349	0.175835
$(0.4, 0.4)$	0.780838	0.979655	0.146083
$(0.5, 0.2)$	0.376031	1.63884	0.20602
$(0.5, 0.3)$	0.576	1.30132	0.176569
$(0.5, 0.4)$	0.776015	0.986434	0.146768
$(0.6, 0.2)$	0.371279	1.64547	0.206663
$(0.6, 0.3)$	0.571207	1.30907	0.177301
$(0.6, 0.4)$	0.771202	0.993205	0.147453
Nominal 0.5, 0.3, step: 0.15			
$(0.35, 0.15)$	0.283215	1.78037	0.21769

Table X. Continued.

$(0.35, 0.3)$	0.583216	1.28954	0.175467
$(0.35, 0.45)$	0.883275	0.842432	0.132094
$(0.5, 0.15)$	0.276094	1.78788	0.218496
$(0.5, 0.3)$	0.576	1.30132	0.176569
Points	Maximum Slope	CAMS	Sqrt Sectional
$(0.5, 0.45)$	0.87603	0.851451	0.133047
$(0.65, 0.15)$	0.269047	1.794	0.219278
$(0.65, 0.3)$	0.568816	1.31291	0.177667
$(0.65, 0.45)$	0.868805	0.860488	0.134003

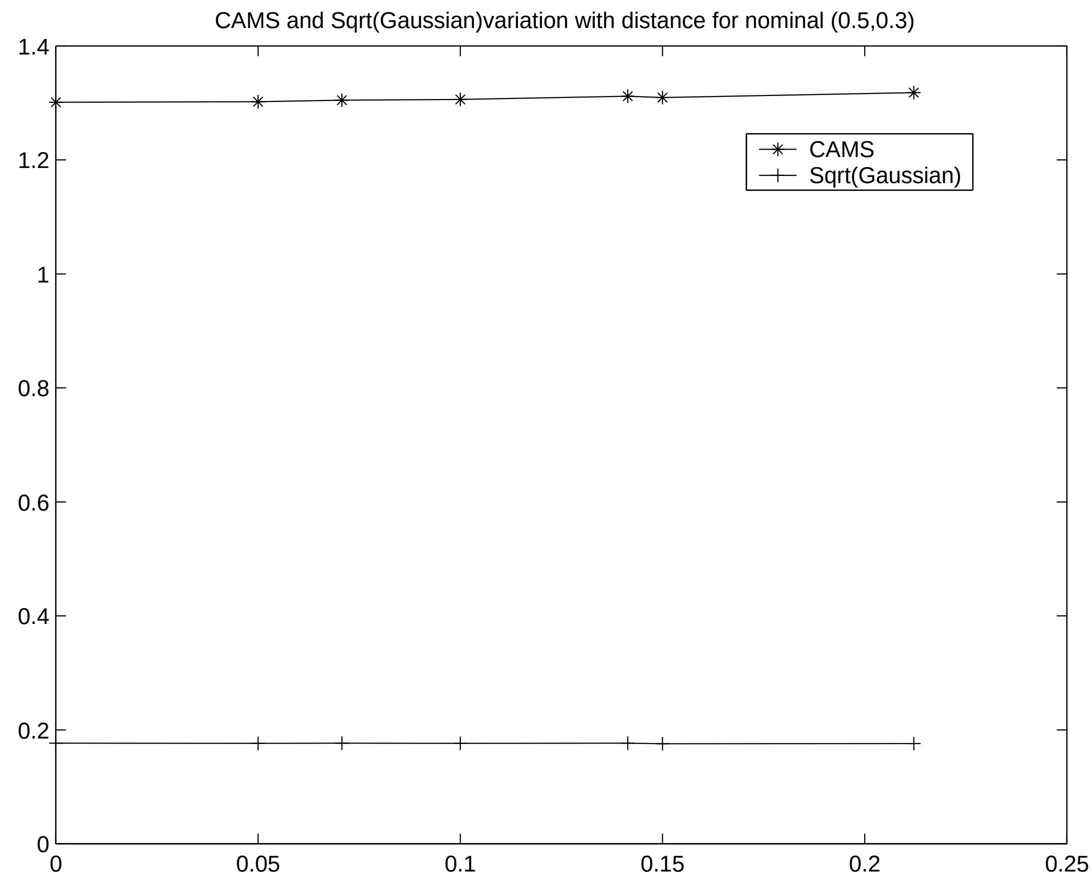

Fig. 26. Surface 2 Curvatures and Slopes for Nominal 0.5,0.3

From the tables and graphs, it is obvious that CAMS is always much more dominant than the square-root of Gaussian curvature, which is not surprising. However we do have an example that shows the square-root of Gaussian being much higher than the CAMS. From this we can conclude that the two curvatures are highly dependent on the surface, but in general the CAMS can be safely taken as the curvature value when used in conjunction with the maximum slope to calculate the worst case change in performance. Another important observation that can be made is that both the plots follow the same pattern almost always. This pattern is not easy to explain and we recommend further research to find if this is specific to certain surfaces or is a general phenomenon.

D. Summary and Conclusion

This research was based on the idea that the performance of a linear estimator (or any system for that matter) can be modelled graphically as a function of the system parameters, which in our case are the elements of the covariance matrix. Further we consider the case where we know some extra information about the system parameters which allows us to represent the performance function as a two-dimensional surface, as it is reduced to a function of two variables. If we have a complex system, in which there are more than two random variables involved, then the performance function will be a multi-dimension manifold. Given this model, we know from the research in the past that the most important parameter to quantify the robustness of the estimator is the slope of the performance surface at any point. Since slope is nothing but the first order derivative of the function at any point, it is clear that the second order derivative or curvature will also be a significant contributor to measure the changes along the surface. The question that we address here is of the type of curvature that should be employed for this analysis. Two types of curvatures are taken into consideration, namely the Curvature Along Max Slope and the Square-root of Gaussian curvature. In order to check the consistency between these two curvatures, we have experimented with two different examples, and for each example, we measured the curvature values for different nominals. The results are shown in tabular and graphical form in the previous section. From the graphs it is clear that the Curvature Along Max Slope is always dominant, i.e., much higher in value than the Square-Root of Gaussian curvature. Which means that in most cases it will be more beneficial to employ CAMS. However there was one case where the Gaussian curvature was higher than CAMS,which implies that it cannot be neglected completely, and for some surfaces it is possible that it is sufficient or more informative than CAMS, (this for example could be the case where slope is very high but the curvature in that direction is low). Also for surfaces in higher dimension or when we do not have a explicit functional relation for the parameters, then Gaussian curvature is much easier to use.

E. Recommendation for Future Research

The most intuitive continuation of the above work is to determine if there is some definite or generalized relation between the two curvatures. Another important result would be to quantify the contribution of either type of curvature in the measurement of robustness. The above work can be more effectively used by varying the method used to choose the nominals and use some well-defined algorithm that can be used for any surface as mentioned in the previous section.

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VITA

Rajeshwary Tayade was born in Amaravati, Maharashtra, India. She received her B.S. degree in electronics and telecomminications from the Government College of Pune, Pune, India in August 1999. She worked as an Assistant Systems Analyst from August 1999 to August 2000. She joined the Masters program in electrical engineering at Texas A&M University in August 2000.

She can be reached at 407, College Main, Apt. 4 , College Station, TX.

The typist for this thesis was Rajeshwary Tayade.