ON SIMPLE MODULES
FOR CERTAIN POINTED HOPF ALGEBRAS

A Dissertation

by

MARIANA PEREIRA LOPEZ

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2006

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Sarah Witherspoon
Committee Members,	Marcelo Aguiar
	Andreas Klappenecker
	Frank Sottile
Head of Department,	Al Boggess

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ABSTRACT

On Simple Modules
for Certain Pointed Hopf Algebras. (December 2006)

Mariana Pereira Lopez, B.S., Universidad de la República, Uruguay;
M.S., University of Massachusetts
Chair of Advisory Committee: Dr. Sarah Witherspoon

In 2003, Radford introduced a new method to construct simple modules for the Drinfel'd double of a graded Hopf algebra. Until then, simple modules for such algebras were usually constructed by taking quotients of Verma modules by maximal submodules. This new method gives a more explicit construction, in the sense that the simple modules are given as subspaces of the Hopf algebra and one can easily find spanning sets for them. I use this method to study the representations of two types of pointed Hopf algebras: restricted two-parameter quantum groups, and the Drinfel'd double of rank one pointed Hopf algebras of nilpotent type. The groups of group-like elements of these Hopf algebras are abelian; hence, they fall among those Hopf algebras classified by Andruskiewitsch and Schneider. I study, in particular, under what conditions a simple module can be factored as the tensor product of a one dimensional module with a module that is naturally a module for a special quotient. For restricted two-parameter quantum groups, given θ a primitive ℓth root of unity, the factorization of simple $\mathfrak{u}_{\theta^y,\theta^z}(\mathfrak{sl}_n)$-modules is possible, if and only if $\gcd((y-z)n,\ell)=1$. I construct simple modules using the computer algebra system Singular::Plural and present computational results and conjectures about bases and dimensions. For rank one pointed Hopf algebras, given the data $\mathcal{D}=(G,\chi,a)$, the factorization of simple $D(H_{\mathcal{D}})$-modules is possible if and only if $|\chi(a)|$ is odd and $|\chi(a)|=|a|=|\chi|$. Under this condition, the tensor product of two simple $D(H_{\mathcal{D}})$-

modules is completely reducible, if and only if the sum of their dimensions is less or equal than $|\chi(a)| + 1$.

To my family

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CHAPTER I

INTRODUCTION AND PRELIMINARIES

I study the simple modules of two types of pointed Hopf algebras: restricted two-parameter quantum groups and the Drinfel'd double of rank one pointed Hopf algebras of nilpotent type. The main tool I use is a construction introduced by Radford [19] where the simple modules for the Drinfel'd double of a Hopf algebra are parametrized by group-like elements of the Drinfel'd double.

The dissertation is organized as follows. In this chapter I give the definitions and notations that I will use and I present Radford's construction for simple modules for the Drinfel'd double of certain Hopf algebras. In Chapter II, I define the two-parameter quantum groups and present a theorem on factorization of their simple modules. In Chapter III, I show the code used to construct these modules using the computer algebra system SINGULAR::PLURAL and I formulate conjectures about their bases and dimensions based on the computational results. In Chapter IV, I present the rank one pointed Hopf algebras of nilpotent type defined by Krop and Radford in [15], and give a theorem about the reducibility of the tensor product of two simple modules for their Drinfel'd doubles.

In what follows $\mathbb{K}$ is a field of characteristic 0. All vector spaces and tensor products are over $\mathbb{K}$. A *map* between vector spaces means a linear transformation. For a map $T : V \to W$ between vector spaces V and W, I will denote the *dual* of T by T^*; that is $T^* : W^* \to V^*$ and $T(f)(v) = f(T(v))$ for all $f \in W^*$ and $v \in V$. For vector spaces V and W, the *twist map* $\tau : V \otimes W \to W \otimes V$ is given by $\tau(v \otimes w) = w \otimes v$.

The journal model is Journal of Algebra.

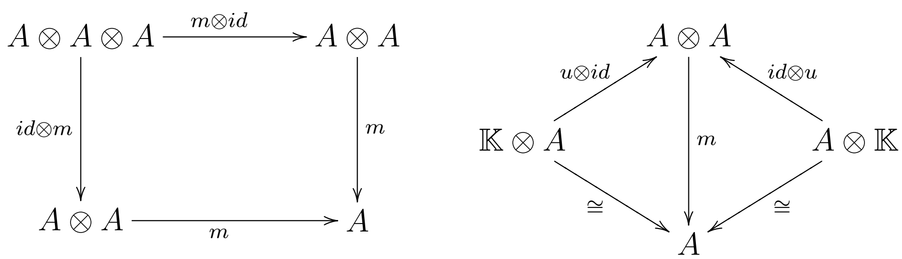
1. **Hopf algebras**

I give a brief introduction to Hopf algebras, summarizing the first chapter of [18].

Definition I.1. An *algebra* is a triple (A, m, u) where A is a vector space and

$$m : A \otimes A \to A \text{ and } u : \mathbb{K} \to A$$

are maps so that the following diagrams commute:



These are the diagrams of *associativity* and *unit* respectively. The map m is called *multiplication* and u is the *unit*.

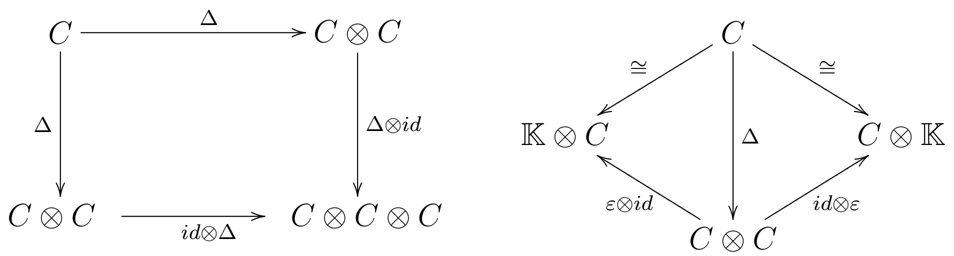
Write $m(a \otimes b) = ab$ and $u(1_{\mathbb{K}}) = 1_A$. With this notation, the commutativity of the diagrams means $(ab)c = a(bc)$ and $a1_A = 1_A a = a$, $\forall a, b, c \in A$. When there is no place for confusion I will say the algebra A instead of (A, m, u).

Now I dualize the notions just defined to define coalgebras.

Definition I.2. A *coalgebra* is a triple (C, Δ, ε) where C is a vector space and

$$\Delta : C \to C \otimes C \text{ and } \varepsilon : C \to \mathbb{K}$$

are maps so that the following diagrams commute:



These are the *coassociativity* and *counit* diagrams respectively. The map Δ is called *comultiplication* and ε is the *counit*.

The following notation was introduced by Heyneman and Sweedler.

Notation. The *sigma notation* for Δ is given as follows: for any $c \in C$, write

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$

The subscripts (1) and (2) are symbolic and do not indicate particular elements of C.

With this notation the coassociativity diagram translates as

$$\sum x_{(1)(1)} \otimes x_{(1)(2)} \otimes x_{(2)} = \sum x_{(1)} \otimes x_{(2)(1)} \otimes x_{(2)(2)}.$$

This element is denoted by

$$\Delta_2(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}.$$

Iterating this process, applying coassociativity $n-1$ times, gives

$$\Delta_{n-1}(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(n)}.$$

The counit diagram says that, for all $c \in C$

$$\sum \varepsilon(c_{(1)})c_{(2)} = c = \sum \varepsilon(c_{(2)})c_{(1)}.$$

Definition I.3. Let (C, Δ, ε) be a coalgebra and I a subspace of C.

1. I is a *left coideal of* C if $\Delta(I) \subset C \otimes I$.
2. I is a *right coideal of* C if $\Delta(I) \subset I \otimes C$.
3. I is a *coideal of* C if $\Delta(I) \subset I \otimes C + C \otimes I$ and $\varepsilon(I) = 0$.

If I is a coideal of (C, Δ, ε), then C/I is a coalgebra with comultiplication and counit induced from Δ and ε respectively.

Example I.4. If (A, m, u) is a finite-dimensional algebra then its dual, A^*, is a coalgebra with $\Delta = m^*$ and $\varepsilon = u^*$. Explicitly, if $f \in A^*$, then $\Delta(f)(a \otimes b) = \sum f_{(1)}(a) f_{(2)}(b) = f(ab)$ for all a and b in A, and $\varepsilon(f) = f(1_A)$.

If (C, Δ, ε) is a coalgebra, then C^* is an algebra with $m = \Delta^*$ and $u = \varepsilon^*$. That is, for f and g in C^*, $(fg)(c) = \sum f(c_{(1)}) g(c_{(2)})$ for all $c \in C$ and $1_{C^*} = \varepsilon$.

Definition I.5. A *bialgebra* is a quintuple $(B, m, u, \Delta, \varepsilon)$ where (B, m, u) is an algebra, (B, Δ, ε) is a coalgebra, and the maps Δ and ε are algebra morphisms (or equivalently, m and u are coalgebra morphisms).

Example I.6. If $(B, m, u, \Delta, \varepsilon)$ is a bialgebra, then so are $B^{\mathrm{op}} = (B, m^{\mathrm{op}}, u, \Delta, \varepsilon)$ and $B^{\mathrm{coop}} = (B, m, u, \Delta^{\mathrm{op}}, \varepsilon)$, with $m^{\mathrm{op}} = m \circ \tau$ and $\Delta^{\mathrm{op}} = \tau \circ \Delta$. If $m^{\mathrm{op}} = m$ then then B is *commutative*, and if $\Delta^{\mathrm{op}} = \Delta$ it is *cocommutative.*

Definition I.7. Let (A, m, u) be an algebra and (C, Δ, ε) a coalgebra. Then $\mathrm{Hom}_{\mathbb{K}}(C, A)$, the set of linear maps from C to A , is an algebra with the *convolution product*

$$f * g := m \circ (f \otimes g) \circ \Delta$$

for all $f, g \in \mathrm{Hom}_{\mathbb{K}}(C, A)$; *i.e.*

$$(f * g)(x) = \sum f\left(x_{(1)}\right) g\left(x_{(2)}\right), \forall x \in C.$$

The unit element in $\mathrm{Hom}_{\mathbb{K}}(C, A)$ is $u\varepsilon$.

From now on, when I say the algebra $\mathrm{Hom}_{\mathbb{K}}(C, A)$, I mean $(\mathrm{Hom}_{\mathbb{K}}(C, A), *, u \circ \varepsilon)$. In particular, if $(B, m, u, \Delta, \varepsilon)$ is a bialgebra, then $\mathrm{Hom}_{\mathbb{K}}(B, B)$ is an algebra with the structure just described. The map id_B is invertible in $\mathrm{Hom}_{\mathbb{K}}(B, B)$ if and only if there exists a map $S : B \to B$ such that $S * \mathrm{id}_B = \mathrm{id}_B * S = u \circ \varepsilon$. In other words,

$$\sum S\left(x_{(1)}\right) x_{(2)} = \sum x_{(1)} S\left(x_{(2)}\right) = \varepsilon\left(x\right) 1_B, \; \forall\, x \in B.$$

Such a map S is called an *antipode* in B. If an antipode exists in $(B, m, u, \Delta, \varepsilon)$, it is unique.

Definition I.8. A *Hopf algebra* is a sextuple $(H, m, u, \Delta, \varepsilon, S)$ where $(H, m, u, \Delta, \varepsilon)$ is a bialgebra and $S : H \to H$ is an antipode in H.

A subspace I of H is a *Hopf ideal* of H, if it is both an ideal and a coideal and $S(I) \subseteq I$. If I is a Hopf ideal of H, then H/I is a Hopf algebra with the structure induced from H.

Example I.9. If $(G, \cdot, e)$ is a group, let $\mathbb{K}G$ be the vector space with basis G. Then $\mathbb{K}G$ is a Hopf algebra with the operations defined by

$$m(g \otimes g') = g \cdot g' \text{ and } u(1) = e, \; \forall\, g, g' \in G,$$

$$\Delta(g) = g \otimes g, \; \varepsilon(g) = 1, \text{ and } S(g) = g^{-1}, \; \forall g \in G.$$

The algebra $\mathbb{K}G$ is called *the group algebra of G*.

For any coalgebra C, an element $c \in C$ is called *group-like* if

$$\Delta(c) = c \otimes c \text{ and } \varepsilon(c) = 1.$$

Denote by $G(C)$ the set of group-like elements of C. Then $\mathbb{K}G(C)$ is a subcoalgebra of C.

Example I.10. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. The *universal enveloping algebra* $U(\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal generated by the relations $h \otimes g - g \otimes h - [h, g]$ for all h, g in $\mathfrak{g}$. Then $U(\mathfrak{g})$ is a Hopf algebra with:

$$\Delta(h) = h \otimes 1 + 1 \otimes h, \ \varepsilon(h) = 0, \text{ and } S(h) = -h, \forall h \in \mathfrak{g}.$$

Example I.11. If H is a finite-dimensional Hopf algebra with antipode S, then H^* with the structures described in I.4 is a Hopf algebra with antipode S^*.

Example I.12. In [21] Taft constructed a family of finite-dimensional non-commutative, non-cocommutative Hopf algebras: let $\ell \in \mathbb{Z}_{>0}$, and θ a primitive ℓth root of unity. The Taft algebra T_θ is generated as an algebra by elements x and a, subject to the relations:

$$x^\ell = 0, \quad a^\ell = 1, \quad ax = \theta xa.$$

The coalgebra structure and the antipode are determined by:

$$\Delta(a) = a \otimes a, \quad \epsilon(a) = 1, \quad S(a) = a^{-1} = a^{\ell-1},$$

$$\Delta(x) = x \otimes a + 1 \otimes x. \quad \epsilon(x) = 0, \quad S(x) = -xa^{-1}.$$

The set $\{a^i x^j : 0 \leq i, j < \ell\}$ is a linear basis for T_θ.

The Hopf algebras that I will study are generalizations of the Taft algebras, and they will all be graded Hopf algebras, as defined next.

Definition I.13. A Hopf algebra H is *graded* if $H = \oplus_{n=0}^{\infty} H_n$ and

1. H is a graded algebra, *i.e.* $1 \in H_0$ and $H_m H_n \subseteq H_{m+n}$.

2. H is a graded coalgebra, *i.e.* $\Delta(H_n) \subseteq \sum_{i=0}^{n} H_{n-i} \otimes H_i$ and $\varepsilon(H_n) = 0$, $\forall n > 0$.

3. $S(H_n) \subseteq H_n$, $\forall n \geq 0$.

The Taft algebra T_θ is a graded Hopf algebra with $(T_\theta)_n = \mathbb{K}\{a^i x^n : 0 \le i < \ell\}$ if $n < \ell$, and $(T_\theta)_n = (0)$ for $n \ge \ell$.

Another property of the Taft algebras is that they are pointed Hopf algebras, as defined next.

Definition I.14. A coalgebra is called *simple* if it has no proper subcoalgebras. For a coalgebra C, the *coradical* $C_{(0)}$ of C, is the sum of the simple subcoalgebras of C. If $C_{(0)} = \mathbb{K}G(C)$ (in other words, every simple subcoalgebra of C is one-dimensional), C is *pointed*.

Now I give a definition that will be used in the following chapters. Given any Hopf algebra H and L a subset of H, let

$$L^+ = L \cap \operatorname{Ker} \varepsilon.$$

Note that if L is a subcoalgebra of H, then L^+ is a coideal and hence H/L^+ is a coalgebra. Morover, let $\langle L^+ \rangle = HL^+H$ be the two-sided ideal generated by L^+, then $H/\langle L^+ \rangle$ is a bialgebra. I will use this construction in the particular case where $L \subset Z(H)$, the center of H, in which case $\langle L^+ \rangle = HL^+$ and so H/HL^+ is a bialgebra. If in addition $S(L^+) \subset L^+$, then H/HL^+ is a Hopf algebra. A simple calculation shows that if $L = \mathbb{K}J$ with J a subgroup of $G(H)$, the group of group-like elements of H, then

$$L^+ = \mathbb{K}\left\{g - 1 : g \in J\right\}.$$

Remark I.15. In [20] H.-J. Schneider strengthened the Nichols-Zoeller theorem and showed that if H is a finite-dimensional Hopf algebra and L is a Hopf subalgebra of H, then $H \simeq H/HL^+ \otimes L$ as right L-modules [20] . In particular

$$\dim(H/HL^+) = \frac{\dim(H)}{\dim(L)}.$$

Definition I.16. For H a finite-dimensional Hopf algebra, let

$$G_C(H) = G(H) \cap Z(H)$$

denote the group of central group-like elements of H and let

$$\overline{H} = H/H(\mathbb{K}G_C(H))^+.$$

Then $\overline{H}$ is a Hopf algebra, and by Remark I.15

$$\dim(\overline{H}) = \frac{\dim(H)}{|G_C(H)|}.$$

2. Modules, comodules and Yetter-Drinfel'd modules

Definition I.17. Let A be an algebra. A *left A-module* is a pair (M, ρ), where M is a vector space and $\rho : A \otimes M \to M$ is a map so that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{m \otimes id} & A \otimes M \\ {\scriptstyle id \otimes \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ A \otimes M & \xrightarrow[\rho]{} & A \end{array}, \qquad \begin{array}{ccc} \mathbb{K} \otimes M & \xrightarrow{u \otimes id} & A \otimes M \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \rho} \\ & & M \end{array}.$$

A map ρ as above is called an *action*. Write $\rho(a \otimes m) = a \cdot m$. With this notation the diagrams become

$$a \cdot (b \cdot m) = (ab) \cdot m \text{ and } 1_A \cdot m = m, \ \forall a, b \in A, \ m \in M.$$

There is an analogous definition of *right modules*; since all the modules I will consider will be left modules, I will say *module* for left module.

If (M, ρ_M) and (N, ρ_N) are A-modules, a map $f : M \to N$ is a *morphism of modules* if $f(a \cdot m) = a \cdot f(m), \ \forall m \in M, a \in A$.

Definition I.18. Let (M, ρ_M) be an A-module and N a subspace of M; N is an

A-submodule of M if $A \cdot N \subset N$. An A-module M is *simple* if its only submodules are 0 and M. A modules is *completely reducible* if it is the direct sum of its simple submodules.

Dualizing the previous definitions we get the analogous notions for coalgebras.

Definition I.19. Let C be a coalgebra. A *right C-comodule* is a pair (M, δ) where M is a vector space and $\delta : M \to M \otimes C$ is a map so that the following diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\delta} & M \otimes C \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle id\otimes\Delta} \\ M \otimes C & \xrightarrow[\delta\otimes id]{} & M \otimes C \otimes C \end{array} \quad , \quad \begin{array}{ccc} M & \xrightarrow{\delta} & M \otimes C \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle id\otimes\varepsilon} \\ & & M \otimes \mathbb{K} \end{array} .$$

The map δ is called *coaction*. Write

$$\delta(m) = \sum m_{(0)} \otimes m_{(1)},$$

where $m_{(0)} \in M$, $m_{(1)} \in C$.

Remark I.20. With this notation, the diagrams above translate as

$$\sum m_{(0)(0)} \otimes m_{(0)(1)} \otimes m_{(1)} = \sum m_{(0)} \otimes m_{(1)(1)} \otimes m_{(1)(2)} \tag{I.1}$$

and

$$\sum m_{(0)}\varepsilon(m_{(1)}) = m,$$

for all $m \in M$. The element of the equation (I.1) will be denoted $\sum m_{(0)} \otimes m_{(1)} \otimes m_{(2)}$.

Definition I.21. Given (M, δ) and (N, η) two C-comodules, a map $f : M \to N$ is a *comodule morphism* if the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \delta_M} & & \downarrow{\scriptstyle \delta_N} \\ M \otimes C & \xrightarrow[f\otimes id]{} & N \otimes C \end{array} .$$

That is, if $\sum f\left(m_{(0)}\right) \otimes m_{(1)} = \sum f(m)_{(0)} \otimes f(m)_{(1)}, \forall m \in M$.

There is an analogous definition of *left comodule*; since all the comodules will be right comodules, I will say *comodule* for right comodule.

Definition I.22. Let (M, δ) be a C-comodule and N a subspace of M; N is a C-*subcomodule of* M if $\delta(N) \subset N \otimes C$.

Remark I.23. If $(B, m, u, \Delta, \varepsilon)$ is a bialgebra and M and N are B-modules, then $M \otimes N$ is also B-module with action given by

$$b \cdot (m \otimes n) = \sum b_{(1)} \cdot m \otimes b_{(2)} \cdot n, \ \forall\, b \in B,\ m \in M, n \in N.$$

If M and N are B-comodules then $M \otimes N$ is a B-comodule with coaction

$$\delta(m \otimes n) = \sum m_{(0)} \otimes n_{(0)} \otimes m_{(1)} n_{(1)}, \ \forall\, m \in M, n \in N.$$

Definition I.24. Let H be a finite-dimensional Hopf algebra over $\mathbb{K}$ with antipode S. The *Drinfel'd double of* H, $D(H)$, is

$$D(H) = (H^*)^{\text{coop}} \otimes H$$

as a coalgebra. The algebra structure is given by

$$(g \otimes h)(f \otimes k) = \sum g\left(h_{(1)} \rightharpoonup f \leftharpoonup S^{-1}(h_{(3)})\right) \otimes h_{(2)} k,$$

for all $g, f \in H^*$ and $h, k \in H$; where $(a \rightharpoonup f)(b) = f(ba)$ and $(f \leftharpoonup a)(b) = f(ab)$, for all $a, b \in H$ and $f \in H^*$.

This construction is due to Drinfel'd [10] where he showed that if H is a finite-dimensional Hopf algebra, then $D(H)$ is a Hopf algebra. Furthermore, if M and N are $D(H)$-modules, then

$$M \otimes N \simeq N \otimes M.$$

Explicitly, if $\{h_i\}$ is a basis for H and $\{h_i^*\}$ is the corresponding dual basis of H^*, let

$$R = \sum_i (\varepsilon_H \otimes h_i) \otimes (h_i^* \otimes 1_H) \in D(H) \otimes D(H).$$

Then $M \otimes N \simeq N \otimes M$ via $m \otimes n \mapsto R^{-1}(n \otimes m)$. Drinfel'd doubles are examples of *quasitriangular bialgebras*, which are bialgebras B equipped with invertible elements $R \in B \otimes B$, satisfying certain conditions, and for which the symmetry of tensor products of modules is realized via R^{-1}.

If M is a $D(H)$-module, then it is both an H-module and an $(H^*)^{\mathrm{coop}}$-module. The action of H^* gives rise to an H-comodule structure on M such that if $\delta(m) = \sum m_{(0)} \otimes m_{(1)}$ then $f \cdot m = \sum \langle f, m_{(1)} \rangle m_{(0)}$ for all $f \in H^*$.

Definition I.25. For any bialgebra H, a *left-right Yetter-Drinfel'd module* is a $\mathbb{K}$-vector space M which is both a left H-module and a right H-comodule, and satisfies the compatibility condition

$$\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)} m_{(1)} = \sum (h_{(2)} \cdot m)_{(0)} \otimes (h_{(2)} \cdot m)_{(1)} h_{(1)}.$$

The category of left-right Yetter-Drinfel'd modules over a bialgebra H will be denoted by ${}_H\mathcal{Y}D^H$.

Proposition I.26 (Majid [17])**.** *Let H be a finite dimensional Hopf algebra. Then $D(H)$-modules are left-right Yetter-Drinfel'd modules and conversely. Explicitly, if M is a left-right Yetter-Drinfel'd module, then it is a $D(H)$-module with the same* *action of H and the action of H^* given by*

$$f \cdot m = \sum f(m_{(1)}) m_{(0)}, \tag{I.2}$$

for all f in H^* and m in M.

Remark I.27. If $M, N \in {}_H\mathcal{Y}D^H$, by the last proposition M and N are $D(H)$-modules. Since $D(H)$ is a bialgebra, by Remark I.23 $M \otimes N$ is also a $D(H)$-module and hence a Yetter-Drinfel'd module over H. The Yetter-Drinfel'd structure is given by

$$h \cdot (m \otimes n) = \sum h_{(1)} \cdot m \otimes h_{(2)} \cdot n$$

and

$$\delta(m \otimes n) = \sum m_{(0)} \otimes n_{(0)} \otimes n_{(1)} m_{(1)}.$$

An alternative definition of the Drinfel'd double is $D'(H) = H \otimes (H^*)^{\text{coop}}$ as coalgebras, and multiplication given by

$$(k \otimes f)(h \otimes g) = \sum k f_{(1)}(S^{-1}(h_{(1)})) f_{(3)}(h_{(3)}) h_{(2)} \otimes f_{(2)} g,$$

where $(\Delta^{\text{op}} \otimes \text{id}\,)\Delta^{\text{op}}(f) = \sum f_{(1)} \otimes f_{(2)} \otimes f_{(3)}$. I will need both definitions of the Drinfel'd double since two of the papers I will be using [6, 19] use these different definitions. The following lemma gives the relationship between these two definitions of the Drinfel'd double.

Lemma I.28. *$D'(H) \simeq D(H^*)^{\text{coop}}$ as Hopf algebras.*

Proof. As $H^{**} \simeq H$, we have $D(H^*) \cong H^{\text{coop}} \otimes H^*$, with multiplication

$$\begin{aligned}(k \otimes f)(h \otimes g) &= \sum k \left(f_{(1)} \rightharpoonup h \leftharpoonup (S^*)^{-1}(f_{(3)}) \right) \otimes f_{(2)} g \\ &= \sum k \left(f_{(1)}(h_{(2)}) h_{(1)} \leftharpoonup (f_{(3)} \circ S^{-1}) \right) \otimes f_{(2)} g \\ &= \sum k f_{(1)}(h_{(3)}) (f_{(3)}(S^{-1}(h_{(1)})) h_{(2)} \otimes f_{(2)} g,\end{aligned}$$

where $(\Delta^{\mathrm{op}} \otimes \mathrm{id}\,)\Delta^{\mathrm{op}}(f) = \sum f_{(3)} \otimes f_{(2)} \otimes f_{(1)}$. So $D'(H) \simeq D(H^*)$ as algebras. As coalgebras $D(H^*) \simeq H^{\mathrm{coop}} \otimes H^* = (H \otimes (H^*)^{\mathrm{coop}})^{\mathrm{coop}} = D'(H)^{\mathrm{coop}}$. □

3. Radford's construction

In this section I describe results from [19]. Although Radford's results are more general, I will only write them for $\mathbb{K}$ an algebraically closed field of characteristic 0. This is the main tool I will use to study representations of Drinfel'd doubles. For algebras A and B, the set of algebra maps from A to B will be denoted by $\mathrm{Alg}\,(A, B)$. It is not hard to see that if H is a finite dimensional algebra, then $\mathrm{Alg}\,(H, \mathbb{K}) = G(H^*)$, the set of group-like elements of H^*.

Lemma I.29 (Radford [19])**.** *Let H be a bialgebra over $\mathbb{K}$ and suppose H^{op} is a Hopf algebra with antipode $\overline{S}$. If $\beta \in \mathrm{Alg}\,(H, \mathbb{K})$, then $H_\beta = (H, \bullet_\beta, \Delta) \in {}_H\mathcal{Y}D^H$, where*

$$h \bullet_\beta a = \sum \beta(h_{(3)}) h_{(2)} a \overline{S}(h_{(1)}), \tag{I.3}$$

for all h, a in H.

If $\beta : H \to \mathbb{K}$ is an algebra map and N is a right coideal of H, then the H-submodule of H_β generated by N, $H \bullet_\beta N$, is a Yetter-Drinfel'd H-submodule of H_β. If $g \in G(H)$, then $\mathbb{K}g$ is a right coideal and $H \bullet_\beta \mathbb{K}g = H \bullet_\beta g$ is a Yetter-Drinfel'd submodule of H_β. For M a Yetter-Drinfel'd module over H, $[M]$ will denote the the isomorphism class of M.

Proposition I.30 (Radford [19])**.** *Let $H = \bigoplus_{n=0}^{\infty} H_n$ be a graded Hopf algebra over $\mathbb{K}$. Suppose that $H_0 = \mathbb{K}G$ where G is a finite abelian group and $H_n = H_{n+1} = \cdots = (0)$ for some $n > 0$. Then*

$$(\beta, g) \mapsto [H \bullet_\beta g]$$

is a bijective correspondence between the Cartesian product of sets $\mathrm{Alg}\,(H,\mathbb{K})\times G$ *and the set of isomorphism classes of simple Yetter-Drinfel'd* H*-modules.*

Let $H = \bigoplus_{n=0}^{\infty} H_n$ be a graded coalgebra and $h = h_0 + \cdots + h_n$ a group-like element of H with $h_i \in H_i$ and $h_n \neq 0$. The coalgebra grading implies that $\Delta(h) \in \sum_{m=0}^{n} (\sum_{i=0}^{m} H_{m-i} \otimes H_i)$, but since h is a group-like element $\Delta(h) = h \otimes h = \sum_{i,j=0}^{n} h_i \otimes h_j \notin \sum_{m=0}^{n} (\sum_{i=0}^{m} H_{m-i} \otimes H_i)$ unless $n = 0$. Hence $G(H) = G(H_0)$. In the case where $H_0 = \mathbb{K}G$ we have $G(H) = G(H_0) = G(\mathbb{K}G) = G$, the last equality holding since distinct group-like elements are linearly independent. If H is as in the last proposition is also finite dimentional, then $\mathrm{Alg}\,(H,\mathbb{K}) \times G = G(H^*) \times G(H)$.

Remark I.31. Let $H = \bigoplus_{n=0}^{\infty} H_n$ be a graded Hopf algebra with $H_m = H_{m+1} = \cdots = (0)$ for some $m > 0$ and $H_0 = \mathbb{K}G(H)$ where $G(H) = G$ is a finite group. If $\beta : H \to \mathbb{K}$ is an algebra map and $i > 0$, since $H_i^m = (0)$ we have that $\beta_{|H_i} = 0$. Then β is determined by its restriction to $H_0 = \mathbb{K}G$. Since G is a finite group, $1 = \beta(g^{|G|}) = \beta(g)^{|G|}$ and so $\beta(g) \neq 0$ for all $g \in G$. Let

$$\widehat{G} = \mathrm{Hom}\,(G, \mathbb{K}^{\times}), \tag{I.4}$$

the set of group homomorphisms from G to $\mathbb{K}^{\times} = \mathbb{K} - \{0\}$. Then, to give an algebra map $\beta : H \to \mathbb{K}$, is equivalent to giving a map in $\widehat{G}$; when no confusion arises, the corresponding map in $\widehat{G}$ will also be called β.

Example I.32. Let $H = H_0 = \mathbb{K}G$ with G a finite abelian group. If $\beta \in \widehat{G}$ and $g, h \in G$, then $h \bullet_\beta g = \beta(h) g$ and so $H \bullet_\beta g = \mathbb{K}g$. In this case $D(H) = \mathbb{K}\widehat{G} \otimes \mathbb{K}G$ with multiplication given by $(\alpha \otimes h)(\beta \otimes g) = \alpha\beta \otimes hg$. A pair $(\beta, g) \in \widehat{G} \times G$ is then a character of $G \times \widehat{G}$ via $(\beta, g)((h, \alpha)) = \beta(h)\alpha(g)$, $\forall h \in G$ and $\alpha \in \widehat{G}$. The simple Yetter-Drinfel'd module $H \bullet_\beta g$ is then a $D(H)$-module with action

$$(\alpha \otimes h) \cdot g = \alpha(g)\beta(h) g = (\beta, g)((h, \alpha)) g.$$

4. Some general results

I first start by presenting some general results on the tensor product of Yetter-Drinfel'd modules. Throughout this section $H = \bigoplus_{n=0}^{\infty} H_n$ is a graded Hopf algebra over an algebraically closed field $\mathbb{K}$, $H_0 = \mathbb{K}G$ where G is a finite abelian group and $H_m = H_{m+1} = \cdots = (0)$ for some $m > 0$.

Proposition I.33. *Let β, $\beta' \in \operatorname{Alg}(H, \mathbb{K})$ and g, $g' \in G(H)$. If $H \bullet_\beta g \otimes H \bullet_{\beta'} g'$ is a simple Yetter-Drinfel'd module, then*

$$H \bullet_\beta g \otimes H \bullet_{\beta'} g' \simeq H \bullet_{\beta * \beta'} g g'$$

Proof. Since $H \bullet_\beta g \otimes H \bullet_{\beta'} g'$ is a simple Yetter-Drinfel'd module, by Proposition I.30, there exist unique $\beta'' \in \operatorname{Alg}(H, \mathbb{K})$ and $g'' \in G(H)$ such that

$$H \bullet_\beta g \otimes H \bullet_{\beta'} g' \simeq H \bullet_{\beta''} g''$$

as Yetter-Drinfel'd modules. Let $\Phi : H \bullet_\beta g \otimes H \bullet_{\beta'} g' \to H \bullet_{\beta''} g''$ be such an isomorphism. Since Φ is a comodule map, we have

$$\begin{aligned} (\Phi \otimes \mathrm{id}\,) \circ \delta(g \otimes g') &= \delta \circ \Phi(g \otimes g') \quad \Rightarrow \\ (\Phi \otimes \mathrm{id}\,) \left(\sum g_{(0)} \otimes g'_{(0)} \otimes g'_{(1)} g_{(1)} \right) &= \Delta(\Phi(g \otimes g')). \end{aligned}$$

Then

$$\Phi(g \otimes g') \otimes g' g = \Delta(\Phi(g \otimes g')). \tag{I.5}$$

This implies that $\mathbb{K}\Phi(g \otimes g')$ is a (simple) right coideal of $H \bullet_{\beta''} g''$. In [19] it was shown that if N is a simple right coideal of H, then the only coideal contained in $H \bullet_\beta N$ is N. Therefore $\mathbb{K}\Phi(g \otimes g') = \mathbb{K}g''$ and so $g'' = \lambda \Phi(g \otimes g')$ for some $0 \neq \lambda \in \mathbb{K}$; we may assume that $\lambda = 1$. Applying $\varepsilon \otimes \mathrm{id}$ to both sides of Equation (I.5), we get that

$\Phi(g \otimes g') = \varepsilon(\Phi(g \otimes g'))g'g$. We then have:

$$g'' = \Phi(g \otimes g') = \varepsilon(\Phi(g \otimes g'))g'g.$$

Since distinct group-like elements are linearly independent, this implies that $g'' = g'g$.

Since $(H_i)^m = (0)$ for all $i \geq 1$ we have that $\beta * \beta'(H_i) = (0) = \beta''(H_i)$ for all $i \geq 1$. To show that $\beta'' = \beta * \beta'$ it is then enough to show that they agree on G. Let $h \in G$, then

$$\beta''(h)gg' = h{\bullet_{\beta''}}gg' = h{\bullet_{\beta''}}(\Phi(g \otimes g')) = \Phi(h \cdot (g \otimes g')) =$$

$$= \Phi(h{\bullet_\beta}g \otimes h{\bullet_{\beta'}}g') = \Phi(\beta(h)\beta'(h)g \otimes g') = (\beta * \beta')(h)gg',$$

and so $\beta''(h) = (\beta * \beta')(h)$ for all h in G. □

If H is any Hopf algebra and $\gamma : H \to \mathbb{K}$ is an algebra map, then γ has an inverse in $\mathrm{Hom}\,(H, \mathbb{K})$ given by $\gamma^{-1}(h) = \gamma(S(h))$, since

$$(\gamma * (\gamma \circ S))(h) = \sum \gamma(h_{(1)})\gamma(S(h_{(2)})) = \sum \gamma(h_{(1)}S(h_{(2)})) = \gamma(\epsilon(h)1_H) = \varepsilon(h)1_{\mathbb{K}}.$$

Let $N = \mathbb{K}n$ be a one-dimensional H-module. Then there is an algebra homomorphism $\gamma : H \to \mathbb{K}$ such that $h \cdot n = \gamma(h)n$ for all $h \in H$. Let $\mathbb{K}_\gamma$ be $\mathbb{K}$ as a vector space with the action given by $h \cdot 1 = \gamma(h)$, and so $N \simeq \mathbb{K}_\gamma$ as H-modules.

If M is any H-module and $\gamma : H \to \mathbb{K}$ is an algebra morphism, then the natural vector space isomorphism $M \otimes \mathbb{K}_\gamma \simeq M$ endows M with a new module structure, $\cdot'$, given by $h \cdot' m = \sum \gamma(h_{(2)})h_{(1)} \cdot m$. I will denote this module by M_γ.

Note that $\mathbb{K}_\gamma \otimes \mathbb{K}_{\gamma^{-1}} \simeq \mathbb{K}_\epsilon$ as H-modules, and therefore for any H-module M,

$$(M_\gamma)_{\gamma^{-1}} = M_\epsilon = M.$$

Remark I.34. Let H be any Hopf algebra and $\gamma : H \to \mathbb{K}$ an algebra map. If

M is an H-module and N is a submodule of M, then N_γ is a submodule of M_γ. In particular, M is simple if and only if M_γ is simple.

Let $\mathrm{Soc}(M)$ denote the socle of M, that is, $\mathrm{Soc}(M) = \oplus N$, the sum over all simple submodules of M. Then, by the last remark, we have that

$$\mathrm{Soc}(M_\gamma) = \left(\mathrm{Soc}(M)\right)_\gamma .$$

CHAPTER II

TWO-PARAMETER QUANTUM GROUPS

In 1985 Drinfel'd and Jimbo independently introduced the algebra $U_\theta(\mathfrak{g})$, a one-parameter deformation of the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ [9, 13]. They were first used to construct solutions to the quantum Yang-Baxter equations and have applications in various areas of mathematics and physics. For θ a root of unity, Lusztig defined the restricted one-parameter quantum group $\mathfrak{u}_\theta(\mathfrak{g})$, a finite-dimensional quotient of $U_\theta(\mathfrak{g})$. In what follows, I give the definitions of the two-parameter versions, $U_{r,s}(\mathfrak{g})$ and $\mathfrak{u}_{r,s}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_n$, the Lie algebra of $n \times n$ matrices of trace 0. These algebras are examples of the algebras constructed by Andruskiewitsch and Schneider in their classification of pointed Hopf algebras with abelian groups of group-like elements. In section 2, I give a theorem about factorization of simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules.

1. Definition of restricted quantum groups

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and let $\{\epsilon_1, \dots, \epsilon_n\}$ denote an orthonormal basis of an Euclidean space $E = \mathbb{R}^n$ with an inner product $\langle\ ,\ \rangle$. Let $\alpha_j = \epsilon_j - \epsilon_{j+1}$ $(j = 1, \dots, n-1)$. Let $r, s \in \mathbb{K}^\times$ be roots of unity with $r \neq s$ and ℓ be the least common multiple of the orders of r and s. Let θ be a primitive ℓth root of unity and y and z be nonnegative integers such that $r = \theta^y$ and $s = \theta^z$. Takeuchi defined the following Hopf algebra [22].

Definition II.1. The algebra $U = U_{r,s}(\mathfrak{sl}_n)$ is the unital associative $\mathbb{K}$-algebra generated by $\{e_j,\ f_j,\ \omega_j^{\pm 1},\ (\omega_j')^{\pm 1},\ \ 1 \leq j < n\}$, subject to the following relations.

(R1) The $\omega_i^{\pm 1}$, $(\omega_j')^{\pm 1}$ all commute with one another and $\omega_i \omega_i^{-1} = \omega_j'(\omega_j')^{-1} = 1$, (R2) $\omega_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} e_j \omega_i$ and $\omega_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} s^{-\langle \epsilon_{i+1}, \alpha_j \rangle} f_j \omega_i$,

(R3) $\omega_i' e_j = r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle} e_j \omega_i'$ and $\omega_i' f_j = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} f_j \omega_i'$,

(R4) $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(\omega_i - \omega_i')$.

(R5) $[e_i, e_j] = [f_i, f_j] = 0$ if $|i-j| > 1$,

(R6) $e_i^2 e_{i+1} - (r+s) e_i e_{i+1} e_i + rs e_{i+1} e_i^2 = 0$,

$e_i e_{i+1}^2 - (r+s) e_{i+1} e_i e_{i+1} + rs e_{i+1}^2 e_i = 0$,

(R7) $f_i^2 f_{i+1} - (r^{-1}+s^{-1}) f_i f_{i+1} f_i + r^{-1}s^{-1} f_{i+1} f_i^2 = 0$,

$f_i f_{i+1}^2 - (r^{-1}+s^{-1}) f_{i+1} f_i f_{i+1} + r^{-1}s^{-1} f_{i+1}^2 f_i = 0$,

for all $1 \le i, j < n$.

The following coproduct, counit, and antipode give U the structure of a Hopf algebra:

$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega_i',$$

$$\epsilon(e_i) = 0, \qquad \epsilon(f_i) = 0,$$

$$S(e_i) = -\omega_i^{-1} e_i, \qquad S(f_i) = -f_i (\omega_i')^{-1},$$

and ω_i, ω_i' are group-like, for all $1 \le i < n$.

Let U^0 be the group algebra generated by all $\omega_i^{\pm 1}$, $(\omega_i')^{\pm 1}$ and let U^+ (respectively, U^-) be the subalgebra of U generated by all e_i (respectively, f_i). Let

$$\mathcal{E}_{j,j} = e_j \quad \text{and} \quad \mathcal{E}_{i,j} = e_i \mathcal{E}_{i-1,j} - r^{-1} \mathcal{E}_{i-1,j} e_i \quad (i > j),$$

$$\mathcal{F}_{j,j} = f_j \quad \text{and} \quad \mathcal{F}_{i,j} = f_i \mathcal{F}_{i-1,j} - s \mathcal{F}_{i-1,j} f_i \quad (i > j).$$

The algebra U has a triangular decomposition $U \cong U^- \otimes U^0 \otimes U^+$ (as vector spaces), and the subalgebras U^+, U^- respectively have monomial Poincaré-Birkhoff-Witt

(PBW) bases [14, 4]

$$\mathcal{E} := \{\mathcal{E}_{i_1,j_1}\mathcal{E}_{i_2,j_2}\cdots\mathcal{E}_{i_p,j_p} \mid (i_1,j_1) \le (i_2,j_2) \le \cdots \le (i_p,j_p) \text{ lexicographically}\}, \quad \text{(II.1)}$$

$$\mathcal{F} := \{\mathcal{F}_{i_1,j_1}\mathcal{F}_{i_2,j_2}\cdots\mathcal{F}_{i_p,j_p} \mid (i_1,j_1) \le (i_2,j_2) \le \cdots \le (i_p,j_p) \text{ lexicographically}\}. \quad \text{(II.2)}$$

It is shown in [6] that all $\mathcal{E}_{i,j}^{\ell}$, $\mathcal{F}_{i,j}^{\ell}$, $\omega_i^{\ell} - 1$, and $(\omega_i')^{\ell} - 1$ $(1 \le j \le i < n)$ are central in $U_{r,s}(\mathfrak{sl}_n)$. The ideal I_n generated by these elements is a Hopf ideal [6, Thm. 2.17], and so the quotient

$$\mathfrak{u} = \mathfrak{u}_{r,s}(\mathfrak{sl}_n) = U_{r,s}(\mathfrak{sl}_n)/I_n \quad \text{(II.3)}$$

is a Hopf algebra, called the *restricted two-parameter quantum group.* Examination of the PBW-bases (II.1) and (II.2) shows that $\mathfrak{u}$ is finite-dimensional and Benkart and Witherspoon showed that $\mathfrak{u}$ is pointed [6, Prop. 3.2].

Let $\mathcal{E}_{\ell}$ and $\mathcal{F}_{\ell}$ denote the sets of monomials in $\mathcal{E}$ and $\mathcal{F}$ respectively, in which each $\mathcal{E}_{i,j}$ or $\mathcal{F}_{i,j}$ appears as a factor at most $\ell - 1$ times. Identifying cosets in $\mathfrak{u}$ with their representatives, we may assume $\mathcal{E}_{\ell}$ and $\mathcal{F}_{\ell}$ are basis for the subalgebras of $\mathfrak{u}$ generated by the elements e_i and f_i respectively.

Let $\mathfrak{b}$ be the Hopf subalgebra of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ generated by $\{\omega_i, e_i : 1 \le i < n\}$, and $\mathfrak{b}'$ the subalgebra generated by $\{\omega_i',\, f_i : 1 \le i < n\}$.

Benkart and Witherspoon showed that, under some conditions on the parameters r and s, $\mathfrak{b}^* \simeq (\mathfrak{b}')^{\text{coop}}$ as Hopf algebras ([6, Lemma 4.1]). This implies that $\mathfrak{b} \simeq ((\mathfrak{b}')^{\text{coop}})^*$; I present the lemma using the dual isomorphism of the original one.

Lemma II.2. [6, Lemma 4.1] *If* $\gcd(y^{n-1} - y^{n-2}z + \cdots + (-1)^{n-1}z^{n-1}, \ell) = 1$ *and* rs^{-1} *is a primitive* ℓ*th root of unity, then* $\mathfrak{b} \simeq ((\mathfrak{b}')^{\text{coop}})^*$ *as Hopf algebras. Such an* *isomorphism is given by*

$$\langle \omega_i, \omega_j' \rangle = r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} \quad \textit{and} \quad \langle \omega_i, f_j \rangle = 0, \tag{II.4}$$

and

$$\langle e_i, f_j^a g \rangle = \delta_{i,j}\delta_{1,a} \quad \forall\, g \in G(\mathfrak{b}'). \tag{II.5}$$

Proposition II.3. [6, Thm. 4.8] *Assume $r = \theta^y$ and $s = \theta^z$, where θ is a primitive ℓth root of unity, and*

$$\gcd(y^{n-1} - y^{n-2}z + \cdots + (-1)^{n-1}z^{n-1}, \ell) = 1.$$

Then there is an isomorphism of Hopf algebras $\mathfrak{u}_{r,s}(\mathfrak{sl}_n) \cong D'(\mathfrak{b}) \cong D((\mathfrak{b}')^{\mathrm{coop}})^{\mathrm{coop}}$.

In the special case $r = \theta$, a primitive ℓth root of unity, and $s = \theta^{-1}$, $\mathfrak{u} = \mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_n)$ is isomorphic to $D'((\mathfrak{b}')^{\mathrm{coop}})$ when n and ℓ are relatively prime.

Under the assumption that $\gcd(y^{n-1} - y^{n-2}z + \cdots + (-1)^{n-1}z^{n-1}, \ell) = 1$, by Proposition II.3, $\mathfrak{u}_{r,s}(\mathfrak{sl}_n) = (D((\mathfrak{b}')^{\mathrm{coop}}))^{\mathrm{coop}}$ and so $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules are Yetter-Drinfel'd modules for $(\mathfrak{b}')^{\mathrm{coop}}$ (only the algebra structure of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ plays a role when studying $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules, hence $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules are $D((\mathfrak{b}')^{\mathrm{coop}})$-modules). For simplicity I will denote $H = (\mathfrak{b}')^{\mathrm{coop}}$. Then $G = G(H) = \langle \omega_i' : 1 \leq i < n \rangle$ and H is a graded Hopf algebra with $\omega_i' \in H_0$ and $f_i \in H_1$ for all $1 \leq i < n$ and $H_j = (0)$ if $j \geq 2\ell$. Therefore Proposition I.30 applies to H and isomorphism classes of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules (or simple Yetter-Drinfel'd H-modules) are in one to one correspondence with $\mathrm{Alg}\,(H, \mathbb{K}) \times G(H)$.

2. Factorization of simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules

In this section I study under what conditions a simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-module can be factored as the tensor product of a one-dimensional module and a simple module which is also a module for $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)} = \mathfrak{u}_{r,s}(\mathfrak{sl}_n)/\mathfrak{u}_{r,s}(\mathfrak{sl}_n)(\mathbb{K}G_C(\mathfrak{u}_{r,s}(\mathfrak{sl}_n)))^+$. Let ℓ, n, y and z be fixed and θ be a primitive ℓth root of unity. Let A be the $(n-1)\times(n-1)$ matrix

$$A = \begin{pmatrix} y-z & z & 0 & 0 & \cdots & 0 \\ -y & y-z & z & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -y & y-z & z \\ 0 & \cdots & \cdots & 0 & -y & y-z \end{pmatrix}$$

The determinant of A is $y^{n-1} - y^{n-2}z + \cdots + (-1)^{n-1}z^{n-1}$. Throughout this section, assume that $\gcd(y^{n-1} - y^{n-2}z + \cdots + (-1)^{n-1}z^{n-1}, \ell) = 1$, and so $\det(A)$ is invertible in $\mathbb{Z}/\ell\mathbb{Z}$. I start by describing the set of central group-like elements in $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$. Clearly $G(\mathfrak{u}_{r,s}(\mathfrak{sl}_n)) = \langle \omega_i, \omega_i' : 1 \le i < n\rangle$.

Proposition II.4. *A group-like element $g = \omega_1^{a_1}\cdots\omega_{n-1}^{a_{n-1}}\omega_1'^{b_1}\cdots\omega_{n-1}'^{b_{n-1}}$ is central in $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ if and only if*

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A^{-1}A^t \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

in $(\mathbb{Z}/\ell\mathbb{Z})^{n-1}$.

Proof. The element g is central in $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ if and only if $ge_k = e_kg$ and $gf_k = f_kg$ for all $k = 1, \cdots, n-1$. By the relations (R2) and (R3) of the definition of $U_{r,s}(\mathfrak{sl}_n)$, for all $k = 1, \cdots, n-1$ we have that

$$\begin{aligned} ge_k &= \prod_{i=1}^{n-1}\left(r^{\langle\epsilon_i,\alpha_k\rangle}s^{\langle\epsilon_{i+1},\alpha_k\rangle}\right)^{a_i}\prod_{j=1}^{n-1}\left(r^{\langle\epsilon_{j+1},\alpha_k\rangle}s^{\langle\epsilon_j,\alpha_k\rangle}\right)^{b_j} e_kg \text{ and} \\ gf_k &= \prod_{i=1}^{n-1}\left(r^{-\langle\epsilon_i,\alpha_k\rangle}s^{-\langle\epsilon_{i+1},\alpha_k\rangle}\right)^{a_i}\prod_{j=1}^{n-1}\left(r^{-\langle\epsilon_{j+1},\alpha_k\rangle}s^{-\langle\epsilon_j,\alpha_k\rangle}\right)^{b_j} f_kg. \end{aligned}$$

Then g is central if and only if

$$\begin{aligned} 1 &= \prod_{i=1}^{n-1}\left(r^{\langle\epsilon_i,\alpha_k\rangle}s^{\langle\epsilon_{i+1},\alpha_k\rangle}\right)^{a_i}\prod_{j=1}^{n-1}\left(r^{\langle\epsilon_{j+1},\alpha_k\rangle}s^{\langle\epsilon_j,\alpha_k\rangle}\right)^{b_j} \\ &= s^{a_{k-1}}r^{a_k}s^{-a_k}r^{-a_{k+1}}r^{b_{k-1}}r^{-b_k}s^{b_k}s^{-b_{k+1}},\ \forall k=1,\cdots,n-1, \end{aligned}$$

where $a_0 = a_n = 0 = b_0 = b_n$. Since $r = \theta^y$ and $s = \theta^z$, the last equation holds if and only if

$$za_{k-1} + (y-z)a_k - ya_{k+1} = (-yb_{k-1} + (y-z)b_k + zb_{k+1}) \mod \ell, \tag{II.6}$$

for all $k = 1, \cdots, n-1$. The matrix of coefficients of the left hand side of this system of equations is

$$\begin{pmatrix} y-z & -y & 0 & 0 & \cdots & 0 \\ z & y-z & -y & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & z & y-z & -y \\ 0 & \cdots & \cdots & 0 & z & y-z \end{pmatrix} = A^{\mathrm{t}}$$

and the matrix of coefficients of the right hand side is

$$\begin{pmatrix} y-z & z & 0 & 0 & \cdots & 0 \\ -y & y-z & z & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -y & y-z & z \\ 0 & \cdots & \cdots & 0 & -y & y-z \end{pmatrix} = A.$$

We then have that g is central if and only if

$$A^{\mathrm{t}}\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = A\begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}$$

in $(\mathbb{Z}/\ell\mathbb{Z})^{n-1}$. □

Example II.5. For $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_n)$ ($y = 1$ and $z = \ell - 1$), the matrix A is symmetric. Therefore, a group-like element $g = \omega_1^{a_1} \cdots \omega_{n-1}^{a_{n-1}} \omega_1'^{b_1} \cdots \omega_{n-1}'^{b_{n-1}}$ is central if and only if $b_i = a_i$ for all $i = 1, \cdots, n-1$.

In general, $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)(\mathbb{K}G_C(\mathfrak{u}_{r,s}(\mathfrak{sl}_n)))^+ = \mathfrak{u}_{r,s}(\mathfrak{sl}_n)\{g - 1 : g \in G_C(\mathfrak{u}_{r,s}(\mathfrak{sl}_n))\}$. In particular, by the last example, we have that $\mathfrak{u}_{\theta,\theta^{-1}}(\mathbb{K}G_C(\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_n)))^+$ is generated by $\left\{\omega_i^{-1} - \omega_i' : i = 1, \ldots, n-1\right\}$. This gives $\overline{\mathfrak{u}_{\theta,\theta^{-1}}} \simeq \mathfrak{u}_\theta(\mathfrak{sl}_n)$, the one parameter quantum group.

Henceforth r and s are such that rs^{-1} is also a primitive ℓth root of unity, that is, $\gcd(y - z, \ell) = 1$.

Remark II.6. If $\beta \in G(H^*)$ and $g = \omega_1'^{c_1} \cdots \omega_{n-1}'^{c_{n-1}} \in G(H)$, by Proposition I.26, the Yetter-Drinfel'd module $H \bullet_\beta g$ is also a $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-module where the action of $H^* = \mathfrak{b}$ is given by

$$f \cdot h = \sum \langle f, h_{(2)} \rangle h_{(1)},$$

for all h in $H = (\mathfrak{b}')^{coop}$ and f in $H^* = \mathfrak{b}$. In particular,

$$\omega_i \cdot g = \langle \omega_i, g \rangle g = \prod_{j=1}^{n-1} \langle \omega_i, \omega_j' \rangle^{c_j} g = \prod_{j=1}^{n-1} \left(r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} \right)^{c_j} g.$$

Proposition II.7. *Let $\beta \in G(H^*)$ be defined by $\beta(\omega_i') = \theta^{\beta_i}$ and $g = \omega_1'^{c_1} \cdots \omega_{n-1}'^{c_{n-1}}$.*

The simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$*-module* $H\bullet_\beta g$ *is naturally a* $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)}$*-module if and only if*

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix} = -A^t \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{II.7}$$

in $(\mathbb{Z}/\ell Z)^{n-1}$.

Proof. $H\bullet_\beta g$ is a $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)}$-module if and only if $(h-1)\cdot m = 0$ for all h in $G_C(\mathfrak{u}_{r,s}(\mathfrak{sl}_n))$ and m in $H\bullet_\beta g$. If $h \in G_C(\mathfrak{u}_{r,s}(\mathfrak{sl}_n))$, $h\cdot m = m$ for all m in $H\bullet_\beta g$ if and only if $h\cdot g = g$.

Let $h = \omega_1^{a_1}\cdots\omega_{n-1}^{a_{n-1}}\omega_1'^{b_1}\cdots\omega_{n-1}'^{b_{n-1}} \in G_C(\mathfrak{u}_{r,s}(\mathfrak{sl}_n))$; then by Proposition II.4

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A^{-1}A^t \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}.$$

We have

$$\begin{aligned} \omega_1'^{b_1}\cdots\omega_{n-1}'^{b_{n-1}}\bullet_\beta g &= \beta(\omega_1'^{b_1}\cdots\omega_{n-1}'^{b_{n-1}})g \\ &= \theta^{b_1\beta_1+\cdots+b_{n-1}\beta_{n-1}}g \end{aligned} \tag{II.8}$$

and

$$\begin{aligned} \omega_1^{a_1}\cdots\omega_{n-1}^{a_{n-1}}\cdot g &= \langle\omega_1^{a_1}\cdots\omega_{n-1}^{a_{n-1}}, g\rangle g \\ &= \prod_{i=1}^{n-1}\langle\omega_i, g\rangle^{a_i} g \\ &= \prod_{i=1}^{n-1}\prod_{j=1}^{n-1}\left(r^{\langle\epsilon_i,\alpha_j\rangle}s^{\langle\epsilon_{i+1},\alpha_j\rangle}\right)^{a_ic_j} g \\ &= \prod_{i=1}^{n-1}\left(r^{-c_{i-1}}\left(rs^{-1}\right)^{c_i}s^{c_{i+1}}\right)^{a_i} g \\ &= \theta^x g \end{aligned} \tag{II.9}$$

where $c_0 = c_n = 0$ and $x = \sum_{i=1}^{n-1} \left(-yc_{i-1} + (y-z)c_i + zc_{i+1}\right) a_i$. From (II.8) and (II.9) we get that

$$h \cdot g = \theta^{x + \sum_{i=1}^{n-1} b_i \beta_i} g$$

for all $\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$, where $\begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A^{-1} A^t \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$. Now

$$\sum_{i=1}^{n-1} \left(-yc_{i-1} + (y-z)c_i + zc_{i+1}\right) a_i + \sum_{i=1}^{n-1} b_i \beta_i = 0 \bmod \ell$$

if and only if

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}^{\mathrm{t}} A \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = - \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}^{\mathrm{t}} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix} \text{ in } \mathbb{Z}/\ell\mathbb{Z}.$$

We then have that $H \bullet_\beta g$ is a $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)}$-module, if and only if

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}^{\mathrm{t}} A \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = - \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}^{\mathrm{t}} A \left(A^{-1}\right)^{\mathrm{t}} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix},$$

for all $(a_1, \cdots, a_{n-1})$ in $(\mathbb{Z}/\ell\mathbb{Z})^{n-1}$. This occurs if and only if

$$\begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = - \left(A^{-1}\right)^{\mathrm{t}} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix} \text{ in } (\mathbb{Z}/\ell\mathbb{Z})^{n-1}.$$

□

Given $g = (\omega_1')^{c_1} \cdots (\omega_{n-1}')^{c_{n-1}} \in G(H)$, let $\beta_1, \ldots, \beta_n$ be defined as in Equation

(II.7). I will denote by β_g the algebra map given by $\beta_g(\omega_i') = \theta^{\beta_i}$.

For any Hopf algebra H, let $\mathcal{S}_H$ denote the denote the set of isomorphism classes of simple H-modules. Then $\mathcal{S}_{\overline{H}}$ can be identified as the subset of $\mathcal{S}_H$ consisting of the H-modules that are naturally $\overline{H}$-modules. Combining the last proposition with Proposition I.30, we get

Corollary II.8. *The correspondence $G(H) \to \mathcal{S}_{\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)}}$ given by*

$$g \mapsto [H\bullet_{\beta_g} g]$$

is a bijection.

Example II.9. In the $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_2)$ case, the matrix A is $A = (2)$. Then, the simple $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_2)$-modules that are naturally $\mathfrak{u}_\theta(\mathfrak{sl}_2)$-modules, are of the form $H\bullet_\beta(\omega')^c$ with $\beta(\omega') = \theta^{-2c}$.

Example II.10. Using the last Proposition in the case $n = 3$, we have that the $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-module $H\bullet_\beta(\omega_1')^{c_1}(\omega_2')^{c_2}$ is a $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_3)}$-module if and only if, $\beta(\omega_1') = \theta^{(z-y)c_1+yc_2}$ and $\beta(\omega_2') = \theta^{-zc_1+(z-y)c_2}$. In particular, for the $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_3)$-modules, the condition is $\beta(\omega_1') = \theta^{-2c_1+c_2}$ and $\beta(\omega_2') = \theta^{c_1-2c_2}$.

For an algebra map $\chi : \mathfrak{u}_{r,s}(\mathfrak{sl}_n) \to \mathbb{K}$, let $\mathbb{K}_\chi$ be the 1-dimensional $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-module given by $h \cdot 1 = \chi(h)1$. Since $e_i^\ell = 0 = f_i^\ell$ we have that $\chi(e_i) = \chi(f_i) = 0$, and this together with (R4) of the Definition II.1 of $U_{r,s}(\mathfrak{sl}_n)$, gives that $\chi(\omega_i) = \chi(\omega_i')$. For each $i = 1, \dots, n-1$, since $\omega_i^\ell = 1$, $\chi(\omega_i) = \theta^{\chi_i}$ for some $0 \le \chi_i < \ell$.

Proposition II.11. *For $\chi : \mathfrak{u}_{r,s}(\mathfrak{sl}_n) \to \mathbb{K}$ an algebra map, we have that $\mathbb{K}_\chi \simeq H\bullet_{\chi_{|H}} g_\chi$, where*

$$g_\chi = \omega_1'^{d_1} \cdots \omega_{n-1}'^{d_{n-1}}, \text{ with}$$

$$\begin{pmatrix} d_1 \\ \vdots \\ d_{n-1} \end{pmatrix} = A^{-1} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} \ in\ (\mathbb{Z}/\ell\mathbb{Z})^{n-1}.$$

Proof. Since $\mathbb{K}_\chi$ is a simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-module, we have that $\mathbb{K}_\chi \simeq H\bullet_\beta g$ for some unique $\beta \in G(H^*)$ and $g \in G(H)$. Let $\phi : \mathbb{K}_\chi \to H\bullet_\beta g$ be an isomorphism of Yetter-Drinfel'd modules. We may assume that $g = \phi(1)$; then

$$\beta(\omega_i')g = \omega_i'\bullet_\beta g = \omega_i'\bullet_\beta(\phi(1)) = \phi\left(\omega_i' \cdot 1\right) = \phi(\chi(\omega_i')1) = \chi(\omega_i')g.$$

Therefore $\beta(\omega_i') = \chi(\omega_i')$ and since $\beta(f_i) = 0 = \chi(f_i)$ for all $i = 1, \cdots, n-1$, we have $\beta = \chi_{|_H}$.

We have that

$$\begin{aligned} \omega_i \cdot g &= \langle \omega_i, g\rangle g \\ &= \left(\prod_{j=1}^{n-1} \langle \omega_i, \omega_i'\rangle^{d_j}\right) g \\ &= \left(\prod_{j=1}^{n-1} \left(r^{\langle \epsilon_i, \alpha_j\rangle} s^{\langle \epsilon_{i+1}, \alpha_j\rangle}\right)^{d_j}\right) g \\ &= r^{-d_{i-1}}\left(rs^{-1}\right)^{d_i} s^{d_{i+1}} g \\ &= \theta^{y(d_i - d_{i-1}) + z(d_{i+1} - d_i)} g. \end{aligned} \tag{II.10}$$

On the other hand

$$\omega_i \cdot g = \omega_i \cdot \phi(1) = \phi(\omega_i \cdot 1) = \phi(\theta^{\chi_i} 1) = \theta^{\chi_i} g. \tag{II.11}$$

By (II.10) and (II.11) we have that

$$\begin{aligned} -yd_{i-1} + (y-z)d_i + zd_{i+1} &= \chi_i \bmod \ell, \ \forall i = 1, \cdots, n-1; \text{ and so} \\ A \begin{pmatrix} d_1 \\ \vdots \\ d_{n-1} \end{pmatrix} &= \begin{pmatrix} \chi_i \\ \vdots \\ \chi_{n-1} \end{pmatrix} \text{ in } (\mathbb{Z}/\ell\mathbb{Z})^{n-1}. \end{aligned}$$

□

For any Hopf algebra H, let $\mathcal{S}_H^1 = \{[N] \in \mathcal{S}_H \ : \dim(N) = 1\}$. Combining the last proposition and Proposition I.30 we get

Corollary II.12. *The correspondence* $G(\mathfrak{u}_{r,s}(\mathfrak{sl}_n)^*) \to \mathcal{S}^1_{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)}$ *given by*

$$\chi \mapsto [H \bullet_{\chi_{|H}} g_\chi]$$

is a bijection.

Theorem II.13. *The map* $\Phi : \mathcal{S}_{\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)}} \times \mathcal{S}^1_{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)} \to \mathcal{S}_{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)}$ *given by*

$$\Phi([M],[N]) = [M \otimes N]$$

is a bijection if and only if $\gcd((y-z)n, \ell) = 1$.

Proof. By the last corollary we have that 1-dimensional simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-modules are of the form $H \bullet_{\chi_{|H}} g_\chi$ with $\chi \in G(\mathfrak{u}_{r,s}(\mathfrak{sl}_n)^*)$. Also by Corollary II.8, simple $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)}$-modules are of the form $H \bullet_{\beta_g} g$ for $g \in G(H)$. Furthermore by Proposition I.33, we have that $H \bullet_{\beta_g} g \otimes H \bullet_{\chi_{|H}} g_\chi \simeq H \bullet_{\beta_g * \chi} g g_\chi$. Then Φ is a bijection if and only if

$$\Psi : \{(g, \beta_g) : g \in G(H)\} \times \{(g_\chi, \chi) : \chi \in G(\mathfrak{u}_{r,s}(\mathfrak{sl}_n)^*)\} \to G(H) \times G(H^*)$$

given by $\Psi\left((g,\beta_g),(g_\chi,\chi)\right) = (gg_\chi, \beta_g * \chi_{|H})$ is a bijection. The latter holds if and only if for all $h = \omega_1'^{b_1} \cdots \omega_{n-1}'^{b_{n-1}}$ and γ given by $\gamma(\omega_i') = \theta^{\gamma_i}$, there exist unique

$g = \omega_1'^{c_1} \cdots \omega_{n-1}'^{c_{n-1}}$ and χ with $\chi(w_i) = \chi(\omega_i') = \theta^{\chi_i}$, so that $h = gg_\chi$ and $\gamma = \beta_g * \chi_{|H}$. If $\beta_g(\omega_i') = \theta^{\beta_i}$ and $g_\chi = \omega_1'^{d_1} \cdots \omega_{n-1}'^{d_{n-1}}$, then

$$gg_\chi = \omega_1'^{c_1} \cdots \omega_{n-1}'^{c_{n-1}} \omega_1'^{d_1} \cdots \omega_{n-1}'^{d_{n-1}} \text{ and } (\beta_g * \chi_{|H})(\omega_i') = \theta^{\beta_i + \chi_i}.$$

Then Ψ is bijective if and only if the system of equations

$$\begin{pmatrix} c_1 + d_1 \\ \vdots \\ c_{n-1} + d_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}$$
$$\begin{pmatrix} \beta_1 + \chi_1 \\ \vdots \\ \beta_{n-1} + \chi_{n-1} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{pmatrix}$$

subject to

$$\begin{pmatrix} d_1 \\ \vdots \\ d_{n-1} \end{pmatrix} = A^{-1} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$$
$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix} = -A^{\mathrm{t}} \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}$$

has a unique solution for all $(b_1. \cdots, b_{n-1})$, $(\gamma_1, \cdots \gamma_{n-1})$. The last four vector equations are equivalent to

$$\begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} + A^{-1} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}$$

$$-A^{\mathrm{t}} \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} + \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{pmatrix}$$

which can be written as

$$\begin{pmatrix} \mathsf{id} & A^{-1} \\ -A^{\mathrm{t}} & \mathsf{id} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \\ \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{pmatrix}.$$

This last system has a unique solution if and only if the matrix

$$M = \begin{pmatrix} \mathsf{id} & A^{-1} \\ -A^{\mathrm{t}} & \mathsf{id} \end{pmatrix}$$

is invertible in $\mathrm{M}_{(n-1)\times(n-1)}(\mathbb{Z}/\ell\mathbb{Z})$, or equivalently, if $\gcd(\det(M), \ell) = 1$. By row-reducing M we have that

$$\det\begin{pmatrix} \mathsf{id} & A^{-1} \\ -A^{\mathrm{t}} & \mathsf{id} \end{pmatrix} = \det\begin{pmatrix} A & \mathsf{id} \\ -A^{\mathrm{t}} & \mathsf{id} \end{pmatrix}$$
$$= \det\begin{pmatrix} A+A^{\mathrm{t}} & 0 \\ -A^{\mathrm{t}} & \mathsf{id} \end{pmatrix}$$
$$= \det(A+A^{\mathrm{t}}).$$

Now

$$A+A^{\mathrm{t}} = \begin{pmatrix} 2(y-z) & z-y & 0 & 0 & \cdots & 0 \\ z-y & 2(y-z) & z-y & 0 & \cdots & 0 \\ 0 & z-y & 2(y-z) & z-y & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & z-y & 2(y-z) & z-y \\ 0 & \cdots & \cdots & 0 & z-y & 2(y-z) \end{pmatrix}$$

$$= (y-z)\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Therefore $\det(A+A^{\mathrm{t}}) = (y-z)^{n-1}n$. We then have that Φ is a bijection if and only if $\gcd\left((y-z)n, \ell\right) = 1$. □

CHAPTER III

COMPUTATIONAL RESULTS

In this chapter I present how I used the computer algebra system Singular::Plural [12] to construct simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules. These computations were begun as part of a joint project with G. Benkart and S. Witherspoon to understand the information obtained by Radford's method about $\mathfrak{u}_\theta(\mathfrak{sl}_n)$-modules [5]. To reduce computations, I use Proposition II.13 and construct only the $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules that are also modules for the quotient $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_3)}$ via the quotient map; that is, I only look at the cases when $\gcd((y-z)3,\ell) = 1$. According to Example II.10, we only need to construct the modules $H \bullet_\beta (\omega_1')^{c_1}(\omega_2')^{c_2}$ where $\beta(\omega_1') = \theta^{(z-y)c_1+yc_2}$ and $\beta(\omega_2') = \theta^{-zc_1+(z-y)c_2}$.

1. *G*-algebras

The system Singular::Plural allows us to do computations on G-algebras, which are algebras given by generators and re-writing relations where Gröbner basis computations can be done. I will give the precise definition of G-algebras and show that $H = (\mathfrak{b}')^{\text{coop}}$ is a quotient of a G-algebra. The notion of G-algebras was introduced by Apel in [2] and later refined by Levandovskyy in [16], and is a generalization of commutative polynomial rings.

Let $T = \mathbb{K}\langle x_1, \dots x_m\rangle$, the associative algebra generated by $x_1, \dots x_m$. The *standard monomials* in A, are elements from the set

$$\mathrm{Mon}_{\mathrm{S}}(A) = \{x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m} : \alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m\}.$$

A relation $<_A$ on $\mathrm{Mon}_{\mathrm{S}}(A)$ is called a *monomial ordering on* $\mathrm{Mon}_{\mathrm{S}}(A)$ if the following relations hold:

- $<_A$ is a total well-ordering.
- If $x^\alpha <_A x^\beta$ and $x^\gamma \in \mathrm{Mon_S}(A)$, then $x^{\alpha+\gamma} <_A x^{\beta+\gamma}$.

The *degree* of a monomial $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in \mathrm{Mon_S}(A)$ is $\deg(x^\alpha) = \alpha_1 + \cdots + \alpha_m$. For an element $0 \neq f \in \mathbb{K}\mathrm{Mon_S}(A)$, the leading monomial of f with respect to $<_A$ will be denoted by $\mathrm{lm}(f)$. An example of a monomial ordering is the *degree lexicographic* order, $<_{\mathrm{dlex}}$ which is defined by $x^\alpha <_{\mathrm{dlex}} x^\beta$ if $\deg(x^\alpha) < \deg(x^\beta)$ or if $\deg(x^\alpha) = \deg(x^\beta)$ and the left-most nonzero entry of $\beta - \alpha$ is positive. With this order we have $x_1 >_{\mathrm{dlex}} x_2 >_{\mathrm{dlex}} \cdots >_{\mathrm{dlex}} x_m$.

Definition III.1. Let $\mathbb{K}$ be a field and A be an algebra given in terms of generators and relations:

$$A = \mathbb{K}\langle x_1, \ldots, x_k | \, x_j x_i = C_{ij} x_i x_j + D_{ij}, \forall \, 1 \leq i < j \leq k \rangle,$$

where the $C_{ij} \in \mathbb{K}^\times$ and $D_{ij} \in \mathbb{K}\mathrm{Mon_S}(A)$. A is a G-algebra if the following conditions hold:

- There is a monomial well-ordering on $\mathrm{Mon}_s(A)$, $<_A$, such that $\mathrm{lm}(D_{ij}) <_A x_i x_j$ for all $1 \leq i < j \leq m$.
- $C_{ik} C_{jk} D_{ij} x_k - x_k D_{ij} + C_{jk} x_j D_{ik} - C_{ij} D_{ik} x_j + D_{jk} x_i - C_{ij} C_{ik} x_i D_{jk} = 0$, $\forall \, 1 \leq i < j < k \leq m$ (non-degeneracy conditions).

If A is a G-algebra, then the set $\{x_j x_i - C_{ij} x_i x_j - D_{ij}, \, 1 \leq i < j \leq m\}$ is a Gröbner basis for the ideal it generates in $\mathbb{K}\langle x_1, \ldots x_m \rangle$ [16]. Also, if A is an algebra with PBW basis, then the non-degeneracy conditions are automatically satisfied.

Let $\mathcal{B}'$ be the subalgebra of $U_{r,s}(\mathfrak{sl}_3)$ generated by $\{f_1, f_2, \omega_1', \omega_2'\}$. Adding the element $\mathcal{F}_{21} = f_2 f_1 - s f_1 f_2$ to the generating set, $\mathcal{B}'$ is generated by $\{f_1, \mathcal{F}_{21}, f_2, \omega_1', \omega_2'\}$ subject to the relations

1. $\mathcal{F}_{21}f_1 = rf_1\mathcal{F}_{21}$ and $f_2\mathcal{F}_{21} = r\mathcal{F}_{21}f_2$,

2. $f_2f_1 = sf_1f_2 + \mathcal{F}_{21}$,

3. $\omega_1'\mathcal{F}_{21} = s^{-1}\mathcal{F}_{21}\omega_1'$ and $\omega_2'\mathcal{F}_{21} = r\mathcal{F}_{21}\omega_2'$,

4. the second type of relations (R3) from Definition II.1,

   (a) $\omega_1' f_1 = rs^{-1}f_1\omega_1'$,

   (b) $\omega_2' f_1 = sf_1\omega_2'$,

   (c) $\omega_1' f_2 = r^{-1}f_2\omega_1'$,

   (d) $\omega_2' f_2 = rs^{-1}f_2\omega_2'$, and

5. $\omega_1'\omega_2' = w_2'\omega_1'$.

Therefore $\mathcal{B}'$ is generated by $\{x_1 = f_1,\ x_2 = \mathcal{F}_{21},\ x_3 = f_2,\ x_4 = \omega_1',\ x_5 = \omega_2'\}$, subject to relations $\{x_jx_i = C_{ij}x_ix_j + D_{ij}, 1 \leq i < j \leq 5\}$ where the coefficients C_{ij} and polynomials D_{ij} are given by the relations above; that is $D_{ij} = 0$ if $(i,j) \neq (1,3)$ and

1. $C_{12} = r$ and $C_{23} = r$,

2. $C_{13} = s$ and $D_{13} = \mathcal{F}_{21}$,

3. $C_{24} = s^{-1}$ and $C_{25} = r$,

4. (a) $C_{14} = rs^{-1}$,

   (b) $C_{15} = s$,

   (c) $C_{34} = r^{-1}$,

   (d) $C_{35} = rs^{-1}$, and

5. $C_{45} = 1$.

Recall from Chapter II that $\{f_1^{\alpha_1}\mathcal{F}_{21}^{\alpha_2}f_2^{\alpha_3}(\omega_1')^{\alpha_4}(\omega_2')^{\alpha_5}\}$ is a PBW basis for $\mathcal{B}'$; hence the non-degeneracy conditions are satisfied. If we take $<_{\mathcal{B}'}$ to be the degree lexicographic order with $f_1 > \mathcal{F}_{21} > f_2 > \omega_1' > \omega_2'$, then $\mathrm{lm}(D_{13}) = \mathcal{F}_{21} < f_1 f_2$ since $\deg(\mathcal{F}_{21}) = 1 < 2 = \deg(f_1 f_2)$. Hence $\mathcal{B}'$ is a G-algebra. Let I be the two-sided ideal of $\mathcal{B}'$ generated by the set

$$\left\{(\omega_1')^\ell - 1,\, (\omega_2')^\ell - 1,\, f_1^\ell,\, \mathcal{F}_{21}^\ell,\, f_2^\ell\right\},$$

we have that $H = (\mathfrak{b}')^{\mathrm{coop}} = \mathcal{B}'/I$.

2. **The code**

I now present how I defined $\mathfrak{b}'$ in Singular::Plural. The input and output are displayed in `typewriter` font and the output begins with the Singular comment characters (//). For simplicity I wrote `W(i)` for ω_i' and `Q` for θ. The library `linalg.lib` contains the function `mat_rk` that calculates the rank of a matrix; from the library `matrix.lib` I use the command `gauss_col` which transforms a matrix into its column-reduced Gauss normal form. The library `qhmoduli.lib` contains the functions `Max` and `Min` which compute the maximum and minimum of a list of integers.

```
LIB "linalg.lib";
LIB "matrix.lib";
LIB "qhmoduli.lib";
```

For ℓ, y and z positive integers with $\gcd(y - z, \ell) = 1$, I define the ring `B`. I write the code in terms of parameters `l`, `y` and `z`; the values of these parameters can be fixed in a preamble as will be shown in Example III.3.

```
ring B = (0,Q), (F(1), F(21), F(2), W(1), W(2)), Dp;
minpoly = rootofUnity(l);
```

The underlying coefficient field has characteristic 0 and it contains `Q`, which is a primitive ℓth root of unity and is generated by the elements `F(1), F(21), F(2), L(1), L(2)` (which correspond to $f_1, \mathcal{F}_{21}, f_2, \omega_1'$ and ω_2' respectively). The monomial ordering `Dp` is the degree lexicographical order. I write the elements C_{ij} and D_{ij} that define the relations in $\mathcal{B}'$; these are given with upper-triangular matrices `C` and `D`, and only the non-zero elements need to be given.

```
matrix C[5][5];
matrix D[5][5];
C[1,2] = Q^y; C[1,3] = Q^z; C[1,4] = Q^(y-z); C[1,5] = Q^z;
C[2,3] = Q^y; C[2,4] = Q^(-z); C[2,5] = Q^y;
C[3,4] = Q^(-y); C[3,5] = Q^(y-z);
C[4,5] = 1;
D[1,3] = F(21);
```

The command `ncalgebra(C,D)` creates the G-algebra with the relations given by `C` and `D`, and sets it as the base ring. I then give the generators of the ideal I.

```
ncalgebra(C,D);
option(redSB); option(redTail);
ideal I = F(1)^l, F(2)^l, W(1)^l - 1 , W(2)^l - 1, (F(21))^l;
qring B = twostd(I);
```

The last command sets the base ring to be the quotient of the previous ring by the ideal `I` (the ideal has to be given by a two-sided Gröbner basis, and so I applied `twostd` to it). We now have $\mathfrak{b}'$ as the base ring. The option `redSB` forces Singular to work with reduced Gröbner basis, and `redTail` forces the reduction of the tails of polynomials during Gröbner basis computations. Next I describe how I generate the simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules. Combining the definition of the $\bullet_\beta$ action (Equation (I.3) in

Lemma I.29), together with the coproduct formulas in $H = (\mathfrak{b}')^{\mathrm{coop}}$ we have that for all $x \in H$ and $g \in G(H)$,

$$f_i \bullet_\beta x = -xS^{\mathrm{op}}(f_i) + \beta(\omega_i')f_i x(\omega_i')^{-1} = -xf_i(\omega_i')^{-1} + \beta(\omega_i')f_i x(\omega_i')^{-1} \qquad \text{(III.1)}$$

and

$$\omega_i' \bullet_\beta g = \beta(\omega_i')w_i' g(\omega_i')^{-1} = \beta(\omega_i')g.$$

The second equation shows that if $g \in G(H)$, then $H \bullet_\beta g$ is generated by

$$\left\{(f_1^k \mathcal{F}_{21}^t f_2^m)\bullet_\beta g \,:\, 0 \leq k,\, t,\, m < \ell\right\}.$$

Recall from Chapter II that

$$\mathcal{F}_\ell = \{f_1^k \mathcal{F}_{21}^t f_2^m \,:\, 0 \leq k,\, t,\, m < \ell\}$$

and so

$$H \bullet_\beta g = \mathbb{K}\{f \bullet_\beta g \,:\, f \in \mathcal{F}_\ell\}.$$

Using Equation (III.1) I define the procedures `Beta1` and `Beta2`, so that `Beta1(a,h)` gives $f_1 \bullet_\beta h$ if $\beta(f_1) = \theta^a$ and `Beta(b,h)` gives $f_2 \bullet_\beta h$ if $\beta(f_2) = \theta^b$. Since $\mathcal{F}_{21} = f_2 f_1 - s f_1 f_2$, I define the procedure `Beta21` from the previous ones. For the results to be linear combinations of monomials where each generator appears as a factor at most ℓ times, I have to reduce the answer with respect to the ideal `std(0)`.

```
proc Beta1(int a, poly h)
{poly X;
 X = reduce((-h)*F(1)*W(1)^(l-1) + Q^a*F(1)*h*W(1)^(l-1), std(0));
 return(X);}


proc Beta2(int b, poly h)
```

```
{poly X;
 X = reduce((-h)*F(2)*W(2)^(l-1) + Q^b*F(2)*h*W(2)^(l-1), std(0));
 return(X);}


proc Beta21(int a, int b, poly h)
{return(Beta2(b, Beta1(a,h)) - Q^(z) * Beta1(a,Beta2(b,h)));}
```

Using compositions of these last procedures, I define the procedures `PBeta1`, `PBeta2` and `PBeta21`, so that if $k \in \mathbb{N}$, $h \in H$ and $\beta(f_1) = \theta^a$ then `PBeta1(a,h, k)` gives $f_1^k \bullet_\beta h$, and similarly for $f_2^k \bullet_\beta h$ and $\mathcal{F}_{21}^k \bullet_\beta h$.

```
proc PBeta1(int a, poly h, int k)
{ poly Y = h;
  for(int n=1;n<=k;n++)
   { Y = Beta1( a, Y);}
  return(Y); }


proc PBeta2(int b, poly h, int k)
{ poly Y = h;
  for(int n=1;n<=k;n++)
   { Y = Beta2( b, Y);}
  return(Y);}


proc PBeta21(int a, int b,  poly h, int k)
{ poly Y = h;
  for(int n=1;n<=k;n++)
   { Y = Beta21( a, b, Y);}
```

```
return(Y);}
```

Combining these procedures I define the procedure `Beta` so that if $0 \leq k, t, m < \ell$, $h \in H$ and $\beta : H \to \mathbb{K}$ is an algebra map given by $\beta(f_1) = \theta^a$ and $\beta(f_2) = \theta^b$, then `Beta(a,b,k,t,m,h)` gives $(f_1^k \mathcal{F}_{21}^t f_2^m) \bullet_\beta h$.

```
proc Beta( int a , int b , int k, int t, int m, poly h)
{return( PBeta1( a, PBeta21( a, b, PBeta2(b,g,m) , t), k)) ;}
```

Fix a group-like element $g = (\omega_1')^c(\omega_2')^d \in H$. In what follows I will construct a basis and compute the dimensions for the module $H \bullet_\beta g$, where $\beta(\omega_1') = \theta^{(z-y)c+yd}$ and $\beta(\omega_2') = \theta^{-zc+(z-y)d}$. The basic idea is to consider the linear map $T_\beta : \mathbb{K}\mathcal{F}_\ell \to H$ given by $T_\beta(f) = f \bullet_\beta g$, and construct the matrix M representing T_β in the basis $\mathcal{F}_\ell$ and $\{fh : f \in \mathcal{F}_\ell, h \in G(H)\}$ of $\mathbb{K}\mathcal{F}_\ell$ and H respectively. Then $\dim(H \bullet_\beta g) = \operatorname{rank}(M)$, and the non-zero columns of the column-reduced Gauss normal form of M give the coefficients for the elements of a basis of $H \bullet_\beta g$. The problem with this method is that since $\dim(H) = \ell^5$ and $\dim(\mathbb{K}\mathcal{F}_\ell) = \ell^3$, the size of M is $\ell^5 \times \ell^3$. Computing the Gauss normal form of these matrices is an expensive calculation even for small values of ℓ such as $\ell = 5$. However, by some reordering of $\mathcal{F}_\ell$ and of the PBW basis of H, M is block diagonal. I proceed to show how this is done.

For a monomial $h = f_1^{\alpha_1} \mathcal{F}_{21}^{\alpha_2} f_2^{\alpha_3} (\omega_1')^{\alpha_5} (\omega_2')^{\alpha_6}$ let $\deg_1(h) = \alpha_1 + \alpha_2$ and $\deg_2(h) = \alpha_2 + \alpha_3$. Note that Equation (III.1) implies that $h \bullet_\beta x$ is a linear combination of monomials m with $\deg_i(m) = \deg_i(h) + \deg_i(x)$. For all $0 \leq u, v < 2\ell$, let

$$D_{(u,v)} = \{h \in \mathcal{F}_\ell : \deg_1(h) = u \text{ and } \deg_2(h) = v\}$$

and

$$R_{(u,v)} = \{f(\omega_1')^{-u}(\omega_2')^{-v} g : f \in D_{(u,v)}\}.$$

Then for all $h \in D_{(u,v)}$, $h \bullet_\beta g \in \mathbb{K}R_{(u,v)}$. The possible pairs (u,v) are such that $0 \leq u, v \leq 2(\ell-1)$ and since $|v-u|$ is the maximum power of $\mathcal{F}_{21}$ that can be a factor of a monomial in $D_{(u,v)}$, we must have $|v-u| \leq \ell-1$; that is $u-(\ell-1) \leq v \leq u+\ell-1$. Another way of describing the sets $D_{(u,v)}$ and $R_{(u,v)}$ is as follows.

$$
\begin{aligned}
D_{(u,v)} &= \{f_1^{u-i}\mathcal{F}_{21}^i f_2^{v-i}, \forall i \in \mathbb{N} : 0 \leq u-i, i, v-i \leq \ell-1\} \\
&= \{f_1^{u-i}\mathcal{F}_{21}^i f_2^{v-i}, \forall i \in \mathbb{N} : n_{u,v} \leq i \leq m_{u,v}\}
\end{aligned}
$$

where $n_{u,v} = \max(0, \ell-1-u, \ell-1-v)$ and $m_{u,v} = \min(\ell-1, u, v)$. Since $(\omega_i')^{-1} = (\omega_i')^{\ell-1}$, if $g = (\omega_1')^c(\omega_2')^d$ we also have

$$
R_{(u,v)} = \{f(\omega_1')^{(\ell-1)u+c}(\omega_2')^{(\ell-1)v+d} : f \in D_{(u,v)}\}.
$$

Remark III.2. It is clear that $\mathcal{F}_\ell = \bigcup D_{(u,v)}$, the union disjoint, and that $H \bullet_\beta g = \oplus \mathbb{K}R_{(u,v)}$. Therefore a basis for $H \bullet_\beta g$ is a disjoint union of the bases for $\mathbb{K}D_{(u,v)} \bullet_\beta g$ for all possible pairs (u,v), and $\dim(H \bullet_\beta g) = \sum_{(u,v)} \dim(\mathbb{K}D_{(u,v)} \bullet_\beta g)$.

For ideals I_1 and I_2 given by a list of their generators, the command `coeffs` applied to the pair (I_1, I_2) returns a matrix A such that $I_2 A = I_1$, where the ideals I_1 and I_2 are thought of as one-row matrices whose entries are their generators. Therefore, for given u and v, if $M_{u,v}$ is the result of applying `coeffs` to the pair $(D_{(u,v)}, R_{(u,v)})$, then $\text{rank}(M_{(u,v)}) = \dim(\mathbb{K}D_{(u,v)} \bullet_\beta g)$, and if $N_{(u,v)}$ is the column-reduced Gauss normal form of $M_{(u,v)}$, the non-zero columns of $D_{(u,v)}N_{(u,v)}$ form a basis of $\mathbb{K}D_{(u,v)} \bullet_\beta g$.

I define the procedure `Submod`, where the output of `Submod(a,b,u,v)` is a list `L`, where the first component of the list is a basis for $D_{(u,v)} \bullet_\beta g$ and the second component is $\dim(D_{(u,v)} \bullet_\beta g)$.

```
proc Submod(int c, int d, int u, int v)
{ list L;
```

```
    ideal D;
    ideal R;
    list e = u-(l-1),v-(l-1),0; int n= Max(e);
    list f = u,v, l-1; int m= Min(f);
    int a = (z-y)*c+ y*d; int b= -z*c+(z-y)*d;
  for(int i= n; i<= m; i++)
  {
  D[i+1-n] = Beta(a, b , u-i, i, v-i , W(1)^c * W(2)^d);
  R[i+1-n] = F(1)^(u-i)* F(21)^i* F(2)^(v-i)*
             W(1)^(((l-1)*u+c) mod l)* W(2)^(((l-1)*v+d) mod l);}
  matrix M = coeffs(D,R);
  matrix N = gauss_col(M);
  matrix K[1][m-n+1] = R;
  matrix S = K*N;
  L[1] = compress(S);
  L[2] = mat_rk(N);
  return(L);}
```

The command `compress` deletes the zero columns of a matrix. For $g = (\omega_1')^c(\omega_2')^d$ the procedure `Totalbasis(c,d)` returns $\dim(H\bullet_\beta g)$ and a basis for $H\bullet_\beta g$, and is defined using Remark III.2.

```
  proc Totalbasis(int c , int d)
   { list T; matrix A; int t; t = 0;
     for(int u = 0; u<=2*(l-1); u++)
       { list e = 0, u-(l-1);
         list f = u+(l-1), 2*(l-1);
```

```
        for(int v = Max(e); v <= Min(f); v++)
          { list M = Submod(c,d, u,v);
            A = compress(concat(A, M[1]));
            t = t + M[2];
          }
      }
    T[1] = A; T[2] = t; return(T);
  }
```

Example III.3. For $\ell = 5$, $y = 1$ and $z = 4$, for $g = (\omega_1')^4(\omega_2')^2$, I construct the module $H{\bullet_\beta}g$ as follows. To give SINGULAR:PLURAL the values of ℓ, y and z, I write at the beginning of the code

```
ring r0 = 0,x,dp;
int l = 1;
int y = 4;
int z = 1;
```

Then the command

```
Totalbasis(4,2);
```

returns

```
// [1]:
//    _[1,1]=W(1)^4*W(2)^2
//    _[1,2]=F(1)*W(1)^3*W(2)^2
//    _[1,3]=(-Q^3-Q^2-2*Q-1)*F(1)*F(2)*W(1)^3*W(2)+F(21)*W(1)^3*W(2)
// [2]:
//      3
```

which tells us that $\dim(H\bullet_\beta((\omega_1')^4(\omega_2')^2)) = 3$. In this case $\beta(\omega_i') = \theta^{3.4+2} = \theta^4$ and $\beta(\omega_2') = \theta^{-4.4+3.2} = 1$. A basis for $H\bullet_\beta g$ is $\{1\bullet_\beta g, f_1\bullet_\beta g, \mathcal{F}_{21}\bullet_\beta g\}$ since

```
Beta(4,0,0,0,0,W(1)^4*W(2)^2);
Beta(4,0,1,0,0,W(1)^4*W(2)^2)/(-Q^3-Q^2-2*Q-1);
Beta(4,0,0,1,0,W(1)^4*W(2)^2)/(-Q^3-Q^2-2*Q-1);
```

returns

```
// W(1)^4*W(2)^2
// F(1)*W(1)^3*W(2)^2
// (-Q^3-Q^2-2*Q-1)*F(1)*F(2)*W(1)^3*W(2)+F(21)*W(1)^3*W(2)
```

3. Computational results and conjectures

For $\ell = 5$, y and z such that $\gcd(3(y^2 - yz + z^2)(y - z), \ell) = 1$ and $g = (\omega_1')^c(\omega_2')^d$ $(0 \le c, d < 5)$ the corresponding $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_3)}$-module $H\bullet_\beta g$ has dimension $\dim(c, d)$, where $\dim(c, d)$ is the entry in position $(c + 1, d + 1)$ of the symmetric matrix:

$$\mathrm{DIM} = \begin{pmatrix} 1 & 60 & 90 & 15 & 18 \\ 60 & 8 & 10 & 15 & 39 \\ 90 & 10 & 19 & 35 & 3 \\ 15 & 15 & 35 & 63 & 6 \\ 18 & 39 & 3 & 6 & 125 \end{pmatrix}.$$

For $\ell = 7$, the results are analogous to the case $\ell = 5$, with matrix

$$\mathrm{DIM} = \begin{pmatrix} 1 & 105 & 162 & 210 & 24 & 42 & 33 \\ 105 & 8 & 10 & 273 & 21 & 36 & 75 \\ 162 & 10 & 27 & 35 & 28 & 63 & 114 \\ 210 & 273 & 35 & 37 & 71 & 3 & 6 \\ 24 & 21 & 28 & 71 & 117 & 154 & 15 \\ 42 & 36 & 63 & 3 & 154 & 215 & 15 \\ 33 & 75 & 114 & 6 & 15 & 15 & 343 \end{pmatrix}.$$

By looking at these results, and the results obtained for other values of ℓ, I formulate the following conjecture:

Conjecture III.4. *Let y and z be integers such that $\gcd(3(y^2-yz+z^2)(y-z),\ell) = 1$ and set $r = \theta^y$ and $s = \theta^z$. For integers $0 \leq c,\, d < \ell$ let $g = (\omega_1')^c(\omega_2')^d \in G(H)$ and $\beta : H \to \mathbb{K}$ be the algebra map given by $\beta(f_1) = \theta^{(z-y)c+d}$ and $\beta(f_2) = \theta^{-zc+(z-y)d}$. Let m_1 and m_2 be defined by*

$$m_1 \equiv (2c-d+1) mod\ \ell, \quad m_2 \equiv (2d-c+1) mod\ \ell \quad and \quad 0 < m_i \leq \ell.$$

If $m_1 + m_2 \leq \ell$ then

$$dim(H \bullet_\beta g) = \frac{m_1 m_2 (m_1 + m_2)}{2}.$$

If $m_1 + m_2 > \ell$, let $x = m_1 + m_2 - \ell$, then

$$dim(H \bullet_\beta g) = \frac{m_1 m_2 (m_1 + m_2)}{2} - \frac{(m_1 - x)(m_2 - x)(m_1 + m_2 - 2x)}{2}.$$

In the particular case when $y = 1$ and $z = \ell - 1$, the formulas above for the dimensions of the simple $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_3)$-modules appeared in a work by Dobrev [8], where he calculated the dimensions of the simple modules for $U_\theta(\mathfrak{sl}_3)$, the infinite dimensional one-parameter quantum group. By analyzing the results of the calculations in SINGULAR::PLURAL I formulate the following conjecture about simple $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_3)$-modules.

Conjecture III.5. *For* $g = (\omega_1')^c(\omega_2')^d \in G(H)$, *let* $m_1 \equiv (2c - d + 1) mod\, \ell$ *and* $m_2 \equiv (2d - c + 1) mod\, \ell$, $0 < m_i \leq \ell$. *Let* $\beta : H \to \mathbb{K}$ *be the algebra map defined by* $\beta(f_1) = \theta^{-2c+d} = \theta^{-m_1+1}$ *and* $\beta(f_2) = \theta^{-c+2d} = \theta^{-m_2+1}$ *so that* $H \bullet_\beta g$ *is a* $\overline{\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_3)}$-*module.*

If $m_1 + m_2 \leq \ell$, *then the set*

$$\left\{ f_1^i \mathcal{F}_{21}^j f_2^k \bullet_\beta g : 0 \leq i < m_1,\ 0 \leq j < \ell,\ 0 \leq k < m_2 \text{ and } i + j + k \leq m_1 + m_2 - 2 \right\}$$

is a basis for $H \bullet_\beta g$.

The conjecture was checked in PLURAL for $\ell = 5, 7, 11$, and calculations show that it holds when $m_2 = 1$.

CHAPTER IV

POINTED HOPF ALGEBRAS OF RANK ONE

Recently Andruskiewitsch and Schneider classified the pointed Hopf algebras with abelian groups of group-like elements, over an algebraically closed field of characteristic 0 [1]. Earlier, in 2005, Krop and Radford classified the pointed Hopf algebras of rank one, where $\mathrm{rank}(H)+1$ is the rank of $H_{(1)}$ as an $H_{(0)}$-module and H is generated by $H_{(1)}$ as an algebra, where $H_{(1)}$ is the first term of the coradical filtration of H [15]. They also studied the representation theory of $D(H)$ in a fundamental case. Using Radford's construction of simple modules, in Theorem IV.18, I give necessary and sufficient conditions for the tensor product of two $D(H)$-modules to be completely reducible.

1. **Pointed Hopf algebras of rank one of nilpotent type**

Let G be a finite abelian group, $\mathbb{K}$ an algebraically closed field of characteristic zero, $\chi : G \to \mathbb{K}$ a character and $a \in G$; we call the triple $\mathcal{D} = (G, \chi, a)$ *data*. Let $\ell := |\chi(a)|$, $N := |a|$ and $M = |\chi|$; note that ℓ divides both N and M. In [15] Krop and Radford defined the following Hopf algebra.

Definition IV.1. Let $\mathcal{D} = (G, \chi, a)$ be data. The Hopf algebra $H_{\mathcal{D}}$ is generated by G and x as a $\mathbb{K}$-algebra, with relations:

1. $x^\ell = 0$.
2. $xg = \chi(g)gx$, for all $g \in G$.

The coalgebra structure is given by $\Delta(x) = x \otimes a + 1 \otimes x$ and $\Delta(g) = g \otimes g$ for all $g \in G$.

The Hopf algebra $H_{\mathcal{D}}$ is pointed of rank one. Let $\Gamma = \mathrm{Hom}\,(G, \mathbb{K}^\times)$, the set of group homomorphisms from G to $\mathbb{K}^\times$ also written $\widehat{G}$.

Proposition IV.2 (Krop and Radford [15])**.** *As a $\mathbb{K}$-algebra, $H_{\mathcal{D}}^*$ is generated by Γ and ξ subject to relations:*

1. $\xi^\ell = 0$.
2. $\xi\gamma = \gamma(a)\gamma\xi$, *for all* $\gamma \in \Gamma$.

The coalgebra structure of $H_{\mathcal{D}}^*$ is determined by $\Delta(\xi) = \xi\otimes\chi + 1\otimes\xi$ and $\Delta(\gamma) = \gamma\otimes\gamma$ for all $\gamma \in \Gamma$.

Proposition IV.3 (Krop and Radford [15])**.** *The double $D(H_{\mathcal{D}})$ is generated by G, x, Γ, ξ subject to the relations defining $H_{\mathcal{D}}$ and $H_{\mathcal{D}}^*$ and the following relations:*

1. $g\gamma = \gamma g$ *for all* $g \in G$ *and* $\gamma \in \Gamma$.
2. $\xi g = \chi^{-1}(g)g\xi$ *for all* $g \in G$.
3. $[x, \xi] = a - \chi$.
4. $\gamma(a)x\gamma = \gamma x$ *for all* $\gamma \in \Gamma$.

Recall that the coalgebra structure of $H_{\mathcal{D}}^*$ in $D(H_{\mathcal{D}})$ is the co-opposite to the one in H^*. Then in $D(H_{\mathcal{D}})$, $\Delta(\xi) = \chi\otimes\xi + \xi\otimes 1$. Note that $H_{\mathcal{D}}$ satisfies the hypothesis of Proposition I.30, where elements in G have degree 0 and x has degree 1. Therefore, simple $D(H_{\mathcal{D}})$-modules are of the form $H\bullet_\beta g$, for $g \in G$ and $\beta \in G(H^*) = \Gamma$.

2. Factorization of simple $D(H_{\mathcal{D}})$-modules

In this section I study under what conditions a simple $D(H_{\mathcal{D}})$-module can be factored as the tensor product of a one-dimensional module with a simple module which is also a module for $\overline{D(H_{\mathcal{D}})} = D(H_{\mathcal{D}})/D(H_{\mathcal{D}})(\mathbb{K}G_C(D(H_{\mathcal{D}})))^+$. I also study, under certain conditions on the parameters, the reducibility of the tensor product of two simple $D(H_{\mathcal{D}})$-modules.

I start by describing the central group-like elements of $D(H_{\mathcal{D}})$. It is clear that $G(D(H_{\mathcal{D}})) = G \times \Gamma$. An element $(g, \gamma) \in G \times \Gamma$ will be denoted by $g\gamma$. An element $g\gamma$ is central in $D(H_{\mathcal{D}})$ if and only if $(g\gamma)x = x(g\gamma)$ and $(g\gamma)\xi = \xi(g\gamma)$. Using the relations of $D(H_D)$, we have that

$$g\gamma x = \gamma(a)gx\gamma = \chi^{-1}(g)\gamma(a)xg\gamma,$$

and

$$g\gamma\xi = \gamma g\xi = \chi(g)\gamma\xi g = \chi(g)\gamma(a)^{-1}\xi\gamma g.$$

Hence, $g\gamma$ is central if only if $\chi^{-1}(g)\gamma(a) = 1$. Let $\mathrm{ev}_{\chi^{-1}a} : G \times \Gamma \to \mathbb{K}^\times$ be the character given by $\mathrm{ev}_{\chi^{-1}a}(g\gamma) = \chi^{-1}(g)\gamma(a)$; we just showed the following lemma:

Lemma IV.4. $G_C(D(H_{\mathcal{D}})) = \mathrm{Ker}\,(ev_{\chi^{-1}a})$.

For $\alpha : D(H_{\mathcal{D}}) \to \mathbb{K}$ an algebra map, let $\mathbb{K}_\alpha$ be the one-dimensional module defined by $h \cdot k = \alpha(h)k$ for all $h \in D(H_{\mathcal{D}})$ and $k \in \mathbb{K}$. Note that α being an algebra map implies that $\alpha(x) = \alpha(\eta) = 0$ (because $0 = x^\ell = \xi^\ell$) and $\alpha(a) = \alpha(\chi)$ (by the third relation in Proposition IV.3). Since $\alpha(x) = \alpha(\eta) = 0$, we can think of α as a group homomorphism $\alpha : G \times \Gamma \to \mathbb{K}^\times$, that is, $\alpha \in \widehat{G \times \Gamma} \simeq \Gamma \times G$. Let $\beta_\alpha \in \Gamma$ and $g_\alpha \in G$ so that $\alpha = \beta_\alpha g_\alpha$; that is $\alpha(g\gamma) = \beta_\alpha(g)\gamma(g_\alpha)$ for all $g\gamma$ in $G \times \Gamma$. If we extend β_α to $H_{\mathcal{D}}$ by setting $\beta_\alpha(x) = 0$ and also call this extension β_α (as no confusion will arise), we have $\beta_\alpha = \alpha_{|H_{\mathcal{D}}}$.

Proposition IV.5. *$\mathbb{K}_\alpha \simeq H_{\mathcal{D}} \bullet_{\beta_\alpha} g_\alpha$ as Yetter-Drinfel'd $H_{\mathcal{D}}$-modules.*

Proof. Since $\mathbb{K}_\alpha$ is a simple Yetter-Drinfel'd module, there exists an isomorphism of Yetter-Drinfel'd modules $\Phi : \mathbb{K}_\alpha \to H_{\mathcal{D}} \bullet_\beta g$ for some algebra map $\beta : H_{\mathcal{D}} \to \mathbb{K}$ and some $g \in G$. We may assume that $\Phi(1) = g$. Let $h \in G$, we have

$$h{\bullet_\beta}g = \beta(h)g.$$

Since Φ is a module map,

$$\begin{aligned} h{\bullet_\beta}g &= h{\bullet_\beta}\Phi(1) = \Phi(h \cdot 1) = \Phi(\alpha(h)) \\ &= \alpha(h)\Phi(1) = \beta_\alpha(h)g. \end{aligned}$$

We then have $\beta(h) = \beta_\alpha(h)$ for all h in G, and since $\beta(x) = \beta_\alpha(x) = 0$, $\beta = \beta_\alpha$.

If $\gamma \in \Gamma$, then

$$\gamma{\bullet_\beta}g = \gamma(g)g.$$

On the other hand,

$$\gamma{\bullet_\beta}g = \gamma{\bullet_\beta}\Phi(1) = \Phi(\gamma \cdot 1) = \Phi(\alpha(\gamma)1) = \alpha(\gamma)\Phi(1) = \gamma(g_\alpha)g.$$

Then $\gamma(g) = \gamma(g_\alpha)$ for all $\gamma \in \Gamma$, hence $g = g_\alpha$. □

For simplicity let $K = \mathrm{Ker}\,(ev_{\chi^{-1}a})$. If $\alpha = \beta_\alpha g_\alpha \in \Gamma \times G$, the condition $\alpha(a) = \alpha(\chi)$ is $\beta_\alpha(a) = \chi(g_\alpha)$ or $\chi^{-1}(g_\alpha)\beta_\alpha(a) = 1$. Hence, α in $\Gamma \times G$ defines a one-dimensional module if and only if $g_\alpha\beta_\alpha \in \mathrm{Ker}\,(\mathrm{ev}_{\chi^{-1}a}) = K$. This, together with the previous proposition, shows

Corollary IV.6. *The set $\mathcal{S}^1_{D(H_\mathcal{D})}$ of isomorphism classes of one dimensional $D(H_\mathcal{D})$-modules is in one to one correspondence with K.*

Recall that $\overline{D(H_\mathcal{D})} = D(H_\mathcal{D})/D(H_\mathcal{D})(\mathbb{K}G_C(D(H_\mathcal{D})))^+$. Since $G_C(D(H_\mathcal{D})) = K$,

$$D(H_\mathcal{D})(\mathbb{K}G_C(D(H_\mathcal{D})))^+ = D(H)\{g\gamma - 1 :\ g\gamma \in K\}.$$

For a group A and a subgroup $B \subset A$, let

$$B^{\perp} = \{f \in \widehat{A} : f(b) = 1 \text{ for all } b \in B\}.$$

Note that $K^{\perp} \subset \widehat{G \times \Gamma} \simeq \Gamma \times G$.

Proposition IV.7. *For $\beta \in G(H_{\mathcal{D}}^*) = \Gamma$ and $g \in G$, the simple $D(H_{\mathcal{D}})$-module $H_{\mathcal{D}} \bullet_{\beta} g$ is also a $\overline{D(H_D)}$-module via the quotient map, if and only if $\beta g \in K^{\perp}$.*

Proof. $H_{\mathcal{D}} \bullet_{\beta} g$ is a $\overline{D(H_{\mathcal{D}})}$-module, if and only if $f\gamma \cdot (h \bullet_{\beta} g) = h \bullet_{\beta} g$, for all $f\gamma \in K$ and $h \in H_{\mathcal{D}}$. Since $K \subset \mathcal{Z}(D(H_{\mathcal{D}}))$, if $f\gamma \in K$ then $f\gamma \cdot (h \bullet_{\beta} g) = (f\gamma h) \cdot g = (hf\gamma) \cdot g = h \bullet_{\beta} ((f\gamma) \cdot g)$. Thus, $H_{\mathcal{D}} \bullet_{\beta} g$ is a $\overline{D(H_{\mathcal{D}})}$-module, if and only if $f\gamma \cdot g = g$, for all $f\gamma \in K$. Now $f\gamma \cdot g = f \bullet_{\beta} \gamma(g) g = \gamma(g)\beta(f) g$. And so, $H_{\mathcal{D}} \bullet_{\beta} g$ is a $\overline{D(H_{\mathcal{D}})}$-module, if and only if $\gamma(g)\beta(f) = 1$ for all $f\gamma \in K$; that is, if and only if, $\beta g \in K^{\perp}$. □

Lemma IV.8. $K^{\perp} = \langle ev_{\chi^{-1}a} \rangle$.

Proof. Since $K^{\perp} \simeq \widehat{\left(\frac{G \times \Gamma}{K}\right)}$, we have $|K^{\perp}| = |\frac{G \times \Gamma}{K}| = |\operatorname{Im} \operatorname{ev}_{\chi^{-1}a}| = |\operatorname{ev}_{\chi^{-1}a}|$; the last equality holding as $\operatorname{Im} \operatorname{ev}_{\chi^{-1}a}$ is cyclic (since it is a finite subgroup of $\mathbb{K}^{\times}$). By the definitions of K and $K^{\perp}$, $\operatorname{ev}_{\chi^{-1}a} \in K^{\perp}$, hence $K^{\perp} = \langle \operatorname{ev}_{\chi^{-1}a} \rangle$. □

It will be convenient to think of $K^{\perp}$ as a subgroup of $G \times \Gamma$ via the identification $\widehat{G \times \Gamma} \simeq \widehat{G} \times \widehat{\Gamma} \simeq \Gamma \times G \simeq G \times \Gamma$. Under this identification we have $K^{\perp} = \langle a\chi^{-1} \rangle$.

Remark IV.9. We can restate Proposition IV.7 as follows: the simple $D(H_{\mathcal{D}})$-modules that are also $\overline{D(H_{\mathcal{D}})}$-modules are of the form $H_{\mathcal{D}} \bullet_{(\chi^{-c})} a^c$, for $c = 1, \ldots, |a\chi^{-1}|$.

Recall that $\mathcal{S}_{D(H_{\mathcal{D}})}$ denotes the set of isomorphism classes of simple $D(H_{\mathcal{D}})$-modules. Combining Proposition I.33, Corollary IV.6 and Proposition IV.7, we get that the map

$$\Phi : \mathcal{S}_{\overline{D(H_{\mathcal{D}})}} \times \mathcal{S}^1_{D(H_{\mathcal{D}})} \to \mathcal{S}_{D(H_{\mathcal{D}})}$$

given by $\Phi(U,V) = U \otimes V$, is equivalent to the multiplication map

$$\mu : K^{\perp} \times K \to G \times \Gamma,$$

under the identification of simple $D(H_{\mathcal{D}})$-modules with elements of $G \times \Gamma$.

Theorem IV.10. *The map Φ as above is a bijection if and only if ℓ is odd and $\ell = M = N$.*

Proof. By the last remark, Φ is an bijection, if an only if $G \times \Gamma = K^{\perp} \times K$, that is $G \times \Gamma = K^{\perp}K$ and $K \cap K = \{1\}$. Now $|K^{\perp}| = |\frac{G\times\Gamma}{K}| = \frac{|G\times\Gamma|}{|K|}$, and so $|K^{\perp}K| = \frac{|K^{\perp}||K|}{|K^{\perp}\cap K|} = \frac{|G\times\Gamma|}{|K^{\perp}\cap K|}$. We then have that $K^{\perp}K = G \times \Gamma$ if and only if $K^{\perp} \cap K = \{1\}$. If $\ell = M = N$, then $|a| = |\chi| = \ell$ and so $|a\chi^{-1}| = \ell$. Since $K^{\perp} \cap K \subset K^{\perp} = \langle a\chi^{-1}\rangle$, we have that $K^{\perp} \cap K = \langle (a\chi^{-1})^r \rangle$ for some $r \in \{1, \cdots, \ell\}$. Since $(a\chi^{-1})^r \in K = \mathrm{Ker}\,(\mathrm{ev}_{\chi^{-1}a})$, $1 = \mathrm{ev}_{\chi^{-1}a}\left((a\chi^{-1})^r\right) = \left(\chi^{-1}(a)\right)^{2r}$ and so $\ell | \, 2r$. If ℓ is odd, then $\ell | \, r$ and so $(a\chi^{-1})^r = 1$, giving $K^{\perp} \cap K = \{1\}$.

Conversely, if $K^{\perp} \cap K = \{1\}$, let $n = |a\chi^{-1}|$. Then for all $r \in \{1, \cdots, n-1\}$, $(a\chi^{-1})^r \notin K$. If either $M \neq \ell$ or $N \neq \ell$, then $n > \ell$ and so $(a\chi^{-1})^{\ell} \notin K$, which is a contradiction since $\mathrm{ev}_{\chi^{-1}a}((a\chi^{-1})^{\ell}) = \chi^{-1}(a)^{2\ell} = 1$. Hence, $\ell = M = N$. If ℓ is even, then $(a\chi^{-1})^{\frac{\ell}{2}} \notin K$, which is again a contradiction since $\mathrm{ev}_{\chi^{-1}a}((a\chi^{-1})^{\frac{\ell}{2}}) = \chi^{-1}(a)^{\ell} = 1$. Hence ℓ is odd. $\square$

Next I describe the structure of $\overline{D(H_{\mathcal{D}})}$ under the hypothesis of the last Theorem.

Proposition IV.11. *If ℓ is odd and $\ell = N = M$, then $\overline{D(H_{\mathcal{D}})} \simeq \mathfrak{u}_{\theta}(\mathfrak{sl}_2)$ as Hopf algebras, where $\theta = \chi(a)^{-\frac{1}{2}}$.*

Proof. Recall that $\mathfrak{u}_{\theta}(\mathfrak{sl}_2) = \mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_2)/\langle(\omega_1')^{-1} - \omega_1\rangle$. Since there is only one generator of each kind, I will omit the subindex 1; we then have that $\mathfrak{u}_{\theta}(\mathfrak{sl}_2)$ is generated by e,

f and ω, with relations:

$$e^\ell = 0 = f^\ell, \quad \omega^\ell = 1, \quad \omega e = \theta^2 e\omega, \quad \omega f = \theta^{-2} f\omega \quad \text{and} \quad [e,f] = \frac{1}{\theta-\theta^{-1}}\omega - \omega^{-1}.$$

In the proof of the previous proposition, we showed that if ℓ is odd and $\ell = N = M$, then $G\times\Gamma = \langle a\chi^{-1}\rangle K$, and so $\langle \chi^{-1}a\rangle$ is a complete set of representatives of the classes in $\frac{G\times\Gamma}{K}$. Let $\psi : D(H_{\mathcal{D}}) \to \mathfrak{u}_\theta(\mathfrak{sl}_2)$ be the algebra map such that

- $\psi(g\gamma) = \omega^{-2c}$ if $g\gamma \in (a\chi^{-1})^c K$, $\forall g\gamma \in G\times\Gamma$,
- $\psi(\xi) = e$ and
- $\psi(x) = (\theta-\theta^{-1})f$.

For ψ to be defined, it must commute with the defining relations of $D(H_D)$ (from Definition IV.1 and Propositions IV.2 and IV.3). This is the case by the following calculations:

1. $\psi(x)\psi(g) = \chi(g)\psi(g)\psi(x),$ for all $g\in G$:

   Let $g\in G$; if $g\in(a\chi^{-1})^cK$, then $g = (a\chi^{-1})^c g_K\chi^c$, with $g_K\chi^c \in K$. Hence $\chi^c(a)\chi^{-1}(g_K) = 1$ and so $\chi(g_K) = \chi^c(a) = q^c$. Therefore

   $$\chi(g) = \chi(a^c g_K) = \chi(a^c)\chi(g_K) = q^{2c}.$$

   Then,

   $$\begin{aligned}\psi(x)\psi(g) &= (\theta-\theta^{-1})f\omega^{-2c} = (\theta-\theta^{-1})\theta^{-4c}\omega^{-2c}f = \chi(g)\omega^{-2c}(\theta-\theta^{-1})f\\ &= \chi(g)\psi(g)\psi(x).\end{aligned}$$

2. $\psi(\xi)\psi(\gamma) = \gamma(a)\psi(\gamma)\psi(\xi),$ for all $\gamma\in\Gamma$:

   Let $\gamma\in\Gamma$, in a similar way as in the previous relation, it can be shown that if

$\gamma \in (a\chi^{-1})^c K$, then $\gamma(a) = q^{-2c}$. We then have

$$\psi(\xi)\psi(\gamma) = e\omega^{-2c} = \theta^{4c}\omega^{-2c}e = \gamma(a)\psi(\gamma)\psi(\xi).$$

3. $[\psi(x), \psi(\xi)] = \psi(a) - \psi(\chi)$:

   To prove this, we first need to know the images of a and χ under ψ. Since ℓ is odd, let $c \in \mathbb{Z}$ be such that $2c = 1 \bmod \ell$. Then, $a = (a\chi^{-1})^c(a\chi)^c$, and since $a\chi \in K$, we have that

$$\psi(a) = \omega^{-2c} = \omega^{-1}. \tag{IV.1}$$

   Similarly, $\chi = (a\chi^{-1})^{-c}(a\chi)^c$ and so $\psi(\chi) = \omega$. Now

$$\begin{aligned} [\psi(x), \psi(\xi)] &= (\theta - \theta^{-1})[f, e] = -(\theta - \theta^{-1})[e, f] = -\frac{\theta - \theta^{-1}}{\theta - \theta^{-1}}(\omega - \omega^{-1}) \\ &= \omega^{-1} - \omega = \psi(a) - \psi(\chi). \end{aligned}$$

Clearly $\psi(x)^\ell = 0 = \psi(\xi)^\ell$ and $\psi(g)\psi(\gamma) = \psi(\gamma)\psi(g)$ for all $g \in G$ and $\gamma \in \Gamma$. The other relations follow in a similar way as 1 and 2 above.

Next we need to show that ψ is a map of coalgebras. Group-like elements in $D(H_{\mathcal{D}})$ are mapped to group-like elements in $\mathfrak{u}_\theta(\mathfrak{sl}_2)$. Moreover,

$$\begin{aligned} \psi \otimes \psi(\Delta(x)) &= \psi \otimes \psi(x \otimes a + 1 \otimes x) = (\theta - \theta^{-1})\left(f \otimes \omega^{-1} + 1 \otimes f\right) \\ &= (\theta - \theta^{-1})\Delta(f) = \Delta(\psi(x)) \end{aligned}$$

and

$$\psi \otimes \psi(\Delta(\xi)) = \psi \otimes \psi(\chi \otimes \xi + \xi \otimes 1) = (\omega \otimes e + e \otimes 1) = \Delta(e) = \Delta(\psi(\xi)).$$

Therefore ψ is a map of Hopf algebras.

Recall that $D(H_{\mathcal{D}})(\mathbb{K}K)^+ = D(H_{\mathcal{D}})\{k - 1 \,:\, k \in K\}$. Note that $\psi(K) = \{1\}$ and so $\psi(\{k - 1 \,:\, k \in K\}) = 0$. Therefore $D(H_{\mathcal{D}})(\mathbb{K}K)^+ \subset \operatorname{Ker}(\psi)$ and the map ψ

induces a Hopf algebra map $\overline{\psi} : \overline{D(H_{\mathcal{D}})} \to \mathfrak{u}_\theta(\mathfrak{sl}_2)$. Since ℓ is odd, $\langle \omega \rangle = \langle \omega^{-2} \rangle$, and so $\overline{\psi}$ is surjective.

By Remark I.15,

$$\begin{aligned} \dim(\overline{D(H_{\mathcal{D}})}) &= \frac{\dim(D(H_{\mathcal{D}}))}{\dim(\mathbb{K}K)} = \frac{|G \times \Gamma|\ell^2}{|K|} = |K^{\perp}|\ell^2 = |\langle a\chi^{-1} \rangle|\ell^2 = \ell^3 \\ &= \dim(\mathfrak{u}_\theta(\mathfrak{sl}_2)). \end{aligned}$$

Hence, $\overline{\psi}$ is an isomorphism.

□

Remark IV.12. Let $\mathfrak{b}'$ be (as in Chapter II) the subalgebra of $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_2)$ generated by f and ω' and $H = (\mathfrak{b}')^{\mathrm{coop}}$. Via the isomorphism $\overline{\psi}$ defined in the proof of Proposition IV.11, a simple $D(H_{\mathcal{D}})$-module of the form $H_{\mathcal{D}}\bullet_{(\chi^{-c})}(a^c)$ is also a $\mathfrak{u}_\theta(\mathfrak{sl}_2)$-module. Explicitly, for $h \in \mathfrak{u}_\theta(\mathfrak{sl}_2) = \overline{\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_2)}$ and $m \in H_{\mathcal{D}}\bullet_{(\chi^{-c})}(a^c)$, $h \cdot m = \overline{\psi}^{-1}(h) \cdot m$. Therefore, as $\mathfrak{u}_\theta(\mathfrak{sl}_2)$-modules, $H_{\mathcal{D}}\bullet_{(\chi^{-c})}(a^c) \simeq H\bullet_\beta(\omega'^d)$ with $\beta(\omega') = \theta^{-2d}$ for some $d \in \mathbb{Z}$. By analyzing the action of ω' on both of this modules, it follows that $d = -c$. Conversely, a simple $\mathfrak{u}_\theta(\mathfrak{sl}_2)$-module $H\bullet_\beta(w')^d$ becomes a simple $D(H_{\mathcal{D}})$-module via $\overline{\psi}$, and is isomorphic to $H_{\mathcal{D}}\bullet_{(\chi^d)}(a^{-d})$ as $D(H_{\mathcal{D}})$-modules.

I finish this section by studying the reducibility of tensor products of simple $D(H_{\mathcal{D}})$-modules when $n = M = N$ is odd.

In [19], Radford used his construction to describe simple modules for the Drinfel'd Double of the Taft algebra, which is isomorphic to $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_2)$ when ℓ is odd (ℓ is the order of θ). Translating his result to our notation ($H = (\mathfrak{b}')^{\mathrm{coop}}$, generated by ω' and f and the corresponding relations) we have

Proposition IV.13 (Radford [19])**.** *For $g = (\omega')^c$ and $\beta : H \to \mathbb{K}$ an algebra morphism, let $r \geq 0$ be minimal such that $\beta(\omega') = \theta^{2(c-r)}$. Then the simple $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_2)$-module $H\bullet_\beta g$ is $(r+1)-$dimensional with basis $\{g, f\bullet_\beta g, \ldots, f_c^r\bullet_\beta g\}$ and $f^{r+1}\bullet_\beta g = 0$.*

In [7], H-X. Chen studied the reducibility of tensor products of these simple modules:

Proposition IV.14 (Chen [7]). *Given* $g = (\omega')^c$, $g' = (\omega')^{c'}$ *in* $G(H)$ *and* $\beta, \beta' \in G(H^*)$, *let* $r, r' \in \{0, \dots, \ell - 1\}$ *be such that* $\beta(\omega') = \theta^{2(c-r)}$ *and* $\beta'(\omega') = \theta^{2(c'-r')}$. *Then the* $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_2)$*-module* $H \bullet_\beta g \otimes H \bullet_{\beta'} g'$ *is completely reducible if and only if* $r + r' < \ell$. *Moreover, let*

$$g_j = gg'(\omega')^{-j} \quad \text{and} \quad \beta_j(\omega') = \theta^{2j}\beta(\omega')\beta'(\omega');$$

if $r + r' < \ell$ *then*

$$H \bullet_\beta g \otimes H \bullet_{\beta'} g' \simeq \bigoplus_{j=0}^{\min(r,r')} H \bullet_{\beta_j} g_j.$$

If $r + r' \geq \ell$, *let* $t = r + r' - \ell + 1$; *then*

$$\mathrm{Soc}\,(H \bullet_\beta g \otimes H \bullet_{\beta'} g') \simeq \bigoplus_{j=\left[\frac{t+1}{2}\right]}^{\min(r,r')} H \bullet_{\beta_j} g_j.$$

Remark IV.15. By Example II.9, if $H \bullet_\beta (\omega')^c$ is naturally a $\mathfrak{u}_\theta(\mathfrak{sl}_2)$-module, then $\beta = \beta_g$, *i.e.* $\beta(\omega') = \theta^{-2c} = \theta^{2(c-2c)}$. Then the number r from Proposition IV.13 is $r = 2c \bmod \ell$, with $0 \leq r < \ell$. I will denote such number by r_c.

We get the following corollary for simple $\mathfrak{u}_\theta(\mathfrak{sl}_2)$-modules:

Corollary IV.16. *Given* $g = (\omega')^c$ *and* $g' = (\omega')^{c'}$ *in* $G(H)$. *If* $r_c + r_{c'} < \ell$ *then*

$$H \bullet_{\beta_g} g \otimes H \bullet_{\beta_{g'}} g' \simeq \bigoplus_{j=0}^{\min(r_c, r_{c'})} H \bullet_{\beta_j} g_j,$$

as $\mathfrak{u}_\theta(\mathfrak{sl}_2)$*-modules, where* $g_j = gg'(\omega')^{-j}$ *and* $\beta_j = \beta_{g_j}$.

Remark IV.17. This last corollary is a particular case of a more general formula for simple modules for the non-restricted quantum group $U_q(\mathfrak{sl}_2)$, that appears as an exercise in [3].

We have an analogous result to Proposition IV.14 for $D(H_{\mathcal{D}})$-modules:

Theorem IV.18. *If $\ell = M = N$ is odd and $g\beta$, $g'\beta' \in G \times \Gamma = G(D(H_{\mathcal{D}}))$, let c and $c' \in \mathbb{Z}$ such that $(a^{-1}\chi)^c$ and $(a^{-1}\chi)^{c'}$ are representatives of the classes of $g\beta$ and $g'\beta'$ in $G \times \Gamma/K$ respectively. Then the $D(H_{\mathcal{D}})$-module $H_{\mathcal{D}\bullet\beta}g \otimes H_{\mathcal{D}\bullet\beta'}g'$ is completely reducible if and only if $r_c + r_{c'} < \ell$. Moreover, let*

$$g_j = gg'a^j \quad \textit{and} \quad \beta = \chi^{-j}\beta\beta';$$

if $r_c + r_{c'} < \ell$ then

$$H_{\mathcal{D}\bullet\beta}g \otimes H_{\mathcal{D}\bullet\beta'}g' \simeq \bigoplus_{j=0}^{\min(r_c, r_{c'})} H_{\mathcal{D}\bullet\beta_j}g_j.$$

If $r_c + r_{c'} \geq \ell$, then

$$\mathrm{Soc}\,(H_{\mathcal{D}\bullet\beta}g \otimes H_{\mathcal{D}\bullet\beta'}g') \simeq \bigoplus_{j=\left[\frac{t+1}{2}\right]}^{\min(r_c, r_{c'})} H_{\mathcal{D}\bullet\beta_j}g_j,$$

where $t = r_c + r_{c'} - \ell + 1$.

Proof. Let $g_K\beta_K$ and $g'_K\beta'_K \in K$ such that $g\beta = (a^{-1}\chi)^c g_K\beta_K$ and $g'\beta' = (a^{-1}\chi)^{c'} g'_K\beta'_K$. By Proposition IV.10, $H_{\mathcal{D}\bullet\beta}g \simeq H_{\bullet_{\chi^c}}a^{-c} \otimes H_{\mathcal{D}\bullet\beta_K}g_K$, the first factor in $\mathcal{S}_{\overline{D(H_{\mathcal{D}})}}$, and the second factor in $\mathcal{S}^1_{D(H_D)}$. Similarly $H_{\mathcal{D}\bullet\beta'}g' = H_{\mathcal{D}\bullet_{\chi^{c'}}}a^{-c'} \otimes H_{\mathcal{D}\bullet\beta'_K}g'_K$. Then

$$\begin{aligned} H_{\mathcal{D}\bullet\beta}g \otimes H_{\mathcal{D}\bullet\beta'}g' &\simeq \left(H_{\mathcal{D}\bullet_{\chi^c}}a^{-c} \otimes H_{\mathcal{D}\bullet\beta_K}g_K\right) \otimes \left(H_{\mathcal{D}\bullet_{\chi^{c'}}}a^{-c'} \otimes H_{\mathcal{D}\bullet\beta'_K}g'_K\right) \\ &\simeq \left(H_{\mathcal{D}\bullet_{\chi^c}}a^{-c} \otimes H_{\mathcal{D}\bullet_{\chi^{c'}}}a^{-c'}\right) \otimes \left(H_{\mathcal{D}\bullet\beta_K}g_K \otimes H_{\mathcal{D}\bullet\beta'_K}g'_K\right) \\ &\simeq \left(H_{\mathcal{D}\bullet_{\chi^c}}a^{-c} \otimes H_{\mathcal{D}\bullet_{\chi^{c'}}}a^{-c'}\right) \otimes H_{\mathcal{D}\bullet_{\beta_K * \beta'_k}}g_Kg'_K; \end{aligned}$$

the second isomorphism by symmetry of tensor products of modules for $D(H_{\mathcal{D}})$, and the third by combining Propositions I.33 and I.34. Let $\gamma, \gamma' : H \to \mathbb{K}$ be the algebra maps given by $\gamma(\omega') = \theta^{-2c}$ and $\gamma'(\omega') = \theta^{-2c'}$. If $r_c + r_{c'} < \ell$, we have the following isomorphisms of $\mathfrak{u}_\theta(\mathfrak{sl}_2)$-modules:

$$H_{\mathcal{D}^\bullet \chi^c} a^{-c} \otimes H_{\mathcal{D}^\bullet \chi^{c'}} a^{-c'} \simeq H_{\bullet\gamma}(\omega')^c \otimes H_{\bullet\gamma'}(\omega')^{c'} \simeq \bigoplus_{j=0}^{\min(r,r')} H_{\bullet\beta_j} g_j,$$

where $g_j = (\omega')^{c+c'-j}$and $\gamma_j(\omega') = \theta^{-2(c+c'-j)}$, the first isomorphism following from the Remark IV.12 and the second from Corollary IV.16. Again by the Remark IV.12, the j^{th} summand of the last module is isomorphic to $H_{\mathcal{D}^\bullet \chi^{-c_j}} a^{c_j}$ as $D(H_{\mathcal{D}})$-modules, where $c_j = -(c + c' - j)$. Then

$$H_{\mathcal{D}^\bullet\beta} g \otimes H_{\mathcal{D}^\bullet\beta'} g' \simeq \left(\bigoplus_{j=0}^{\min(r,r')} H_{\mathcal{D}^\bullet \chi^{-c_j}} a^{c_j} \right) \otimes H_{\mathcal{D}^\bullet \beta_K * \beta'_k} g_K g'_K \simeq \bigoplus_{j=0}^{\min(r,r')} H_{\mathcal{D}^\bullet \gamma_j} g_j,$$

where

$$g_j = a^{c_j} g_K g'_K = a^{-c} g_K a^{-c'} g'_K a^j = g g' a^j$$

and

$$\gamma_j = \chi^{-c_j} \beta_K \beta'_K = \chi^c \beta_K \chi^{c'} \beta'_K \chi^{-j} = \beta\beta'\chi^{-j}.$$

If $r_c + r_{c'} \geq \ell$, we have

$$H_{\mathcal{D}^\bullet\beta} g \otimes H_{\mathcal{D}^\bullet\beta'} g' \simeq \left(H_{\mathcal{D}^\bullet \chi^{-c}} a^c \otimes H_{\mathcal{D}^\bullet \chi^{-c'}} a^{c'} \right)_{\beta_K \beta'_K}$$

and by Remark I.34 we have

$$\operatorname{Soc}\left(H_{\mathcal{D}^\bullet\beta} g \otimes H_{\mathcal{D}^\bullet\beta'} g'\right) \simeq \left(\operatorname{Soc}\left(H_{\mathcal{D}^\bullet \chi^{-c}} a^c \otimes H_{\mathcal{D}^\bullet \chi^{-c'}} a^{c'} \right)\right)_{\beta_K \beta'_K}.$$

With a similar reasoning as before, we get that

$$\operatorname{Soc}\left(H_{\mathcal{D}^\bullet \chi^{-c}} a^c \otimes H_{\mathcal{D}^\bullet \chi^{-c'}} a^{c'} \right) \simeq \bigoplus_{j=\left[\frac{t+1}{2}\right]}^{\min(r_c, r_{c'})} H_{\bullet \chi^{-c_j}} a^{c_j},$$

where $c_j = -(c+c'-j)$. Therefore

$$\begin{aligned}
\operatorname{Soc}\left(H_{\mathcal{D}}\bullet_{\beta} g \otimes H_{\mathcal{D}}\bullet_{\beta'} g'\right) &\simeq \left(\bigoplus_{j=\left[\frac{t+1}{2}\right]}^{\min(r_c, r_{c'})} H\bullet_{\chi^{-c_j}} a^{c_j}\right)_{\beta_K \beta'_K} \\
&\simeq \left(\bigoplus_{j=\left[\frac{t+1}{2}\right]}^{\min(r_c, r_{c'})} H\bullet_{\chi^{-c_j}} a^{c_j}\right) \otimes H\bullet_{\beta_K * \beta'_K} g_K g'_K \\
&\simeq \bigoplus_{j=\left[\frac{t+1}{2}\right]}^{\min(r_c, r_{c'})} H\bullet_{\beta_j} g_j,
\end{aligned}$$

where $g_j = h_j g_K g'_K = a^{-c-c'+j} g_K g'_K = g g' a^j$ and $\beta_j = \gamma_j * \beta_K * \beta'_K = \beta\beta'\chi^{-j}$.

□

In [11], the authors studied the representation theory of the Drinfel'd double of a family of Hopf algebras that generalize the Taft algebra. In their case, the order of the generating group-like element need not be the same as the order of the root of unity. They give a similar decomposition of tensor products as in Theorem IV.18. Although the algebras $H_{\mathcal{D}}$ generalize their Hopf algebras, Theorem IV.18 does not generalize their result since I require $|a| = |q|$. However, since G need not be cyclic, Theorem IV.18 generalizes Chen's result for Taft algebras.

CHAPTER V

CONCLUSION

In this dissertation I used Radford's method to construct simple modules for the Drinfel'd double of a graded Hopf algebra, to get information about the structure of these modules. I worked with two different classes of Hopf algebras: the restricted two-parameter quantum groups (of type A) defined by Benkart and Witherspoon in [6], and the rank one pointed Hopf algebras of nilpotent type introduced by Krop and Radford in [15].

For the two-parameter quantum groups, I presented necessary and sufficient conditions on the parameters r and s, for a simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$-module to be factored as the tensor product of a one-dimensional module with a module that is naturally a module for $\overline{\mathfrak{u}_{r,s}(\mathfrak{sl}_n)}$, the quotient of $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ by group-like central elements (Theorem II.13). In Chapter III, I introduced the code used in Singular::Plural to construct simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules, and presented conjectures about bases and dimensions based on the computational results.

In Chapter IV, for $H_{\mathcal{D}}$ a rank one pointed Hopf algebra of nilpotent type, I gave necessary and sufficient conditions on $\mathcal{D}$ for a simple $D(H_{\mathcal{D}})$-module to factor as the tensor product of a one-dimensional module with a module that is naturally a module for $\overline{D(H_{\mathcal{D}})}$ (Theorem IV.10). Using this result, I studied the complete reducibility of the tensor product of two simple $D(H_{\mathcal{D}})$-modules (Theorem IV.18). This result is a generalization of the work of Chen on the Drinfel'd double of the Taft algebra [7].

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VITA

Name: Mariana Pereira Lopez

Education:

M.S., Mathematics, University of Massachusetts, Amherst, Massachusetts, 2003

B.S., Mathematics, Universidad de la República, Montevideo, Uruguay, 2001

Ph.D., Mathematics, Texas A&M University, College Station, Texas, 2006

Address: c/o Dr. Sarah Witherspoon, Department of Mathematics, Texas A&M University, College Station, TX 77843-3368

e-mail: mpereira@math.tamu.edu