ON THE SYNTHESIS OF FIXED ORDER STABILIZING CONTROLLERS

A Dissertation

by

SIN CHEON KANG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2005

Major Subject: Mechanical Engineering

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Approved by:

Chair of Committee,	D.V.A.H.G Swaroop
Committee Members,	W. J. Kim
	R. Langari
	S. P. Bhattacharyya
Head of Department,	Dennis O' Neal

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ABSTRACT

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Sin Cheon Kang, B.S., Korea University;

M.S., Korea Advanced Institute of Science and Technology

Chair of Advisory Committee: Dr. Darbha Swaroop

In this dissertation, we consider two problems concerning the synthesis of fixed order controllers for Single Input, Single Output systems. The first problem deals with the synthesis of absolutely stabilizing fixed order controllers for Lure-Postnikov systems. The second problem deals with the synthesis of fixed order stabilizing controllers directly from the empirical frequency response data and from some coarse information of the plant.

Lure-Postnikov systems are frequently encountered in mechanical engineering applications. Analytical tools for synthesizing stabilizing fixed structure controllers, such as the PID controllers examining the absolute stability of Lure-Postnikov systems, have recently been studied in the literature. However, tools for synthesizing controllers of arbitrary order have not been studied yet. We propose a systematic method for synthesizing absolutely stabilizing controllers of arbitrary order for the Lure-Postnikov systems. Our approach is based on recent results in the literature on approximation of the set of stabilizing controller parameters that render a family of real and complex polynomials Hurwitz. We provide an example of a robotic system to illustrate the procedure developed.

Exact analytical models of plants may not be readily available for controller design. The current approach is to synthesize controllers through the identification of the analytical model of the plant from empirical frequency response data. In this dissertation, we depart from this conventional approach. We seek to synthesize controllers directly (i.e. without resort to identification) from the empirical frequency response data of the plant and coarse information about it. The coarse information required is the number of non-minimum phase zeros of the plant(or the number of poles of the plant with positive real parts) and the frequency range beyond which the phase response of the LTI plant does not change appreciably and the amplitude response goes to zero. We also assume that the LTI plant does not have purely imaginary zeros or poles. The method of synthesizing stabilizing controllers involves the use of generalized Hermite-Biehler theorem for counting the roots of rational functions and the use of recently developed Sum-of-Squares techniques for checking the nonnegativity of a polynomial in an interval through the Markov-Lucaks theorem. The method does not require an explicit analytical model of the plant that must be stabilized or the order of the plant, rather, it only requires the empirical frequency response data of the plant. The method also allows for measurement errors in the frequency response of the plant. We illustrate the developed procedure with an example. Finally, we extended the technique to the synthesis of controllers of arbitrary order that also guarantee performance specifications such as the phase margin and gain margin.

To Danjoo Kim, Jaeseok Kang, Minseok Kang

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CHAPTER I

INTRODUCTION

This dissertation deals with two problems of fixed order controller synthesis for Single Input, Single Output(SISO) systems. The first problem is that of synthesizing a fixed order controller that absolutely stabilizes a Lure-Postnikov system. The second problem deals with the synthesis of fixed order stabilizing controllers directly from the empirical frequency response data and some coarse information about the SISO system being controlled.

One encounters nonlinear systems in engineering applications where the nonlinearities are sector-bounded. Such nonlinear systems are typically referred to as Lure-Postnikov systems which are important and common, see [1, 2]. Linear control systems which have actuator/sensor nonlinearity(saturation) also can be represented as Lure-Postnikov systems [3]. Analytical tools examining the absolute stability of Lure-Postnikov systems exist [4, 5, 6, 7, 8, 9, 10, 11, 12]. However, the tools for synthesizing stabilizing controllers, especially that of fixed structure such as the PID or low-order controllers, have not received as much attention. Since the (sufficient) conditions for stabilizing a Lure-Postnikov system involve the Positive Realness (PR ness) or Strict Positive Realness (SPR ness) of the product of two transfer functions - one describing the linear part of the Lure-Postnikov System and the other a multiplier, it is conceivable that a direct parametric method may be employed. It is this approach that was adopted recently by Ho and Lu [2] for synthesizing PID controllers for Lure-Postnikov systems. The systematic synthesis of PID controllers exploits the special structure of the characteristic polynomial [13]. Although a first order controller also has three control parameters, the method for PID controller synthesis cannot

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be directly applied for the synthesis of a first order controller. Recently, Malik, Darbha and Bhattacharyya have proposed a systematic method for approximating the set of controller parameters that render a family of real and complex polynomials Hurwitz [14, 15, 16]. This method involves separating the roots of the real and imaginary parts of the characteristic polynomial to systematically construct linear programs in the controller parameters - the union of the feasible sets of the linear programs constructed is an approximation to the set of controller parameters that can enable a certain transfer function either to be SPR or to have a $\mathcal{H}_\infty$ less than a specified value. Based on the results of Malik, Darbha and Bhattacharyya [14, 15, 16], we propose a method to construct sets of fixed order stabilizing controllers of arbitrary order for Lure-Postnikov systems.

The synthesis of fixed order/structure controllers for LTI plants is an important open problem with a wide variety of practical applications [17, 18]. It is also widely recognized that an accurate analytical model of the plant may not be available to a control designer. However, it is reasonable in many applications that one will have an empirical model of the plant in terms of its frequency response data and from physical considerations or from the empirical time response data, one may have some coarse information about the plant such as the number of non-minimum phase zeros of the plant etc. In view of this, we consider the problem of synthesizing sets of stabilizing controllers directly from the empirical data and such coarse information about the plant.

A systematic attempt to synthesize PID and first order controllers for delay-free SISO Linear Time Invariant (LTI) plants was first presented in [19]. However, we are unaware of any systematic attempt at synthesizing sets of stabilizing controllers of arbitrary order from the frequency response data and this work is a first attempt in that direction. The proposed method also allows for measurement errors in the frequency response of the plant. The method requires the computation of a set of parameters that guarantee the nonnegativity of a polynomial on interval when the coefficients of the polynomial are affinely dependent on the parameters. Recently, the Sum-of-Squares technique has received significant attention for checking the nonnegativity of a polynomial. For example, Roh and Vandenberghe have presented a new semidefinte programming(SDP) formulation of sum of squares representations of nonnegative polynomials of one variable in [20]. This method is based on the Markov-Lucaks theorem and discrete polynomial transform and can be solved by using of the well developed SDP solvers such as SeDuMi [21]. We used the procedure formulated in [20] for checking the nonnegativity of a polynomial in any interval.

A. Contributions of the Dissertation

Recently, Ho and Lu [2] proposed a synthesis method to design fixed structure(PID) controllers for Lure-Postnikov systems which is common and important in mechanical applications. Since the synthesis of PID controllers exploits the special structure of the characteristic polynomial the same method cannot be directly applied to fixed order controllers. Bhattacharyya and Keel [22] have developed a method for synthesizing first order controllers. However, this method does not easily extended to the synthesis of controllers of order greater than one. In this dissertation, we propose a new method that can be extended to the synthesis of controllers of order greater than one for Lure-Postnikov systems using a recent systematic method for approximating the set of controller parameters that render a family of real and complex polynomials Hurwitz proposed by Malik, Darbha and Bhattacharyya [14, 15, 16].

We also propose a novel method for synthesizing set of fixed order stabilizing controllers of strictly proper, SISO LTI plants directly from their empirical frequency response data and with two pieces of information about them. One is the number of non minimum phase zeros of the plant and the other is frequency range beyond which the phase response of the LTI plant does not change appreciably and the amplitude response goes to zero. The method does not require an explicit analytical model of the plant that must be stabilized or the order of the plant, rather, it only requires the empirical frequency response data of the plant. The method also allows for measurement errors in the frequency response of the plant. It is remarkable that these results indicate the possibility of fixed order controller synthesis using only frequency response measurements. The proposed method can also be extended to the synthesis of controllers of arbitrary order that guarantee performance specifications such as the gain/phase margin and upper bound on the $\mathcal{H}_\infty$ norm.

B. Organization of the Dissertation

In chapter II, a method to synthesize the fixed order/structure controllers that absolutely stabilize a Lure-Postnikov system is proposed. We also provide an example of Lure-Postnikov system(one-link robot with a flexible joint) and construct the set of fixed structure(PID) and first order controllers which absolutely stabilize the example system. In chapter III, we review a technique which can check the nonnegativity of a real polynomial on an interval using SDPs. We use the well known Chebyshev polynomials to approximate the real and imaginary parts of the frequency response of the LTI plant and we provide a brief review of Chebyshev approximation. In chapter IV, we formulate the problem of fixed order controller as the feasibility of a robust SDP, based on the methods reviewed in chapter III. The proposed formulation does not require an explicit analytical model of the plant that must be stabilized or the order of the plant, rather, it only requires the empirical frequency response data of the plant. The method of synthesizing stabilizing controllers involves the use of generalized Hermite-Biehler theorem for rational functions for counting the roots and the nonnegativity of a polynomial in some intervals. We also show in some case the nonnegativity of a polynomial in some intervals can be replaced as nonnegativity of the end points of the intervals. In chapter V, a method to synthesize a controller that make a system guaranteeing certain level of performance as well as stability with finite frequency response data is proposed. Those performance criteria can be gain margin, phase margin, upper bound on the $\mathcal{H}_\infty$ norm of a weighted sensitivity transfer function, or a requirement that a certain closed loop transfer function be SPR etc. The results of this dissertation are summarized and recommendations for future work are presented in In chapter VI.

CHAPTER II

ON THE SYNTHESIS OF FIXED ORDER CONTROLLERS FOR LURE-POSTNIKOV NONLINEAR SYSTEMS

A. Introduction

One encounters nonlinear systems in engineering applications where the nonlinearities are sector-bounded, for example, see [1, 2]. A one-link robot with a flexible joint in [1], as will be seen later, is an example of a Lure-Postnikov system. Linear systems with actuator/sensor nonlinearity can also be represented as Lure-Postnikov systems. Such nonlinear systems are typically referred to as Lure-Postnikov systems.

Analytical tools examining the absolute stability of Lure-Postnikov systems exist, see [4, 5, 6, 7, 8, 9, 10, 11, 12]. However, the tools for synthesizing stabilizing controllers, especially that of fixed structure such as the PID or low-order controllers, have not received as much attention.

The synthesis of PI controllers for general nonlinear systems was considered by Desoer and Lin [23]. In this work, the nonlinear system is assumed to be stabilized exponentially through some means and an integral action is provided in the feedforward path of the outer loop so that step inputs could be tracked with zero steady state error. Using this method, the problem of synthesizing a stabilizing controller can be performed in two steps - the first one involves the synthesis of a stabilizing controller and the second one involves the design of a PI controller in the outer loop. The problem of stabilizing a general nonlinear system with output feedback is a daunting task. For this reason, we restrict ourselves to this class of important nonlinear system. One encounters nonlinear systems which consist of a linear system in the feed forward path and a output nonlinearity in the feedback path. If the output nonlinearity is sector bounded, such systems are referred to as Lure-Postnikov systems (see Narendra and Taylor [5]). The problem of stabilizing Lure-Postnikov systems has received significant attention since it was posed in early 1940s. The first solution (a sufficient condition) was provided by Popov and subsequently various other sufficient conditions were provided [4, 5]. All the sufficient conditions involve the positive realness (PR) or strictly positive realness (SPR) of the product of two transfer functions - one related to the linear part of the transfer function and the other a multiplier of a certain class. Three characterizations of SPR transfer functions have been developed - In state space form, the KYP lemma and its variants provide conditions on a transfer function being SPR. In the frequency domain, a transfer function is SPR if it is analytic in the RHP and the Nyquist plot of the transfer function is always in the 1^{st} and 4^{th} quadrants of the complex plane. In the parametric approach, a transfer function is SPR if (1) the DC gain is positive (2) the numerator is Hurwitz and a function of complex polynomial is Hurwitz. It is the latter characterization that has recently been used by Ho and Lu (2005) to synthesize stabilizing PID controllers for Lure-Postnikov systems. The first use of parametric approach to analyze robustness of an absolutely stabilizing controller is given in [24]. We adopt the parametric approach for synthesizing stabilizing controllers for Lure-Postnikov systems in much the same way as Ho and Lu have recently used.

Central to the method of Ho and Lu [2] are two recent ideas: (1) the systematic synthesis of PID controllers [25] for SISO systems that exploit interlacing properties of real and complex Hurwitz polynomials, and (2) the reduction of the SPR condition of transfer function to that of rendering Hurwitz a one-parameter family of complex polynomials [26]. Using the circle criterion [3, 27], Ho and Lu convert the problem of PID controller synthesis to that of synthesis of PID gains that render a family of complex polynomials Hurwitz [2]. The advantage of the parametric approach is that the set of all stabilizing PID controllers that make a specified transfer function SPR can be approximated computationally and be made available graphically to the control algorithm designer who may be faced with other constraints.

The systematic synthesis of PID controllers exploits the special structure of the characteristic polynomial. Although a first order controller also has three control parameters, the same method cannot be directly applied. Bhattacharyya and Keel [22] have developed a method for synthesizing first order controllers based on the D-decomposition technique; however, this method does not readily extend to the synthesis of controllers of order greater than one. Recently, Malik, Darbha and Bhattacharyya have proposed a systematic method for approximating the set of controller parameters that render a family of real and complex polynomials Hurwitz [14, 15, 16]. This method involves separating the roots of the real and imaginary parts of the characteristic polynomial to systematically construct linear programs in the controller parameters - the union of the feasible sets of the linear programs constructed is an approximation to the set of controller parameters. The criteria for the rational function either to be SPR or to have a $\mathcal{H}_\infty$ less than a specified value can be posed as the determination of controller parameters that render a family of complex polynomials Hurwitz. In this chapter, we use this method for constructing sets of stabilizing controllers for Lure-Postnikov systems.

This chapter is organized as follows: section B provides a review of the relevant mathematical preliminaries, section C details the systematic methodology for the construction of stabilizing controllers, in section D, an example of a Lure-Postnikov system (one-link robot with a flexible joint) is considered and the set of PID and first order stabilizing controllers for the example system are constructed and graphically illustrated.

B. Preliminaries

Consider a SISO Lure-Postnikov system with saturation nonlinearity as shown in Figure 1. The nonlinear scalar function $\psi(y)$ is assumed to satisfy the sector bound $0 \leq y\psi(y) \leq$

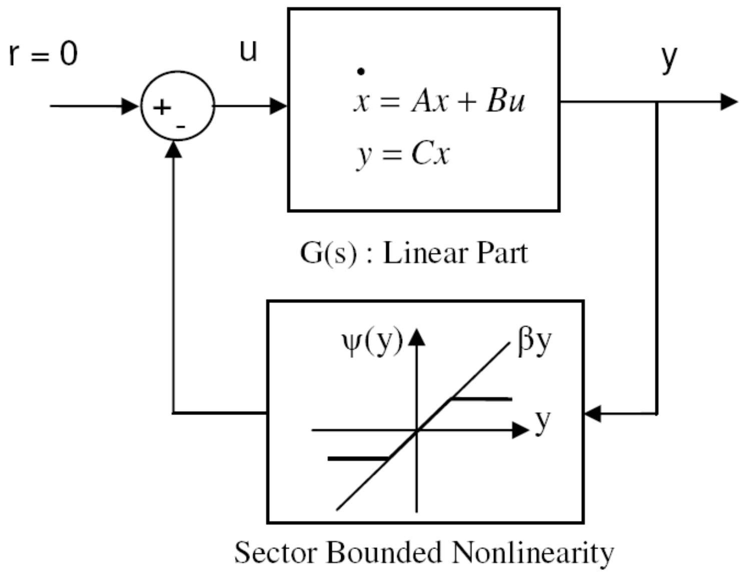

Fig. 1.: *An Example of a Lure-Postnikov System*

βy^2. Many physical systems can be represented by the feedback connection of Figure 1 with the sector bounded nonlinearity [1, 2].

In general, a Lure-Postnikov system can be represented by :

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= -\psi(y) \\ G(s) &= C(sI - A)^{-1}B + D \end{aligned}$$

The absolute stability for Lure-Postnikov systems can be defined as follows [5]:

Definition II.1. *Absolute Stability*

If the equilibrium solution on $x \equiv 0$ is asymptotically stable for every nonlinearity satisfying the sector bound, then $x \equiv 0$ is absolutely stable (Lure-Postnikov system is absolutely stable).

In this chapter, we deal with the synthesis of absolutely stabilizing fixed order controllers of arbitrary order for Lure-Postnikov systems.

An important condition that arises in the solution of this absolute stability problem is the property of strictly positive realness (SPR) of a transfer function. The SPR property is defined as follows [26] :

Definition II.2. *(SPR)*

A proper, rational, scalar, transfer function $G(s)$ is SPR if

1. *$G(s)$ has no poles in the closed right half plane.*
2. *$Re[G(jw)] > 0, \;\; \forall w \in (-\infty, +\infty)$.*

The following results are well-established in the literature [4, 5, 6, 7, 8].

Theorem II.1. *(Circle Criterion)*

Consider the Lure-Postnikov system in Figure 1. If $G(s) + \frac{1}{\beta}$ is SPR, then the equilibrium points $x \equiv 0$ is asymptotically stable for every sector bounded nonlinearity ψ satisfying $0 \leq y\psi(y) \leq \beta y^2$.

Another sufficient condition for absolutely stability is given through the Popov criterion.

Theorem II.2. *(Popov)*

If the linear part of the Lure-Postnikov system is described by the transfer function $G(s)$, where

$$G(s) = \frac{d}{s} + c(sI - A)^{-1}B \tag{2.1}$$

with $d > 0$, A Hurwitz, and the triplet (A, B, c) is minimal, then the equilibrium solution $x = 0$ of the Lure-Postnikov system is globally asymptotically stable if the transfer function $(\gamma_1 s + \gamma_0)(G(s) + \frac{1}{\beta})$ is SPR for some $\gamma_1 \geq 0$ and $\gamma_0 > 0$.

Without any loss of generality, one may set $\gamma_0 = 1$. The term $(\gamma_1 s + \gamma_0)$ is referred to as a multiplier.

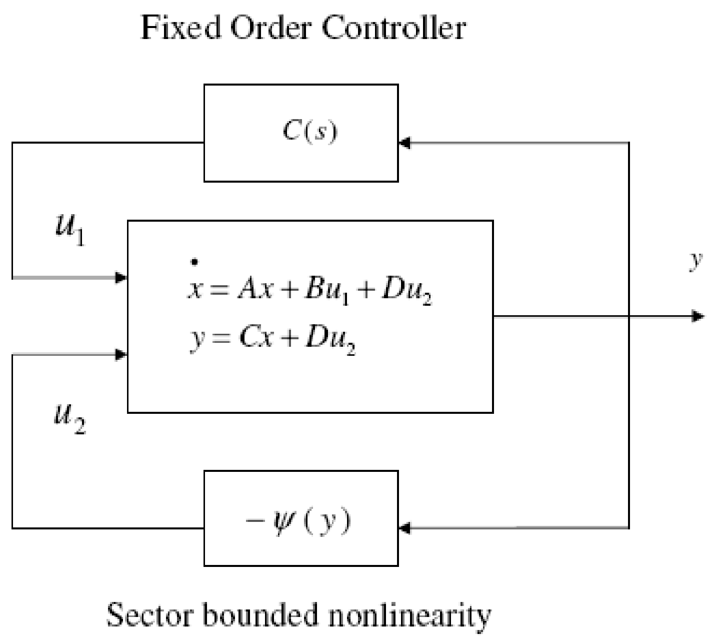

Fig. 2.: *Controller Synthesis*

For monotone nonlinearities which form a subset of the sector bounded nonlinearities and are described by $0 \leq (y_1 - y_2)(\psi(y_1) - \psi(y_2) \leq \bar{M}(y_1 - y_2)^2$, the sufficient conditions that guarantee the global asymptotic stability of the equilibrium require a transfer function of the form $M(s)G(s)$ to be PR, where $M(s) = \frac{s+\gamma_0}{s+\gamma_1}$, $\gamma_0 > 0$ and $\gamma_1 > 0$ [5]. In general, the sufficient conditions require $M(s)G(s)$ to be SPR for some multiplier transfer function of a certain class.

The problem of synthesizing a stabilizing controller for a Lure-Postnikov problem can be understood from the block diagram as shown in Figure 2.

If one were to apply the Popov's stability criterion or any other criteria for absolute stability, one requires checking if a certain transfer function $M(s)G_{cl}(s)$ is SPR, where G_{cl} is a upper linear fractional transformation obtained by closing the loop. As such, the coefficients of the numerator and denominator of $G_{cl}(s)$ will affinely depend on the controller coefficients. Application of a Routh-like procedure, due to Siljak [28] will result in a system of polynomial inequalities; at present, the methods such as Tarski and Seidenberg Theory [29, 30] that check the feasibility of a system of polynomial inequalities are computationally difficult.

There are difficulties with synthesizing controllers using other characterizations :

1. Kalman-Yakubovich-Popov (KYP) Lemma states that
   $G_{cl}(s) = C(sI - A_{cl})^{-1}B$ is SPR if there exists a positive definite symmetric matrix,P, matrix L and $\epsilon > 0$ such that $A_{cl}^T P + PA_{cl} < -LL^T - \epsilon P$ and $B^T P = C$. We have an unknown controller vector K and an unknown positive definite symmetric matrix P. A_{cl} is affine in K. The closed loop matrix inequality is bilinear in K and P and currently there is no general algorithm to solve this type of bilinear matrix inequalities.

2. We also can consider in frequency domain characterization for synthesizing controllers. Essentially this will require

$$\begin{aligned} G_{cl}(s,K) &= \frac{N(s,K)}{D(s,K)}, \quad Re\left[G_{cl}(jw,K)\right] > 0 \\ Re\left[G_{cl}(jw,K)\right] &= \frac{D_r(w,K)N_r(w,K) - D_i(w,K)N_i(w,K)}{D_r^2(w,K) + D_i^2(w,K)} > 0, \quad \forall w \in \Re \end{aligned}$$

   The numerator polynomial of $Re\left[G_{cl}(jw,K)\right]$ has coefficients that are quadratic in the controller vector K. As we will see in chapter III, the nonnegativity of a polynomial can be checked using a linear matrix inequality in the coefficients. This implies that the frequency domain approach leads to quadratic matrix inequalities. There are no general algorithms for solving them at current time.

   For this reason, we will use the results in [26], which provides the following characterization of SPR transfer functions:

Theorem II.3. *(SPR)*
$G_T(s,K) = \frac{N_T(s,K)}{D_T(s,K)}$ *is SPR if and only if*

(1) $G_T(0,K) > 0$*,*

(2) $N_T(s,K)$ *is Hurwitz, and*

(3) $\Delta(\alpha,s,K) = D_T(s,K) + j\alpha N_T(s,K)$ *is Hurwitz for every* $\alpha \in \Re$*.*

This characterization is useful for our work because $G_T(0,K)$, $N_T(s,K)$ and $\Delta(\alpha,s,K)$ are affine in controller vector K with fixed α. We will show that this will lead to linear inequalities. For this reason, the third characterization of SPR in [26] is useful in constructing absolutely stabilizing controllers through the solution of a family of linear programs.

Essentially, for the purposes of controller synthesis, this result reduces the problem to determining the set of controller parameters, $K = (k_1, \ldots, k_n)$ that render Hurwitz (1) a real polynomial, N, of the following form :

$$N(s,K) = N_0(s) + k_1 N_1(s) + \cdots + k_n N_n(s).$$

and (2) a complex polynomial, Δ, of the following form :

$$\Delta(s,K) = \Delta_0(s) + k_1 \Delta_1(s) + \cdots + k_l \Delta_l(s).$$

The polynomials $N_i, i = 0, \ldots, n$ and $\Delta_j, j = 0, \ldots, l$ may be assumed known (from the plant data and the structure of the controller chosen).

The Hermite-Biehler theorem for a real polynomial provides a characterization when a real polynomial is Hurwitz [31, 32]. If $N(s,K)$ is a real polynomial of degree n and $N(jw)$ may be expressed as $N_e(w^2,K) + jwN_o(w^2,K)$ for some real polynomials $N_e(w^2)$ and $N_o(w^2)$. The degrees of polynomials N_e and N_o are n_e and n_o respectively in w^2; specifically, if n is odd, $n_e = n_o = \frac{n-1}{2}$ and if n is even, $n_e = \frac{n}{2}$ and $n_o = n_e - 1$. Let $w_{e,i}$, $w_{o,i}$ denote the i^{th} positive real roots of N_e and N_o respectively.

The Hermite-Biehler theorem for real polynomials may be stated as in Theorem II.4. For the sake of clarity, and for the general case, the dependence on K is suppressed.

Theorem II.4. *Hermite-Biehler Theorem for real polynomials*

A real polynomial $N(s)$ is Hurwitz iff

1. *The constant coefficients of $N_e(w^2)$ and $N_o(w^2)$ are of the same sign,*
2. *All roots of $N_e(w^2)$ and $N_o(w^2)$ are real and distinct; the positive roots interlace according to the following:*
   - *if n is even:*

$$0 < w_{e,1} < w_{o,1} < \cdots < w_{o,n_e-1} < w_{e,n_e}$$

   - *if n is odd:*

$$0 < w_{e,1} < w_{o,1} < \cdots < w_{e,n_e} < w_{o,n_e}$$

A proof of the Hermite-Biehler theorem can be found in [26].

The following version [14, 15, 16] of the Hermite-Biehler theorem poses the problem of rendering $N(s, K)$ Hurwitz through a choice of $n-1$ frequencies. Let $C_k, S_k, k = 1, 2, 3, 4$ denote diagonal matrices of dimension n; the $(m+1)^{st}$ diagonal elements of C_k and S_k are respectively $\cos((2k-1)\frac{\pi}{4} + m\frac{\pi}{2})$ and $\sin((2k-1)\frac{\pi}{4} + m\frac{\pi}{2})$. By way of notation, we represent the polynomials N_e and N_o compactly in the following form, owing to the affine dependence of their coefficients on the controller parameter vector K.

Note that $N_e(0, K)$ and $N_o(0, K)$ denote constant coefficients of $N_e(w^2, K)$ and $N_o(w^2, K)$ respectively.

Theorem II.5.

There exists a real control parameter vector $K = (k_1,\ k_2,\ \cdots,\ k_l)$ so that the real *polynomial*

$$\begin{aligned} N(s,K) &:= N_0(s) + k_1 N_1(s) + \ldots + k_l N_l(s) \\ &= n_n(K)s^n + n_{n-1}(K)s^{n-1} + \cdots + n_0(K) \end{aligned}$$

is Hurwitz iff there exists a set of $n-1$ frequencies, $0 = w_0 < w_1 < w_2 < w_3 < \cdots < w_{n-1}$, so that one of the two Linear Programs (LPs) corresponding to $k = 1$ and $k = 3$ is feasible:

$$C_k \begin{bmatrix} N_e(0,K) \\ N_e(w_1^2,K) \\ \vdots \\ N_e(w_{n-1}^2,K) \end{bmatrix} > 0 \; and \; S_k \begin{bmatrix} N_o(0,K) \\ N_o(w_1^2,K) \\ \vdots \\ N_o(w_{n-1}^2,K) \end{bmatrix} > 0. \tag{2.2}$$

The union of the feasible sets of the above LPs corresponding to all such sets of frequencies ($0 < w_1 < w_2 < \ldots < w_{n-1}$) is the set of all stabilizing controllers.

Proof.

The first condition of the Hermite-Biehler theorem requires that the constant coefficients of N_e and N_o be of the same sign. This condition implies that $N_e(0,K) > 0, N_o(0,K) > 0$ or $N_e(0,K) < 0, N_o(0,K) < 0$. The second condition of the Hermite-Biehler theorem is equivalent to the existence of $n-1$ frequencies, $0 < w_1 < w_2 < \cdots < w_{n-1}$ such that the roots of the even polynomial, N_e, lie in $(0,w_1),(w_2,w_3),(w_4,w_5),\ldots,$ while the roots of the odd polynomial, N_o, lie in $(w_1,w_2),(w_3,w_4),\ldots.$
If $N_e(0,K) > 0, N_o(0,K) > 0$, then the placement of roots will require $N_e(w_1^2,K) < 0, N_e(w_2^2,K) < 0, N_e(w_3^2,K) > 0, N_e(w_4^2,K) > 0, \ldots$ and $N_o(w_1^2,K) > 0, N_o(w_2^2,K) < 0, N_o(w_3^2,K) < 0, N_o(w_4^2,K) > 0, \ldots$ as shown in Figure 3, where $N(s)$ is of degree 8 and the constant coefficient of $N(s)$ is positive. In other words, the signs of $N_e(w_i^2,K)$ and

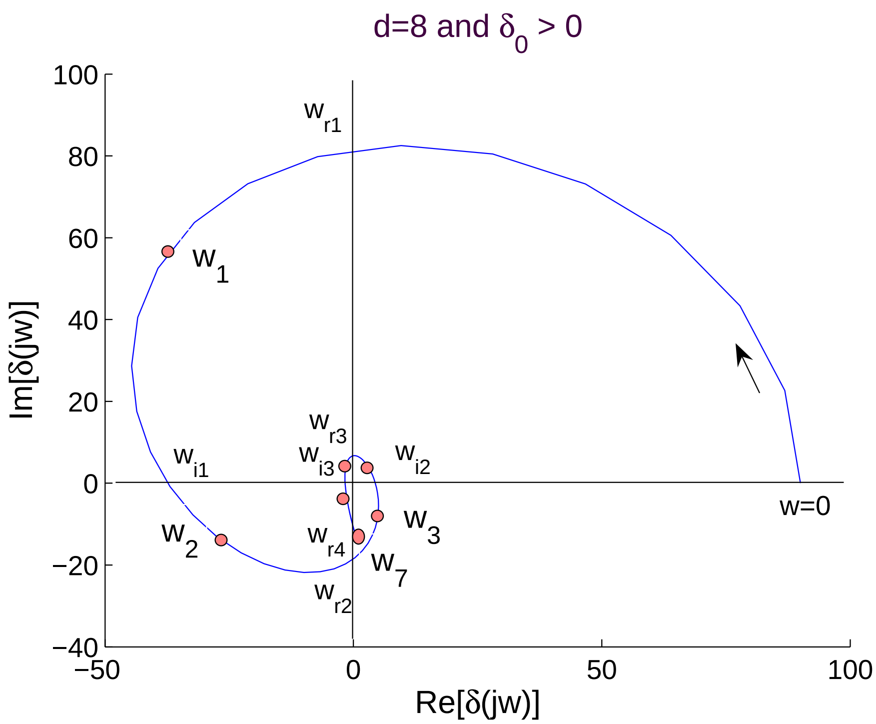

Fig. 3.: *Phase Property for Hurwitz Polynomials*

$N_o(w_i^2, K)$ *are the same as that of* $\cos(\frac{\pi}{4} + i\frac{\pi}{2})$ *and* $\sin(\frac{\pi}{4} + i\frac{\pi}{2})$ *respectively. Therefore, for the case when* $N_e(0, K) > 0, N_o(0, K) > 0$*, we have*

$$\cos(\frac{\pi}{4} + i\frac{\pi}{2})N_e(w_i^2, K) > 0 \textit{ and } \sin(\frac{\pi}{4} + i\frac{\pi}{2})N_o(w_i^2, K) > 0.$$

Similarly when $N_e(0, K) < 0, N_o(0, K) < 0$*, we have*

$$\cos(\frac{5\pi}{4} + i\frac{\pi}{2})N_e(w_i^2, K) > 0 \textit{ and } \sin(\frac{5\pi}{4} + i\frac{\pi}{2})N_o(w_i^2, K) > 0$$

Putting the inequality conditions together, there exists a stabilizing controller K *iff there exists a set of* $(n-1)$ *frequencies* $0 < w_1 < \ldots < w_{n-1}$ *such that one of the two Linear Programs (LPs) given by equations (2.2) is feasible.* $\nabla\nabla\nabla$

The third condition for SPR in Theorem II.3 requires that a complex polynomial be Hurwitz. A characterization of a complex Hurwitz polynomial is presented in [26, 31, 33].

For the sake of completeness we provide a characterization below. If $\Delta(s)$ is a complex polynomial of degree n and $\Delta(jw)$ may be expressed as $\Delta_r(w) + j\Delta_i(w)$ for some real polynomials $\Delta_r(s)$ and $\Delta_i(s)$. Without any loss of generality, one may assume that Δ_r and Δ_i to be of degree n. Let $w_{r,1}, w_{r,2}, \ldots, w_{r,n}$ be the roots of Δ_r and $w_{i,1}, w_{i,2}, \ldots, w_{i,n}$ are the roots of Δ_i.

Theorem II.6. *(Hermite-Biehler Theorem for complex polynomials)*
The polynomial $\Delta(s)$ is Hurwitz if and only if all roots of Δ_r and Δ_i are real and interlace according to the following:

- *If the leading coefficients of Δ_r and Δ_i are of the same sign, then*

$$-\infty < w_{r,1} < w_{i,1} < w_{r,2} < w_{i,2} < \cdots < w_{r,n} < w_{i,n} < \infty,$$

  and

- *if the leading coefficients of Δ_r and Δ_i are of opposite sign, then*

$$-\infty < w_{i,1} < w_{r,1} < w_{i,2} < w_{r,2} < \cdots < w_{i,n} < w_{r,n} < \infty.$$

If $\Delta(s, K)$ is a complex polynomial whose coefficients are affine in K, then the coefficients of $\Delta_i(w, K)$ and $\Delta_r(w, K)$ are also affine in controller parameters in K. Let $\delta_{r,n}$ and $\delta_{i,n}$ denote the leading coefficients of Δ_r and Δ_i respectively. Let $C_k, S_k, k = 1, 2, 3, 4$ denote diagonal matrices of dimension $2n$; the $(m+1)^{st}$ diagonal elements of C_k and S_k are respectively $\cos((2k-1)\frac{\pi}{4} + m\frac{\pi}{2})$ and $\sin((2k-1)\frac{\pi}{4} + m\frac{\pi}{2})$.

The following result [14, 15, 16] exploits the interlacing property of Hurwitz polynomials, as described by the Hermite-Biehler theorem, to provide conditions for the existence of a controller parameter K that renders a complex $\Delta(s, K)$ Hurwitz in terms of the existence of separating frequencies and the feasibility of linear programs:

Theorem II.7.

There exists a stabilizing controller parameter vector K *such that* $\Delta(s, K)$ *is Hurwitz if and only if there exists a set of separating frequencies* $-\infty < w_1 < w_2 < \cdots < w_{2n-1} < \infty$ *such that at least one of the four linear programs corresponding to* $k = 1, 2, 3, 4$ *is feasible:*

$$C_k \begin{bmatrix} \delta_{r,n}(K) \\ \Delta_r(w_1, K) \\ \vdots \\ \Delta_r(w_{2n-1}, K) \end{bmatrix} > 0 \; and \; S_k \begin{bmatrix} \delta_{i,n}(K) \\ \Delta_i(w_1, K) \\ \vdots \\ \Delta_i(w_{2n-1}, K) \end{bmatrix} > 0. \tag{2.3}$$

The proofs follow the same pattern as that of Theorem II.5 as shown in [14, 15, 16].

In the next section, we will combine all the results to provide a computational method for an inner approximation of the set of absolutely stabilizing controllers of a fixed order for a SISO Lure-Postnikov System.

C. Main Results

Consider a Lure-Postnikov system of Figure 2. The linear part of the system may be described by the following equation:

$$Y(s) = G_1(s)U_1(s) + G_2(s)U_2(s), \tag{2.4}$$

where $G_1(s)$ is the transfer function relating the control input, $u_1(t)$ to the output, $y(t)$ and the transfer function $G_2(s)$ relates how the disturbance $u_2(t)$ affects the output $y(t)$. We assume $G_1(s), G_2(s)$ to be proper rational transfer functions. If a controller, $-C(s)$ of order r is used to stabilize the system, then $U_1(s) = -C(s)Y(s)$, and the relation from the disturbance, u_2, to the output y may be described as:

$$Y(s) = \frac{G_2}{1+G_1C(s)}U_2(s) = G(s)U_2(s), \quad G(s) = \frac{G_2}{1+G_1C(s)} \tag{2.5}$$

If $C(s)$ is expressed as

$$C(s) = \frac{n_0 + n_1 s + \ldots + n_r s^r}{d_0 + d_1 s + \ldots + d_{r-1}s^{r-1} + s^r},$$

then the coefficients of the numerator and denominator polynomials of the transfer function $G(s)$ are affine functions of the controller parameter vector, $K := (n_0, n_1, \ldots, n_r,\ d_0, \ldots, d_{r-1})$.

Since $G(s)$ depends on K, we will highlight the dependence through the use of K as an additional argument as $G(s,K)$. We will express $G(s,K)$ as $\frac{N(s,K)}{D(s,K)}$ and any multiplier $M(s)$ as $\frac{N_M(s)}{D_M(s)}$.

Clearly, from the absolute stability theory, if $M(s)G(s,K)$ is SPR for an appropriate multiplier $M(s)$, then the closed loop system is absolutely stable. We will consider a family of polynomials, $\mathcal{F}$ as:

$$\mathcal{F} := \{\Delta(s,\alpha,K) := D(s,K)D_M(s) + j\alpha N(s,K)N_M(s), \ \ \alpha \in \Re\} \tag{2.6}$$

Let the degree of each polynomial be n. We will write $\Delta(jw,\alpha,K)$ as $\Delta_r(w,\alpha,K) + j\Delta_i(w,\alpha,K)$. The terms $\delta_{r,n}(\alpha,K)$ and $\delta_{i,n}(\alpha,K)$ denote the leading coefficients of Δ_r and Δ_i respectively.

We now formally state the main result:

Theorem II.8.

There exists an absolutely stabilizing controller $C(s)$ of order r if there exists a K that renders

1. $\frac{N_M(0)}{D_M(0)}\frac{N(0,K)}{D(0,K)} \ > \ 0$

2. $N_M(s)N(s,K)$ *is Hurwitz and*

3. *Each member of the family $\mathcal{F}$ of polynomials Hurwitz, i.e., for every $\alpha \in \Re$, there exists a set of frequencies $-\infty < w_1(\alpha) < w_2(\alpha) < \cdots < w_{2n-1}(\alpha) < \infty$, such that K is in the feasible set of at least one of $k = 1, 2, 3, 4$:*

$$C_k \begin{bmatrix} \delta_{r,n}(\alpha, K) \\ \Delta_r(w_1, \alpha, K) \\ \vdots \\ \Delta_r(w_{2n-1}, \alpha, K) \end{bmatrix} > 0 \; and \; S_k \begin{bmatrix} \delta_{i,n}(\alpha, K) \\ \Delta_i(w_1, \alpha, K) \\ \vdots \\ \Delta_i(w_{2n-1}, \alpha, K) \end{bmatrix} > 0. \tag{2.7}$$

D. Example

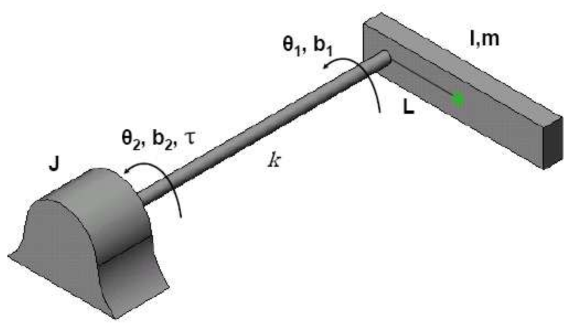

Fig. 4.: *One-Link Robot with a Flexible Joint*

We will consider a one-link robot with a flexible joint as an example of Lure-Postnikov systems as shown in Figure 4 [1].

$$\begin{aligned} I\ddot{\theta}_1 + b_1\dot{\theta}_1 + mgL\sin\theta_1 + k(\theta_1 - \theta_2) &= 0 \\ J\ddot{\theta}_2 + b_2\dot{\theta}_2 - k(\theta_1 - \theta_2) &= \tau \end{aligned} \tag{2.8}$$

We can obtain a state space representation of the system (2.8) by choosing state variables :

$$x_1 = \theta_1 \quad x_2 = \dot{\theta}_1$$
$$x_3 = \theta_2 \quad x_4 = \dot{\theta}_2 \tag{2.9}$$

Then :

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{I}\{-k(\theta_1 - \theta_2) - b_1\dot{\theta}_1 - mgL\sin\theta_1\} \\
&= -\frac{k}{I}x_1 - \frac{b_1}{I}x_2 + \frac{k}{I}x_3 - \frac{mgL}{I}\sin x_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{J}\{k(\theta_1 - \theta_2) - b_2\dot{\theta}_2 + \tau\} \\
&= \frac{k}{J}x_1 - \frac{k}{J}x_3 - \frac{b_2}{J}x_4 + \frac{\tau}{J}
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\dot{x} &= Ax + B_1u - B\psi(y) \\
y &= Cx,
\end{aligned} \tag{2.11}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{I} & -\frac{b_1}{I} & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & -\frac{b_2}{J} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{2.12}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.13}$$

$$\psi(y) = \frac{mgL}{I}\sin y, \quad u = \frac{\tau}{J} \tag{2.14}$$

Suppose the joint system parameters are given as follows :

$$J = 0.5kg \cdot m^2, b1 = 0.0N - m \cdot s/rad, k = 50.0N - m/rad$$
$$I = 25.0kg \cdot m^2, b2 = 1.0N - m \cdot s/rad, m = 1.0kg, L = 5.0m$$

1. PID Controller

Let us consider a PID controller :

$$C(s) = k_p + \frac{k_i}{s} + k_d s \tag{2.15}$$

$$u = k_p(r - y) + k_d(\dot{r} - \dot{y}) + k_i w \tag{2.16}$$

$$\dot{w} = r - y, \tag{2.17}$$

where $C(s)$ is the PID controller, w is the integral of the error and r is reference which is set to be 0. Figure 5 shows a control structure for the one-link robot with a flexible joint which has a sector-bounded nonlinearity.

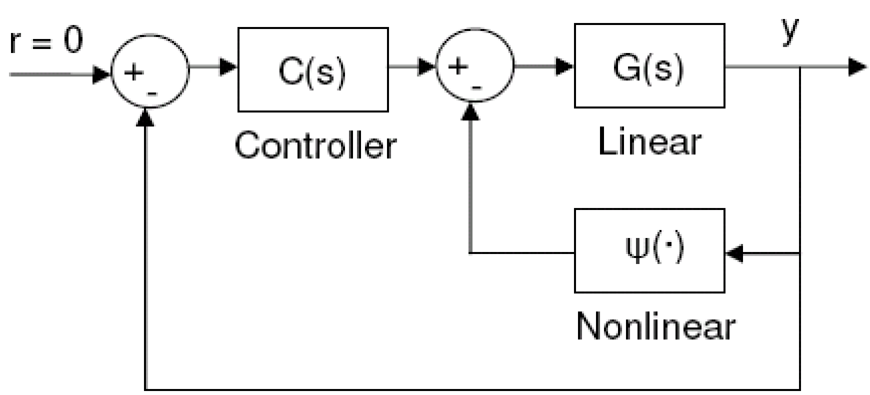

Fig. 5.: *Control Structure of One-Link Robot with a Flexible Joint*

Now, the overall system can be represented as a augmented system as follows :

$$\dot{z} = \mathbf{A}z - \mathbf{B}\psi(y) \tag{2.18}$$

$$y = \mathbf{C}z, \tag{2.19}$$

where $z = \begin{bmatrix} x & w \end{bmatrix}'$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{I} & -\frac{b_1}{I} & \frac{k}{I} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k}{J} - k_p & -k_d & -\frac{k}{J} & -\frac{b_2}{J} & k_i \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.20}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.21}$$

$$\psi(y) = \frac{mgL}{I} \sin y, \tag{2.22}$$

$$\begin{aligned} G(s) = &= \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} \\ &= \frac{N_{cl}(s)}{D_{cl}(s)} \\ &= \frac{s^3 + 2s^2 + 100s}{s^5 + 2s^4 + 102s^3 + (4 + 2k_d)s^2 + 2k_p s + 2k_i} \end{aligned} \tag{2.23}$$

From the popov theorem, above system is absolutely stable if there is $\eta \geq 0$, with $-\frac{1}{\eta}$ not an eigenvalue of A such that $G_T(s) = \frac{N_{GT}(s)}{D_{GT}(s)} = 1+(1+\eta s)\beta G(s) = \frac{D_{cl}(s)+(1+\eta s)\beta N_{cl}(s)}{D_{cl}(s)}$ is strictly positive real [3].

For strictly positive realness of the $G_T(s)$, the following conditions should be held from Theorem II.3.

1. $G_T(0) = \frac{N_{GT}(s)}{D_{GT}(s)} > 0$,

2. $N_{GT}(s) = D_{cl}(s) + (1 + \eta s)\beta N_{cl}(s))$ is Hurwitz for some $\eta \geq 0$, and

3. $P(s, K) = D_{GT}(s) + j\alpha N_{GT}(s) = D_{cl}(s) + j\alpha\{D_{cl}(s) + (1+\eta s)\beta N_{cl}(s)\}$ is Hurwitz for some $\eta \geq 0, \quad \forall\alpha \in \Re$.

We will illustrate how to find the set of all controllers so that above SPR conditions satisfy under $\eta = 1, \ \beta = 2$.

1. For condition 1:

$$\begin{aligned} G_T(s) &= \frac{N_{GT}(s)}{D_{GT}(s)} \\ &= \frac{D_{cl}(s) + (1 + \eta s)\beta N_{cl}(s))}{D_{cl}(s)} \\ &= \frac{s^5 + 4s^4 + 108s^3 + (208 + 2k_d)s^2 + (2k_p + 200)s + 2k_i}{s^5 + 2s^4 + 102s^3 + (4 + 2k_d)s^2 + 2k_p s + 2k_i} \end{aligned}$$

   and we clearly see that $G_T(0) = 1 > 0$

2. For condition 2:

$$\begin{aligned} N_{GT}(s) &= D_{cl}(s) + (1 + \eta s)\beta N_{cl}(s)) \\ &= s^5 + 4s^4 + 108s^3 + (208 + 2k_d)s^2 + (2k_p + 200)s + 2k_i \end{aligned}$$

   The real and imaginary parts of the N_{GT} at jw are given by

$$\begin{aligned} N_{GT}(jw, K) &= N_{GT,e}(w, K) + jwN_{GT,o}(w, K) \\ N_{GT,e}(w, K) &= 4w^4 - (208 + 2k_d)w^2 + 2k_i \\ N_{GT,o}(w, K) &= w^4 - 108w^2 + 200 + 2k_p \end{aligned}$$

For the polynomial N_{GT} to be Hurwitz, there must exist a set of frequencies $0 = w_0 < w_1 < w_2 < w_3 < w_4$ for either C_1 and S_1 or C_3 and S_3

$$C_k \begin{bmatrix} 1 & 0 & 0 \\ 1 & w_1^2 & w_1^4 \\ 1 & w_2^2 & w_2^4 \\ 1 & w_3^2 & w_3^4 \\ 1 & w_4^2 & w_4^4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ -208 & 0 & 0 & -2 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0,$$

and

$$S_k \begin{bmatrix} 1 & 0 & 0 \\ 1 & w_1^2 & w_1^4 \\ 1 & w_2^2 & w_2^4 \\ 1 & w_3^2 & w_3^4 \\ 1 & w_4^2 & w_4^4 \end{bmatrix} \begin{bmatrix} 200 & 2 & 0 & 0 \\ -108 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0$$

Figure 6 shows the set of controller which hold SPR condition 2.

3. For condition 3:

$$\begin{aligned} P(s) &= D_{GT}(s) + j\alpha N_{GT}(s) \\ &= D_{cl} + j\alpha\{D_{cl}(s) + (1+\eta s)\beta N_{cl}(s)\} \\ &= (1+j\alpha)s^5 + (2+j4\alpha)s^4 + (102+j108\alpha)s^3 + \{4+2k_d+j(208+2k_d)\alpha\}\, s^2 \\ &\quad + \{2k_p + j(2k_p+200)\alpha\}\, s + 2k_i + j2\alpha k_i \end{aligned}$$

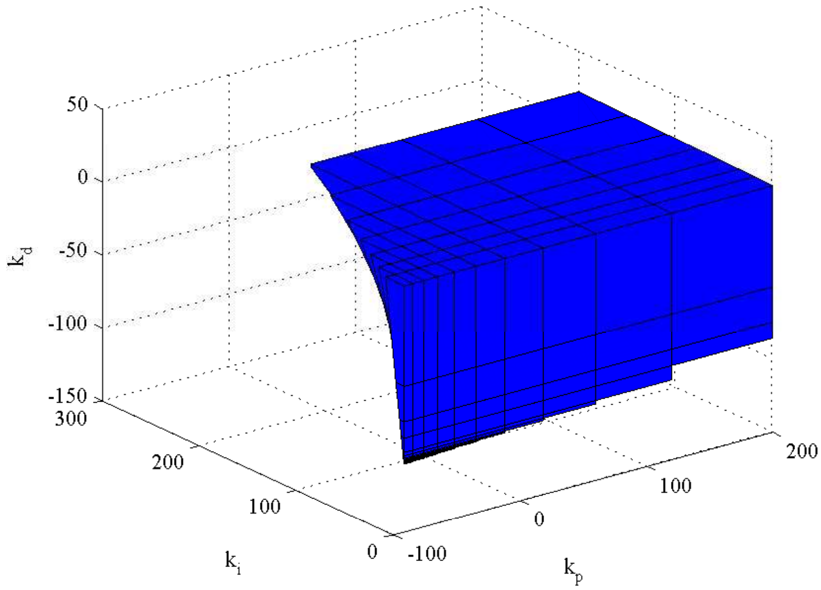

Fig. 6.: *Set of PID Controllers Satisfying SPR Condition* 2

$$
\begin{aligned}
P(jw,K) &= P_r(w,K) + jP_i(w,K) \\
P_r(w,K) &= -\alpha w^5 + 2w^4 + 108\alpha w^3 - 2(2+k_d)w^2 - 2\alpha(100+k_p)w + 2k_i \\
P_i(w,K) &= w^5 + 4\alpha w^4 - 102w^3 - 2\alpha(104+k_d)w^2 + 2k_p w + 2\alpha
\end{aligned}
$$

$$
C_k \begin{bmatrix} 0 & 0 & \dots & -1 \\ 1 & w_1 & \dots & w_1^5 \\ 1 & w_2 & \dots & w_2^5 \\ \vdots & & & \\ \vdots & & & \\ 1 & w_9 & \dots & w_9^5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ -200\alpha & -2\alpha & 0 & 0 \\ -4 & 0 & 0 & -2 \\ 108\alpha & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0,
$$

and

$$S_k \begin{bmatrix} 0 & 0 & \dots & -1 \\ 1 & w_1 & \dots & w_1^5 \\ 1 & w_2 & \dots & w_2^5 \\ \vdots & & & \\ \vdots & & & \\ 1 & w_9 & \dots & w_9^5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2\alpha & 0 \\ 0 & 2 & 0 & 0 \\ -208\alpha & 0 & 0 & -2\alpha \\ -102 & 0 & 0 & 0 \\ 4\alpha & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0$$

Figure 7 shows the set of controller for which the SPR condition 3 holds.

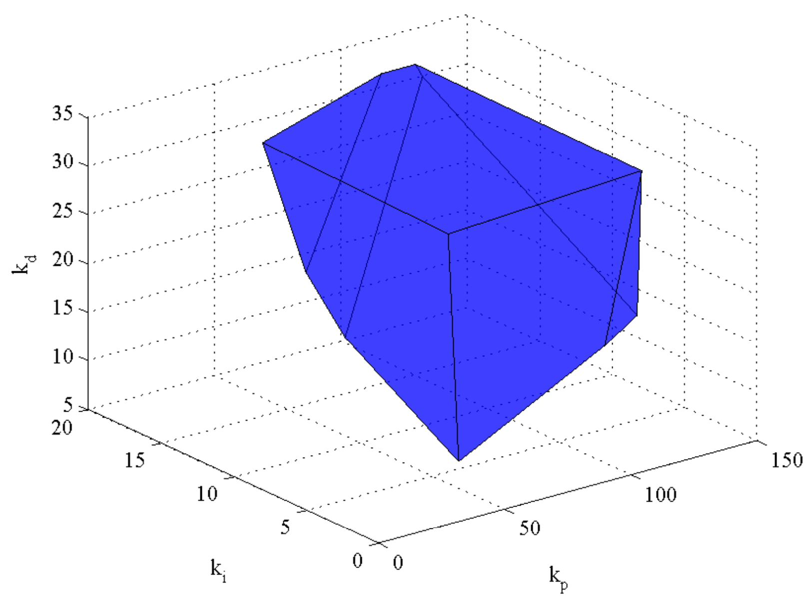

Fig. 7.: *Set of PID Controllers Satisfying SPR Condition* 3

Figure 8 shows the set of controller for which the transfer function is SPR and this set of controller absolutely stabilize the one-link robot with a flexible joint.

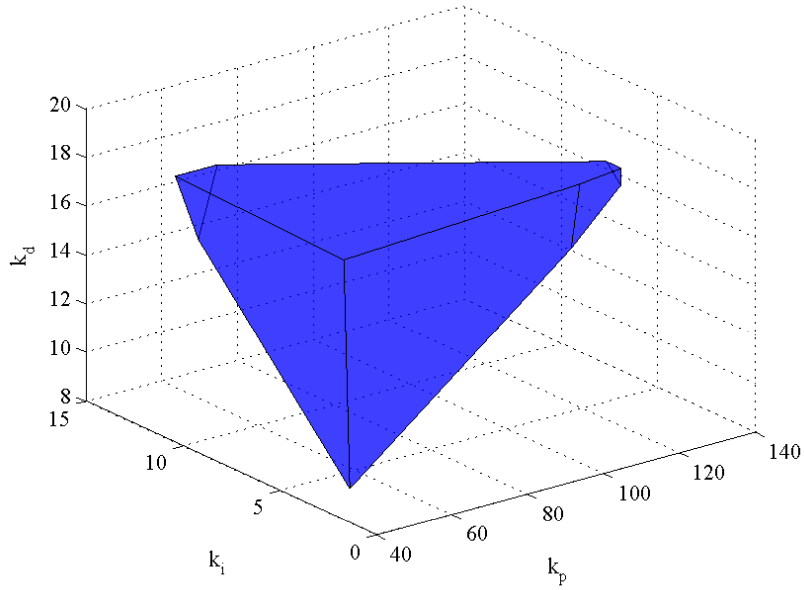

Fig. 8.: *Set of Absolutely Stabilizing PID Controllers*

From the admissible region shown in Figure 8, we selected the PID gain values to be $k_p = 50,\ k_i = 5$, and $k_d = 15$. Figure 9 shows the response for the one-link robot system with the selected PID controller.

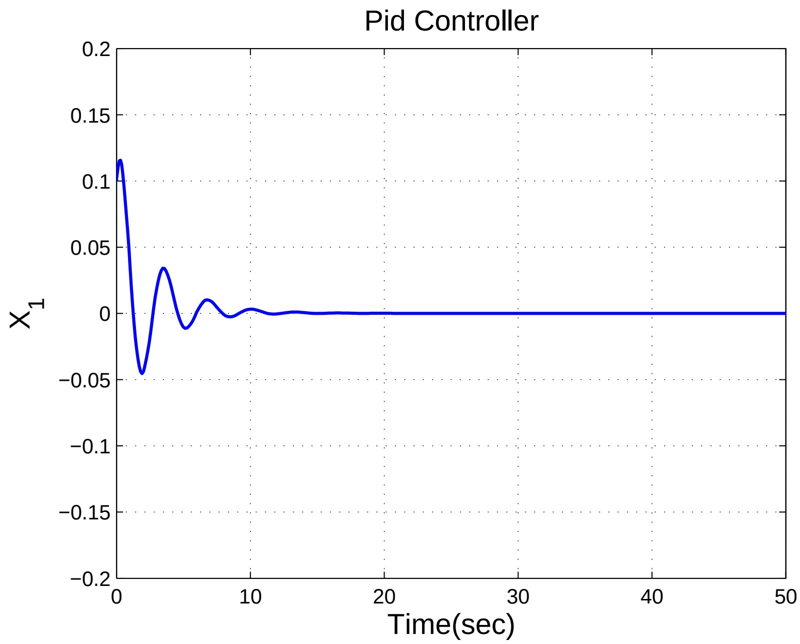

Fig. 9.: *Response of the Robot's Angular Position with a PID Controller (*$k_p = 50,\ k_i = 5,\ k_d = 15$*)*

2. First Order Controller

Let us consider the first order controller

$$C(s) = \frac{k_2 s + k_1^*}{s + k_3} \tag{2.24}$$

$$u = k_1 w + k_2 y \tag{2.25}$$

$$\dot{w} = -k_3 w + y$$

$$k_1^* = k_1 + k_2 k_3,$$

where $C(s)$ is the first order controller, w is a output filter and r is reference which is set to be 0. Now, the overall system can be represented as a augmented system.

$$\begin{aligned} \dot{z} &= \mathbf{A}z - \mathbf{B}\psi(y) \\ y &= \mathbf{C}z, \end{aligned} \tag{2.26}$$

where $z = \begin{bmatrix} x \\ w \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{I} & -\frac{b_1}{I} & \frac{k}{I} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k}{J} + k_2 & 0 & -\frac{k}{J} & -\frac{b_2}{J} & k_1 \\ 1 & 0 & 0 & 0 & -k_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.27}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.28}$$

$$\psi(y) = \frac{mgL}{I} \sin y, \tag{2.29}$$

$$\begin{aligned} G(s) &= \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} \\ &= \tfrac{N_{cl}(s)}{D_{cl}} \\ &= \tfrac{s^3+(2+k_3)s^2+(100+2k_3)s+100k_3}{s^5+(2+k_3)s^4+(150+2k_3)s^3+(150k_3+100)s^2+(-50k_2+100k_3)s-50k_1^*}, \end{aligned}$$

where $k_1^* = k_1 + k_2k_3$

From the Popov theorem, the above system is absolutely stable if there is $\eta \geq 0$, with $-\frac{1}{\eta}$ not an eigenvalue of A such that $G_T(s) = \frac{N_{GT}(s)}{D_{GT}(s)} = 1 + (1 + \eta s)\beta G(s) = \frac{D_{cl}(s)+(1+\eta s)\beta N_{cl}(s)}{D_{cl}(s)}$ is strictly positive real.

We will illustrate how to find the set of all first order controllers so that SPR conditions satisfy under $\eta = 1,\ \beta = 2$ as before.

1. For condition 1: $G_T(s) = \frac{N_{GT}(s)}{D_{GT}(s)}$
   $= \frac{D_{cl}(s)+(1+\eta s)\beta N_{cl}(s))}{D_{cl}(s)}$

$$= \frac{s^5+(4+k_3)s^4+(156+4k_3)s^3+(304+156k_3)s^2+(-50k_2+304k_3+200)s-50k_1^*+200k_3}{s^5+(2+k_3)s^4+(150+2k_3)s^3+(150k_3+100)s^2+(-50k_2+100k_3)s-50k_1^*}$$

$$G_T(0) = 1 - \frac{4k_3}{k_1^*} > 0$$

Figure 10 shows the set of controller which satisfying SPR condition 1 of Theorem II.3.

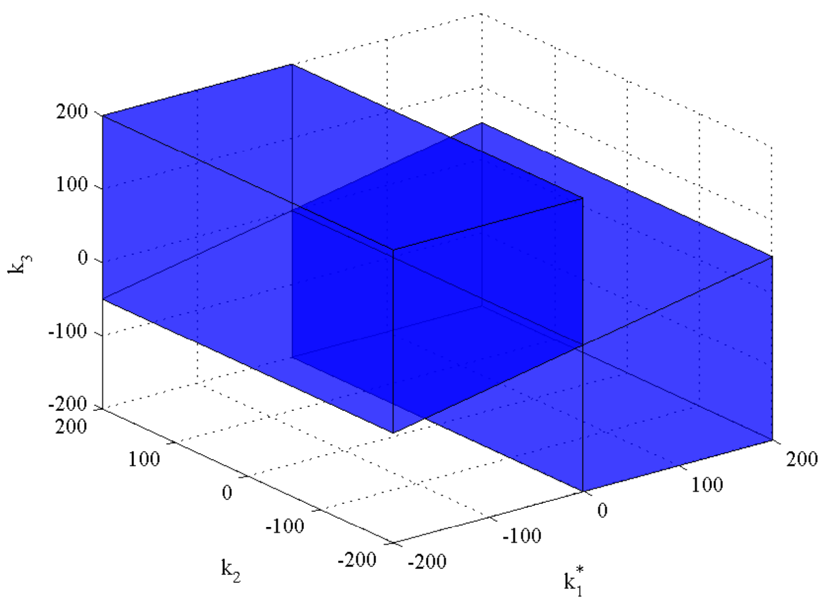

Fig. 10.: *Set of First Order Controllers Satisfying SPR Condition* 1

2. For condition 2:

$$\begin{aligned} N_{GT}(s) &= D_{cl}(s) + (1+\eta s)\beta N_{cl}(s)) \\ &= s^5 + (4+k_3)s^4 + (156+4k_3)s^3 + (304+156k_3)s^2 \\ &\quad +(-50k_2+304k_3+200)s - 50k_1^* + 200k_3 \end{aligned}$$

$$\begin{aligned} N_{GT}(jw,K) &= N_{GT,e}(w,K) + jwN_{GT,o}(w,K) \\ N_{GT,e}(w,K) &= (4+k_3)w^4 - (304+156k_3)w^2 - 50k_1^* + 200k_3 \\ N_{GT,o}(w,K) &= w^4 - (156+4k_3)w^2 - 50k_2 + 304k_3 + 200 \end{aligned}$$

$$C_k \begin{bmatrix} 1 & 0 & 0 \\ 1 & w_1^2 & w_1^4 \\ 1 & w_2^2 & w_2^4 \\ 1 & w_3^2 & w_3^4 \\ 1 & w_4^2 & w_4^4 \end{bmatrix} \begin{bmatrix} 0 & -50 & 0 & 200 \\ -304 & 0 & 0 & -156 \\ 4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ k_1^* \\ k_2 \\ k_3 \end{bmatrix} > 0,$$

and

$$S_k \begin{bmatrix} 1 & 0 & 0 \\ 1 & w_1^2 & w_1^4 \\ 1 & w_2^2 & w_2^4 \\ 1 & w_3^2 & w_3^4 \\ 1 & w_4^2 & w_4^4 \end{bmatrix} \begin{bmatrix} 200 & 0 & -50 & 304 \\ -156 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_1^* \\ k_2 \\ k_3 \end{bmatrix} > 0$$

Figure 11 shows the set of first order controller for which satisfying the SPR condition 2.

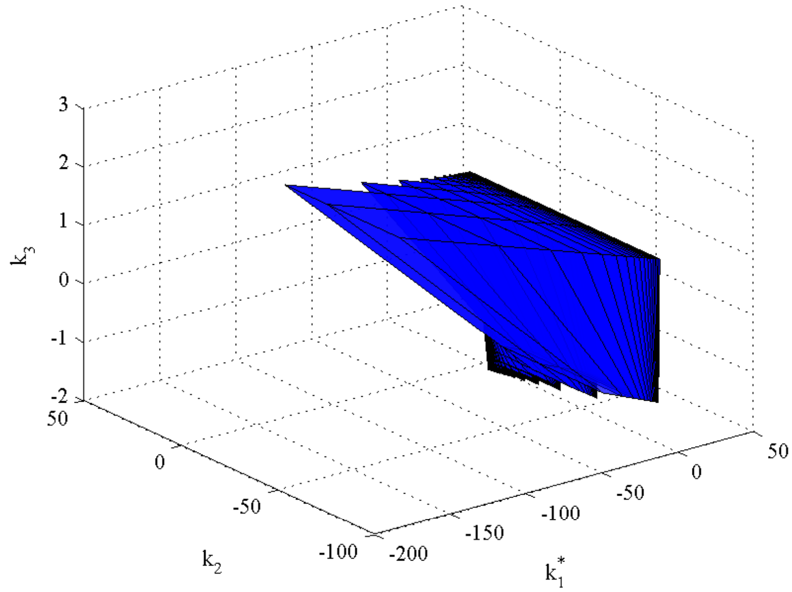

Fig. 11.: *Set of First Order Controllers Satisfying SPR Condition* 2

3. For condition 3:

$$
\begin{aligned}
P(s) &= D_{GT}(s) + j\alpha N_{GT}(s) \\
&= D_{cl} + j\alpha\{D_{cl}(s) + (1+\eta s)\beta N_{cl}(s)\} \\
&= (1+j\alpha)s^5 + \{2+k_3+j(4+k_3)\alpha\}\, s^4 + \{150+2k_3+j(156+4k_3)\alpha\}\, s^3 \\
&\quad + \{100+150k_3+j(304+156k_3)\alpha\}\, s^2 \\
&\quad + \{-50k_2+100k_3+j(-50k_2+304k_3+200)\alpha\}\, s \\
&\quad -50k_1^* + j(-50k_1^*+200k_3)\alpha
\end{aligned}
$$

$$
\begin{aligned}
P(jw,K) &= P_r(w,K) + jP_i(w,K) \\
P_r(w,K) &= -\alpha w^5 + (2+k_3)w^4 + 4\alpha(39+k_3)w^3 - 50(2+3k_3)w^2 \\
&\quad +\alpha(-200+50k_2-304k_3)w - 50k_1^* \\
P_i(w,K) &= w^5 + \alpha(4+k_3)w^4 - 2(75+k_3)w^3 + \alpha(-304-156k_3)w^2 \\
&\quad +50(-k_2+2k_3)w - 50\alpha(k_1^*-4k_3)
\end{aligned}
$$

$$
C_k \begin{bmatrix} 0 & 0 & \dots & -1 \\ 1 & w_1 & \dots & w_1^5 \\ 1 & w_2 & \dots & w_2^5 \\ \vdots & & & \\ \vdots & & & \\ 1 & w_9 & \dots & w_9^5 \end{bmatrix} \begin{bmatrix} 0 & -50 & 0 & 0 \\ -200\alpha & 0 & 50\alpha & -304\alpha \\ -100 & 0 & 0 & -150 \\ 156\alpha & 0 & 0 & 4\alpha \\ 2 & 0 & 0 & 1 \\ -\alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_1^* \\ k_2 \\ k_3 \end{bmatrix} > 0,
$$

and

$$
S_k \begin{bmatrix} 0 & 0 & \dots & -1 \\ 1 & w_1 & \dots & w_1^5 \\ 1 & w_2 & \dots & w_2^5 \\ \vdots & & & \\ \vdots & & & \\ 1 & w_9 & \dots & w_9^5 \end{bmatrix} \begin{bmatrix} 0 & -50\alpha & 0 & 200\alpha \\ 0 & 0 & -50 & 100 \\ -304\alpha & 0 & 0 & -156\alpha \\ -150 & 0 & 0 & -2 \\ 4\alpha & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_1^* \\ k_2 \\ k_3 \end{bmatrix} > 0
$$

Figure 12 shows the set of first order controller for which the SPR condition 3 holds.

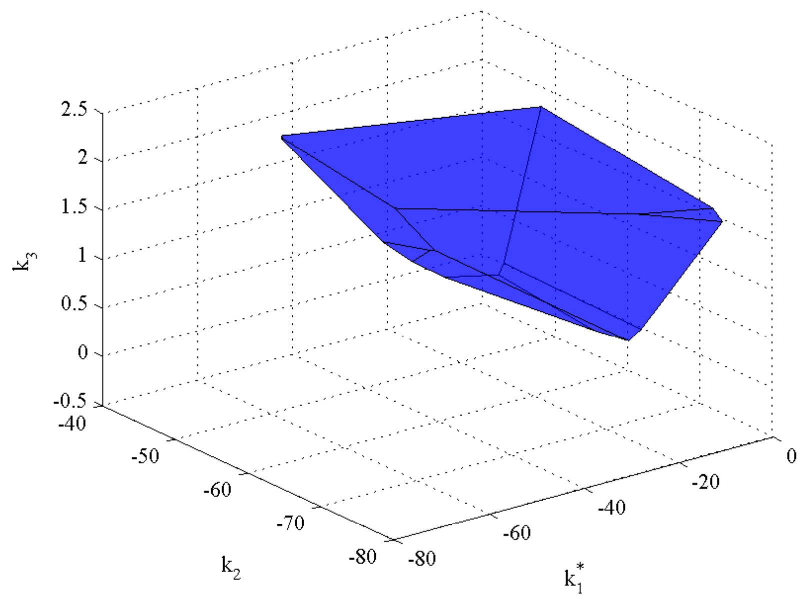

Fig. 12.: *Set of First Order Controllers Satisfying SPR Condition* 3

Figure 13 shows the set of the first order controller for which all the conditions hold and this set of controller absolutely stabilize the one-link robot with a flexible joint. From the admissible region shown in Figure 13, we selected the first order gain values to be $k_1^* = -20,\ k_2 = -50$ and $k_3 = 1$. Figure 14 shows the response for the one-link robot system with the selected first order controller.

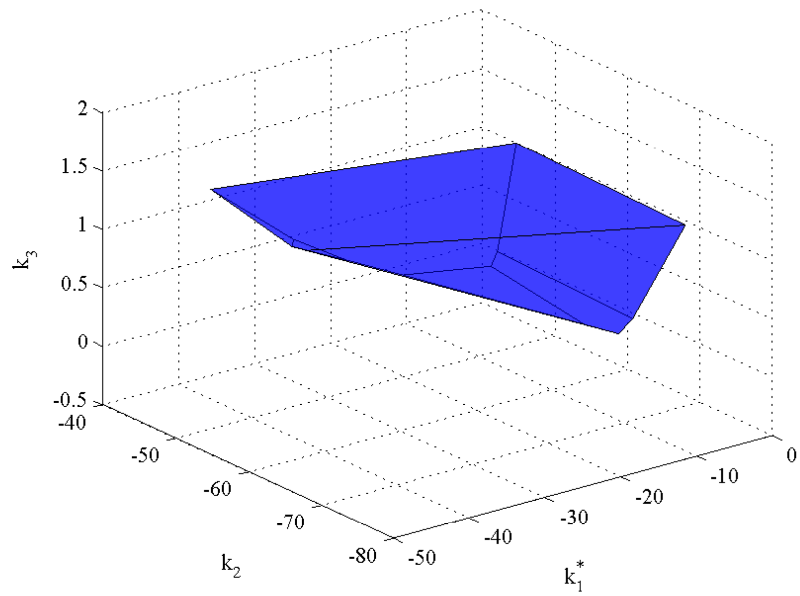

Fig. 13.: *Set of Absolutely Stabilizing First Order Controllers*

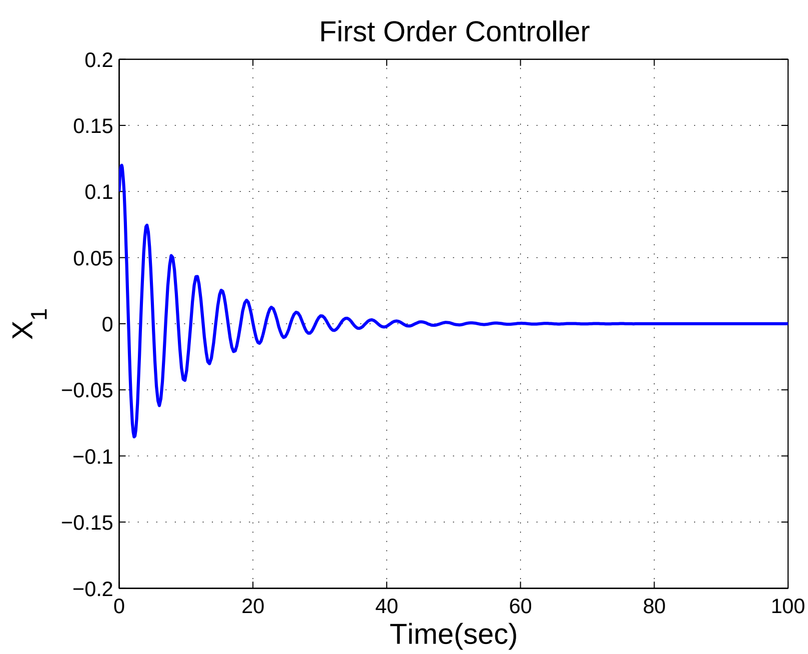

Fig. 14.: *Response of the Robot's Angular Position with a First Order Controller ($k_1^* = -20,\ k_2 = -50,\ k_3 = 1$)*

CHAPTER III

SUM-OF-SQUARES REPRESENTATIONS OF NONNEGATIVE POLYNOMIALS AND SEMIDEFINITE PROGRAMMING

A. Introduction

A method for synthesizing of fixed order stabilizing controllers directly from the empirical frequency response data and some coarse information about the SISO system will be proposed in chapter IV.

We utilize the well known Chevyshev polynomial [34, 35, 36, 37, 38, 39] to approximate the frequency response function in chapter IV. For this reason, we will provide a brief review of Chebyshev approximation, in section B. We also apply some recent results that sums of squares can be formulated as a linear inequality over the cone of positive semidefinite matrices(LMI) [20, 40, 41, 42, 43, 44].

The method to be proposed requires the nonnegativity of a real polynomial on some intervals. In section C of this chapter, we review the formulation for checking the nonnegativity of a real polynomial on an interval as a semidefinte program(SDP) through the use of Markov-Lucaks theorem [20].

B. Polynomial Approximation of Continuous Functions

1. Chebyshev Polynomials of the First and Second Kinds

We start with Weierstrass's result on approximation of a continuous function by polynomials.

Theorem III.1. *(Weierstrass Approximation)*

If f is a continuous real-valued function on $[a, b]$ and if any $\epsilon > 0$ is given, then there exists *a polynomial P on $[a,b]$ such that*

$$| \ f(x) - P(x) \ | < \ \epsilon \ \ \forall \ x \ \in \ [a,b] \tag{3.1}$$

In words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy. Proofs of the Weierstrass approximation theorem can be found in [45, 46].

The algebraic polynomials $T_n(x)$ satisfying

$$T_n(\cos x) = \cos(nx), \quad for \ n = 0, 1, 2, \ldots \tag{3.2}$$

are called the Chebyshev polynomials of the first kind [34, 35, 36, 37, 38]. This formula uniquely defines T_n as a polynomial of degree exactly n. The Chebyshev polynomial T_n is of degree n and its leading coefficient is 1 if $n = 0$, and 2^{n-1} if $n \geq 1$. We describe some properties of Chebyshev polynomials of the first kind.

1. Since $\cos(nx) = 2\cos x \ \cos(n-1)x - \cos(n-2)x$, $T_n(x)$ has the following recurrence relation [36].

   $$T_{n+1}(x) = 2xT_n(x) - T_{n-1}, \ \ n \ \geq \ 2,$$

   where $T_0(x) = 1, \ T_1(x) = x$. This recurrence relation may be taken as a definition for the Chebyshev polynomial of the first kind.

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
&\vdots
\end{aligned}$$

2. Chebyshev polynomials of the first kind are orthogonal with respect to the weight function $(1-x^2)^{-1/2}$ on the interval $(-1, 1)$.

$$\begin{aligned}
\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}} dx &= \int_0^{\pi} \cos n\theta \; \cos m\theta \; d\theta \\
&= \left\{ \begin{array}{c} 0, \quad n \neq m \\ \pi, \quad n = m = 0 \\ \pi/2, \quad n = m \neq 0 \end{array} \right\}
\end{aligned}$$

3. The polynomial $T_n(x)$ has n zeros in the interval $[-1,\ 1]$, and they are located at the points

$$x = \cos\left(\frac{\pi(k-1/2)}{n}\right), \quad k = 1, 2 \ldots, n \tag{3.3}$$

4. The Chebyshev polynomials also satisfy a discrete orthogonal property. If x_k $(k =$

$1, 2, \ldots, m)$ are the m zeros of $T_m(x)$ given by (3.3) and if $i, j \; < \; m$, then

$$\sum_{k=1}^{m} T_i(x_k)T_j(x_k) = \left\{ \begin{array}{c} 0, \quad i \neq j \\ m, \quad i = j = 0 \\ m/2, \quad i = j \neq 0 \end{array} \right\} \tag{3.4}$$

The Chebyshev polynomials $U_n(x)$ of the second kind are some polynomials of degree n in x and are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta \tag{3.5}$$

This formula uniquely defines $U_n(x)$ as a polynomial of degree exactly n.

We describe some properties of Chebyshev polynomials of the second kind.

1. Since $\sin(n+1)\theta + \sin(n-1)\theta = 2\cos\theta\sin n\theta$, $U_n(x)$ has the following recurrence relation [36].

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x), \quad n \; \geq \; 2,$$

where $U_0(x) = 1, \; U_1(x) = 2x$. This recurrence relation may be taken as a definition for the Chebyshev polynomials of the second kind.

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \\ U_4(x) &= 16x^4 - 12x^2 + 1 \\ U_5(x) &= 32x^5 - 32x^3 + 6x \\ &\vdots \end{aligned}$$

2. Chebyshev polynomials of the second kind are orthogonal with respect to the weight function $(1-x^2)^{1/2}$ on the interval $(-1,1)$.

3. The derivative of Chebyshev polynomials of the first kind can be represented as Chebyshev polynomials of the second kind:

$$\begin{aligned}\frac{dT_n(x)}{dx} &= -\frac{1}{\sin\theta}\frac{\cos n\theta}{d\theta}\\ &= \frac{\cos n\theta}{d\theta}/\frac{\cos\theta}{d\theta}\\ &= \frac{n\sin n\theta}{\sin\theta}\\ &= nU_{n-1}(x) \qquad (3.6)\end{aligned}$$

2. Chebyshev Approximation

The Chebyshev approximation uses Chebyshev polynomials as a basis for the approximating polynomials [39].

Theorem III.2. *(Chebyshev Approximation)*
Let $f(x)$ be an arbitrary continuous function in the interval $[-1,1]$ then $f(x)$ can be approximated using the first "$N+1$" Chebyshev polynomials as:

$$p_n(x) = \left[\sum_{k=0}^{N} C_k T_k(x)\right] - \frac{1}{2}C_0, \qquad (3.7)$$

where

$$\begin{aligned}C_j &\equiv \frac{2}{N}\sum_{k=1}^{N} f(x_k)T_j(x_k)\\ &= \frac{2}{N}\sum_{k=1}^{N} f\left[\cos\left(\frac{\pi(k-\frac{1}{2})}{N}\right)\right]\cos\left(\frac{\pi j(k-\frac{1}{2})}{N}\right), \qquad (3.8)\end{aligned}$$

It is not difficult to verify theorem III.2 with (3.3) and (3.4).

For Chebyshev polynomial approximation, it is necessary to normalize the frequency range $w \in [a, b]$ to $x \in [-1, 1]$ as follows:

$$x = -1 + 2\frac{w-a}{b-a}, \quad w \in [a, b] \tag{3.9}$$

Let $\delta(s) = \Delta(s)\frac{N_p(-s)}{D_p(s)D_p(-s)}$, where $\Delta(s)$ is characteristic polynomial for a system and $D_p(s)$ and $N_p(s)$ are the denominator and numerator of a plant will be seen in chapter IV. Now, we are ready to approximate $\frac{\delta_r(jw,K)}{|D_p(jw)|^2}$ and $\frac{\delta_i(jw,K)}{|D_p(jw)|^2}$ with finite frequency data to the Chebyshev polynomial of degree N.

1. The approximation of the real part $f_r(x, K) \approx \frac{\delta_r(jw,K)}{|D_p(jw)|^2}$

$$\begin{aligned} \frac{\delta_r(jw, K)}{\mid D_p(jw) \mid^2} &= \Delta_r(w, |G(jw)|^2, G_r(w), G_i(w))\,[K'] \\ f_r(x, K) &= C_0^r(K)T_0(x) + C_1^r(K)T_1(x) + \ldots + C_N^r(K)T_N(x) \end{aligned} \tag{3.10}$$

2. The approximation of the imaginary part $f_i(x, K) \approx \frac{\delta_i(jw,K)}{|D_p(jw)|^2}$

$$\begin{aligned} \frac{\delta_i(jw, K)}{\mid D_p(jw) \mid^2} &= \Delta_i(w, |G(jw)|^2, G_r(w), G_i(w))\,[K'] \\ f_i(x, K) &= C_0^i(K)T_0(x) + C_1^i(K)T_1(x) + \ldots + C_N^i(K)T_N(x) \end{aligned} \tag{3.11}$$

3. To approximate derivative of the imaginary part $f_i(x, K)$, we use the relation between $\frac{dT_n(x)}{dx}$ and $U_n(x)$ as shown in equation (3.6).

$$\frac{dT_n(x)}{dx} = nU_{n-1}(x)$$

$$\frac{df_i(x, K)}{dx} = C_0^d(K)U_0(x) + C_1^d(K)U_1(x) + \ldots + C_{N-1}^d(K)U_{N-1}(x),$$

where $C_j^r(K),\ C_j^i(K),\ j = 1, N$ and $C_k^d(K),\ k = 1, N-1$ are affine in K.

3. Discrete Polynomial Transforms

Let $p_n(x)$ be a orthogonal and normalized polynomial on a bounded or unbounded interval $I \subseteq \Re$, with respect to a nonnegative weight function $w(x)$.

$$\int_I p_n(x)p_m(x)w(x)dx = \begin{cases} 0, & \text{n} \neq \text{m} \\ 1, & \text{n=m.} \end{cases}$$

The Chebyshev polynomials of the first kind are orthogonal on the interval $(-1, 1)$ with respect to a weight function $(1 - x^2)^{-1/2}$.

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \left\{ \begin{array}{cc} 0, & n \neq m \\ \pi, & n = m = 0 \\ \pi/2, & n = m \neq 0 \end{array} \right\} \tag{3.12}$$

Then the normalized Chebyshev polynomials are as follows:

$$\begin{aligned} p_0(x) &= \sqrt{\frac{1}{\pi}}\, T_0(x) \\ p_1(x) &= \sqrt{\frac{2}{\pi}}\, T_1(x) \\ p_2(x) &= \sqrt{\frac{2}{\pi}}\, T_2(x) \\ &\vdots \end{aligned} \tag{3.13}$$

Now, the approximation polynomials $f_r(x, K)$ or $f_i(x, K)$ can be rewritten as $p_i(x)$.

$$f(x, K) = C_0^p(K)p_0(x) + C_1^p(K)p_1(x) + \ldots + C_N^p(K)p_N(x) \tag{3.14}$$

We define the discrete polynomial transforms V_p for $f(x) = C_0^p p_0(x) + C_1^p p_1(x) + \ldots + C_N^p p_N(x)$ which offers a way to map the coefficients of a polynomial to its polynomial values.

Definition III.1.

Let $\lambda_0, \lambda_1, \ldots, \lambda_N$ are the roots of p_{N+1}. then we define V_p [20, 35].

$$V_p = \begin{bmatrix} p_0(\lambda_0) & p_1(\lambda_0) & \cdots & p_N(\lambda_0) \\ p_0(\lambda_1) & p_1(\lambda_1) & \cdots & p_N(\lambda_1) \\ \vdots & \vdots & & \vdots \\ p_0(\lambda_N) & p_1(\lambda_N) & \cdots & p_N(\lambda_N) \end{bmatrix} \tag{3.15}$$

The linear transformation V_p maps the coefficients of the polynomial

$$f(x) = C_0^p p_0(x) + C_1^p p_1(x) + \ldots + C_N^p p_N(x) \tag{3.16}$$

to $N+1$ values at $\lambda_0, \lambda_1, \cdots, \lambda_N$ and vice-versa.

$$y = V_p C^p = \begin{bmatrix} p_0(\lambda_0) & p_1(\lambda_0) & \cdots & p_N(\lambda_0) \\ p_0(\lambda_1) & p_1(\lambda_1) & \cdots & p_N(\lambda_1) \\ \vdots & \vdots & & \vdots \\ p_0(\lambda_N) & p_1(\lambda_N) & \cdots & p_N(\lambda_N) \end{bmatrix} \begin{bmatrix} C_0^p \\ C_1^p \\ \vdots \\ C_N^p \end{bmatrix}, \tag{3.17}$$

where $y = [f(\lambda_0),\ f(\lambda_1)\ , \ldots, f(\lambda_N)]^T$.

Then the coefficients C^p can be determined as follows:

$$C^p = W_p^T y, \tag{3.18}$$

where W_p is such that $W_P^T V_p = I$.

We can similarly define $q(x)$, V_q, W_q and C^q for $\frac{df_i(x,K)}{dx}$ corresponding to $p(x)$, V_p, W_p and C^p respectively.

$$\frac{df_i(x,K)}{dx} = C_1^q q_0(x) + C_2^q q_1(x) + \ldots + C_N^q q_{N-1}(x), \tag{3.19}$$

where $q_i(x) = \sqrt{\frac{2}{\pi}} U_i(x), \quad i = 0, \ldots, N-1$.

C. Semidefinite Representations for Nonnegative Polynomials

It is well known that nonnegative polynomials can be represented as sums of squares(SOS) [47, 48]. The condition that a polynomial is sums of squares can be formulated as a linear inequality over the cone of positive semidefinite matrices(LMI) [20, 40, 41, 42, 43, 44].

1. Sum of Squares

A basic problem that appears in many areas of control and optimization is that of checking global, or local nonnegativity of a function of several variables [43, 44].

Theorem III.3.

If a real polynomial $f(x)$ of degree n is nonnegative for all $x \in \Re$, then $f(x)$ can be written as sum of squares.

$$f(x) = f_1^2(x) + f_2^2(x) \tag{3.20}$$

for some polynomials f_1 and f_2 such that $deg(f_1) \leq n/2$ and $deg(f_2) \leq n/2$

Proof.

If $f(x) \geq 0 \; \forall \; x \in \Re$ then it cannot have real roots. this implies $f(x)$ must be even degree, i.e. $n = 2m$ for some m. Let $\sigma_i + jw_i, \;\; i = 1, m$ be the $2m$ roots of $f(x)$. In the factored form:

$$\begin{aligned}
f(x) &= \left\{\sqrt{\alpha}\prod_{i=1}^{m}(x - \sigma_i - jw_i)\right\}\left\{\prod_{i=1}^{m}(x - \sigma_i + jw_i)\sqrt{\alpha}\right\} \\
&= [R(x) + jI(x)]\,[R(x) - jI(x)] \\
&= R^2(x) + I^2(x)
\end{aligned}$$

$\nabla\nabla\nabla$

2. Semidefinite Representations

Let $g_1, g_2, \ldots, g_s$ be all monomials of degree r or less. A monomial is a product of positive integer powers of a fixed set of variables.

Theorem III.4.

A polynomial $f(x)$ of degree n is a sum of squares if and only if there exist a positive semidefinite matrix X and a vector of monomials $g(x)$, each row of degree no more than $n/2$ such that

$$f(x) = g^T(x)Xg(x), \quad \textit{for some } X \succeq 0 \tag{3.21}$$

Proof.

Let $q(x) = [q_1(x)\ q_2(x)\ \ldots]^T = Lg(x)$. L is a compatible coefficient matrix and $g(x)$ is a vector of monomials containing all monomials in $q(x)$. Then

$$f(x) = q^T(x)q(x) = g^T(x)L^T Lg(x)$$

and $X = L^T L$, $X \succeq 0$. Now suppose there exists $f(x) = g^T(x)Xg(x)$. A positive semidefinite matrix X can be represented by the eigenvalue decomposition $X = M^T \Lambda M$ as shown in [44]. Then

$$f(x) = g^T(x)M^T \Lambda M g(x) = \sum_{i=1} \lambda_i (Mg(x))_i^2$$

$\nabla\nabla\nabla$

Since $f(x)$ being sum of squares is equivalent to $X \succeq 0$, the problem to find a X which proves that $f(x)$ is a sum of squares can be put a linear matrix inequality. We can show this through an example.

Example

Consider a fourth-order polynomial $f(x)$ and define $g(x) = [x^2\ x\ 1]^T$.

$$\begin{aligned}
f(x) &= 2x^4 + 4x^3 + 10x^2 - 8x + 5 \\
&= \begin{bmatrix} x^2 & x & 1 \end{bmatrix} X \begin{bmatrix} x^2 & x & 1 \end{bmatrix}^T \\
&= \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x^2 & x & 1 \end{bmatrix}^T \\
&= \alpha_{11}x^4 + (\alpha_{11} + \alpha_{21})x^3 + (\alpha_{13} + \alpha_{22} + \alpha_{31})x^2 + (\alpha_{23} + \alpha_{32})x + \alpha_{33}
\end{aligned}$$

Comparing the coefficients, we can get followings:

$$X = \begin{bmatrix} 2 & -2 & \alpha_{13} \\ -2 & 10 - 2\alpha_{13} & -4 \\ \alpha_{13} & -4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 10 & -4 \\ 0 & -4 & 5 \end{bmatrix} + \alpha_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Now, $f(x)$ can be decomposed as a sum of squares by searching for α_{13} such that $X \succeq 0$. In other words, $X \succeq 0$ if and only if f is sums of squares. In particular, for $\alpha_{13} = 3$, the matrix X will be positive semidefinite and we have

$$\begin{aligned}
X &= \begin{bmatrix} 2 & -2 & 3 \\ -2 & 4 & -4 \\ 3 & -4 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 2 \end{bmatrix} \\
&= LL^T
\end{aligned}$$

This yields a sum of squares decomposition.

$$f(x) = (x^2 + 1)^2 + (x^2 - 2x + 2)^2.$$

Since we have to find nonnegative conditions of a real polynomial in the specific frequency intervals, local nonnegativity of a polynomial has to be considered.

Theorem III.5. *(Markov-Lucaks)*
Let f be a polynomial of degree n with real coefficients. Suppose $f(x) \geq 0$ for all $x \in [a, b]$, then one of the following holds.

1. *If $deg(f) = n = 2m$ is even, then*

$$f(x) = f_1^2(x) + (x - a)(b - x)f_2^2(x) \tag{3.22}$$

for some polynomials f_1 and f_2 such that $deg(f_1) \leq m$ and $deg(f_2) \leq m - 1$

2. *If $deg(f) = n = 2m + 1$ is odd, then*

$$f(x) = (x - a)f_1^2(x) + (b - x)f_2^2(x) \tag{3.23}$$

for some polynomials f_1 and f_2 such that $deg(f_1) \leq m$ and $deg(f_2) \leq m$

Proofs can be found in [47, 48, 49].

Recently, Roh and Vandenberghe provided a technique for checking the local nonnegativity problem through the feasibility of a positive semidefinite matrix satisfying a set of Linear Matrix Inequalities. The proof requires the following definition and the use of Markov-Lucaks Theorem.

Definition III.2.
$A\ o\ B$ denotes the Hadamard product of two matrices A and B of the same dimension, i.e., the matrix with elements $(A\ o\ B)_{ik} = A_{ik}B_{ik}$. The same notation is used for vectors

$(x\ o\ y)_i = x_i y_i$. *For real matrices* $sqr(A) = A\ o\ A$*, For complex matrices* $sqr(A) = A\ o\ \bar{A}$, $(\bar{A}$ *is complex conjugate of* $A)$.

Theorem III.6. *(Roh and Vandenberghe)*

$f(x) \geq 0$ *for* $x \in [x_1, x_2]$ *iff there exist* $X_1 \in S^{m_1+1}, X_2 \in S^{m_2+1}$ *such that*

$$C^p(K) = W_p^T \left[d_1\ o\ diag(V_1 X_1 {V_1}^T) + d_2\ o\ diag(V_2 X_2 {V_2}^T) \right], X_1 \succeq 0, X_2 \succeq 0 \quad (3.24)$$

$m_1 = \lfloor N/2 \rfloor$, $m_2 = \lfloor \frac{N-1}{2} \rfloor$. *The matrices* V_1 *and* V_2 *are formed by the first* $m_1 + 1$ *and* $m_2 + 1$ *columns of* V_p *respectively.* $\lfloor z \rfloor$ *is the largest integer which does not exceed* z. *The vectors* $d_1, d_2 \in \Re^{N+1}$ *are defined as*

$$d_1 = \left\{ \begin{array}{cc} \bar{1}, & for\ even\ N \\ \lambda - x_1\bar{1}, & for\ odd\ N \end{array} \right\}, \quad d_2 = \left\{ \begin{array}{cc} (\lambda - x_1\bar{1})\ o\ (x_2\bar{1} - \lambda), & for\ even\ N \\ x_2\bar{1} - \lambda, & for\ odd\ N \end{array} \right\} \quad (3.25)$$

$\lambda = [\lambda_0\ \lambda_1\ \ldots\ \lambda_N]^T$ *are the roots of* p_{N+1}, *the normalized Chebyshev polynomial of degree* $N + 1$.

Proof.

Let us suppose the degree of $f(x)$ *is even* $(N = 2m)$. *Then by Markov-Lucaks Theorem, the nonnegative* $f(x) \geq 0$ *for* $x \in [x_1, x_2]$ *can be represented as sums of squares:*

$$f(\lambda) = g^2(\lambda) + (\lambda - x_1)(x_2 - \lambda)h^2(\lambda)$$

$$\begin{bmatrix} f(\lambda_0) \\ f(\lambda_1) \\ \vdots \\ f(\lambda_N) \end{bmatrix} = \begin{bmatrix} g^2(\lambda_0) + (\lambda_0 - x_1)(x_2 - \lambda_0)h^2(\lambda_0) \\ g^2(\lambda_1) + (\lambda_1 - x_1)(x_2 - \lambda_1)h^2(\lambda_1) \\ \vdots \\ g^2(\lambda_N) + (\lambda_N - x_1)(x_2 - \lambda_N)h^2(\lambda_N) \end{bmatrix}$$

Let the polynomials g *and* h *be such that* $deg(g) \leq m$ *and* $deg(h) \leq m - 1$. *Let* $g(\lambda)$

has the form

$$g(\lambda) = \sum_{i=0}^{m} u_i p_i(\lambda)$$

Let $\bar{u} = [u_0 \; u_1 \; \ldots \; u_m]^T$ *and* $p_i(\lambda), \quad i = 0, m$ *be a orthogonal polynomial.*

$$\begin{bmatrix} g(\lambda_0) \\ g(\lambda_1) \\ \vdots \\ g(\lambda_N) \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{m} u_i p_i(\lambda_0) = \bar{u}^T p(\lambda_0) \\ \sum_{i=0}^{m} u_i p_i(\lambda_1) = \bar{u}^T p(\lambda_1) \\ \vdots \\ \sum_{i=0}^{m} u_i p_i(\lambda_N) = \bar{u}^T p(\lambda_N) \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} g^2(\lambda_0) \\ g^2(\lambda_1) \\ \vdots \\ g^2(\lambda_N) \end{bmatrix} &= \begin{bmatrix} \{\bar{u}^T p(\lambda_0)\}^T \{\bar{u}^T p(\lambda_0)\} = p(\lambda_0)^T \bar{u}\bar{u}^T p(\lambda_0) \\ \{\bar{u}^T p(\lambda_1)\}^T \{\bar{u}^T p(\lambda_1)\} = p(\lambda_1)^T \bar{u}\bar{u}^T p(\lambda_1) \\ \vdots \\ \{\bar{u}^T p(\lambda_N)\}^T \{\bar{u}^T p(\lambda_N)\} = p(\lambda_N)^T \bar{u}\bar{u}^T p(\lambda_N) \end{bmatrix} \\ &= diag(V_1 X_1 V_1^T) \end{aligned}$$

The matrix V_1 *is x formed by the first* $m+1$ *columns of* V_p *and* $X_1 \in \mathbf{S}^{m+1}$ *is a positive semidefinite matrix.*

Similarly $h(\lambda)$ *can be represented*

$$\begin{bmatrix} h^2(\lambda_0) \\ h^2(\lambda_1) \\ \vdots \\ h^2(\lambda_N) \end{bmatrix} = diag(V_2 X_2 V_2^T),$$

where V_2 *is a matrix formed by the first* m *columns of* V_p *and* $X_2 \in \mathbf{S}^m$ *is a positive* *semidefinite matrix. Then* $f(\lambda)$ *can be written as:*

$$f(\lambda) = \bar{1}\ o\ diag(V_1 X_1 V_1^T) + (\lambda - x_1 \bar{1})\ o\ (x_2 \bar{1} - \lambda) diag(V_2 X_2 V_2^T)$$

Finally, we can get the equation (3.24) in Theorem III.6 by $C^p = W_p^T f(\lambda)$.

$$C^p(K) = W_p^T \left[\bar{1}\ o\ diag(V_1 X_1 V_1^T) + (\lambda - x_1 \bar{1})\ o\ (x_2 \bar{1} - \lambda) diag(V_2 X_2 V_2^T)\right]$$

$\nabla\nabla\nabla$

In essence, the problem of checking if a polynomial is nonnegative on an interval can be accomplished by ascertaining the existence of two positive semidefinite matrices, the entries of which constrained by the coefficients of the polynomials through linear equality constraints.

CHAPTER IV

SYNTHESIS OF FIXED ORDER STABILIZING CONTROLLERS USING FREQUENCY RESPONSE MEASUREMENTS

A. Introduction

It is widely recognized that an accurate analytical model of the plant may not be available to a control designer. However, it is reasonable in many applications that one will have an empirical model of the plant in terms of its frequency response data and from physical considerations or from the empirical time response data, one may have some coarse information about the plant such as the number of non-minimum phase zeros of the plant etc. In view of this, we consider here the problem of synthesizing sets of stabilizing controllers directly from the empirical data and such coarse information about the plant.

The frequency response information can have a variety of applications for the analysis and design of control systems. For example, the Nyquist stability criterion enables us to investigate both the absolute and relative stabilities of linear closed-loop systems from the knowledge of their open-loop frequency response characteristics [50]. The frequency response information can also be used for system identification and controller design in time domain [51].

There are many techniques for synthesizing controllers from empirical data of the plant, see [52, 53, 54, 55, 56, 57, 58]. In [55], Oaki used the frequency response information to determine force control parameters in a robot manipulator force control. A PID controller design method based on frequency-response data for process control was introduced in [58]. A systematic attempt to synthesize PID and first order controllers for LTI plants using frequency response measurements was first presented in [19]. However, we are unaware of any systematic attempt at synthesizing sets of stabilizing controllers of arbitrary order directly from the frequency response data and this work is a first attempt in that direction. We propose a new method to synthesize a stabilizing fixed order controller with finite frequency response data and the number of non-minimum zeroes of a plant. We pose the problem of synthesizing the sets of stabilizing controllers as that of sets of controllers satisfying some robust SDPs. The robust SDPs take into account measurement errors in frequency response.

The following are the standing assumptions about the plant:

Assumption IV.1.

1. *The transfer function $G(s)$ of the plant is rational and strictly proper, i.e., $G(s) = \frac{N_p}{D_p}(s)$, for some co-prime polynomials, $N_p(s)$ and $D_p(s)$, with the degree n of $D_p(s)$ greater than the degree m of $N_p(s)$. We may not know either m or n.*
2. *There are no poles and zeros of the plant on the imaginary axis, i.e., $D_p(jw) \neq 0$ and $N_p(jw) \neq 0$ for every $w \in \Re$.*
3. *This assumption and the following assumptions concern the knowledge of frequency response of the plant: There is a frequency w_b beyond which the phase of the plant does not change appreciably and the amplitude response of the plant is negligible. To quantify this statement, let $G(jw)$ be expressed as $G_r(w) + jwG_i(w)$, where G_r and G_i are real, rational functions of w. For some known $\epsilon > 0$, we assume that $|G(jw)| \leq \epsilon \; \forall w \geq w_b$. This is a reasonable assumption since the plant is strictly proper.*
4. *The relative degree $n-m$ is known. This can be inferred from the amplitude response of the plant at sufficiently high frequencies.*
5. *We will assume that the functions $|G(jw)|^2$, $G_r(w)$, $G_i(w)$ have been approximated using polynomials $P_0(w)$, $P_1(w)$, $P_2(w)$ respectively and the maximum estimation*

errors are bounded by μ_0, μ_1, μ_2 and the maximum derivatives of the estimation errors are bounded by η_0, η_1, η_2 respectively. Mathematically, for all $w \in [0, w_b]$, we have

$$\begin{aligned}
||G(jw)|^2 - P_0(w)| &\leq \mu_0, \\
|G_r(w) - P_1(w)| &\leq \mu_1, \\
|G_i(w) - P_2(w)| &\leq \mu_2, \\
|\frac{d(|G_p(jw)|^2 - P_0(w)|)}{dw} &\leq \eta_0, \\
|\frac{d(G_r(w) - P_1(w))}{dw}| &\leq \eta_1, \\
|\frac{d(G_i(w) - P_2(w))}{dw}| &\leq \eta_2.
\end{aligned}$$

We assume that $\mu_i,\ \ \eta_i,\ \ i = 0, 1, 2$ and the polynomials $P_0(w), P_1(w), P_2(w)$ are known.

6. *We will assume that the number of non-minimum phase zeros, z_r of the plant are known.*

We are interested in synthesizing a rational, proper stabilizing controller $C(s)$, i.e., for some monic polynomial $D_c(s)$ of degree r and a polynomial $N_c(s)$ of degree at most r, $C(s) = \frac{N_c}{D_c}(s)$. Let $N_c(s) = n_0 + n_1 s + \ldots + n_r s^r$ and $D_c(s) = d_0 + d_1 s + \ldots + d_{r-1} s^{r-1} + s^r$. Let K be the vector of controller coefficients:

$$\begin{bmatrix} n_0 & n_1 & \ldots & n_r & d_0 & d_1 & \ldots & d_{r-1} \end{bmatrix}^T.$$

The determination of the vector K is equivalent to the determination of the stabilizing controller $C(s)$.

This chapter is organized as follows: In section B, we provide basic ideas to derive main results. In section C, we present the main results and provide a numerical example.

In section D, we propose some robust SDPs to handle the measurement errors. In section E, we deal with a special case in which case the nonnegativity in intervals can be posed to linear inequalities.

B. Basic Ideas

The basic ideas used in the construction of stabilizing sets are as follows:

1. We first construct a rational function:

$$\delta(s) = G(s)G(-s)N_c(s) + G(-s)D_c(s) \tag{4.1}$$

   In fact, if $\Delta(s) := N_p(s)N_c(s) + D_p(s)D_c(s)$ is the characteristic polynomial of the closed loop system, then it is easy to see that:

$$\delta(s) = \Delta(s)\frac{N_p(-s)}{D_p(s)D_p(-s)} \tag{4.2}$$

   If $\Delta(s)$ has coefficients that are affine in the controller coefficients, then the rational function, $\delta(s)$, is also affine in the controller coefficients.

2. All controllers, $C(s)$, that stabilize $\Delta(s)$, are such that the total phase accumulation of $\delta(jw)$ as w varies from 0 to ∞ is the same and equals $(n - m + r + 2z_r)\frac{\pi}{2}$.

$$\angle\delta(jw)|_{w=0}^{w=\infty} = \frac{\pi}{2}(n - m + r + 2z_r) \tag{4.3}$$

   Since $n - m$, r and z_r are known, the total desired phase accumulation is known.

3. Let $\delta(jw) = \delta_r(w) + jw\delta_i(w)$, where $\delta_r(w)$ and $\delta_i(w)$ are real, rational functions. In Lemma IV.2, we relate how the total accumulation of phase is related to the roots of $\delta_i(w)$ and the sign of $\delta_r(w)$ at those roots.

Essentially, the numerator of $\delta(s)$ must have a certain number of roots with negative real parts. This can happen only if the Nyquist plot of $\delta(s)$ is one of finitely many patterns, where each pattern can be identified with the signs of the real part of the Nyquist plot when the imaginary part is zero. The set of such patterns can be characterized using the generalized phase formula developed in [25, 59].

4. The existence of a stabilizing controller for the plant can be expressed in terms of the existence of an appropriate set of frequency intervals which admit exactly one or zero roots of the imaginary part of the Nyquist plot and no roots of the real part. This is shown in Theorem IV.1. For every set of frequency intervals, these conditions can be translated into linear inequality constraints or linear matrix inequality (LMI) constraints involving the controller parameters. This step involves the Chebyshev approximation of the frequency response in the frequency band $[0, w_b]$. It subsequently involves the use of Markov-Lucaks theorem to convert the conditions into a LMI form.

C. Main Results

Let $\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_d s^d$ be a real polynomial. Then the following Lemma relates the net phase change $\angle\delta(jw)$ as w increases from zero to infinity [25, 32].

Lemma IV.1. *(Net Phase Change Property for real polynomials)*

1. *The phase of the real Hurwitz polynomial,* $\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_d s^d$ *monotonically increases as* $w : 0 \ \rightarrow \ +\infty$*. The plot of* $\delta(jw)$ *moves strictly counterclockwise and goes through* d *quadrants as* $w : 0 \ \rightarrow \ +\infty$*.*

2. *The plot of the* d^{th} *order real polynomial (not necessary Hurwitz),* $\delta(jw) = \delta_r(w) + jw\delta_i(w)$ *goes through* $l(\delta(s)) - r(\delta(s))$ *quadrants as* $w : 0 \ \rightarrow \ +\infty$*.*

$$\angle\delta(jw)|_{w=0}^{w=\infty} = \frac{\pi}{2}\left[l(\delta) - r(\delta)\right], \tag{4.4}$$

where $l(\delta), r(\delta)$ denote the numbers of roots of $\delta(s)$ in the left half plane and in the right half plane respectively.

Following the outline of the basic ideas presented in the earlier section, we begin with a generalization of Hermite-Biehler theorem for rational functions in Lemma IV.2.

Lemma IV.2.

Consider $\delta(s) = \frac{\Delta(s)N_p(-s)}{D_p(s)D_p(-s)}$. Let the nonnegative real roots of $\delta_i(w)$ be $w_1, \ldots, w_l$ and the sign of $\delta_r(w)$ at these frequencies be correspondingly $i_1, \ldots, i_l$. Then $\Delta(s)$ is Hurwitz if and only if

1. *for $n - m + r$: even*

$$n - m + r + 2z_r = sgn[\delta_i(0)]\{i_0 - 2i_1 + \ldots + 2(-1)^l i_l + (-1)^{l+1} i_{l+1}\} \tag{4.5}$$

2. *for $n - m + r$: odd*

$$n - m + r + 2z_r = sgn[\delta_i(0)]\{i_0 - 2i_1 + 2i_2 + \ldots + 2(-1)^l i_l\} \tag{4.6}$$

Proof.

We first note that the degree of the polynomial $\Delta(s)N_p(-s)$ is $n+r+m$. Hence, the parity of the degree of the polynomial $\Delta(s)N_p(-s)$ is the same as that of $n - m + r$.

Let the sign of $\frac{d(w\delta_i(w))}{dw}$ at $w = w_l$ be I_l. The change in the phase of $\delta(jw)$ from w_l to w_{1+1} is given by: $I_l(i_l - i_{l+1})\frac{\pi}{2}$. Let $w_0 = 0$ and $w_{l+1} = \infty$. Since $I_i = -I_{i-1}$ for $i = 1, 2, \ldots, l$, the phase change in $\delta(jw)$ from $w = w_0$ to $w = w_l$ can be expressed as:

$$\angle\delta(jw)|_{w=0}^{w=w_l} = I_0\left\{(i_0 - i_1) - (i_1 - i_2) + (i_2 - i_3) + \ldots + (-1)^{l-1}(i_{l-1} - i_l)\right\}\frac{\pi}{2}$$

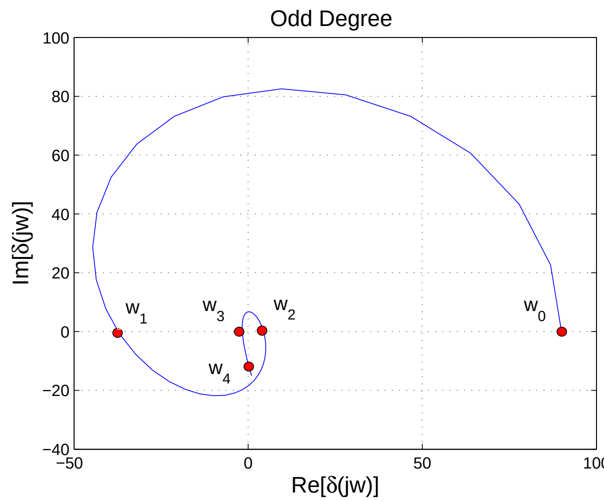

Fig. 15.: *Phase Change Property for an Odd Degree Polynomial*

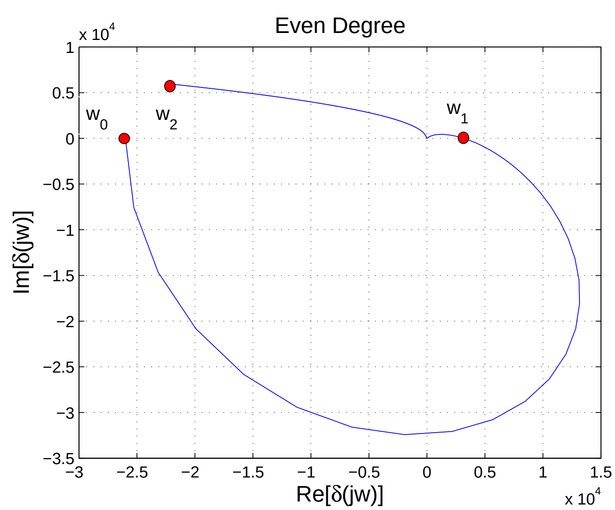

Fig. 16.: *Phase Change Property for an Even Degree Polynomial*

The phase change in $\delta(jw)$ from $w = w_l$ to ∞ will depend on the degree of the polynomial $\Delta(s)N_p(-s)$; if the degree is odd, it will be $I_l \frac{\pi}{2} i_l$ as shown in Figure 15, and if the degree is even , it will be $I_l(i_l - i_{l+1})\frac{\pi}{2}$ as also shown in Figure 16. Since $I_0 = sign(\delta_i(0))$ and $I_l = (-1)^l I_0$, we have the change in the phase of $\delta(jw)$ as w changes from 0 to ∞ is:

1. *for $n - m + r$: even*

$$\angle\delta(jw)|_{w=0}^{w=\infty} = sgn[\delta_i(0)]\{i_0 - 2i_1 + \ldots + (-1)^l 2i_l + (-1)^{l+1} i_{l+1}\}\frac{\pi}{2} \tag{4.7}$$

2. *for $n - m + r$: odd*

$$\angle\delta(jw)|_{w=0}^{w=\infty} = sgn[\delta_i(0)]\{i_0 - 2i_1 + 2i_2 + \ldots + (-1)^l 2i_l\}\frac{\pi}{2} \tag{4.8}$$

Since $D_p(s)$ does not have any zeros on the imaginary axis, the phase change in $\delta(jw)$ as w changes from 0 to ∞ is the same as that of $\Delta(jw)N_p(-jw)$ as w changes from 0 to ∞. The accumulation or change of phase of $\Delta(jw)N_p(-jw)$ is $(n-m+r+2z_r)\frac{\pi}{2}$ if and only if $\Delta(s)$ is Hurwitz. With this observation $(n-m+r+2z_r)$ equals the quantity expressed in equations (4.5) or (4.6). $\nabla\nabla\nabla$

The following theorem uses Lemma IV.2 to characterize a stabilizing controller of a fixed order in terms of frequency response of the plant.

Theorem IV.1.

A controller $C(s)$ stabilizes the plant if and only if

1. *There exists a sequence $i_0, i_1, \ldots, i_l$ satisfying equation (4.5) or (4.6), and*
2. *For the sequence of integers $i_1, \ldots, i_l$, there exists correspondingly l disjoint frequency bands or intervals, $[w_{p,1}, w_{p,2}], p = 1, \ldots, l$ such that*

(a) there exists exactly one root of $\delta_i(w)$ *in* $(w_{p,1}, w_{p,2})$*,*

(b) the sign of $\delta_r(w)$ *in* $[w_{p,1}, w_{p,2}]$ *is the same as that of* i_p*,* $p = 1, l$ *and* $i_0\delta_r(0) > 0$*, and*

(c) there is no sign change of $\delta_i(w)$ *in the disjoint intervals* $[0, w_{1,1}]$*,* $[w_{l,2}, \infty]$ *and* $[w_{p,2}, w_{p+1,1}],\ p = 1, \ldots, l-1.$

Proof.

Let the root of $\delta_i(w)$ *in* $(w_{p,1}, w_{p,2})$ *be* w_p*. Since the sign of* $\delta_r(w)$ *at* w_p *is* i_p*, the change in phase of* $\delta(jw)$ *as* w *varies from* 0 *to* ∞ *is* $(n-m+r+2z_r)\frac{\pi}{2}$*, indicating that* $\Delta(s)N_p(-s)$ *has* $m-z_r$ *roots with positive real part. However, this is the case if and only if* $\Delta(s)$ *is Hurwitz.* $\nabla\nabla\nabla$

Remark IV.1.

1. *We first observe that* $\delta(s)$ *may be expressed as* $\delta_0(s) + \sum_{p=1}^{2r+1} \delta_p(s)k_p$*, where* k_p *is the* p^{th} *component of the controller vector,* K*, and* $\delta_0, \delta_1, \ldots, \delta_{2r+1}$ *are rational functions, which can be determined once the expression for* $G(s)$ *is known. Similarly,* δ_r *and* δ_i *are affinely dependent on the controller parameter vector,* K*. To emphasize the dependence on* K*, we will use the notation* $\delta_r(w, K)$ *and* $\delta_i(w, K)$ *as appropriate. One may express the affine dependence of* $\delta_r(w, K)$ *and* $\delta_i(w, K)$ *as:*

$$\delta_r(w, K) = \Delta_r(w, |G(jw)|^2, G_r(w), G_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix} \tag{4.9}$$

$$\delta_i(w, K) = \Delta_i(w, |G(jw)|^2, G_r(w), G_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix} \tag{4.10}$$

for some vectors Δ_r and Δ_i that depend affinely on $|G(jw)|^2$, $G_r(w)$ and $G_i(w)$.

2. *The conditions in Theorem IV.1 may be replaced as follows:*

   (a) *The condition (a) of 2 may be replaced by: $\delta_i(w_{p_1}, K)\delta_i(w_{p,2}, K) < 0$ and $\frac{d\delta_i(w,K)}{dw}$ has the sign $I_0(-1)^p$ in $[w_{p,1}, w_{p,2}]$. This ensures that $\delta_i(w, K)$ has exactly one root in the interval of interest. If the frequency response at frequencies, $w_{p,1}$ and $w_{p,2}$ are known, we note that the first condition $\delta_i(w_{p_1}, K)\delta_i(w_{p_2}, K) < 0$ can be written as two sets of linear inequalities.*

   (b) *The conditions (b) and (c) of 2 may similarly be replaced as:*

$$i_0\delta_r(0) > 0,\ w = 0 \tag{4.11}$$

$$i_p\delta_r(w) > 0,\ \forall w \in [w_{p,1}, w_{p,2}] \tag{4.12}$$

$$I_0\delta_i(w) > 0,\ \forall w \in [0, w_{1,1}] \tag{4.13}$$

$$(-1)^l I_0\delta_i(w) > 0,\ \forall w \in [w_{l,2}, \infty) \tag{4.14}$$

$$(-1)^q I_0\delta_i(w) > 0,\ \forall w \in [w_{q,2}, w_{q+1,1}], \tag{4.15}$$

   where $p = 1, 2, \ldots, l$, $q = 1, \ldots, l-1$ and dependence on K is suppressed.

If $G(jw)$ is exactly known, the condition of $(-1)^p I_0 \frac{d\delta_i(w,K)}{dw}$ being nonnegative in $[w_{p,1}, w_{p,2}]$ can be posed as a SDP using Markov-Lucaks theorem.

The nonnegativity of $f(x)$ for $x \in [x_1, x_2]$ becomes a feasibility problem as shown in chapter III and in [20, 40]. We also have to consider an additional condition to satisfy the nonnegativity of the $f(x)$ at specific value of frequency. Finally, This leads to a new feasibility problem combined with the nonnegativity of the derivatives of $f(x)$ in an interval of frequency.

Remark IV.2.

The above Linear Matrix Inequality (LMI) conditions for nonnegativity of polynomials can be used to synthesize the controller $C(s)$ in Theorem IV.1.

1. *The constraint $i_0\delta_r(0) > 0$ is a linear inequality constraint on K.*
2. *All other constraints can be posed as nonnegativity of a polynomial on an interval. As seen in the previous chapter, constraints (4.12 - 4.15) lead to LMIs; the feasibility problem of a controller satisfying LMIs.*

This problem can be solved by applying interior point methods (Feasibility and Phase 1 method) with SeDuMi [21].

1. Examples

Let us suppose we have the frequency response data for a plant $G(s)$ by experiments as shown in Figure 17 and the actual plant transfer function is

$$\begin{aligned} G(s) &= \frac{N_p(s)}{D_p(s)} \\ &= \frac{s^4 + 4s^3 + 23s^2 + 46s - 12}{s^5 + s^4 + 20s^3 + 36s^2 + 99s + 100} \end{aligned} \tag{4.16}$$

For our simulations we assume that the plant structure is not known. We collect frequency response measurements from this ‘unknown’ plant.

We can know that $n - m = 1$ from the magnitude rate with respect to frequency at high frequency and assume that $z_r = 1$ is known. If we know that $p_r = 2$ instead of z_r then $z_r = 1$ can be determined from the equation (4.27) in section E. and the net accumulated phase change as $w = 0 \to \infty$.

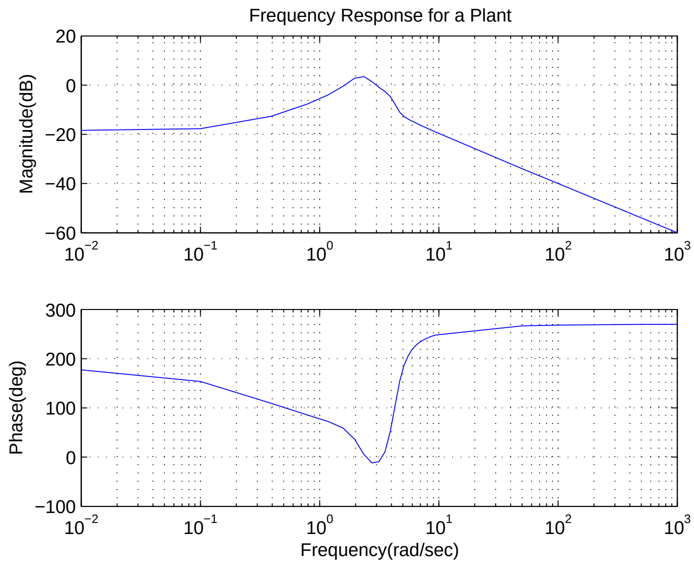

Fig. 17.: *Frequency Response of a Plant*

$$\begin{aligned} \angle G(jw)|_{w=0}^{w=\infty} &= -\frac{\pi}{2}\left[(n-m)+2\{z_r-p_r\}\right] \\ \frac{\pi}{2} &= -\frac{\pi}{2}\left[1+2\{z_r-2\}\right] \end{aligned} \tag{4.17}$$

The frequency data is upper bounded by $w_b = 10$.

We aim to find a first order controller which stabilizes the closed loop system. The controller is given as:

$$C(s) = \frac{k_1 s + k_2}{s + k_3}$$

Currently, we do not consider any measurement errors.

Let us multiply $N_p(-s)$ on characteristic equation, $\Delta(s) = D_p(s)D_c(s)+N_p(s)N_c(s)$.

$$\begin{aligned} \delta(s) &= \Delta(s)N_p(-s) \\ &= D_p(s)N_p(-s)D_c(s) + N_p(s)N_p(-s)N_c(s) \end{aligned}$$

For stable $\Delta(s)$, $\delta(jw)$ have to satisfy the following condition for the net phase change since all the roots of $\Delta(s)$ should be in the left half plane.

$$\begin{aligned}\angle\delta(jw)|_{w=0}^{w=\infty} &= \frac{\pi}{2}\left[l(\delta)-r(\delta)\right]\\ &= \frac{\pi}{2}\left[n+r-z_l+z_r\right]\\ &= \frac{\pi}{2}\left[n-m+r+2z_r\right]\\ &= \frac{\pi}{2}\left[1+1+2(1)\right]\\ &= \frac{\pi}{2}\left[4\right]\end{aligned}$$

We do not know the maximum number of the real, nonnegative, distinct finite roots of $\delta_i(w)$ since we have no information of the degree of the plant.

Let us start with in case of $l=1$. The signature for even $n-m+r=2$ will be as follows.

$$\begin{aligned}\sigma(\delta(s,K)) &= sgn[\delta_i(0)]\{i_0-2i_1+2i_2+\cdots+(-1)^{l-1}2i_{l-1}+(-1)^l i_l\}\cdot(-1)^{l-1}\\ &= -sgn[\delta_i(0)]\{i_0-2i_1+i_2\}\cdot\\ &= 4\end{aligned}$$

The set of feasible string A_f becomes.

$$A_f=\left\{\begin{array}{c}\{-1\quad+1\quad-1\}\\ \{+1\quad-1\quad+1\}\end{array}\right\}$$

For this example, Theorem IV.1 can be interpreted as follows.

There exists a real control parameter vector K that render $\Delta(s,K)$ Hurwitz if and only if there exists : (1) a sequence of i_0, i_1, i_2 such that the equation (4.5) in Lemma IV.2 holds, and (2) there exists a set of frequencies $0=w_{0,1}<w_{0,2}<w_{1,1}<w_{1,2}<w_{2,1}=w_b$ such that the following inequalities hold:

1. $\delta_i(w_{1,1},K)\cdot\delta_i(w_{1,2},K)<0$
   $-I_0\delta_i(w_{1,2},K)\cdot\frac{d\delta_i(w,K)}{dw}\geq 0$ for all $w\in[w_{1,1},w_{1,2}]$

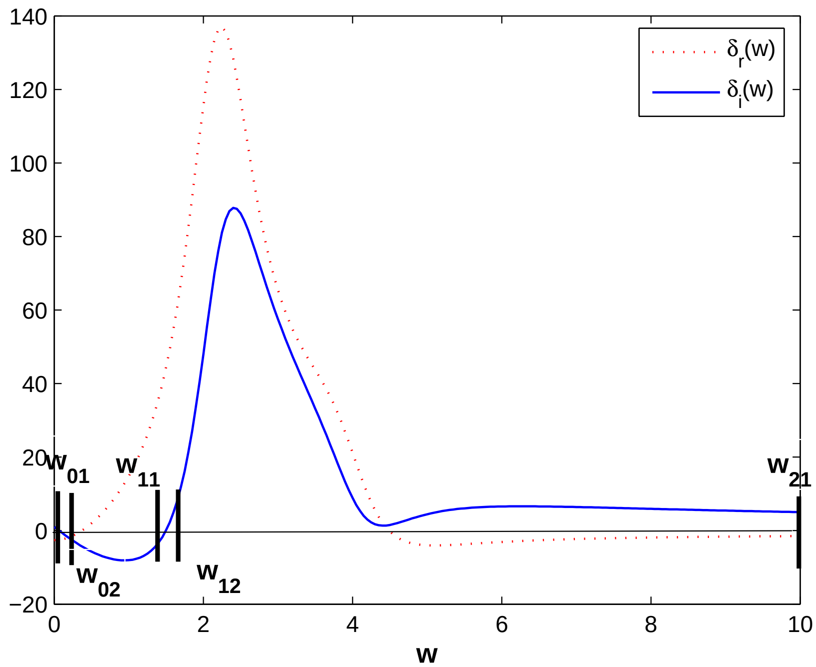

Fig. 18.: *Signs of Real and Imaginary Part*

2. $i_0 \cdot \delta_r(w, K) > 0$ for $w = w_{0,1}$

   $i_1 \cdot \delta_r(w, K) > 0$ for all $w \in [w_{1,1}, w_{1,2}]$

   $i_2 \cdot \delta_r(w, K) > 0$ for $w = w_{2,1}$

3. $\delta_i(w_{0,2}, K) \cdot \delta_i(w, K) > 0$ for all $w \in (w_{0,2}, w_{1,1})$

   $\delta_i(w_{1,2}, K) \cdot \delta_i(w, K) > 0$ for all $w \in (w_{1,2}, w_{2,1})$

We can illustrate the above conditions graphically using Figure 18.

We consider the frequency information at 31 discrete points corresponding to the Chebyshev's nodes. The procedure introduced in the previous section was used to solve the SDP [20]. The computer packages SeDuMi [21] and YALMIP [60] are used to obtain a solution.

The following stabilizing controller was obtained in 13 iterations:

$$C(s) = \frac{16.4329s + 41.4416}{s + 26.6348}.$$

For this controller, the roots of closed loop are at $(-40.9354,\ -1.1790\pm1.4901i,\ -0.0228\pm 4.4853i,\ -0.7285)$.

A projection algorithm was used to obtain an idea about the feasible set of the SDP and hence find a set of stabilizing controllers. This set is shown in Figure 19. In the set shown, controllers on or near the surface boundary may not be stabilizing and might have unstable poles very close to the imaginary axis. These are numerical issues which needs to be overcome.

D. Robustness

If the frequency response data, $G(jw)$ is approximately known as is typically the case when fitting a rational function approximation to the given data contaminated with noise, the nonnegativity condition can be posed as a robust SDP.

In the pursuit of posing nonnegativity conditions of the polynomial approximations of rational functions, we will require Lemma IV.3. To prepare for Lemma IV.3, let $\tilde{P}_0 := |G(jw)|^2 - P_0(w),\ \tilde{P}_1 := G_r(w) - P_1(w),\ \tilde{P}_2 := G_i(w) - P_2(w)$, where $P_1(w), P_2(w), P_3(w)$ are approximate polynomials stated in Assumption IV.1 and let $\tilde{Q}_i := \frac{d\tilde{P}_i}{dw},\ i = 0, 1, 2$. Let B_μ be the box, $|\tilde{P}_i| \leq \mu_i,\ i = 0, 1, 2$ and B_η be the box, $|\tilde{Q}_i| \leq \eta_i,\ i = 0, 1, 2$. We will define $w_{0,1} = w_{0,2} = 0$ and $w_{l+1,1} = \infty$.

The following lemma provides a sufficient condition for checking the nonnegativity of a rational function through its polynomial approximation and the approximation error

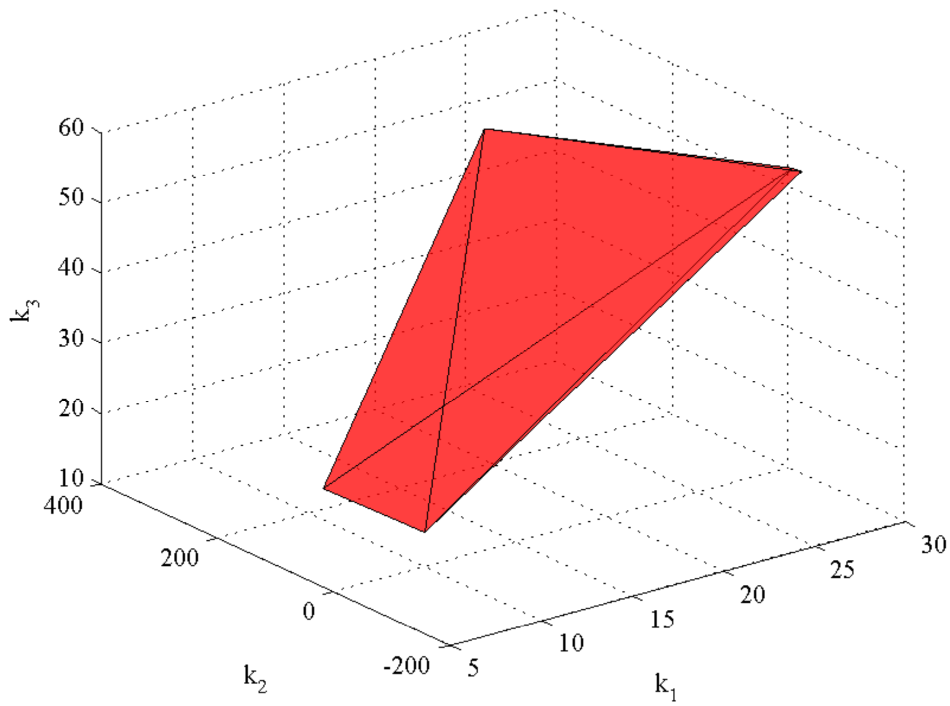

Fig. 19.: *Set of Stabilizing First Order Controllers.*

bounds. Let $\Delta_r^*(w, \mu_{0,e}, \mu_{1,e}, \mu_{2,e})$ represent $\Delta_r(w, P_0(w) + \mu_{0,e}, P_1(w) + \mu_{1,e}, P_2(w) + \mu_{2,e})$, where $\mu_{i,e},\ i = 0, 1, 2$ are the vertices of the box B_μ.

Lemma IV.3.

Let $[w_{low}, w_{high}] \subset [0, w_b]$. Let K be such that for all vertices of $(\mu_{0,e}, \mu_{1,e}, \mu_{2,e})$ the box B_μ,

$$\Delta_r^*(w, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \quad \forall w \in [w_{low}, w_{high}] \tag{4.18}$$

Then, K satisfies.

$$\delta_r(w, K) = \Delta_r(w, |G(jw)|^2, G_r(w), G_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0 \ \ \forall w \in [w_{low}, w_{high}] \tag{4.19}$$

Proof.

The proof is by contraposition. Suppose $\delta_r(\bar{w}, K) < 0$ for some $\bar{w} \in [w_{low}, w_{high}]$.
Set $\tilde{\mu}_i = \tilde{P}_i(\bar{w}),\ i = 0, 1, 2$. Therefore, we have

$$\Delta_r^*(\bar{w}, \tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2) \begin{bmatrix} K \\ 1 \end{bmatrix} < 0.$$

Since $|\tilde{\mu}_i| \leq \mu_i$, and since Δ_r depends affinely on $\tilde{\mu}_i,\ i = 0, 1, 2$, it must be that at some vertex $(\mu_{0,e}, \mu_{1,e}, \mu_{2,e})$ of the box $|\tilde{P}_i| \leq \mu_i$,

$$\Delta_r^*(\bar{w}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} < 0.$$

$\nabla\nabla\nabla$

Remark IV.3.

The polynomial condition given by equation (4.18) is a sufficient condition for the the *rational function* $\delta_r(w, K)$ *to be nonnegative on the interval* $[w_{low}, w_{high}]$ *for the given value of* K*. In particular, the set of* K*'s that satisfy the polynomial condition at every vertex of the box also render the rational function* $\delta_r(w, K)$ *to be nonnegative on* $[w_{low}, w_{high}]$*. The set of* K*'s satisfying the polynomial condition at a vertex of the box can be written as a SDP; for example, one may use the recent formulation of [20] or that of [40]. Since there are only eight vertices for the box* $|\tilde{P}_i| \leq \mu_0$*, this means that the set of* K*'s that simultaneously satisfy eight SDP's (which can be cast as a bigger SDP) also render the rational function* $\delta_r(w, K)$ *to be nonnegative on* $[w_{low}, w_{high}]$*.*

Similar conditions can be derived for the nonnegativity of rational functions $\delta_i(w, K)$ *and* $\frac{d\delta_i(w,K)}{dw}$*.*

The following lemma deals with the non negativity of $\delta_i(w, K)$ on $[w_b, \infty)$, where the polynomial approximation does not hold.

Lemma IV.4.

Let $B_\epsilon := \{(\epsilon_0, \epsilon_1, \epsilon_2) : 0 \le \epsilon_0 < \epsilon^2,\ \max\{|\epsilon_1|, |\epsilon_2|\} < \epsilon\}$*. Let* $(\epsilon_{0,e}, \epsilon_{1,e}, \epsilon_{2,e}),\ e = 1, \ldots, 8$ *be the vertices of the box* B_ϵ*. If, for some* K *and* l *and for* $e = 1, \ldots, 8$*, we have*

$$(-1)^l \Delta_i(w, \epsilon_{0,e}, \epsilon_{1,e}, \epsilon_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \ \ \forall w \in [w_b, \infty)$$

then

$$(-1)^l \Delta_i(w, |G(jw)|^2, G_r(w), G_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \ \ \forall w \in [w_b, \infty)$$

The proof for this lemma is similar to that of Lemma IV.3.

We will require the last lemma before stating our main result. Let $\Delta_i^*(w_{p,1}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) = \Delta_i(w_{p,1}, P_0(w_{p,1}) + \mu_{0,e}, P_1(w_{p,1}) + \mu_{1,e}, P_2(w_{p,1}) + \mu_{2,e}))$.

Lemma IV.5.

Let $[w_{p,1}, w_{p,2}] \subset [0, w_b]$ *and* $I_0 \in \{-1, +1\}$*. If, for some* K*, and for all* $e = 1, \ldots, 8$*, we have*

$$\begin{aligned} I_0(-1)^{p-1}\Delta_i^*(w_{p,1}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} &> 0 \\ I_0(-1)^{p}\Delta_{i,e}(w_{p,2}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} &> 0 \end{aligned}$$

then

$$I_0(-1)^{p-1}\delta_i(w_{p,1}) > 0, \ \ I_0(-1)^{p}\delta_i(w_{p,2}) > 0.$$

Proof.

The proof is by contraposition. Suppose $I_0(-1)^{p-1}\delta_i(w_{p,1}, K) < 0$*. Let* $\tilde{\mu}_i := \tilde{P}_i(w_{p,1})$*.*

Then,

$$I_0(-1)^{p-1}\Delta_i(w_{p,1},\tilde{\mu}_0,\tilde{\mu}_1,\tilde{\mu}_2) > 0.$$

However, this cannot happen unless at some vertex e*, we have*

$$I_0(-1)^{p-1}\Delta_i(w_{p,1},\mu_{0,e},\mu_{1,e},\mu_{2,e}) > 0.$$

A similar reasoning can be applied to the second condition in the Lemma IV.5 to complete the proof. $\nabla\nabla\nabla$

Remark IV.4.

If $I_0, p, w_{p,1}, w_{p,2}$ *are known in the above lemma, the sufficient conditions are linear inequalities in* K*. In particular, every* K *that satisfies the system of linear inequalities at the vertices of the box* B_μ*, also satisfies the linear inequalities* $(-1)^{p-1}I_0\delta_i(w_{p,1},K) > 0$ *and* $(-1)^p I_0\delta_i(w_{p,2},K) > 0$*. We emphasize that* $\delta_i(w,K)$ *is not required to be known exactly, but only polynomial approximations of* $|G(jw)|^2,\ G_r(w),\ G_i(w)$ *are available.*

Since the second condition in Theorem IV.1 also requires the nonnegativity of $\frac{d\delta_i}{dw}$, we will first express it as:

$$\Delta_{d,i}(w,|G(jw)|^2,G_r(w),G_i(w),\frac{d|G(jw)|^2}{dw},\frac{dG_r}{dw},\frac{dG_i}{dw})\begin{bmatrix} K \\ 1 \end{bmatrix}$$

for some array $\Delta_{d,i}$ that is polynomial in w and is dependent affinely on $|G(jw)|^2, G_r(w), G_i(w)$ and its derivatives.

The following is the main result and provides a sufficient condition for the direct synthesis of sets of stabilizing controllers from the frequency response data:

Theorem IV.2.

Let $i_0, i_1, \ldots, i_l$ *be a sequence of integers from the set* $\{-1,1\}$ *satisfying equations (4.5) or*

(4.6). Let $(\mu_{0,e}, \mu_{1,e}, \mu_{2,e}),\ e = 1, \ldots, 8$ *be the vertices of box* B_μ *and* $(\eta_{0,f}, \eta_{1,f}, \eta_{2,f}),\ f = 1, \ldots, 8$ *be the vertices of the box* B_η*. Let* K *satisfy the every constraint in the following set of constraints for* $I_0 = -1$ *or for* $I_0 = +1$ *and for every* $e = 1, \ldots, 8$ *and* $f = 1, \ldots, 8$*:*

$$I_0(-1)^{p-1}\Delta_i^*(w_{p,1}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \quad p = 1, \ldots, l \tag{4.20}$$

$$I_0(-1)^{p}\Delta_i^*(w_{p,2}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \quad p = 1, \ldots, l \tag{4.21}$$

$$I_0(-1)^{p}\Delta_{d,i}^*(w, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}, \eta_{0,f}, \eta_{1,f}, \eta_{2,f}) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \tag{4.22}$$
$$\forall w \in [w_{p,1}, w_{p,2}],\ p = 1, \ldots, l$$

$$I_0(-1)^{p}\Delta_i^*(w, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0,\ \forall w \in [w_{p,2}, w_{p+1,1}],\ p = 0, 1, \ldots, l \tag{4.23}$$

$$i_p\Delta_r^*(w, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0,\ \forall w \in [w_{p,1}, w_{p,2}],\ p = 1, \ldots, l \tag{4.24}$$

$$(-1)^{l}\Delta_i(w, \epsilon_{0,e}, \epsilon_{1,e}, \epsilon_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0,\ \forall w \in [w_b, \infty) \tag{4.25}$$

Then, K *is a stabilizing controller for the plant.*

This theorem covers all the cases discussed in this section and provides a sufficient condition for the synthesis of sets of stabilizing controllers.

1. Equations (4.20) and (4.21) together ensures that $\delta_i(w_{p,1}, K)\delta_i(w_{p,2}, K) < 0$. This follows from Lemma IV.5.

2. Equation (4.22) guarantees that $\frac{d\delta_i(w,K)}{dw}$ has the sign $I_0(-1)^p$ in $[w_{p,1}, w_{p,2}]$. This is

an application of Lemma IV.3 to $\frac{d\delta_i(w,K)}{dw}$.

3. Equations (4.20), (4.21) and (4.22) provide the condition for $\delta_i(w, K)$ to have only one real root in the interval $[w_{p,1}, w_{p,2}]$.

4. Equations (4.23) and (4.25) provide the condition for the real roots of $\delta_i(w, K)$ to not lie outside the intervals $[w_{p,1}, w_{p,2}]$. This is necessary for the correct application of Lemma IV.2. This condition is satisfied by ensuring that the polynomial is either positive or negative in the complete range of $[w_{p,2}, w_{p+1,1}]$.

5. Equation (4.24) ensures that at the real roots of $\delta_i(w, K)$, the sign of $\delta_r(w, K)$ is correct and is given by the sequence of integers satisfying equations (4.5) or (4.6).

We consider the same plant as 4.16. The frequency response data, $G(jw)$ is approximately known. The controller is found using the robust SDPs procedure outlined in Theoem IV.2.

The following robust stabilizing controller was obtained in 15 iterations:

$$C(s) = \frac{70.4268s + 8.2073}{s + 114.4617}.$$

For this controller, the roots of closed loop are at $(-183.69,\ -0.05 \pm 4.23i,\ -0.32 \pm 1.5i,\ -1.46)$.

E. Special Case

1. Special Case ($z_l = 0, 1$)

The nonnegativity conditions which have to be satisfied in some intervals in Theorem IV.1 or Theorem IV.2 can be replaced by linear inequalities if additional assumptions are made.

Assumption IV.2.

1. *The degree of a plant,* n *is known.*
2. *The number,* z_l*, of non-minimum phase zeroes of the plant is at-most one.*

Since we assumed the degree of a plant be n, the degree of the polynomial of the numerator of $\delta(s)$, i.e. $\Delta(s)N_p(-s)$ is:

$$d = n + m + r = 2n - (n - m) + r \tag{4.26}$$

$n - m$ is the relative degree of a plant $G(s)$ which can be determined from the frequency response data of the plant. r is the degree of controller. Now, let l be the number of nonnegative distinct real roots of the $\delta_i(w)$.

We explain why the interval conditions can be relaxed to linear inequalities in detail when $z_l < 2$. We observe the following:

Let us suppose d is even. Then the maximum number of nonnegative distinct real roots of the $\delta_r(w)$ and $\delta_i(w)$ are $d/2$ and $l = d/2 - 1$ respectively.

1. For the equation (4.5) to hold when $z_l = 1$, l nonnegative distinct real roots of the $\delta_r(w)$ are required.

$$\begin{aligned} n - m + r + 2z_r &= n + m + r - 2z_l \\ &= n + m + r - 2 \\ &= sgn[\delta_i(0)]\{i_0 - 2i_1 + \ldots + 2(-1)^l i_l + (-1)^{l+1} i_{l+1}\} \end{aligned}$$

   - When $d = 6$, l becomes 2. Then

$$n + m + r - 2 = 4 = sgn[\delta_i(0)]\{i_0 - 2i_1 + 2i_2 - i_3\}$$

For positive $\delta_i(0)$, the set of feasible string A_f becomes.

$$A_f = \left\{ \begin{array}{c} A_1(i_0 \;\; i_1 \;\; i_2 \;\; i_3) \\ A_2(i_0 \;\; i_1 \;\; i_2 \;\; i_3) \end{array} \right\} = \left\{ \begin{array}{c} \{+1 \;\; -1 \;\; +1 \;\; +1\} \\ \{-1 \;\; -1 \;\; +1 \;\; -1\} \end{array} \right\}$$

Since there are 2 sign changes in $i_p, \;\; (p = 0, 3)$, 2 nonnegative distinct real roots of the $\delta_r(w)$ are required. It can be easily verified that the same number of nonnegative distinct real roots of the $\delta_r(w)$ are required for negative $\delta_i(0)$.

- When $d = 8$, l becomes 3. Then

$$n + m + r - 2 = 6 = sgn[\delta_i(0)]\{i_0 - 2i_1 + 2i_2 - 2i_3 + i_4\}$$

For positive $\delta_i(0)$, the set of feasible string A_f becomes.

$$A_f = \left\{ \begin{array}{c} A_1(i_0 \;\; i_1 \;\; i_2 \;\; i_3 \;\; i_4) \\ A_2(i_0 \;\; i_1 \;\; i_2 \;\; i_3 \;\; i_4) \end{array} \right\} = \left\{ \begin{array}{c} \{+1 \;\; -1 \;\; +1 \;\; -1 \;\; -1\} \\ \{-1 \;\; -1 \;\; +1 \;\; -1 \;\; +1\} \end{array} \right\}$$

Since there are 3 sign changes in $i_p, \;\; (p = 0, 4)$, 3 nonnegative distinct real roots of the $\delta_r(w)$ are required. It can be easily verified that the same number of nonnegative distinct real roots of the $\delta_r(w)$ are required for negative $\delta_i(0)$.

For a real polynomial $f(w)$, if $f(w_1)$ and $f(w_2)$ are of same sign then in the interval $w \in [w_1, w_2]$ there exist either no roots or an even number of roots. Since we have at most $l + 1$ nonnegative distinct real roots of $\delta_r(w)$, if $\delta_r(w_1)$ and $\delta_r(w_2)$ are of same sign then there exist no roots in the interval $w \in [w_1, w_2]$.

2. When $z_l = 0$

   There are no roots of $\delta_r(w)$ between roots of $\delta_i(w)$ since $\delta(s)$ becomes Hurwitz polynomial for $z_l = 0$.

For odd d, this can be explained on similiar lines.

Now, we can simplify the theorem IV.1 under Assumption IV.2.

Theorem IV.3.

There exists a real control parameter vector $K = (k_1,\ k_2,\ \cdots,\ k_q)$ *such that the real closed-loop characteristic polynomial* $\Delta(s,K)$ *is Hurwitz if (1) There exists a sequence* $i_0, i_1, \ldots, i_l$ *satisfying equations (4.5) or (4.6) and for the sequence of integers* $i_0, i_1, \ldots, i_l$*, there exists there exists a set of frequencies,* $0 = w_{0,1} < w_{0,2} < w_{1,1} < w_{1,2} < \cdots < w_{l,1} < w_{l,2} < w_{l+1,1} = \infty$ *for* $\delta(s,K) = \delta_d(K)s^d + \delta_{d-1}(K)s^{d-1} + \cdots + \delta_0(K)$*, the number of nonnegative, distinct, real roots of* $\delta_i(w)$*,* l *is the smallest integer greater than or equal to* $d/2-1$*, so that the following sets of linear inequality conditions hold :*

1. $\delta_i(w_{p,1},K) \cdot \delta_i(w_{p,2},K) < 0, \ \ p = 1,\ldots,l$
2. $i_p \cdot \delta_r(w_{p,1},K) > 0$ *and* $i_p \cdot \delta_r(w_{p,2},K) > 0, \ \ p = 1,\ldots,l$
3. $i_0 \cdot \delta_r(w_{0,2},K) > 0$ *and* $i_{l+1,1} \cdot \delta_r(w_{l+1,1},K) > 0$

We can rewrite the LPs in terms of frequency response data, by dividing them with $|D_p(jw)|^2$. Let us suppose we are considering a first order controller as an example of fixed order controller and separate the controller real and imaginary parts.

$$\begin{aligned} C(jw) = \frac{N_c(jw,K)}{D_c(jw,K)} &= \frac{N_{c,r}(w,K) + jN_{c,i}(w,K)}{D_{c,r}(w,K) + jD_{c,i}(w,K)} \\ &= \frac{jk_1w + k_2}{jw + k_3} \end{aligned}$$

The frequency response of a plant can be expressed as $G(jw) = G_r(w) + jG_i(w)$. We can represent the polynomials $\frac{\delta_r}{|D_p(jw)|^2}$ and $\frac{\delta_i}{|D_p(jw)|^2}$ compactly in the following form, owing to the affine dependence of their coefficients on the controller parameter vector K.

$$\begin{aligned}
\frac{\delta_r}{|D_p(jw)|^2} &= |G(jw)|^2 N_{c,r}(w,K) + G_r(w)D_{c,r}(w,K) - G_i(w)D_{c,i}(w,K) \\
&= |G(jw)|^2 k_2 + G_r(w)k_3 - wG_i(w) \\
&= \left[\begin{array}{cccc} wG_i(w) & 0 & |G(jw)|^2 & G_r(w) \end{array}\right] \left[\begin{array}{c} 1 \\ K \end{array}\right] \\
&= \left[\begin{array}{c} \Delta_r(w, |G(jw)|^2, G_r(w), G_i(w)) \end{array}\right] \left[\begin{array}{c} K' \end{array}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\delta_i}{|D_p(jw)|^2} &= |G(jw)|^2 N_{c,i}(w,K) + G_r(w)D_{c,i}(w,K) - G_i(w)D_{c,r}(w,K) \\
&= |G(jw)|^2 k_1 w + wG_r(w) - G_i(w)k_3 \\
&= \left[\begin{array}{cccc} wG_r(w) & w|G(jw)|^2 & 0 & -G_i(w) \end{array}\right] \left[\begin{array}{c} 1 \\ K \end{array}\right] \\
&= \left[\begin{array}{c} \Delta_i(w, |G(jw)|^2, G_r(w), G_i(w)) \end{array}\right] \left[\begin{array}{c} K' \end{array}\right],
\end{aligned}$$

where

1. $\Delta_r(w, |G(jw)|^2, G_r(w), G_i(w)) = \left[\begin{array}{cccc} wG_i(w) & 0 & |G(jw)|^2 & G_r(w) \end{array}\right]$
2. $\Delta_i(w, |G(jw)|^2, G_r(w), G_i(w)) = \left[\begin{array}{cccc} wG_r(w) & w|G(jw)|^2 & 0 & -G_i(w) \end{array}\right]$
3. $K' = [1 \;\; k_1 \;\; k_2 \;\; k_3]^T$.

Then, the LP conditions can be put in a nice compact matrix form, which only involves the set of frequencies chosen and the frequency response data at those points.

Lemma IV.6.

In case of $z_l < 2$, there exists a real control parameter vector $K = (k_1, \; k_2, \; \cdots, \; k_q)$ so that the real closed-loop characteristic polynomial $\Delta(s, K)$ is Hurwitz if there exists *a set of frequencies,* $0 = w_{0,2} < w_{1,1} < w_{1,2} < \cdots < w_{l,1} < w_{l,2} < w_{l+1,1} = \infty$ *for* $\delta(s, K) = \delta_d(K)s^d + \delta_{d-1}(K)s^{d-1} + \cdots + \delta_0(K)$*, where* l *is the smallest integer greater than or equal to* $d/2-1$*, so that one of the following two Linear Programs(LPs) is feasible:*

$$
\begin{aligned}
\textit{LP 1 :}\quad & \textit{(a)} \begin{bmatrix} I(w_{p,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \quad \textit{and} \quad \begin{bmatrix} I(w_{p,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} < 0, \;\; p = 1, \ldots, l \\
& \textit{(b)}\; i_0 \cdot \begin{bmatrix} R(w_{0,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \;\; \textit{and}\; i_{l+1} \cdot \begin{bmatrix} R(w_{l+1,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \\
& \textit{(c)}\; i_p \cdot \begin{bmatrix} R(w_{p,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \;\; \textit{and}\; i_p \cdot \begin{bmatrix} R(w_{p,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0\,, \;\; p = 1, \ldots, l \\
\textit{LP 2 :}\quad & \textit{(a)} \begin{bmatrix} I(w_{p,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} < 0 \quad \textit{and} \quad \begin{bmatrix} I(w_{p,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0, \;\; p = 1, \ldots, l \\
& \textit{(b)}\; i_0 \cdot \begin{bmatrix} R(w_{0,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \;\; \textit{and}\; i_{l+1} \cdot \begin{bmatrix} R(w_{l+1,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \\
& \textit{(c)}\; i_p \cdot \begin{bmatrix} R(w_{p,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \;\; \textit{and}\; i_p \cdot \begin{bmatrix} R(w_{p,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0\,, \;\; p = 1, \ldots, l
\end{aligned}
$$

2. Examples

Let us suppose we have the frequency response data for a plant $G(s)$ by experiments as shown in Figure 20 and the actual plant transfer function is as follows

$$
\begin{aligned}
G(s) &= \frac{N_p(s)}{D_p(s)} \\
&= \frac{s^3 + 3s^2 + s + 8}{s^4 + 2s^3 + 3s^2 + 7s + 14}, \quad z_l = 1
\end{aligned}
$$

We can know that $n - m = 1$ from the magnitude rate with respect to frequency at high frequency. Suppose we know that $n = 4$ and $z_r = 2$. If we know $p_r = 2$ instead of z_r then z_r can be determined from the equation (4.27) and the net accumulated phase change as $w = 0 \to \infty$.

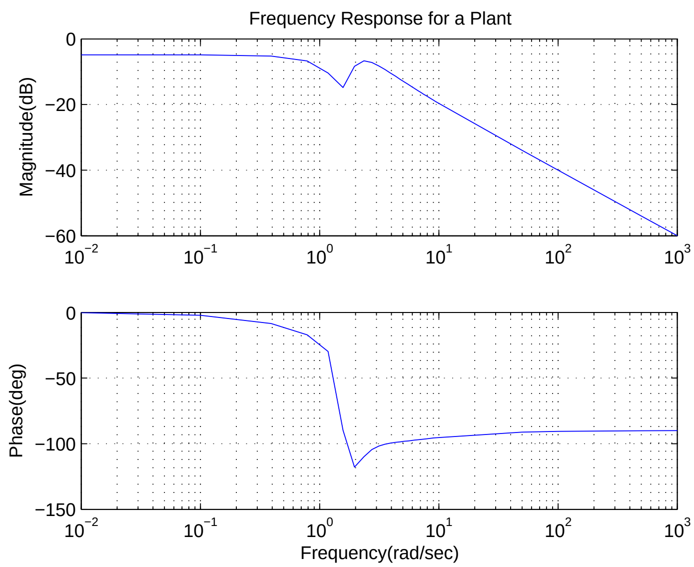

Fig. 20.: *Frequency Response of a Plant*

$$\begin{aligned} \angle G(jw)|_{w=0}^{w=\infty} &= -\frac{\pi}{2}\left[(n-m)+2(z_r-p_r)\right] \\ -\frac{\pi}{2} &= -\frac{\pi}{2}\left[1+2(z_r-2)\right] \end{aligned} \tag{4.27}$$

Now, we consider the first order controller which stabilizes the closed loop characteristic polynomial.

$$\begin{aligned} C(s) &= \frac{N_c(s)}{D_c(s)} \\ &= \frac{k_1 s + k_2}{s + k_3} \end{aligned}$$

Let us multiply $N_p(-s)$ on characteristic equation, $\Delta(s) = D_p(s)D_c(s)+N_p(s)N_c(s)$.

$$\begin{aligned} \delta(s) &= \Delta(s)N_p(-s) \\ &= D_p(s)N_p(-s)D_c(s) + N_p(s)N_p(-s)N_c(s) \end{aligned}$$

For stable $\Delta(s)$, $\delta(jw)$ have to satisfy the following condition for the net phase change since all the roots of $\Delta(s)$ should be in the left half plane.

$$\begin{aligned}
\angle\delta(jw)|_{w=0}^{w=\infty} &= \frac{\pi}{2}\left[l(\delta)-r(\delta)\right] \\
&= \frac{\pi}{2}\left[n+r-z_l+z_r\right] \\
&= \frac{\pi}{2}\left[n-m+r+2z_r\right] \\
&= \frac{\pi}{2}\left[1+1+2(2)\right] \\
&= \frac{\pi}{2}\left[6\right]
\end{aligned}$$

The maximum number of the real, positive, distinct finite roots of $\delta_i(w)$ with odd multiplicities, l becomes 3 since $\delta(s)$ is of order 8 and an even polynomial. The signature for even polynomial having 3 real, positive, distinct finite roots will be as follows.

$$\begin{aligned}
\sigma(\delta(s,K)) &\doteq sgn[\delta_i(0)]\{i_0-2i_1+2i_2+\cdots+(-1)^l 2i_l+(-1)^{l+1}i_{l+1}\} \\
&= sgn[\delta_i(0)]\{i_0-2i_1+2i_2-2i_3+i_4\} \\
&= 6
\end{aligned}$$

The set of feasible string A_f becomes.

$$\begin{aligned}
A_f &= \left\{\begin{array}{l} A_1(i_0\ \ i_1\ \ i_2\ \ i_3\ \ i_4) \\ A_2(i_0\ \ i_1\ \ i_2\ \ i_3\ \ i_4) \\ A_3(i_0\ \ i_1\ \ i_2\ \ i_3\ \ i_4) \\ A_4(i_0\ \ i_1\ \ i_2\ \ i_3\ \ i_4) \end{array}\right\} \\
&= \left\{\begin{array}{l} \{+1\ \ +1\ \ -1\ \ +1\ \ -1\} \\ \{-1\ \ +1\ \ -1\ \ +1\ \ +1\} \\ \{+1\ \ -1\ \ +1\ \ -1\ \ -1\} \\ \{-1\ \ -1\ \ +1\ \ -1\ \ +1\} \end{array}\right\}
\end{aligned}$$

Now, the problem becomes to find a real stabilizing control parameter vector $K=(k_1,\ k_2,\ k_3)$

so that the real closed-loop characteristic polynomial $\Delta(s, K)$ is Hurwitz.

If there exists a set of frequencies, $0 = w_{0,2} < w_{1,1} < w_{1,2} < \cdots < w_{4,1} = w_b$ for $\delta(s, K) = \delta_d(K)s^d + \delta_{d-1}(K)s^{d-1} + \cdots + \delta_0(K)$, where l is the smallest integer greater than(or equal to) $= d/2 - 1$, so that one of the following two *Linear Programs(LPs)* for any feasible set of strings is feasible:

$$\text{LP 1}: \quad \text{(a)} \begin{bmatrix} I(w_{p,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} I(w_{p,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} < 0, \;\; p = 1, 2, 3$$

$$\text{(b)}\; i_0 \cdot \begin{bmatrix} R(w_{0,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \;\text{ and }\; i_4 \cdot \begin{bmatrix} R(w_{4,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0$$

$$\text{(c)}\; i_p \cdot \begin{bmatrix} R(w_{p,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \;\text{ and }\; i_p \cdot \begin{bmatrix} R(w_{p,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0\,, \;\; p = 1, 2, 3$$

$$\text{LP 2}: \quad \text{(a)} \begin{bmatrix} I(w_{p,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} < 0 \quad \text{and} \quad \begin{bmatrix} I(w_{p,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0, \;\; p = 1, 2, 3$$

$$\text{(b)}\; i_0 \cdot \begin{bmatrix} R(w_{0,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \;\text{ and }\; i_4 \cdot \begin{bmatrix} R(w_{4,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0$$

$$\text{(c)}\; i_p \cdot \begin{bmatrix} R(w_{p,1}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0 \;\text{ and }\; i_p \cdot \begin{bmatrix} R(w_{p,2}) \end{bmatrix} \begin{bmatrix} K' \end{bmatrix} > 0\,, \;\; p = 1, 2, 3,$$

where

1. $\Delta_r(w, |G(jw)|^2, G_r(w), G_i(w)) = \begin{bmatrix} wG_i(w) & 0 & |G(jw)|^2 & G_r(w) \end{bmatrix}$
2. $\Delta_i(w, |G(jw)|^2, G_r(w), G_i(w)) = \begin{bmatrix} wG_r(w) & w|G(jw)|^2 & 0 & -G_i(w) \end{bmatrix}$
3. $K' = [1 \;\; k_1 \;\; k_2 \;\; k_3]^T$.

The set of stabilizing first order controller was obtained with 22 frequency response data at $w = \{0.1,\ 0.2,\ 0.4,\ 0.6,\ \ldots,\ 3.6,\ 3.8,\ 4.0,\ 1000\}$ as shown in Figure 21.

Figure 21 also shows the set of stabilizing first order controller obtained with 41 frequency response data at $w = \{0.1,\ 0.2,\ 0.3,\ \ldots,\ 3.9,\ 4.0,\ 1000\}$. As we expect, it is observed that we can get more accurate results(a larger set of stabilizing controller) by taking more frequency response data.

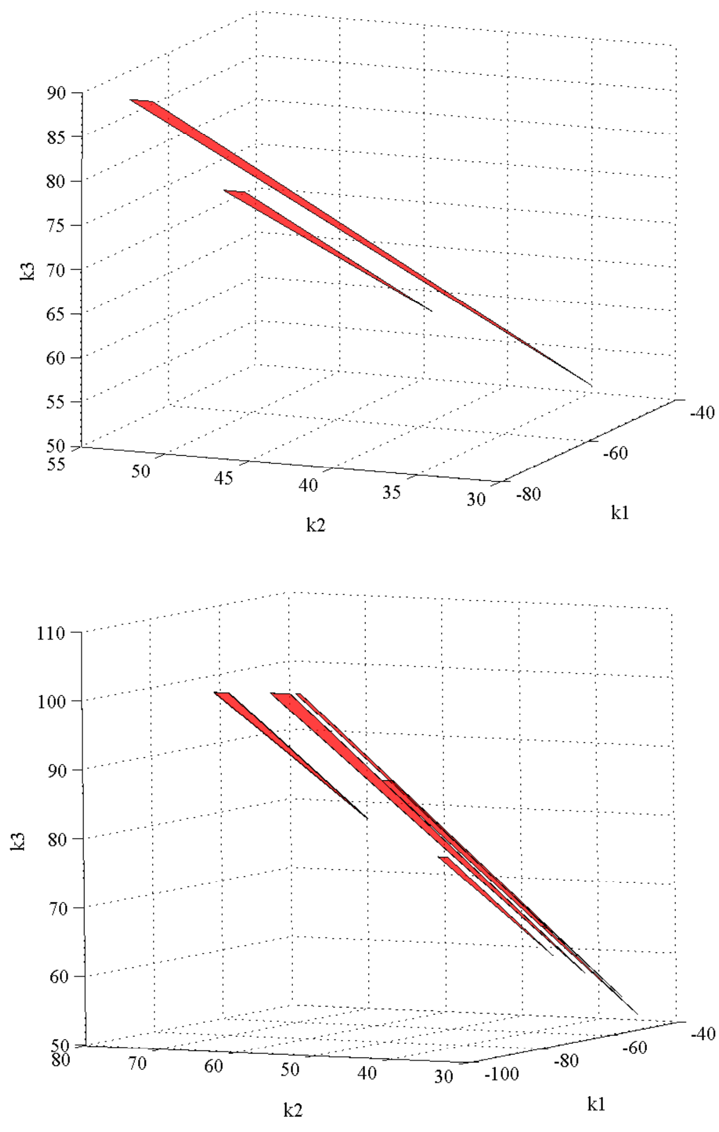

Fig. 21.: *Set of Stabilizing First Order Controller with 20 and 41 Data Points*

CHAPTER V

CONTROLLER DESIGN WITH PERFORMANCE SPECIFICATIONS

A. Introduction

In this chapter, we consider the problem of synthesizing sets of controllers which satisfy some performance criteria using the frequency response measurements under the similiar assumptions that were made in chapter III. Those performance criteria can be gain margin, phase margin, upper bound on the $\mathcal{H}_\infty$ norm of a weighted sensitivity transfer function, or a requirement that a certain closed loop transfer function be SPR etc. A large class of performance problems such as those listed here can be reduced to the problem of determining a set of stabilizing controllers that render a set of complex polynomials Hurwitz [15, 26].

1. The criterion for guaranteeing a gain margin for a SISO plant with a transfer function $\frac{N_p}{D_p}(s)$ stabilized by a fixed order controller, $\frac{N_c}{D_c}(s)$ is that, for every $K_g \in (K_g^-, K_g^+)$, the polynomial

$$D_p(s)D_c(s) + K_g N_p(s)N_c(s)$$

must be Hurwitz.

2. The criterion for guaranteeing a phase margin of ϕ for a SISO plant with a transfer function $\frac{N_p}{D_p}(s)$ stabilized by a fixed order controller, $\frac{N_c}{D_c}(s)$ is that, for every $\theta \in (-\phi, \phi)$, the polynomial

$$D_p(s)D_c(s) + e^{j\theta} N_p(s)N_c(s)$$

must be Hurwitz.

3. For the same controller to achieve a $\mathcal{H}_\infty$ norm of the complementary sensitivity transfer function less than γ is equivalent to having the following family of complex poly-

nomials

$$\gamma\{D_p(s)D_c(s) + N_p(s)N_c(s)\} + e^{j\theta}N_p(s)N_c(s)$$

is Hurwitz, for every $\theta \in (0, 2\pi)$.

4. A real proper transfer function $G(s, K) = \frac{N(s,K)}{D(s,K)}$ is Strictly Positive Real (SPR) if and only if the following three conditions are satisfied:

   (a) $G(0, K) > 0$,

   (b) $N(s, K)$ is Hurwitz, and

   (c) $D(s, K) + j\alpha N(s, K)$ is Hurwitz for every $\alpha \in \Re$.

   In fact, this problem arises in guaranteeing *absolute stability*, that is, robust stability to sector bounded nonlinearities, as was shown in chapter II.

Thus to satisfy the performance criteria, we need to make complex polynomials Hurwitz. For example, in the performance criterion 3, we have to make the complex polynomial Hurwitz for all the possible values of $\theta \in [0, 2\pi]$. In most cases, it suffices to check whether the polynomial is Hurwitz for a few values of θ. This is because the set of the control vector which make polynomials Hurwitz change smoothly with respect to θ.

In this chapter, we propose a method to synthesize a controller that make a system guaranteeing certain level of performance with the frequency response measurements under similar assumptions as before.

By way of notation, we denote the transfer function of the plant to be $G(s)$. The following are the assumptions about the plant:

Assumption V.1.

1. *The transfer function $G(s)$ of the plant is rational and strictly proper, i.e., $G(s) = \frac{N_p}{D_p}(s)$, for some co-prime polynomials, $N_p(s)$ and $D_p(s)$, with the degree, n, of $D_p(s)$ greater than the degree m of $N_p(s)$. We may not know either m or n.*

2. *There are no poles and zeros of the plant on the imaginary axis, i.e., $D_p(jw) \neq 0$, $N_p(jw) \neq 0$ for every $w \in \Re$.*

3. *This assumption and the following assumptions concern the knowledge of frequency response of the plant: There are frequency bounds w_{lb} for lower bound and w_{ub} for upper bound beyond which the phase of the plant does not change appreciably and the amplitude response of the plant is negligible. To quantify this statement, let $G(jw)$ be expressed as $G_r(w)+jG_i(w)$, where G_r and G_i are real, rational functions of w. For a known $\epsilon > 0$, we assume that $|G(jw)| \leq \epsilon$ for all $w \geq w_{ub}$ and $w \leq w_{lb}$. This is a reasonable assumption since the plant is strictly proper.*

4. *The relative degree $n-m$ is known. This can be inferred from the amplitude response of the plant at sufficiently high frequencies.*

5. *We will assume that the functions $|G(jw)|^2$, $G_r(w)$, $G_i(w)$ have been approximated using polynomials $P_0(w)$, $P_1(w)$, $P_2(w)$ respectively and the maximum estimation errors are bounded by μ_0, μ_1, μ_2 and the maximum derivatives of the estimation errors are bounded by η_0, η_1, η_2 respectively.*

6. *We will assume that the number of non minimum phase zeros, z_r of the plant are known.*

Now let's suppose we design a fixed order controller $C(s)$ satisfying H_∞ specification under Assumption V.1.

This chapter is organized as follows: In section B, we provide basic ideas to derive main results. In section C, we present the main results. In section D, we deal with a special case in which case the nonnegativity in intervals can be relaxed to the nonnegativity of the end points of the interval and give a numerical example.

B. Basic Ideas

The basic ideas used in the construction of sets which satisfy some performance criteria are as follows:

1. The problem to design the controller to achieve a $\mathcal{H}_\infty$ norm of the complementary sensitivity transfer function less than γ is equivalent to having the following family of complex polynomials $\Delta_c(s) = \gamma \{D_p(s)D_c(s) + N_p(s)N_c(s)\} + e^{j\theta}N_p(s)N_c(s)$ is Hurwitz, for every $\theta \in (0, 2\pi)$ as shown before.

2. We construct a rational function

$$\begin{aligned} \delta(s) &= \Delta_c(s)\frac{N_p(-s)}{D_p(s)D_p(-s)} \\ &= \gamma(G(s)G(-s)N_c(s) + G(-s)D_c(s)) + e^{j\theta}G(s)G(-s)N_c(s) \end{aligned} \tag{5.1}$$

   If $\Delta_c(s)$ has coefficients that are affine in the controller coefficients, then the rational function, $\delta(s)$ is also affine in the controller coefficients.

3. All controllers, $C(s)$, that make $\Delta_c(s)$ Hurwitz, are such that the total phase accumulation of $\delta(jw)$ as w varies from $-\infty$ to $+\infty$ is the same and equals $(n{-}m{+}r{+}2z_r)\pi$.

$$\angle\delta(jw)|_{w=-\infty}^{w=+\infty} = \pi(n - m + r + 2z_r) \tag{5.2}$$

   Since $n - m$, r and z_r are known, the total desired accumulation of phase is known.

4. Let $\delta(jw) = \delta_r(w) + j\delta_i(w)$, where $\delta_r(w)$ and $\delta_i(w)$ are real, rational functions. In Lemma V.2, we relate how the total accumulation of phase is related to the roots of $\delta_i(w)$ and the sign of $\delta_r(w)$ at those roots.

5. The existence of a stabilizing controller for the complex polynomial can be expressed in terms of the existence of an appropriate set of frequency intervals which admit exactly one or zero roots of the imaginary part of the Nyquist plot and no roots of the real part. This is shown in Theorem V.1. For every set of frequency intervals, these conditions can be translated into linear inequality constraints or linear matrix inequality (LMI) constraints involving the controller parameters. This step involves the Chebyshev approximation of the frequency response in the frequency band $[w_{lb}, w_{ub}]$. It subsequently involves the use of Markov-Lucaks theorem to convert the conditions into a LMI form.

We are interested in synthesizing a rational, proper controller $C(s)$ satisfying H_∞ specification, i.e., for some monic polynomial $D_c(s)$ of degree r and a polynomial $N_c(s)$ of degree at most r, $C(s) = \frac{N_c}{D_c}(s)$. Let $N_c(s) = n_0 + n_1 s + \ldots + n_r s^r$ and $D_c(s) = d_0 + d_1 s + \ldots + d_{r-1}s^{r-1} + s^r$. Let K be the vector of controller coefficients:

$$\begin{bmatrix} n_0 & n_1 & \ldots & n_r & d_0 & d_1 & \ldots & d_{r-1} \end{bmatrix}^T$$

. The determination of the vector K is equivalent to the determination of the stabilizing controller $C(s)$.

C. Main Results

The net phase change property for complex polynomials can be represented similarly with that for real polynomials as follows [25, 32].

Lemma V.1.

1. *The phase of the complex Hurwitz polynomial,* $\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_d s^d$ *monotonically increases as* $w : -\infty \ \rightarrow \ +\infty$ *and the plot of* d^{th} *order complex Hurwitz polynomial* $\delta(jw) = \delta_r(w) + j\delta_i(w)$ *has to move strictly counterclockwise and go through* $2d$ *quadrants as* $w : -\infty \ \rightarrow \ +\infty$*.*
2. *The plot of the* d^{th} *order complex polynomial(not necessary Hurwitz),* $\delta(jw) = \delta_r(w) + j\delta_i(w)$ *has to go through* $2\{l(\delta(s)) - r(\delta(s))\}$ *quadrants as* $w : -\infty \ \rightarrow +\infty$*.*

$$\angle\delta(jw)|_{w=-\infty}^{w=\infty} = \pi\left[l(\delta) - r(\delta)\right], \tag{5.3}$$

where $l(\delta), r(\delta)$ denote the numbers of roots of $\delta(s)$ in the left half plane and in the right half plane respectively.

Following the outline of the basic ideas of the chapter presented in the earlier section, we begin with a generalization of Hermite-Biehler theorem for rational functions in Lemma V.2.

Lemma V.2.

Consider $\delta(s) = \Delta_c(s)\frac{N_p(-s)}{D_p(s)D_p(-s)}$*. Let the real roots of* $\delta_i(w)$ *be* $w_1, \ldots, w_l$*,* $w_0 = -\infty,\ w_{l+1} = +\infty$ *and the sign of* $\delta_r(w)$ *at these frequencies be correspondingly* $i_0, i_1, \ldots, i_l, i_{l+1}$*. Then* $\Delta_c(s)$ *is Hurwitz if and only if*

1. *for* $n - m + r$ *: even*

$$n - m + r + 2z_r = \frac{1}{2}sgn[\delta_i(w_0)]\{i_0 - 2i_1 + \ldots + (-1)^l 2i_l + (-1)^{l+1} i_{l+1}\} \tag{5.4}$$

2. *for* $n-m+r$ *: odd*

$$n-m+r+2z_r = -sgn[\delta_i(w_0)]\{i_1 - i_2 + \ldots + (-1)^{l-1} i_l\} \qquad (5.5)$$

Proof.

We first note that the degree of the polynomial $\Delta_c(s)N_p(-s)$ *is* $n+r+m$*. Hence, the parity of the degree of the polynomial* $\Delta_c(s)N_p(-s)$ *is the same as that of* $n-m+r$*.*

Let the sign of $\frac{d\delta_i(w)}{dw}$ *at* $w = w_l$ *be* I_l*. The change in the phase of* $\delta(jw)$ *from* w_l *to* w_{1+1} *is given by:* $I_l(i_l - i_{l+1})\frac{\pi}{2}$ *when* i_l *and* i_{l+1} *are the roots of* $\delta_i(w)$*. Let* $w_0 = -\infty$ *and* $w_{l+1} = +\infty$*. Since* $I_i = -I_{i-1}$ *for* $i = 2, 3, \ldots, l$ *, the phase change in* $\delta(jw)$ *from* $w = w_1$ *to* $w = w_l$ *can be expressed as:*

$$\begin{aligned}
\angle\delta(jw)|_{w=w_1}^{w=w_l} &= \frac{\pi}{2}\left\{I_1(i_1 - i_2) + I_2(i_2 - i_3) + \ldots + I_{l-1}(i_{l-1} - i_l)\right\} \\
&= \frac{\pi}{2} I_1 \left\{i_1 - 2i_2 + 2i_3 - \ldots + (-1)^{l-2} 2i_{l-1} + (-1)^{l-1} i_l\right\}
\end{aligned}$$

We note that the degree of the polynomial $\Delta_c(s)N_p(-s)$ *is* $n+r+m$*. Hence, the parity of the degree of the polynomial* $\Delta_c(s)N_p(-s)$ *is the same as that of* $n-m+r$*.*

1. *for* $n-m+r$ *: even*

   The phase change in $\delta(jw)$ *from* $w = w_0 = -\infty$ *to* $w = w_1$ *will depend on the degree of the polynomial* $\Delta_c(s)N_p(-s)$*. If the degree of the is even,*

   $$\angle\delta(jw)|_{w=-\infty}^{w=w_l} = -\frac{\pi}{2} I_1(i_0 - i_1)$$

   The phase change in $\delta(jw)$ *from* $w = w_l$ *to* $w = w_{l+1} = +\infty$ *becomes*

   $$\begin{aligned}
   \angle\delta(jw)|_{w=w_l}^{w=+\infty} &= \frac{\pi}{2} I_l(i_l - i_{l+1}) \\
   &= \frac{\pi}{2}(-1)^{l+1} I_1(i_l - i_{l+1})
   \end{aligned}$$

Finally, we get the phase change in $\delta(jw)$ *from* $w = w_0$ *to* $w = w_{l+1}$ *with* $I_1 = -sign(\delta_i(0))$ *as follows :*

$$\begin{aligned}
\angle\delta(jw)|_{w=-\infty}^{w=+\infty} &= \angle\delta(jw)|_{w=-\infty}^{w=w_1} + \angle\delta(jw)|_{w=w_1}^{w=w_l} + \angle\delta(jw)|_{w=w_l}^{w=+\infty} \\
&= -\frac{\pi}{2}\left[I_1(i_0 - i_1) - I_1\left\{i_1 - 2i_2 + \ldots + (-1)^{l-2}2i_{l-1} + (-1)^{l-1}i_l\right\}\right] \\
&\quad -\frac{\pi}{2}\left[-(-1)^{l+1}I_1(i_l - i_{l+1})\right] \\
&= -\frac{\pi}{2}I_1\left\{i_0 - 2i_1 + 2i_2 - \ldots + (-1)^l 2i_l + (-1)^{l+1}i_{l+1}\right\} \\
n - m + r + 2z_r &= \frac{1}{2}sgn[\delta_i(w_0)]\{i_0 - 2i_1 + \ldots + (-1)^l 2i_l + (-1)^{l+1}i_{l+1}\}
\end{aligned}$$

2. *for* $n - m + r$ *: odd*

   If the degree is odd, The phase change in $\delta(jw)$ *from* $w = w_0$ *to* $w = w_1$ *becomes*

$$\angle\delta(jw)|_{w=-\infty}^{w=w_l} = \frac{\pi}{2}I_1 i_1$$

The phase change in $\delta(jw)$ *from* $w = w_l$ *to* $w = w_{l+1}$ *becomes*

$$\angle\delta(jw)|_{w=w_l}^{w=+\infty} = \frac{\pi}{2}I_l i_l = \frac{\pi}{2}(-1)^{l-1}I_1 i_l$$

We can get the phase change in $\delta(jw)$ *from* $w = w_0$ *to* $w = w_{1+1}$ *with* $I_1 = -sign(\delta_i(0))$ *as follows :*

$$\begin{aligned}
\angle\delta(jw)|_{w=-\infty}^{w=+\infty} &= \angle\delta(jw)|_{w=-\infty}^{w=w_1} + \angle\delta(jw)|_{w=w_1}^{w=w_l} + \angle\delta(jw)|_{w=w_l}^{w=+\infty} \\
&= \frac{\pi}{2}\left[I_1 i_1 + I_1\left\{i_1 - 2i_2 + \ldots + (-1)^{l-2}2i_{l-1} + (-1)^{l-1}i_l\right\}\right] \\
&\quad +\frac{\pi}{2}\left[(-1)^{l-1}I_1 i_l\right] \\
&= \pi I_1\left\{i_1 - i_1 + i_2 - \ldots + (-1)^{l-1}i_l\right\} \\
n - m + r + 2z_r &= -sgn[\delta_i(w_0)]\left\{i_1 - i_2 + \ldots + (-1)^{l-1}i_l\right\}
\end{aligned}$$

Since $D_p(s)$ *does not have any zeros on the imaginary axis, the phase change in* $\delta(jw)$ *as* w *changes from* $-\infty$ *to* $+\infty$ *is the same as that of* $\Delta_c(jw)N_p(-jw)$ *as* w *changes from*

$-\infty$ to $+\infty$. The accumulation or change of phase of $\Delta_c(jw)N_p(-jw)$ is $(n-m+r+2z_r)\pi$ if and only if $\Delta_c(s)$ is Hurwitz. With this observation $(n-m+r+2z_r)$ equals the quantity expressed in equations (5.4) or (5.5). $\nabla\nabla\nabla$

The following theorem will use Lemma V.2 to characterize a stabilizing controller of a fixed order in terms of frequency response of the plant.

Theorem V.1.

A controller $C(s)$ stabilizes $\Delta_c(s,K)$ if and only if for a given γ and every $\theta \in [0, 2\pi]$,

1. *There exists a sequence $i_0, i_1, \ldots, i_l$ satisfying equation (5.4) or (5.5), and*
2. *For the sequence of integers $i_0, i_1, \ldots, i_l$, there exists correspondingly l disjoint frequency bands, $[w_{p,1}, w_{p,2}], p = 1, \ldots, l$ such that*
   (a) *there exists exactly one root of $\delta_i(w)$ in $(w_{p,1}, w_{p,2})$,*
   (b) *the sign of $\delta_r(w)$ in $[w_{p,1}, w_{p,2}]$ is the same as that of i_p, and*
   (c) *there is no sign change of $\delta_i(w)$ in the disjoint intervals $[-\infty, w_{1,1}]$, $[w_{l,2}, \infty]$ and $[w_{p,2}, w_{p+1,1}],\ p = 1, \ldots, l-1$.*

Proof.

Let the root of $\delta_i(w)$ in $(w_{p,1}, w_{p,2})$ be w_p. Since the sign of $\delta_r(w)$ at w_p is i_p, the change in phase of $\delta(jw)$ as w varies from $-\infty$ to $+\infty$ is $(n-m+r+2z_r)\pi$, indicating that $\Delta_c(s)N_p(-s)$ has $m-z_r$ roots with positive real part. However, this is the case if and only if $\Delta_c(s)$ is Hurwitz. $\nabla\nabla\nabla$

Remark V.1.

1. *We first observe that* $\delta(s)$ *may be expressed as* $\delta_0(s) + \sum_{p=1}^{2r+1} \delta_p(s)k_p$*, where* k_p *is the* p^{th} *component of the controller vector,* K*, and* $\delta_0, \delta_1, \ldots, \delta_{2r+1}$ *are rational functions, which can be determined once the expression for* $G(s)$ *is known. Similarly,* δ_r *and* δ_i *are affinely dependent on the controller parameter vector,* K*. To emphasize the dependence on* K*, we will use the notation* $\delta_r(w, K)$ *and* $\delta_i(w, K)$ *as appropriate. One may express the affine dependence of* $\delta_r(w, K)$ *and* $\delta_i(w, K)$ *as:*

$$\delta_r(w, K) = \Delta_r(w, |G(jw)|^2, G_r(w), G_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix} \quad (5.6)$$

$$\delta_i(w, K) = \Delta_i(w, |G(jw)|^2, G_r(w), G_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix} \quad (5.7)$$

   for some vectors Δ_r *and* Δ_i *that depend affinely on* $|G(jw)|^2$*,* $G_r(w)$ *and* $G_i(w)$*.*

2. *The conditions in Theorem V.1 may be replaced as follows:*

   (a) *The condition (a) of 2 may be replaced by:* $\delta_i(w_{p_1}, K)\delta_i(w_{p,2}, K) < 0$ *and* $\frac{d\delta_i(w,K)}{dw}$ *has the sign* $(-1)^{p+1}I_1$ *in* $[w_{p,1}, w_{p,2}]$*. This ensures that* $\delta_i(w, K)$ *has exactly one root in the interval of interest. If the frequency response at frequencies,* $w_{p,1}$ *and* $w_{p,2}$ *are known, we note that the first condition* $\delta_i(w_{p_1}, K)\delta_i(w_{p_2}, K) < 0$ *can be written as two sets of linear inequalities.*

   (b) *The conditions (b) and (c) of 2 may similarly be replaced as:*

$$
\begin{aligned}
i_p\delta_r(w) &> 0,\ \forall w \in [w_{p,1}, w_{p,2}] && (5.8)\\
-I_1\delta_i(w) &> 0,\ \forall w \in [-\infty, w_{1,1}] && (5.9)\\
(-1)^{l+1}I_1\delta_i(w) &> 0,\ \forall w \in [w_{l,2}, +\infty) && (5.10)\\
(-1)^{q+1}I_1\delta_i(w) &> 0,\ \forall w \in [w_{q,2}, w_{q+1,1}], && (5.11)
\end{aligned}
$$

where $p = 1, 2, \ldots, l$, $q = 1, \ldots, l-1$ *and dependence on* K *is suppressed.*

If $G(jw)$ is exactly known, the condition of $(-1)^{p+1}I_1\frac{d\delta_i(w,K)}{dw}$ being nonnegative in $[w_{p,1}, w_{p,2}]$ can be posed as a SDP using Markov-Lucaks theorem as shown in chapter III and in [20, 40]. If $G(jw)$ is approximately known as is typically the case when fitting a rational function approximation to the given data contaminated with noise, the nonnegativity condition can be posed as a robust SDP as shown in chapter IV.

CHAPTER VI

CONCLUSIONS AND FUTURE WORK

In this dissertation, we have addressed the problem of synthesizing fixed order controllers which absolutely stabilize a Lure-Postnikov system. We have also proposed a method to synthesize sets of stabilizing controllers of strictly proper, delay-free, SISO LTI plants directly from their empirical frequency response data and some coarse information about them.

Analytical tools for synthesizing stabilizing fixed structure controllers such as the PID or low-order controllers examining the absolute stability of Lure-Postnikov systems which have sector-bounded nonlinearities have been studied in the literature, but tools for synthesizing higher order controllers have not been studied as yet. We have proposed a systematic method designing fixed higher order controllers which absolutely stabilize Lure-Postnikov systems with the recent results which approximate the set of controller parameters that render a family of real and complex polynomials and provided an example.

The advantage of the proposed approach is that sets of absolutely stabilizing controllers can be presented to a control engineer. The control engineer may further based on other constrains

It is widely recognized that an accurate analytical model of the plant may not be available to a control designer. However, it is reasonable in many applications that one will have an empirical model of the plant in terms of its frequency response data and from physical considerations or from the empirical time response data, one may have some coarse information about the plant such as the number of non minimum phase zeros of the plant etc. We have proposed a systematic method to synthesize arbitrary order controllers for delay-free SISO LTI plants from the frequency response data and the number of non minimum phase zeros of the plant. We posed the problem of synthesizing the sets of stabilizing controllers as that of sets of controllers satisfying some robust SDPs considering the frequency response measurement errors. It indicates the possibility of fixed order controller synthesis using only frequency response measurements.

A. Summary of Results

In chapter II, we have proposed a method to synthesize fixed order controllers as well as PID controllers that absolutely stabilize a Lure-Postnikov system. We also provided an example of Lure-Postnikov system(one-link robot with flexible joint) and constructed the set of PID and first order controllers which absolutely stabilize the example system.

In chapter III, the recently developed Sum-of-Squares techniques for checking the nonnegativity of a real polynomial in an interval have been reviewed.

In chapter IV, we have proposed a method for synthesizing sets of stabilizing controllers of strictly proper, delay-free, SISO LTI plants directly from their empirical frequency response data and from some coarse information about them. The coarse information that is required is the following: the number of non minimum phase zeros of the plant and the frequency range beyond which the phase response of the LTI plant does not change appreciably and the amplitude response goes to zero. The proposed method in this chapter involves nonnegativity of real polynomials in some intervals. We also posed the problem of synthesizing the sets of stabilizing controllers as that of sets of controllers satisfying some robust SDPs considering the frequency response measurement errors.

In chapter V, the problem of fixed order stabilizing controller design has been extended to the design of controllers which guarantee some performance criteria. Those performance criteria can be gain margin, phase margin, upper bound on the $\mathcal{H}_\infty$ norm of a weighted sensitivity transfer function, or a requirement that a certain closed loop transfer function be SPR etc.

B. Future Work

- The proposed methods are computationally intensive. Hence, efficient numerical algorithms that exploit the structure of SDPs resulting from those problems will be practically very useful.
- The extension of the developed techniques to multivariable systems is a challenging problem.

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VITA

Sin Cheon Kang was born in Kyeong-Joo, Korea. He received his Bachelor of Science in mechanical engineering from Korea University, Seoul in 1989. He received an M.S. degree in mechanical engineering from Korea Advanced Institute of Science and Technology in 1991. He worked for the Agency for Defense Development as a researcher from 1991 to 2000. In January 2001, he started his Ph.D. in mechanical engineering at Texas A&M University and graduated in December 2005. He may be contacted through Dr. Darbha Swaroop at the Department of mechanical engineering, Texas A&M University, College Station, Texas 77843.