OPERATOR VALUED HARDY SPACES
AND RELATED SUBJECTS

A Dissertation

by

TAO MEI

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2006

Major Subject: Mathematics

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Chair of Committee,	Gilles Pisier
Committee Members,	William Johnson
	Roger Smith
	Daren Cline
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ABSTRACT

Operator Valued Hardy Spaces

and Related Subjects. (August 2006)

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Chair of Advisory Committee: Dr. Gilles Pisier

We give a systematic study of the Hardy spaces of functions with values in the non-commutative L^p-spaces associated with a semifinite von Neumann algebra $\mathcal{M}$. This is motivated by matrix valued harmonic analysis (operator weighted norm inequalities, operator Hilbert transform), as well as by the recent development of non-commutative martingale inequalities. Our non-commutative Hardy spaces are defined by non-commutative Lusin integral functions. It is proved in this dissertation that they are equivalent to those defined by the non-commutative Littlewood-Paley G-functions.

We also study the L^p boundedness of operator valued dyadic paraproducts and prove that their L^q boundedness implies their L^p boundedness for all $1 < q < p < \infty$.

To my parents

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CHAPTER I

INTRODUCTION

This dissertation gives a systematic study of matrix valued (and more generally, operator valued) Hardy spaces. Our motivations come from two closely related directions. The first one is matrix valued harmonic analysis, which deals with extending results from classical harmonic analysis to the operator valued setting. We should emphasize that such extensions not only are interesting in themselves but also have applications to other domains such as prediction theory and rational approximation. A central subject in this direction is the study of "operator valued" Hankel operators (i.e. Hankel matrices with operator entries). As in the scalar case, this is intimately linked to operator valued weighted norm inequalities, operator valued Carleson measures, operator valued Hardy spaces.... Much research in this direction has been done, notably by F. Nazarov, S. Treil and A. Volberg; see, for instance, the recent works [8], [28], [31], [30], [34]).

The second direction which motivates this dissertation is non-commutative martingale theory. This theory was initiated already in the 70's. For example, I. Cuculescu ([3]) proved a non-commutative analogue of the classical Doob weak type (1,1) maximal inequality. This has immediate applications to the almost sure convergence of non-commutative martingales (see also [11], [12]). The new input into the theory is the recent development of non-commutative martingale inequalities. This has been largely influenced and inspired by operator space theory. Many inequalities in classical martingale theory have been transferred into the non-commutative setting. These include the non-commutative Burkholder-Gundy inequalities, the non-

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This dissertation follows the style of C. R. Math. Acad. Sci. Paris.

commutative Doob inequality, the non-commutative Burkholder-Rosenthal inequalities and the boundedness of the non-commutative martingale transforms (see [33], [14], [17], [18], [36]).

One common important object in the two directions above is the non-commutative analogue of the classical BMO space. Because of the non-commutativity, there are now two non-commutative BMO spaces, column BMO and row BMO. As expected, these non-commutative BMO spaces are proved to be the duals of some non-commutative H^1 spaces. To be more precise and to go into some details, we introduce these spaces in the case of matrix valued functions. Let $\mathcal{M}_d$ be the algebra of $d \times d$ matrices with its usual trace tr. Then the column BMO space is defined by

$$\mathrm{BMO_c}(\mathbb{R}, \mathcal{M}_\mathrm{d}) = \left\{\varphi : \mathbb{R} \to \mathcal{M}_\mathrm{d}, \|\varphi\|_{\mathrm{BMO_c}} < \infty\right\}$$

where

$$\|\varphi\|_{\mathrm{BMO}_c} = \sup_h \left\{ \|\varphi(\cdot)h\|_{\mathrm{BMO}(l_2^d)}, h \in l_2^d, \|h\|_{l_2^d} \leq 1 \right\}.$$

Similarly, the row BMO space is

$$\mathrm{BMO}_r(\mathbb{R}, \mathcal{M}_d) = \left\{\varphi : \mathbb{R} \to \mathcal{M}_d, \|\varphi\|_{\mathrm{BMO}_r} = \|\varphi^*\|_{\mathrm{BMO}_c} < \infty\right\}.$$

We will also need the intersection of these BMO spaces, which is

$$\mathrm{BMO}_{cr}(\mathbb{R}, \mathcal{M}_d) = \mathrm{BMO}_c(\mathbb{R}, \mathcal{M}_d) \cap \mathrm{BMO}_r(\mathbb{R}, \mathcal{M}_d)$$

equipped with the norm $\|\varphi\|_{\mathrm{BMO}_{cr}} = \max\{\|\varphi\|_{\mathrm{BMO}_c}, \|\varphi\|_{\mathrm{BMO}_r}\}$. When $d = 1$, all these BMO spaces coincide with the classical BMO space which is well known to be the dual of the classical Hardy space H^1. This result can be extended to the case of $d < \infty$ very easily. Let

$$H^1(\mathbb{R}, S_d^1) = \left\{ f : \mathbb{R} \to S_d^1; \int \sup_{y>0} \|f(x,y)\|_{S_d^1} \, dx < \infty \right\},$$

where S_d^1 is the trace class over l_d^2, and $f(x,y)$ denotes the Poisson integral of f corresponding to the point $x+iy$. Then

$$(H^1(\mathbb{R},\mathcal{S}_d^1))^* = \mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}_d)$$

and

$$c_d^{-1}\left\|\varphi\right\|_{\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}_d)} \leq \left\|\varphi\right\|_{(H^1(\mathbb{R},\mathcal{S}_d^1))^*} \leq c_d\left\|\varphi\right\|_{\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}_d)}.$$

Here the constant $c_d \to +\infty$ as $d \to +\infty$. Thus this duality between $H^1(\mathbb{R},\mathcal{S}_d^1)$ and $\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}_d)$ fails for the infinite dimensional case. One of our goals is to find a natural predual space of BMO_{cr} with relevant constants independent of d.

In the case of non-commutative martingales, this natural dual of BMO_{cr} was already introduced by Pisier and Xu in their work on the non-commutative Burkholder-Gundy inequality. To define the right space $\mathcal{H}^1$, they considered a non-commutative analogue of the classical square function for martingales. Motivated by their work, we will introduce a new definition of H^1 for matrix valued functions by considering a non-commutative analogue of the classical Lusin integral (recall that, in the classical case, a scalar valued function is in H^1 if and only if its Lusin integral is in L^1, see [5], [37]). For a matrix valued function f, $f \in L^1((\mathbb{R},\frac{dt}{1+t^2}),\mathcal{M}_d)$, $1 \leq p < \infty$, let

$$\|f\|^p_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M}_d)} = tr\int_{-\infty}^{+\infty}(\iint\limits_{\Gamma}|\nabla f(t+x,y)|^2dxdy)^{\frac{p}{2}}dt,$$

where $\Gamma = \{(x,y)\in\mathbb{R} : |x| < y, y > 0\}$ and

$$|\nabla f|^2 = (\frac{\partial f}{\partial x})^*\frac{\partial f}{\partial x} + (\frac{\partial f}{\partial y})^*\frac{\partial f}{\partial y}.$$

Then we define

$$\mathcal{H}_c^p(\mathbb{R},\mathcal{M}_d) = \left\{f:\mathbb{R}\to\mathcal{M}_d; \|f\|_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M}_d)} < \infty\right\}.$$

Similarly, set

$$\mathcal{H}_r^p(\mathbb{R},\mathcal{M}_d)=\left\{f:\mathbb{R}\to\mathcal{M}_d;\|f\|_{\mathcal{H}_r^p(\mathbb{R},\mathcal{M}_d)}=\|f^*\|_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M}_d)}<\infty\right\}.$$

Finally, if $1\le p<2$, we define

$$\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M}_d)=\mathcal{H}_c^p(\mathbb{R},\mathcal{M}_d)+\mathcal{H}_r^p(\mathbb{R},\mathcal{M}_d)$$

equipped with the norm

$$\|f\|_{\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M}_d)}=\inf\{\|g\|_{\mathcal{H}_c^p}+\|h\|_{\mathcal{H}_r^p}:f=g+h,g\in\mathcal{H}_c^p(\mathbb{R},\mathcal{M}_d),h\in\mathcal{H}_r^p(\mathbb{R},\mathcal{M}_d)\}.$$

If $p\ge 2$, let

$$\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M}_d)=\mathcal{H}_c^p(\mathbb{R},\mathcal{M}_d)\cap\mathcal{H}_r^p(\mathbb{R},\mathcal{M}_d)$$

equipped with the norm

$$\|f\|_{\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M}_d)}=\max\{\|f\|_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M}_d)},\|f\|_{\mathcal{H}_r^p(\mathbb{R},\mathcal{M}_d)}\}.$$

One of our main results is the identification of $\mathrm{BMO}_c(\mathbb{R},\mathcal{M}_d)$ as the dual of $\mathcal{H}_c^1(\mathbb{R},\mathcal{M}_d)$: $(\mathcal{H}_c^1(\mathbb{R},\mathcal{M}_d))^*=\mathrm{BMO}_c(\mathbb{R},\mathcal{M}_d)$ with equivalent norms, where the relevant equivalence constants are universal. Similarly, $\mathrm{BMO}_r(\mathbb{R},\mathcal{M}_d)$ (resp. $\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}_d)$) is the dual of $\mathcal{H}_c^1(\mathbb{R},\mathcal{M}_d)$ (resp. $\mathcal{H}_{cr}^1(\mathbb{R},\mathcal{M}_d)$). Another result is the equality $\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M}_d)$ $=L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}_d)$ with equivalent norms for all $1<p<\infty$. This is the function space analogue of the non-commutative Burkholder-Gundy inequality in [33]. It is also closely related to the recent work ([16]) by Junge, Le Merdy and Xu on the Littlewood-Paley theory for semigroups on non-commutative L^p-spaces.

We also prove the analogue of the classical Hardy-Littlewood maximal inequality. Our approach to this inequality for functions consists in reducing it to the same inequality for dyadic martingales. It is very simple and seems new even in the scalar case. The same idea allows us to write BMO as an intersection of two dyadic BMO. This latter result plays an important role in this dissertation. It permits us to reduce many problems involving BMO (or its variant BMO^q, which is the dual of $\mathcal{H}^p$ for $1 \le p < 2, \frac{1}{p} + \frac{1}{q} = 1$) to dyadic BMO, that is, to BMO of dyadic non-commutative martingales. For instance, we use this reduction for the interpolation problems on our non-commutative Hardy spaces.

All the results mentioned above remain valid for a general semifinite von Neumann algebra $\mathcal{M}$ in place of the matrix algebras.

We now explain the organization of this dissertation. Chapter II (the next one) contains preliminaries, definitions and notations used throughout the dissertation. There we define the two non-commutative square functions which are the non-commutative analogues of the Lusin area integral and Littlewood-Paley g-function. These square functions allow us to define the corresponding non-commutative Hardy spaces $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$, where $\mathcal{M}$ is a semifinite von Neumann algebra. This chapter also contains the definition of $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ and some elementary properties of these spaces.

The main result of Chapter III is the analogue in our setting of the famous Fefferman duality theorem between $\mathcal{H}^1$ and BMO. As in the classical case, this result implies an atomic decomposition for our Hardy spaces $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ (as well as $\mathcal{H}_r^1(\mathbb{R}, \mathcal{M}), \mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M})$). Another consequence is the characterization of functions in $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ (as well as $\mathrm{BMO}_r(\mathbb{R}, \mathcal{M}), \mathrm{BMO}_{cr}(\mathbb{R}, \mathcal{M})$) via operator valued Carleson measures.

The objective of Chapter IV is the non-commutative Hardy-Littlewood maximal inequality. As already mentioned above, our approach to this is to reduce this inequality to the corresponding maximal inequality for dyadic martingales. To this end, we construct two "separate" increasing filtrations $\mathcal{D} = \{\mathcal{D}_n\}_{n\in\mathbb{Z}}$ and $\mathcal{D}' = \{\mathcal{D}'_n\}_{n\in\mathbb{Z}}$ of dyadic σ-algebras. One of them is just the usual dyadic filtration on $\mathbb{R}$, while the other is a kind of translation of the first. The main point is that any interval of $\mathbb{R}$ is contained in an atom of some $\mathcal{D}_n$ or $\mathcal{D}'_n$ with comparable size. This approach will be repeatedly used in the subsequent chapters. We also prove the non-commutative Poisson maximal inequality and the non-commutative Lebesgue differentiation theorem.

In Chapter V, we define the L^p-space analogues of the BMO spaces introduced in Chapter II, denoted by $\mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$, $\mathrm{BMO}_r^q(\mathbb{R},\mathcal{M})$, $\mathrm{BMO}_{cr}^q(\mathbb{R},\mathcal{M})$. These spaces are proved to be the duals of the respective Hardy spaces $\mathcal{H}_c^p(\mathbb{R},\mathcal{M})$, $\mathcal{H}_r^p(\mathbb{R},\mathcal{M})$, $\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M})$ for $1<p<2$ ($q=\frac{p}{p-1}$). The proof of this duality is also valid for $p=1$. In that case, we recover the duality theorem in Chapter III. However, for $1<p<2$, we need, in addition, the non-commutative maximal inequality from Chapter IV. This is one of the two reasons why we have decided to present these two duality theorems separately. Another is that the reader may be more familiar with the duality between H^1 and BMO and those only interested in this duality can skip the case $1<p<2$. It is also proved in this chapter that $\mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})=\mathcal{H}_c^q(\mathbb{R},\mathcal{M})$ with equivalent norms for all $2<q<\infty$. The third result of Chapter V is the following: Regarded as a subspace of $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma})),\mathcal{H}_c^p(\mathbb{R},\mathcal{M})$ is complemented in $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma}))$ for all $1<p<\infty$. This result is the function space analogue of the non-commutative Stein inequality in [33]. This chapter is largely inspired by the recent work of M. Junge and Q. Xu, where the above results for non-commutative martingales were obtained.

In Chapter VI, we further exploit the reduction idea introduced in Chapter IV, in order to describe $\mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$ as $\mathrm{BMO}_c^{q,\mathcal{D}}(\mathbb{R},\mathcal{M})\cap$ $\mathrm{BMO}_c^{q,\mathcal{D}'}(\mathbb{R},\mathcal{M})$. These two latter BMO spaces are those of dyadic non-commutative martingales. Among the consequences given in this chapter, we mention the equivalence of $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})$

and $\mathcal{H}^p_{cr}(\mathbb{R}, \mathcal{M})$ for all $1 < p < \infty$.

Chapter VII deals with the interpolation for our Hardy spaces. As expected, these spaces behave very well with respect to the complex and real interpolations. This chapter also contains a result on Fourier multipliers.

In Chapter VIII, by using the interpolation results got in Chapter VII, we prove the noncommutative analogue of the classical John-Nirenberg theorem in our setting.

In Chapter IX, we consider the dyadic paraproducts for matrix valued functions and prove that their L^q boundedness implies their L^p boundedness for all $1 < q < p < \infty$.

We close this introduction by mentioning that throughout the dissertation the letter c will denote an absolute positive constant, which may vary from line to line, and c_p a positive constant depending only on p.

CHAPTER II

PRELIMINARIES

2.1. The non-commutative spaces $L^p(\mathcal{M}, L_c^2(\Omega))$

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $S_{\mathcal{M}}^{+}$ be the set of all positive x in $\mathcal{M}$ such that $\tau(\text{supp } x) < \infty$, where supp x denotes the support of x, that is, the least projection $e \in \mathcal{M}$ such that $ex = x$ (or $xe = x$). Let $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^{+}$. We define

$$\|x\|_p = (\tau|x|^p)^{\frac{1}{p}}, \quad \forall x \in S_{\mathcal{M}}$$

where $|x| = (x^*x)^{\frac{1}{2}}$. One can check that $\|\cdot\|_p$ is well-defined and is a norm on $S_{\mathcal{M}}$ if $1 \leq p < \infty$. The completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ is denoted by $L^p(\mathcal{M})$ which is the usual non-commutative L^p space associated with $(\mathcal{M}, \tau)$. For convenience, we usually set $L^{\infty}(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$. The elements in $L^p(\mathcal{M}, \tau)$ can also be viewed as closed densely defined operators on H (H being the Hilbert space on which $\mathcal{M}$ acts). We refer to [4] for more information on non-commutative L^p spaces.

Let (Ω, μ) be a measurable space. We say h is a $S_{\mathcal{M}}$-valued simple function on (Ω, μ) if it can be written as

$$h = \sum_{i=1}^{n} m_i \cdot \chi_{A_i} \tag{2.1}$$

where $m_i \in S_{\mathcal{M}}$ and A_i's are measurable disjoint subsets of Ω with $\mu(A_i) < \infty$. For such a function h we define

$$\|h\|_{L^p(\mathcal{M}, L_c^2(\Omega))} = \left\| \left(\sum_{i=1}^{n} m_i^* m_i \cdot \mu(A_i) \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{M})}$$

and

$$\|h\|_{L^p(\mathcal{M},L^2_r(\Omega))} = \left\| \left(\sum_{i=1}^n m_i m_i^* \cdot \mu(A_i) \right)^{\frac{1}{2}} \right\|_{L^p(\mathcal{M})}$$

This gives two norms on the family of all such h's. To see that, denoting by $B(L^2(\Omega))$ the space of all bounded operators on $L^2(\Omega)$ with its usual trace $tr,$ we consider the von Neumann algebra tensor product $\mathcal{M} \otimes B(L^2(\Omega))$ with the product trace $\tau \otimes tr$. Given a set $A_0 \subset \Omega$ with $\mu(A_0) = 1$, any element of the family of h's above can be regarded as an element in $L^p\left(\mathcal{M} \otimes B(L^2(\Omega))\right)$ via the following map:

$$h \mapsto T(h) = \sum_{i=1}^n m_i \otimes (\chi_{A_i} \otimes \chi_{A_0}) \tag{2.2}$$

and

$$\|h\|_{L^p(\mathcal{M};L^2_c(\Omega))} = \|T(h)\|_{L^p(\mathcal{M}\otimes B(L^2(\Omega)))}$$

Therefore, $\|\cdot\|_{L^p(\mathcal{M};L^2_c(\Omega))}$ defines a norm on the family of the h's. The corresponding completion (for $1 \leq p < \infty$) is a Banach space, denoted by $L^p(\mathcal{M}; L^2_c(\Omega))$. Then $L^p(\mathcal{M}; L^2_c(\Omega))$ is isometric to the column subspace of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$. For $p = \infty$ we let $L^\infty(\mathcal{M}; L^2_c(\Omega))$ be the Banach space isometric by the above map T to the column subspace of $L^\infty(\mathcal{M} \otimes B(L^2(\Omega)))$.

Similarly to $\|\cdot\|_{L^p(\mathcal{M};L^2_c(\Omega))}$, $\|\cdot\|_{L^p(\mathcal{M};L^2_r(\Omega))}$ is also a norm on the family of $S_{\mathcal{M}}$-valued simple functions and it defines the Banach space $L^p(\mathcal{M}; L^2_r(\Omega))$ which is isometric to the row subspace of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$.

Alternatively, we can fix an orthonormal basis of $L^2(\Omega)$. Then any element of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$ can be identified with an infinite matrix with entries in $L^p(\mathcal{M})$. Accordingly, $L^p(\mathcal{M}; L^2_c(\Omega))$ (resp. $L^p(\mathcal{M}; L^2_r(\Omega))$) can be identified with the subspace of $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$ consisting of matrices whose entries are all zero except those in the first column (resp. row).

Proposition 2.1 *Let* $f \in L^p(\mathcal{M}; L^2_c(\Omega)), g \in L^q(\mathcal{M}; L^2_c(\Omega))(1 \leq p, q \leq \infty), \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *Then* $\langle g, f\rangle$ *exists as an element in* $L^r(\mathcal{M})$ *and*

$$\|\langle g, f\rangle\|_{L^r(\mathcal{M})} \leq \|g\|_{L^q(\mathcal{M}; L^2_c(\Omega))} \|f\|_{L^p(\mathcal{M}; L^2_c(\Omega))},$$

where $\langle\, ,\, \rangle$ *denotes the scalar product in* $L^2_c(\Omega)$. *A similar statement also holds for row spaces.*

Proof. This is clear from the discussion above via the matrix representation of $L^p(\mathcal{M}; L^2_c(\Omega))$ (in an orthonormal basis of $L^2(\Omega)$). ∎

Remark. Note that if f and g are $S_{\mathcal{M}}$-valued simple functions, then

$$\langle g, f\rangle = \int_\Omega g^* f d\mu.$$

For general f and g as in Proposition 2.1, if one of p and q is finite, one can easily prove that $\langle g, f\rangle$ is the limit in $L^r(\mathcal{M})$ of a sequence $(\langle g_n, f_n\rangle)_n$ with $S_{\mathcal{M}}$-valued simple functions $f_n,\ g_n$. Consequently, we can define $\int_\Omega g^* f d\mu$ as the limit of $\int_\Omega g_n^* f_n d\mu$. If both p and q are infinite, this limit procedure is still valid but only in the w*-sense.

Convention. Throughout this paper whenever we are in the situation of Proposition 2.1, we will write $\langle g, f\rangle$ as the integral $\int_\Omega g^* f d\mu$. Notationally, this is clearer. Moreover, by the proceding remark this indeed makes sense in many cases.

Observe that the column and row subspaces of $L^p(\mathcal{M}\otimes B(L^2(\Omega)))$ are 1-complemented subspaces. Therefore, from the classical duality between $L^p(\mathcal{M} \otimes B(L^2(\Omega)))$ and $L^q(\mathcal{M} \otimes B(L^2(\Omega)))$ ($\frac{1}{p} + \frac{1}{q} = 1, 1 \leq p < \infty$) we deduce that

$$\left(L^p(\mathcal{M}; L^2_c(\Omega))\right)^* = L^q(\mathcal{M}; L^2_c(\Omega))$$

and

$$\left(L^p(\mathcal{M}; L^2_r(\Omega))\right)^* = L^q(\mathcal{M}; L^2_r(\Omega))$$

isometrically via the antiduality

$$(f,g) \mapsto \tau(\langle g,f\rangle) = \tau \int_\Omega g^* f d\mu.$$

Moreover, it is well known that (by the same reason), for $0<\theta<1$ and $1 \leq p_0, p_1, p_\theta \leq \infty$ with $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we have isometrically

$$\left(L^{p_0}(\mathcal{M}; L_c^2(\Omega)), L^{p_1}(\mathcal{M}; L_c^2(\Omega))\right)_\theta = L^{p_\theta}(\mathcal{M}; L_c^2(\Omega)). \tag{2.3}$$

In the following, we are mainly interested in the spaces $L^p(\mathcal{M}; L_c^2(\Omega))$ (resp. $L^p(\mathcal{M}; L_r^2(\Omega))$) with $(\Omega,\mu) = \widetilde{\Gamma} = (\Gamma, dxdy) \times (\{1,2\},\sigma)$, where $\Gamma = \{(x,y) \in \mathbb{R}_+^2, |x| < y\}$, $\sigma\{1\} = \sigma\{2\} = 1$. (This cone Γ is a fundamental subject used in the classical harmonic analysis, see [6], [5], [21], [37] or any book on Hardy spaces). The presence of $\{1,2\}$ corresponds to our two variables x,y, see below. We then denote them by $L^p(\mathcal{M}, L_c^2(\widetilde{\Gamma}))$ (resp. $L^p(\mathcal{M}, L_r^2(\widetilde{\Gamma}))$). For simplicity, we will abbreviate them as $L^p(\mathcal{M}, L_c^2)$ (resp. $L^p(\mathcal{M}, L_r^2)$) if no confusion can arise.

2.2. Operator valued Hardy spaces

Let $1 \leq p < \infty$. For any $S_{\mathcal{M}}$-valued simple function f on $\mathbb{R}$, we also use f to denote its Poisson integral on the upper half plane $\mathbb{R}_+^2 = \{(x,y) | y>0\}$,

$$f(x,y) = \int_{\mathbb{R}} P_y(x-s) f(s) ds, \quad (x,y) \in \mathbb{R}_+^2\ ,$$

where $P_y(x)$ is the Poisson kernel (i.e. $P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$). Note that $f(x,y)$ is a harmonic function still with values in $S_{\mathcal{M}}$, and so in $\mathcal{M}$. Define the $\mathcal{H}_c^p(\mathbb{R},\mathcal{M})$ norm of f by

$$\|f\|_{\mathcal{H}_c^p} = \|\nabla f(x+t,y)\chi_\Gamma(x,y)\|_{L^p(L^\infty(\mathbb{R},dt)\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))}\ ,$$

where ∇f is the gradient of the Poisson integral $f(x,y)$ and $\widetilde{\Gamma}$ is defined as in the end of Section 2.1. In this dissertation, we will always regard $\nabla f(x+t,y)\chi_\Gamma(x,y)$ as functions defined on $\mathbb{R}\times\widetilde{\Gamma}$ with $t\in\mathbb{R},(x,y)\in\Gamma$ and

$$\nabla f(x+t,y)(1)=\frac{\partial f}{\partial x}(x+t,y),\quad \nabla f(x+t,y)(2)=\frac{\partial f}{\partial y}(x+t,y).$$

And set

$$|\nabla f(x+t,y)|^2=|\frac{\partial f}{\partial x}(x+t,y)|^2+|\frac{\partial f}{\partial y}(x+t,y)|^2.$$

Define the $\mathcal{H}_r^p(\mathbb{R},\mathcal{M})$ norm of f by

$$\|f\|_{\mathcal{H}_r^p}=\|\nabla f(x+t,y)\chi_\Gamma\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M},L_r^2)}\,.$$

Set $\mathcal{H}_c^p(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}_r^p(\mathbb{R},\mathcal{M})$) to be the completion of the space of all $S_\mathcal{M}$-valued simple function f's with finite $\mathcal{H}_c^p(\mathbb{R},\mathcal{M})$(resp. $\mathcal{H}_r^p(\mathbb{R},\mathcal{M})$) norm. Equipped respectively with the previous norms, $\mathcal{H}_c^p(\mathbb{R},\mathcal{M})$ and $\mathcal{H}_r^p(\mathbb{R},\mathcal{M})$ are Banach spaces. Define the non-commutative analogues of the classical Lusin integral by

$$S_c(f)(t) = (\iint_\Gamma |\nabla f(x+t,y)|^2dxdy)^{\frac{1}{2}} \tag{2.4}$$

$$S_r(f)(t) = (\iint_\Gamma |\nabla f^*(x+t,y)|^2dxdy)^{\frac{1}{2}}. \tag{2.5}$$

Note that

$$|\nabla f(x,y)|^2=\int_{\{1,2\}}|\nabla f(x,y)(i)|^2d\sigma(i).$$

Then, for $f\in\mathcal{H}_c^p(\mathbb{R},\mathcal{M})$,

$$\|f\|_{\mathcal{H}_c^p}=\|S_c(f)\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})}$$

and the similar equality holds for $\mathcal{H}_r^p(\mathbb{R},\mathcal{M})$. $S_c(f)$ and $S_r(f)$ are the non-commutative analogues of the classical Lusin square function. We will need the non-commutative analogues of the classical Littlewood-Paley g-function, which are defined by

$$G_c(f)(t) = (\int_{\mathbb{R}_+} |\nabla f(t,y)|^2 y dy)^{\frac{1}{2}} \tag{2.6}$$

$$G_r(f)(t) = (\int_{\mathbb{R}_+} |\nabla f^*(t,y)|^2 y dy)^{\frac{1}{2}} \tag{2.7}$$

We will see, in Chapters III and V, that

$$\|S_c(f)\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \backsimeq \|G_c(f)\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})}$$

$$\|S_r(f)\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \backsimeq \|G_r(f)\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})}$$

for all $1 \le p < \infty$.

Define the Hardy spaces of non-commutative functions f as follows: if $1 \le p < 2$,

$$\mathcal{H}^p_{cr}(\mathbb{R},\mathcal{M}) = \mathcal{H}^p_c(\mathbb{R},\mathcal{M}) + \mathcal{H}^p_r(\mathbb{R},\mathcal{M}) \tag{2.8}$$

equipped with the norm

$$\|f\|_{\mathcal{H}^p_{cr}} = \inf\{\|g\|_{\mathcal{H}^p_c} + \|h\|_{\mathcal{H}^p_r} : f = g + h, g \in \mathcal{H}^p_c(\mathbb{R},\mathcal{M}), h \in \mathcal{H}^p_r(\mathbb{R},\mathcal{M})\}$$

and if $2 \le p < \infty$,

$$\mathcal{H}^p_{cr}(\mathbb{R},\mathcal{M}) = \mathcal{H}^p_c(\mathbb{R},\mathcal{M}) \cap \mathcal{H}^p_r(\mathbb{R},\mathcal{M}) \tag{2.9}$$

equipped with the norm

$$\|f\|_{\mathcal{H}^p_{cr}} = \max\{\|f\|_{\mathcal{H}^p_c}, \|f\|_{\mathcal{H}^p_r}\}.$$

Remark. We have

$$\mathcal{H}^2_c(\mathbb{R},\mathcal{M}) = \mathcal{H}^2_r(\mathbb{R},\mathcal{M}) = \mathcal{H}^2_{cr}(\mathbb{R},\mathcal{M}) = L^2(L^\infty(\mathbb{R}) \otimes \mathcal{M}).$$

In fact, notice that $\triangle|f|^2 = 2|\nabla f|^2$ and $f(x,y)(|x|+y) \to 0, \nabla f(x,y)(|x|+y)^2 \to 0$

as $|x|+y\to 0$, for $S_{\mathcal{M}}$-valued simple function f's. By Green's theorem

$$\begin{aligned}
& ||\nabla f(t+x,y)\chi_\Gamma||^2_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M},L^2_c)} \\
= \ & 2\tau\iint\limits_{\mathbb{R}^2_+}|\nabla f|^2 y dx dy \\
= \ & \tau\iint\limits_{\mathbb{R}^2_+}\triangle|f|^2 y dx dy \\
= \ & \tau\int_{\mathbb{R}}|f|^2 ds = \|f\|^2_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})}.
\end{aligned} \tag{2.10}$$

Similarly, $||f||_{\mathcal{H}^2_r}=\|f^*\|_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})}=\|f\|_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})}$.

Note we have also the following polarized version of (2.10),

$$2\iint\limits_{\mathbb{R}^2_+}\nabla f(x,y)\nabla g(x,y) y dx dy=\int_{\mathbb{R}}f(s)g(s)ds \tag{2.11}$$

for $S_{\mathcal{M}}$-valued simple function f,g's.

We will repeatedly use the following consequence of the convexity of the operator valued function: $x\mapsto|x|^2$ (This convexity follows from the convexity of $x\mapsto\langle x^*xh,h\rangle=\|xh\|^2$ for any h). Letting $f:(\Omega,\mu)\to\mathcal{M}$ be a weak-* integrable function, we have

$$|\int_A f(t)d\mu(t)|^2\le\mu(A)\int_A|f(t)|^2 d\mu(t),\quad\forall A\subset\Omega \tag{2.12}$$

In particular, set $d\mu(t)=g^2(t)dt$,

$$|\int_A f(t)g(t)dt|^2\le\int_A|f(t)|^2dt\int_A g^2(t)dt,\quad\forall A\subset\mathbb{R} \tag{2.13}$$

for every measurable function g on $\mathbb{R}$, and

$$|\int_A f(t)dt|^2\le\int_A|f(t)|^2g^{-1}(t)dt\int_A g(t)dt,\quad\forall A\subset\mathbb{R} \tag{2.14}$$

for every positive measurable function g on $\mathbb{R}$.

Let $H^p(\mathbb{R})$ $(1 \leq p < \infty)$ denote the classical Hardy space on $\mathbb{R}$. It is well known that

$$H^p(\mathbb{R}) = \{f \in L^p(\mathbb{R}) : S(f) \in L^p(\mathbb{R})\},$$

where $S(f)$ is the classical Lusin integral function ($S(f)$ is equal to $S_c(f)$ above by taking $\mathcal{M} = \mathbb{C}$). In the following, $H^p(\mathbb{R})$ is always equipped with the norm $\|S(f)\|_{L^p(\mathbb{R})}$.

Proposition 2.2 *Let $1 \leq p < \infty$, $f \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ and $m \in L^q(\mathcal{M})$ (with q the index conjugate to p). Then $\tau(mf) \in H^p(\mathbb{R})$ and*

$$\|\tau(mf)\|_{H^p} \leq \|m\|_{L^q(\mathcal{M})} \|f\|_{\mathcal{H}_c^p}.$$

Proof. Note that

$$\nabla(\tau(mf) * P) = \tau(m(f * \nabla P)) = \tau(m\nabla f),$$

here P is the Poisson kernel (i.e. $P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$). By (2.13), we have

$$\begin{aligned}
&\|\tau(mf)\|_{H^p}^p \\
= &\int_{\mathbb{R}} (\iint_{\Gamma} |\tau(m\nabla f(x+t,y))|^2 dxdy)^{\frac{p}{2}} dt \\
\leq &\int_{\mathbb{R}} \sup_{\|g\|_{L^2(\tilde{\Gamma})}\leq 1} \left| \iint_{\Gamma} g\tau(m\nabla f(x+t,y))dxdy \right|^p dt \\
= &\int_{\mathbb{R}} \sup_{\|g\|_{L^2(\tilde{\Gamma})}\leq 1} \left| \tau\left[m \iint_{\Gamma} g_1 \frac{\partial f}{\partial x}(x+t,y) + g_2 \frac{\partial f}{\partial y}(x+t,y)dxdy \right] \right|^p dt \\
\leq &\int_{\mathbb{R}} \sup_{\|g\|_{L^2(\tilde{\Gamma})}\leq 1} \|m\|_{L^q(\mathcal{M})}^p \left\| \iint_{\Gamma} g_1 \frac{\partial f}{\partial x}(x+t,y) + g_2 \frac{\partial f}{\partial y}(x+t,y)dxdy \right\|_{L^p(\mathcal{M})}^p dt
\end{aligned}$$

$$\begin{aligned}
&\le \|m\|^p_{L^q(\mathcal{M})}\int_{\mathbb{R}}\sup_{\|g\|_{L^2(\widetilde{\Gamma})}\le 1}\left\|\Big(\iint_{\Gamma}|g|^2dxdy\Big)^{\frac{1}{2}}\Big(\iint_{\Gamma}|\nabla f(x+t,y)|^2dxdy\Big)^{\frac{1}{2}}\right\|^p_{L^p(\mathcal{M})}dt\\
&\le \|m\|^p_{L^q(\mathcal{M})}\tau\int_{\mathbb{R}}\Big(\iint_{\Gamma}|\nabla f(x+t,y)|^2dxdy\Big)^{\frac{p}{2}}dt\\
&= \|m\|^p_{L^q(\mathcal{M})}\ \|f\|^p_{\mathcal{H}^p_c}. \ \blacksquare
\end{aligned}$$

Remark. We should emphasize that for two functions g, f defined on $\widetilde{\Gamma}$, we always set

$$gf(z) = g(z)(1)f(z)(1) + g(z)(2)f(z)(2).$$

Then in the above formula $|\tau(m\nabla f(x+t,y))|^2$ and $g\tau(m\nabla f(x+t,y))$ etc. are functions defined on Γ. We will use very often such a product for ($\mathcal{M}$-valued) functions defined on $\widetilde{\Gamma}$.

Remark. (i) $\int f dt = 0, \forall f \in \mathcal{H}^1_c(\mathbb{R},\mathcal{M})$. In fact, if $f \in \mathcal{H}^1_c(\mathbb{R},\mathcal{M})$, by Proposition 1.2 and the classical property of H^1(see [37], p.128), we have $\tau(m\int f dt) = 0, \forall m \in \mathcal{M}$. Thus $\int f dt = 0$.

(ii) The collection of all $S_{\mathcal{M}}$-valued simple functions f such that $\int f dt = 0$ is a dense subset of $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})(1 < p < \infty)$. Note that

$$\lim_{N\to\infty}\left\|\frac{m}{N}\chi_{[-N,N]}(t)\right\|_{\mathcal{H}^p_c(\mathbb{R},\mathcal{M})} = 0, \quad \forall m \in S_{\mathcal{M}}.$$

For a simple function f, let $f_N = f - \frac{\int f dt}{N}\chi_{[-N,N]}$. Then $\int f_N = 0$ and $f_N \to f$ in $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$.

Remark. See [5] and [37] for discussions on the classical Lusin integral and the Littlewood-Paley g-function and the fact that a scalar valued function is in H^1 if and only if its Lusin integral is in L^1. We define the non-commutative Hardy spaces $\mathcal{H}^p_{cr}(\mathbb{R},\mathcal{M})$ differently for the case $1 \le p < 2$ and $p \ge 2$ (respectively by (2.8) and (2.9)) as Pisier and Xu did for non-commutative martingales in [18]. This is to get the expected equivalence between $\mathcal{H}^p_{cr}(\mathbb{R},\mathcal{M})$ and $L^p(\mathbb{R},\mathcal{M})$ for $1<p<\infty$ (see Chapter VI). And $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ or $\mathcal{H}^p_r(\mathbb{R},\mathcal{M})$ alone could be very far away from $L^p(\mathbb{R},\mathcal{M})$ for $p\neq 2$.

2.3. Operator valued BMO spaces

Now, we introduce the non-commutative analogue of BMO spaces. For any interval I on $\mathbb{R}$, we will denote its center by C_I and its Lebesgue measure by $|I|$. Let $\varphi \in L^\infty(\mathcal{M}, L^2_c(\mathbb{R}, \frac{dt}{1+t^2}))$. By Proposition 2.1 (and our convention), for every $g\in L^2(\mathbb{R},\frac{dt}{1+t^2})$, $\int_{\mathbb{R}} g\varphi\frac{dt}{1+t^2}\in\mathcal{M}$. Then the mean value of φ over I $\varphi_I := \frac{1}{|I|}\int_I \varphi(s)ds$ exists as an element in $\mathcal{M}$. And the Poisson integral of φ

$$\varphi(x,y)=\int_{\mathbb{R}} P_y(x-s)\varphi(s)ds$$

also exists as an element in $\mathcal{M}$. Set

$$\|\varphi\|_{\mathrm{BMO}_c}=\sup_{I\subset\mathbb{R}}\left\{\left\|\left(\frac{1}{|I|}\int_I|\varphi-\varphi_I|^2d\mu\right)^{\frac{1}{2}}\right\|_{\mathcal{M}}\right\} \tag{2.15}$$

where again $|\varphi-\varphi_I|^2=(\varphi-\varphi_I)^*(\varphi-\varphi_I)$ and the supremum runs over all intervals $I\subset\mathbb{R}$.(see Let H be the Hilbert space on which $\mathcal{M}$ acts. Obviously, we have

$$\|\varphi\|_{\mathrm{BMO}_c}=\sup_{e\in H,\|e\|=1}\|\varphi e\|_{\mathrm{BMO}_2(\mathbb{R},H)} \tag{2.16}$$

where $\mathrm{BMO}_2(\mathbb{R},H)$ is the usual H-valued BMO space on $\mathbb{R}$. Thus $\|\cdot\|_{\mathrm{BMO}_c}$ is a norm modulo constant functions. Set $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ to be the space of all $\varphi\in L^\infty(\mathcal{M},L^2_c(\mathbb{R},\frac{dt}{1+t^2}))$ such that $\|\varphi\|_{\mathrm{BMO}_c}<\infty$. $\mathrm{BMO}_r(\mathbb{R},\mathcal{M})$ is defined as the space of all φ's such that $\varphi^*\in\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ with the norm $\|\varphi\|_{\mathrm{BMO}_r}=\|\varphi^*\|_{\mathrm{BMO}_c}$. We define $\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M})$ as the intersection of these two spaces

$$\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M})=\mathrm{BMO}_c(\mathbb{R},\mathcal{M})\cap\mathrm{BMO}_r(\mathbb{R},\mathcal{M})$$

with the norm

$$\|\varphi\|_{\mathrm{BMO}_{cr}} = \max\{\|\varphi\|_{\mathrm{BMO}_c}, \|\varphi\|_{\mathrm{BMO}_r}\}.$$

As usual, the constant functions are considered as zero in these BMO spaces, and then these spaces are normed spaces (modulo constants).

Given an interval I, we denote by $2^k I$ the interval $\{t : |t - C_I| < 2^{k-1}|I|\}$. The technique used in the proof of the following Proposition is classical (see [37]).

Proposition 2.3 *Let* $\varphi \in \mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$. *Then*

$$\|\varphi\|_{L^\infty(\mathcal{M}, L^2_c(\mathbb{R}, \frac{dt}{1+t^2}))} \leq c(\|\varphi\|_{\mathrm{BMO}c} + \left\|\varphi_{I_1}\right\|_{\mathcal{M}})$$

where $I_1 = (-1, 1]$. *Moreover,* $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M}), \mathrm{BMO}_r(\mathbb{R}, \mathcal{M}), \mathrm{BMO}_{cr}(\mathbb{R}, \mathcal{M})$ *are Banach spaces.*

Proof. Let $\varphi \in \mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ and I be an interval. Using (2.12), (2.14) we have

$$\begin{aligned} |\varphi_{2^n I} - \varphi_I|^2 &\leq n \sum_{k=0}^{n-1} |\varphi_{2^k I} - \varphi_{2^{k+1} I}|^2 \\ &= n \sum_{k=0}^{n-1} \left| \frac{1}{|2^k I|} \int_{2^k I} (\varphi(s) - \varphi_{_{2^{k+1} I}}) ds \right|^2 \\ &\leq n \sum_{k=0}^{n-1} \frac{2}{|2^{k+1} I|} \int_{2^{k+1} I} |\varphi(s) - \varphi_{_{2^{k+1} I}}|^2 ds \\ &\leq 2n \|\varphi\|^2_{\mathrm{BMO}_c}. \end{aligned} \tag{2.17}$$

By (2.14), (2.17),

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \frac{|\varphi(t)|^2}{1+t^2} dt \right\|_{\mathcal{M}} \\ = &\left\| \int_{I_1} \frac{|\varphi(t)|^2}{1+t^2} dt + \sum_{k=0}^{\infty} \int_{2^{k+1} I_1 / 2^k I_1} \frac{|\varphi(t)|^2}{1+t^2} dt \right\|_{\mathcal{M}} \\ \leq &\ 2 \left\| \int_{I_1} (|\varphi(t) - \varphi_{_{I_1}}|^2 + |\varphi_{I_1}|^2) dt \right\|_{\mathcal{M}} \end{aligned}$$

$$+4\left\|\sum_{k=0}^{\infty}\int_{2^{k+1}I_1/2^kI_1}\frac{|\varphi(t)-\varphi_{_{2^{k+1}I_1}}|^2+|\varphi_{2^{k+1}I_1}-\varphi_{I_1}|^2+|\varphi_{I_1}|^2}{2^{2k}}dt\right\|_{\mathcal{M}}$$
$$\leq \ c(\left\||\varphi_{I_1}|^2\right\|_{\mathcal{M}}+\|\varphi\|^2_{\mathrm{BMO}_c}) \tag{2.18}$$

Thus

$$\|\varphi\|_{L^\infty(\mathcal{M},L^2_c(\mathbb{R},\frac{dt}{1+t^2}))}=\left\|(\int_{\mathbb{R}}\frac{|\varphi(t)|^2}{1+t^2}dt)^{\frac{1}{2}}\right\|_{\mathcal{M}}\leq c(\left\|\varphi_{I_1}\right\|_{\mathcal{M}}+\|\varphi\|_{\mathrm{BMO}_c})$$

And then $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ is complete. Consequently, $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$, $\mathrm{BMO}_r(\mathbb{R},\mathcal{M})$, $\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M})$ are Banach spaces. ∎

It is classical that BMO functions are related with Carleson measures(See [6], [21]). The same relation still holds in the present non-commutative setting. We say that an $\mathcal{M}$-valued measure $d\lambda$ on $\mathbb{R}^2_+$ is a Carleson measure if

$$N(\lambda)=\sup_I\left\{\frac{1}{|I|}\left\|\iint\limits_{T(I)}d\lambda\right\|_{\mathcal{M}}:I\in\mathbb{R}\text{ interval}\right\}<\infty,$$

where, as usual, $T(I)=I\times(0,|I|]$.

Lemma 2.4 *Let $\varphi\in\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$. Then $d\lambda\varphi=|\nabla\varphi|^2ydxdy$ is an $\mathcal{M}$-valued Carleson measure on $\mathbb{R}^2_+$ and $N(\lambda\varphi)\leq c\,\|\varphi\|^2_{\mathrm{BMO}_c}$.*

Proof. The proof is very similar to the scalar situation (see [37], p.160). For any interval I on $\mathbb{R}$, write $\varphi=\varphi_1+\varphi_2+\varphi_3$, where $\varphi_1=(\varphi-\varphi_{2I})\chi_{2I},\varphi_2=(\varphi-\varphi_{2I})\chi_{(2I)^c}$ and $\varphi_3=\varphi_{2I}$. Set

$$d\lambda_{\varphi_1}=|\nabla\varphi_1|^2ydxdy,d\lambda_{\varphi_2}=|\nabla\varphi_2|^2ydxdy.$$

Thus

$$N(\lambda_\varphi)\leq 2(N(\lambda_{\varphi_1})+N(\lambda_{\varphi_2})).$$

We treat $N(\lambda_{\varphi_1})$ first. Notice that $\triangle|\varphi_1|^2=2|\nabla\varphi_1|^2$ and $\varphi_1(x,y)(|x|+y)\rightarrow$

$0, \nabla\varphi_1(x,y)(|x|+y)^2 \to 0$ as $|x|+y \to 0$. By Green's theorem

$$\begin{aligned}\frac{1}{|I|}\left\|\iint\limits_{T(I)} |\nabla\varphi_1|^2 y dx dy\right\|_{\mathcal{M}} &\leq \frac{1}{|I|}\left\|\iint\limits_{\mathbb{R}_2^+} |\nabla\varphi_1|^2 y dx dy\right\|_{\mathcal{M}} \\ &= \frac{1}{2|I|}\left\|\int_{\mathbb{R}} |\varphi_1|^2 ds\right\|_{\mathcal{M}} \\ &= \frac{1}{2|I|}\left\|\int_{2I} |\varphi - \varphi_{2I}|^2 ds\right\|_{\mathcal{M}} \leq \|\varphi\|^2_{\mathrm{BMO}_c}\end{aligned} \tag{2.19}$$

To estimate $N(\lambda_{\varphi_1})$, we note

$$|\nabla P_y(x-s)|^2 \leq \frac{1}{4(x-s)^4} \leq \frac{1}{4|I|^4 2^{4k}}, \quad \forall s \in 2^{k+1}I/2^kI, \quad (x,y) \in T(I),$$

by (2.14) and (2.17)

$$\begin{aligned}&\frac{1}{|I|}\left\|\iint\limits_{T(I)} |\nabla\varphi_2|^2 y dx dy\right\|_{\mathcal{M}} \\ = \;& \frac{1}{|I|}\left\|\iint\limits_{T(I)} |\nabla \int_{-\infty}^{+\infty} P_y(x-s)\varphi_2(s) ds|^2 y dx dy\right\|_{\mathcal{M}} \\ \leq \;& \frac{1}{|I|}\iint\limits_{T(I)} \sum_{k=1}^{\infty} \int\limits_{2^{k+1}I/2^kI} |\nabla P_y(x-s)|^2 2^{2k} ds \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \left\|\int\limits_{2^{k+1}I} |\varphi_2|^2 ds\right\|_{\mathcal{M}} y dx dy \\ \leq \;& \frac{c}{|I|}\iint\limits_{T(I)} \frac{1}{|I|^2} \|\varphi\|^2_{\mathrm{BMO}_c} y dx dy \leq c\,\|\varphi\|^2_{\mathrm{BMO}_c}\end{aligned}$$

Therefore $N(\lambda_{\varphi_i}) \leq c\,\|\varphi\|^2_{\mathrm{BMO}_c}, i = 1, 2$, and then $N(\lambda_\varphi) \leq c\,\|\varphi\|^2_{\mathrm{BMO}_c}$. ∎

Remark. We will see later (Corollary 3.6) that the converse to lemma 2.4 is also true.

We will need the following elementary fact to make our later applications of Green's theorem rigorous in Chapters III and V.

Lemma 2.5 *Suppose* $\varphi \in \mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ *and suppose* I *is an interval such that* $\varphi_I =$

0. *Let* $3I$ *be the interval concentric with* I *having length* $3|I|$. *Then there is* $\psi \in \mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ *such that* $\psi = \varphi$ *on* $I, \psi = 0$ *on* $\mathbb{R} \backslash 3I$ *and*

$$\|\psi\|_{\mathrm{BMO}_c} \leq c \, \|\varphi\|_{\mathrm{BMO}_c} .$$

Proof. This is well known for the classical BMO and a proof is outlined in [6], p. 269. One can check that the method to construct ψ mentioned there works as well for $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$. ∎

Remark. We have seen that the non-commutative $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ are well adapted to many generalizations of classical results, such as Proposition 2.3 and Lemma 2.4, 2.5. We will also prove an analogue of the classical Fefferman duality between $\mathcal{H}^1$ and BMO in the next chapter. However, unlike the classical case, we could not replace the power 2 by p in the definition of the non-commutative BMO norm ((2.15)). In fact, $\sup_{I \subset \mathbb{R}} \left\| \left(\frac{1}{|I|} \int_I |\varphi - \varphi_I|^p d\mu \right)^{\frac{1}{p}} \right\|_{\mathcal{M}}$ may not be a norm for $p \neq 2$ in the non-commutative case (Note we do not have $|x_1 + x_2| \leq |x_1| + |x_2|$ in general for $x_1, x_2 \in \mathcal{M}$). See the remark at the end of Chapter VIII for more information.

CHAPTER III

THE DUALITY BETWEEN $\mathcal{H}^1$ AND **BMO**

The main result (Theorem 3.4) of this chapter is the analogue in our setting of the famous Fefferman duality theorem between H^1 and BMO.

3.1. The bounded map from $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c)$ to $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$

As in the classical case, we will embed $\mathcal{H}^1_c(\mathbb{R},\mathcal{M})$ into a larger space $L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c)$, which requires the following maps Φ,Ψ.

Definition 3.1 *We define a map Φ from $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ $(1\le p<\infty)$ to $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ by*

$$\Phi(f)(x,y,t)=\nabla f(x+t,y)\chi_\Gamma(x,y)$$

and a map Ψ for a sufficiently nice $h\in L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ $(1\le p\le\infty)$ by

$$\Psi(h)(s)=\int_{\mathbb{R}}\iint_\Gamma h(x,y,t)Q_y(x+t-s)dydxdt;\quad \forall s\in\mathbb{R} \tag{3.1}$$

where, $Q_y(x)$ is defined as a function on $\mathbb{R}\times\widetilde{\Gamma}$ by

$$Q_y(x)(1)=\frac{\partial P_y(x)}{\partial x},\quad Q_y(x)(2)=\frac{\partial P_y(x)}{\partial y};\forall(x,y)\in\Gamma. \tag{3.2}$$

Note that Φ is simply the natural embedding of $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ into $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$. On the other hand, Ψ is well defined for sufficiently nice h, more precisely "nice" will mean that $h(x,y,t)=\sum_{i=1}^n m_if_i(t)\chi_{A_i}$ with $m_i\in S_{\mathcal{M}}, A_i\in\widetilde{\Gamma}, |A_i|<\infty$ and with scalar valued simple functions f_i. In this case, it is easy to check that $\Psi(h)\in L^p(\mathcal{M}, L^2_c(\mathbb{R},\frac{dt}{1+t^2}))$.

We will prove that Ψ extends to a bounded map from $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ to $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ (see Lemma 3.2) and also from $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ to $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$

for all $1 < p < \infty$ (see Theorem 5.8). The following proposition, combined with Theorem 5.8 in Chapter V, implies that Ψ is a projection of $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ onto $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ if we identify $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ with a subspace of $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ via Φ.

Proposition 3.1 *For any $f\in\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ $(1\le p<\infty)$,*

$$\Psi\Phi(f) = f$$

Proof. We have

$$\begin{aligned}&\int_{-\infty}^{+\infty}\iint\limits_{\Gamma}\Phi(f)\nabla g(t+x,y)dydxdt\\ = \ &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\iint\limits_{\Gamma}\Phi(f)Q_y(x+t-s)dydxdtg(s)ds.\end{aligned}$$

On the other hand, by (2.11) we have

$$\int_{-\infty}^{+\infty}\iint\limits_{\Gamma}\Phi(f)\nabla g(t+x,y)dydxdt = \int_{-\infty}^{+\infty} f(s)g(s)ds$$

for every g good enough. Therefore

$$\int_{-\infty}^{+\infty}\iint\limits_{\Gamma}\Phi(f)Q_y(x+t-s)dydxdt = f(s)$$

almost everywhere. This is $\Psi\Phi(f) = f$. ∎

We can also prove $\Psi\Phi(\varphi)=\varphi$ by showing directly the Poisson integral of $\Psi\Phi(\varphi)$ coincides with that of φ, namely

$$\int_{\mathbb{R}}\Psi\Phi(\varphi)(w)P_v(u-w)dw = \int_{\mathbb{R}}\varphi(w)P_v(u-w)dw,\quad \forall(u,v)\in\mathbb{R}^2_+. \tag{3.3}$$

Indeed, using elementary properties of the Poisson kernel, we have

$$\int_{\mathbb{R}}\Psi\Phi(\varphi)(h)P_v(u-h)dh$$

$$
\begin{aligned}
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\iint\limits_{\Gamma}\int_{\mathbb{R}} \varphi(s)\nabla P_y(x+t-s)ds\nabla P_y(x+t-h)dydxdtP_v(u-h)dh \\
&= \int_{\mathbb{R}}\varphi(s)\iint\limits_{\Gamma}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\partial}{\partial y}P_y(x+t-s)\frac{\partial}{\partial y}P_y(x+t-h)P_v(u-h)dtdhdxdyds \\
&\quad + \int_{\mathbb{R}}\varphi(s)\iint\limits_{\Gamma}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\partial}{\partial x}P_y(x+t-s)\frac{\partial}{\partial x}P_y(x+t-h)P_v(u-h)dtdhdxdyds \\
&= \int_{\mathbb{R}}\varphi(s)\int_{\mathbb{R}}\iint\limits_{\mathbb{R}^2_+}\frac{\partial}{\partial y}P_y(x-s)\frac{\partial}{\partial y}P_y(x-h)2ydydxP_v(u-h)dhds \\
&\quad + \int_{\mathbb{R}}\varphi(s)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\partial}{\partial s}P_y(x-s)\frac{\partial}{\partial u}P_{y+v}(x-u)2ydxdyds \\
&= \int_{\mathbb{R}}\varphi(s)\int_0^{\infty}2y\frac{\partial^2}{\partial v^2}P_{v+2y}(u-s)dyds - \int_{\mathbb{R}}\varphi(s)\int_0^{\infty}2y\frac{\partial^2}{\partial u^2}P_{v+2y}(u-s)dyds \\
&= \int_{\mathbb{R}}\varphi(s)\int_0^{\infty}y\frac{\partial^2}{\partial y^2}P_{v+2y}(u-s)dyds \\
&= \int_{\mathbb{R}}\varphi(s)(0-\int_0^{\infty}\frac{\partial}{\partial y}P_{v+2y}(u-s)dy)ds \\
&= \int_{\mathbb{R}}\varphi(s)P_v(u-s)ds. \ \blacksquare
\end{aligned}
$$

Lemma 3.2 *Ψ extends to a bounded map from $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ to $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ of norm controlled by a universal constant.*

Proof. Let $\mathcal{S}$ be the family of all $L^\infty(\mathbb{R})\otimes\mathcal{M}$-valued simple functions h which can written as $h(x,y,t) = \sum_{i=1}^n m_i f_i(t)\chi_{A_i}(x,y)$ with $m_i \in S_{\mathcal{M}},\ f_i \in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ and compact $A_i \subset \widetilde{\Gamma}$. (By compact A_i we mean that the two components of A_i are compact subsets in Γ.) Note that $\mathcal{S}$ is w*-dense in $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ (in fact, the unit ball of $\mathcal{S}$ is w*-dense in the unit ball of $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma})))$. We will first show that

$$||\Psi(h)||_{\mathrm{BMO}_c} \leq c\,\|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L^2_c)}\ ,\ \forall\, h\in\mathcal{S}. \tag{3.4}$$

Fix $h \in \mathcal{S}$ and let $\varphi = \Psi(h)$. Then $\varphi \in L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ by Proposition 2.3. To estimate the BMO$_c$-norm of φ, we fix an interval I and set $h = h_1 + h_2$ with

$$\begin{aligned} h_1(x,y,t) &= h(x,y,t)\chi_{2I}(t) \\ h_2(x,y,t) &= h(x,y,t)\chi_{(2I)^c}(t). \end{aligned}$$

Let

$$B_I = \int_{-\infty}^{+\infty} \iint\limits_{\Gamma} Q_I h_2 dydxdt$$

with the notation $Q_I(x,t) = \frac{1}{|I|}\int_I Q_y(x+t-s)ds$. Now

$$\begin{aligned} & \frac{1}{|I|}\int_I |\varphi(s) - B_I|^2 ds \\ \leq\ & \frac{2}{|I|}\int_I \Big| \int_{(2I)^c} \iint\limits_{\Gamma} (Q_y(x+t-s) - Q_I)hdxdydt\Big|^2 ds \\ & + \frac{2}{|I|}\int_I \Big| \int_{-\infty}^{+\infty} \iint\limits_{\Gamma} Q_y(x+t-s)h_1 dxdydt\Big|^2 ds \\ =\ & A + B \end{aligned}$$

Notice that

$$\begin{aligned} \iint\limits_{\Gamma} |Q_y(x+t-s) - Q_I|^2 dxdy &\leq c\iint\limits_{\Gamma} \Big(\frac{|I|}{(|x+t-s|+y)^3}\Big)^2 dxdy \\ &\leq c|I|^2(t - C_I)^{-4} \end{aligned} \tag{3.5}$$

for every $t \in (2I)^c$ and $s \in I$. By (2.14)

$$\left|\iint\limits_{\Gamma} (Q_y(x+t-s) - Q_I)hdxdy\right|^2 \leq c|I|^2(t-C_I)^{-4}\iint\limits_{\Gamma} h^*hdxdy$$

and by (2.14) again,

$$\|A\|_{\mathcal{M}}$$

$$
\begin{aligned}
&\leq \quad c|| \int_{(2I)^c} (t-C_I)^{-2} dt \int_{(2I)^c} (t-C_I)^2 \iint\limits_{\Gamma} h^* h dx dy |I|^2 (t-C_I)^{-4} dt ||_{\mathcal{M}} \\
&\leq \quad || \frac{c}{|I|} \int_{(2I)^c} |I|^2 (t-C_I)^{-2} \iint\limits_{\Gamma} h^* h dx dy dt ||_{\mathcal{M}} \\
&\leq \quad c \left\| h \right\|^2_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c)}
\end{aligned}
$$

For the second term B, we have

$$
\begin{aligned}
& \left\| B \right\|_{\mathcal{M}} \\
\leq \quad & \frac{2}{|I|} || \int_{\mathbb{R}} | \int_{\mathbb{R}} \iint\limits_{\Gamma} Q_y(x+t-s) h_1 dx dy dt|^2 ds ||_{\mathcal{M}} \\
= \quad & \frac{2}{|I|} \sup_{\tau|a|=1} \tau(|a| \int_{\mathbb{R}} | \int_{\mathbb{R}} \iint\limits_{\Gamma} Q_y(x+t-s) h_1 dx dy dt|^2 ds) \\
= \quad & \frac{2}{|I|} \sup_{\tau|a|=1} \tau \int_{\mathbb{R}} | \int_{\mathbb{R}} \iint\limits_{\Gamma} Q_y(x+t-s) h_1 |a|^{\frac{1}{2}} dx dy dt|^2 ds \\
= \quad & \frac{2}{|I|} \sup_{\tau|a|=1} \sup_{||f||_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})}=1} (\tau \int_{\mathbb{R}} f(s) \int_{\mathbb{R}} \iint\limits_{\Gamma} Q_y(x+t-s) h_1 |a|^{\frac{1}{2}} dx dy dt ds)^2 \\
= \quad & \frac{2}{|I|} \sup_{\tau|a|=1} \sup_{||f||_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})}=1} (\tau \int_{\mathbb{R}} \iint\limits_{\Gamma} \nabla f(t+x,y) h_1 |a|^{\frac{1}{2}} dx dy dt)^2
\end{aligned}
$$

Hence by Cauchy-Schwartz inequality and (2.10)

$$
\begin{aligned}
\left\| B \right\|_{\mathcal{M}} \quad &\leq \quad \frac{2}{|I|} \sup_{\tau|a|=1} \tau \int_{\mathbb{R}} \iint\limits_{\Gamma} h_1^* h_1 |a| dx dy dt \\
&\leq \quad \frac{2}{|I|} || \int_{\mathbb{R}} \iint\limits_{\Gamma} h_1^* h_1 dx dy dt ||_{\mathcal{M}} \\
&= \quad \frac{2}{|I|} || \int_{2I} \iint\limits_{\Gamma} h^* h dx dy dt ||_{\mathcal{M}} \\
&\leq \quad 4 \left\| h \right\|^2_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c)}
\end{aligned}
$$

Thus

$$
||\varphi||_{\mathrm{BMO}_c} \leq c \left\| h \right\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c)} .
$$

In particular, by Proposition 2.3,

$$\|\varphi\|_{L^\infty(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2}))} \leq c\|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2)}.$$

Thus we have proved the boundedness of Ψ from the w*-dense vector subspace $\mathcal{S}$ of $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma}))$ to $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$. Now we extend Ψ to the whole $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma}))$. To this end we first extend Ψ to a bounded map from $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma}))$ into $L^\infty(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2}))$. By the discussion above, Ψ is also bounded from $\mathcal{S}$ to $L^\infty(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2}))$. Let H_0^1 be the subspace of all $f\in H^1(\mathbb{R})$ such that $(1+t^2)f(t)\in L^2(\mathbb{R})$. Let $L^1(\mathcal{M})\otimes H_0^1$ denote the algebraic tensor product of $L^1(\mathcal{M})$ and H_0^1. Note that

$$L^1(\mathcal{M})\otimes H_0^1\subset\mathcal{H}_c^1(\mathbb{R},\ \mathcal{M}),\quad L^1(\mathcal{M})\otimes H_0^1\subset L^1(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2}))$$

and $L^1(\mathcal{M})\otimes H_0^1$ is dense in both of the latter spaces. Moreover, it is easy to see that for any $h\in\mathcal{S}$ and $f\in L^1(\mathcal{M})\otimes H_0^1$

$$\tau\int_{-\infty}^{+\infty}\iint\limits_{\Gamma} h^*(x,y,t)\nabla f(t+x,y)dydxdt=\tau\int_{-\infty}^{+\infty}\Psi(h)^*(s)f(s)ds.$$

Then it follows that Ψ is continuous from $(\mathcal{S},\ \sigma(\mathcal{S},L^1(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma}))))$ to $(L^\infty(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2})),\ \sigma(L^\infty(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2})),L^1(\mathcal{M})\otimes H_0^1))$.

Now given $f\in L^1(\mathcal{M})\otimes H_0^1$ we define $\Psi_*(f):\mathcal{S}\to\mathbb{C}$ by

$$\Psi_*(f)(h)=\tau\int_{-\infty}^{+\infty}\Psi(h)^*(s)f(s)ds.$$

Then $\Psi_*(f)$ is an anti-linear functional on $\mathcal{S}$ continuous with respect to the w*-topology; hence $\Psi_*(f)$ extends to a w*-continuous anti-linear functional on $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma})))$, i.e. an element in $L^1(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma})))$, still denoted by $\Psi_*(f)$. By the w*-density of $\mathcal{S}$ in $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma})))$, this extension is unique. Therefore, we have defined a map

$$\Psi_* : L^1(\mathcal{M}) \otimes H_0^1 \to L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma})).$$

The above uniqueness of the extension $\Psi_*(f)$ for any given f implies that Ψ_* is linear. On the other hand, by what we already proved in the previous part, we have

$$\begin{aligned} |\Psi_*(f)(h)| &\leq \|f\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))} \|\Psi(h)\|_{L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))} \\ &\leq c\, \|f\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))} \|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2)} \ . \end{aligned}$$

Since the unit ball of $\mathcal{S}$ is w*-dense in the unit ball of $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma})))$, it follows that

$$\Psi_* : (L^1(\mathcal{M}) \otimes H_0^1, \|\cdot\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))}) \to L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))$$

is bounded and its norm is majorized by c. This, together with the density of $L^1(\mathcal{M}) \otimes H_0^1$ in $L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ implies that Ψ_* extends to a unique bounded map from $L^1(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ into $L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma})))$, still denoted by Ψ_*. Consequently, the adjoint $(\Psi_*)^*$ of Ψ_* is bounded from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma})))$ to $L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ (noting that this adjoint is taken with respect to the anti-dualities). By the very definition of Ψ_*, we have

$$(\Psi_*)^*|_{\mathcal{S}} = \Psi.$$

This shows that $(\Psi_*)^*$ is an extension of Ψ from $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))$ to $L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$, which we denote by Ψ again. Being an adjoint, Ψ is w*-continuous.

It remains to show that the so extended map Ψ really takes values in $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$. Given a bounded interval $I \subset \mathbb{R}$, the w*-topology of $L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ induces a topology in $L^\infty(\mathcal{M}, L_c^2(I))$ equivalent to the w*-topology in $L^\infty(\mathcal{M}, L_c^2(I))$. Then by the w*-continuity of Ψ, we deduce that, for every $\varepsilon > 0, I \subset \mathbb{R}, f \in L^1(\mathcal{M}, L_c^2(I))$, there exists a $h \in \mathcal{S}$ such that

$$\begin{aligned} & \tau \int_I f^*(\Psi(g)(t) - \Psi(g)_I)dt \\ \leq & \ \tau \int_I f^*(\Psi(h)(t) - \Psi(h)_I)dt + \varepsilon \\ \leq & \ \|\Psi(h)(t) - \Psi(h)_I\|_{L^\infty(\mathcal{M}, L_c^2(I))} \|f\|_{L^1(\mathcal{M}, L_c^2(I))} + \varepsilon \end{aligned} \tag{3.6}$$

and

$$\|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))} \leq \|g\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))} + \varepsilon \tag{3.7}$$

Combining (3.6), (3.7) and (3.4) we get

$$\begin{aligned} & \int_I f^*(\Psi(g)(t) - \Psi(g)_I)dt \\ \leq & \ c|I| \, \|h\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))} \|f\|_{L^1(\mathcal{M}, L_c^2(I))} + \varepsilon \\ \leq & \ c|I|(\|g\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))} + \varepsilon) \|f\|_{L^1(\mathcal{M}, L_c^2(I))} + \varepsilon \end{aligned}$$

By letting $\varepsilon \to 0$ and taking supremum over all $\|f\|_{L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))} \leq 1$ and $I \subset \mathbb{R}$, we get $\Psi(g) \in \mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ and

$$||\Psi(g)||_{\mathrm{BMO}_c} \leq c \, \|g\|_{L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)} .$$

Therefore, we have extended Ψ to a bounded map from $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))$ to $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$, thus completing the proof of the lemma. ∎

Remark. We sketch an alternate proof of the fact that $\varphi = \Psi(h)$ is in $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ for $h \in \mathcal{S}$. Let H be the Hilbert space on which $\mathcal{M}$ acts. Recall that $\mathcal{M}_*$ is a quotient space of $B(H)_*$ by the preannihilator of $\mathcal{M}$. Denote the quotient map by q. For every $a, b \in H$, denote $[a \otimes b] = q(a \otimes b)$. Note that $\tau(m^*[a \otimes b]) = \tau([m^*(a \otimes b)]) = \langle m(b), \overline{a} \rangle, \forall m \in \mathcal{M}$. From (2.16) and the classical duality between $\mathrm{BMO}(\mathbb{R}, H)$ and

$H^1(\mathbb{R}, H)$,

$$\begin{aligned}
||\varphi||_{\mathrm{BMO}_c(\mathbb{R},\mathcal{M})} &= \sup_{e\in H,\|e\|_H=1} ||\varphi e||_{\mathrm{BMO}(\mathbb{R},H)}. \\
&\leq c \sup_{e\in H,\|e\|_H=1} \sup_{\|g\|_{H^1(\mathbb{R},H)}=1} \left| \int_{-\infty}^{+\infty} \langle \varphi(e), \overline{g} \rangle \, dt \right| \\
&= c \sup_{e\in H,\|e\|_H=1} \sup_{\|g\|_{H^1(\mathbb{R},H)}=1} \left| \tau \int_{-\infty}^{+\infty} \varphi^*[g\otimes e] dt \right| \qquad (3.8)
\end{aligned}$$

Let $f = [g \otimes e]$. Noting that

$$|\nabla f|^2 = \langle \nabla g, \nabla g \rangle \, [e\otimes e] = |\nabla g|^2 [e\otimes e],$$

we get

$$\tau\left(S_c(f)(t)\right) = (\iint\limits_{\Gamma} |\nabla g(t+x,y)|^2 \, dxdy)^{\frac{1}{2}}. \qquad (3.9)$$

Thus $\|f\|_{\mathcal{H}_c^1(\mathbb{R},\mathcal{M})} = 1$ if $\|g\|_{H^1(\mathbb{R},H)} = 1$ and $\|e\|_H = 1$. Therefore

$$\begin{aligned}
||\varphi||_{\mathrm{BMO}_c(\mathbb{R},\mathcal{M})} &\leq c \sup_{\|f\|_{\mathcal{H}_c^1(\mathbb{R},\mathcal{M})}=1} \left| \tau \int_{-\infty}^{+\infty} \varphi^* f dt \right| \\
&= c \sup_{\|f\|_{\mathcal{H}_c^1(\mathbb{R},\mathcal{M})}=1} \left| \tau \int_{-\infty}^{+\infty} \iint\limits_{\Gamma} h^*(x,y,t) \nabla f(t+x,y) dydxdt \right| \\
&\leq c \, \|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2)}.
\end{aligned}$$

Corollary 3.3 *Let $f \in L^1(\mathcal{M}, L_c^2(\mathbb{R}, (1+s^2)ds))$ with $\int f ds = 0$. Then $f \in \mathcal{H}_c^1(\mathbb{R},\mathcal{M})$ and*

$$\|f\|_{\mathcal{H}_c^1} \leq c \, \|f\|_{L^1(\mathcal{M}, L_c^2(\mathbb{R},(1+s^2)ds))}$$

Proof. By Lemma 3.2, the assumption that $\int f ds = 0$ and Proposition 2.3, we have

$$\begin{aligned}
\|f\|_{\mathcal{H}_c^1} &= ||\nabla f(t+x,y)\chi_\Gamma||_{L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2)} \\
&= \sup_{\|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2)}\leq 1} \left| \tau \int \iint\limits_{\Gamma} h^* \nabla f(t+x,y) dxdydt \right|
\end{aligned}$$

$$\begin{aligned}
&= \sup_{\|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L^2_c)}\le 1} \left|\tau\int_{\mathbb{R}}(\Psi(h))^*(s)f(s)ds\right| \\
&\le c \sup_{\|\varphi\|_{\mathrm{BMO}_c(\mathbb{R},\mathcal{M})}\le 1} \left|\tau\int_{\mathbb{R}}\varphi^*(s)f(s)ds\right| \\
&\le c \sup_{\|\varphi\|_{L^\infty(\mathcal{M},L^2_c(\mathbb{R},\frac{ds}{1+s^2}))}\le 1} \left|\tau\int_{\mathbb{R}}\varphi^*(s)(1+s^2)f(s)\frac{ds}{1+s^2}\right| \\
&\le c\left\|(1+s^2)f(s)\right\|_{L^1(\mathcal{M},L^2_c(\mathbb{R},\frac{ds}{1+s^2}))} \\
&= c\left\|f\right\|_{L^1(\mathcal{M},L^2_c(\mathbb{R},(1+s^2)ds))}. \blacksquare
\end{aligned}$$

Remark. In particular, every $S_{\mathcal{M}}$-valued simple function f with $\int f ds = 0$ is in $\mathcal{H}^1_c(\mathbb{R},\mathcal{M})$. Consequently, by the remark before Proposition 2.3, $\mathcal{H}^1_c(\mathbb{R},\mathcal{M})\cap\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ is dense in $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ $(p>1)$.

3.2. The duality theorem of operator valued $\mathcal{H}^1$ and BMO

Denote by $\mathcal{H}^1_{c0}(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}^1_{r0}(\mathbb{R},\mathcal{M})$) the family of functions f in $\mathcal{H}^1_c(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}^1_r(\mathbb{R},\mathcal{M}),\mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M})$) such that $f\in L^1(\mathcal{M},L^2_c(\mathbb{R},(1+t^2)dt))$ (resp. $L^1(\mathcal{M}, L^2_r(\mathbb{R},(1+t^2)dt))$. It is easy to see that $\mathcal{H}^1_{c0}(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}^1_{r0}(\mathbb{R},\mathcal{M})$) is a dense subspace of $\mathcal{H}^1_c(\mathbb{R},\mathcal{M})$ (resp. $\mathcal{H}^1_r(\mathbb{R},\mathcal{M})$)). Let

$$\mathcal{H}^1_{cr0}(\mathbb{R},\mathcal{M})=\mathcal{H}^1_{c0}(\mathbb{R},\mathcal{M})+\mathcal{H}^1_{r0}(\mathbb{R},\ \mathcal{M}).$$

Then $\mathcal{H}^1_{cr0}(\mathbb{R},\mathcal{M})$ is a dense subspace of $\mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M})$. Recall that we have proved in Chapter II that $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})\subseteq L^\infty(\mathcal{M},L^2_c(\mathbb{R},\frac{dt}{1+t^2}))$. Thus by Proposition 1.1 $\langle\varphi,f\rangle=\int_{-\infty}^{+\infty}\varphi^* f dt$ exists in $L^1(\mathcal{M})$ for all $\varphi\in\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ and $f\in\mathcal{H}^1_{c0}(\mathbb{R},\mathcal{M})$ (see our convention after Proposition 2.1).

Theorem 3.4 *(a) We have* $(\mathcal{H}^1_c(\mathbb{R},\mathcal{M}))^*=\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ *with equivalent norms.*

More precisely, every $\varphi \in \mathrm{BMO}_c(\mathcal{M})$ *defines a continuous linear functional on* $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ *by*

$$l\varphi(f) = \tau \int_{-\infty}^{+\infty} \varphi^* f dt; \qquad \forall f \in \mathcal{H}_{c0}^1(\mathbb{R}, \mathcal{M}). \tag{3.10}$$

Conversely, every $l \in (\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}))^*$ *can be given as above by some* $\varphi \in \mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$*. Moreover, there exists a universal constant* $c > 0$ *such that*

$$c^{-1} \|\varphi\|_{\mathrm{BMO}_c} \leq \|l\varphi\|_{(\mathcal{H}_c^1)^*} \leq c \|\varphi\|_{\mathrm{BMO}_c} \ .$$

Thus $(\mathcal{H}_c^1(\mathbb{R}, \mathcal{M}))^* = \mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ *with equivalent norms.*

(b) Similarly, $(\mathcal{H}_r^1(\mathbb{R}, \mathcal{M}))^* = \mathrm{BMO}_r(\mathbb{R}, \mathcal{M})$ *with equivalent norms.*

(c) $(\mathcal{H}_{cr}^1(\mathbb{R}, \mathcal{M}))^* = \mathrm{BMO}_{cr}(\mathbb{R}, \mathcal{M})$ *with equivalent norms.*

Our proof of Theorem 3.4 requires two technical variants of the square functions $G_c(f)$ and $S_c(f)$. These are operator valued functions defined as follows:

$$G_c(f)(x, y) = (\int_y^\infty |\nabla f(x, s)|^2 s ds)^{\frac{1}{2}}, \tag{3.11}$$

$$S_c(f)(x, y) = (\iint_{\Gamma(0,y)} |\nabla f(t + x, s)|^2 dt ds)^{\frac{1}{2}} \tag{3.12}$$

where $y \geq 0, \Gamma(0, y) = \{(t, s) : |t| < s - y, s \geq y\}$ and f is $S_{\mathcal{M}}$-valued simple function. Note that $G_c(f)(x, 0)$ and $S_c(f)(x, 0)$ are just $G_c(f)$ and $S_c(f)$ defined in Chapter II.

Lemma 3.5

$$G_c(f)(x, y) \leq 2\sqrt{2} S_c(f)(x, \frac{y}{2}) \ .$$

Proof. It suffices to prove this inequality for $x = 0$. Let us denote by B_s the ball centered at $(0, s)$ and tangent to the boundary of $\Gamma(0, \frac{y}{2}), \forall s > y$. By the harmonicity of ∇f, we get

$$\nabla f(0, s) = \frac{2}{\pi (s - \frac{y}{2})^2} \int_{B_s} \nabla f(x, u) dx du$$

By (2.12),

$$|\nabla f(0,s)|^2 \leq \frac{8}{\pi s^2}\int_{B_s}|\nabla f(x,u)|^2 dxdu$$

Integrating this inequality, we obtain

$$\int_y^\infty s|\nabla f(0,s)|^2 ds \leq \int_y^\infty \frac{8}{\pi s}\int_{B_s}|\nabla f(x,u)|^2 dxduds \tag{3.13}$$

However $(x,u) \in B_s$ clearly implies that $\frac{u}{2} \leq s \leq 4u$. Thus, the right hand side of (3.13) is majorized by

$$\int_{\Gamma(0,\frac{y}{2})}|\nabla f(x,u)|^2 \int_{\frac{u}{2}}^{4u}\frac{8}{\pi s}dsdxdu \leq 8S_c^2(f)(0,\frac{y}{2})$$

Therefore $G_c(f)(0,y) \leq 2\sqrt{2}S_c(f)(0,\frac{y}{2})$. ∎

Proof of Theorem 3.4. (i) We will first prove

$$|l_\varphi(f)| \leq c\,\|\varphi\|_{\mathrm{BMO}_c}\,\|f\|_{\mathcal{H}_c^1} \tag{3.14}$$

when both f and φ have compact support. Once this is done, by Lemma 2.5, we can see (3.14) holds for any $\varphi \in \mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ and any compactly supported $f \in \mathcal{H}_{c0}^1(\mathbb{R},\mathcal{M})$. Then recall that by Proposition 2.3

$$\mathrm{BMO}_c(\mathbb{R},\mathcal{M}) \subset L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$$

and by Corollary 3.3

$$\|f\|_{\mathcal{H}_c^1} \leq c\,\|f\|_{L^1(\mathcal{M},L_c^2(\mathbb{R},(1+t^2)dt))}\,,\ \forall f \in \mathcal{H}_{c0}^1(\mathbb{R},\mathcal{M}),$$

we deduce (3.14) for all $\varphi \in \mathrm{BMO}_c(\mathbb{R},\mathcal{M}), f \in \mathcal{H}_{c0}^1(\mathbb{R},\mathcal{M})$ by choosing compactly supported $f_n \in \mathcal{H}_{c0}^1(\mathbb{R},\mathcal{M}) \to f$ in $L^1(\mathcal{M}, L_c^2(\mathbb{R},(1+t^2)dt))$. Finally, from the density of $\mathcal{H}_{c0}^1(\mathbb{R},\mathcal{M})$ in $\mathcal{H}_c^1(\mathbb{R},\mathcal{M})$, l_φ defined in (3.10) extends to a continuous functional on $\mathcal{H}_c^1(\mathbb{R},\mathcal{M})$.

Let us now prove (3.14) for compactly supported $f \in \mathcal{H}^1_{c0}(\mathbb{R}, \mathcal{M})$ and compactly supported $\varphi \in \mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$. By approximation we may assume that τ is finite and $G_c(f)(x,y)$ is invertible in $\mathcal{M}$ for every $(x,y) \in \mathbb{R}^2_+$. Recall that $\triangle(\varphi^* f) = 2\nabla\varphi^*\nabla f$. By Green's theorem and the Cauchy-Schwarz inequality

$$\begin{aligned}
& |l\varphi(f)| \\
= \ & 2|\tau \iint_{\mathbb{R}^2_+} \nabla\varphi^*\nabla f y dy dx| \\
\leq \ & 2(\tau \iint_{\mathbb{R}^2_+} G_c^{-\frac{1}{2}}(f)|\nabla f|^2 G_c^{-\frac{1}{2}}(f) y dy dx)^{\frac{1}{2}} (\tau \iint_{\mathbb{R}^2_+} G_c^{\frac{1}{2}}(f)|\nabla\varphi|^2 G_c^{\frac{1}{2}}(f) y dy dx)^{\frac{1}{2}} \\
= \ & 2(\tau \iint_{\mathbb{R}^2_+} G_c^{-1}(f)|\nabla f|^2 y dy dx)^{\frac{1}{2}} (\tau \iint_{\mathbb{R}^2_+} G_c(f)|\nabla\varphi|^2 y dy dx)^{\frac{1}{2}} \\
= \ & 2I \bullet II,
\end{aligned}$$

Note here $G_c(f)$ is the function of two variables defined by (3.11), which is differentiable in the weak-* sense. For I we have

$$\begin{aligned}
I^2 &= \tau \int_{-\infty}^{+\infty} \int_0^{\infty} -G_c^{-1}(f)\frac{\partial G_c^2(f)}{\partial y} dy dx \\
&= \tau \int_{-\infty}^{+\infty} \int_0^{\infty} (-G_c^{-1}(f)\frac{\partial G_c(f)}{\partial y} G_c(f) - \frac{\partial G_c(f)}{\partial y}) dy dx \\
&= 2\tau \int_{-\infty}^{+\infty} \int_0^{\infty} -\frac{\partial G_c(f)}{\partial y} dy dx \\
&= 2\tau \int_{-\infty}^{+\infty} G_c(f)(x,0) dx \\
&\leq 4\sqrt{2}\tau \int_{-\infty}^{+\infty} S_c(f)(x,0) dx \\
&= 4\sqrt{2}\, \|f\|_{\mathcal{H}^1_c}.
\end{aligned}$$

To estimate II, we create a square net partition in $\mathbb{R}^2_+$ as follows:

$$\sigma(i,j) = \{(x,y) : (i-1)2^j < x \leq i2^j, 2^j \leq y < 2^{j+1}\}, \quad \forall i,j \in \mathbb{Z}.$$

Let $C_{i,j}$ denote the center of $\sigma(i,j)$. Define

$$\begin{aligned}\widetilde{S}_c(f)(x,y) &= S_c(f)(C_{i,j}), \quad \forall (x,y)\in\sigma(i,j),\\ d_k(x) &= \widetilde{S}_c(f)(x,2^k)-\widetilde{S}_c(f)(x,2^{k+1}), \quad \forall x\in\mathbb{R}.\end{aligned}$$

It is easy to check that

$$\begin{aligned}S_c(f)(x,2y) &\leq \widetilde{S}_c(f)(x,y)\leq S_c(f)(x,\frac{y}{2}),\\ d_k(x) &\geq 0, \quad \forall x\in\mathbb{R},\\ \widetilde{S}_c(f)(x,y) &= \sum_{k=j}^{\infty} d_k(x), \quad \forall 2^j\leq y<2^{j+1},\\ S_c(f)(x,0) &= \sum_{k=-\infty}^{\infty} d_k(x). \qquad (3.15)\end{aligned}$$

Now by Lemma 3.5 and (3.15)

$$\begin{aligned}II^2 &= \tau\int_{-\infty}^{+\infty}\int_0^{\infty} G_c(f)(x,y)|\nabla\varphi|^2 ydydx\\ &\leq 2\sqrt{2}\tau\int_{-\infty}^{+\infty}\int_0^{\infty}\widetilde{S}_c(f)(x,\frac{y}{4})|\nabla\varphi|^2 ydydx\\ &= 2\sqrt{2}\tau\int_{-\infty}^{+\infty}\sum_{k=-\infty}^{\infty}\widetilde{S}_c(f)(x,2^k)\int_{2^{k+2}}^{2^{k+3}}|\nabla\varphi|^2 ydydx\\ &= 2\sqrt{2}\tau\int_{-\infty}^{+\infty}\sum_{k=-\infty}^{\infty}(\sum_{j=k}^{\infty} d_j(x))\int_{2^{k+2}}^{2^{k+3}}|\nabla\varphi|^2 ydydx\\ &= 2\sqrt{2}\tau\int_{-\infty}^{+\infty}\sum_{j=-\infty}^{\infty} d_j(x)\int_0^{2^{j+3}}|\nabla\varphi|^2 ydydx\\ &= 2\sqrt{2}\tau\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} d_j(i2^j)\int_{(i-1)2^j}^{i2^j}\int_0^{2^{j+3}}|\nabla\varphi|^2 ydydx\end{aligned}$$

Hence by Lemma 2.4

$$II^2 \leq c\tau\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} d_j(i2^j)2^j\,\|\varphi\|^2_{\mathrm{BMO}_c}$$

$$
\begin{aligned}
&= c\,\|\varphi\|^2_{\mathrm{BMO}_c}\,\tau \sum_{j=-\infty}^{\infty} \int_{-\infty}^{+\infty} d_j(x)dx \\
&= c\,\|\varphi\|^2_{\mathrm{BMO}_c}\,\tau \int_{-\infty}^{+\infty} S_c(f)(x,0)dx \\
&= c\,\|\varphi\|^2_{\mathrm{BMO}_c}\,\|f\|_{\mathcal{H}^1_c}\,.
\end{aligned}
$$

Combining the preceding estimates on I and II, we get

$$|l\varphi(f)| \leq c\,\|\varphi\|_{\mathrm{BMO}_c}\,\|f\|_{\mathcal{H}^1_c}\,.$$

Therefore, $l\varphi$ defines a continuous functional on $\mathcal{H}^1_c$ of norm smaller than $c\,\|\varphi\|_{\mathrm{BMO}_c}\,.$

(ii) Now suppose $l \in (\mathcal{H}^1_c(\mathbb{R},\mathcal{M}))^*$. Then by the Hahn-Banach theorem l extends to a continuous functional on $L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ of the same norm. Thus by

$$(L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma})))^* = L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$$

there exists $g \in L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ such that

$$||g||^2_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L^2_c(\widetilde{\Gamma}))} = \sup_{t\in\mathbb{R}} || \iint\limits_{\Gamma} g^*(x,y,t)g(x,y,t)dydx||_{L^\infty(\mathbb{R})\otimes\mathcal{M}} = ||l||^2$$

and

$$l(f) = \tau \int_{-\infty}^{+\infty} \iint\limits_{\Gamma} g^*(x,y,t)\nabla f(t+x,y)dydxdt,\ \forall\ f \in \mathcal{H}^1_{c0}(\mathbb{R},\mathcal{M}).$$

Let $\varphi = \Psi(g)$, where Ψ is the extension given by Lemma 2.2. By that lemma $\varphi \in \mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ and

$$||\varphi||_{\mathrm{BMO}_c} \leq c||g||_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M},L^2_c(\widetilde{\Gamma}))} = c\|l\|.$$

Then we must show that

$$l(f) = \tau \int_{-\infty}^{+\infty} \varphi^*(s)f(s)ds,\ \forall\ f \in \mathcal{H}^1_{c0}(\mathbb{R},\mathcal{M}).$$

But this follows from the second part of the proof of Lemma 3.2 in virtue of the w*-continuity of Ψ. Therefore, we have accomplished the proof of the theorem concerning $\mathcal{H}_c^1(\mathbb{R},\mathcal{M})$ and $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$. Passing to adjoints yields the part on $\mathcal{H}_r^1(\mathbb{R},\mathcal{M})$ and BMO_r. Finally, the duality between $\mathcal{H}_{cr}^1(\mathbb{R},\mathcal{M})$ and $\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M})$ is obtained by the classical fact that the dual of a sum is the intersection of the duals. ∎

Corollary 3.6 $\varphi \in \mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ *if and only if* $d\lambda\varphi = |\nabla\varphi|^2 ydxdy$ *is an* $\mathcal{M}$*-valued Carleson measure on* $\mathbb{R}_+^2$*, and* $c^{-1}N(\lambda\varphi) \leq \|\varphi\|_{\mathrm{BMO}_c}^2 \leq cN(\lambda\varphi)$*.*

Proof. From the first part of the proof of Theorem 3.4, if φ is such that $d\lambda_\varphi = |\nabla\varphi|^2 ydxdy$ is an $\mathcal{M}$-valued Carleson measure, then φ defines a continuous linear functional $l_\varphi = \tau\int_{-\infty}^{+\infty}\varphi^* f dt$ on $\mathcal{H}_{c0}^1(\mathbb{R},\mathcal{M})$ and

$$\|l_\varphi\|_{(\mathcal{H}_c^1)^*} \leq cN^{\frac{1}{2}}(\lambda_\varphi)$$

Therefore by Theorem 3.4 again there exists a function $\varphi' \in \mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ with $\|\varphi'\|_{\mathrm{BMO}_c}^2 \leq \|l_\varphi\|_{(\mathcal{H}_c^1)^*}^2 \leq cN(\lambda_\varphi)$ such that

$$\tau\int_{-\infty}^{+\infty}\varphi^* f dt = \tau\int_{-\infty}^{+\infty}\varphi'^* f dt.$$

Thus $\varphi = \varphi'$ and $\varphi \in \mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ with $\|\varphi\|_{\mathrm{BMO}_c}^2 \leq cN(\lambda_\varphi)$. The converse had been already proved in Lemma 2.4. ∎

Corollary 3.7 *For* $f \in \mathcal{H}_c^1(\mathbb{R},\mathcal{M})$*, we have*

$$c^{-1}\left\|G_c(f)\right\|_1 \leq \left\|S_c(f)\right\|_1 \leq c\left\|G_c(f)\right\|_1$$

Proof. By Theorem 3.4 and the first part of its proof, we have

$$\left\|S_c(f)\right\|_1 = \|f\|_{\mathcal{H}_c^1} \leq c \sup_{\|\varphi\|_{\mathrm{BMO}_c}=1}\left|\tau\int f\varphi^* dt\right| \leq c\left\|G_c(f)\right\|_1^{\frac{1}{2}}\left\|S_c(f)\right\|_1^{\frac{1}{2}}$$

Therefore

$$\|S_c(f)\|_1 \leq c\,\|G_c(f)\|_1$$

The converse is an immediate consequence of Lemma 3.5. ∎

Remark. The technique used in the proof of Lemma 3.5 is classical (see [38]). The method to prove Theorem 3.4 is inspired by the analogous one for martingales (see [7], [10], [33]).

3.3. The atomic decomposition of operator valued $\mathcal{H}^1$

As in the classical case, the duality between $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ and $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ implies an atomic decomposition of $\mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$. The rest of this chapter is devoted to this atomic decomposition. We say that a function $a \in L^1(\mathcal{M}, L_c^2(\mathbb{R}))$ is an $\mathcal{M}_c$-atom if

(i) a is supported in a bounded interval I;

(ii) $\int_I a dt = 0$;

(iii) $\tau(\int_I |a|^2 dt)^{\frac{1}{2}} \leq |I|^{-\frac{1}{2}}$.

Let $\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M})$ be the space of all f which admit a representation of the form

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i,$$

where the a_i's are $\mathcal{M}_c$-atoms and $\lambda_i \in \mathbb{C}$ are such that $\sum_{i \in \mathbb{N}} |\lambda_i| < \infty$. We equip $\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M})$ with the following norm

$$\|f\|_{\mathcal{H}_c^{1,at}} = \inf\{\sum_{i \in \mathbb{N}} |\lambda_i|; f = \sum_{i \in \mathbb{N}} \lambda_i a_i; a_i \text{ are } \mathcal{M}_c\text{-atoms}, \lambda_i \in \mathbb{C}\}$$

Similarly, we define $\mathcal{H}_r^{1,at}(\mathbb{R}, \mathcal{M})$. Then we set

$$\mathcal{H}_{cr}^{1,at}(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M}) + \mathcal{H}_r^{1,at}(\mathbb{R}, \mathcal{M}).$$

Theorem 3.8 *$\mathcal{H}_c^{1,at}(\mathbb{R}, \mathcal{M}) = \mathcal{H}_c^1(\mathbb{R}, \mathcal{M})$ with equivalent norms.*

Proof. It is enough to prove $(\mathcal{H}_c^{1,at}(\mathbb{R},\mathcal{M}))^* = \mathrm{BMO}_c(\mathbb{R},\mathcal{M})$. Now, for any $\varphi \in \mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ and $f \in \mathcal{H}_c^{1,at}(\mathbb{R},\mathcal{M})$ with $f = \sum_{i\in\mathbb{N}} \lambda_i a_i$ as above, by the Cauchy-Schwartz inequality we have

$$\begin{aligned} |\tau \int \varphi^* f dt| &\leq \sum_{i\in\mathbb{N}} |\lambda_i \tau \int_{I_i} (\varphi - \varphi_{I_i})^* a_i dt| \\ &\leq \sum_{i\in\mathbb{N}} |\lambda_i| \tau (\int_{I_i} |a_i|^2 dt)^{\frac{1}{2}} \left\| (\int_{I_i} |\varphi - \varphi_{I_i}|^2 dt)^{\frac{1}{2}} \right\|_{\mathcal{M}} \\ &\leq \|\varphi\|_{\mathrm{BMO}_c} \sum_{i\in\mathbb{N}} |\lambda_i|. \end{aligned}$$

Thus $\mathrm{BMO}_c(\mathbb{R},\mathcal{M}) \subset (\mathcal{H}_c^{1,at}(\mathbb{R},\mathcal{M}))^*$ (a contractive inclusion). To prove the converse inclusion, we denote by $L_0^1(\mathcal{M}, L_c^2(I))$ the space of functions $f \in L^1(\mathcal{M}, L_c^2(I))$ with $\int f dt = 0$. Notice that $L_0^1(\mathcal{M}, L_c^2(I)) \in \mathcal{H}_c^{1,at}(\mathbb{R},\mathcal{M})$ for every bounded I. Thus, every continuous functional l on $\mathcal{H}_c^{1,at}(\mathbb{R},\mathcal{M})$ induces a continuous functional on $L_0^1(\mathcal{M}, L_c^2(I))$ with norm smaller than $|I|^{\frac{1}{2}} \|l\|_{(\mathcal{H}_c^{1,at})^*}$. Consequently, we can choose a sequence $(\varphi_n)_{n\geq 1}$ satisfying the following conditions:

$$l(a) = \tau \int \varphi_n^* a dt, \quad \forall \mathcal{M}_c\text{- atom } a \text{ with } \operatorname{supp} a \subset (-n, n],$$

$$\|\varphi_n\|_{L^\infty(\mathcal{M}, L_c^2((-n,n]))} \leq c\sqrt{n} \|l\|_{(\mathcal{H}_c^{1,at})^*};$$

$$\varphi_n|_{(-m,m]} = \varphi_m, \ \forall n > m.$$

Let $\varphi(t) = \varphi_n(t), \forall t \in (-n, -n+1] \cup (n-1, n], n > 0$. We then have $\varphi \in L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ and

$$l(a) = \tau \int \varphi^* a dt, \quad \forall \mathcal{M}_c\text{- atom } a.$$

Considering $[g \otimes e]$ as defined in the remark after Lemma 2.2, by (3.8) and (3.9) we have

$$\|\varphi\|_{\mathrm{BMO}_c} \leq c \sup_{e\in H, \|e\|_H = 1} \sup_{\|g\|_{H^1(\mathbb{R},H)}=1} \left| \tau \int_{-\infty}^{+\infty} \varphi^* [g \otimes e] dt \right|$$

$$\begin{aligned} &\leq \sup_{\|f\|_{\mathcal{H}_c^{1,at}}=1} \left| \tau \int_{-\infty}^{+\infty} \varphi^* f dt \right| \\ &= \|l\|_{(\mathcal{H}_c^{1,at})^*} . \blacksquare \end{aligned}$$

Corollary 3.9 *$\mathcal{H}_r^{1,at}(\mathbb{R},\mathcal{M}) = \mathcal{H}_r^1(\mathbb{R},\mathcal{M})$ and $\mathcal{H}_{cr}^{1,at}(\mathbb{R},\mathcal{M}) = \mathcal{H}_{cr}^1(\mathbb{R},\mathcal{M})$ with equivalent norms.*

Remark. The $\mathcal{M}$-atom considered in this section is a non-commutative analogue of the classical 2-atom for H^1 space. It seems difficult to consider the non-commutative analogues of the classical $p-$atom for $p \neq 2$.

Remark. We only considered the functions defined on $\mathbb{R}$ in this chapter. However, one can check that all the proofs work well for the functions defined on $\mathbb{R}^n$. And the analogous results can be proved similarly for the functions defined on $\mathbb{T}^n$, where $\mathbb{T}$ is the unit circle. Moreover, the relevant constants are independent of n.

CHAPTER IV

THE MAXIMAL INEQUALITY

4.1. The non-commutative Hardy-Littlewood maximal inequality

We recall the definition of the noncommutative maximal norm introduced by Pisier (see [32]) and Junge (see [14]). Let $0 < p \leq \infty$, and let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of elements in $L^p(\mathcal{M})$. Set

$$\left\|\sup_{n\in\mathbb{Z}} |a_n|\right\|_{L^p(\mathcal{M})} = \inf_{a_n = a y_n b} \|a\|_{L^{2p}(\mathcal{M})} \|b\|_{L^{2p}(\mathcal{M})} \sup_n \|y_n\|_{\mathcal{M}} \tag{4.1}$$

where the infimum is taken over all $a, b \in L_{2p}(\mathcal{M})$ and all bounded sequences $(y_n)_{n\in\mathbb{Z}} \in \mathcal{M}$ such that $a_n = a y_n b$. By convention, if $(a_n)_{n\in\mathbb{Z}}$ does not have such a representation , we define $\|\sup_{n\in\mathbb{Z}} |a_n|\|_{L^p(\mathcal{M})}$ as $+\infty$.

If $p \geq 1$ and $(a_n)_{n\in\mathbb{Z}}$ is a sequence of positive elements, it was proved by Junge and Xu (see [14], Remark 3.7; [19], Proposition 2.1) that (with q the index conjugate to p)

$$\left\|\sup_{n\in\mathbb{Z}} |a_n|\right\|_{L^p(\mathcal{M})} = \sup\left\{\sum_{n\in\mathbb{Z}} \tau(a_n b_n) : b_n \in L^q(\mathcal{M}), b_n \geq 0, \left\|\sum_{n\in\mathbb{Z}} b_n\right\|_{L^q(\mathcal{M})} \leq 1\right\}. \tag{4.2}$$

In this case, $\|\sup_{n\in\mathbb{Z}} |a_n|\|_{L^p(\mathcal{M})} < \infty$ if and only if there exists $a \in L^p(\mathcal{M}), a > 0$ and a sequence of positive contractions y_n such that $a_n = a^{\frac{1}{2}} y_n a^{\frac{1}{2}}$, $\forall n \in \mathbb{Z}$, and moreover,

$$\left\|\sup_n |a_n|\right\|_{L^p(\mathcal{M})} = \inf\{||a||_{L^p(\mathcal{M})} : a > 0, a_n \leq a, \forall n \in \mathbb{Z}\}.$$

We define similarly $\|\sup_{\lambda\in\Lambda} |a(\lambda)|\|_p$ if Λ is a countable set. If Λ is uncountable we set

$$\left\|\sup_{\lambda\in\Lambda} |a(\lambda)|\right\|_{L^p(\mathcal{M})} = \sup_{(\lambda_n)_{n\in\mathbb{Z}}\in\Lambda} \left\|\sup_{n\in\mathbb{Z}} |a(\lambda_n)|\right\|_{L^p(\mathcal{M})}. \tag{4.3}$$

Please note that $\sup_\lambda |a(\lambda)|$ does not make any sense in the noncommutative setting and $\|\sup_{\lambda\in\Lambda} |a(\lambda)|\|_{L^p(\mathcal{M})}$ is just a notation. Also note that

$$\left\|\sup_{\lambda\in\Lambda} |a(\lambda)|\right\|_{L^\infty(\mathcal{M})} = \sup_{\lambda\in\Lambda} \|a(\lambda)\|_{L^\infty(\mathcal{M})}. \tag{4.4}$$

and for $1 \leq p \leq \infty$,

$$\left\|\sup_{\lambda\in\Lambda} |a(\lambda)|\right\|_{L^p(\mathcal{M})} = \sup_{J\subset\Lambda \text{ finite}} \left\|\sup_{n\in J} |a(\lambda_n)|\right\|_{L^p(\mathcal{M})}. \tag{4.5}$$

The main result of this chapter is the non-commutative Hardy-Littlewood maximal inequality. We will reduce it to the non-commutative Doob maximal inequality for martingales already established by M. Junge [9]. To this end, we need to introduce two increasing filtration of dyadic $\sigma-$algebras on $\mathbb{R}$. The key property of these $\sigma-$algebras is that any interval of $\mathbb{R}$ is contained in an atom belonging to one of these $\sigma-$algebras with a comparable size (see Proposition 3.1 below). This approach is very simple. And we will need it later when prove $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ is the intersection of two dyadic BMO spaces. That is one of the reasons that we do not follow the classical ways to dominate Hardy-Littlewood maximal functions by the correspondent dyadic ones.

The two increasing filtrations of dyadic $\sigma-$algebras $\mathcal{D} =\{\mathcal{D}_n\}_{n\in\mathbb{Z}}, \mathcal{D}' = \{\mathcal{D}'_n\}_{n\in\mathbb{Z}}$ that we will need are defined as follows: The first one, $\mathcal{D} =\{\mathcal{D}_n\}_{n\in\mathbb{Z}}$, is simply the usual dyadic filtration, that is, $\mathcal{D}_n$ is the $\sigma-$algebra generated by the atoms

$$D_n^k = (k2^{-n}, (k+1)2^{-n}]; \quad k \in \mathbb{Z}.$$

The definition of $\mathcal{D}' = \{\mathcal{D}'_n\}_{n\in\mathbb{Z}}$ is a little more complicated. For an even integer n, the atoms of $\mathcal{D}'_n$ are given by

$$D_n'^k = ((k+\frac{1}{3})2^{-n}, (k+\frac{4}{3})2^{-n}], \quad k \in \mathbb{Z};$$

while for an odd integer n, $\mathcal{D}'_n$ is generated by the atoms

$$D'^k_n = ((k+\frac{2}{3})2^{-n}, (k+\frac{5}{3})2^{-n}], \quad k \in \mathbb{Z}.$$

It is easy to see that $\mathcal{D}' = \{\mathcal{D}'_n\}_{n\in\mathbb{Z}}$ is indeed an increasing filtration.

The following simple observation is the key of our approach.

Proposition 4.1 *For any interval $I \subset \mathbb{R}$, there exist $k_I, N \in \mathbb{Z}$ such that $I \subset D^{k_I}_N$ and $|D^{k_I}_N| \leq 6|I|$ or $I \subset D'^{k_I}_N$ and $|D'^{k_I}_N| \leq 6|I|$, the constant N only depends on the length of I.*

Proof. To see this, choose $N \in \mathbb{Z}$ such that $\frac{2^{-N-1}}{3} \leq |I| < \frac{2^{-N}}{3}$. Denote

$$A_N = \{(k2^{-N}); k \in \mathbb{Z}\}, \quad A'_N = \{((k+\frac{1}{3})2^{-N}, (k+\frac{2}{3})2^{-N}); k \in \mathbb{Z}\}.$$

Note that for any two points $a, b \in A_N \cup A'_N$, we have $|a-b| \geq \frac{1}{3}2^{-N} > |I|$. Thus there is no more than one element of $A_N \cup A'_N$ in I. Then $I \cap A_N = \phi$ or $I \cap A'_N = \phi$. Therefore, I must be contained in some $D^{k_I}_N$ or $D'^{\,k_I}_N$. ∎

Remark. See [24] for a generalization of Proposition 4.1.

Remark. If an $\mathcal{M}_c$-atom defined in Chapter III admits its supporting interval as D^k_N (resp. $D'^{\,k}_N$) for some $k, N \in \mathbb{Z}$, we call it $\mathcal{M}_c$-$\mathcal{D}$-atom (resp. $\mathcal{M}_c$-$\mathcal{D}'$-atom). Proposition 4.1 implies that an $\mathcal{M}_c$-atom is either an $\mathcal{M}_c$-$\mathcal{D}$-atom or an $\mathcal{M}_c$–$\mathcal{D}'$-atom up to a fixed factor. Therefore the atomic Hardy space $\mathcal{H}^{1,at}_c(\mathbb{R}, \mathcal{M})$ defined in Chapter III can be characterized only by $\mathcal{M}_c$-$\mathcal{D}$-atoms and $\mathcal{M}_c$-$\mathcal{D}'$-atoms. A similar remark applies to the atomic row Hardy space $\mathcal{H}^{1,at}_r(\mathbb{R}, \mathcal{M})$. See Chapter VI for more results of this type.

The proof of the following Proposition (as well as that of Theorem 3.3) illustrates well our approach to reduce problems on functions to those on martingales. Put

$$f_h(t) = \frac{1}{h_1 + h_2} \int_{t-h_1}^{t+h_2} f(x)dx, \ \forall h = (h_1, h_2) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Proposition 4.2 *Let $(a_n)_{n\in\mathbb{Z}}$ be a positive sequence in $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $h_n = (h_{n,1}, h_{n,2}) \in \mathbb{R}^+ \times \mathbb{R}^+, n \in \mathbb{Z}$.*

(i) If $1 \leq p < \infty$,

$$\left\| \sum_{n\in\mathbb{Z}} (a_n)_{h_n} \right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq c_p \left\| \sum_{n\in\mathbb{Z}} a_n \right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})}. \tag{4.6}$$

(ii) If $1 < p \leq \infty$,

$$\left\| \sup_{n\in\mathbb{Z}} |(a_n)_{h_n}| \right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq c_p \left\| \sup_{n\in\mathbb{Z}} |a_n| \right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})}. \tag{4.7}$$

Proof. From Proposition 4.1, $\forall n \in \mathbb{Z}$, for every $t \in \mathbb{R}$, there exist some $k_t, N_n \in \mathbb{Z}$ such that $(t - h_{n,1}, t + h_{n,2})$ is contained in $D_{N_n}^{k_t}$ or ${D'}_{N_n}^{k_t}$ and

$$|D_{N_n}^{k_t}| = |{D'}_{N_n}^{k_t}| \leq 6(h_{n,1} + h_{n,2}).$$

Thus

$$(a_n)_{h_n} \leq 6(E(a_n|\mathcal{D}_{N_n}) + E(a_n|\mathcal{D}'_{N_n})), \quad \forall n \in \mathbb{Z}, \tag{4.8}$$

where $E(\cdot\ |\mathcal{D}_{N_n})$(resp. $E(\cdot\ |\mathcal{D}'_{N_n})$) denotes the conditional expectation with respect to $\mathcal{D}_{N_n}$ (resp. $\mathcal{D}'_{N_n}$). Then (4.6) follows from Theorem 0.1 of [14]. By (4.2) and (4.6),

$$\begin{aligned}
&\left\| \sup_{n\in\mathbb{Z}} |(a_n)_{h_n}| \right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
= \ & \sup\{\sum_{n\in\mathbb{Z}} \tau \int_{\mathbb{R}} \frac{1}{h_{n,1} + h_{n,2}} \int_{t-h_{n,1}}^{t+h_{n,2}} a_n(x)dx b_n(t)dt : \left\| \sum_{n\in\mathbb{Z}} b_n \right\|_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq 1\} \\
= \ & \sup\{\sum_{n\in\mathbb{Z}} \tau \int_{\mathbb{R}} \frac{1}{h_{n,1} + h_{n,2}} \int_{x-h_{n,2}}^{x+h_{n,1}} b_n(t)dt a_n(x)dx : \left\| \sum_{n\in\mathbb{Z}} b_n \right\|_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq 1\}
\end{aligned}$$

$$\leq \sup\{\sum_{n\in\mathbb{Z}} \tau \int_{\mathbb{R}} b_n(x)a_n(x)dx : \left\|\sum_{n\in\mathbb{Z}} b_n\right\|_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq c_p\}$$

$$\leq c_p \left\|\sup_{n\in\mathbb{Z}} |a_n|\right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})}$$

This is (4.7). ∎

The following is our non-commutative Hardy-Littlewood maximal inequality. Denote by $\mathcal{P}(\mathcal{M})$ the family of all projections of a von Neumann algebra $\mathcal{M}$.

Theorem 4.3 *(i) Let $f \in L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $\lambda > 0$. Then there exists $e^\lambda \in \mathcal{P}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that*

$$\sup_{h\in\mathbb{R}^+\times\mathbb{R}^+} \left\|e^\lambda f_h e^\lambda\right\|_{L^\infty(\mathbb{R})\otimes\mathcal{M}} \leq \lambda, \quad \left[\tau \otimes \int\right](1-e^\lambda) < \frac{c_1 \left\|f\right\|_1}{\lambda}. \tag{4.9}$$

(ii) Let $1 < p \leq \infty$ and $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. Then

$$\left\|\sup_{h\in\mathbb{R}^+\times\mathbb{R}^+} |f_h|\right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq c_p \left\|f\right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})}. \tag{4.10}$$

Moreover, for every positive $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, there exists a positive $F \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that $f_h \leq F$ for all h and

$$\left\|F\right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq c_p \left\|f\right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})}. \tag{4.11}$$

Proof. By decomposing $f = f_1 - f_2 + i(f_3 - f_4)$ with positive f_k, we can assume f positive. To prove (i), for given $f, \lambda, (h_n)_{n\in\mathbb{Z}} \in \mathbb{R}^+\times\mathbb{R}^+$, let $\mathcal{D}_{N_n}, \mathcal{D}'_{N_n}$ be as in the proof of Proposition 3.2. By the weak type (1,1) inequality of non-commutative martingales in [3] we have $\forall\lambda > 0$, $\exists e^\lambda, e'^\lambda \in \mathcal{P}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that

$$\sup_n \left\|e^\lambda E(f|\mathcal{D}_{N_n})e^\lambda\right\|_{L^\infty(\mathbb{R})\otimes\mathcal{M}} \leq \frac{\lambda}{12}, \quad \tau \otimes \int (1-e^\lambda) < \frac{c\left\|f\right\|_1}{\lambda}$$

and

$$\sup_n \left\|e^\lambda E(f|\mathcal{D}'_{N_n})e^\lambda\right\|_{L^\infty(\mathbb{R})\otimes\mathcal{M}} \leq \frac{\lambda}{12}, \quad \tau \otimes \int (1-e'^\lambda) < \frac{c\left\|f\right\|_1}{\lambda}$$

for every $f \in L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $(h_n)_{n\in\mathbb{Z}} \in \mathbb{R}^+ \times \mathbb{R}^+$. Let $\widetilde{e^\lambda} = e^\lambda \wedge e'^\lambda$, then

$$\tau \otimes \int (1 - \widetilde{e^\lambda}) < \frac{2c \, \|f\|_1}{\lambda}.$$

By Proposition 4.1, we have

$$\widetilde{e^\lambda} f_{h_n} \widetilde{e^\lambda} \leq 6(e^\lambda E(f|\mathcal{D}_{N_n})e^\lambda + e'^\lambda E(f|\mathcal{D}'_{N_{h_n}})e'^\lambda).$$

Therefore,

$$\begin{aligned}
& \sup_{h \in \mathbb{R}^+ \times \mathbb{R}^+} \left\| \widetilde{e^\lambda} f_h \widetilde{e^\lambda} \right\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\
= & \sup_{(h_n)_{n\in\mathbb{Z}}} \sup_n \left\| \widetilde{e^\lambda} f_{h_n} \widetilde{e^\lambda} \right\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\
\leq & \ 6 \sup_n \left\| e'^\lambda E(f|\mathcal{D}'_{N_n}) e'^\lambda \right\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} + 6 \sup_n \left\| e^\lambda E(f|\mathcal{D}_{N_n}) e^\lambda \right\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\
\leq & \ \lambda.
\end{aligned}$$

This is (4.9). To prove (4.10), consider the two filtrations $\mathcal{D}, \mathcal{D}'$ introduced above. By Theorem 0.2 of [14], there exist two positive $F_1, F_2 \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that $\|F_1\|_{L^p}, \|F_2\|_{L^p} \leq c_p \|f\|_{L^p}$, and

$$E(f|\mathcal{D}_n) \leq F_1, \qquad \text{and} \qquad E(f|\mathcal{D}'_n) \leq F_2, \quad \forall n \in \mathbb{Z}. \tag{4.12}$$

Thus, similar to (4.8), we have (by Proposition 4.1), for every $h \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$f_h \leq 6(F_1 + F_2) \tag{4.13}$$

Let $F = 6(F_1 + F_2)$, we proved (4.11). (4.10) follows immediately by decomposing $f = f_1 - f_2 + i(f_3 - f_4)$ with positive f_k. ∎

Using standard arguments and Theorem 3.3 we can easily obtain the non-commutative analogue of the classical non-tangential maximal inequality. Recall, as in Chapter II, we also use f to denote its Poisson integral on the upper half plane.

Theorem 4.4 *(i) Let $f \in L^1(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. Then $\forall \lambda > 0, \exists\, e^\lambda \in \mathcal{P}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, such that*

$$\sup_{(t,y)\in\Gamma} \left\| e^\lambda f(x+t,y) e^\lambda \right\|_{L^\infty(\mathbb{R})\otimes\mathcal{M}} \le \lambda, \quad \tau \otimes \int (1-e^\lambda) < \frac{c_1 \|f\|_1}{\lambda}, \forall \lambda > 0 \tag{4.14}$$

(ii) Let $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}), 1 < p \le \infty$. Then

$$\left\| \sup_{(t,y)\in\Gamma} |f(x+t,y)| \right\|_p \le c_p \|f\|_p . \tag{4.15}$$

Moreover, for every positive $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, there exists a positive $F \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that $f(\cdot + t, y) \le F$ for all $(t,y) \in \Gamma$ and

$$\|F\|_p \le c_p \|f\|_p . \tag{4.16}$$

Proof. Notice that

$$P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2} \le \frac{1}{\pi}\frac{1}{2^{2(k-1)}y + y}, \ \forall 2^{k-1}y \le |x|.$$

We have, for every positive f and any $(t,y) \in \Gamma$,

$$\begin{aligned}
& f(x+t,y) \\
= \ & \int_{\mathbb{R}} f(s) P_y(x+t-s)ds \\
\le \ & \frac{1}{\pi}\int_{|x+t-s|\le y} f(s)\frac{1}{y}ds + \frac{1}{\pi}\sum_{k=1}^{\infty}\int_{2^{k-1}y\le|x+t-s|\le 2^k y} f(s)\frac{1}{2^{2(k-1)}y+y}ds \\
\le \ & \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{8}{2^k}\frac{1}{2^{k+1}y}\int_{|x+t-s|\le 2^k y} f(s)ds.
\end{aligned} \tag{4.17}$$

Considering $h_{k,y} = (2^k y - t, 2^k y + t) \in \mathbb{R}^+ \times \mathbb{R}^+$, we get (4.16) from (4.11). And by (4.10),

$$\begin{aligned}
\left\| \sup_{(t,y)\in\Gamma} |f(x+t,y)| \right\|_p & \le \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{8}{2^k}\left\| \sup_{h_{k,y}} |f_{h_{k,y}}| \right\|_p \\
& \le c_p \|f\|_p .
\end{aligned}$$

Decomposing $f = f_1 - f_2 + i(f_3 - f_4)$ with positive f_k, we get (4.15) for all $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. We can prove (4.14) similarly. ∎

4.2. The non-commutative Lebesgue differentiation theorem and non-tangential limit of Poisson integrals

We end this chapter with the non-commutative Lebesgue differentiation theorem and non-tangential limit of Poisson integrals. These are consequences of Theorem 4.3 and Theorem 4.4. To this end, we first need to recall the non-commutative version of the almost everywhere convergence. Let $(f_\lambda)_{\lambda\in\lambda}$ be a family of elements in $L^p(\mathcal{M}, \tau)$. We say $(f_\lambda)_{\lambda\in\lambda}$ converges to f almost uniformly, abbreviated as $f_\lambda \overset{a.u}{\to} f$, if for every $\varepsilon > 0$, there exists $e_\varepsilon \in \mathcal{P}(\mathcal{M})$ such that $\tau(1 - e_\varepsilon) < \varepsilon$ and

$$\lim_{\lambda\to\lambda_0} \|e_\varepsilon(f_\lambda - f)\|_\infty = 0.$$

Moreover, we say $(f_\lambda)_{\lambda\in\lambda}$ converges to f bilaterally almost uniformly, abbreviated as $f_\lambda \overset{b.a.u}{\to} f$, if for every $\varepsilon > 0$, there exists $e_\varepsilon \in \mathcal{P}(\mathcal{M})$ such that $\tau(1 - e_\varepsilon) < \varepsilon$ and

$$\lim_{\lambda\to\lambda_0} \|e_\varepsilon(f_\lambda - f)e_\varepsilon\|_\infty = 0.$$

Obviously, $f_\lambda \overset{a.u}{\to} f$ implies $f_\lambda \overset{b.a.u}{\to} f$.

Recall that the map $x \mapsto x^p$ $(1 \le p \le 2)$ is convex on the positive cone $\mathcal{M}_+$ of $\mathcal{M}$ (see [2]). Thus, for $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ $(1 \le p \le 2)$, we get

$$\int_A |f|dt \le (\int_A |f|^p dt)^{\frac{1}{p}}, \quad \forall A \subseteq \mathbb{R},\ |A| = 1. \tag{4.18}$$

Note that for any $x, y \in \mathcal{M}_+$, $x \le y$ implies $x^q \le y^q, \forall 0 < q \le 1$. Using (4.18) successively, we get the following Lemma.

Lemma 4.5 *For* $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$, $1 \le p < \infty$,

$$\int_A |f| dt \le (\int_A |f|^p dt)^{\frac{1}{p}}, \quad \forall A \subseteq \mathbb{R},\ |A| = 1. \tag{4.19}$$

Recall that for any bounded linear operators a, b on a Hilbert space H, a positive and $\|b\| \le 1$, if T is an operator monotone function defined for positive operators (for example, $T(a) = a^{\frac{1}{p}}, p \ge 1$) then

$$b^* T(a) b \le T(b^* a b). \tag{4.20}$$

This is the so-called Hansen's inequality (see [9]). In particular, we have

$$b^* a b \le (b^* a^p b)^{\frac{1}{p}}. \tag{4.21}$$

Theorem 4.6 *(i) Let* $1 \le p < 2$. *We have* $f_h \stackrel{b.a.u}{\to} f$ *as* $h \to 0$ *for any* $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$.

(ii) Let $2 \le p < \infty$. *We have* $f_h \stackrel{a.u}{\to} f$ *as* $h \to 0$ *for any* $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$.

Proof. (i) Without loss of generality, we can assume f selfadjoint. For any given $f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ and $\varepsilon > 0$, choose $f^n = \sum_{k=1}^{N_n} \varphi_k x_k,$ where $x_k \in S^+_{\mathcal{M}}$ and where $\varphi_k : \mathbb{R} \to \mathbb{C}$ are continuous functions with compact support, such that

$$\| |f - f^n|^p \|_1 = \|f - f^n\|_p^p < (\frac{1}{2^n})^p \frac{\varepsilon}{2^n}. \tag{4.22}$$

Choose $e^\varepsilon_{1,n} \in \mathcal{P}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ such that

$$\tau \otimes \int (1 - e^\varepsilon_{1,n}) < \frac{\varepsilon}{2^n} \quad \text{and} \quad \left\| e^\varepsilon_{1,n} |f^n - f|^p e^\varepsilon_{1,n} \right\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} < (\frac{1}{2^n})^p.$$

Set $e^\varepsilon_1 = \wedge_n e^\varepsilon_{1,n}$. We have $\tau \otimes \int (1 - e^\varepsilon_1) < \varepsilon$ and by (4.21),

$$\begin{aligned} \|e^\varepsilon_1 (f^n - f) e^\varepsilon_1\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} &\le \|e^\varepsilon_1 |f^n - f| e^\varepsilon_1\|_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \\ &\le \|e^\varepsilon_1 |f^n - f|^p e^\varepsilon_1\|^{\frac{1}{p}}_{L^\infty(\mathbb{R}) \otimes \mathcal{M}} \end{aligned}$$

$$< \frac{1}{2^n}, \quad \forall n \geq 1. \tag{4.23}$$

On the other hand, by (4.9) and (4.22) we can find a sequence $(e^{\varepsilon}_{2,n})_{n\geq 0} \subset \mathcal{P}(L^{\infty}(\mathbb{R})\otimes \mathcal{M})$ such that

$$\begin{aligned} \tau \otimes \int (1 - e^{\varepsilon}_{2,n}) &< \frac{\varepsilon}{2^n} \\ \left\| e^{\varepsilon}_{2,n}(|f^n - f|^p)_h e^{\varepsilon}_{2,n} \right\|_{L^{\infty}(\mathbb{R})\otimes\mathcal{M}} &< (\frac{1}{2^n})^p, \quad \forall h \in \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned} \tag{4.24}$$

Set $e^{\varepsilon}_2 = \wedge_n e^{\varepsilon}_{2,n}$, we have $\tau \otimes \int (1 - e^{\varepsilon}_2) < \varepsilon$. By (4.19), (4.21) and (4.24)

$$\begin{aligned} \| e^{\varepsilon}_2 (f^n_h - f_h) e^{\varepsilon}_2 \|_{L^{\infty}(\mathbb{R})\otimes\mathcal{M}} &\leq \left\| e^{\varepsilon}_{2,n}(|f^n - f|)_h e^{\varepsilon}_{2,n} \right\|_{L^{\infty}(\mathbb{R})\otimes\mathcal{M}} \\ &\leq \left\| e^{\varepsilon}_{2,n}(|f^n - f|^p)_h^{\frac{1}{p}} e^{\varepsilon}_{2,n} \right\|_{L^{\infty}(\mathbb{R})\otimes\mathcal{M}} \\ &\leq (\left\| e^{\varepsilon}_{2,n}(|f^n - f|^p)_h e^{\varepsilon}_{2,n} \right\|_{L^{\infty}(\mathbb{R})\otimes\mathcal{M}})^{\frac{1}{p}} \\ &< \frac{1}{2^n}, \qquad \forall n \geq 0, h \in \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned} \tag{4.25}$$

Recall that by the classical Lebesgue differentiation theorem,

$$\lim_{h\to 0} \|\varphi_h - \varphi\|_{\infty} = 0$$

if $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous with compact support. Then by the choice of f_n we deduce that

$$\lim_{h\to 0} \|f^n_h - f^n\|_{L^{\infty}(\mathbb{R})\otimes\mathcal{M}} = 0, \forall n \geq 1.$$

Let $e^{\varepsilon} = e^{\varepsilon}_1 \wedge e^{\varepsilon}_2$, then $\tau \otimes \int (1 - e^{\varepsilon}) < 2\varepsilon$. For any $n > 0$, choose $S_n > 0$ such that $\|f^n_h - f^n\|_{\infty} < \frac{1}{2^n}$ for any $h \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $h_1 + h_2 < S_n$. Then, for any $h \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $h_1 + h_2 < S_n$,

$$\begin{aligned} \|e^{\varepsilon}(f_h - f)e^{\varepsilon}\|_{\infty} &\leq \|e^{\varepsilon}(f^n - f)e^{\varepsilon}\|_{\infty} + \|f^n_h - f^n\|_{\infty} + \|e^{\varepsilon}(f^n_h - f_h)e^{\varepsilon}\|_{\infty} \\ &\leq \|e^{\varepsilon}_1(f^n - f)e^{\varepsilon}_1\|_{\infty} + \|f^n_h - f^n\|_{\infty} + \|e^{\varepsilon}_2(f^n_h - f_h)e^{\varepsilon}_2\|_{\infty} \end{aligned}$$

$$\leq \frac{3}{2^n}.$$

Thus $\lim_{h\to 0}\|e^\varepsilon(f_h - f)e^\varepsilon\|_\infty \to 0.$ This completes the proof of (i).

(ii) The proof of (i) works well for the part (ii) of the theorem with some minor changes. Let $(f^n)_{n\in\mathbb{N}}$ and $e_1^\varepsilon, e_2^\varepsilon, e^\varepsilon$ be as above. Since $p \geq 2$, instead of (4.23), (4.25), by (4.19) and (4.21) we have

$$\|e_1^\varepsilon(f^n - f)\|_\infty = \left\|e_1^\varepsilon|f^n - f|^2 e_1^\varepsilon\right\|_\infty^{\frac{1}{2}} \leq \|e_1^\varepsilon|f^n - f|^p e_1^\varepsilon\|_\infty^{\frac{1}{p}} < \frac{1}{2^n}, \ \forall n \geq 1; \tag{4.26}$$

and also

$$\begin{aligned} \|e_2^\varepsilon(f_h^n - f_h)\|_\infty &= \left\|e_2^\varepsilon|f_h^n - f_h|^2 e_2^\varepsilon\right\|_\infty^{\frac{1}{2}} \\ &\leq \left(\left\|e_2^\varepsilon(|f^n - f|^2)_h e_2^\varepsilon\right\|_\infty\right)^{\frac{1}{2}} \\ &\leq (\|e_2^\varepsilon(|f^n - f|^p)_h e_2^\varepsilon\|_\infty)^{\frac{1}{p}} < \frac{1}{2^n}, \quad \forall n \geq 1. \end{aligned} \tag{4.27}$$

Then we can conclude as in the proof of (i). ∎

Theorem 4.7 *(i) Let $1 \leq p < 2, f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. We have $f(\cdot + u, y) \overset{b.a.u}{\to} f$ as $\Gamma \ni (u,y) \to 0$.*

(ii) Let $2 \leq p < \infty, f \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$. We have $f(\cdot + u, y) \overset{a.u}{\to} f$ as $\Gamma \ni (u,y) \to 0$.

Proof. We can assume $f \geq 0$ by decomposing f into four positive parts. Given $\varepsilon > 0$, let $f^n, e_{i,n}^\varepsilon, e_i^\varepsilon$ $(i = 1, 2)$ be as in the proof of Theorem 3.6. We use the same notation f^n for the Poisson integral of f^n. It is easy to see that

$$\lim_{(u,y)\to 0.} \|f^n(\cdot + u, y) - f^n\|_\infty \to 0, \qquad \forall n \geq 0, \quad (u,y) \in \Gamma$$

Let $e^\varepsilon = e_1^\varepsilon \wedge e_2^\varepsilon$. For any $n > 0$, choose $Y_n > 0$ such that

$$\|f^n(\cdot + u, y) - f^n\|_\infty < \frac{1}{2^n}$$

for any $(u, y) \in \Gamma, |u| + y \leq Y_n$. To prove (i), from (4.23), (4.25) we have, for any $(u, y) \in \Gamma, |u| + y \leq Y_n$,

$$
\begin{aligned}
& \left\|e^{\varepsilon}(f(\cdot + u, y) - f(\cdot))e^{\varepsilon}\right\|_{\infty} \\
\leq\ & \left\|e^{\varepsilon}(f^n - f)e^{\varepsilon}\right\|_{\infty} + \left\|f^n(\cdot + u, y) - f^n\right\|_{\infty} \\
& + \left\|e^{\varepsilon}(\int_{\mathbb{R}} (f - f^n)(s)P_y(x + u - s)ds)e^{\varepsilon}\right\|_{\infty} \\
\leq\ & \frac{1}{2^n} + \frac{1}{2^n} + \sum_{k=0}^{\infty} \left\|e^{\varepsilon}(\int_{|x+u-s|\leq 2^k y} |f - f^n| \frac{2}{2^{2(k-1)}y + y} ds)e^{\varepsilon}\right\|_{\infty} \\
\leq\ & \frac{2}{2^n} + \sum_{k=0}^{\infty} \frac{8}{2^k} \left\|e_2^{\varepsilon}(\frac{1}{2^k y} \int_{|x+u-s|\leq 2^k y} |f - f^n| ds)e_2^{\varepsilon}\right\|_{\infty} \\
\leq\ & \frac{2}{2^n} + \sum_{k=0}^{\infty} \frac{8}{2^k} \left\|e_2^{\varepsilon}(|f - f^n|)_{h_{k,y}} e_2^{\varepsilon}\right\|_{\infty} \\
\leq\ & \frac{2}{2^n} + \frac{8}{2^n},
\end{aligned}
$$

where $h_{k,y} = (2^k y - t, 2^k y + t) \in \mathbb{R}^+ \times \mathbb{R}^+$. Thus

$$
\lim_{(u,y)\to 0} \left\|e^{\varepsilon}(f(\cdot + ty, y) - f)e^{\varepsilon}\right\|_{\infty} = 0, \forall \varepsilon > 0,
$$

and then $f(\cdot + u, y) \stackrel{b.a.u}{\rightarrow} f$ when $\Gamma \ni (u, y) \to 0$. This is (i). Using (4.26) and (4.27) instead of (4.23) and (4.25), we can prove (ii) similarly. ∎

Remark. When $p = \infty$, the corresponding convergence problems discussed in this section are still open.

CHAPTER V

THE DUALITY BETWEEN $\mathcal{H}^P$ AND BMO^Q, $1 < P < 2$

In this chapter, we describe the dual of $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$, which is $\mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M})$ (q being the conjugate index of p), the latter is the L^q-space analogue of BMO space already considered in Chapters II and III. These $\mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M})$ spaces not only are used to describe the dual of $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ but also play an important role for all results in the sequel. In particular, we will use it to prove the map Ψ introduced in Chapter IV extends to a bounded map from $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))$ to $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ for all $1 < p < \infty$. Consequently, $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ can be considered as a complemented subspace of $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))$. For the most part, our results in this chapter are extension to the function space setting of results proved for non-commutative martingales in [18].

5.1. Operator valued BMO^q ($q > 2$)

We will now introduce a useful operator inequality. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, let $a, b \in B(H)$, then

$$|a + b|^2 \leq (1 + t)|a|^2 + (1 + \frac{1}{t})|b|^2, \forall t > 0, t \in \mathbb{R}. \tag{5.1}$$

In fact, by the Cauchy-Schwarz inequality, we have, for every $h \in H$,

$$\begin{aligned} \langle |a + b|^2 h, h \rangle &= \langle (a + b)h, (a + b)h \rangle \\ &\leq \langle ah, ah \rangle + \langle bh, bh \rangle + 2\langle ah, ah \rangle^{\frac{1}{2}} \langle bh, bh \rangle^{\frac{1}{2}} \\ &\leq (1 + t)\langle |a|^2 h, h \rangle + (1 + \frac{1}{t})\langle |b|^2 h, h \rangle; \quad \forall t > 0, t \in \mathbb{R}. \end{aligned}$$

Let $\varphi \in L^q(\mathcal{M}, L^2_c(\mathbb{R}, \frac{dt}{1+t^2}))$. For $h \in \mathbb{R}^+ \times \mathbb{R}^+$, denote $I_{h,t} = (t - h_1, t + h_2]$. Let

$$\varphi_h^{\#}(t) = \frac{1}{h_1 + h_2} \int_{I_{h,t}} |\varphi(x) - \varphi_{I_{h,t}}|^2 dx$$

Set, for $2 < q \leq \infty$,

$$\|\varphi\|_{\mathrm{BMO}_c^q} = \left\| \sup_{h \in \mathbb{R}^+ \times \mathbb{R}^+} |\varphi_h^{\#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^{\frac{1}{2}}$$

and

$$\|\varphi\|_{\mathrm{BMO}_r^q} = \|\varphi^*\|_{\mathrm{BMO}_c^q}.$$

It is easy to check by (5.1) that $\|\cdot\|_{\mathrm{BMO}_r^q}$ and $\|\cdot\|_{\mathrm{BMO}_c^q}$ are norms. Let $\mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M})$ (resp. $\mathrm{BMO}_r^q(\mathbb{R}, \mathcal{M})$) be the space of all $\varphi \in L^q(\mathcal{M}, L^2_c(\mathbb{R}, \frac{dt}{1+t^2}))$ (resp. $L^q(\mathcal{M}, L^2_r(\mathbb{R}, \frac{dt}{1+t^2}))$) such that $\|\varphi\|_{\mathrm{BMO}_c^q} < \infty$(resp. $\|\varphi\|_{\mathrm{BMO}_r^q} < \infty$). $\mathrm{BMO}_{cr}^q(\mathbb{R}, \mathcal{M})$ is defined as the intersection of these two spaces

$$\mathrm{BMO}_{cr}^q(\mathbb{R}, \mathcal{M}) = \mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M}) \cap \mathrm{BMO}_r^q(\mathbb{R}, \mathcal{M})$$

equipped with the norm

$$\|\varphi\|_{\mathrm{BMO}_{cr}^q} = \max\{\|\varphi\|_{\mathrm{BMO}_c^q}, \|\varphi\|_{\mathrm{BMO}_r^q}\}.$$

If $q = \infty$, all these spaces coincide with those introduced in Chapter III. And if $\mathcal{M} = \mathbb{C}$, all these spaces coincide with the classical BMO^q. As in the case of $\mathrm{BMO}(\mathbb{R}, \mathcal{M})$, we regard $\mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M})$ (resp. $\mathrm{BMO}_r^q(\mathbb{R}, \mathcal{M})$, $\mathrm{BMO}_r^q(\mathbb{R}, \mathcal{M})$) as normed spaces modulo constants. The following is the analogue for $\mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M})$ of Proposition 2.3. Recall that $I_t^n = (t - 2^{n-1}, t + 2^{n-1}]$ for $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Note that we have trivially

$$\left\| \frac{1}{2^k} \int_{I_t^k} |\varphi(s) - \varphi_{I_t^k}|^2 ds \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^{\frac{1}{2}} \leq \|\varphi\|_{\mathrm{BMO}_c^q} \tag{5.2}$$

Proposition 5.1 *Let $2 < q \leq \infty$. Let $\varphi \in \mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$. Then*

$$\|\varphi\|_{L^q(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2}))} \leq c\left(\|\varphi\|_{\mathrm{BMO}_c^q} + \left\|\varphi_{I_0^1}\right\|_{L^q(\mathcal{M})}\right).$$

Moreover, $\mathrm{BMO}_c^q(\mathbb{R},\mathcal{M}), \mathrm{BMO}_r^q(\mathbb{R},\mathcal{M}), \mathrm{BMO}_{cr}^q(\mathbb{R},\mathcal{M})$ *are Banach spaces.*

Proof. The proof is similar to that of Proposition 2.3. By (2.12) we have

$$\begin{aligned}
|\varphi_{I_t^n} - \varphi_{I_0^1}|^2 &\leq n(\sum_{k=3}^n |\varphi_{I_t^k} - \varphi_{I_t^{k-1}}|^2 + |\varphi_{I_t^2} - \varphi_{I_0^1}|^2) \\
&\leq n(\sum_{k=3}^n \frac{1}{2^{k-1}} \int_{I_t^{k-1}} |\varphi(s) - \varphi_{I_t^k}|^2 ds + \frac{1}{2}\int_{I_0^1} |\varphi(s) - \varphi_{I_t^2}|^2 ds) \\
&\leq n(\sum_{k=3}^n \frac{2}{2^k} \int_{I_t^k} |\varphi(s) - \varphi_{I_t^k}|^2 ds + \frac{2}{4}\int_{I_t^2} |\varphi(s) - \varphi_{I_t^2}|^2 ds) \\
&= 2n\sum_{k=2}^n \frac{1}{2^k}\int_{I_t^k} |\varphi(s) - \varphi_{I_t^k}|^2 ds, \quad \forall n > 1, t \in [-1,1]. \qquad (5.3)
\end{aligned}$$

Thus by (5.2)

$$\left\||\varphi_{I_t^n} - \varphi_{I_0^1}|^2\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq 2n^2 \|\varphi\|_{\mathrm{BMO}_c^q}^2, \quad \forall n > 1, t \in [-1,1]. \qquad (5.4)$$

To control φ's $L^q(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2}))$ norm by its BMO_c^q norm, we write

$$\begin{aligned}
&\|\varphi\|^2_{L^q(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2}))} \\
= &\left\|\int_{\mathbb{R}} \frac{|\varphi(s)|^2}{1+s^2} ds\right\|_{L^{\frac{q}{2}}(\mathcal{M})} \\
= &\left\|\chi_{[-\frac{1}{2},\frac{1}{2}]}(t)\int_{\mathbb{R}} \frac{|\varphi(s)|^2}{1+s^2} ds\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq &\left\|\chi_{[-\frac{1}{2},\frac{1}{2}]}(t)(\sum_{n=0}^\infty \int_{I_t^{n+1}/I_t^n} \frac{|\varphi(s)|^2}{1+s^2} ds + \int_{I_0^1} \frac{|\varphi(s)|^2}{1+s^2} ds)\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq &c(\left\|\chi_{[-\frac{1}{2},\frac{1}{2}]}(t)(\sum_{n=2}^\infty \int_{I_t^n} \frac{|\varphi(s)|^2}{2^{2n}} ds + \int_{I_0^1} |\varphi(s)|^2 ds)\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})}
\end{aligned}$$

hence by (5.4)

$$\begin{aligned}\|\varphi\|^2_{L^q(\mathcal{M},L^2_c(\mathbb{R},\frac{dt}{1+t^2}))} &\le c\Big(\Big\|\sum_{n=2}^{\infty}\chi_{[-\frac{1}{2},\frac{1}{2}]}(t)\int_{I_t^n}\frac{|\varphi(s)-\varphi_{I_t^n}|^2}{2^{2n}}ds\Big\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\ &\quad+\Big\|\sum_{n=1}^{\infty}\frac{|\varphi_{I_0^1}|^2}{2^n}\Big\|_{L^{\frac{q}{2}}(\mathcal{M})}+\sum_{n=1}^{\infty}\frac{n^2\,\|\varphi\|^2_{\mathrm{BMO}_c^q}}{2^n} \\ &\le c\sum_{n=1}^{\infty}\frac{(n^2+1)\,\|\varphi\|^2_{\mathrm{BMO}_c^q}}{2^n}+c\left\|\varphi_{I_0^1}\right\|^2_{L^q(\mathcal{M})} \\ &< \infty. \end{aligned} \tag{5.5}$$

Thus $\mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$ is a Banach space. Passing to adjoints we get that $\mathrm{BMO}_r^q(\mathbb{R},\mathcal{M})$ is a Banach spaces and then so is $\mathrm{BMO}_{cr}^q(\mathbb{R},\mathcal{M})$. ∎

Put

$$\lambda_\varphi^{n,\#}(t)=\frac{1}{2^n}\iint\limits_{T(I_t^n)}|\nabla\varphi|^2 y\,dx\,dy.$$

Lemma 5.2 *Let* $\varphi\in\mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$ $(2<q<\infty)$. *Then* $\exists c>0$ *such that*

$$\Big\|\sup_{n\in\mathbb{Z}}|\lambda\varphi^{n,\#}|\Big\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})}\le c\,\|\varphi\|^2_{\mathrm{BMO}_c^q}.$$

Proof. The proof is similar to that of Lemma 2.4 but more complicated. For any $n\in\mathbb{Z}, t\in\mathbb{R}$, write $\varphi=\varphi_1^{n,t}+\varphi_2^{n,t}+\varphi_3^{n,t}$, where $\varphi_1^{n,t}=(\varphi-\varphi_{I_t^{n+1}})\chi_{I_t^{n+1}}$, $\varphi_2^{n,t}=(\varphi-\varphi_{I_t^{n+1}})\chi_{(I_t^{n+1})^c}$, and $\varphi_3^{n,t}=\varphi_{I_t^{n+1}}$. Set

$$\lambda_i^{n,\#}(t)=\frac{1}{2^n}\iint\limits_{T(I_t^n)}|\nabla\varphi_i^{n,t}|^2 y\,dx\,dy,\quad i=1,2.$$

Thus

$$\begin{aligned}&\Big\|\sup_{n\in\mathbb{Z}}|\lambda\varphi^{n,\#}|\Big\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\ \le\;& 2\Big\|\sup_{n\in\mathbb{Z}}|\lambda_1^{n,\#}|\Big\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})}+2\Big\|\sup_{n\in\mathbb{Z}}|\lambda_2^{n,\#}|\Big\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})}.\end{aligned}$$

We treat $\lambda_1^{n,\#}$ first. Arguing as earlier for (2.19), by Green's theorem we have

$$\frac{1}{2^n}\iint\limits_{T(I_t^n)}|\nabla\varphi_1^{n,t}|^2 ydxdy \leq \frac{1}{2^n}\int_{-\infty}^{+\infty}|\varphi_1^{n,t}|^2 ds.$$

Therefore,

$$\begin{aligned}
&\left\|\sup_{n\in\mathbb{Z}}|\frac{1}{2^n}\iint\limits_{T(I_t^n)}|\nabla\varphi_1^{n,t}|^2 ydxdy|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})}\\
\leq\ &\left\|\sup_{n\in\mathbb{Z}}|\frac{1}{2^n}\int_{-\infty}^{+\infty}|\varphi_1^{n,t}|^2 ds|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})}\\
=\ &\left\|\sup_{n\in\mathbb{Z}}|\frac{1}{2^n}\int_{I_t^{n+1}}|\varphi-\varphi_{I_t^{n+1}}|^2 ds|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})}\\
\leq\ &2\left\|\varphi\right\|^2_{\mathrm{BMO}_c^q}
\end{aligned}\tag{5.6}$$

To deal with $\lambda_2^{n,\#}$, we note that

$$|\nabla P_y(x-s)|^2 \leq \frac{1}{4(x-s)^4} \leq \frac{c}{2^{4(n+k)}},\quad \forall s\in I_t^{n+k+1}/I_t^{n+k},\quad (x,y)\in T(I_t^n).$$

Let $A_k = I_t^{n+k+1}/I_t^{n+k}$. Then by (2.14), (2.17) and (5.3)

$$\begin{aligned}
&\frac{1}{2^n}\iint\limits_{T(I_t^n)}|\nabla\varphi_2^{n,t}|^2 ydxdy\\
=\ &\frac{1}{2^n}\iint\limits_{T(I_t^n)}|\nabla\int_{-\infty}^{+\infty}P_y(x-s)\varphi_2^{n,t}(s)ds|^2 ydxdy\\
\leq\ &\frac{1}{2^n}\iint\limits_{T(I_t^n)}\left(\sum_{k=1}^{\infty}\int\limits_{A_k}|\nabla P_y(x-s)|^2 2^{2k}ds\sum_{k=1}^{\infty}\int\limits_{A_k}\frac{1}{2^{2k}}|\varphi_2^{n,t}(s)|^2 dsy\right)dxdy\\
\leq\ &\frac{c}{2^n}\iint\limits_{T(I_t^n)}\frac{1}{2^{3n}}\sum_{k=1}^{\infty}\int\limits_{A_k}\frac{1}{2^{2k}}|\varphi-\varphi_{I_t^{n+1}}|^2 dsydxdy\\
\leq\ &\frac{c}{2^n}\sum_{k=1}^{\infty}\int\limits_{A_k}\frac{2}{2^{2k}}(|\varphi-\varphi_{I_t^{n+k+1}}|^2+|\varphi_{I_t^{n+k+1}}-\varphi_{I_t^{n+1}}|^2)ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \ c\sum_{k=1}^{\infty}\frac{1}{2^{2k+n}}\int\limits_{A_k}|\varphi-\varphi_{I_t^{n+k+1}}|^2ds+\sum_{k=1}^{\infty}\frac{c}{2^k}\sum_{i=1}^{k}\frac{2k}{2^{n+i}}\int_{I_t^{n+i}}|\varphi(u)-\varphi_{I_t^{n+i}}|^2du \\
&\leq \ cX_n+cY_n
\end{aligned}
$$

where

$$
\begin{aligned}
X_n &= \sum_{k=1}^{\infty}\frac{1}{2^{2k+n}}\int\limits_{A_k}|\varphi-\varphi_{I_t^{n+k+1}}|^2ds, \\
Y_n &= \sum_{k=1}^{\infty}\frac{k}{2^k}\sum_{i=1}^{k}\frac{1}{2^{n+i}}\int_{I_t^{n+i}}|\varphi(s)-\varphi_{I_t^{n+i}}|^2ds.
\end{aligned}
$$

X_n, Y_n are estimated as follows. For X_n we have

$$
\begin{aligned}
&\left\|\sup_{n\in\mathbb{Z}}|X_n|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
= &\left\|\sup_{n\in\mathbb{Z}}|\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{2^{n+k}}\int\limits_{A_k}|\varphi-\varphi_{I_t^{n+k+1}}|^2ds|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq &\sum_{k=1}^{\infty}\frac{1}{2^k}\left\|\sup_{n\in\mathbb{Z}}|\frac{1}{2^{n+k}}\int\limits_{I_t^{n+k+1}}|\varphi-\varphi_{I_t^{n+k+1}}|^2ds|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq &\ 2\left\|\varphi\right\|^2_{\mathrm{BMO}_c^q}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\left\|\sup_{n\in\mathbb{Z}}|Y_n|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq &\sum_{k=1}^{\infty}\frac{k}{2^k}\sum_{i=1}^{k}\left\|\sup_{n\in\mathbb{Z}}|\frac{1}{2^{n+i}}\int_{I_t^{n+i}}|\varphi(s)-\varphi_{I_t^{n+i}}|^2ds|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq &\sum_{k=1}^{\infty}\frac{k^2}{2^k}\left\|\varphi\right\|^2_{\mathrm{BMO}_c^q} \\
= &\ 6\left\|\varphi\right\|^2_{\mathrm{BMO}_c^q}.
\end{aligned}
$$

Combining the preceding inequalities we get

$$\left\|\sup_{n\in\mathbb{Z}}|\lambda_{\varphi_2}^{n,\#}|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \le c\,\|\varphi\|^2_{\mathrm{BMO}_c^q},$$

which, together with (5.6), yields

$$\left\|\sup_{n\in\mathbb{Z}}|\lambda\varphi^{n,\#}|\right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \le c\,\|\varphi\|^2_{\mathrm{BMO}_c^q}. \quad \blacksquare$$

Set

$$\varphi_n^{\#}(t) = \frac{1}{2^n}\int_{I_t^n}|\varphi(x)-\varphi_{I_t^n}|^2 dx$$

Notice that for every $h\in\mathbb{R}^+\times\mathbb{R}^+$ there exists $n\in\mathbb{Z}$ such that $(t-h_1,t+h_2)\in I_t^n$ for every $t\in\mathbb{R}$ and $2^n\le 4(h_1+h_2)$, we have

$$\frac{1}{4}\|\varphi\|_{\mathrm{BMO}_c^q} \le \left\|\sup_n \varphi_n^{\#}\right\|^{\frac{1}{2}}_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \le \|\varphi\|_{\mathrm{BMO}_c^q}. \tag{5.7}$$

Lemma 5.3 *The operator Ψ defined in Chapter III extends to a bounded map from $L^q(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))$ $(2<q<\infty)$ into $\mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$ and there exists $c_q>0$ such that*

$$\|\Psi(h)\|_{\mathrm{BMO}_c^q} \le c_q\,\|h\|_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2)}. \tag{5.8}$$

Proof. The pattern of this proof is similar to that of Lemma 3.2. One new thing we need is the non-commutative Hardy-Littlewood maximal inequality proved in the previous chapter.

Let $\mathcal{S}$ be the family of functions introduced in the proof of Lemma 3.2. Since $\mathcal{S}$ is dense in $L^q(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))$, we need only to prove (5.8) for all $h\in\mathcal{S}$. Fix $h\in\mathcal{S}$ and set $\varphi=\Psi(h)$. Then $\varphi\in L^q(\mathcal{M}, L_c^2(\mathbb{R},\frac{ds}{1+s^2}))$. Let $u\in\mathbb{R}$ and $n\in\mathbb{Z}$. Set

$$\begin{aligned} h_1^u(x,y,t) &= h(x,y,t)\chi_{I_u^{n+1}}(t),\\ h_2^u(x,y,t) &= h(x,y,t)\chi_{(I_u^{n+1})^c}(t) \end{aligned}$$

and

$$B_{I_u^n} = \int_{-\infty}^{+\infty} \iint_{\Gamma} Q_{I_u^n} h_2^u dydxdt,$$

where

$$Q_{I_u^n}(x,y,t) = \frac{1}{2^n} \int_{I_u^n} Q_y(x+t-s)ds$$

(recall that $Q_y(x)$ is defined by (3.2) as the gradient of the Poisson kernel). Then

$$\begin{aligned}
\varphi_n^{\#}(u) &\leq \frac{4}{2^n} \int_{I_u^n} |\varphi(s) - B_{I_t^n}|^2 ds \\
&\leq \frac{8}{2^n} \int_{I_u^n} | \int_{(I_u^{n+1})^c} \iint_{\Gamma} (Q_y(x+t-s) - Q_{I_u^n}) h dxdydt|^2 ds \\
&\quad + \frac{8}{2^n} \int_{I_u^n} | \int_{-\infty}^{+\infty} \iint_{\Gamma} Q_y(x+t-s) h_1^u dxdydt|^2 ds \\
&= 8A_n + \frac{8}{2^n} \int_{I_u^n} | \int_{I_u^{n+1}} \iint_{\Gamma} Q_y(x+t-s) h dxdydt|^2 ds
\end{aligned}$$

Recall that, as noted earlier in (3.5),

$$\iint_{\Gamma} |Q_y(x+t-s) - Q_{I_u^n}|^2 dxdy \leq c 2^{2n} (t-u)^{-4}$$

for $t \in (I_u^{n+1})^c$ and $s \in I_u^n$. By (2.14), we have

$$\begin{aligned}
A_n &= \frac{1}{2^n} \int_{I_u^n} | \int_{(I_u^{n+1})^c} \iint_{\Gamma} (Q_y(x+t-s) - Q_{I_u^n}) h dxdydt|^2 ds \\
&\leq \int_{(I_u^{n+1})^c} c 2^{2n} (t-u)^{-2} dt \int_{(I_u^{n+1})^c} (t-u)^{-2} \iint_{\Gamma} |h|^2 dxdydt \\
&= c 2^n \int_{(I_u^{n+1})^c} (t-u)^{-2} \iint_{\Gamma} |h|^2 dxdydt
\end{aligned}$$

Then, for any positive $(a_n)_{n\in\mathbb{Z}}$ such that $\left\| \sum_{k\in\mathbb{Z}} a_n \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq 1$,

$$\tau \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} \varphi_n^{\#}(u) a_n(u) du$$

$$
\begin{aligned}
&\leq \sum_{n\in\mathbb{Z}} \tau \int_{-\infty}^{+\infty} c2^n \int_{(I_u^{n+1})^c} (t-u)^{-2} \iint\limits_{\Gamma} |h|^2 dxdydt a_n(u)du \\
&\quad + \sum_{n\in\mathbb{Z}} \tau \int_{-\infty}^{+\infty} \frac{8}{2^n} \int_{I_u^n} |\int_{I_u^{n+1}} \iint\limits_{\Gamma} Q_y(x+t-s)hdxdydt|^2 ds a_n(u)du \\
&= A+B
\end{aligned}
$$

By the non-commutative Hölder inequality,

$$
\begin{aligned}
A &= \sum_{n\in\mathbb{Z}} \tau \int_{-\infty}^{+\infty} c2^n \int_{(I_t^{n+1})^c} (t-u)^{-2} a_n(u)du \iint\limits_{\Gamma} |h|^2 dxdydt \\
&\leq \left\| \iint\limits_{\Gamma} |h|^2 dxdy \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \left\| \sum_{n\in\mathbb{Z}} c2^n \int_{(I_t^n)^c} (t-u)^{-2} a_n(u)du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
&\leq \|h\|^2_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))} \left\| \sum_{n\in\mathbb{Z}} \sum_{k=n}^{+\infty} 2^n \int_{I_t^{k+1}} \frac{1}{2^{2k}} a_n(u)du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})}.
\end{aligned}
$$

Let us estimate the second factor in the last term. By (4.6),

$$
\begin{aligned}
&\left\| \sum_{n\in\mathbb{Z}} \sum_{k=n+1}^{+\infty} 2^n \int_{I_t^{k+1}} \frac{1}{2^{2k}} a_n(u)du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
= &\left\| \sum_{k\in\mathbb{Z}} \frac{1}{2^k} \int_{I_t^{k+1}} \sum_{n=-\infty}^{k-1} \frac{2^n}{2^k} a_n(u)du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq &\ c_q \left\| \sum_{k\in\mathbb{Z}} \sum_{n=-\infty}^{k-1} \frac{2^n}{2^k} a_n \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq &\ c_q \left\| \sum_{n\in\mathbb{Z}} a_n \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq c_q.
\end{aligned}
$$

Thus

$$
A \leq c_q \|h\|^2_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2)}.
$$

For the term B, by (4.6), (2.10) and the Cauchy-Schwarz inequality,

$$
B \leq \sum_{n\in\mathbb{Z}} \tau \int_{\mathbb{R}} \frac{8}{2^n} \int_{\mathbb{R}} |\int_{I_u^{n+1}} \iint\limits_{\Gamma} Q_y(x+t-s)hdxdydt|^2 ds a_n(u)du
$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \frac{8}{2^n} \sup_{\|f\|_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})}=1} (\tau \int_{\mathbb{R}} \int_{I_u^{n+1}} \iint_{\Gamma} Q_y(x+t-s) h a_n^{\frac{1}{2}}(u) dx dy dt f(s) ds)^2 du \\
&= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \frac{8}{2^n} \sup_{\|f\|_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})}=1} (\tau \int_{I_u^{n+1}} \iint_{\Gamma} h a_n^{\frac{1}{2}}(u) \nabla f(t+x,y) dx dy dt)^2 du \\
&\leq \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \frac{8}{2^n} \tau \int_{I_u^{n+1}} \iint_{\Gamma} |h|^2 a_n(u) dx dy dt du \\
&= \sum_{n\in\mathbb{Z}} \tau \int_{\mathbb{R}} \iint_{\Gamma} |h|^2 dx dy \frac{8}{2^n} \int_{I_t^{n+1}} a_n(u) du dt \\
&\leq \|\iint_{\Gamma} |h|^2 dx dy\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \left\| \sum_{n\in\mathbb{Z}} \frac{16}{2^n} \int_{I_t^n} a_n(u) du \right\|_{L^{(\frac{q}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
&\leq c_q \|h\|^2_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c)} .
\end{aligned}
$$

Thus

$$
\left\| \sup_n |\varphi_n^{\#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq c_q \|h\|^2_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c)}
$$

and then

$$
\|\Psi(h)\|_{\mathrm{BMO}^q_c} \leq c_q \|h\|_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c)} . \quad \blacksquare
$$

Remark. It seems difficult to define non-commutative BMO^q for $q < 2$.

5.2. The duality theorem of $\mathcal{H}^p$ and $\mathrm{BMO}^q (1 < p < 2)$

Denote by $\mathcal{H}^p_{c0}(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}^p_{r0}(\mathbb{R}, \mathcal{M})$) the functions f in $\mathcal{H}^p_c(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}^p_r(\mathbb{R}, \mathcal{M})$) such that $f \in L^p(\mathcal{M}, L^2_c(\mathbb{R}, (1+t^2)dt))$ (resp. $L^p(\mathcal{M}, L^2_r(\mathbb{R}, (1+t^2)dt))$ and $\int f dt = 0$. Set

$$
\mathcal{H}^p_{cr0}(\mathbb{R}, \mathcal{M}) = \mathcal{H}^p_{c0}(\mathbb{R}, \mathcal{M}) + \mathcal{H}^p_{r0}(\mathbb{R}, \mathcal{M}).
$$

It is easy to see that $\mathcal{H}^p_{c0}(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}^p_{r0}(\mathbb{R}, \mathcal{M})$, $\mathcal{H}^p_{cr0}(\mathbb{R}, \mathcal{M})$) is a dense subspace of $\mathcal{H}^p_c(\mathbb{R}, \mathcal{M})$ (resp. $\mathcal{H}^p_r(\mathbb{R}$, $\mathcal{H}^p_{cr0}(\mathbb{R}, \mathcal{M})$). By Propositions 2.1 and 5.1, $\int_{-\infty}^{+\infty} \varphi^* f dt$

exists as an element in $L^1(\mathcal{M})$ for any $\varphi \in \mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$ and $f \in \mathcal{H}_{c0}^p(\mathbb{R},\mathcal{M})$.

Theorem 5.4 *Let $1<p<2$, $q=\frac{p}{p-1}$. Then*

(a) $(\mathcal{H}_c^p(\mathbb{R},\mathcal{M}))^* = \mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$ with equivalent norms. More precisely, every $\varphi \in \mathrm{BMO}_c^q(\mathcal{M})$ defines a continuous linear functional on $\mathcal{H}_c^p(\mathbb{R},\mathcal{M})$ by

$$l\varphi(f) = \tau \int_{-\infty}^{+\infty} \varphi^* f dt; \qquad \forall f \in \mathcal{H}_{c0}^p(\mathbb{R},\mathcal{M}) \tag{5.9}$$

Conversely every $l \in (\mathcal{H}_c^p(\mathbb{R},\mathcal{M}))^*$ can be given as above by some $\varphi \in \mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$ and there exist constants $c, c_q > 0$ such that

$$c_q \|\varphi\|_{\mathrm{BMO}_c^q} \leq \|l\varphi\|_{(\mathcal{H}_c^p)^*} \leq c\|\varphi\|_{\mathrm{BMO}_c^q}$$

Thus $(\mathcal{H}_c^p(\mathbb{R},\mathcal{M}))^* = \mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$ with equivalent norms.

(b) Similarly, $(\mathcal{H}_r^p(\mathbb{R},\mathcal{M}))^* = \mathrm{BMO}_r^q(\mathbb{R},\mathcal{M})$ with equivalent norms.

(c) $(\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M}))^* = \mathrm{BMO}_{cr}^q(\mathbb{R},\mathcal{M})$ with equivalent norms.

Proof. (i) Let $\varphi \in \mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})$ and $f \in \mathcal{H}_{c0}^p(\mathbb{R},\mathcal{M})$. As in the proof of Theorem 3.4, we assume φ and f compactly supported. Let $G_c(f)$ and $\widetilde{S}_c(f)$ be as in the proof of Theorem 3.4. Similar to what we have explained there, $G_c(f)(x,y)$ can be assumed to be invertible in $\mathcal{M}$ for every $(x,y) \in \mathbb{R}_+^2$. By Green's theorem and the Cauchy-Schwarz inequality (see the corresponding part of the proof of Theorem 3.4 to see why Green's theorem works well),

$$\begin{aligned} |l\varphi(f)| &= 2|\tau \int_{-\infty}^{+\infty}\int_0^{\infty} \nabla\varphi^*\nabla f y dy dx| \\ &\leq 2(\tau \int_{-\infty}^{+\infty}\int_0^{\infty} G_c^{p-2}(f)(x,y)|\nabla f|^2(x,y) y dy dx)^{\frac{1}{2}} \\ &\quad \bullet (3\tau \int_{-\infty}^{+\infty}\int_0^{\infty} \widetilde{S}_c^{2-p}(f)(x,\frac{y}{4})|\nabla\varphi|^2 y dy dx)^{\frac{1}{2}} \\ &= 2I \bullet II \end{aligned}$$

Noting that $G_c^{p-1}(f)(x,y) \le G_c^{p-1}(f)(x,0)$, we have

$$\begin{aligned}
I^2 &= \tau\int_{-\infty}^{+\infty}\int_0^{\infty} -G_c^{p-2}(f)(x,y)\frac{\partial G_c^2(f)}{\partial y}(x,y)dydx \\
&= \tau\int_{-\infty}^{+\infty}\int_0^{\infty} (-G_c^{p-2}(f)(x,y)\frac{\partial G_c(f)}{\partial y}G_c(f)(x,y) \\
&\quad -G_c^{p-1}(f)\frac{\partial G_c(f)}{\partial y}(x,y))dydx \\
&= 2\tau\int_{-\infty}^{+\infty}\int_0^{\infty} -G_c^{p-1}(f)(x,y)\frac{\partial G_c(f)}{\partial y}dydx \\
&\le 2\tau\int_{-\infty}^{+\infty}\int_0^{\infty} -G_c^{p-1}(f)(x,0)\frac{\partial G_c(f)}{\partial y}(x,y)dxdy \\
&\le 2\tau\int_{-\infty}^{+\infty} G_c^p(f)(x,0)dx \\
&\le 6\tau\int_{-\infty}^{+\infty} S_c^p(f)(x)dx \\
&= 6\,\|f\|_{\mathcal{H}_c^p}^p
\end{aligned}$$

Define

$$\delta^k(x) = \widetilde{S}_c^{2-p}(f)(x,2^k) - \widetilde{S}_c^{2-p}(f)(x,2^{k+1}), \quad \forall x \in \mathbb{R}.$$

Then $\delta^k \in L^{\frac{p}{2-p}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ is positive. Note that $(\frac{q}{2})' = \frac{p}{p-2}$. Moreover,

$$\begin{aligned}
\delta^k(x) &= \delta^k(x'), \forall (i-1)2^j < x, x' \le i2^j \\
\sum_{k=-\infty}^{\infty}\delta^k(x) &= \widetilde{S}_c^{2-p}(f)(x,0)
\end{aligned}$$

Arguing as earlier for Theorem 3.4, we have

$$\begin{aligned}
II^2 &= 3\tau\int_{-\infty}^{+\infty}\sum_{k=-\infty}^{\infty}\widetilde{S}_c^{2-p}(f)(x,2^k)\int_{2^{k+2}}^{2^{k+3}}|\nabla\varphi|^2 ydydx \\
&= 3\tau\int_{-\infty}^{+\infty}\sum_{k=-\infty}^{\infty}(\sum_{j=k}^{\infty}\delta^j(x))\int_{2^{k+2}}^{2^{k+3}}|\nabla\varphi|^2 ydydx \\
&= 3\tau\int_{-\infty}^{+\infty}\sum_{j=-\infty}^{\infty}2^j\delta^j(x)\frac{1}{2^j}\int_0^{2^{j+3}}|\nabla\varphi|^2 ydydx
\end{aligned}$$

$$\begin{aligned}
&\leq 3\tau \int_{-\infty}^{+\infty} \sum_{j=-\infty}^{\infty} \int_{x-2^j}^{x+2^j} \delta^j(t) dt \frac{1}{2^j} \int_0^{2^{j+3}} |\nabla\varphi|^2 y dy dx \\
&= 24\tau \sum_{j=-\infty}^{\infty} \int_{-\infty}^{+\infty} \delta^j(t) \frac{1}{2^{j+3}} \int_{t-2^j}^{t+2^j} \int_0^{2^{j+3}} |\nabla\varphi|^2 y dy dx dt
\end{aligned}$$

hence by (4.2) and Lemma 5.2

$$\begin{aligned}
II^2 &\leq 24 \left\| \sum_{j=-\infty}^{\infty} \delta^j(t) \right\|_{L^{(\frac{q}{2})'}} \left\| \sup_j |\frac{1}{2^{j+3}} \int_{t-2^j}^{t+2^j} \int_0^{2^{j+3}} |\nabla\varphi|^2 y dy dx| \right\|_{L^{\frac{q}{2}}} \\
&\leq c \|f\|_{\mathcal{H}_c^p}^{2-p} \|\varphi\|_{\mathrm{BMO}_c^q}^2 .
\end{aligned}$$

Combining the preceding estimates on I and II, we get

$$|l\varphi(f)| \leq c \|\varphi\|_{\mathrm{BMO}_c^q} \|f\|_{\mathcal{H}_c^p} .$$

Therefore, $l\varphi$ defines a continuous functional on $\mathcal{H}_c^p$ of norm smaller than $c\|\varphi\|_{\mathrm{BMO}_c^q}$.

(ii) Now suppose $l \in (\mathcal{H}_c^p)^*$. Then by the Hahn-Banach theorem l extends to a continuous functional on $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))$ of the same norm. Thus by

$$(L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma})))^* = L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))$$

there exists $h \in L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))$ such that

$$||h||^2_{L^q(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))} = || \iint_\Gamma h^*(x,y,t)h(x,y,t) dy dx||_{L^{\frac{q}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} = ||l||^2$$

and

$$\begin{aligned}
l(f) &= \tau \int_{-\infty}^{+\infty} \iint_\Gamma h^*(x,y,t) \nabla f(t+x,y) dy dx dt \\
&= \tau \int_{-\infty}^{+\infty} \Psi^*(h) f(s) ds.
\end{aligned}$$

Let

$$\varphi = \Psi(h) \tag{5.10}$$

Then

$$l(f) = \tau \int_{-\infty}^{+\infty} \varphi^*(s) f(s) ds$$

and by Lemma 5.3 $||\varphi||_{\mathrm{BMO}_c^q} \leq c_q ||l||$. This finishes the proof of the theorem concerning $\mathcal{H}_c^p$ and BMO_c^q. Passing to adjoints yields the part on $\mathcal{H}_r^p$ and BMO_r^q. Finally, the duality between $\mathcal{H}_{cr}^p$ and BMO_{cr}^q is obtained from the classical fact that the dual of a sum is the intersection of the duals. ∎

Corollary 5.5 $\varphi \in \mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M})$ *if and only if*

$$\left\| \sup_{n \in \mathbb{Z}} |\lambda \varphi^{n,\#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} < \infty$$

and there exist $c, c_q > 0$ *such that*

$$c_q \|\varphi\|_{\mathrm{BMO}_c^q}^2 \leq \left\| \sup_{n \in \mathbb{Z}} |\lambda \varphi^{n,\#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \leq c \|\varphi\|_{\mathrm{BMO}_c^q}^2 .$$

Proof. From the proof of Theorem 5.4, if φ is such that

$$\left\| \sup_n |\lambda_\varphi^{n,\#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} < \infty,$$

then φ defines a continuous linear functional on $\mathcal{H}_{c0}^p$ by $l_\varphi = \tau \int_{-\infty}^{+\infty} \varphi^* f dt$ and

$$\|l_\varphi\|_{(\mathcal{H}_c^p)^*} \leq c \left\| \sup_n |\lambda_\varphi^{n,\#}| \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}^{\frac{1}{2}}$$

and then by Theorem 5.4 again, there exists a function $\varphi' \in \mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M})$ with

$$\|\varphi'\|_{\mathrm{BMO}_c^q}^2 \leq c_q \|l_\varphi\|_{(\mathcal{H}_c^p)^*}^2 \leq c_q \left\| \sup_n \lambda_\varphi^{n,\#} \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}$$

such that

$$\tau \int_{-\infty}^{+\infty} \varphi^* f dt = \tau \int_{-\infty}^{+\infty} \varphi'^* f dt.$$

Thus $\varphi \in \mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M})$ and $\|\varphi\|_{\mathrm{BMO}_c^q}^2 \leq c_q \left\| \sup_n \lambda_\varphi^{n,\#} \right\|_{L^{\frac{q}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})}$. Combining this with Lemma 5.2, we get the desired assertion. ∎

Now we are in a position to show that as in the classical case, the Lusin square function and the Littlewood-Paley g-function have equivalent L^p-norm in the noncommutative setting. The case $p = 1$ was already obtained in Chapter III.

Theorem 5.6 *For $f \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$(resp. $\mathcal{H}_r^p(\mathbb{R}, \mathcal{M})$), $1 \leq p < \infty$, we have*

$$c_p^{-1} \left\| G_c(f) \right\|_p \leq \left\| S_c(f) \right\|_p \leq c_p \left\| G_c(f) \right\|_p ; \tag{5.11}$$

$$c_p^{-1} \left\| G_r(f) \right\|_p \leq \left\| S_r(f) \right\|_p \leq c_p \left\| G_r(f) \right\|_p . \tag{5.12}$$

Proof. We need only to prove the second inequality of (5.11). The case of $p = 2$ is obvious. The case of $p = 1$ is Corollary 3.7 and the part of $1 < p < 2$ can be proved similarly by using the following inequality already obtained during the proof of Theorem 5.4

$$|\tau \int \varphi^* f dt| \leq c \left\| \varphi \right\|_{\mathrm{BMO}_c^q} \left\| G_c(f) \right\|_p^{\frac{p}{2}} \left\| S_c(f) \right\|_p^{1-\frac{p}{2}} .$$

For $p > 2$, let g be a positive element in $L^{(\frac{p}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ with $\|g\|_{(\frac{p}{2})'} \leq 1$. By (4.2) and (4.10) we have

$$\begin{aligned}
& \left| \tau \int_{\mathbb{R}} \iint\limits_{\Gamma} |\nabla f(x+t,y)|^2 dxdy g(t) dt \right| \\
= & \left| \tau \iint\limits_{\mathbb{R}_+^2} |\nabla f(x,y)|^2 y \frac{1}{y} \int_{x-y}^{x+y} g(t) dt dx dy \right| \\
\leq & \; 4 \left| \tau \int_{\mathbb{R}} \sum_{n=-\infty}^{+\infty} \int_{2^{n-1}}^{2^n} |\nabla f(x,y)|^2 y dy \frac{1}{2^{n+1}} \int_{x-2^n}^{x+2^n} g(t) dt dx \right| \\
\leq & \; 4 \left\| \int_{\mathbb{R}_+} |\nabla f(x,y)|^2 y dy \right\|_{L^{\frac{p}{2}}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \left\| \sup_n \left| \frac{1}{2^{n+1}} \int_{x-2^n}^{x+2^n} g(t) dt \right| \right\|_{L^{(\frac{p}{2})'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})} \\
\leq & \; c_p \left\| G_c(f) \right\|_p^2
\end{aligned}$$

Therefore, taking the supremum over all g as above, we obtain

$$\|S_c(f)\|_p^2 \leq c_p \|G_c(f)\|_p^2 . \quad \blacksquare$$

5.3. The equivalence of $\mathcal{H}^q$ and $\mathrm{BMO}^q (q > 2)$

The following is the analogue for functions of a result for non-commutative martingales proved in [18].

Theorem 5.7 $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) = \mathrm{BMO}_c^p(\mathbb{R}, \mathcal{M})$ *with equivalent norms for* $2 < p < \infty$.

Proof. Note that for every $\varphi \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ and every $g \in \mathcal{H}_c^{p'}(\mathbb{R}, \mathcal{M})$ $(p' = \frac{p}{p-1})$

$$\begin{aligned} & |\tau \int_{-\infty}^{+\infty} \iint_{\Gamma} \nabla g(x+t,y) \nabla \varphi^*(x+t,y) dx dy dt| \\ \leq \ & \|\nabla g(x+t,y)\|_{L^{p'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))} \|\nabla \varphi(x+t,y)\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))} \\ \leq \ & \|g\|_{\mathcal{H}_c^{p'}} \|\varphi\|_{\mathcal{H}_c^p} . \end{aligned}$$

Then by Theorem 5.4

$$\|\varphi\|_{\mathrm{BMO}_c^p} \leq c_p \sup_{\|g\|_{\mathcal{H}_c^{p'}} \leq 1} |\tau \int g \varphi^* dt| \leq c_p \|\varphi\|_{\mathcal{H}_c^p} . \tag{5.13}$$

To prove the converse, we consider the following tent space T_c^p. Denote $\widetilde{\mathbb{R}_+^2} = (\mathbb{R}_+^2, \frac{dxdy}{y^2}) \times (\{1,2\}, \sigma)$ with $\sigma\{1\} = \sigma\{2\} = 1$. For $f \in L^p(\mathcal{M}, L_c^2(\widetilde{\mathbb{R}_+^2}))$, set

$$A_c(f)(t) = (\iint_{|x|<y} |f(x+t,y)|^2 dx \frac{dy}{y^2})^{\frac{1}{2}}.$$

Define, for $1 < p < \infty$,

$$T_c^p = \{f \in L^p(\mathcal{M}, L_c^2(\widetilde{\mathbb{R}_+^2})), \|f\|_{T_c^p} = \|A_c(f)\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})} < \infty\}. \tag{5.14}$$

We will prove that, for $p > 2$ and $\varphi \in \mathrm{BMO}_c^p(\mathbb{R}, \mathcal{M})$, φ induces a linear functional on $T_c^{p'}$ defined by

$$l_\varphi(f) = \tau \int\int_{\mathbb{R}_+^2} \nabla\varphi^*(x,y) y f(x,y) dxdy/y$$

and

$$\|\varphi\|_{\mathcal{H}_c^p} \le c_p \|l_\varphi\| \le c_p \|\varphi\|_{\mathrm{BMO}_c^p}. \tag{5.15}$$

We first prove the second inequality of (5.15). Set

$$\begin{aligned} A_c(f)(t,y) &= \Big(\iint\limits_{s>y, |x|<s-y} |f(x+t,s)|^2 dx \frac{ds}{s^2} \Big)^{\frac{1}{2}} \\ \overline{A}_c(f)(t,y) &= \Big(\iint\limits_{s>y, |x|<\frac{s}{4}} |f(x+t,s)|^2 dx \frac{ds}{s^2} \Big)^{\frac{1}{2}}. \end{aligned}$$

It is easy to see that

$$\overline{A}_c^2(f)(t,y) \le \overline{A}_c^2(f)(t,0) \le A_c^2(f)(t), \tag{5.16}$$

$$\overline{A}_c^2(f)(t+x,y) \le A_c^2(f)(t,\frac{y}{2}), \quad \forall |x| < \frac{y}{4}, (t,y) \in \mathbb{R}_+^2. \tag{5.17}$$

For nice f and by approximation, we can assume $A_c(f)(t,y)$ is invertible for all $(t,y) \in \mathbb{R}_+^2$. Thus by the Cauchy-Schwarz inequality

$$\begin{aligned} l_\varphi(f) &= \tau \int\int_{\mathbb{R}_+^2} f(t,y) \nabla\varphi^*(t,y) y dt \frac{dy}{y} \\ &\le \Big(\tau \iint\limits_{\mathbb{R}_+^2} A_c^{p'-2}(f)(t,\frac{y}{2}) |f|^2 y dt \frac{dy}{y^2}\Big)^{\frac{1}{2}} \Big(\tau \iint\limits_{\mathbb{R}_+^2} A_c^{2-p'}(f)(t,\frac{y}{2}) |\nabla\varphi|^2 y dt dy\Big)^{\frac{1}{2}} \\ &= I \cdot II \end{aligned}$$

Similarly to the proof of Theorem 5.4, we have

$$II^2 \le c \|\varphi\|_{\mathrm{BMO}_c^p}^2 \|f\|_{T_c^{p'}}^{2-p'}$$

Concerning the factor I, by (5.17) we have (recall $p' - 2 < 0$)

$$\begin{aligned}
I^2 &\leq \tau \iint\limits_{\mathbb{R}^2_+} 2 \int_{t-\frac{y}{4}}^{t+\frac{y}{4}} \overline{A}_c^{p'-2}(f)(x,y)dx|f(t,y)|^2 dt \frac{dy}{y^2} \\
&\leq 2\tau \iint\limits_{\mathbb{R}^2_+} \overline{A}_c^{p'-2}(f)(x,y) \int_{x-\frac{y}{4}}^{x+\frac{y}{4}} |f(t,y)|^2 dt dx \frac{dy}{y^2} \\
&\leq -2\tau \iint\limits_{\mathbb{R}^2_+} \overline{A}_c^{p'-2}(f)(x,y) \frac{\partial \overline{A}_c^2(f)}{\partial y}(x,y) dy dx \\
&= -4\tau \iint\limits_{\mathbb{R}^2_+} \overline{A}_c^{p'-1}(f)(x,y) \frac{\partial \overline{A}_c(f)}{\partial y}(x,y) dy dx \\
&\leq -4\tau \int_{\mathbb{R}} \overline{A}_c^{p'-1}(f)(x,0) \int_{\mathbb{R}^+} \frac{\partial \overline{A}_c(f)}{\partial y}(x,y) dy dx \\
&\leq 4 \|f\|_{T_c^{p'}}^{p'}
\end{aligned}$$

Thus

$$\|l_\varphi\| \leq c \|\varphi\|_{\mathrm{BMO}_c^p}. \tag{5.18}$$

Next we prove that $\|\varphi\|_{\mathcal{H}_c^p} \leq c_p \|l_\varphi\|$. Since we can regard $T_c^{p'}$ as a closed subspace of $L^{p'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\mathbb{R}^2_+}))$ via the map $f(x,y) \to f(x,y)\chi_{\{|x-t|<y\}}$. l_φ extends to a linear functional on $L^{p'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\mathbb{R}^2_+}))$ with the same norm. Then there exists $h \in L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\mathbb{R}^2_+}))$ such that $\|h\|_{L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\mathbb{R}^2_+}))} \leq \|l_\varphi\|$ and

$$\begin{aligned}
l_\varphi(f) &= \tau \int_{\mathbb{R}} \iint\limits_{|x-t|<y} f(x,y) h^*(x,y,t) dx \frac{dy}{y^2} dt \\
&= \tau \iint\limits_{\mathbb{R}^2_+} f(x,y) \int_{x-y}^{x+y} h^*(x,y,t) dt dx \frac{dy}{y^2}.
\end{aligned}$$

for every $f(x,y) \in T_c^{p'}$. Thus

$$\nabla \varphi(x,y) y = \frac{1}{y} \int_{x-y}^{x+y} h(x,y,t) dt. \tag{5.19}$$

Then

$$
\begin{aligned}
\|\varphi\|^2_{\mathcal{H}^p_c} &= (\tau\int_{\mathbb{R}}(\iint\limits_{\Gamma}|\frac{1}{y}\int_{x+s-y}^{x+s+y}h(x+s,y,t)dt|^2dx\frac{dy}{y^2})^{\frac{p}{2}}ds)^{\frac{2}{p}} \\
&\leq (\tau\int_{\mathbb{R}}(\iint\limits_{\mathbb{R}^2_+}\frac{1}{y}\int_{s-2y}^{s+2y}|h(x,y,t)|^2dtdx\frac{dy}{y^2})^{\frac{p}{2}}ds)^{\frac{2}{p}} \\
&= \left\|\iint\limits_{\mathbb{R}^2_+}\frac{1}{y}\int_{s-2y}^{s+2y}|h(x,y,t)|^2dtdx\frac{dy}{y^2}\right\|_{L^{\frac{p}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})}
\end{aligned}
$$

Notice that, for every positive a with $\|a\|_{L^{(\frac{p}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq 1$, by (4.10) and (4.2) we have

$$
\begin{aligned}
&\tau\int_{\mathbb{R}}\iint\limits_{\mathbb{R}^2_+}\frac{1}{y}\int_{s-2y}^{s+2y}|h(x,y,t)|^2dtdx\frac{dy}{y^2}a(s)ds \\
= \ &\tau\int_{\mathbb{R}}\iint\limits_{\mathbb{R}^2_+}|h(x,y,t)|^2\frac{1}{y}\int_{t-2y}^{t+2y}a(s)dsdx\frac{dy}{y^2}dt \\
\leq \ &8\tau\int_{\mathbb{R}}\sum_{n=-\infty}^{+\infty}\int_{2^{n-2}}^{2^{n-1}}\int_{\mathbb{R}}|h(x,y,t)|^2dx\frac{dy}{y^2}\frac{1}{2^{n+1}}\int_{t-2^n}^{t+2^n}a(s)dsdt \\
\leq \ &8\left\|\iint\limits_{\mathbb{R}^2_+}|h(x,y,t)|^2dx\frac{dy}{y^2}\right\|_{L^{\frac{p}{2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})}\left\|\sup_n|\frac{1}{2^{n+1}}\int_{t-2^n}^{t+2^n}a(s)ds|\right\|_{L^{(\frac{p}{2})'}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq \ &c_p\|h\|^2_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M},L^2_c(\widetilde{\mathbb{R}^2_+}))} \leq c_p\|l_\varphi\|^2
\end{aligned}
$$

Therefore by taking the supremum over all a as above, we obtain

$$\|\varphi\|^2_{\mathcal{H}^p_c} \leq c_p\|l_\varphi\|^2$$

Combining this with (5.18) we get

$$\|\varphi\|_{\mathcal{H}^p_c} \leq c_p\|\varphi\|_{\mathrm{BMO}^p_c}.$$

Then $\|\varphi\|_{\mathcal{H}_c^p} \backsimeq \|\varphi\|_{\mathrm{BMO}_c^p}$ for every $\varphi \in \mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$.

To prove $\mathrm{BMO}_c^p(\mathbb{R}, \mathcal{M})$ and $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ are the same space, it remains to show that the family of $S_{\mathcal{M}}$-simple functions is dense in $\mathrm{BMO}_c^p(\mathbb{R}, \mathcal{M})$. From the proof of Theorem 5.4 we can see that for every $\varphi \in \mathrm{BMO}_c^p(\mathbb{R}, \mathcal{M})$, there exists a $h \in L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)$ such that $\varphi = \Psi(h)$ and $\|\Psi(h)\|_{\mathrm{BMO}_c^p} \leq c\,\|h\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2)}$. Recall that the family of "nice" h's (i.e. $h(x,y,t) = \sum_{i=1}^n m_i f_i(t) \chi_{A_i}$ with $m_i \in S_{\mathcal{M}}, A_i \in \widetilde{\Gamma}, |A_i| < \infty$ and with scalar valued simple functions f_i) is dense in $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)$. Choose "nice" $h_n \to h$ in $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2)$. Let $\varphi_n = \Psi(h_n)$. Then $\varphi_n \to \varphi$ in $\mathrm{BMO}_c^p(\mathbb{R}, \mathcal{M})$. Since the φ_n's are continuous functions with compact support, we can approximate them by simple functions in $\mathrm{BMO}_c^p(\mathbb{R}, \mathcal{M})$. This shows the density of simple functions in $\mathrm{BMO}_c^p(\mathbb{R}, \mathcal{M})$ and thus completes the proof of the theorem. ∎

Remark. By the same idea used in the proof above, we can get the analogue of the classical duality result for the tent spaces: $(T_c^p)^* = T_c^q$ $(1 < p < \infty)$ with equivalent norms, where T_c^p is defined as (5.14).

Theorem 5.8 *(i)* Ψ *extends to a bounded map from* $L^\infty(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))$ *into* $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M})$ *and*

$$\|\Psi(h)\|_{\mathrm{BMO}_c} \leq c\,\|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2)} \tag{5.20}$$

(ii) Ψ *extends to a bounded map from* $L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M}, L_c^2(\widetilde{\Gamma}))$ *into* $\mathcal{H}_c^p(\mathbb{R}, \mathcal{M})$ $(1 < p < \infty)$ *and*

$$\|\Psi(h)\|_{\mathcal{H}_c^p} \leq c_p\,\|h\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2)}\,. \tag{5.21}$$

(iii) The statements (i) and (ii) also hold with column spaces replaced by row spaces.

Proof. (5.20) is Lemma 3.2. The part of (5.21) concerning $p > 2$ follows from Lemma 5.3 and Theorem 5.7. For $1 < p < 2$, by the duality between $\mathcal{H}_c^p$ and BMO_c^q, and

Theorem 5.7, we have

$$\begin{aligned}
\|\Psi(h)\|_{\mathcal{H}_c^p} &\leq c \sup_{\|f\|_{\mathrm{BMO}_c^q}\leq 1} \left| \tau \int_{\mathbb{R}} \Psi(h)(s) f^*(s) ds \right| \\
&\leq \sup_{\|f\|_{\mathcal{H}_c^q}\leq 1} \left| \tau \int_{\mathbb{R}} \int_{\mathbb{R}} \iint_{\Gamma} h(x,y,t) \nabla P_y(x+t-s) dx dy dt f^*(s) ds \right| \\
&= \sup_{\|f\|_{\mathcal{H}_c^q}\leq c} \left| \tau \int_{\mathbb{R}} \iint_{\Gamma} h(x,y,t) \nabla f^*(x+t,y) dx dy dt \right| \\
&\leq c \|h\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2)} . \qquad (5.22)
\end{aligned}$$

When $p = 2$, similarly but taking supremum over $\|f\|_{\mathcal{H}_c^2} \leq 1$ in the formula above, we have$\|\Psi(h)\|_{\mathcal{H}_c^2} \leq \|h\|_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2)}$.

Corollary 5.9 $(\mathcal{H}_c^p(\mathbb{R},\mathcal{M}))^* = \mathcal{H}_c^q(\mathbb{R},\mathcal{M})$ *with equivalent norms for all* $1 < p < \infty$.

CHAPTER VI

REDUCTION OF BMO TO DYADIC BMO

Our approach in Chapter IV towards the maximal inequality is to reduce it to the corresponding maximal inequality for dyadic martingales. In this chapter, we pursue this idea. We will see that BMO spaces can be characterized as intersections of dyadic BMO. This result has many consequences. It will be used in the next chapter for interpolation too.

6.1. BMO is the intersection of two dyadic BMO

Consider an increasing family of σ-algebras $\mathcal{F}=\{\mathcal{F}_n\}_{n\in\mathbb{Z}}$ on $\mathbb{R}$. Assume that each $\mathcal{F}_n$ is generated by a sequence of atoms $\{F_n^k\}_{k\in\mathbb{Z}}$. We are going to introduce the BMO^q spaces for martingales with respect to $\mathcal{F}=\{\mathcal{F}_n\}_{n\in\mathbb{Z}}$. Let $2<q\leq\infty$ and $\varphi\in L^q(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2}))$. Define

$$\varphi^{\#}_{\mathcal{F}_n}(t)=\frac{1}{|F_n^k|}\int_{F_n^k\ni t}|\varphi(x)-\varphi_{F_n^k}|^2dx$$

For $\varphi\in L^q(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2}))$(resp. $L^q(\mathcal{M},L_r^2(\mathbb{R},\frac{dt}{1+t^2}))$), let

$$\|\varphi\|_{\mathrm{BMO}_c^{q,\mathcal{F}}}=\left\|\sup_n|\varphi^{\#}_{\mathcal{F}_n}|\right\|_{\frac{q}{2}}^{\frac12}\quad\text{and}\quad\|\varphi\|_{\mathrm{BMO}_r^{q,\mathcal{F}}}=\|\varphi^*\|_{\mathrm{BMO}_c^{q,\mathcal{F}}}.$$

And set

$$\begin{aligned}\mathrm{BMO}_c^{q,\mathcal{F}}(L^\infty(\mathbb{R})\otimes\mathcal{M}) &= \{\varphi\in L^q(\mathcal{M},L_c^2(\mathbb{R},\frac{dt}{1+t^2})),\ \|\varphi\|_{\mathrm{BMO}_c^{q,\mathcal{F}}}<\infty\},\\ \mathrm{BMO}_r^{q,\mathcal{F}}(L^\infty(\mathbb{R})\otimes\mathcal{M}) &= \{\varphi\in L^q(\mathcal{M},L_r^2(\mathbb{R},\frac{dt}{1+t^2})),\ \|\varphi\|_{\mathrm{BMO}_r^{q,\mathcal{F}}}<\infty\}.\end{aligned}$$

Define $\mathrm{BMO}_{cr}^{q,\mathcal{F}}$ to be the intersection of $\mathrm{BMO}_c^{q,\mathcal{F}}$ and $\mathrm{BMO}_r^{q,\mathcal{F}}$ with the intersection norm $\max\{\|\varphi\|_{\mathrm{BMO}_c^{q,\mathcal{F}}},\|\varphi\|_{\mathrm{BMO}_r^{q,\mathcal{F}}}\}$. These BMO^q spaces were already studied in [18] for general non-commutative martingales.

In the following, we will consider the spaces $\mathrm{BMO}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$, $\mathrm{BMO}_c^{q,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M})$, $\mathrm{BMO}_r^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$, $\mathrm{BMO}_r^{q,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ etc. with respect to the families $\mathcal{D},\mathcal{D}'$ of dyadic σ-algebras defined in Chapter IV.

Theorem 6.1 *Let $2 < q \leq \infty$. With equivalent norms,*

$$\begin{aligned}
\mathrm{BMO}_c^q(\mathbb{R},\mathcal{M}) &= \mathrm{BMO}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})\cap\mathrm{BMO}_c^{q,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M});\\
\mathrm{BMO}_r^q(\mathbb{R},\mathcal{M}) &= \mathrm{BMO}_r^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})\cap\mathrm{BMO}_r^{q,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M});\\
\mathrm{BMO}_{cr}^q(\mathbb{R},\mathcal{M}) &= \mathrm{BMO}_{cr}^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})\cap\mathrm{BMO}_{cr}^{q,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}).
\end{aligned}$$

Proof. From Proposition 4.1, $\forall t\in\mathbb{R}, h\in\mathbb{R}^+\times\mathbb{R}^+$, there exist $k_{t,h}, N_h\in\mathbb{Z}$ such that $I_{h,t} := (t-h_1, t+h_2]$ is contained in $D_{N_h}^{k_{t,h}}$ or $D_{N_h}^{\prime k_{t,h}}$ and

$$|D_{N_h}^{k_{t,h}}| = |D_{N_h}^{\prime k_{t,h}}| \leq 6(h_1+h_2).$$

If $I_{h,t}\subset D_{N_h}^{k_{t,h}}$, then

$$\begin{aligned}
\varphi_h^\#(t) &= \frac{1}{h_1+h_2}\int_{t-h_1}^{t+h_2}|\varphi(x)-\varphi_{I_{h,t}}|^2dx\\
&\leq \frac{4}{h_1+h_2}\int_{t-h_1}^{t+h_2}|\varphi(x)-\varphi_{D_{N_h}^{k_{t,h}}}|^2dx\\
&\leq \frac{24}{|D_{N_h}^{k_{t,h}}|}\int_{D_{N_h}^{k_{t,h}}}|\varphi(x)-\varphi_{D_{N_h}^{k_{t,h}}}|^2dx\\
&\leq 24\varphi_{\mathcal{D}_{N_h}}^\#(t).
\end{aligned}$$

Similarly, if $I_{h,t}\subset D_{N_h}^{\prime k_{t,h}}$, then

$$\varphi_h^\#(t)\leq 24\varphi_{\mathcal{D}'_{N_h}}^\#(t).$$

Thus

$$
\begin{aligned}
\|\varphi\|_{\mathrm{BMO}_c^q} &= \left\|\sup_{h\in\mathbb{R}^+\times\mathbb{R}^+}|\varphi_h^{\#}|\right\|_{\frac{q}{2}}^{\frac{1}{2}}\\
&\leq \sqrt{24}\left\|\sup_n|(\varphi_{\mathcal{D}_n}^{\#}+\varphi_{\mathcal{D}'_n}^{\#})|\right\|_{\frac{q}{2}}^{\frac{1}{2}}\\
&\leq 4\sqrt{3}\max(\|\varphi\|_{\mathrm{BMO}_c^{q,\mathcal{D}}},\|\varphi\|_{\mathrm{BMO}_c^{q,\mathcal{D}'}}).
\end{aligned}
$$

It is trivial that $\max(\|\varphi\|_{\mathrm{BMO}_c^{q,\mathcal{D}}},\|\varphi\|_{\mathrm{BMO}_c^{q,\mathcal{D}'}})\leq\|\varphi\|_{\mathrm{BMO}_c^q}$. Therefore

$$
\mathrm{BMO}_c^q(\mathbb{R},\mathcal{M})=\mathrm{BMO}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})\cap\mathrm{BMO}_c^{q,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M})
$$

with equivalent norms. The two other equalities in the theorem are immediate consequences of this. ∎

6.2. The equivalence of $\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M})$ and $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})(1<p<\infty)$

We denote the non-commutative martingale Hardy spaces defined in [33] and [18] with respect to $\mathcal{D}$ and $\mathcal{D}'$ by $\mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}),\mathcal{H}_c^{p,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ etc.$(1\leq p<\infty)$. Note that

$$
\mathcal{H}_c^2(\mathbb{R},\mathcal{M})=\mathcal{H}_c^{2,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})=\mathcal{H}_c^{2,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M})=L^2(L^\infty(\mathbb{R})\otimes\mathcal{M}).
$$

By Theorems 5.4, 6.1 and the duality equality $(\mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}))^*=\mathrm{BMO}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ proved in [18], we get the following result.

Corollary 6.2 $\mathrm{BMO}_{cr}^q(\mathbb{R},\mathcal{M})=L^q(L^\infty(\mathbb{R})\otimes\mathcal{M})$ *with equivalent norms for* $2<q<\infty$.

Proof. From the inequalities (4.5) and (4.7) of [18] we have

$$
\mathrm{BMO}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})\cap\mathrm{BMO}_r^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})
$$

$$
\begin{aligned}
&= L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\
&= \mathrm{BMO}_c^{q,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathrm{BMO}_r^{q,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})
\end{aligned}
$$

with equivalent norms. Therefore, by Theorem 6.1

$$
\begin{aligned}
&\mathrm{BMO}_{cr}^q(\mathbb{R}, \mathcal{M}) \\
= \;& \mathrm{BMO}_c^q(\mathbb{R}, \mathcal{M}) \cap \mathrm{BMO}_r^q(\mathbb{R}, \mathcal{M}) \\
= \;& \mathrm{BMO}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathrm{BMO}_r^{q,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\
&\quad \cap \mathrm{BMO}_c^{q,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathrm{BMO}_r^{q,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \\
= \;& L^q(L^\infty(\mathbb{R}) \otimes \mathcal{M}). \blacksquare
\end{aligned}
$$

Corollary 6.3 *If* $1 \leq p < 2$, *then*

$$
\begin{aligned}
\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) + \mathcal{H}_c^{p,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \\
\mathcal{H}_r^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_r^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) + \mathcal{H}_r^{p,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \\
\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_{cr}^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) + \mathcal{H}_{cr}^{p,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}).
\end{aligned}
$$

If $p \geq 2$, *then*

$$
\begin{aligned}
\mathcal{H}_c^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_c^{p,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \\
\mathcal{H}_r^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_r^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_r^{p,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}), \\
\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) &= \mathcal{H}_{cr}^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) \cap \mathcal{H}_{cr}^{p,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M}).
\end{aligned}
$$

Corollary 6.4 $\mathcal{H}_{cr}^p(\mathbb{R}, \mathcal{M}) = L^p(L^\infty(\mathbb{R}) \otimes \mathcal{M})$ *with equivalent norms for all* $1 < p < \infty$.

Proof. Recall the result

$$
\mathcal{H}_{cr}^{p,\mathcal{D}}(L^\infty(\mathbb{R}) \otimes \mathcal{M}) = L^p(\mathbb{R}, \mathcal{M}) = \mathcal{H}_{cr}^{p,\mathcal{D}'}(L^\infty(\mathbb{R}) \otimes \mathcal{M})
$$

proved in [33] and [18]. By Corollary 6.3, for $1 < p < 2$, we have

$$\begin{aligned}
\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M}) &= \mathcal{H}_c^p(\mathbb{R},\mathcal{M}) + \mathcal{H}_r^p(\mathbb{R},\mathcal{M}) \\
&= \mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}) + \mathcal{H}_c^{p,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \\
&\qquad +\mathcal{H}_r^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}) + \mathcal{H}_r^{p,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \\
&= \mathcal{H}_{cr}^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}) + \mathcal{H}_{cr}^{p,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \\
&= L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})
\end{aligned}$$

and, for $2 \leq p < \infty$,

$$\begin{aligned}
\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M}) &= \mathcal{H}_c^p(\mathbb{R},\mathcal{M}) \cap \mathcal{H}_c^p(\mathbb{R},\mathcal{M}) \\
&= \mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \cap \mathcal{H}_c^{p,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \\
&\qquad \cap\mathcal{H}_r^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \cap \mathcal{H}_r^{p,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \\
&= \mathcal{H}_{cr}^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \cap \mathcal{H}_{cr}^{p,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \\
&= L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}). \blacksquare
\end{aligned}$$

Remark. In [15] and [16], M. Junge, C. Le Merdy and Q. Xu studied the Littlewood-Paley theory for semigroups on non-commutative L^p-spaces. Among many results, they proved, in particular, that for many nice semigroups, the corresponding non-commutative Hardy spaces defined by the Littlewood-Paley g-function coincide with the underlying non-commutative L^p-spaces $(1 < p < \infty)$. In their viewpoint, the semigroup in the context of our paper is the Poisson semigroup tensorized by the identity of $L^p(\mathcal{M})$. This semigroup satisfies all assumptions of [16]. Thus if we define our Hardy spaces $\mathcal{H}_{cr}^p(\mathbb{R},\mathcal{M})$ by the g-function $G_c(f)$ and $G_r(f)$ (which is the same as that defined by $S_c(f)$ and $S_r(f)$ in virtue of Theorem 5.6), then Corollary 6.4 is a particular case of a general result from [16]. We should emphasize that the method in [16] is completely different from ours. It is based on the H^∞ functional calculus. It seems that the method in [16] does not permit to deal with the Lusin square functions $S_c(f)$ and $S_r(f)$.

CHAPTER VII

INTERPOLATION

In this chapter, we consider interpolation for non-commutative Hardy spaces and BMO. The main results in this chapter are function space analogues of those in [27] for non-commutative martingales. On the other hand, they are also the extensions to the present non-commutative setting of the scalar results in [13]. Recall that the non-commutative L^p spaces associated with a semifinite von Neumann algebra form an interpolation scale with respect to both the complex and real interpolation methods. And, as the column (resp. row) subspaces of $L^p(\mathcal{M}\otimes B(L^2(\Omega)))$, the spaces $L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))$ form an interpolation scale also.

7.1. Complex interpolation

We first consider complex interpolation.

Let $\mathrm{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ and $\mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ (resp. $\mathrm{BMO}_c^{\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ and $\mathcal{H}_c^{p,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M})$) $(1\le p<\infty)$ be the non-commutative martingale BMO spaces and Hardy spaces defined in [18] with respect to the usual dyadic filtration $\mathcal{D}$ (resp. the dyadic filtration $\mathcal{D}'$) described in Chapter IV.

Lemma 7.1 *For $1<p<\infty$, we have*

$$(\mathrm{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}),\mathcal{H}_c^{1,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}} = \mathcal{H}_c^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}), \tag{7.1}$$

$$(\mathrm{BMO}_r^{\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}),\mathcal{H}_r^{1,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}} = \mathcal{H}_r^{p,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}), \tag{7.2}$$

$$(X,Y)_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}). \tag{7.3}$$

where $X=\mathrm{BMO}_{cr}^{\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ or $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M})$ and $Y=\mathcal{H}_{cr}^{1,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ or $L^1(L^\infty(\mathbb{R})\otimes\mathcal{M})$. Moreover, the same results hold for $\mathrm{BMO}_c^{\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M})$

and $\mathcal{H}_c^{p,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M})$.

Proof. For each $k\in\mathbb{N}$ and each projection p of $\mathcal{M}$ with $\tau(p)<\infty$, denote by $\mathcal{H}_c^{q,\mathcal{D}}(L^\infty(-2^k,2^k)\otimes p\mathcal{M}p)$ the subspace of $\mathcal{H}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$ consisting of elements supported on $(-2^k,2^k)$ and with values in $p\mathcal{M}p$. By dualizing Theorem 3.1 of [27] we get, for $1<r\leq q<\infty$,

$$\begin{aligned}&\Big(\mathcal{H}_c^{1,\mathcal{D}}(L^\infty(-2^k,2^k)\otimes p\mathcal{M}p),\mathcal{H}_c^{\frac{r}{r-1},\mathcal{D}}(L^\infty(-2^k,2^k)\otimes p\mathcal{M}p)\Big)_{\frac{r}{q}}\\ =\;&\mathcal{H}_c^{\frac{q}{q-1},\mathcal{D}}(L^\infty(-2^k,2^k)\otimes p\mathcal{M}p).\end{aligned}$$

Note that the union of all these $\mathcal{H}_c^{r,\mathcal{D}}(L^\infty(-2^k,2^k)\otimes p\mathcal{M}p)$ is dense in $\mathcal{H}_c^{r,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})$. By approximation we get

$$(\mathcal{H}_c^{1,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}),\mathcal{H}_c^{\frac{r}{r-1},\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{r}{q}}=\mathcal{H}_c^{\frac{q}{q-1},\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \tag{7.4}$$

Dualizing (7.4) we have

$$(\mathrm{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}),\mathcal{H}_c^{r,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{r}{q}}=\mathcal{H}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}). \tag{7.5}$$

Combining (7.4) and (7.5) we get (7.1) by Wolff's interpolation theorem (see [39]). The equalities (7.2), (7.3) and the arguments for the dyadic filtration $\mathcal{D}'$ can be proved similarly.

Theorem 7.2 *Let $1<p<\infty$. Then with equivalent norms,*

$$\begin{aligned}(\mathrm{BMO}_c(\mathbb{R},\mathcal{M}),\mathcal{H}_c^1(\mathbb{R},\mathcal{M}))_{\frac{1}{p}}&=\mathcal{H}_c^p(\mathbb{R},\mathcal{M}), &(7.6)\\ (\mathrm{BMO}_r(\mathbb{R},\mathcal{M}),\mathcal{H}_r^1(\mathbb{R},\mathcal{M}))_{\frac{1}{p}}&=\mathcal{H}_r^p(\mathbb{R},\mathcal{M}), &(7.7)\\ (X,Y)_{\frac{1}{p}}&=L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}). &(7.8)\end{aligned}$$

where $X=\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M})$ or $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M})$ and $Y=\mathcal{H}_{cr}^1(\mathbb{R},\mathcal{M})$ or $L^1(L^\infty(\mathbb{R})\otimes\mathcal{M})$.

Proof. Note that

$$\mathcal{H}_c^2(\mathbb{R},\mathcal{M}) = \mathcal{H}_c^{2,\mathcal{D}}(\mathbb{R},\mathcal{M}) = \mathcal{H}_c^{2,\mathcal{D}'}(\mathbb{R},\mathcal{M}).$$

Let $2 < q < \infty$. By Theorem 6.1 and Lemma 7.1 we have

$$\begin{aligned}
& (\mathrm{BMO}_c(\mathbb{R},\mathcal{M}),\mathcal{H}_c^2(\mathbb{R},\mathcal{M}))_{\frac{2}{q}} \\
= \;& (\mathrm{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})\cap \mathrm{BMO}_c^{\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}),\mathcal{H}_c^2(\mathbb{R},\mathcal{M}))_{\frac{2}{q}} \\
\subseteq \;& (\mathrm{BMO}_c^{\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}),\mathcal{H}_c^2(\mathbb{R},\mathcal{M}))_{\frac{2}{q}} \cap (\mathrm{BMO}_c^{\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}),\mathcal{H}_c^2(\mathbb{R},\mathcal{M}))_{\frac{2}{q}} \\
\subseteq \;& \mathcal{H}_c^{q,\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})\cap\mathcal{H}_c^{q,\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}) \\
= \;& \mathcal{H}_c^q(\mathbb{R},\mathcal{M}).
\end{aligned}$$

Then by duality

$$(\mathcal{H}_c^1(\mathbb{R},\mathcal{M}),\mathcal{H}_c^2(\mathbb{R},\mathcal{M}))_{\frac{2}{q}} \supseteq \mathcal{H}_c^{q'}(\mathbb{R},\mathcal{M}). \tag{7.9}$$

The converse of (7.9) can be easily proved since the map Φ defined by $\Phi(f) = \nabla f(x+t,y)\chi_\Gamma(x,y)$ is isometric from $\mathcal{H}_c^{q'}(\mathbb{R},\mathcal{M})$ to $L^{q'}(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma}))$ for $q \geq 1$. Thus we have

$$(\mathcal{H}_c^1(\mathbb{R},\mathcal{M}),\mathcal{H}_c^2(\mathbb{R},\mathcal{M}))_{\frac{2}{q}} = \mathcal{H}_c^{q'}(\mathbb{R},\mathcal{M}). \tag{7.10}$$

Dualizing this equality once more, we get

$$(\mathrm{BMO}_c(\mathbb{R},\mathcal{M}),\mathcal{H}_c^2(\mathbb{R},\mathcal{M}))_{\frac{2}{q}} = \mathcal{H}_c^q(\mathbb{R},\mathcal{M}). \tag{7.11}$$

Note that by Proposition 4.1 and Theorem 5.8, $\mathcal{H}_c^q$ is complemented in $L^q(L^\infty(\mathbb{R})\otimes\mathcal{M},L_c^2(\widetilde{\Gamma}))(1 < q < \infty)$ via the embedding Φ. Hence, from the interpolation result (2.3) we have

$$(\mathcal{H}_c^q(\mathbb{R},\mathcal{M}),\mathcal{H}_c^{q'}(\mathbb{R},\mathcal{M}))_{\frac{1}{2}} = \mathcal{H}_c^2(\mathbb{R},\mathcal{M}) \tag{7.12}$$

Combining (7.10), (7.11) and (7.12) we get (7.6) by Wolff's interpolation theorem (see [39]). (7.7) can be proved similarly. For (7.8), by Lemma 7.1 and Theorem 5.1,

$$\begin{aligned}
& (\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}} \\
= \ & (\mathrm{BMO}_{cr}^{\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M})\cap \mathrm{BMO}_{cr}^{\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}} \\
\subseteq \ & (\mathrm{BMO}_{cr}^{\mathcal{D}}(L^\infty(\mathbb{R})\otimes\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}} \\
& \quad \cap(\mathrm{BMO}_{cr}^{\mathcal{D}'}(L^\infty(\mathbb{R})\otimes\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}} \\
= \ & L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})
\end{aligned}$$

On the other hand, since $\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M})\supset L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M})$,

$$\begin{aligned}
& (\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}} \\
\supseteq \ & (L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}} \\
= \ & L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}).
\end{aligned}$$

Therefore,

$$(\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}).$$

By duality we have

$$(L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}), \mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M}))_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}).$$

Finally,

$$\begin{aligned}
(L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}), \mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M}))_{\frac{1}{p}} & \subseteq (\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}), \mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M}))_{\frac{1}{p}} \\
& \subseteq (\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p}}.
\end{aligned}$$

Hence

$$(\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}), \mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M}))_{\frac{1}{p}} = L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}).$$

Thus we have obtained all equalities in the theorem. ∎

Remark. We know little about $(\mathrm{BMO}_c(\mathbb{R},\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M})_{\frac{1}{p}}$ even for $p=2$.

7.2. Real interpolation

The following theorem concerns real interpolation.

Theorem 7.3 *Let $1\le p<\infty$. Then with equivalent norms,*

$$(X,Y)_{\frac{1}{p},p} = L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}). \tag{7.13}$$

where $X=\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M})$ or $L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M})$ and $Y=\mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M})$ or $L^1(L^\infty(\mathbb{R})\otimes\mathcal{M})$.

Proof. By Theorem 4.3 of [27] and Theorem 6.1 we have (using the same argument as above for the complex method)

$$(\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p},p} \subseteq L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}).$$

On the other hand, for $1<p<\infty$,

$$\begin{aligned}(\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p},p} &\supseteq (L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p},p}\\ &= L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}).\end{aligned}$$

Therefore

$$(\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}), L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p},p} = L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}), \quad 1<p<\infty.$$

By duality we have

$$(L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}), \mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M}))_{\frac{1}{p},p} = L^p(L^\infty(\mathbb{R})\otimes\mathcal{M}), \quad 1<p<\infty.$$

Noting again that

$$\begin{aligned}(L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}),\mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M}))_{\frac{1}{p},p} &\subseteq (\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}),\mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M}))_{\frac{1}{p},p}\\ &\subseteq (\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}),L^1(L^\infty(\mathbb{R})\otimes\mathcal{M}))_{\frac{1}{p},p},\end{aligned}$$

we conclude

$$\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}),\mathcal{H}^1_{cr}(\mathbb{R},\mathcal{M}))_{\frac{1}{p},p}=L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})),\quad 1<p<\infty. \quad \blacksquare$$

7.3. Fourier multipliers

We close this chapter with a result on Fourier multipliers. Recall that $H^1(\mathbb{R})$ denotes the classical Hardy space on $\mathbb{R}$. We will also need $H^1(\mathbb{R},H)$, the H^1 on $\mathbb{R}$ with values in a Hilbert space H. Recall that we say a bounded map $M: H^1(\mathbb{R})\rightarrow H^1(\mathbb{R})$ is a Fourier multiplier if there exists a function $m\in L^\infty(\mathbb{R})$ such that

$$\widehat{Mf}=m\widehat{f},\quad \forall f\in H^1(\mathbb{R})$$

where $\widehat{f}$ is the Fourier transform of f.

Theorem 7.4 *Let M be a Fourier multiplier of the classical Hardy space $H^1(\mathbb{R})$. Then M extends in a natural way to a bounded map on $\mathrm{BMO}_c(\mathbb{R},\mathcal{M})$ and $\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ for all $1\le p<\infty$ and*

$$\|M:\mathrm{BMO}_c(\mathbb{R},\mathcal{M})\rightarrow\mathrm{BMO}_c(\mathbb{R},\mathcal{M})\| \le c\,\|M:H^1(\mathbb{R})\rightarrow H^1(\mathbb{R})\|, \tag{7.14}$$

$$\|M:\mathcal{H}^p_c(\mathbb{R},\mathcal{M})\rightarrow\mathcal{H}^p_c(\mathbb{R},\mathcal{M})\| \le c\,\|M:H^1(\mathbb{R})\rightarrow H^1(\mathbb{R})\|. \tag{7.15}$$

Similar assertions also hold for $\mathrm{BMO}_r(\mathbb{R},\mathcal{M}),\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M}),\mathcal{H}^p_c(\mathbb{R},\mathcal{M})$ and $\mathcal{H}^p_{cr}(\mathbb{R},\mathcal{M})$.

Proof. Assume $\|M:H^1(\mathbb{R})\rightarrow H^1(\mathbb{R})\|=1$. Let H be the Hilbert space on which $\mathcal{M}$ acts. We start by showing the (well known) fact that M is bounded on $H^1(\mathbb{R},H)$.

Denote by R the Hilbert transform. Recall that $\|f\|_{H^1(\mathbb{R},H)} \backsimeq \|f\|_{L^1(\mathbb{R},H)}+\|Rf\|_{L^1(\mathbb{R},H)}$ for every $f \in H^1(\mathbb{R},H)$. Denote by $\{e_\lambda\}_{\lambda\in\Lambda}$ the orthogonal normalized basis of H. Then $f = (f_\lambda)_{\lambda\in\Lambda}$ with $f_\lambda = \langle e_\lambda, f\rangle e_\lambda$. Note that if $f \in H^1(\mathbb{R},H)$ then at most countably many f_λ's are non zero. Let $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables on some probability space (Ω, P) such that $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = \frac{1}{2}$, $\forall n \in \mathbb{N}$. Notice that $MR = RM$. Let $f \in H^1(\mathbb{R},H)$. Let $\{\lambda_n : n \in \mathbb{N}\}$ be an enumeration of the λ's such that $f_\lambda \neq 0$. Then by Khintchine's inequality,

$$\begin{aligned}
\|Mf\|_{H^1(\mathbb{R},H)} &\backsimeq \int_{\mathbb{R}}((\sum_{n\in\mathbb{N}}|Mf_{\lambda_n}|^2)^{\frac{1}{2}} + (\sum_{n\in\mathbb{N}}|RMf_{\lambda_n}|^2)^{\frac{1}{2}})dt \\
&\backsimeq \int_{\mathbb{R}}\int_{\Omega}|\sum_{n\in\mathbb{N}}\varepsilon_n Mf_{\lambda_n}|dP(\varepsilon)dt + \int_{\mathbb{R}}\int_{\Omega}|\sum_{n\in\mathbb{N}}\varepsilon_n MRf_{\lambda_n}|dP(\varepsilon)dt \\
&\backsimeq \int_{\Omega}\left\|M(\sum_{n\in\mathbb{N}}\varepsilon_n f_{\lambda_n})\right\|_{H^1(\mathbb{R},H)} dP(\varepsilon) \\
&\leq c\int_{\Omega}\left\|\sum_{n\in\mathbb{N}}\varepsilon_n f_{\lambda_n}\right\|_{H^1(\mathbb{R},H)} dP(\varepsilon) \\
&\leq c\,\|f\|_{H^1(\mathbb{R},H)}
\end{aligned}$$

Therefore, as announced

$$\left\|M : H^1(\mathbb{R},H) \to H^1(\mathbb{R},H)\right\| \leq c_1.$$

Then by transposition

$$\|M : \mathrm{BMO}(\mathbb{R},H) \to \mathrm{BMO}(\mathbb{R},H)\| \leq c_2;$$

whence, in virtue of (2.16),

$$\|M : \mathrm{BMO}_c(\mathbb{R},\mathcal{M}) \to \mathrm{BMO}_c(\mathbb{R},\mathcal{M})\| \leq c_2.$$

Thus by duality

$$\left\|M:\mathcal{H}_c^1(\mathbb{R},\mathcal{M})\to\mathcal{H}_c^1(\mathbb{R},\mathcal{M})\right\|\le c_3.$$

Then by Theorem 7.1 we have

$$\|M:\mathcal{H}_c^p(\mathbb{R},\mathcal{M})\to\mathcal{H}_c^p(\mathbb{R},\mathcal{M})\|\le c_4.$$

Hence we have obtained the assertion concerning the column spaces. The other assertions are immediate consequences of this one. ∎

Very recently, Junge and Musat got a John-Nirenberg theorem for BMO spaces of noncommutative martingales (see [17]). By using Proposition 4.1 and the dyadic trick of this dissertation, they got a John-Nirenberg theorem for noncommutative BMO spaces discussed here (which can also be proved as a consequence of the interpolation results established in this chapter). Unlike the classical case, the value of

$$\sup_{I\subset\mathbb{R}}\left\|\left(\frac{1}{|I|}\int_I|\varphi-\varphi_I|^p d\mu\right)^{\frac{1}{p}}\right\|_{\mathcal{M}} \tag{7.16}$$

for different $p, 0<p<\infty$ are no longer equivalent to each other. In fact, if $\mathcal{M}=M_n$ the algebra of n by n matrices, it can be proved that the best constant c_n such that

$$\sup_{I\subset\mathbb{R}}\left\|\frac{1}{|I|}\int_I|\varphi-\varphi_I|^2 d\mu\right\|_{M_n}^{\frac{1}{2}}\le c_n\sup_{I\subset\mathbb{R}}\left\|\frac{1}{|I|}\int_I|\varphi-\varphi_I|d\mu\right\|_{M_n}, \tag{7.17}$$

holds for all $\varphi\in\mathrm{BMO}_c(\mathbb{R},M_n)$ will be at least $c\log n$ as $n\to\infty$. And the corresponding constant for M_n valued martingales could be $cn^{\frac{1}{2}}$ if no additional assumption on the related filtration. What remains true is the equivalence of

$$\sup_{I\subset\mathbb{R}}\sup_{\tau|a|^p\le 1}|I|^{-\frac{1}{p}}\left\|(f-f_I)a\chi_I\right\|_{L^p(\mathbb{R},\mathcal{M})}+\sup_{I\subset\mathbb{R}}\sup_{\tau|a|^p\le 1}|I|^{-\frac{1}{p}}\left\|a\chi_I(f-f_I)\right\|_{L^p(\mathbb{R},\mathcal{M})} \tag{7.18}$$

for different $p, 2 \leq p < \infty$ (see Theorem 1.2 of [17]) and the equivalence of

$$\sup_{\text{cube } I \subset \mathbb{R}} \sup_{\tau|a|^p \leq 1,} \{|I|^{-\frac{1}{p}} \|(f - f_I) a \chi_I\|_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M})}\} \tag{7.19}$$

for different p, $2 \leq p < \infty$. See [17], [26] for more information on this.

CHAPTER VIII

NONCOMMUTATIVE JOHN-NIRENBERG INEQUALITY

8.1. Introduction and preliminaries

The classical BMO spaces have been successfully extended to the non-commutative setting in the last several years. A lot of work has been done on this subject (see [33], [18], [23], [27], [28]). We recall their definition in the tricial case. Let $\mathcal{M}$ be a von Neuman algebra with a semifinite trace τ. $\mathcal{M}_n$ is an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that $\cup_{n\geq 0}\mathcal{M}_n$ generates $\mathcal{M}$ (in the w$^*-$ topology). Denote by E_n the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$. A sequence $x = (x_k) \in L^p(\mathcal{M})$ is called a non-commutative martingale if $x_k \in L^p(\mathcal{M}_k)$ and $E_k x_m = x_k, \forall k \leq m$. Denote $d_k x = x_k - x_{k-1}$. The BMO spaces of non-commutative martingales are defined for $x = (x_k) \in L^1(\mathcal{M})$ as below:

$$
\begin{aligned}
\mathrm{BMO}_c(\mathcal{M}) &= \{x : ||x||_{\mathrm{BMO}_c(\mathcal{M})} = \sup_n \left\| E_n |\sum_{k=n}^{\infty} d_k x|^2 \right\|_{\mathcal{M}}^{\frac{1}{2}} < \infty\}; \\
\mathrm{BMO}_r(\mathcal{M}) &= \{x : ||x||_{\mathrm{BMO}_r(\mathcal{M})} = ||x^*||_{\mathrm{BMO}_c(\mathcal{M})} < \infty\}; \\
\mathrm{BMO}_{cr}(\mathcal{M}) &= \{x : ||x||_{\mathrm{BMO}_{cr}(\mathcal{M})} = \max\{||x||_{\mathrm{BMO}_c(\mathcal{M})}, ||x||_{\mathrm{BMO}_r(\mathcal{M})}\} < \infty\}.
\end{aligned}
$$

We can also consider BMO spaces for operator valued functions that are defined for $f \in L^\infty(\mathcal{M}, L^2_c(\mathbb{R}, \frac{1}{1+|t|^2}dt))$ as follows:

$$
\begin{aligned}
\mathrm{BMO}_c(\mathbb{R}, \mathcal{M}) &= \{f : \|f\|_{\mathrm{BMO}_c} = \sup_{\text{interval } I\subset\mathbb{R}} \{\left\| \left(\frac{1}{|I|} \int_I (f - f_I)^*(f - f_I) d\mu \right)^{\frac{1}{2}} \right\|_{\mathcal{M}} < \infty\}; \\
\mathrm{BMO}_r(\mathbb{R}, \mathcal{M}) &= \{f : \|f\|_{\mathrm{BMO}r} = \|f^*\|_{\mathrm{BMO}_c} < \infty\}; \\
\mathrm{BMO}_{cr}(\mathbb{R}, \mathcal{M}) &= \{f : \|f\|_{\mathrm{BMO}_{cr}} = \max\{\|f\|_{\mathrm{BMO}_c}, \|f\|_{\mathrm{BMO}r}\} < \infty\}.
\end{aligned}
$$

In [17], a non-commutative version of John-Nirenberg theorem was proved by considering the norms

$$|| \cdot ||_{BMO_c^p(\mathcal{M})} = \sup_{\tau|a|^p \le 1, a \in L^p(\mathcal{M}_n)} \|(x - x_{n-1})a\|_{L^p(\mathcal{M})}$$

. For the convenience of the reader, we state it as follows:

Theorem 8.1 *(Junge, Musat)For* $2 \le p < \infty$,

$$c||x||_{\mathrm{BMO}_{cr}(\mathcal{M})} \le \max\{||x||_{\mathrm{BMO}_c^p(\mathcal{M})}, ||x^*||_{\mathrm{BMO}_c^p(\mathcal{M})}\} \le cp||x||_{\mathrm{BMO}_{cr}(\mathcal{M})}.$$

This theorem does not hold if considering the column case or the row case separately, while we need to work on the column case and the row case separately very often; for example the non-commutative H^1- BMO duality theorem is proved for the column case and the row case separately. In this chapter, we get a non-commutative John-Nirenberg theorem in the column case and the row case separately. We introduce a new series of BMO norms for the non-commutative martingales and non-commutative functions as follows.

Definition 8.2 *For martingale difference* $(d_k x) \in L^1(\mathcal{M}), 2 \le p < \infty$, *we define*

$$\begin{aligned}
||x||_{\mathrm{BMO}_c^{\infty p}(\mathcal{M})} &= \sup_{\tau|a|^p \le 1, a \in L^p(\mathcal{M}_n)} \left\| (\sum_{k=n}^{\infty} d_k x) a \right\|_{\mathcal{H}_c^p(\mathcal{M})}; \\
||x||_{\mathrm{BMO}_r^{\infty p}(\mathcal{M})} &= \sup_{\tau|a|^p \le 1, a \in L^p(\mathcal{M}_n)} \left\| a(\sum_{k=n}^{\infty} d_k x) \right\|_{\mathcal{H}_r^p(\mathcal{M})}; \\
||x||_{\mathrm{BMO}_{cr}^{\infty p}(\mathcal{M})} &= \max\{||x||_{\mathrm{BMO}_c^{\infty p}(\mathcal{M})}, ||x||_{\mathrm{BMO}_r^{\infty p}(\mathcal{M})}\}.
\end{aligned}$$

It is easy to verify that $|| \cdot ||_{\mathrm{BMO}_c^{\infty p}(\mathcal{M})}$ $(|| \cdot ||_{\mathrm{BMO}_r^{\infty p}(\mathcal{M})}, || \cdot ||_{\mathrm{BMO}_c^{\infty p}(\mathcal{M})})$ are norms. When $p = 2$ they coincide with the norms $|| \cdot ||_{\mathrm{BMO}_c(\mathcal{M})}$ $(|| \cdot ||_{\mathrm{BMO}_r(\mathcal{M})}, || \cdot ||_{\mathrm{BMO}_{cr}(\mathcal{M})})$ defined in [33] and [18].

Definition 8.3 *For operator valued functions* $f \in L^\infty(\mathcal{M}, L_c^2(\mathbb{R}^n, \frac{1}{1+|t|^2}dt)),\ 2 \le p <$

∞, *we define*

$$\begin{aligned}
||f||_{\mathrm{BMO}_c^{\infty p}(\mathbb{R},\mathcal{M})} &= \sup_{cube\ I\subset\mathbb{R}} \sup_{\tau|a|^p\leq 1,} |I|^{-\frac{1}{p}} \left\|(f-f_I)a\chi_I\right\|_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M})}; \\
||f||_{\mathrm{BMO}_c^{\infty p}(\mathbb{R},\mathcal{M})} &= \sup_{cube\ I\subset\mathbb{R}} \sup_{\tau|a|^p\leq 1,} |I|^{-\frac{1}{p}} \left\|a(f-f_I)\chi_I\right\|_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M})}; \\
||f||_{\mathrm{BMO}_{cr}^{\infty p}(\mathbb{R},\mathcal{M})} &= \max\{||f||_{\mathrm{BMO}_c^{\infty p}(\mathbb{R},\mathcal{M})}, ||f||_{\mathrm{BMO}_r^{\infty p}(\mathbb{R},\mathcal{M})}\}.
\end{aligned}$$

where $f_I = |I|^{-1}\int_I f ds$.

It is easy to verify that $||\cdot||_{\mathrm{BMO}_c^{\infty p}(\mathbb{R},\mathcal{M})}$ $(||\cdot||_{\mathrm{BMO}_r^{\infty p}(\mathbb{R},\mathcal{M})},\ ||\cdot||_{\mathrm{BMO}_c^{\infty p}(\mathbb{R},\mathcal{M})})$ are norms. When $p=2$ they coincide with the norms $||\cdot||_{\mathrm{BMO}_c(\mathbb{R},\mathcal{M})}$ $(||\cdot||_{\mathrm{BMO}_r(\mathbb{R},\mathcal{M})},\ ||\cdot||_{\mathrm{BMO}_{cr}(\mathbb{R},\mathcal{M})})$ defined in Chapter II. We will prove in the next section that is the case for all $2\leq p<\infty$.

8.2. Main results

Lemma 8.1 *For* $2\leq p<\infty$, *we have*

$$cp^{-1}||b||_{\mathcal{M}} \leq \sup_{\tau|a|^p\leq 1} ||ba||_{\mathcal{H}_c^p(\mathcal{M})} \leq cp^{\frac{1}{2}}||b||_{\mathcal{M}}.$$

Proof. Note $||\cdot||_{\mathcal{H}_c^p(\mathcal{M})} \leq cp^{\frac{1}{2}}||\cdot||_{L^p(\mathcal{M})}$ (see [36], Remark 5.4 as a reference for the constant we use here), we have

$$\sup_{\tau|a|^p\leq 1} ||ba||_{\mathcal{H}_c^p(\mathcal{M})} \leq cp^{\frac{1}{2}} \sup_{\tau|a|^p\leq 1} ||ba||_{L^p(\mathcal{M})} = cp^{\frac{1}{2}}||b||_{\mathcal{M}}.$$

For the first inequality, without loss of generality assume $||b||_{\mathcal{M}}=1$. Note that for self adjoint $x\in\mathcal{M}, ||x||_{L^p(\mathcal{M})}\leq cp||x||_{\mathcal{H}_c^p(\mathcal{M})}$ (see [36], Remark 5.4). Then

$$\begin{aligned}
||b^*||_{\mathcal{M}} &= \sup_{\tau|f|^{2p}\leq 1} ||fb^*||_{L^{2p}} \\
&= \sup_{\tau|f|^{2p}\leq 1} ||b|f|^2b^*||_{L^p}^{\frac{1}{2}}
\end{aligned}$$

$$\leq \quad cp^{\frac{1}{2}} \sup_{\tau|f|^{2p}\leq 1} ||b|f|^2 b^*||^{\frac{1}{2}}_{\mathcal{H}^p_c(\mathcal{M})}$$
$$\leq \quad cp^{\frac{1}{2}} \sup_{\tau|a|^p\leq 1} ||ba||^{\frac{1}{2}}_{\mathcal{H}^p_c(\mathcal{M})}.$$

And then $cp^{-1}||b||_{\mathcal{M}} \leq \sup_{\tau|a|^p\leq 1} ||ba||_{\mathcal{H}^p_c(\mathcal{M})}$.

Theorem 8.2 *For* $2 \leq p < \infty$,

$$cp^{-1}||x||_{\mathrm{BMO}_c^{\infty_2}(\mathcal{M})} \leq ||x||_{\mathrm{BMO}_c^{\infty_p}(\mathcal{M})} \leq cp||x||_{\mathrm{BMO}_c^{\infty_2}(\mathcal{M})}.$$
$$cp^{-1}||x||_{\mathrm{BMO}_r^{\infty_2}(\mathcal{M})} \leq ||x||_{\mathrm{BMO}_r^{\infty_p}(\mathcal{M})} \leq cp||x||_{\mathrm{BMO}_r^{\infty_2}(\mathcal{M})}.$$

Proof. We only prove the inequalities for the column case, the row case can be proved similarly. By the previous lemma and Hölder's inequality, we have

$$\begin{aligned}
&||E_n \sum_{k=n}^{\infty} |d_k x|^2||^2_{\mathcal{M}} \\
\leq\ & \sup_{\tau b\leq 1, b\geq 0} \tau \sum_{k=n}^{\infty} |d_k x|^2 b + ||x_n - x_{n-1}||^2_{\mathcal{M}} \\
=\ & \sup_{\tau b\leq 1, b\geq 0} \tau \sum_{k=n}^{\infty} |(d_k x) b^{\frac{1}{p}}|^2 b^{\frac{p-2}{p}} + cp^2 \sup_{\tau|a|^p\leq 1} ||(x_n - x_{n-1})a||^2_{\mathcal{H}^p_c(\mathcal{M})} \\
\leq\ & \sup_{\tau b\leq 1, b\geq 0} \left\| \sum_{k=n}^{\infty} |(d_k x) b^{\frac{1}{p}}|^2 \right\|_{L^{\frac{p}{2}}} \left\| b^{\frac{p-2}{p}} \right\|_{L^{(\frac{p}{2})'}} + cp^2 \sup_{\tau|a|^p\leq 1} ||(x_n - x_{n-1})a||^2_{\mathcal{H}^p_c(\mathcal{M})} \\
\leq\ & \sup_{\tau b\leq 1, b\geq 0} \left\| (x - x_n) b^{\frac{1}{p}} \right\|_{\mathcal{H}^p_c(\mathcal{M})} + cp^2 \sup_{\tau|a|^p\leq 1} ||(x_n - x_{n-1})a||^2_{\mathcal{H}^p_c(\mathcal{M})} \\
\leq\ & cp^2 \sup_{\tau|a|^p\leq 1, a\in L^p(\mathcal{M}_n)} \|(x - x_{n-1})a\|^2_{\mathcal{H}^p_c(\mathcal{M})} = cp^2 ||x||^2_{\mathrm{BMO}_c^{\infty_p}(\mathcal{M})}.
\end{aligned}$$

By taking the suprem over n, we get the first inequality. Conversely, by the previous lemma,

$$\begin{aligned}
||x||_{\mathrm{BMO}_c^{\infty_p}(\mathcal{M})} &\leq \sup_{\tau|a|^p\leq 1, a\in L^p(\mathcal{M}_n)} \|(x - x_n)a\|_{\mathcal{H}^p_c(\mathcal{M})} + \sup_{\tau|a|^p\leq 1} ||(x_n - x_{n-1})a||_{\mathcal{H}^p_c(\mathcal{M})} \\
&\leq \sup_{\tau|a|^p\leq 1, a\in L^p(\mathcal{M}_n)} \|(x - x_n)a\|_{\mathcal{H}^p_c(\mathcal{M})} + cp^{\frac{1}{2}} ||x_n - x_{n-1}||_{\mathcal{M}} \\
&\leq \sup_{\tau|a|^p\leq 1, a\in L^p(\mathcal{M}_n)} \left\| (d_k x a)_{k=n+1}^{\infty} \right\|_{L^p(\mathcal{M}, l^2_c)} + cp^{\frac{1}{2}} ||x||_{\mathrm{BMO}_c^{\infty_2}(\mathcal{M})}. \qquad (8.1)
\end{aligned}$$

Note, by the Hahn Banach theorem and the duality between $\mathcal{H}_c^1(\mathcal{M})$ and $\mathrm{BMO}_c^{\infty_2}(\mathcal{M})$ (for the general case, see [18]), there exists a $(b_n)_{n=1}^{\infty} \in L^{\infty}(\mathcal{M}, l_c^2)$ such that

$$\|(b_n)_{n=1}^{\infty}\|_{L^p(\mathcal{M},l_c^2)} = ||x||_{\mathrm{BMO}_c^{\infty_2}(\mathcal{M})}.$$

and

$$d_k x = E_k b_k - E_{k-1} b_k.$$

Thus by Hölder's inequality and the Stein inequality for non-commutative martingales:

$$\begin{aligned}
& \sup_{\tau|a|^p \leq 1, a \in L^p(\mathcal{M}_n)} \left\|(d_k x a)_{k=n+1}^{\infty}\right\|_{L^p(\mathcal{M},l_c^2)} \\
\leq & \sup_{\tau|a|^p \leq 1, a \in L^p(\mathcal{M}_n)} \left\|(E_k(b_k a))_{k=n+1}^{\infty}\right\|_{L^p(\mathcal{M},l_c^2)} + \sup_{\tau|a|^p \leq 1, a \in L^p(\mathcal{M}_n)} \|(E_k b_k a)_{k=n}^{\infty}\|_{L^p(\mathcal{M},l_c^2)} \\
\leq & \; cp \sup_{\tau|a|^p \leq 1, a \in L^p(\mathcal{M}_n)} \left\|(b_k a)_{k=n+1}^{\infty}\right\|_{L^p(\mathcal{M},l_c^2)} \\
= & \; cp \sup_{\tau|a|^p \leq 1, a \in L^p(\mathcal{M}_n)} \left\| |a|^{\frac{1}{2}} (\sum_{k=1}^{\infty} |b_k|^2)^{\frac{1}{2}} |a|^{\frac{1}{2}} \right\|_{L^p(\mathcal{M})} \\
\leq & \; cp \left\| \sum_{k=1}^{\infty} |b_k|^2 \right\|_{L^{\infty}(\mathcal{M})}^{\frac{1}{2}} \\
= & \; cp||x||_{\mathrm{BMO}_c^{\infty_2}(\mathcal{M})}.
\end{aligned}$$

Combining this with (8.1) we finishes the proof.

Remark 8.1 *From the proof of this theorem, when considering the regular case (i.e. there exists a positive constant d such that $E_n x \leq d E_{n-1} x$ for all $n \in \mathbb{N}$, $x \in \mathcal{M}_+$.) we can have $||x||_{\mathrm{BMO}_c^{\infty_2}(\mathcal{M})} \leq c||x||_{\mathrm{BMO}_c^{\infty_p}(\mathcal{M})}$ for an absolute constant c.*

By the dyadic trick interpreted in Chapter IV and Chapter VI (Proposition 4.1, Corollary 6.3), we could deduce similar results for the BMO spaces of operator valued functions from the previous theorem. But the constants will not be good when we consider functions defined on $\mathbb{R}$ for big n's. In the following, we give a direct proof for the function case.

Theorem 8.3 *For* $f \in L^\infty(\mathcal{M}, L_c^2(\mathbb{R}, \frac{dt}{1+t^2})$,

$$\begin{aligned} c||f||_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})} &\leq ||f||_{\mathrm{BMO}_c^{\infty_p}(\mathbb{R},\mathcal{M})} \leq cp||f||_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})}; \\ c||f||_{\mathrm{BMO}_r^{\infty_2}(\mathbb{R},\mathcal{M})} &\leq ||x||_{\mathrm{BMO}_r^{\infty_p}(\mathbb{R},\mathcal{M})} \leq cp||f||_{\mathrm{BMO}_r^{\infty_2}(\mathbb{R},\mathcal{M})}. \end{aligned}$$

Proof. We only prove the column case.

To prove $c||f||_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})} \leq ||f||_{\mathrm{BMO}_c^{\infty_p}(\mathbb{R},\mathcal{M})}$, we have

$$\begin{aligned} ||f||_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})} &= \sup_{I\subset\mathbb{R}} |||I|^{-1}\int_I |f-f_I|^2 ds||_{\mathcal{M}}^{\frac{1}{2}} \\ &= \sup_{I\subset\mathbb{R}} \sup_{\tau|a|^2\leq 1, a\geq 0} |I|^{-\frac{1}{2}} \left\|(f-f_I)a\chi_I\right\|_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\ &= \sup_{I\subset\mathbb{R}} \sup_{\tau|a|^2\leq 1, a\geq 0} |I|^{-\frac{1}{2}} \left\|S_c\left((f-f_I)\chi_I a\right)\right\|_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})} \qquad (8.2) \end{aligned}$$

Note for function g, supp $g = I$ with $|I| = N < \infty$, we can choose a constant $c_0 > 0$ such that

$$||S_c(g)||^2_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})} \leq 2\tau \int_{c_0 I} S_c^2(g)dt. \qquad (8.3)$$

In fact (without loss of generality assume $I = (0, N]$), for $t \neq s \in \mathbb{R}$ and some constants $c_1, c_2 > 0$ we have

$$\begin{aligned} \tau \iint_\Gamma |\bigtriangledown g(x+t,y)|^2 dxdy &= \iint_\Gamma \tau|\int_0^N \bigtriangledown P_y(x+t-s)g(s)ds|^2 dxdy \\ &\leq N\tau \iint_\Gamma \int_0^N |\bigtriangledown P_y(x+t-s)|^2|g(s)|^2 dsdxdy \\ &\leq N\tau \int_0^N \iint_\Gamma \frac{c_1}{(x+t-s)^4+y^4} dxdy|g(s)|^2 ds \\ &\leq N\tau \int_0^N \frac{c_2}{(t-s)^2}|g(s)|^2 ds. \end{aligned}$$

Then, for $c_0 > 2c_2 + 2$,

$$\begin{aligned}
\tau \int_{|t|>c_0 N} S_c^2(g)(t)dt &\leq \tau \int_{|t|>c_0 N} N \int_0^N \frac{c_2}{(t-s)^2} |g(s)|^2 ds dt \\
&\leq \tau N \int_0^N \int_{|t|>c_0 N} \frac{c_2}{(t-s)^2} dt |g(s)|^2 ds \\
&\leq \tau N \int_0^N \frac{1}{2N} |g(s)|^2 ds \\
&= \frac{1}{2} ||g||^2_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
&= \frac{1}{2} \tau \int_{\mathbb{R}} S_c^2(g)(t)dt.
\end{aligned}$$

We then get (8.3). Combining it with (8.2), we have

$$\begin{aligned}
& ||f||_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})} \\
\leq\ & \sqrt{2} \sup_{I\subset\mathbb{R}} \sup_{\tau|a|^2\leq 1, a\geq 0} |I|^{-\frac{1}{2}} \left\| S_c\left((f-f_I)\chi_I a\right) \chi_{c_0 I} \right\|_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})}. \\
\leq\ & \sqrt{2} \sup_{I\subset\mathbb{R}} \sup_{\tau|a|^2\leq 1, a\geq 0} |I|^{-\frac{1}{2}} \left\| S_c\left((f-f_I)\chi_I a^{\frac{2}{p}}\right) a^{\frac{p-2}{p}} \chi_{c_0 I} \right\|_{L^2(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq\ & \sqrt{2} \sup_{I\subset\mathbb{R}} \sup_{\tau|a|^2\leq 1, a\geq 0} |I|^{-\frac{1}{2}} \left\| S_c\left((f-f_I)\chi_I a^{\frac{2}{p}}\right) \right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \left\| a^{\frac{p-2}{p}} \chi_{c_0 I} \right\|^{\frac{p-2}{2p}}_{L^{\frac{2p}{p-2}}(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
\leq\ & \sqrt{2} \sup_{I\subset\mathbb{R}} \sup_{\tau|a|^p\leq 1} |I|^{-\frac{1}{p}} \left\| S_c\left((f-f_I)\chi_I a\right) \right\|_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
=\ & \sqrt{2} ||f||_{\mathrm{BMO}_c^{\infty_p}(\mathbb{R},\mathcal{M})}.
\end{aligned}$$

To prove the converse inequality, assume $f \in \mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})$ and by the Hahn Banach theorem and the duality between $\mathcal{H}_c^1(\mathbb{R},\mathcal{M})$ and $\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})$ proved in Chapter III, there exists a $h \in L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))$ such that

$$c^{-1} \|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))} ||f||_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})} \leq c \|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))}$$

and

$$f = \Psi(h) = \int_{\mathbb{R}} \iint_{\Gamma} h(x,y,t) Q_y(x+t-s) dy dx dt \tag{8.4}$$

where $Q_y(x) = \nabla P_y(x)$. Fix an interval I, set

$$\begin{aligned} h_1(x,y,t) &= h(x,y,t)\chi_{2I}(t) \\ h_2(x,y,t) &= h(x,y,t)\chi_{(2I)^c}(t). \end{aligned}$$

Let

$$B_I = \int_{-\infty}^{+\infty} \iint\limits_{\Gamma} Q_I h_2 dydxdt$$

with the notation $Q_I(x,t) = \frac{1}{|I|}\int_I Q_y(x+t-s)ds$. Now, for $a \in L^p(\mathcal{M})$,

$$\begin{aligned} & (f(s) - B_I) \\ = & \int_{(2I)^c} \iint\limits_{\Gamma} (Q_y(x+t-s) - Q_I)hdxdydt \\ & + \int_{-\infty}^{+\infty} \iint\limits_{\Gamma} Q_y(x+t-s)h_1 dxdydt \\ = & A + B. \end{aligned}$$

Notice that

$$\iint\limits_{\Gamma} |Q_y(x+t-s) - Q_I|^2 dxdy \leq c|I|^2(t-C_I)^{-4}$$

for every $t \in (2I)^c$ and $s \in I$. By the proof of Lemma 3.2,

$$\begin{aligned} \|A\|_{L^\infty(\mathbb{R})\otimes\mathcal{M}} &\leq c||\int_{(2I)^c}(t-C_I)^{-2}dt \int_{(2I)^c}(t-C_I)^2 \iint\limits_{\Gamma} h^* hdxdy|I|^2(t-C_I)^{-4}dt||_{\mathcal{M}} \\ &\leq ||\frac{c}{|I|}\int_{(2I)^c}|I|^2(t-C_I)^{-2} \iint\limits_{\Gamma} h^* hdxdydt||_{\mathcal{M}} \\ &\leq c\,\|h\|^2_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L^2_c(\widetilde{\Gamma}))} \leq c\,\|f\|_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})}. \end{aligned}$$

And by duality between $\mathcal{H}_c^p(\mathbb{R},\mathcal{M})$ and $\mathcal{H}_c^{p'}(\mathbb{R},\mathcal{M})$ and Hölder's inequality, for $a \in$

$L^p(\mathcal{M}), \tau|a|^p \leq 1,$

$$\begin{aligned}
||Ba||_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M})} &\leq cp \sup_{||f||_{\mathcal{H}_c^{p'}(\mathbb{R},\mathcal{M})}=1} |\tau \int_{\mathbb{R}} \int_{\mathbb{R}} \iint\limits_{\Gamma} h_1^* a Q_y(x+t-s) dxdydt f(s) ds| \\
&= cp \sup_{||f||_{\mathcal{H}_c^{p'}(\mathbb{R},\mathcal{M})}=1} |\tau \int_{\mathbb{R}} \iint\limits_{\Gamma} h_1^* a \nabla f(t+x,y) dxdydt| \\
&\leq cp(\tau \int_{\mathbb{R}} |(\iint\limits_{\Gamma} h_1^* h_1 dxdy)^{\frac{1}{2}} a|^p dt)^{\frac{1}{p}} \\
&\leq cp||(\iint\limits_{\Gamma} h_1^* h_1 dxdydt)^{\frac{1}{2}}||_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M})} ||a\chi_{2I}||_{L^p(L^\infty(\mathbb{R})\otimes\mathcal{M})} \\
&\leq cp|I|^{\frac{1}{p}} \|h\|_{L^\infty(L^\infty(\mathbb{R})\otimes\mathcal{M}, L_c^2(\widetilde{\Gamma}))} \leq cp|I|^{\frac{1}{p}} \|f\|_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})}
\end{aligned}$$

Combining the estimation on A and B we have

$$\begin{aligned}
||f||_{\mathrm{BMO}_c^{\infty_p}(\mathbb{R},\mathcal{M})} &\leq |I|^{-\frac{1}{p}} \sup_{\tau|a|^p\leq 1} (||Aa\chi_I||_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M})} + ||Ba||_{\mathcal{H}_c^p(\mathbb{R},\mathcal{M})}) \\
&\leq cp^{\frac{1}{2}} ||A||_{L^\infty(\mathbb{R})\otimes\mathcal{M}} + cp \|f\|_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})} \\
&\leq cp \|f\|_{\mathrm{BMO}_c^{\infty_2}(\mathbb{R},\mathcal{M})}
\end{aligned}$$

This completes the proof.

CHAPTER IX

PARAPRODUCTS FOR MATRIX VALUED FUNCTIONS AND NON-COMMUTATIVE MARTINGALES

9.1. Introduction

Denote by M_n the algebra of $n \times n$ matrices. Let $(\mathbb{T}, \mathcal{F}_k, dt)$ be the unit circle with Haar measure and the usual dyadic filtration. Let b be an M_n valued function on $\mathbb{T}$. The matrix valued dyadic paraproduct associated with b, denoted by π_b, is the operator defined as

$$\pi_b(f) = \sum_k (d_k b)(E_{k-1} f), \quad \forall f \in L^2(\ell_n^2). \tag{9.1}$$

Here $E_k f$ is the conditional expectation of f with respect to $\mathcal{F}_k$, i.e. the unique $\mathcal{F}_k$-measurable function such that

$$\int_F E_k f dt = \int_F f dt, \quad \forall F \in \mathcal{F}_k.$$

And $d_k b$ is defined to be $E_k b - E_{k-1} b$.

In the classical case (when b is a scalar valued function), paraproducts are usually considered as dyadic singular integrals and play important roles in the proof of the classical T(1) theorem. It is well known that

$$\|\pi_b\|_{L^2 \to L^2} \backsimeq \|b\|_{BMO_d},$$

where BMO$_d$ denotes the dyadic BMO norm defined as

$$\|b\|_{BMO_d} = \sup_m \|E_m \sum_{k=m}^{\infty} |d_k b|^2 \|\|_{L^\infty}^{\frac{1}{2}}.$$

And by the Calderón-Zygmund decomposition and the Marcinkiewicz interpolation theorem, we have $||\pi_b||_{L^p \to L^p} \backsimeq ||\pi_b||_{L^p \to L^p} \backsimeq ||b||_{BMO_d}$ for all $1 < p < \infty$.

When b is M_n valued, it was proved by Katz ([20]) and independently by Nazarov, Treil and Volberg ([29], see [31] for another proof by Pisier) that

$$\|\pi_b\|_{L^2(\ell_n^2) \to L^2(\ell_n^2)} \leq c \log(n+1) \, \|b\|_{BMO_c} . \tag{9.2}$$

Here $\| \cdot \|_{\mathrm{BMO}_c}$ is the column BMO norm defined by

$$\|b\|_{BMO_c} = \sup_m \left\| E_m \sum_{k=m}^{\infty} (d_k b)^* (d_k b) \right\|_{L^\infty(M_n)}^{\frac{1}{2}},$$

where $(d_k b)^*$ is the adjoint of $d_k b$. Nazarov, Pisier, Treil and Volberg ([28]) proved later that the constant $c \log(n+1)$ in (9.2) is optimal. Thus the BMO_c norm does not dominate $\|\pi_b\|_{L^2(\ell_n^2) \to L^2(\ell_n^2)}$ uniformly over n.

Can we expect something weaker? In particular, does there exist a constant c independent of n such that, for every $n \in \mathbb{N}$,

$$\|\pi_b\|_{L^2(\ell_n^2) \to L^2(\ell_n^2)} \leq c \, \|b\|_{L^\infty(M_n)} ? \tag{9.3}$$

Some known facts made (9.3) look hopeful. For example, the Hankel operator associated with the M_n valued function b has norm equivalent to $||b||_{(H^1(S^1))^*}$. Here $|| \cdot ||_{(H^1(S^1))^*}$ denotes the dual norm on the trace class valued Hardy space $H^1(S^1)$. And S. Petermichl proved a close relation between π_b and the Hankel operators associated with b (see [30]).

In this paper, we prove the following theorem, which shows there does not exist any constant c independent of n such that (9.3) holds.

Theorem 9.1 *For every $n \in \mathbb{N}$, there exists an M_n valued function b with $\|b\|_{L^\infty(M_n)} \leq 1$ but such that*

$$\|\pi_b\|_{L^2(\ell_n^2) \to L^2(\ell_n^2)} \geq c \log(n+1),$$

where $c > 0$ is independent of n.

This also gives a new proof that the constant $c\log(n+1)$ in (9.2) is optimal.

Denote by S^p the Schatten p class on ℓ^2. For $f \in L^p(S^p)$, we define $\pi_b(f)$ as in (9.1) also. As pointed out in [31], it is easy to check that $\|\pi_b\|_{L^2(S^2)\to L^2(S^2)} = \|\pi_b\|_{L^2(\ell^2)\to L^2(\ell^2)}$. For scalar valued b, as we mentioned previously, we have $||\pi_b||_{L^p\to L^p} \backsimeq ||\pi_b||_{L^q\to L^q}$. We wonder if this is still true for matrix valued b, i.e. if π_b's boundedness on $L^p(S^p)$ implies their boundedness on $L^q(S^q)$ for all $1 < p, q < \infty$.

More generally, we can consider paraproducts associated with non-commutative martingales. Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized faithful trace τ. For $1 \leq p < \infty$, we denote by $L^p(\mathcal{M})$ the non-commutative L^p space associated with $(\mathcal{M}, \tau)$. Recall the norm in $L^p(\mathcal{M})$ is defined as

$$\|f\|_p = (\tau|x|^p)^{\frac{1}{p}}, \quad \forall f \in L^p(\mathcal{M}),$$

where $|f| = (f^*f)^{\frac{1}{2}}$. For convenience, we usually set $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$. Let $\mathcal{M}_k$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that $\cup_{k\geq 0}\mathcal{M}_k$ generates $\mathcal{M}$ in the w$^*-$ topology. Denote by E_k the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_k$. E_k is a norm 1 projection of $L^p(\mathcal{M})$ onto $L^p(\mathcal{M}_k)$. For $1 \leq p \leq \infty$, a sequence $f = (f_k)_{k\geq 0}$ with $f_k \in L^p(\mathcal{M}_k)$ is called a bounded non-commutative L^p-martingale, denoted by $(f_k)_{k\geq 0} \in L^p(\mathcal{M})$, if $E_k f_m = f_k, \forall k \leq m$ and

$$||(f_k)_{k\geq 0}||_{L^p(\mathcal{M})} = \sup_k ||f_k||_{L^p(\mathcal{M})} < \infty.$$

Because of the uniform convexity of the space $L^p(\mathcal{M})$, for $1 < p < \infty$, we can and will identify the space of all bounded $L^p(\mathcal{M})$-martingales with $L^p(\mathcal{M})$ itself. In particular, for any $f \in L^p(\mathcal{M})$, set $f_k = E_k f$, then $f = (f_k)_{k\geq 0}$ is a bounded $L^p(\mathcal{M})$-martingale and $||(f_k)_{k\geq 0}||_{L^p(\mathcal{M})} = ||f||_{L^p(\mathcal{M})}$. Denote by $d_k f = E_k f - E_{k-1} f$.

We say an increasing filtration $\mathcal{M}_k$ is "regular" if there exists a constant $c > 0$ such that, for any $m, a \in \mathcal{M}_m, a \geq 0$,

$$||a||_\infty \leq c||E_{m-1}a||_\infty.$$

For $\mathcal{M}$ with a regular filtration $\mathcal{M}_k$, $b \in L^2(\mathcal{M})$, we define paraproducts $\pi_b, \widetilde{\pi}_b$ as operators for bounded $L^p(\mathcal{M})$ $(1 < p < \infty)$-martingales $f = (f_k)_{k\geq 0}$ as

$$\pi_b(f) = \sum_k d_k b f_{k-1}, \qquad \widetilde{\pi}_b(f) = \sum_k f_{k-1} d_k b.$$

We prove the following result for π_b and $\widetilde{\pi}_b$:

Theorem 9.2 *Let $1 < p < q < \infty$, if $\widetilde{\pi}_b$ and π_b are both bounded on $L^p(\mathcal{M})$ then they are both bounded on $L^q(\mathcal{M})$.*

We still do not know what happens when $p > q$.

9.2. Proof of Theorem 9.1 and application to "Sweep" functions.

Denote by tr the usual trace on M_n and $S_n^p (1 \leq p < \infty)$ the Schatten p classes on ℓ_n^2.

Proof of Theorem 9.1. Let $c(n)$ be the best constant such that

$$\|\pi_b\|_{L^2(\ell_n^2)\to L^2(\ell_n^2)} \leq c(n)\, \|b\|_{L^\infty(M_n)}, \quad \forall b \in L^\infty(M_n).$$

Denoting by T the triangle projection on S_n^1, we are going to show

$$\|T\|_{S_n^1 \to S_n^1} \leq c(n).$$

Once this is proved, we are done since $\|T\|_{S_n^1 \to S_n^1} \backsim \log(n+1)$ (see [22]). Note that every A in the unit ball of S_n^1 can be written as

$$A = \sum_m \lambda^{(m)} \alpha^{(m)} \otimes \beta^{(m)}$$

with $\sum_m \lambda^{(m)} \leq 1$, $\sup_m\{||\alpha^{(m)}||_{\ell_n^2}, ||\beta^{(m)}||_{\ell_n^2}\} \leq 1$. Therefore, we only need to show

$$\|T(\alpha \otimes \beta)\|_{S_n^1} \leq c(n) \|\alpha\|_{\ell_n^2} \|\beta\|_{\ell_n^2}, \quad \forall \alpha = (\alpha_k)_k, \beta = (\beta_k)_k \in \ell_n^2. \tag{9.4}$$

Let D be the diagonal M_n valued function defined as

$$D = \sum_{i=1}^n r_i e_i \otimes e_i$$

where r_i is the i-th Rademacher function on $\mathbb{T}$ and $(e_i)_{i=1}^n$ is the canonical basis of ℓ_n^2. Given $\alpha = (\alpha_k)_k, \beta = (\beta_k)_k \in \ell_n^2$, let

$$f = D\alpha, g = D\beta.$$

Then $f, g \in L^2(\ell_n^2)$, and

$$\|f\|_{L^2(\ell_n^2)} = \|\alpha\|_{\ell_n^2}, \|g\|_{L^2(\ell_n^2)} = \|\beta\|_{\ell_n^2}. \tag{9.5}$$

It is easy to verify

$$\sum_k E_{k-1} f \otimes d_k g = D(\sum_{i<j\leq n} \alpha_i \beta_j e_i \otimes e_j)D.$$

and

$$\left\|\sum_k E_{k-1} f \otimes d_k g\right\|_{L^1(S_n^1)} = \left\|\sum_{i<j\leq n} \alpha_i \beta_j e_i \otimes e_j\right\|_{S_n^1} = \|T(\alpha \otimes \beta)\|_{S_n^1}. \tag{9.6}$$

On the other hand, by the duality between $L^1(S_n^1)$ and $L^\infty(M_n)$, we have,

$$\left\|\sum_k E_{k-1} f \otimes d_k g\right\|_{L^1(S_n^1)} = \sup\{ tr \int \sum_k d_k b(E_{k-1} f \otimes d_k g), \|b\|_{L^\infty(M_n)} \leq 1\}$$

$$
\begin{aligned}
&\leq \sup\{ \|\pi_b(f)\|_{L^2(\ell^2_n)} \|g\|_{L^2(\ell^2_n)}, \|b\|_{L^\infty(M_n)} \leq 1\} \\
&\leq c(n) \|f\|_{L^2(\ell^2_n)} \|g\|_{L^2(\ell^2_n)} . \qquad (9.7)
\end{aligned}
$$

Combining (9.7), (9.5) and (9.6) we get (9.4) and the proof is complete. ∎

Recall that the square function of b is defined as

$$S(b) = (\sum_k |d_k b|^2)^{\frac{1}{2}}.$$

The so called "sweep" function is just the square of the square function, for this reason we denote it by $S^2(b)$,

$$S^2(b) = \sum_k |d_k b|^2.$$

In the classical case, we know that

$$
\begin{aligned}
||S(b)||_{BMO_d} &\leq c||b||_{BMO_d} \qquad (9.8) \\
||S^2(b)||_{BMO_d} &\leq c||b||^2_{BMO_d} \qquad (9.9)
\end{aligned}
$$

When considering square functions $S(b)$ for M_n valued functions b, a similar result remains true with an absolute constant.

Proposition 9.3 *For any $n \in \mathbb{N}$, and any M_n valued function b, we have*

$$||S(b)||_{BMO_c} \leq \sqrt{2}||b||_{BMO_c}$$

Proof. Since we are in the dyadic case, we have

$$
\begin{aligned}
||S(b)||^2_{BMO_c} &\leq 2 \sup_m ||E_m[(S(b) - E_m S(b))^*(S(b) - E_m S(b))]||_{L^\infty(M_n)} \\
&= 2 \sup_m ||E_m S^2(b) - (E_m S(b))^2||_{L^\infty(M_n)}
\end{aligned}
$$

Note

$$E_m S^2(b) - \sum_{k=1}^{m} |d_k b|^2 \geq E_m S^2(b) - (E_m S(b))^2 \geq 0.$$

We get

$$\begin{aligned} ||S(b)||^2_{BMO_c} &\leq 2 \sup_m ||E_m S^2(b) - \sum_{k=1}^{m} |d_k b|^2||_{L^\infty(M_n)} \\ &= 2 \sup_m ||E_m \sum_{k=m+1} |d_k b|^2||_{L^\infty(M_n)} \\ &\leq 2||b||^2_{BMO_c}. \blacksquare \end{aligned}$$

Matrix valued sweep functions have been studied in [1], [8] etc. Unlike in the case of square functions, it is proved in [1] that the best constant c_n such that

$$||S^2(b)||_{BMO_c} \leq c_n ||b||^2_{BMO_c} \tag{9.10}$$

is $c \log(n+1)$. The following result shows that the best constant c_n is still $c \log(n+1)$ even if we replace $|| \cdot ||_{BMO_c}$ by the bigger norm $|| \cdot ||_{L^\infty(M_n)}$ in the right side of (9.10).

Theorem 9.4 *For every $n \in \mathbb{N}$, there exists an M_n valued function b with $\|b\|_{L^\infty(M_n)} \leq 1$ but such that*

$$\left\| S^2(b) \right\|_{BMO_c} \geq c \log(n+1).$$

Proof. Consider a function b that works for the statement of Theorem 9.1. Then $\|b\|_{L^\infty(M_n)} \leq 1$ and there exists a function $f \in L^2(S_n^2)$, such that $\|f\|_{L^2(S_n^2)} \leq 1$ and

$$\left\| \sum_k d_k b E_{k-1} f \right\|_{L^2(S_n^2)} \geq c \log(n+1). \tag{9.11}$$

We compute the square of the left side of (9.11) and get

$$\left\| \sum_k d_k b E_{k-1} f \right\|^2_{L^2(S_n^2)}$$

$$
\begin{aligned}
&= tr\int\sum_k |d_kb|^2 E_{k-1}fE_{k-1}f^* \\
&= tr\int\sum_k |d_kb|^2 (\sum_{i<k}|d_if^*|^2 + \sum_{i<k}E_{i-1}fd_if^* + \sum_{i<k}d_ifE_{i-1}f^*) \\
&= tr\int\sum_i (\sum_{k>i}|d_kb|^2)|d_if^*|^2 + tr\int\sum_i(\sum_{k>i}|d_kb|^2)(E_{i-1}fd_if^* + d_ifE_{i-1}f^*) \\
&= I + II
\end{aligned}
$$

For I, note $|d_if^*|^2$ is $\mathcal{F}_i$ measurable, we have

$$
\begin{aligned}
I &= tr\int\sum_i E_i(\sum_{k>i}|d_kb|^2)|d_if^*|^2 \\
&\leq \sup_i ||E_i(\sum_{k>i}|d_kb|^2)||_{L^\infty(M_n)}(tr\int\sum_i|d_if^*|^2) \\
&\leq ||b||^2_{BMO_c}||f||^2_{L^2(S_n^2)} \leq 4
\end{aligned}
$$

For II, note $E_{i-1}fd_if^* + d_ifE_{i-1}f^*$ is a martingale difference and $\sum_{k\leq i}|d_k|^2$ is $\mathcal{F}_{i-1}$ measurable since we are in the dyadic case, we get

$$
\begin{aligned}
II &= tr\int\sum_i S^2(b)(E_{i-1}fd_if^* + d_ifE_{i-1}f^*) \\
&= tr\int\sum_i d_i(S^2(b))(E_{i-1}fd_if^* + d_ifE_{i-1}f^*) \\
&\leq 2||\sum_i d_i(S^2(b))E_{i-1}f||_{L^2(S_n^2)}||f||_{L^2(S_n^2)} \\
&\leq 2||\pi_{S^2(b)}||_{L^2(S_n^2)\to L^2(S_n^2)} \\
&\leq 2c\log(n+1)||S^2(b)||_{BMO_c}.
\end{aligned}
$$

We used (9.2) in the last step. Combining this with (9.11), we get

$$
c\log(n+1) \leq \left\|\sum_k d_kbE_{k-1}f\right\|^2_{L^2(S_n^2)} \leq 4 + 2c\log(n+1)||S^2(b)||_{BMO_c}
$$

Thus

$$
||S^2(b)||_{\mathrm{BMO}_c} \geq c\log(n+1).
$$

This completes the proof. ∎

9.3. Proof of Theorem 9.2.

We keep the notations introduced in the end of Section 9.1. Recall BMO spaces of non-commutative martingales are defined for $x = (x_k) \in L^2(\mathcal{M})$ as below (see [33], [18]):

$$\begin{aligned}
\mathrm{BMO}_c(\mathcal{M}) &= \{x : ||x||_{\mathrm{BMO}_c(\mathcal{M})} = \sup_n \left\| E_n|\sum_{k=n}^{\infty} d_k x|^2 \right\|_{\mathcal{M}}^{\frac{1}{2}} < \infty\}; \\
\mathrm{BMO}_r(\mathcal{M}) &= \{x : ||x||_{\mathrm{BMO}_r(\mathcal{M})} = ||x^*||_{\mathrm{BMO}_c(\mathcal{M})} < \infty\}; \\
\mathrm{BMO}_{cr}(\mathcal{M}) &= \{x : ||x||_{\mathrm{BMO}_{cr}(\mathcal{M})} = \max\{||x||_{\mathrm{BMO}_c(\mathcal{M})}, ||x||_{\mathrm{BMO}_r(\mathcal{M})}\} < \infty\}.
\end{aligned}$$

When $\mathcal{M} = L^\infty(M_n)$, $\mathrm{BMO}_c(\mathcal{M})$ is just BMO_c considered in Section 9.1 and 8.2. In this section, for a non-commutative martingale b, we consider π_b and $\widetilde{\pi}_b$ as operators on bounded non-commutative L^p-martingale spaces introduced in Section 9.1. We will need the following interpolation result and the John-Nirenberg theorem for non-commutative martingales proved by Junge and Musat recently (see [17], [27]).

Theorem 9.5 *(Musat) For $1 \le p \le q < \infty$,*

$$(BMO_{cr}(\mathcal{M}), L_p(\mathcal{M}))_\theta = L_q(\mathcal{M}), \ \ with\ \theta = \frac{p}{q}.$$

Theorem 9.6 *(Junge, Musat) For any $1 \le q < \infty$ and any $g = (g_k)_k \in BMO_{cr}(\mathcal{M})$, there exist $c_q, c_q' > 0$ such that*

$$c_q'||g||_{BMO_{cr}} \le \sup_{m\in\mathbb{N}} \sup_{a\in\mathcal{M}_m, \tau(|a|^q)\le 1} \{||\sum_{k\ge m} d_k g a||_{L^q(\mathcal{M})}, ||\sum_{k\ge m} a d_k g||_{L^q(\mathcal{M})}\} \le c_q||g||_{BMO_{cr}}. \tag{9.12}$$

In fact, the formula above is proved for $q \ge 2$ in [17]. It is not hard to show that it is also true for $1 \le q < 2$. In the following, we give a simpler proof of it in the tracial case.

Proof. Note for any $g \in BMO_{cr}(\mathcal{M})$,

$$||g||_{BMO_{cr}(\mathcal{M})} = \sup_{m \in \mathbb{N}} \sup_{a \in \mathcal{M}_m, \tau(|a|^2) \leq 1} \{||\sum_{k \geq m} d_k g a||_{L^2(\mathcal{M})}, ||\sum_{k \geq m} a d_k g||_{L^2(\mathcal{M})}\}.$$

We get $c_2 = c_2' = 1$. Note for p, r, s with $1/p = 1/r + 1/s$ and $a \in L^p(\mathcal{M}), ||a||_{L^p(\mathcal{M})} \leq 1$, there exist b, c such that $a = bc$ and $||b||_{L^r(\mathcal{M})} \leq 1, ||c||_{L^s(\mathcal{M})} \leq 1$. By Hölder's inequality we then get $c_q = 1$ for $1 \leq q < 2$ and $c_q' = 1$ for $2 < q < \infty$. Thus for $2 < q < \infty$, we only need to prove the second inequality of (9.12). And, for $1 \leq q < 2$, we only need to prove the first inequality of (9.12). Fix $g \in BMO_{cr}(\mathcal{M})$, $m \in \mathbb{N}$, consider the left multiplier L_m and the right multiplier R_m defined as

$$L_m(a) = \sum_{k \geq m} d_k g a \text{ and } R_m(a) = \sum_{k \geq m} a d_k g, \quad \forall a \in \mathcal{M}_m.$$

It is easy to check that

$$\begin{aligned}
\sup_m ||L_m||_{L^2(\mathcal{M}_m) \to L^2(\mathcal{M})} &= ||g||_{BMO_c}, \\
\sup_m ||L_m||_{L^\infty(\mathcal{M}_m) \to BMO_{cr}} &\leq ||g||_{BMO_{cr}}; \\
\sup_m ||R_m||_{L^2(\mathcal{M}_m) \to L^2(\mathcal{M})} &= ||g||_{BMO_r}, \\
\sup_m ||R_m||_{L^\infty(\mathcal{M}_m) \to BMO_{cr}} &\leq ||g||_{BMO_{cr}}.
\end{aligned}$$

Thus L_m, R_m extend to bounded operators from $L^2(\mathcal{M}_m)$ to $L^2(\mathcal{M})$, as well as from $L^\infty(\mathcal{M}_m)$ to $BMO_{cr}(\mathcal{M})$. By Musat's interpolation result Theorem 9.5, we get L_m and R_m are bounded from $L^q(\mathcal{M}_m)$ to $L^q(\mathcal{M})$ and their operator norms are smaller than $c_q||g||_{BMO_{cr}}$, for all $2 \leq q < \infty$. By taking the supremum over m, we prove the second inequality of (9.12) for $q \geq 2$.

For $1 \leq q < 2$, by interpolation again, for $\theta = \frac{q}{2}$ and some $c_q'' > 0$,

$$||L_m||_{L^2(\mathcal{M}_m) \to L^2(\mathcal{M})} \leq c_q'' ||L_m||_{L^q(\mathcal{M}_m) \to L^q(\mathcal{M})} \theta ||L_m||^{1-\theta}_{L^\infty(\mathcal{M}_m) \to BMO_{cr}}$$

$$
\begin{aligned}
&\leq c''_q ||L_m||_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})}\theta ||g||^{1-\theta}_{BMO_{cr}}, \\
||R_m||_{L^2(\mathcal{M}_m)\to L^2(\mathcal{M})} &\leq c''_q ||R_m||_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})}\theta ||R_m||^{1-\theta}_{L^\infty(\mathcal{M}_m)\to BMO_{cr}} \\
&\leq c''_q ||R_m||_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})}\theta ||g||^{1-\theta}_{BMO_{cr}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
||g||_{BMO_{cr}} &= \max\{\sup_m ||L_m||_{L^2(\mathcal{M}_m)\to L^2(\mathcal{M})}, \sup_m ||R_m||_{L^2(\mathcal{M}_m)\to L^2(\mathcal{M})}\} \\
&\leq c''_q ||g||^{1-\theta}_{BMO_{cr}} \sup_m \{||L_m||_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})}\theta, ||R_m||_{L^q(\mathcal{M}_m)\to L^q(\mathcal{M})}\theta\}.
\end{aligned}
$$

This gives the first inequality of (9.12) with $c'_q = (c''_q)^{-\frac{1}{\theta}}$ for $1 \leq q < 2$. ∎

Recall that we say a filtration $\mathcal{M}_k$ is "regular" if, for some $c > 0$, $||a||_\infty \leq c||E_{m-}a||_\infty$, $\forall m \in \mathbb{N}, a \geq 0, a \in \mathcal{M}_m$.

Lemma 9.7 *For any regular filtration $\mathcal{M}_k$, we have*

$$
||b||_{BMO_{cr}(\mathcal{M})} \leq c_p \max\{||\pi_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, ||\widetilde{\pi}_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\}, \quad \forall 1 \leq p < \infty. \tag{9.13}
$$

Proof. Note, for any $b \in BMO_{cr}(\mathcal{M})$ with respect to the regular filtration $\mathcal{M}_k$,

$$
||b||_{BMO_{cr}(\mathcal{M})} \leq c \sup_{m\in\mathbb{N}} \sup_{\tau a^2 \leq 1, a\in\mathcal{M}_m} \{||\sum_{k>m} d_k ba||_{L^2(\mathcal{M})}, ||\sum_{k>m} ad_k b||_{L^2(\mathcal{M})}\}.
$$

Similar to the proof of Theorem 9.6, we can get,

$$
c'_q ||b||_{BMO_{cr}} \leq \sup_{m\in\mathbb{N}} \sup_{a\in\mathcal{M}_m, \tau|a|^q \leq 1} \{||\sum_{k>m} d_k ba||_{L^q(\mathcal{M})}, ||\sum_{k>m} ad_k b||_{L^q(\mathcal{M})}\} \leq c_q ||b||_{BMO_{cr}}. \tag{9.14}
$$

On the other hand, by considering $\pi_b(a), \widetilde{\pi}_b(a)$ for $a \in \mathcal{M}_m$, $||a||_{L^p(\mathcal{M})} \leq 1$, we have

$$
\begin{aligned}
&\sup_{a\in\mathcal{M}_m, \tau|a|^q \leq 1} \{||\sum_{k>m} d_k ba||_{L^p(\mathcal{M})}, ||\sum_{k>m} ad_k b||_{L^p(\mathcal{M})}\} \\
&\quad \leq 2 \max\{||\pi_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, ||\widetilde{\pi}_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\}.
\end{aligned}
$$

Taking supremum over m in the inequality above, we get (9.13) by (9.14) . ∎

Lemma 9.8 *For* $1 < p < \infty$, *we have*

$$\|\pi_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}(\mathcal{M})} \leq c_p(\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||b||_{BMO_r(\mathcal{M})}). \tag{9.15}$$

$$\|\widetilde{\pi}_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}(\mathcal{M})} \leq c_p(\|\widetilde{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||b||_{BMO_c(\mathcal{M})}). \tag{9.16}$$

Proof. We prove (9.15) only. Fix a $f \in L^\infty(\mathcal{M})$ with $\|f\|_{L^\infty(\mathcal{M})} \leq 1$. We have

$$\begin{aligned}
&\left\| E_m \sum_{k\geq m} |d_k b E_{k-1} f|^2 \right\|_{L^\infty(\mathcal{M})} \\
= \ & \sup\{\tau E_m \sum_{k\geq m} |d_k b E_{k-1} f|^2 a, \ a \in \mathcal{M}_m, a \geq 0, \tau a \leq 1\} \\
= \ & \sup\{\tau \sum_{k\geq m} (d_k b E_{k-1} f a^{\frac{1}{p}})^* (d_k b E_{k-1} f a^{\frac{1}{q}}), \ a \in \mathcal{M}_m, a \geq 0, \tau a \leq 1\} \\
\leq \ & \sup_a \left\| d_m b E_{m-1} f a^{\frac{1}{p}} + \sum_{k>m} d_k b E_{k-1}(f a^{\frac{1}{p}}) \right\|_{L^p(\mathcal{M})} \left\| \sum_{k\geq m} d_k b E_{k-1} f a^{\frac{1}{q}} \right\|_{L^q(\mathcal{M})}
\end{aligned}$$

Note $||d_m b E_{m-1} f a^{\frac{1}{p}}||_{L^p(\mathcal{M})} \leq ||d_m b||_{\mathcal{M}} \leq ||b||_{BMO_r}$. By (9.12) we get

$$\left\| E_m \sum_{k\geq m} |d_k b E_{k-1} f|^2 \right\|_{L^\infty(\mathcal{M})} \leq c_q(||b||_{BMO_r} + \|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}) \|\pi_b(f)\|_{BMO_{cr}(\mathcal{M})} \tag{9.17}$$

Taking the supremum over m in (9.17), we get

$$\|\pi_b(f)\|^2_{BMO_c(\mathcal{M})} \leq c_q(||b||_{BMO_r} + \|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}) \|\pi_b(f)\|_{BMO_{cr}(\mathcal{M})}.$$

On the other hand, since $(E_{m-1}f)(E_{m-1}f)^* \leq 1$, we have

$$\|\pi_b(f)\|_{BMO_r(\mathcal{M})} \leq \|b\|_{BMO_r(\mathcal{M})}.$$

Thus,

$$\|\pi_b(f)\|^2_{BMO_{cr}(\mathcal{M})} \leq (c_q + 1)(\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||b||_{BMO_r(\mathcal{M})}) \|\pi_b(f)\|_{BMO_{cr}(\mathcal{M})},$$

Therefore

$$\|\pi_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}(\mathcal{M})} \leq (c_q+1)(\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||b||_{BMO_r(\mathcal{M})}). \quad \blacksquare$$

Proof of Theorem 9.2. By Lemma 8.7 and Lemma 8.8 we get immediately that

$$\begin{aligned} &\max \quad \{\|\pi_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}}, \|\widetilde{\pi}_b\|_{L^\infty(\mathcal{M})\to BMO_{cr}}\} \\ \leq c_p &\max \quad \{\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, \quad \|\widetilde{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\} \end{aligned}$$

By the interpolation results on non-commutative martingales (Theorem 8.5), we get

$$\begin{aligned} &\max \quad \{\|\pi_b\|_{L^q(\mathcal{M})\to L^q(\mathcal{M})}, \|\widetilde{\pi}_b\|_{L^q(\mathcal{M})\to L^q(\mathcal{M})}\} \\ \leq c_p &\max \quad \{\|\pi_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}, \|\widetilde{\pi}_b\|_{L^p(\mathcal{M})\to L^p(\mathcal{M})}\}, \end{aligned}$$

for all $1 < p < q < \infty$.

Question : Assume $\pi_b, \widetilde{\pi}_b$ are of type (p,p), are they of weak type $(1,1)$? More precisely, assume $||\pi_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})} + ||\widetilde{\pi}_b||_{L^p(\mathcal{M})\to L^p(\mathcal{M})} < \infty$, does there exist a constant $C > 0$ such that, for any $f \in L^1(\mathcal{M})$, $\lambda > 0$, there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) \leq C\frac{||f||_{L^1(\mathcal{M})}}{\lambda} \quad and \quad ||e\pi_b(f)e||_{L^\infty(\mathcal{M})} + ||e\widetilde{\pi}_b(f)e||_{L^\infty(\mathcal{M})} \leq \lambda?$$

We have the following corollary by applying results of this section to matrix valued dyadic paraproducts discussed in Section 9.1 and Section 9.2. Note M_n valued dyadic martingales on the unit circle are non-commutative martingales associated with the von Neuman algebra $\mathcal{M} = L^\infty(\mathbb{T})\otimes M_n$ and the filtration $\mathcal{M}_k = L^\infty(\mathbb{T}, \mathcal{F}_k)\otimes M_n$.

Corollary 9.9 *Let $1 < p < \infty$, denote by $c_p(n)$ the best constant such that*

$$\|\pi_b\|_{L^p(S_n^p)\to L^p(S_n^p)} \leq c_p(n)\, \|b\|_{L^\infty(M_n)}\,,\ \forall b.$$

Then

$$c_p(n) \backsim \log(n+1).$$

Proof. Note in the proof of Theorem 9.1, if we see f as a column matrix valued function and g as a row matrix valued function, we will have

$$||f||_{L^p(S_n^p)} = ||\alpha||_{\ell_n^2},\quad ||g||_{L^q(S_n^q)} = ||\beta||_{\ell_n^2}.$$

By the same method, we can prove $c_p(n) \geq c\log(n+1)$ for all $1 < p < \infty$. For the inverse relation, by (9.2) we have $c_2(n) \leq c\log(n+1)$. Then, by (9.15), we get

$$\begin{aligned} \|\pi_b\|_{L^\infty(M_n)\to BMO_{cr}} &\leq c_2(c_2(n)\,\|b\|_{L^\infty(M_n)} + ||b||_{BMO_{cr}}) \\ &\leq c\log(n+1)||b||_{L^\infty(M_n)},\quad \forall b \in L^\infty(M_n) \end{aligned} \tag{9.18}$$

Denote by π_b^* the adjoint operator of the dyadic paraproduct π_b, then

$$\pi_b^*(f) = \sum_k (d_k b)^* E_{k-1} f.$$

Note we have the decomposition

$$\pi_b^*(f) = b^* f - \pi_{b^*}(f) - (\pi_{f^*}(b))^*.$$

By (9.18), we get

$$\begin{aligned} \|\pi_b^*\|_{L^\infty(M_n)\to BMO_{cr}} &\leq ||b^*||_{L^\infty(M_n)} + c\log(n+1)||b^*||_{L^\infty(M_n)} + c\log(n+1)||b||_{L^\infty(M_n)} \\ &\leq c\log(n+1)\,\|b\|_{L^\infty(M_n)}\,. \end{aligned} \tag{9.19}$$

By (9.18), (9.19) and the interpolation result Theorem 3.5, we get

$$\|\pi_b\|_{L^p(S_n^p)\to L^p(S_n^p)} \leq c_p \log(n+1)\,\|b\|_{L^\infty(M_n)}\,, \quad \forall 1 < p < \infty.$$

Therefore, we can conclude $c_p(n) \backsim \log(n+1)$. ∎

CHAPTER X

SUMMARY

In this chapter, we give a summary of the results of this dissertation in the matrix valued case.

Operator Valued Hardy Spaces

The Hardy spaces are very important objects in classical analysis. Among several equivalent definitions, one is as follows:

$$H^p(\mathbb{R}) = \{f \in L^p(\mathbb{R}), \|f\|_{H^p} = ||f||_{L^p} + ||Hf||_{L^p} < \infty\}, \quad \text{for } 1 \leq p < \infty,$$

where $H(f)$ is the Hilbert transform of f. Fruitful results on Hardy spaces (such as interpolation results, equivalence between H^p and L^p for $1 < p < \infty$) have been developed during last century, that turned H^p theory into an important branch of classical analysis. One of the most remarkable results of H^p theory is the Fefferman-Stein duality theorem, which says in particular that the dual of $H^1(\mathbb{R})$ is another well known space, the BMO space defined as follows

$$\mathrm{BMO}(\mathbb{R}) = \{f \in L^1_{loc}(\mathbb{R}), \|f\|_{\mathrm{BMO}} = \sup_{I\subset\mathbb{R}} \frac{1}{|I|}\int_I |f(t) - f_I|dt < \infty\},$$

where $f_I = \frac{1}{|I|}\int_I f(t)dt$.

We constructed H^p spaces for operator valued functions by considering the non-commutative Littlewood-Paley G-functions. The non-commutativity is, of course, the main difficulty of our study and the main difference between operator valued Hardy spaces and the vector valued ones. One analogue of classical results we proved is that our H^1's are preduals of the non-commutative BMO spaces defined in recent works on matrix valued harmonic analysis and non-commutative martingale inequalities (see

[20], [28], [29], [33]).

For convenience, we will conclude the results only in the matrix valued case. Because of the non-commutativity, there are now two non-commutative BMO spaces, the column BMO and row BMO. Let $\mathcal{M}_n$ be the algebra of $n \times n$ matrices with its usual trace tr. For $A \in M_n$, denote by $||A||_{M_n}$ the operator norm of A on ℓ_n^2. Then the column BMO space is defined by

$$\mathrm{BMO}_c(\mathbb{R}, \mathcal{M}_n) = \left\{\varphi : \mathbb{R} \to \mathcal{M}_n, \|\varphi\|_{\mathrm{BMO}_c} < \infty\right\}$$

where

$$\|\varphi\|_{\mathrm{BMO}_c} = \sup_{I \subset \mathbb{R}} \|\frac{1}{|I|} \int_I (\varphi(t) - \varphi_I)^* (\varphi(t) - \varphi_I) dt\|_{M_n}^{\frac{1}{2}},$$

and $\varphi_I = \frac{1}{|I|} \int_I \varphi(t) dt$. Similarly, the row BMO space is

$$\mathrm{BMO}_r(\mathbb{R}, \mathcal{M}_n) = \left\{\|\varphi\|_{\mathrm{BMO}_r} = \| \ \varphi^* \|_{\mathrm{BMO}_c} < \infty\right\}.$$

Note that these two norms are not equivalent uniformly over n. Denote by S_n^p $(1 < p < \infty)$ the Schatten p classes on ℓ_n^2. For $f \in L^1((\mathbb{R}, \frac{dt}{1+t^2}), \mathcal{S}_n^1)$, let F denote its Poisson integral. We define the non-commutative G-function as

$$G_{f,c}(x) = (\int_0^\infty |\nabla F(t,y)|^2 y dy)^{\frac{1}{2}}$$

where

$$|\nabla F(t,y)|^2 = |\frac{\partial F}{\partial t}|^2 + |\frac{\partial F}{\partial y}|^2 \quad \text{and} \quad |\frac{\partial F}{\partial t}|^2 = (\frac{\partial F}{\partial t})^* (\frac{\partial F}{\partial t})$$

Define $\mathcal{H}_c^p[n]$ (resp. $\mathcal{H}_r^p[n]$) $(1 < p < \infty)$ to be the space of all f such that $G_{f,c}(x) \in L^p(\mathbb{R}, S_n^p)$ (resp. $G_{f,r}(x) \in L^p(\mathbb{R}, S_n^p)$) and set

$$\|f\|_{\mathcal{H}_c^p} = \|G_{f,c}(x)\|_{L^p(\mathbb{R}, S_n^p)} \quad (\text{resp. } \|f\|_{\mathcal{H}_r^p} = \|f^*\|_{\mathcal{H}_c^p}).$$

When $n = 1$, all these spaces coincide with the classical Hardy spaces.

Theorem *(Non-commutative generalization of Fefferman's duality theorem)*

(a)$(\mathcal{H}^1_c[n])^* = \mathrm{BMO}_c(\mathbb{R}, \mathcal{M}_n)$ *with equivalent norms independent of* n.

(b) Similarly, $(\mathcal{H}^1_r[n])^* = \mathrm{BMO}_r(\mathbb{R}, \mathcal{M}_n)$ *with equivalent norms independent of* n.

And as in the classical case, the duality between $\mathcal{H}^1_c[n]$ *and* $\mathrm{BMO}_c(\mathbb{R}, \mathcal{M}_n)$ *implies an atomic decomposition of* $\mathcal{H}^1_c[n]$.

Remark *Note that the trace class valued Hardy space* $H^1(S^1)$ *has a different dual than the above.*

Theorem *(Equivalence between* $\mathcal{H}^p$ *and* L^p*)*

$\mathcal{H}^p_c[n] + \mathcal{H}^p_r[n] = L^p(\mathbb{R}, S^p_n)$ *with equivalent norms for all* $1 < p \leq 2$

$\mathcal{H}^p_c[n] \cap \mathcal{H}^p_r[n] = L^p(\mathbb{R}, S^p_n)$ *with equivalent norms for all* $2 < p < \infty$. *The equivalence constants are independent of* n.

Theorem *(Interpolation) Let* $1 < p < \infty$. *Then with equivalent norms,*

$$(X, Y)_{\frac{1}{p}} \quad = \quad L^p(\mathbb{R}, S^p_n)$$

where $X = \mathrm{BMO}_c(\mathbb{R}, \mathcal{M}_n) \cap \mathrm{BMO}_r(\mathbb{R}, \mathcal{M}_n)$ *or* $L^\infty(\mathbb{R}, M_n)$, $Y = \mathcal{H}^1_c[n] + \mathcal{H}^1_r[n]$ *or* $L^1(\mathbb{R}, S^1_n)$ *and the equivalence constants are independent of* n.

Matrix valued dyadic paraproduct

Let $(\mathbb{T}, \mathcal{F}_k)$ be the unit circle with the usual dyadic filtration. Let b be an $\mathcal{M}_n$ valued function on $\mathbb{T}$. The matrix valued dyadic paraproduct associated with b, denoted by π_b, is the operator on $L^p(\mathbb{T}, S^p_n)$ defined as

$$\pi_b(f) = \sum_k (d_k b)(E_{k-1} f), \quad \forall f \in L^p(\mathbb{T}, S^p_n),$$

where E_k is the conditional expectation with respect to $\mathcal{F}_k$, and $d_k b = E_k b - E_{k-1} b$. In the classical case (when b is a scalar valued function), it is well known that

$$\|\pi_b\|_{L^2 \to L^2} \backsimeq \|b\|_{BMO_d},$$

where BMO_d denotes the usual dyadic BMO norm.

Note π_b is usually considered as a dyadic singular integral and plays an important role in the proof of the classical T(1) theorem. Also note its relation with the Hankel operator with symbol b (see [30]), which has a norm equivalent to $||b||_{(H^1(S_n^1))^*}$ in the matrix valued case. We may ask two natural questions as follows:

Q(1) Does there exist a constant $c > 0$ independent of n such that, for all $1 < p, q < \infty$,

$$\|\pi_b\|_{L^q(\mathbb{T}, S_n^q) \to L^q(\mathbb{T}, S_n^q)} \le c \|\pi_b\|_{L^p(\mathbb{T}, S_n^p) \to L^p(\mathbb{T}, S_n^p)}?$$

Q(2) Can we dominate $\|\pi_b\|_{L^2(\mathbb{T}, S_n^2) \to L^2(\mathbb{T}, S_n^2)}$ uniformly over n by some reasonable BMO norm? (Note we have various candidates for BMO norms in the matrix valued case. Nazarov, Pisier, Treil, Volberg proved that this is not true if we consider BMO_c norm defined in Section 2.3.)

In this dissertation, we gave a partial positive answer to Q(1) and proved that there exists a constant $c > 0$ independent of n such that, for all $1 < p < q < \infty$,

$$\max\{\|\pi_b\|_{L^q(S_n^q) \to L^q(S_n^q)}, \|\pi_{b^*}\|_{L^q(S_n^q) \to L^q(S_n^q)}\} \le c \max\{\|\pi_b\|_{L^p(S_n^p) \to L^p(S_n^p)}, \|\pi_{b^*}\|_{L^p(S_n^p) \to L^p(S_n^p)}\},$$

where b^* denotes the adjoint of b. We still do not know what happens when $p > q$. We gave a negative answer to Q(2) and proved that even $||b||_{L^\infty(\mathbb{T}, M_n)}$ does not dominate $||\pi_b||_{L^2(\mathbb{T}, S_n^2) \to L^2(\mathbb{T}, S_n^2)}$ uniformly over n (see Chapter IX).

Difficulties and some useful techniques

Non-commutativity. We lose some nice classical properties in the operator valued case because of the non-commutativity. For example, we will no longer have a "good" John-Nirenberg theorem for operator valued BMO (see Chapter VIII and [17], [34]).

Absence of maximal element. A straightforward definition of the maximal function in the operator valued case is not possible. However, using Pisier's non-commutative vector valued spaces we may partially overcome this problem in many situations. In fact, we proved a non-commutative Hardy-Littlewood maximal inequality for operator valued functions (see Chapter IV), which is based on Junge's work on Doob's maximal inequality for non-commutative martingales(see [14]).

Non-commutative Martingale inequalities. As in the classical case, we could borrow some ideas from the study of non-commutative martingales when studying operator valued functions. In particular, Pisier and Xu's work on the non-commutative Burkholder-Gundy inequalities (see [33]) inspired us to consider the non-commutative analogue of the classical Littlewood-Paley G function to define our operator valued H^p spaces. Moreover, we used Junge's work on Doob's maximal inequality (see [14]) to prove our non-commutative Hardy-Littlewood maximal inequality mentioned above. However, it seems difficult to convert results from operator valued martingales to operator valued functions by following the classical methods (Brownian martingales or distribution functions). In Chapter VI, we gave a trick to treat some special situations. The following is an analogue of Theorem 6.1 of this dissertation.

Theorem *([24]) Let $\mathbb{T}$ be the unit circle. Denote by $\mathrm{BMO}(\mathbb{T})$ the scalar valued BMO space and denote by $\mathrm{BMO}_d(\mathbb{T})$ the scalar valued usual dyadic BMO space on $\mathbb{T}$. We have*

$$\|\varphi\|_{\mathrm{BMO}(\mathbb{T})} \leq 6(\|\varphi\|_{\mathrm{BMO}_d(\mathbb{T})} + \left\|\varphi(\cdot - \frac{2\pi}{3})\right\|_{\mathrm{BMO}_d(\mathbb{T})}).$$

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