ERROR ANALYSIS FOR RANDOMIZED UNIAXIAL STRETCH TEST ON HIGH STRAIN MATERIALS AND TISSUES

A Dissertation

by

CHOON-SIK JHUN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2005

Major Subject: Biomedical Engineering

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Approved as to style and content by:

John C. Criscione (Chair of Committee)	William A. Hyman (Member)
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	Gerard L. Coté (Head of Department)

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ABSTRACT

Error Analysis for Randomized Uniaxial Stretch Test

on High Strain Materials and Tissues. (May 2005)

Choon-Sik Jhun, B.S., Kon-Kuk University;

M.S., Texas A&M University

Chair of Advisory Committee: Dr. John C. Criscione

Many people have readily suggested different types of hyperelastic models for high strain materials and biotissues since the 1940's without validating them. But, there is no agreement for those models and no model is better than the other because of the ambiguity. The existence of ambiguity is because the error analysis has not been done yet (Criscione, 2003). The error analysis is motivated by the fact that no physical quantity can be measured without having some degree of uncertainties. Inelastic behavior is inevitable for the high strain materials and biotissues, and validity of the model should be justified by understanding the uncertainty due to it.

We applied the fundamental statistical theory to the data obtained by randomized uniaxial stretch-controlled tests. The goodness-of-fit test ($\overline{R}^2$) and test of significance (***t**-test*) were also employed. We initially presumed the factors that give rise to the inelastic deviation are *time spent testing*, *stretch-rate*, and *stretch history*. We found that these factors characterize the inelastic deviation in a systematic way. A huge amount of inelastic deviation was found at the stretch ratio of 1.1 for both specimens. The significance of this fact is that the inelastic uncertainties in the low

stretch ranges of the rubber-like materials and biotissues are primarily related to the entropy. This is why the strain energy can hardly be determined by the experimentation at low strain ranges and there has been a deficiency in the understanding of the exclusive nature of the strain energy function at low strain ranges of the rubber-like materials and biotissues (Criscione, 2003). We also found the answers for the significance, effectiveness, and differences of the presumed factors above.

Lastly, we checked the predictive capability by comparing the unused deviation data to the predicted deviation. To check if we have missed any variables for the prediction, we newly defined the *prediction deviation* which is the difference between the observed deviation and the point forecasting deviation. We found that the prediction deviation is off in a random way and what we have missed is random which means we didn’t miss any factors to predict the degree of inelastic deviation in our fitting.

In memory of my father, Jong-Pahl Jhun and brother, Sik Jhun
who are in Heaven with Lord.

ACKNOWLEDGMENTS

I wish to express my deepest appreciation to my dissertation chairman, Dr. John C. Criscione. His great endurance, bright guidance, sincere endeavor and support helped this work through its many challenges. No words can express my deepest appreciation to him. I'm also grateful to my thesis committee members, Dr. William A. Hyman, Dr. Hsin-i Wu and Dr. Jay R. Walton for their support and understanding. I would also like to thank my laboratory colleagues, Akhilesh A. Kotiya and Bharati K. Hegde for their unconditional help and meaningful discussions. Financial support from Texas A&M University is gratefully acknowledged.

I would like to express my deepest sense of gratitude to my beautiful wife Hyun-Ju Park for her endless love and support. She is a never-ending source of reason and motivation of my life. I would not be where I am today if it were not for her love, sincere prayer, encouragement and support. I must thank my sons Moses H. Jhun and Joshua Y. Jhun who are another never-ending source of joy and happiness.

I would like to thank my parents Jong-Pahl Jhun and Oak-Soon Kim and parents-in-law, Yong-Nam Park and Myung-Ja Nam who in many ways supported my family's staying at Texas A&M University for both the M.S. and Ph.D studies. In addition, I'm willing to give my special thanks to my elder brother Hyung-Bae Jhun and sister Ji-Young Jhun for their heartfelt support and encouragement. Finally, I want to express deep gratitude to my Lord and Savior Jesus Christ.

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CHAPTER I

INTRODUCTION

During the last half of the century, many people have readily suggested different types of hyperelastic models for high strain materials and biotissues without showing their validities. Even the well-known experts in this area such as Mullins, Mooney, Rivlin, Fung, Ogden, and so on have published their own models without discussing the uncertainties in them. They got the results without finding out what the errors in the experimental data were. They also didn't show how their models deviated from the hyperelastic assumption. It should be noted that no hyperelastic models have been suggested without doing experiments and none of the measurements can be made without having some degree of uncertainties. This fact implies that it is extremely important to know if there is an error in any model. We need to know the uncertainty of a model if we are going to use it. A model is useful only if the uncertainty in it is quantified and if the uncertainty is acceptable for the particular application of the model. The particular application means if we are building some structures made out of steels, the linear elastic model will be acceptable for building it even though the steel is not perfectly elastic. We can accept that the uncertainty that comes from the assumption of linear elasticity for the steel is insignificant. Note that no model is acceptable for all applications. If the error analysis, which should have been done in the 1940's, is not performed, a model cannot be evaluated. In the study of biomechanics (how the stress relates to the strain in biology or vice versa) and

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The Journal of Biomechanics has been used as a model for style and format.

mechanobiology (how the stresses/strains influence the biological procedure), it is crucial to develop a reliable constitutive model that quantifies the uncertainty in it. It is understood that the strain guides the growth and remodeling of the tissues. It has been also suggested that the strain pattern is atypical for unhealthy tissues and hence are the growth and remodeling. To get a better understanding of the strain pattern, modeling of a better constitutive relation is an essential requirement and this can be done by characterizing the uncertainty in it. There may be some other factors that go unnoticed and hence we may get underestimated uncertainties. Thus we have to carefully decide and find all probable causes of errors and estimate their effects.

The error analysis is motivated by the fact that no physical quantity can be measured without having some degree of uncertainties. Although many different types of models for hyperelasticity have been developed since the 1940's, there are very few publications that discussed the errors in the data for the constitutive modeling. Since having some degree of uncertainties is inevitable especially in experiments on high strain material and biotissues, validity of the model should be justified by understanding the uncertainty in the data. Every constitutive relation in hyperelasticity, therefore, should be modeled after characterizing the uncertainties in the data. Only after doing this, can the model be regarded as an appropriate one for an application. The fundamental statistical theory, for this study, has been used to find the systematic significant factors and random errors as well as to characterize the uncertainties. The systematic factors that give rise to the deviation from hyperelasticity were presumed as *time spent testing*, *stretch-rate*, and *stretch history*.

They gave rise to the deviation of elastomers and tissues from hyperelasticity as evident in the randomized uniaxial stretch-controlled tests. By considering all measurements that would make uncertainties such as measurement errors, instrument errors, and extra randomized factors that make additional uncertainties, we found out which ones are significant factors that make those uncertainties in the context of error propagation. After we realized the significant factors that make the uncertainty, we applied fundamental statistical theories such as multivariable linear regression analysis to those factors to better understand the uncertainty in the data obtained by uniaxial (randomized stretch-controlled protocols) stretch tests on high strain materials and biotissues. This has been done for each stretch level λ = 1.1, 1.3, 1.5, 1.7, and 1.9. Since we have assumed that the highly suspected factors that deviate from the hyperelasticity are *time spent testing*, *stretch-rate*, and *stretch history*, they have been treated as independent variables and the *deviation* from hyperelasticity has been treated as the dependent variable for the multivariable linear regression analysis. To test whether the regression model is proper or not, the goodness-of-fit test by using $\overline{R}^2$ (*coefficient of determination*) and test of significance (***t****-test*) have been employed. Finally we have answered for the following three questions;

a) Are they (*time spent testing*, *stretch-rate*, and *stretch history*) really significant factors that make considerable amount of uncertainties from hyperelasticity?
b) Are they effective to all the stretch levels?
c) If not, how do they differ?

Lastly, we checked the predictive capability by comparing the unused (deviation) data in the randomized stretch-controlled protocol to the predicted deviation obtained by the regression models. The test of the predictability should be involved in the final step in any regression model to evaluate how well the derived regression model forecasts the intermediate or future values. It is important to note that the best way to accomplish this work is to compare predicted result to data that were not used in the formulation of the regression models.

Hyperelastic Models for Rubber-like Materials and Biotissues

Hyperelastic Models for Rubber-like Materials: Many people in the area of the hyperelasticity have been struggling to find general constitutive relations in terms of strain energy function *W* which is a scalar function. In spite of all their efforts, most of the models are only fitted to the specified conditions and environments when they were derived. Individual models can predict the stress/strain relations only within the specified ranges of strain. They also applied huge assumptions such as isotropy, hyperelasticity, or pseudoelasticity which ignores the hysteresis that is an inevitable phenomenon in high strain materials and biotissues. While it is acceptable to use these assumptions, the deviation from hyperelasticity that arises from these assumptions has never been considered. It's been suggested that the Mooney model (1940) shown in equation (1.1) has good agreement with the expected value when the materials undergo the stretch ratio λ more than 1.4.

$$\begin{aligned} W &= C_1\left(\lambda_1^{\ 2} + \lambda_2^{\ 2} + \lambda_3^{\ 3} - 3\right) + C_2\left(\frac{1}{\lambda_1^{\ 2}} + \frac{1}{\lambda_2^{\ 2}} + \frac{1}{\lambda_3^{\ 2}} - 3\right) \\ &= C_1(I_1 - 3) + C_2(I_2 - 3) \end{aligned} \tag{1.1}$$

where C_i, i = 1, 2 is a material constant, I_1 = tr$\boldsymbol{C}$ and $I_2 = 1/2[(\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}\boldsymbol{C}^2]$ are called the first and second principal invariants, respectively. The right Cauchy-Green tensor $\boldsymbol{C}$ is described with the deformation gradient $\boldsymbol{F}$ as $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$.

Treloar (1943, 1948) introduced the following form of strain energy function called neo-Hookean model as

$$W = C(I_1 - 3) \tag{1.2}$$

where C is a material constant. For the biaxial swollen rubber sheet, he suggested the stress/strain relations shown in equation (1.3) as

$$\sigma_i = G(\lambda_i^{\ 2} - \lambda_3^{\ 2}),\ i = 1,\ 2 \tag{1.3}$$

where σ_i is a stress and G is a material constant.

Mullins (1947) suggested the stress/strain relations for the simple tensile test as

$$\sigma = G\left\{\lambda - \left(\frac{1}{\lambda}\right)^2\right\} \tag{1.4}$$

where σ is a stress and G is a material constant. Equation (1.4) is only valid for the equilibrium state of stress/strain or low strain rate which can be regarded as quasi-equilibrium state. Like other model developers, Mullins didn't mention the difference between loading and unloading stress/strain curves. The equation (1.5) suggested by Rivlin and Saunders, 1951 mentioned that other forms of strain energy functions for elastomers would be well-fitted while they suggested their own model.

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(I_2 - 3)^2 \tag{1.5}$$

In 1958, Treloar suggested the general form of stored strain energy function as

$$W = \sum_{i,j=0}^{\infty} C_{ij}(I_1 - 3)^i (I_2 - 3)^j \tag{1.6}$$

There are a few other models that have been as shown below, but there are too many to introduce all of the models that have been developed since 1940's.

$$\begin{aligned} \sigma_i &= \frac{1}{\sqrt{J}}\lambda_i \frac{\partial W}{\partial \lambda_i} = \frac{1}{\sqrt{J}}\lambda_i w'(\lambda_i),\ i = 1,\ 2,\ 3 \\ \text{where,}\ \ W &= w(\lambda_1) + w(\lambda_2) + w(\lambda_3) \\ &= 2\mu \sum_{i=1}^{3} \lambda_i \left(\ln \lambda_i - 1\right) \end{aligned} \tag{1.7}$$

Equation (1.7) was suggested by Valanis and Landel, 1967, and J = det$\boldsymbol{F}$ (= dv/dV) which represents the ratio of volume changes of current and reference bodies and μ is a material constant. Obata et al (1970) also suggested the hyperelastic model for rubber-like materials as following.

$$\begin{aligned} \sigma_i &= 2\left[\left(\lambda_i^{\,2} - \frac{1}{\lambda_1^{\,2}\lambda_2^{\,2}}\right)\frac{\partial W}{\partial I_1} - \left(\frac{1}{\lambda_i^{\,2}} - \lambda_1^{\,2}\lambda_2^{\,2}\right)\frac{\partial W}{\partial I_2}\right],\ i = 1,\ 2 \\ \text{where,}\ \ \frac{\partial W}{\partial I_1} &= a_0 + \frac{a_1}{(I_1 - 3)^2} - \frac{a_2\left(I_2 - 3\right)}{\left(I_1 - 3\right)^{2.5}} \\ \frac{\partial W}{\partial I_2} &= A_0 - \frac{A_1}{\left(I_2 - 3\right)} + \frac{2a_2}{3\left(I_1 - 3\right)^{1.5}} \end{aligned} \tag{1.8}$$

Although many models for rubber-like materials and biotissues have been proposed so far, there is no agreement for those models; no model is better than the other. This means that there is ambiguity for developing the hyperelastic models of high strain materials as well as of biotissues. The reason for this ambiguity is because the error analysis has not been done yet (Criscione, 2003).

Hyperelastic Models for Biotissues: Experimentally developed hyperelastic models for biotissues have also been developed by well-known experts in the area of hyperelasticity. Again, those models have been fitted to only a certain range of strain. Fung (1993) suggested that there are three regions of stretch ratio for a tendon. They are toe region, fairly linear region, and nonlinear region. The toe region is defined as the region that the load is exponentially increasing over the stretch, i.e., the constitutive relation is nonlinear. The fairly linear region is the region located in between the toe region and nonlinear region. The stress/strain relationship in the region is regarded as linear. The nonlinear region is the region where the stress and stretch relationship is not linear and rupture occurs within that range of 10 to 15 % of stretch (Fung, 1993). Fung suggested following equation to see the stress/strain relationship for the toe region of the tendon.

$$T = C\left(e^{\alpha\lambda} - e^{\alpha}\right), \quad 1 < \lambda \le \lambda_0 \tag{1.9}$$

where T is a stress, α is an elastic stiffness and C is a material constant. Although, by the inspection and assumption of Fung, a Hooke's law or neo-Hookean law could be used for the fairly linear region, Johnson et al (1992) suggested the following equation for both the toe and fairly linear regions where C_0 and μ are material constants.

$$T = C_0\left(1 + \mu\frac{1}{\lambda}\right)\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{1}{\lambda}, \; 1 < \lambda \le \lambda_0 \tag{1.10}$$

A.Viidik (1987) introduces Wertheim's constitutive relation (1847) for tendons by following equation (1.11).

$$\varepsilon^2 = A\sigma^2 + B\sigma \tag{1.11}$$

where ε is strain, A and B are material constants. Morgan (1960), Kenedi et al (1964), Ridge and Wright (1964) suggested the following constitutive relations shown in equation (1.12), (1.13), and (1.14), respectively.

$$\varepsilon = C_1\sigma^a \tag{1.12}$$

$$\begin{aligned} \sigma &= C_2 \varepsilon^b \\ \sigma &= C_3\left(e^{ce} - 1\right) \end{aligned} \tag{1.13}$$

$$\begin{aligned} \varepsilon &= K + L \ln \sigma \\ \varepsilon &= C_4 + C_5 \sigma^d \end{aligned} \tag{1.14}$$

where *a*, *b*, *c*, *d*, *K* and *L* are constants. Those empirically established equations again are valid for the anticipation of material properties of the biotissues only under the same experimental situations and limited ranges. In addition, many people ignore the higher order terms in deriving the hyperelastic models without knowing that the deviation comes from omission of the higher order terms. Here we considered all the points of motion profile of stretch-controlled protocol and then averaged them which means we didn't omit the higher order terms so that we considered a preferably true hyperelastic model but have a deviation provoked by the time spending, stretch-rate and stretch history.

Why Such Hyperelastic Models are Useful for the High Strain Materials/Biotissues

The usefulness of a model is in its providing of insights into the mechanisms underlying the mechanical behavior of the materials. The reason that we are doing the error analysis for the hyperelastic models of high strain materials and biotissues is that, even though they are never perfectly correct, those models are comprehensively being used and regarded as useful in the area of biomechanics and mechanobiology. The elastic models are never exact but they are useful because they guide our directions. If there is no model for a certain application, one may have to have some trial and error. The elastic models help us look at what is the relationship between stress and strain, how the materials respond relative to the stress and strain. For example, an elastic model for steel is useful for building structures such as bridges and skyscrapers. Although steel is not elastic and has some inelastic behavior (Kliman and Bily, 1984., Tong et al, 1989., Wittke et al, 1997., Ohno and Karim, 2000., Sablik et al, 2004), a linear elastic model for steel gives mostly the right answer so that we can model steel as an elastic material by the first order approximation. Although the linear elastic model for steel gives mostly the right answer, it is not truly right. There are deviations for second order and higher order models as well. We should look at the deviation and include this deviation in the models. But it is not necessary to include the higher order terms for steel for most applications because the first order model gets very close enough to be used for the applications. It is the same with rubber and tissue. In this case the hyperelastic models help us understand what is the relationship between stress and strain, how they grow relative to the stress or strain.

Why We Need to Know the Uncertainty in the Models

If we have an infinite number of higher order terms, then the model would be perfect under the assumption of that the Error-of-measurement is ignorable relative to the Error-of-definition. In spite of that assumption, a perfect model is unfeasible because we can't have infinite number of data points. If we have only first order, or second order terms, then it is approximation. We need to look for the uncertainty due to the approximation or omission of the rest of the higher order terms. By considering all points of motion profile of stretch-controlled protocol and then average, we consider a perfect hyperelastic model but have a deviation caused by the time spending, stretch rate and stretch history. It will be great to have a perfect model for tissue if we are looking at the role of stress in the growth and remodeling of a tissue. But in lieu of perfect model, hyperelastic models would be the best choice for biotissues. Since, for steels, the second and higher order terms are not used, it may not be correct to say that the first order model for steel is linear elastic and the first order model for biotissues is hyperelastic. But it is the first approximation. Approximation is mostly useful but we need to know the error induced by the approximation (called ***Error-of-Definition***) or by neglecting the other terms so that we can estimate if it is good enough. For steel, we already know that the error is small so that we can neglect it for most applications. But, for biotissues, the error comes from the omission of nonlinear terms that have never been specified. Most of the models introduced in prior section have been developed within the high strain ranges where they have low deviations. Even in the low range of low strain which has huge amount of deviation from elasticity, they neglected it and just applied the assumption of hyperelasticity, isotropy, or/and Fung used the pseudoelasticity for the biotissues which is false as the name implies (Skalak and Chien, 1987). So if we use the hyperelastic models for the biotissues, we need to know how it deviates from the hyperelasticity before we can say whether we can neglect the deviation or not because we cannot neglect the deviation prior to the understanding of the deviation. We need to know how the deviation induced by neglected terms will affect our results. It should be answered what if we didn't know by this amount/how the approximation can be proposed to our uncertainty in the final answer. The uncertainty will be looked at by the fundamental statistical theory. It is good to know what degree of uncertainty the hyperelastic models for high strain materials and biotissues have. It is suggested that the degree of hyperelasticity of rubber is very high, about 98% hyperelastic.

Why We Need a Random Protocol

The completely randomized stretch controlled protocol is indispensable for error analysis. By the assumption that a hyperelastic model for the high strain materials and biotissues is a function of three variables such as time *T*, stretch-rate *S* and stretch history *H*, the stress σ can be described as

$$\sigma = \sigma(T, S, H) \tag{1.15}$$

To satisfy equation (1.15), those three variables should be completely independent of each other so that they are not coupled or correlated. Thus, the randomized uniaxial stretch- controlled protocol enables us to look at the three variables as independent variables as well as the major factors that cause the deviation from the hyperelasticity. Since all randomized points that meet the stretch ratio 1.1, 1.3, 1.5, 1.7 and 1.9 of the entire protocol have been used and averaged together, a hyperelastic model that can potentially be made didn't exclude the higher order terms, i.e., it didn't force the stress/strain relationship to be linear, quadratic, cubic or higher than that by allowing all higher order terms. The deviation due to the time spent, stretch-rate and stretch history has been found from the averaged stress.

CHAPTER II

METHODS

Preparation of Test Specimens

For this study, rubber fibers as high strain materials and longitudinal strips of pulmonary artery from adult swine as tissues were used. The initial length of the rubber specimen was chosen as 25.4 mm with radius of 0.3 mm. The maximum limit of stretch ratio that the rubber can go up to is 1.9. The tissue specimen for this study was strips of pulmonary artery of adult swine. The dimension of the tissue was 2.5 mm × 2 mm as the width × the length. The tissue specimens have less collagen ("a protein consisting of bundles of tiny reticular fibrils that combine to form the white glistening inelastic fibers of the tendons, the ligaments, and the fascia, Mosby's Medical, Nursing, & Allied Health Dictionary", Anderson, 1998) but have much elastin ("a protein that forms the principal substance of yellow elastic tissue fibers", Mosby's Medical, Nursing, & Allied Health Dictionary, Anderson, 1998) so that they experienced very high strain. To use the same randomized stretch-controlled protocols for both the rubber and tissue, we checked the maximum stretch ratio that the tissue can go without over limiting the range of the force-transducer (Its maximum is 50 gram). According to the preliminary test to ensure the maximum stretch ratio for the tissue, its maximum stretch ratio was 1.97 at 10.25 voltage output of force-transducer. Thus, we decided the maximum stretch ratio for the tissue as 1.9 and then used the same length of the tissue to use the same randomized stretch-controlled protocol used for the rubber, i.e., we used the same displacements for both the rubber and tissue.

$$(\lambda_r)_{\max} = \frac{L_r + \Delta L_r}{L_r} = 1 + \frac{\Delta L_r}{L_r}, \quad \frac{\Delta L_r}{L_r} = (\lambda_r)_{\max} - 1, \quad \Delta L_r = L_r\{(\lambda_r)_{\max} - 1\}$$
$$(\lambda_t)_{\max} = \frac{L_t + \Delta L_t}{L_t} = 1 + \frac{\Delta L_t}{L_t}, \quad \frac{\Delta L_t}{L_t} = (\lambda_t)_{\max} - 1, \quad \Delta L_t = L_t\{(\lambda_t)_{\max} - 1\} \tag{2.1}$$

Since the $\Delta L_r = \Delta L_t$,

$$L_t = \frac{L_r\{(\lambda_r)_{\max} - 1\}}{(\lambda_t)_{\max} - 1} \tag{2.2}$$

To use the same protocol, the length of the tissue can be calculated by using equation (2.2). Since the maximum stretch ratio was the same as 1.9 for both the rubber and tissue, the length of the tissue specimen was chosen as 1 inch. Figure 2.1 shows the nonlinear behavior of the strips of pulmonary artery of adult swine.

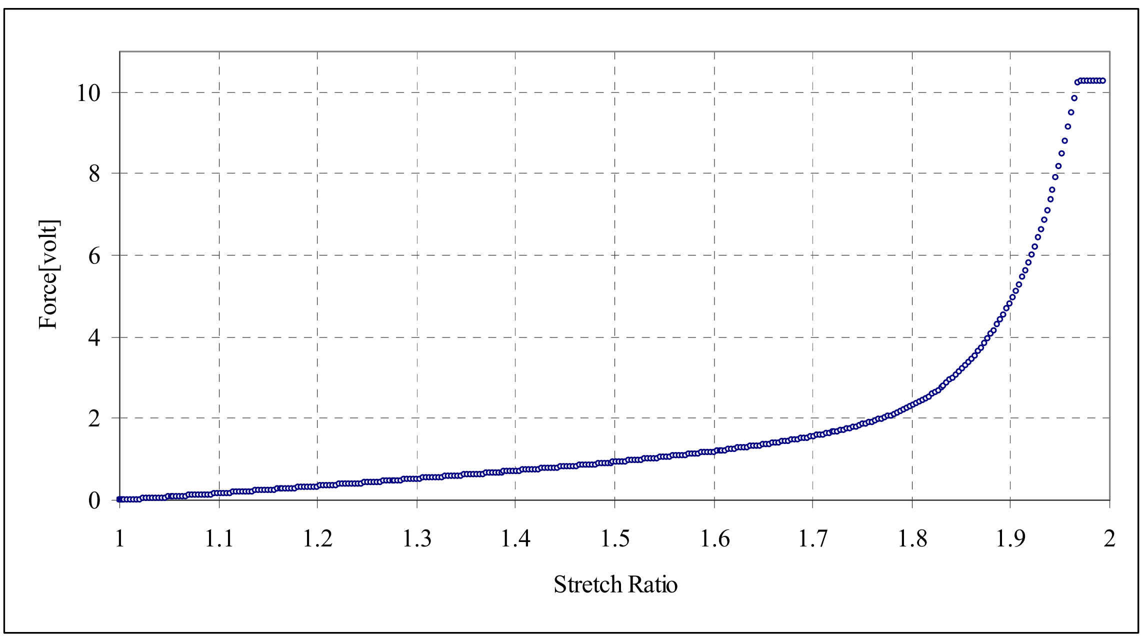

Fig.2.1. Nonlinear behavior of the tissue (strips of pulmonary artery of adult swine). It shows linear behavior until the stretch ratio 1.6 and then nonlinear behavior from the stretch ratio 1.6. Since the force transducer used for this study can detect the force range up to 50 gram, it didn't detect the force of the corresponding stretch ratios which is higher than 1.97.

The tissues have been kept in a phosphate buffered saline (PBS). Although most of the tissues are fixed by formalin, formaldehyde, or glutaraldehyde, elastin is kept in a PBS to hold on to its elasticity (Fung, 1993). The fixation agents such as formalin, formaldehyde, or glutaraldehyde change the property of tissue by cross linking the collagens. It makes the tissue stiff and loses extensible. PBS was sprayed on the tissues to keep them moist during the randomized uniaxial stretch-controlled test. To keep bacteria from growing we kept them on ice. All the experiments have done at room temperature which is around 23 °C.

Uniaxial randomized stretch-controlled tests were executed for both the rubber fiber and tissues and the data obtained by the experiment were analyzed based on the assumption that there are certain factors that cause the deviation from the hyperelasticity. Highly suspected factors that cause the deviation were expected to be strain-rate, strain history and time spent testing. For the stretch-controlled randomized protocol, stretch ratios and stretch rates were randomly generated as for nodal values and they were interpolated with the C^1 continuity by the custom codes. All codes were developed by using LabVIEW. For each nodal interval, maximum and minimum slopes meant by maximum and minimum velocities have been checked. If the maximum slope was higher than the maximum velocity of the actuators then time-interval was expanded to decrease the slope. Newport's Universal Motion Controller/Driver (ESP7000) and its compatible actuators (CMA-25CCCL) that have high precision have been employed for the randomized stretch-controlled uniaxial stretch test. These CMA actuators are capable of having minimum incremental motion of the order of sub-micron (Resolution = 0.048828 μm, Speed = 50 – 400 μm/sec). These motions were controlled by the LabVIEW based algorithms developed by us. After the experiment, it has shown how the error was looked at in 1-D. For each stretch level, it has been checked what factors were the most effective to make the deviation from the hyperelatsticity. It was highly suspected there would be different factors for the different stretch levels that make uncertainties.

Randomized Stretch-Controlled Protocol

Ideal randomized stretch-controlled protocol has been achieved by using cubic Hermite interpolation. Randomly generated nodal values for the interpolation were defined as stretch ratio (displacement) and velocity of the actuators. We, first, got the 71 raw random points for both the stretch ratio and velocity. They were used as nodal values for the interpolation. Figure 2.2 and figure 2.3 show the 71 raw random points for the stretch ratio and interpolated protocol, respectively. Algorithm for cubic Hermite interpolation and motion for the actuators were programmed by using software language LabVIEW (National Instrument, Inc). Figure 2.4 and figure 2.5 show the 71 raw random points for the velocity of actuator and interpolated velocity protocol, respectively. By following the stretch-controlled protocol as shown in figure 2.3 and figure 2.5 below, the specimens (a rubber fiber and a strip of pulmonary artery of an adult swine) were stretched with a specified velocity. Data has been acquired by using Newport Universal Motion Controller/Driver (ESP 7000).

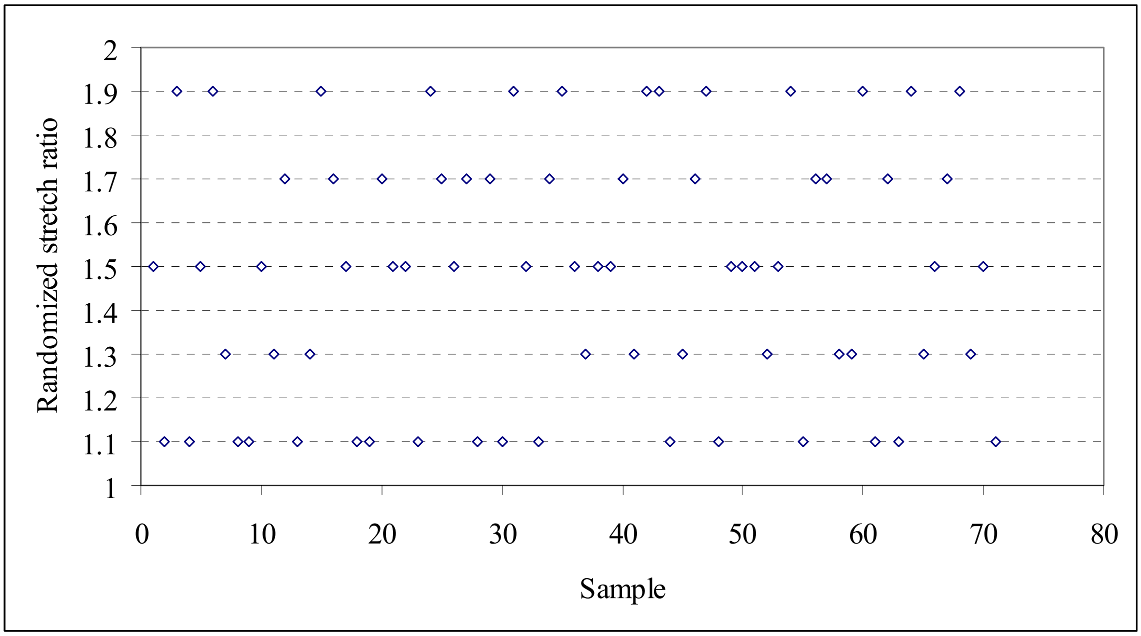

Fig.2.2. Randomly generated raw data points of stretch ratio for the cubic Hermite interpolation.

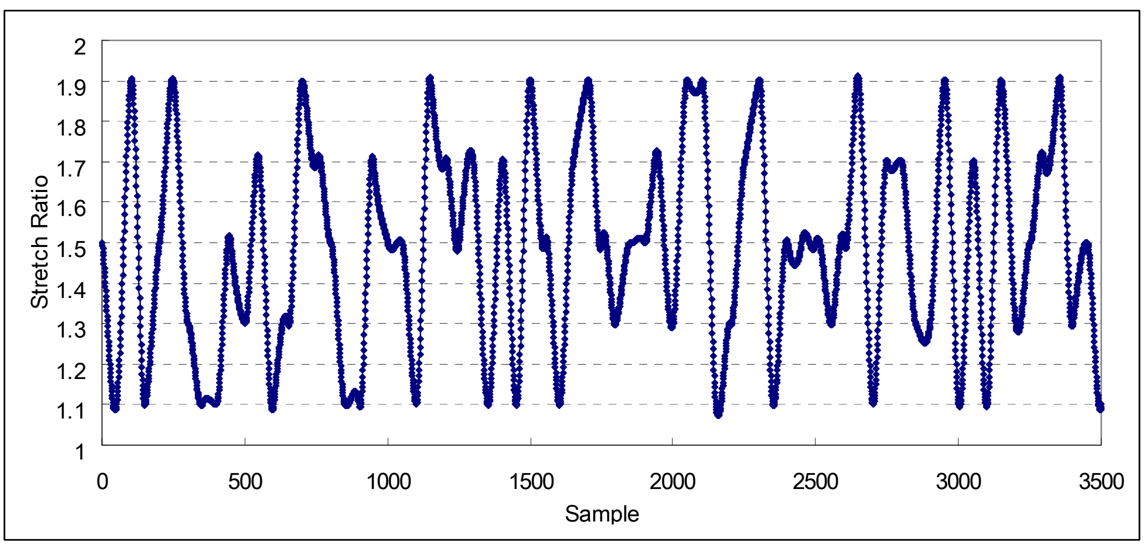

Fig.2.3. Interpolated ideal stretch-controlled protocol.

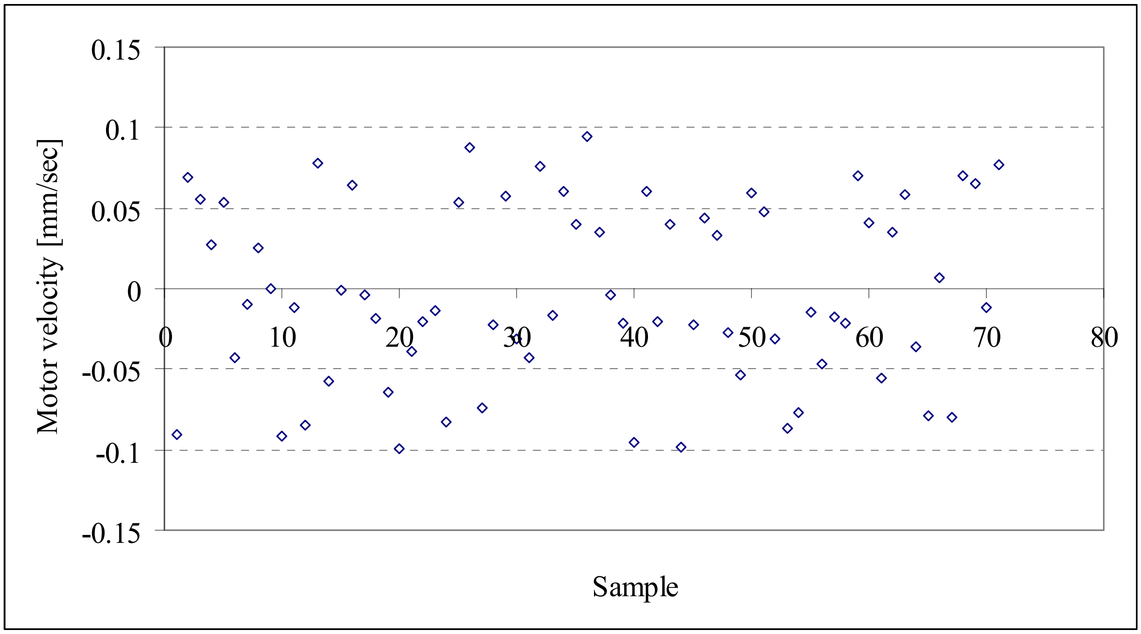

Fig.2.4. Randomly generated raw data points of the actuator velocity for the cubic Hermite interpolation.

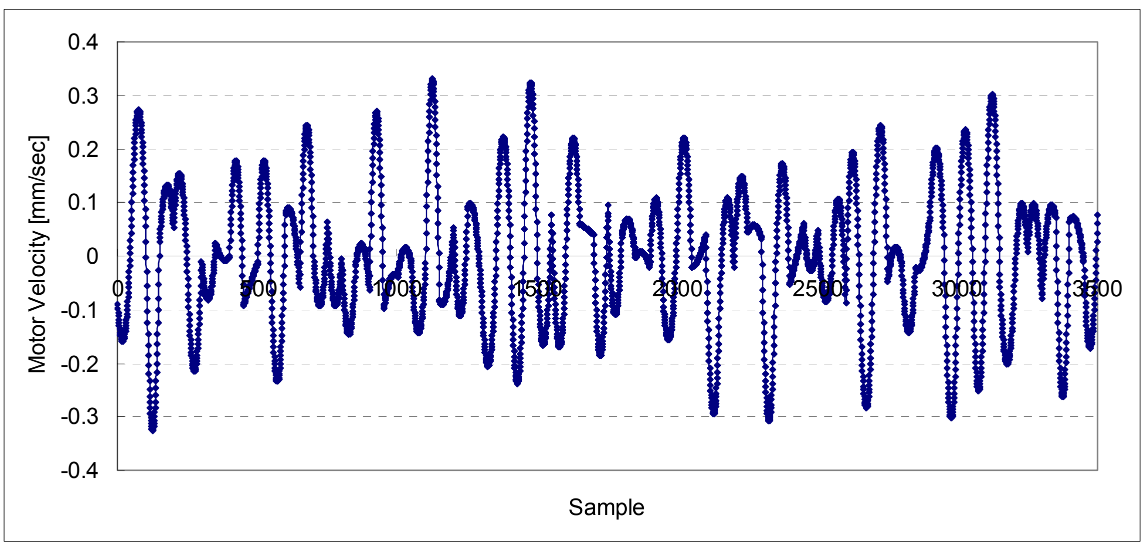

Fig.2.5. Interpolated ideal velocity protocol of an actuator (motor).

Cubic Hermite Interpolation

In the area of numerical method of mechanics such as Finite Element Analysis, a third-order curve called cubic Hermite polynomial which has C^1 continuity property is widely being used. It is a spatial (interpolation with space, *x*, *y*, *z*) or temporal (interpolation with time, *t*) interpolation of nodal values. It is a processing of estimation of the nodal values that are unknown by using the nodal values that are already given as, in this study, stretch ratios and velocities (as stretch-rate) within a given range. The basic form of the cubic Hermite interpolation function for 1-D is

$$\lambda(\xi) = H_0^1(\xi)\lambda\Big|_{N_1} + H_1^1(\xi)\frac{\partial \lambda}{\partial \xi}\bigg|_{N_1} + H_0^2(\xi)\lambda\Big|_{N_2} + H_1^2(\xi)\frac{\partial \lambda}{\partial \xi}\bigg|_{N_2} \tag{2.3}$$

$$\frac{\partial \lambda(\xi)}{\partial \xi} = \frac{\partial H_0^1(\xi)}{\partial \xi}\lambda\Big|_{N_1} + \frac{\partial H_1^1(\xi)}{\partial \xi}\frac{\partial \lambda}{\partial \xi}\bigg|_{N_1} + \frac{\partial H_0^2(\xi)}{\partial \xi}\lambda\Big|_{N_2} + \frac{\partial H_1^2(\xi)}{\partial \xi}\frac{\partial \lambda}{\partial \xi}\bigg|_{N_2} \tag{2.4}$$

where $H_0^1(\xi)$, $H_1^1(\xi)$, $H_0^2(\xi)$, and $H_1^2(\xi)$ are the interpolation functions or shape functions. Cubic Hermite polynomial, as shown above, has the form that has two prearranged points $\lambda\big|_{N_1}$ and $\lambda\big|_{N_2}$ and two prearranged tangents $\left(\frac{\partial \lambda}{\partial \xi}\right)\bigg|_{N_1}$ and $\left(\frac{\partial \lambda}{\partial \xi}\right)\bigg|_{N_2}$.

They are given at element nodes N_1 and N_2. The element, therefore, has four degree of freedoms (i.e., two degree of freedom per node) and they are given by scalar values as stretch ratios and velocities in this study. To get the cubic Hermite polynomials $H_0^1(\xi)$, $H_1^1(\xi)$, $H_0^2(\xi)$, and $H_1^2(\xi)$, we applied the interpolation properties shown below.

From $\lambda(\xi = 0) = \lambda\big|_{N_1}$, we have

$$
\begin{aligned}
H_0^1(0) &= 1 \\
H_1^1(0) &= 0 \\
H_0^2(0) &= 0 \\
H_1^2(0) &= 0
\end{aligned}
\tag{2.5}
$$

From $\lambda(\xi = 1) = \lambda\big|_{N_2}$, we have

$$
\begin{aligned}
H_0^1(1) &= 0 \\
H_1^1(1) &= 0 \\
H_0^2(1) &= 1 \\
H_1^2(1) &= 0
\end{aligned}
\tag{2.6}
$$

From $\dfrac{\partial\lambda(\xi = 0)}{\partial\xi} = \dfrac{\partial\lambda}{\partial\xi}\bigg|_{N_1}$, we have

$$
\begin{aligned}
\frac{\partial H_0^1(0)}{\partial\xi} &= 0 \\
\frac{\partial H_1^1(0)}{\partial\xi} &= 1 \\
\frac{\partial H_0^2(0)}{\partial\xi} &= 0 \\
\frac{\partial H_1^2(0)}{\partial\xi} &= 0
\end{aligned}
\tag{2.7}
$$

From $\dfrac{\partial\lambda(\xi = 1)}{\partial\xi} = \dfrac{\partial\lambda}{\partial\xi}\bigg|_{N_2}$, we have

$$
\begin{aligned}
\frac{\partial H_0^1(1)}{\partial\xi} &= 0 \\
\frac{\partial H_1^1(1)}{\partial\xi} &= 0 \\
\frac{\partial H_0^2(1)}{\partial\xi} &= 0 \\
\frac{\partial H_1^2(1)}{\partial\xi} &= 1
\end{aligned}
\tag{2.8}
$$

The polynomials used to represent the *H(ξ)* must have all terms beginning with a constant terms up to the highest order. Since there are four conditions (two per node) for each *H(ξ)*, a four-parameter polynomial should be chosen for *H(ξ)* which is a cubic polynomial. These four values are given for setting the nodal conditions.

By using the four conditions above, we have

$$\begin{aligned} H_0^1(\xi) &= 1-3\xi^2+2\xi^3 \\ H_1^1(\xi) &= \xi^3-2\xi^2+\xi \\ H_0^2(\xi) &= 3\xi^2-2\xi^3 \\ H_1^2(\xi) &= \xi^3-\xi^2 \end{aligned} \tag{2.9}$$

where $\xi \in [0, 1]$.

If we consider an 1-D cubic Hermite interpolation function in the global coordinate t which is the coordinate of the problem, it will be expressed as

$$\lambda(t) = H_0^1(t)\lambda\big|_e + H_1^1(t)\frac{\partial\lambda}{\partial t}\bigg|_e + H_0^2(t)\lambda\big|_{e+1} + H_1^2(t)\frac{\partial\lambda}{\partial t}\bigg|_{e+1} \tag{2.10}$$

where,

$$\begin{aligned} H_0^1(t) &= 1-3\left(\frac{\bar{t}}{h_e}\right)^2+2\left(\frac{\bar{t}}{h_e}\right)^3 \\ H_1^1(t) &= \bar{t}\left(1-\frac{\bar{t}}{h_e}\right)^2 \\ H_0^2(t) &= 3\left(\frac{\bar{t}}{h_e}\right)^2-2\left(\frac{\bar{t}}{h_e}\right)^3 \\ H_1^2(t) &= \bar{t}\left[\left(\frac{\bar{t}}{h_e}\right)^2-\left(\frac{\bar{t}}{h_e}\right)\right] \end{aligned} \tag{2.11}$$

and, $\bar{t} = t - t_e$. Schematic for the derivation of cubic Hermite interpolation function is shown in the figure 2.6. It shows the relationships between global, local and generalized coordinates where e-1, e, e+1, and e+2 are global node numbers, $\xi = (t - t_e)/h_e$ and $\partial\xi/\partial t = 1/h_e$. The length of the element h_e is the scalar factor and used for chain rule such as $\left.\frac{\partial\lambda}{\partial\xi}\right|_e = \left.\frac{\partial\lambda}{\partial t}\right|_e \cdot \frac{\partial t}{\partial\xi} = \left.\frac{\partial\lambda}{\partial t}\right|_e \cdot h_e$.

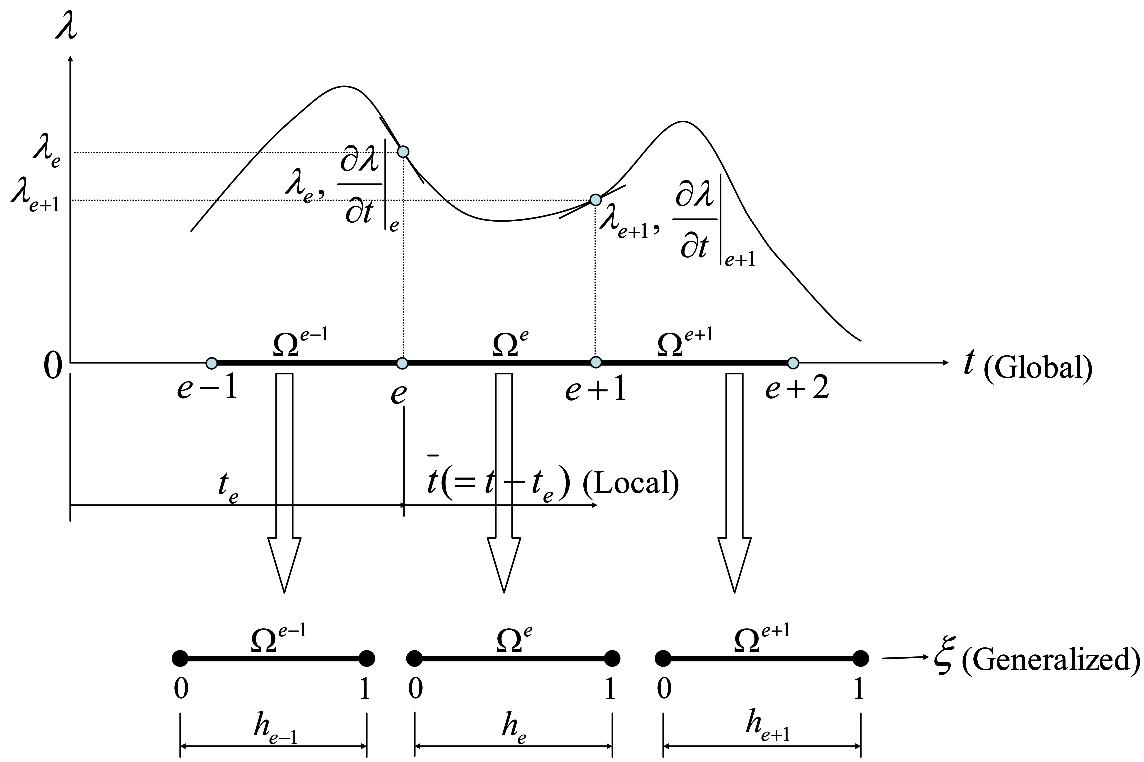

Fig.2.6. Schematic for the derivation of cubic Hermite interpolation function. It shows the relationships between global, local and generalized coordinates where e-1, e, e+1, and e+2 are global node numbers, $\xi = (t - t_e)/h_e$ and $\partial\xi/\partial t = 1/h_e$. The length of the element h_e is the scalar factor and used for chain rule such as $\left.\frac{\partial\lambda}{\partial\xi}\right|_e = \left.\frac{\partial\lambda}{\partial t}\right|_e \cdot \frac{\partial t}{\partial\xi} = \left.\frac{\partial\lambda}{\partial t}\right|_e \cdot h_e$.

Verification of Randomness

The significance of randomness in this study is that it provides reliable results from error analysis. It should be evident that there is no correlation amid the time spent, stretch-rate and stretch history before we do the experiment of randomized stretch-controlled protocol to guarantee, if there is any, the validity of the correlation after the experiment. Thus it can be answered for the question of validity of randomness of the original raw data which have obtained by random number generation command in LabVIEW (National Instrument, Inc). The randomness is defined by the ignorance of cause and effect, i.e., any event should be caused by chance alone and uncontrollable if it can be regarded as random. Thus, a random signal, also known as a white noise, doesn't have any recurring of same patterns in the signal, i.e., there is no correlation between the signal values. White noise, by definition ideally, has all frequency components which have the same powers throughout a defined frequency domain. Thus, by using the properties of white noise, the randomness of the randomized stretch-controlled protocol can be verified by investigating the frequency components and their amplitudes in the protocols. Discrete Fourier Transformation (DFT) and Wavelet Transformation (WT) have been employed to see whether the protocols satisfy the properties of the white noise. Figure 2.7, figure 2.8 and figure 2.9 show the DFT, WT (2-D) and WT (3-D) of the stretch protocol, respectively. Figure 2.10, figure 2.11 and figure 2.12 show the DFT, WT (2-D) and WT (3-D) of the velocity protocol, respectively. The spectral analysis of the protocols by using wavelet transformations (Time-Frequency domain) shown in figure 2.8, 2.9, 2.11, and 2.12 show the more detail distributions of the frequency components over time. If we see a specific frequency component, there is the highest pick of the frequency component. The frequency components that we see in FFT (Frequency domain) are those that have the highest picks for each frequency components. They show the every frequency components that are consist of within the maximum frequency range (0.5Hz) and the spectrums are fairly even. In other words, since there is no dominant frequency, most of the frequencies are represented of similar magnitudes, and spectral representations show that they have all spectrums in the range, they are affectively random. Thus, we can conclude that we prohibited most of the factors that can possibly cause any correlations before we conducted the experiment and error analysis. In reality, we can't have perfectly flat spectrums because it is impossible to have infinite number of points to develop a perfectly flat spectral representation. If we execute the cyclic loading test, we will have dominant spectrums.

If we have more random data, the resolution will get finer and it will seem to mostly follow the properties of white noise for the spectrum analysis. The more random points we have the higher spectral resolution will be achieved. At this moment, we have 71 data points to be analyzed. Figure 2.13 and figure 2.14 show the frequency spectrum of total interpolated stretch ratio and velocity of an actuator data points. They show the much higher resolution of frequency spectrums because of involving a number of data points (the number of total interpolated points is 3500).

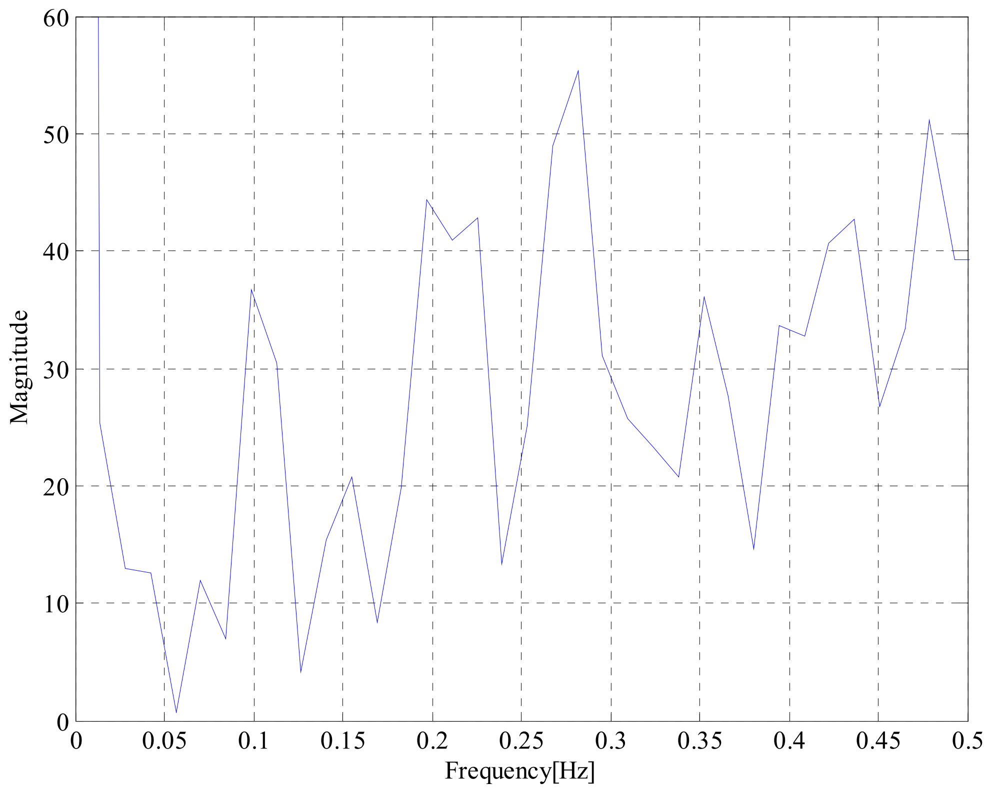

Fig.2.7. Fourier spectral representation of the randomized stretch ratio data points.

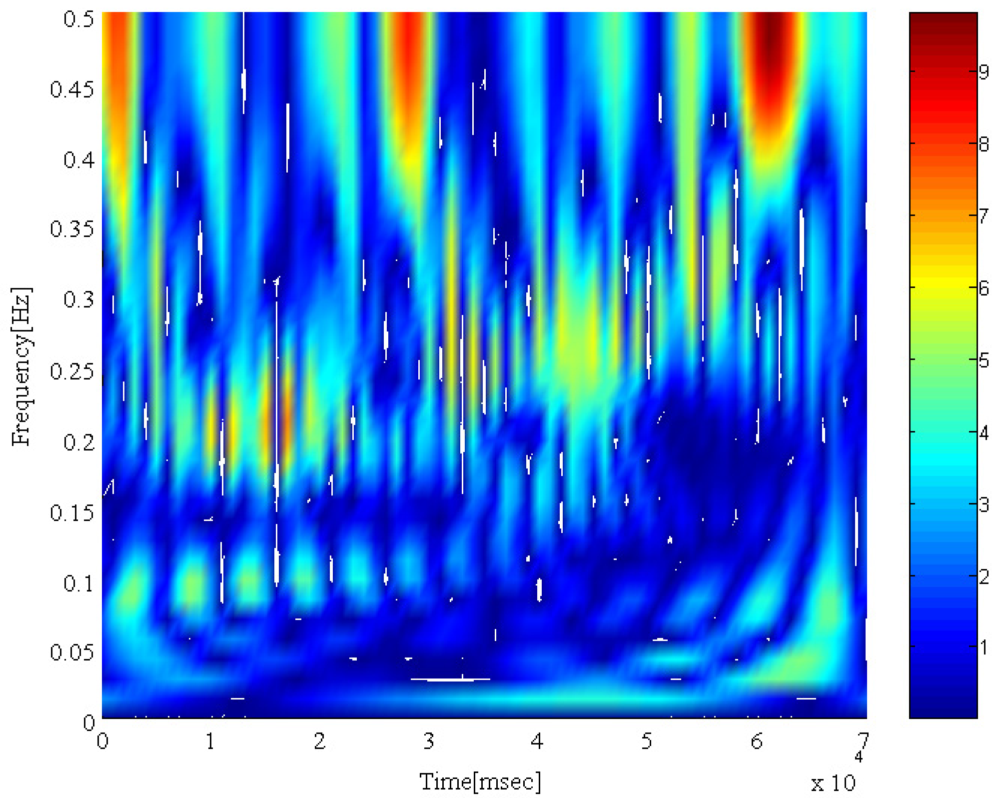

Fig.2.8. Frequency spectrum of randomized stretch ratio data points viewed in time-frequency domain (2-D) using wavelet transformation.

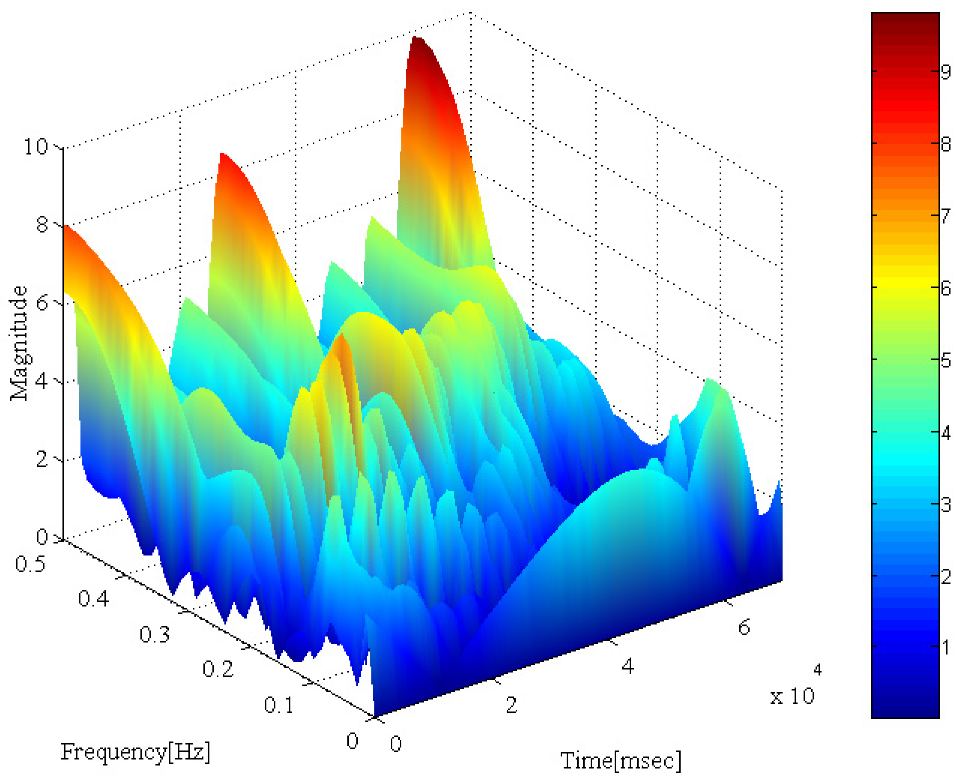

Fig.2.9. Frequency spectrum of randomized stretch ratio data points viewed in time-frequency domain (3-D) using wavelet transformation.

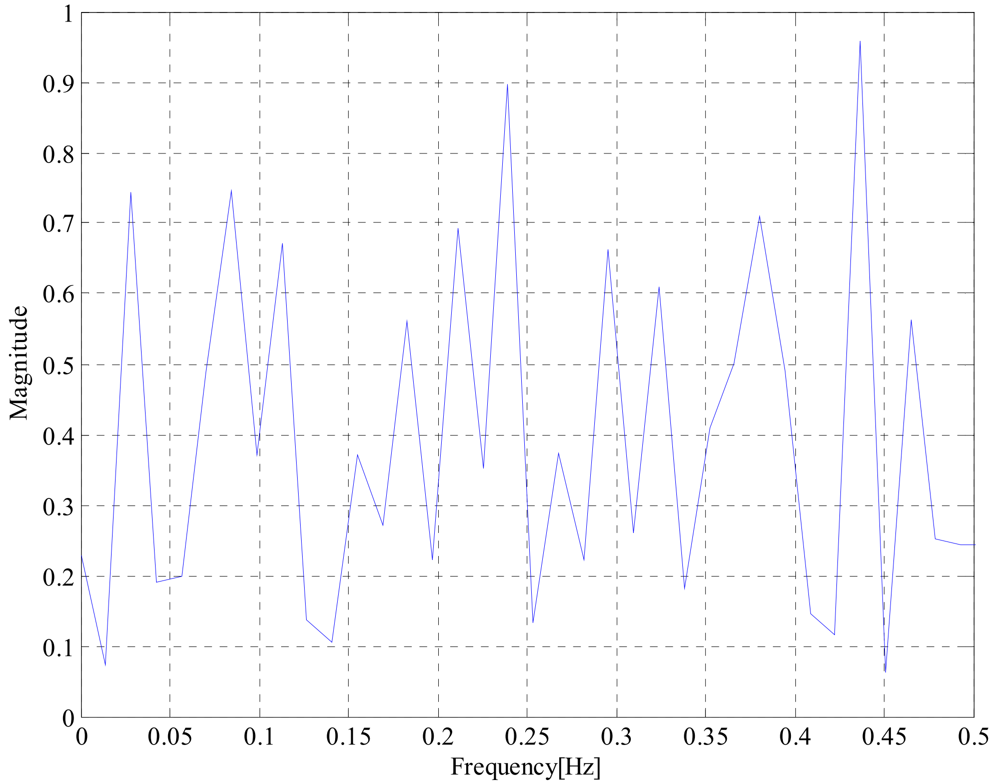

Fig.2.10. Fourier spectral representation of the randomized velocity data points.

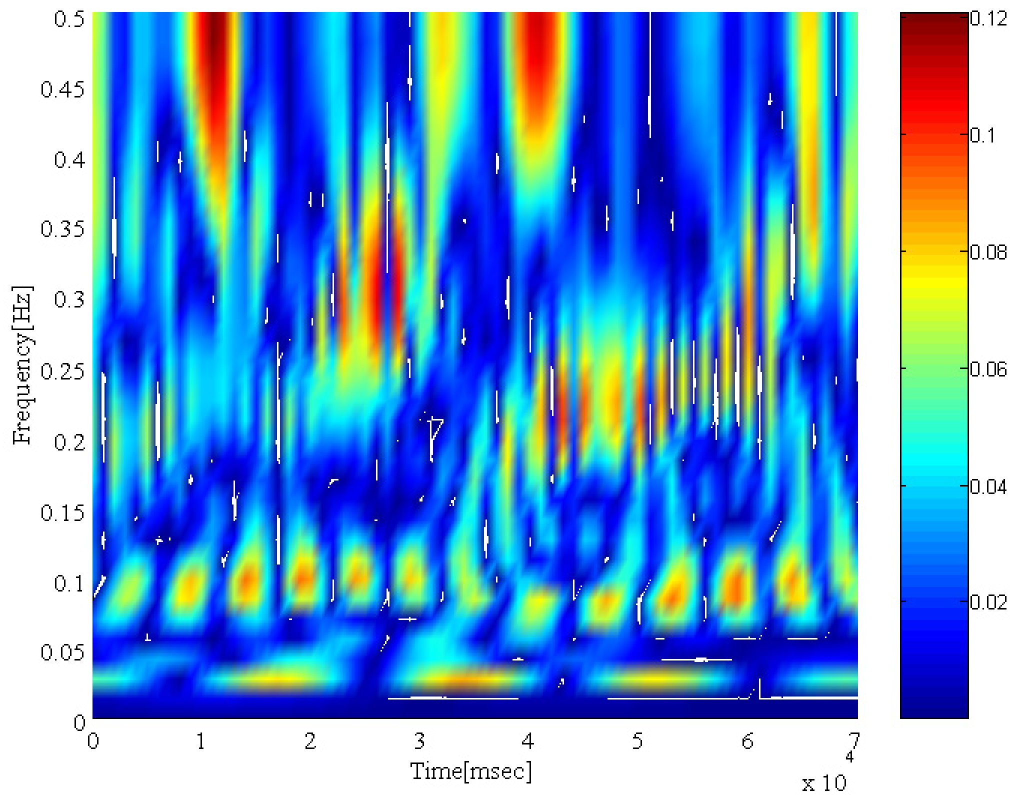

Fig.2.11. Frequency spectrum of randomized velocity data points viewed in time-frequency domain (2-D) using wavelet transformation.

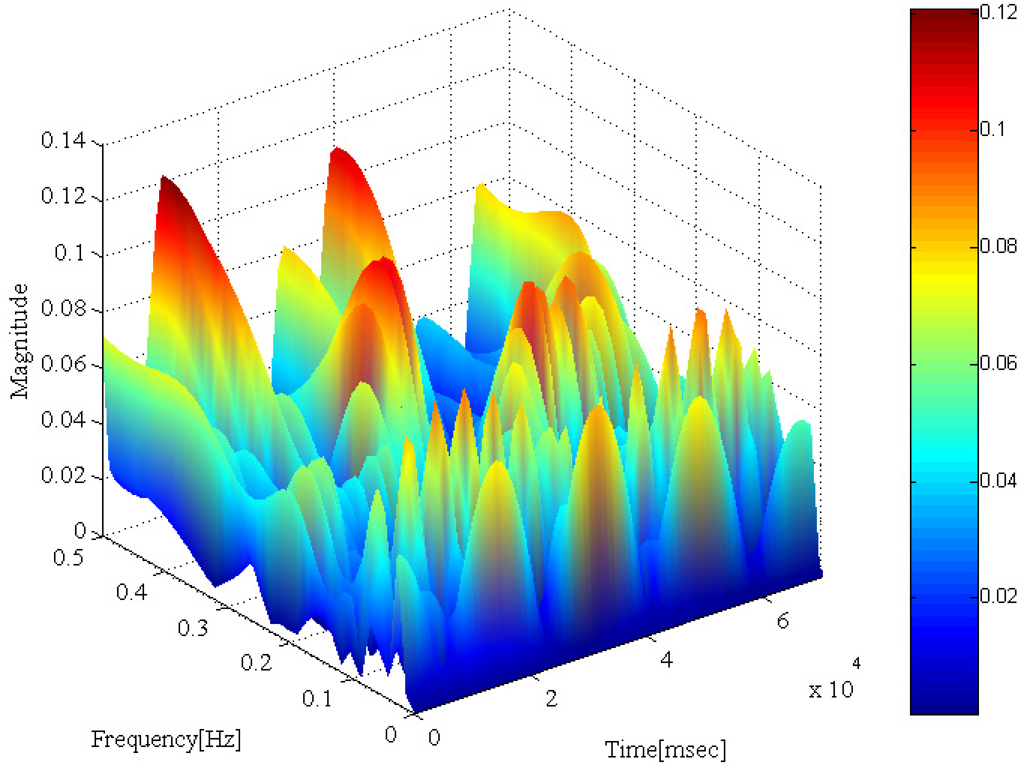

Fig.2.12. Frequency spectrum of randomized velocity data points viewed in time-frequency domain (3-D) using wavelet transformation.

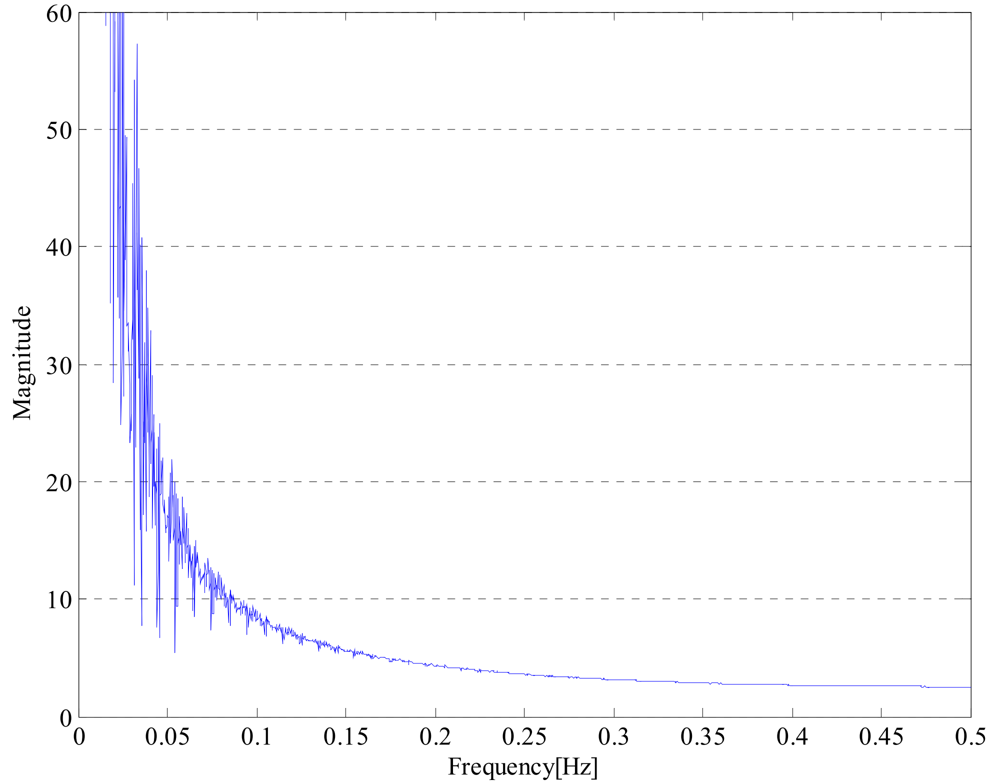

Fig.2.13. Frequency spectrum of total interpolated stretch data points.

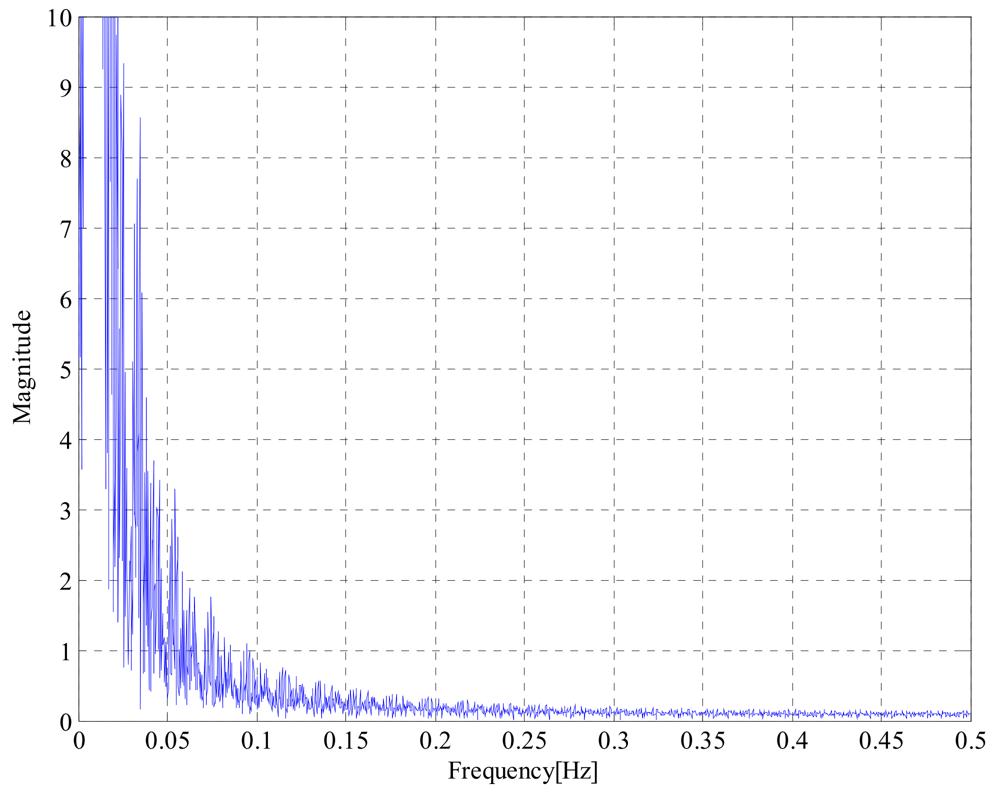

Fig.2.14. Frequency spectrum of total interpolated velocity data points.

Data Acquisition and Conversion

The motions of actuators (CMA-25CCCL, Newport) were controlled and monitored by the Newport's Universal Motion Controller/Driver (ESP7000). The data of the motions for the randomized stretch-controlled protocol also have been obtained by the controller. It gets the time, position of the actuators for a corresponding time, and analog output as data for every 0.1 second. An analog output in this study was a voltage output for the corresponding force input whenever the specimens (rubber/ tissue) were stretched. The force transducer (Harvard Apparatus, Inc) was connected between the specimens and the motion controller/driver through the data acquisition board. It detected the voltage changes according to the motion variations. Figure 2.15 shows the experimental setup for the uniaxial randomized stretch-controlled protocol. Because the controller/driver didn't give the decimal configuration of the numbers for the data, the data obtained by the controller/driver should be converted by proper gains which have not been supported by the Newport. Table 2.1 shows the conversion table for converting the non-decimal configuration of the data obtained by the controller/driver to decimal configuration of the data. Because of the oscillation of the signal (voltage) from the force transducer, a low-pass filter has been made and employed into the experimental system to get more stable signal output from the force transducer. 2nd order Active Low-Pass Butterworth Filter of -40dB/decade has been chosen as for the low-pass filter because it makes the closed-loop gain to be 1 as close as possible within the pass band (Coughlin and Driscoll, 1982). To design the filter, we chose the cutoff frequency f_c as 5 Hz (note that a sampling frequency f_s is 10 Hz) $R1 = R2 = R = 10$ kΩ, and $Rf = 2R = 20$ kΩ. By using following formula

$$C1 = \frac{0.707}{f_c R} = \frac{0.707}{5 \times 10 \times 10^3} = 2.25 \mu F$$
$$C2 = 2C1 = 4.5 \mu F \tag{2.12}$$

we selected the proper capacitors and resistors for designing the filter. Figure 2.16 shows a basic circuit diagram for the 2nd order Active Low-Pass Butterworth Filter of -40dB/decade.

Table.2.1. Conversion table for non-decimal data of force, position of actuator, and time.

Data Conversion		
Analog Input	Position	Time
1 volt = 3200	1 mm = 20480	1 sec = 2560

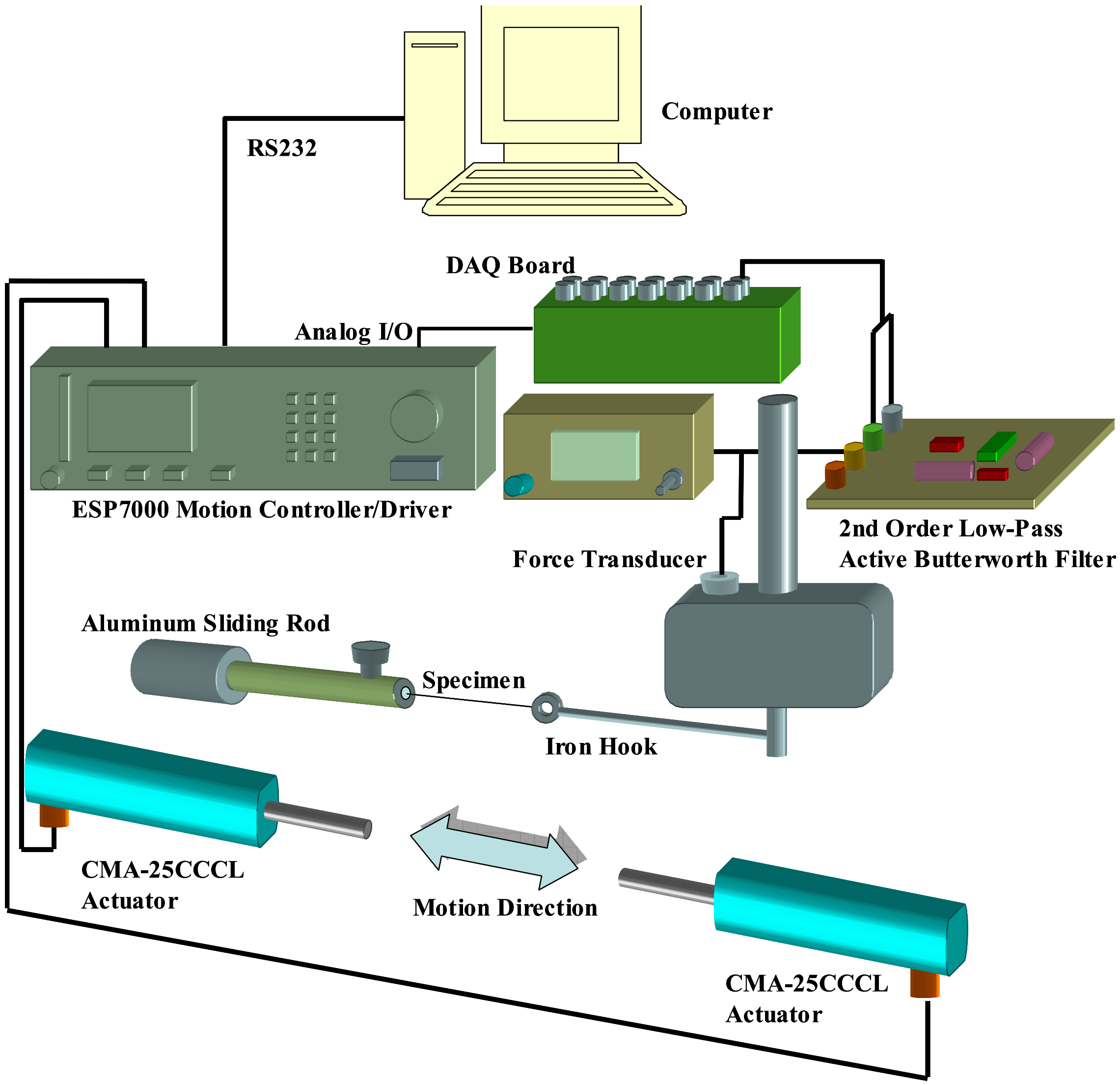

Fig.2.15. Schematic of the experimental setup. Two CMA-25CCCL Actuators are connected to the Aluminum sliding rod and the force transducer through the sliding tables which are not shown in the figure.

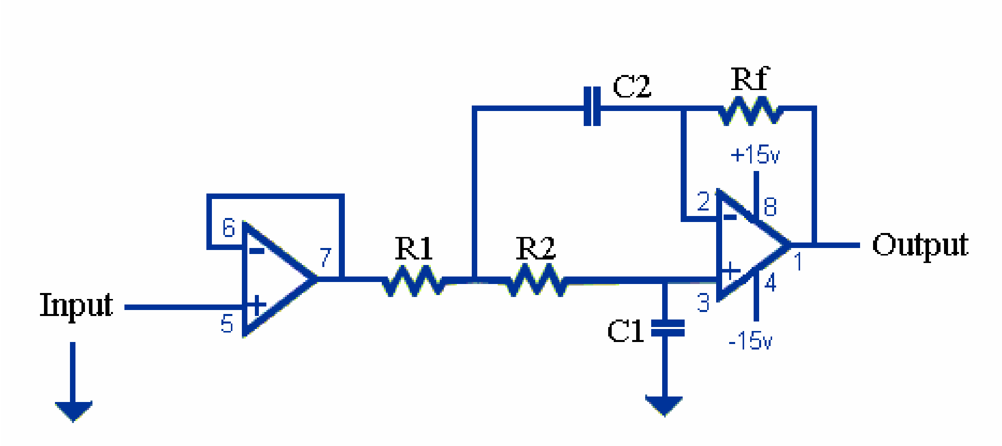

Fig.2.16. Circuit diagram of 2nd Order Low-Pass Active Butterworth Filter of dB/decade.

Stretch History

Since the change in mechanical properties of the rubber-like materials and biotissues are highly dependent on the previous stretches, it is very important to understand the stretch history for the materials. Stretch history tells us what has come before. Thus stretch history says that what types of experiences the specimens have undertaken before. One aspect of stretch history is a quantity that we refer to as ***affective (or average) stretch history*** which is the average of the stretches for the prior time interval of duration t_h - a suitable chosen time constant obtained by the relaxation spectrums. To get the relaxation spectrum, we have executed the stress relaxation tests on the testing materials.

The reason that we have to perform the stress relaxation tests on them can be explained that, for example, if there are three different stretch-controlled protocols, even if the stretch ratio and stretch-rate are the same at a certain time, stretch histories are different, i.e., the areas under the curves are different. Thus, by integrating the $\lambda(t)$ and dividing it by the last amount of time t_h that is a suitably chosen time constant obtained from the relaxation spectrums by the stress relaxation tests, we got averaged stretch ratio for t_h. Figure 2.17 shows the basic concept of stretch history. From the figure 2.17, we saw that it was very important to decide a reasonable t_h for achieving the reasonable stretch history. Because if t_h gets smaller and smaller, $H(t)$ approaches to $\lambda(t)$ therefore it doesn't give us much history. Equation (2.13) shows the relationship between stretch history function $H(t)$ and stretch ratio $\lambda(t)$ which is a function of time.

$$H(t) = \frac{\int_{t-t_h}^{t} \lambda(t)dt}{t_h} \tag{2.13}$$

If t_h is zero, it doesn't give any history. In contrast, as t_h gets bigger and bigger, it gets close to the average of whole stretch ratios of entire motion which is not consequential. Thus finding reasonable t_h is important to figure out the uncertainty due to the stretch history. To find the t_h, we performed the stress relaxation test for the specimens so that we could capture the relaxation spectrum for the specimens and then figured out the fast decay and slow decay of the stress relaxation.

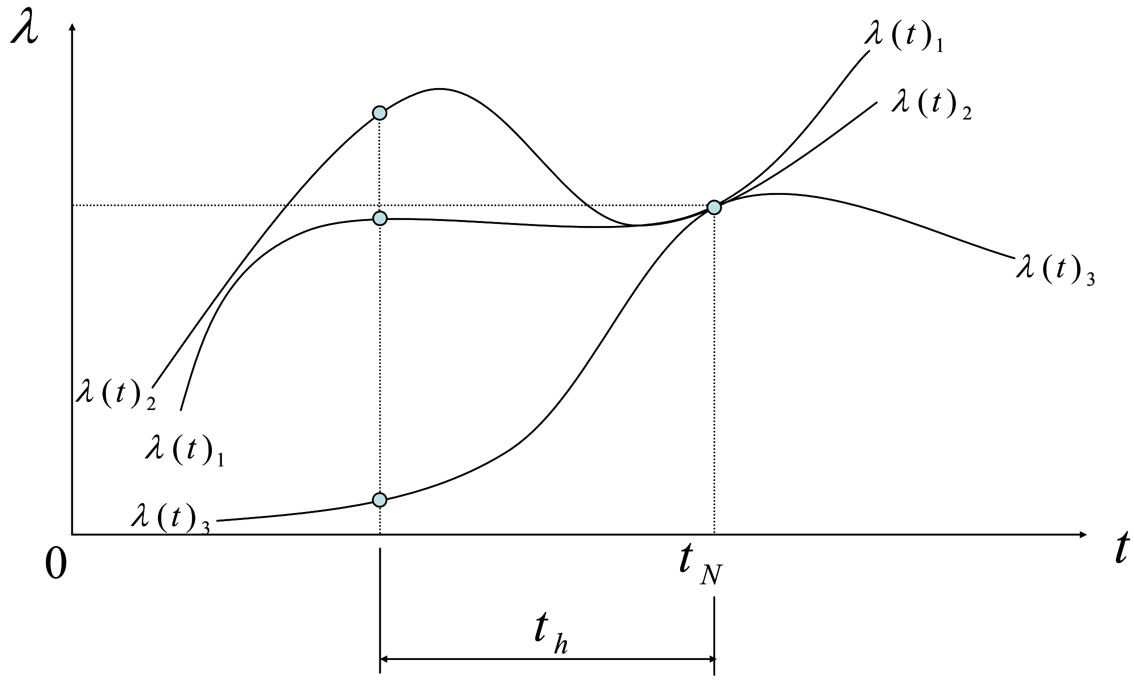

Fig.2.17. Although the stretch ratio functions $\lambda(t)_1$, $\lambda(t)_2$, and $\lambda(t)_3$ that are function of time give the same value at time t_N, the average stretch ratios for the time duration t_h for each of the functions are different.

Stress Relaxation Tests

Rubber: A stretch history, as mentioned in prior section, tells what types of experiences the specimens have for the previous times. To get the appropriate stretch history which is an average stretch ratio for the specified time duration t_h, we executed the stress relaxation tests on the rubber. Figure 2.18 shows the stress relaxation data curve and fitting curve for the rubber. By using the constrained nonlinear optimization in the Matlab (Mathworks, Inc), we obtained the acceptable fitting curve corresponding to the raw stress relaxation data for the rubber. The equation for the optimized fitting curve could be expressed by a summation of two exponential functions as following.

$$V(t) = A + Be^{-\frac{t}{t_1}} + Ce^{-\frac{t}{t_2}} \tag{2.14}$$

where A = 2.756, B = 0.097, C = 0.025, t_1 = 403, t_2 = 8.99, and V is a voltage output which is equivalently a force. For the rubber, the stress history cutoffs (or relaxation spectrums), t_h, have been chosen as t_h = 10 sec and t_h = 400 sec for early (fast) decay and late (slow) decay, respectively.

It is very important to note that the stretch history obtaining by the history cutoff, t_h =10 sec is rely on the direction of approach because it is a relatively short duration. If a stretch rate is positive at a certain point of time t_N, the average stretch ratio which is within relaxation spectrum t_h = 10 sec will be less than the stretch ratio at the point t_N. Since the slope (stretch rate $\dot{\lambda}(t)$) is changing gradually and acceleration of stretch ratio $\ddot{\lambda}(t)$ relatively low, stretch rate $\dot{\lambda}(t)$ tells what was immediately happening before. In rubber case, what happens 10 seconds prior is given by the rate.

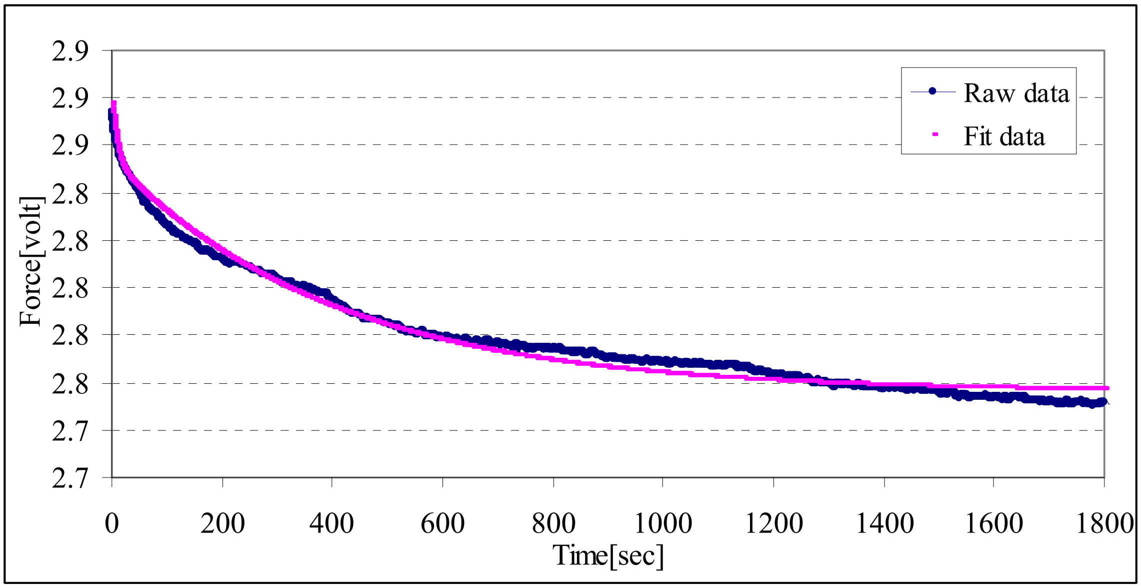

Fig.2.18. Stress relaxation test on the rubber fiber. Fitting shows that it is well-fitted by summation of two exponential functions.

This tells that the relationship between the stretch history scanned by the relaxation spectrum t_h =10 sec and the stretch-rate is exactly inverse. If stretch rate $\dot{\lambda}(t)$ is positive, it means the randomized stretch-controlled protocol move in positive directions and vice versa. Thus, we can have following relationships described in equation (2.15).

$$
\begin{aligned}
&\text{if} \quad \dot{\lambda}(t)\Big|_{t_N} > 0,\ H(t) = \left.\frac{\int_{t_N - t_h}^{t_N} \lambda(t)dt}{t_h}\right|_{t_h = 10} < \lambda(t)\Big|_{t_N} \\
&\text{if} \quad \dot{\lambda}(t)\Big|_{t_N} < 0,\ H(t) = \left.\frac{\int_{t_N - t_h}^{t_N} \lambda(t)dt}{t_h}\right|_{t_h = 10} > \lambda(t)\Big|_{t_N}
\end{aligned} \tag{2.15}
$$

Moreover, the correlation coefficients for both the stretch-rate and stretch history scanned by t_h = 10 sec have exactly the same absolute values but opposite signs. For the tissue, it can be understood likewise. Figure 2.19 shows the schematic of the relationship between the stretch-rate and stretch history scanned by early decay of relaxation spectrums.

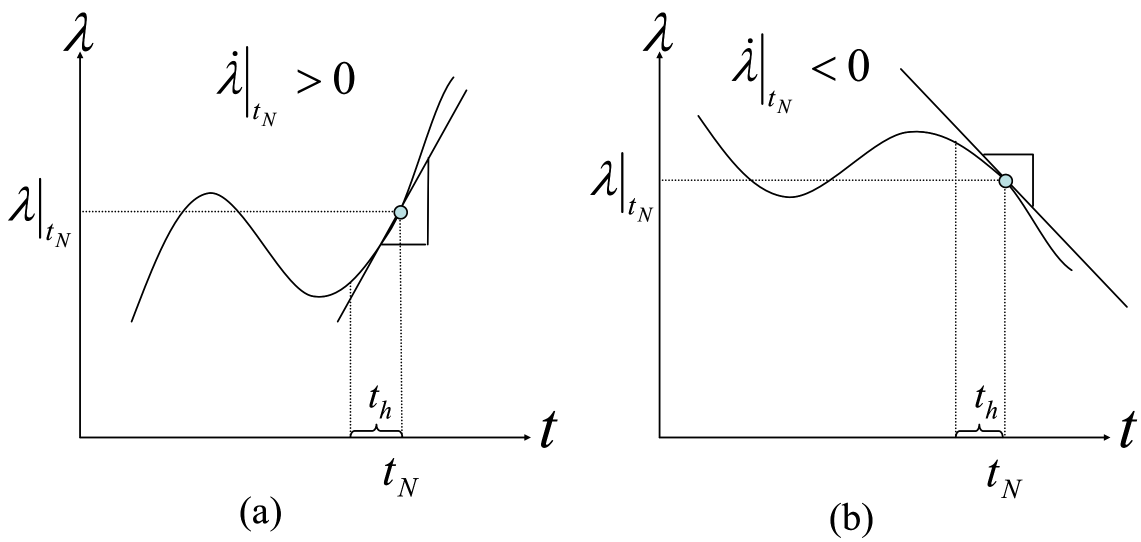

Fig.2.19. When the stretch rate is positive, the stretch history scanned by $\boldsymbol{t_h}$ =10 sec is less than the present stretch ratio. (b) When the stretch rate is negative, the stretch history scanned by $\boldsymbol{t_h}$ =10 sec is bigger than the present stretch ratio $\lambda(t_N)$.

Biotissues: Unlikely to the rubber, the stress relaxation curve for the tissue was well-fitted by the summation of four exponential functions as follows.

$$V(t) = 5.5 + 1.5e^{-t/400} + 0.5e^{-t/30} + 1.3e^{-t/10} + 0.5e^{-t} \tag{2.16}$$

For the tissue, t_h = 10 sec has been used chosen as fast decay of the relaxation spectrums to get the rate-related stretch history function. Although the t_h = 400 sec would be the reasonable slow history cutoff of the relaxation spectrums to get the stretch history function for a long decay, we figured out from the figure 2.20 that that there is a longer relaxation spectrum that was t_h = 1000 sec. So we employed one more variable which was the new stretch history function *H(t)* scanned by history cut-off, t_h = 1000 sec, for the multivariable linear regression analysis for the tissue. But, it was hardly affective to the deviation from the hyperelasticity as will be shown in the results.

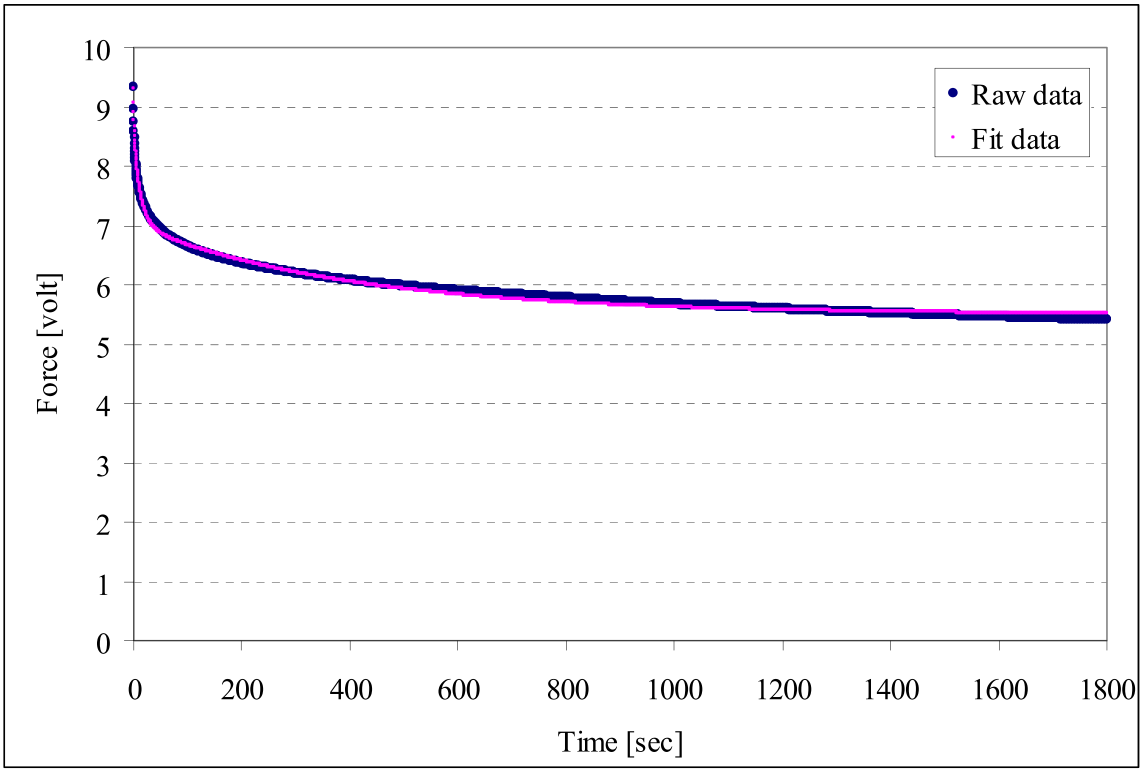

Fig.2.20. Stress relaxation test on the strip of pulmonary artery of the adult swine. Data have been obtained at every 0.4 second for 1 hour. After a half an hour, it seems asymptotically stable.

Types of Uncertainties

There are two types of errors; random error and systematic error. Here the term error is being regarded as an uncertainty. Random errors appear during the time of any measurement in random manner so they are statistically unpredictable. This type of error can be reduced by repeating the same procedure for all measurements under the identical experimental situation. But systematic error cannot be reduced in this way. Systematic errors affect the results in a systematic way, to be exact, they make a certain degree of bias from the true value in the same way. They are, in general, hard to detect and estimate. Here we defined the uncertainties due to the stretch-rate, stretch history, and time spent testing as systematic errors because they behave systematically. We have found that as stretch history and time get bigger the deviations get smaller. But as stretch-rate gets bigger, deviation also gets bigger which intuitively make sense.

Because there is no hyperelastic material in the world, whenever we use the hyperelastic model for a constitutive modeling of elastomers and biotissues, the error-of-definition becomes an issue. Thus if the elastic constitutive relation is to be determined by the experiment, the assumption of hyperelasticity should be considered as a part of errors. Error-of-definition stands for a degree of uncertainty that we are uncertain of the elastic stresses in rubber fibers and biotissues. This tells that we have to consider the inelastic behavior as an experimental error. In contrast, an Error-of-measurement is issued whenever the error comes from any measurement and instrument. More specifically, hyperelastic assumption for the rubber fiber gives rise to the Error-of-definition. The measuring of force (or voltage), radius of the rubber fiber, resolution of force transducer, motion controller, and actuators will give rise to the

Error-of-measurement. Note that the Error-of-Measurement sometimes includes the Error-of-Definition. For example, let say we are measuring a temperature of the beaker and the temperature of the beaker is, for example, 400 K. For measuring the temperature, we normally take the average. By saying that the beaker has a temperature, we expect that there is an Error-of-Definition. Now, if we think about the thermometer, we may not be able to read the thermometer because of the unscaled part of it. This gives rise to the instrument error which is a part of the Error-of-Measurement. Another example is a measurement of the dimension of the specimens. If we measure a width, a, of the tissue, we don't have just one value for a. The measurement of a certain specimen has both the Error-of-Definition and the Error-of-Measurement. To get rid of the Error-of-Definition which is dependent on the location, we measured many times on it and then averaged them. By doing this we got just one value which was finally representing the Error-of-Measurement. The randomness goes away if we measure a lot of times. If we have a specimen that has systematically increasing width, we don't have the Error-of-Definition any more.

Since the error-of-measurement is able to be reduced as much as possible as long as we are using extremely accurate measuring devices, it won't be consequential for this study. Note that, however, a certain amount of error (deviation) comes from the inelastic behavior can't be reduced whatsoever as shown in later chapter. Now we should figure out what causes this inelastic behavior by using the concept of error analysis. By making the assumption that highly suspected factors causing this inelasticity are stretch history, stretch- rate and time, the error analysis has been done by focusing on these three factors. Note that stretch ratio and stretch-rate are instantaneous and they have nothing to do with history obtained by relatively long relaxation spectrum. Some of the uncertainty could be caused by the edge effect. But since we used the long specimens, the deviation due to the edge effect has been found to be inconsequential.

Error Propagation

To quantify the uncertainty due to the inelasticity of rubber fiber and biotissue, we assumed the following relation as

$$F_M = F_W \pm \Delta F \tag{2.17}$$

where F_M is force measure at a certain stretch ratio. F_W is the average of the force measures and it is a continuous function that satisfies the hyperelastic assumption. Since the experiments were performed in 1-D and there was no symmetry group, we didn't need to assume the isotropy. If we assume hyperelasticity, there should be only one value of force for each corresponding deformation or stretch ratio (one-to-one mapping of stress and strain). The average uncertainty or standard deviation ΔF of F_M represents a degree of uncertainty that how much the measurements of forces are deviated from the hyperelastic point of view. The fundamental question for finding ΔF was where the deviation came from. It has been assumed that the deviation ΔF consists of as following;

$$\Delta F = \Delta F_{sr} + \Delta F_{sh} + \Delta F_t + \Delta F_{rf} \tag{2.18}$$

where ΔF_{sr}, ΔF_{sh}, ΔF_t and ΔF_{rf} are uncertainties due to the stretch-rate, stretch history, time spent testing, and extra random factors. Extra random factors would be random noises and errors due to the inaccuracies of measurement of cross-sectional area (related to a measurement of a radius) of the rubber fiber, measurement of cross-sectional area (related to a measurement of a width and a length) of a strip of pulmonary artery of adult swine, voltage output for corresponding stretch levels, reference weight, voltage output for the reference weight and stretch ratios. Because the Cauchy stresses should be calculated after measuring a force, a radius of rubber fiber, a width, and a length of the strip of pulmonary artery of adult swine, and stretch ratios of specimens (rubber and tissue), those measurements have been done first. But each measurement has their errors and hence there would be resulting error which suggests the error propagation. Since, for a rubber specimen, we had principally three measurements such as force f, radius of rubber fiber r, and stretch ratio λ, the Cauchy stress was $t = t(f, r, \lambda)$ for a certain function. In addition, the force measure can be $f = f(v, w, p)$ where v is a voltage output for each deformation (or stretch ratio) during the motion period, w is a reference weight, and p is a corresponding voltage output for reference weight. For the tissue specimen, we have primarily four measurements, force f, a width a, a length b, and stretch ratio λ, therefore the Cauchy stress was $t = t(f, a, b, \lambda)$ for a certain function. To find an uncertainty due to the resolution of force transducer, repeated measurements have been done and standard deviation was obtained. Before we got the resulting error of the Cauchy stress measures, we had to calculate the resulting error due to the process of force measurement. For a calculation of error-propagation, we used the general formula shown by the equation (2.19) (John R. Taylor, 1997) without proof.

$$\begin{aligned}\delta g &= \sqrt{\left(\frac{\partial g}{\partial x}\delta x\right)^2 + \cdots + \left(\frac{\partial g}{\partial z}\delta z\right)^2} \\ &= \sqrt{(\delta g_x)^2 + \cdots + (\delta g_z)^2}\end{aligned} \tag{2.19}$$

where, δg is the uncertainty in $g = g(x,\ldots, z)$. The variables $x,\ldots, z$ are measured values and $\delta x,\ldots, \delta z$ are uncertainties in measuring of $x,\ldots, z$. Note that $\delta g_x,\ldots, \delta g_z$ are the uncertainties in a function g due to the δx alone and δz alone, respectively. Although the equation (2.19) which is for calculating the absolute uncertainty in the function g is not our interest, we used it to figure out how the relative uncertainty which is our true interest affects the absolute uncertainty.

For rubber specimen, since the Cauchy stress is $t = t(f, r, \lambda)$, the uncertainty in function t is expressed as

$$\begin{aligned}\delta t &= \sqrt{\left(\frac{\partial t}{\partial f}\delta f\right)^2 + \left(\frac{\partial t}{\partial r}\delta r\right)^2 + \left(\frac{\partial t}{\partial \lambda}\delta \lambda\right)^2} \\ &= \sqrt{(\delta t_f)^2 + (\delta t_r)^2 + (\delta t_\lambda)^2}\end{aligned} \tag{2.20}$$

Again, δt_f is an uncertainty in t due to the δf alone, δt_r is an uncertainty in t due to the δr alone, and δt_λ is an uncertainty in t due to the $\delta \lambda$ alone. It is extremely important to know that the uncertainty in t due to the δf which is δt_f includes the uncertainty due to the inelastic behavior of the rubber and the biotissues. The uncertainties in the measured forces are not related to any instrumentally measured values such as r, a, and b for the rubber fiber. In other word, the inelastic behavior of the rubber fiber has nothing to do with those measurements. They are just static measurement so that they never vary. They are not independent variables. Again, we didn't look at the absolute error but relative error. For the tissue, it can be understood likewise.

For a strip of pulmonary artery of an adult swine, the Cauchy stress is a function of four variables which is $t = t(f, a, b, \lambda)$. Thus the uncertainty in the function t for biotissues is expressed as

$$\begin{aligned}\delta t &= \sqrt{\left(\frac{\partial t}{\partial f}\delta f\right)^2 + \left(\frac{\partial t}{\partial a}\delta a\right)^2 + \left(\frac{\partial t}{\partial b}\delta b\right)^2 + \left(\frac{\partial t}{\partial \lambda}\delta \lambda\right)^2} \\ &= \sqrt{(\delta t_f)^2 + (\delta t_a)^2 + (\delta t_b)^2 + (\delta t_\lambda)^2}\end{aligned} \tag{2.21}$$

Since the Cauchy stress is related to the 1st Piola-Kirchhoff as

$$t = \frac{1}{J}FP \tag{2.22}$$

where, t is a Cauchy stress, J is a volume change ratio which is $d(vol)_{current}/d(Vol)_{reference}$, and equivalently det($F$), F is a measure of finite deformation called deformation gradient, and P is a 1st Piola-Kirchhoff stress which called nominal or engineering stress and it is a force in deformed body on the area of undeformed body. Since we were studying the uniaxial behaviors of nearly incompressible materials, the equation (2.22) can be simplified as

$$t = \lambda \frac{f}{A_0} \tag{2.23}$$

The force f can be calculated by using the following equation.

$$f = v\frac{w}{p} \tag{2.24}$$

By combining the equation (2.23) and equation (2.24), we get

$$
\begin{aligned}
t &= \frac{\lambda v w}{A_0 p} \\
&= \frac{\lambda v w}{\pi r^2 p} \quad (for\ rubber) \\
&= \frac{\lambda v w}{abp} \quad (for\ biotissue)
\end{aligned}
\tag{2.25}
$$

Note that the equation (2.25) includes both the uncertainty due to the measurements which called the Error-of-measurement and the uncertainty due to the inelastic behavior of the specimens which called the Error-of-definition.

With the equation (2.25) for a rubber, the uncertainty in function t can be expressed as

$$
\begin{aligned}
\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial r}\delta r\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\
&= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_r)^2 + (\delta t_p)^2}
\end{aligned}
\tag{2.26}
$$

and

$$
\delta t_\lambda = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg} w_{avg}}{\pi (r_{avg})^2 p_{avg}}\right|\delta\lambda \tag{2.27}
$$

$$
\delta t_v = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg} w_{avg}}{\pi (r_{avg})^2 p_{avg}}\right|\delta v \tag{2.28}
$$

$$
\delta t_w = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg} v_{avg}}{\pi (r_{avg})^2 p_{avg}}\right|\delta w \tag{2.29}
$$

$$
\delta t_r = \left|\frac{\partial t}{\partial r}\right|\delta r = \left|\frac{-2\lambda_{avg} v_{avg} w_{avg}}{\pi (r_{avg})^3 p_{avg}}\right|\delta r \tag{2.30}
$$

$$
\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{\pi (r_{avg})^2 (p_{avg})^2}\right|\delta p \tag{2.31}
$$

The variables λ, v, w, r, and p are measured values and these measured values are expressed as

$$\begin{aligned}
\lambda &= \lambda_{avg} \pm \delta\lambda \\
v &= v_{avg} \pm \delta v \\
w &= w_{avg} \pm \delta w \\
r &= r_{avg} \pm \delta r \\
p &= p_{avg} \pm \delta p
\end{aligned} \tag{2.32}$$

Therefore, we can get the uncertainty in t for the rubber specimen by using the equations (2.26) – (2.32). Again, the uncertainty includes both the uncertainty due to the error-of-measurement and the uncertainty due to the error-of-definition. Similarly, for the tissue specimen, the uncertainty in function t can be expressed as

$$\begin{aligned}
\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial a}\delta a\right)^2 + \left(\frac{\partial t}{\partial b}\delta b\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\
&= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_a)^2 + (\delta t_b)^2 + (\delta t_p)^2}
\end{aligned} \tag{2.33}$$

and

$$\delta t_\lambda = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg} w_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta\lambda \tag{2.34}$$

$$\delta t_v = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg} w_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta v \tag{2.35}$$

$$\delta t_w = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg} v_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta w \tag{2.36}$$

$$\delta t_a = \left|\frac{\partial t}{\partial a}\right|\delta a = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{(a_{avg})^2 b_{avg} p_{avg}}\right|\delta a \tag{2.37}$$

$$\delta t_b = \left|\frac{\partial t}{\partial b}\right|\delta b = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{a_{avg}(b_{avg})^2 p_{avg}}\right|\delta b \tag{2.38}$$

$$\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{\pi(r_{avg})^2 (p_{avg})^2}\right|\delta p \tag{2.39}$$

All the variables except the *a* and *b* are the same as for the rubber specimen. The reference area can be obtained by using the measured width *a* and length *b* of the tissue specimen.

$$\begin{aligned} a &= a_{avg} \pm \delta a \\ b &= b_{avg} \pm \delta b \end{aligned} \tag{2.40}$$

Thus, the uncertainty in *t* for the tissue specimen can be obtained by using the equations (2.33) – (2.40).

Fundamental Statistical Approaches; Multiple Linear Regression Analysis

To check the assumption that the deviations are mostly caused by stretch rate, stretch history and time spent, fundamental theory of statistics has been used. Statistical theory included the multivariable regression analysis, goodness-of-fit test and test of significance called *t*-test as shown in below. Throughout the experiments we obtained the data of time spent and force for the corresponding stretch ratios 1.1, 1.3, 1.5, 1.7, and 1.9. By using those data we got groups of data such as the *deviation*, *time spent testing*, *stretch-rate*, and *stretch history*.

Since the multivariable linear regression analysis is a linear regression tool, the relationship between the deviated value of force measurements and independent variables such as time spent, stretch history, and stretch-rate is only viewed by linearly.

Note that, instead of using the stretch-rate calculated by simply dividing the change in stretch ratio by change in time spent, we, for the error analysis of stretch history parts, used the stretch history H_{t_2} obtained by using the stress relaxation spectrum t_2 corresponding to the early exponential decay because the speed of the actuators was not our control variable which means motor didn't allow us to control the velocity. We defined the H_{t_2} as rate-related stretch history. The relaxation spectrums were obtained after the stress relaxation tests for the specimens. The constrained nonlinear optimization fitting has been used on the data of stress relaxation test to get the relaxation spectrums. The variables that have been employed for the multivariable linear regression analysis are the Deviation D, Time spent T, rate-related Stretch History H_{t_2} which can be regarded as a stretch-rate, long-time Stretch history H_{t_1} obtained by using the stress relaxation spectrum t_1 corresponding to the long exponential decay.

We also found out how strongly/weakly they were related. Note that we assumed that there were linear relationships between them. We found out how the deviation D which is dependent variable changes with independent variables T, H_{t_2}, and H_{t_1}. The existence of several variables forced us to use the multivariable linear regression model for the error analysis. The linear regression model searches for the first order functional relationship of the independent and dependent variables under several assumptions for the residuals.

The sample model for multivariable linear regression is

$$D_i = \hat{\alpha} + \hat{\beta}_1 T_i + \hat{\beta}_2 \left(H_{t_2}\right)_i + \hat{\beta}_3 \left(H_{t_1}\right)_i + e_i \tag{2.41}$$

Where, D_i represents the variables for a deviation, T_i represents the variables for a time, $\left(H_{t_2}\right)_i$ represents the variables for a rate-related stretch history, $\left(H_{t_1}\right)_i$ represents the variables for a long-time stretch history and e_i is called residual or regression error and it represents a random error. Note that the deviation D_i is identical to the ΔF in the equation (2.17). Thus, the equation (2.17) can be rewritten as $F_M = F_W + D_i$. The partial regression coefficients $\hat{\beta}_i$, $i = 1$, 2 and 3 describe how the independent variables T_i, $\left(H_{t_2}\right)_i$, and $\left(H_{t_1}\right)_i$ affect the dependent variable D_i. Specifically, the partial regression coefficient β_1 describes how much the time variable T affect the D_i when the other variables are assumed to be constant and only T is varying. Other partial regression coefficients can be understood likewise. The sample regression line for the multivariable is given by

$$\hat{D}_i = \hat{\alpha} + \hat{\beta}_1 T_i + \hat{\beta}_2 \left(H_{t_2}\right)_i + \hat{\beta}_3 \left(H_{t_1}\right)_i \tag{2.42}$$

where $\hat{\alpha}$ and $\hat{\beta}_i$, $i = 1$, 2, and 3 are assumed to be the best guessed values obtained by least square method (LSM). Then the sample model for multivariable regression can be rewritten as

$$D_i = \hat{D}_i + e_i \tag{2.43}$$

From the equation (2.43), we get $e_i = D_i - \hat{D}_i$. It represents the difference between the observed deviation D_i and best guessed value $\hat{D}_i$. To get $\hat{\alpha}$ and $\hat{\beta}_i$, i = 1, 2, and 3, we have to minimize the regression error using LSM as following.

$$\begin{aligned} Min\sum {e_i}^2 &= Min\sum \left(D_i - \hat{D}_i\right)^2 \\ &= Min\sum \left(D_i - \hat{\alpha} - \hat{\beta}_1 T_i - \hat{\beta}_2 \left(H_{t_2}\right)_i - \hat{\beta}_3 \left(H_{t_1}\right)_i\right)^2 \end{aligned} \tag{2.44}$$

$$\frac{\partial \sum {e_i}^2}{\partial \hat{\alpha}} = \frac{\partial \sum {e_i}^2}{\partial \hat{\beta}_1} = \frac{\partial \sum {e_i}^2}{\partial \hat{\beta}_2} = \frac{\partial \sum {e_i}^2}{\partial \hat{\beta}_3} = 0 \tag{2.45}$$

Using matrix notation, if we have n measures at least but preferably many more for each variables, the equations above can be described as

$$\underbrace{\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_n \end{bmatrix}}_{Deviation} - \hat{\alpha} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{Intercept} - \hat{\beta}_1 \underbrace{\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_n \end{bmatrix}}_{Time} - \hat{\beta}_2 \underbrace{\begin{bmatrix} \left(H_{t_2}\right)_1 \\ \left(H_{t_2}\right)_2 \\ \left(H_{t_2}\right)_3 \\ \vdots \\ \left(H_{t_2}\right)_n \end{bmatrix}}_{\substack{rate-related \\ Stretch\ History}} - \hat{\beta}_3 \underbrace{\begin{bmatrix} \left(H_{t_1}\right)_1 \\ \left(H_{t_1}\right)_2 \\ \left(H_{t_1}\right)_3 \\ \vdots \\ \left(H_{t_1}\right)_n \end{bmatrix}}_{\substack{long-time \\ Stretch\ History}} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{bmatrix} \tag{2.46}$$

These equations were used for each stretch level from λ=1.1 to λ=1.9 to determine $\hat{\alpha}$ and $\hat{\beta}_i$, i = 1, 2, 3. Actually it was trivial to find a value $\hat{\alpha}$ and regression coefficients $\hat{\beta}_i$, i = 1, 2, 3. After getting the partial regression coefficients there should be two important questions to be asked. The first question is how well the derived sample regression line for the multivariable T, H_{t_2}, and H_{t_1} describes the observed variable D which can be justified by the goodness-of-fit test. The second question is how much the regression coefficients are significant which can be verified by the test of significance called *t*-test.

Goodness-of-fit Test: As mentioned above the goodness-of-fit test tells how well the derived sample regression line explains the linear relationship between the independent variables *T*, H_{t_2} , and H_{t_1} and the observed dependent variable *D*. For this test, coefficient of determination R^2 is being used. For this test, consider the following identity that satisfies for all the observed data.

$$D_i - \overline{D} = \underbrace{\left(D_i - \hat{D}_i\right)}_{=e_i:residual} + \left(\hat{D}_i - \overline{D}\right) \tag{2.47}$$

The equation (2.47) shows the decomposition of the observed value. The value $D_i - \overline{D}$ says the difference between the observed value and the average of D_i. The value $D_i - \hat{D}_i$ represents the difference between the observed value and the best-guessed value of *D* which called residual e_i which is related to the unexplainable portion of the deviation. The value $\hat{D}_i - \overline{D}$ represents the difference between the best-guessed value of *D* and the average of *D* which is related to the explainable portion of the deviation. Figure 2.21 shows the basic concept of the decomposition.

Now, let's define the variation of D_i as

$$\begin{aligned} D_{\text{var}} &= \sum\left(D_i - \overline{D}\right)^2 \\ &= \sum\left(D_i - \hat{D}_i\right)^2 + \sum\left(\hat{D}_i - \overline{D}\right)^2 + 2\sum\underbrace{\left(D_i - \hat{D}_i\right)}_{=e_i}\left(\hat{D}_i - \overline{D}\right) \\ &= \sum\left(D_i - \hat{D}_i\right)^2 + \sum\left(\hat{D}_i - \overline{D}\right)^2 \qquad \left(\because \sum e_i = 0\right) \end{aligned} \tag{2.48}$$

Thus, we have noticed that D_{var} is separated into two categories; residual variation (or unexplained variation) and explained variation such that

$$\underbrace{\sum\left(D_i - \overline{D}\right)^2}_{=SST} = \underbrace{\sum\left(D_i - \hat{D}_i\right)^2}_{=SSE} + \underbrace{\sum\left(\hat{D}_i - \overline{D}\right)^2}_{=SSR} \tag{2.49}$$

where D_i is the observed values, $\overline{D}$ is the average of D_i, $\hat{D}_i$ is the best guessed value of D_i.

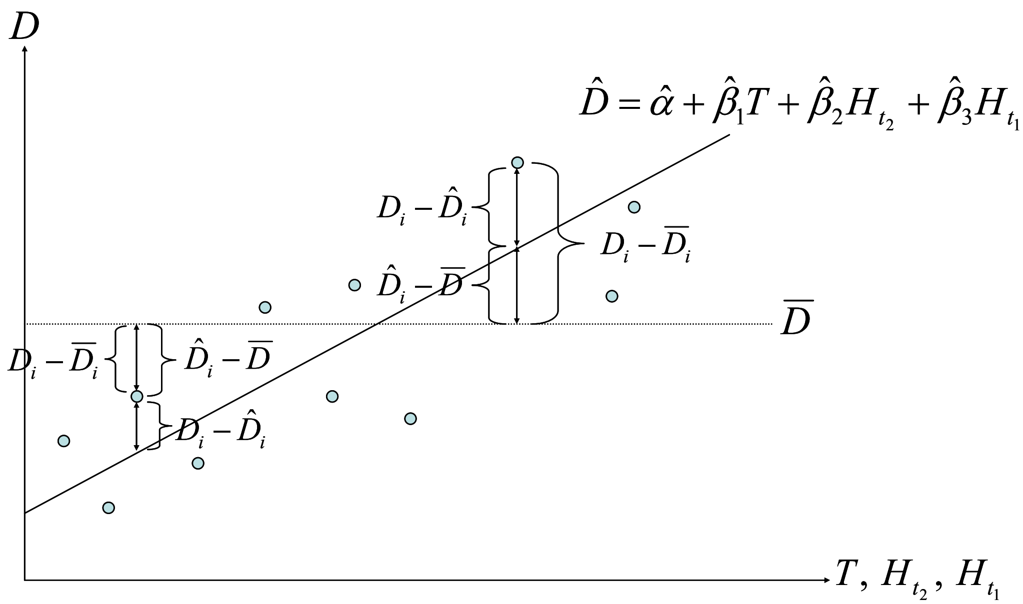

Fig.2.21. Decomposition of the value which is the difference between the observed value D_i and the average of observed value $\overline{D}$.

The equation (2.49) tells that squared sum of difference between the observed value of D and the average of D is equal to sum of squared sum of residual e_i and squared sum of difference between best-guessed value of D and the average of D. Physically, it describes the level of scattering of D_i. The equation (2.49) can be rewritten as

$$\text{SST} = \text{SSE} + \text{SSR} \tag{2.50}$$

SST, SSE, and SSR are abbreviations of sum of squared total, sum of squared error which is related to the unexplained deviation, and sum of squared regression which is related to the explained deviation, respectively. If we normalize the equation (2.50) by dividing by SST, then we have following equation.

$$1 = \frac{\text{SSE}}{\text{SST}} + \underbrace{\frac{\text{SSR}}{\text{SST}}}_{=R^2} \tag{2.51}$$

From the equation (2.51), last term in right hand side SSR/SST is defined as R^2 called coefficient of determination. From the following equation (2.52), it can easily be seen that the range of R^2 is in value between 0 and 1.

$$R^2 = \frac{\text{SSR}}{\text{SST}} = \frac{\text{SST}}{\text{SST}} - \frac{\text{SSE}}{\text{SST}} = 1 - (1 - R^2) \tag{2.52}$$

$$R^2 = \frac{\text{SST} - \text{SSE}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}} = 1 - \frac{\sum e_i{}^2}{\sum \left(D_i - \overline{D}\right)^2} \tag{2.53}$$

Since R^2 is representing the ratio of sum of squared regression in the sum of squared total, the regression model can be regarded as good-fit as R^2 closes to unity. The obscurity arises, however, because of the degree of freedom, when the R^2 is used to test the goodness-of-fit. R^2 is only related to explained (regression) and unexplained (error) variation in D and hence it doesn't justify the number of degree of freedom. To resolve this problem, modified R-squared $\overline{R}_2$ is being used using variances of e_i and D_i as follows.

$$\overline{R}^2 = 1 - \frac{Var(e_i)}{Var(D_i)} = 1 - \frac{\dfrac{\sum e_i{}^2}{n-k-1}}{\dfrac{\sum \left(D_i - \overline{D}\right)^2}{n-1}} \tag{2.54}$$

where, n is the number of data, k is the number of independent variables. Although we have a sample regression model for the multivariable, independent variables T_i, H_{t_2}, and H_{t_1} cannot describe the model perfectly because of four of the following major reasons. First one is the random factors that cannot be explained whatsoever. Second one is the measurement error. Third one is the omission of the significant variables. Last one is the nonlinearity of the independent and dependent variables.

Test of Significance: *t*-test: Although the multivariable regression model has a good fit, each independent variable *T*, H_{t_2}, and H_{t_1} has to be tested to check whether they are significant enough for explaining the variation of dependent variable *D* or not. If one of the partial regression coefficients $\hat{\beta}_i$, $i = 1, 2, 3$ is zero or statistically not significant, then it means the regression model is not proper. For example, if $\hat{\beta}_1$ is zero, then it means that the independent variable *T* scarcely explain the variation of dependent variable *D* and hence the time is not effective variable to cause the deviation. For the *t*-test we generally set two hypotheses; Null-Hypothesis H_0 and Alternative-Hypothesis H_A. So we have done the hypothesis testing for these two hypotheses by looking at the *t* value. Let's assume that we are looking at the significance of the time variable *T*. Then t value can be obtained by using the equation (2.55). The *t* values for the other independent variables can be determined likewise.

$$t_{n-k-1} = \frac{\hat{\beta}_1 - \beta}{S_{\hat{\beta}_1}} = \frac{\hat{\beta}_1 - \beta}{\dfrac{S_e}{\sqrt{\sum\left(T_i - \overline{T}\right)^2}}} = \frac{\hat{\beta}_1 - \beta}{\dfrac{\sqrt{\dfrac{\sum\left(D_i - \hat{D}_i\right)^2}{n-k-1}}}{\sqrt{\sum\left(T_i - \overline{T}\right)^2}}} = \frac{\hat{\beta}_1 - \beta}{\dfrac{\sqrt{\dfrac{\text{SSE}}{n-k-1}}}{\sqrt{\sum\left(T_i - \overline{T}\right)^2}}} \tag{2.55}$$

where *n* is the number of samples, *k* is the number of independent variables, $S_{\hat{\beta}_i}$ is a standard error of estimate for the partial regression coefficient $\hat{\beta}_1$, and S_e is a standard error of estimate for the observed values D_i. For testing the hypotheses, we first have to set the confidence interval. The confidence interval is associated with the level of significance α. The confidence interval (C.I) is calculated as

$$\text{C.I} = 100(1-\alpha) \tag{2.56}$$

This represents the probability that the partial regression coefficient $\hat{\beta}_i$ is likely to be contained within that interval. Confidence interval is used for testing the hypothesis and evaluating the significance of the derived regression coefficients. The hypotheses that we are using is

$$\begin{aligned} &i)\ H_0 : \beta = 0 \\ &ii)\ H_A : \beta \neq 0 \end{aligned} \tag{2.57}$$

Null hypothesis H_0 represents no effect for a specific independent variable. On the other hand, alternative hypothesis H_A represents an effect for a specific independent variable. Now we have to compare calculated *t*-value t_{n-k-1} using equation (2.55) with the critical value t_c obtained from the table by considering the degree of freedom (n-*k*-*1*) and the level of significance α. If the null hypothesis is rejected, alternative hypothesis is accepted which means the independent variable corresponding to the partial regression coefficient is significant. It is important to know that the significance level α should carefully be decided because it depends on the researchers and models that are being developed. We can get 90 % confidence interval, for example, with a 10 % level of significance such that

$$prob\left(-t_c < \frac{\hat{\beta}-\beta}{S_{\hat{\beta}}} < t_c\right) = 0.9 \tag{2.58}$$

Thus, we can obtain the 90 % confidence interval for β from equation (2.58) such that

$$prob\left(\hat{\beta} - t_c S_{\hat{\beta}} < \beta < \hat{\beta} + t_c S_{\hat{\beta}}\right) = 0.9 \qquad (2.59)$$

$$\hat{\beta} \pm t_c S_{\hat{\beta}} \qquad (2.60)$$

It physically means that, for the unknown value β, the true value $\hat{\beta}$ will fall into the range of equation (2.60) 90 times out of the 100. Provided that the critical value t_c of the *t* distribution is properly selected, the confidence interval can be decided for any level of significance.

Basic Assumptions for Least Square Method: There are seven basic assumptions that we can use the least square method to get a sample regression line. These seven assumptions should be satisfied so that the goodness-of-fit test ($\overline{R}^2$) and test of significance (*t*-test) can reasonably be applicable. These assumptions are

1) $E[\varepsilon_i] = 0$
2) $E[\varepsilon_i^2] = \sigma_\varepsilon^2$ for all i
3) $E[\varepsilon_i \varepsilon_j] = 0, i \neq j$
4) $E[\varepsilon_i X_i] = X_i \cdot E[\varepsilon_i] = 0$
5) $\varepsilon_i \approx N(E(\varepsilon_i), Var(\varepsilon_i))$ or $\varepsilon_i \approx N(0, \sigma_\varepsilon{}^2)$
6) $n>k+1$
7) $\rho(X_i, X_j) \neq \pm 1$

First assumption means the average of error terms is zero. The second assumption states that all error terms have same variances called homoscedasticity. Third assumption states that all error terms are linearly independent for different error terms. Fourth assumption is that X_i is nonstochastic variable and can be treated as constant. Fifth assumption is that the error term is normally distributed. Sixth assumption is that the number of observed variables should be bigger than one plus number of independent variables. Seventh assumption is that correlation coefficients between the independent variables shouldn't be ±1.

All the assumptions have been checked to see if they are satisfied. The first assumption which is related to the linearity has been checked by seeing the scatter plots of the variables. It is a basic condition for calculating process of LSM for one's convenience. For the second assumption, we have checked the scatter plot of residuals with each independent variable. The physical meaning the homoscedasticity is that the error terms have constant standard deviation, i.e., SD(e_i) = σ for all i. The third assumption which is related to the autocorrelation has been confirmed by plotting the residuals in order and checking the patterns. The autocorrelation may be at hand if observations have a natural sequential order, for example, time. In general, however, it can hardly be expected that this assumption is perfectly satisfied. The fourth assumption is saying that the independent variable Xi will be treated as nonrandom variable or constant. It is understood that this assumption is not problematic for using the LSM. The fifth assumption has been checked by plotting the histogram of residuals. The sixth assumption has been easily checked by seeing the number of data and independent variables. The last assumption which is related to the multicollinearity has been checked by seeing the correlation coefficients between the independent variables.

CHAPTER III

RESULTS FOR RUBBER

Error-of-Measurement and Error-of-Definition

In general, a stretch ratio of 2 of a certain material can be regarded as highly deformed status. But for the experiment of the rubber fiber in this study, since the stretch ratio of 2 was much lower than the maximum stretch ratio that the rubber fiber can acquire without damaging or breaking the cross-linked polymer chains in it, there was no spiky peak in the output protocol. This means the cross-linked polymer chains in the rubber fiber didn't go up to the states in a tight straight line. Approximately a stretch ratio of 5 was the maximum stretch ratio that the rubber fiber could acquire with no damage of it. The results obtained from the randomized uniaxial stretch-controlled protocol and corresponding force output profiles are shown in the figure 3.1 and the figure 3.2. The output profile has smoothly followed the input stretch protocol. It took about 5600 seconds to finish the entire protocol.

As we expected, the experiment of the randomized stretch-controlled motion of the rubber specimen showed that there is a noticeable inelastic deviation from the hyperelasticity. The figure 3.3 shows that there are behaviors of nonlinearity as well as inelasticity. If the hyperelastic assumption is perfectly satisfied, there would be only one point for each stretch ratio.

To find the uncertainty due to the measurements, all manual measurements (radius r, reference weight w, voltage output p for a fixed weight) have been done for 10 times. It is important to note that the uncertainty due to the manual measurement can be reduced as much as possible as long as we use the extremely accurate measuring devices or equipments. Since the uncertainty due to the manual measurements is fixed or static for entire stretch levels, it can be ignored for the error analysis which was focusing on the uncertainty due to the inelastic behaviors of the specimen. Note that the uncertainty due to the inelastic behavior varies. It was explained in detail in later chapter.

To check the absolute Cauchy stress, first, we measured the reference area of the rubber fiber. The radius of the rubber specimen has been measured for 10 times. As for the reference weight, the weight of the paper clip has been measured for 10 times. To check the resolution of the force transducer, we dangled up a paper clip on the tip of the force transducer and saw the voltage output, and with the same paper clip, we checked the corresponding voltage output for 10 times. The results of those measurements are shown in the table 3.1. It also shows the standard deviations and fractional uncertainties for each measurement.

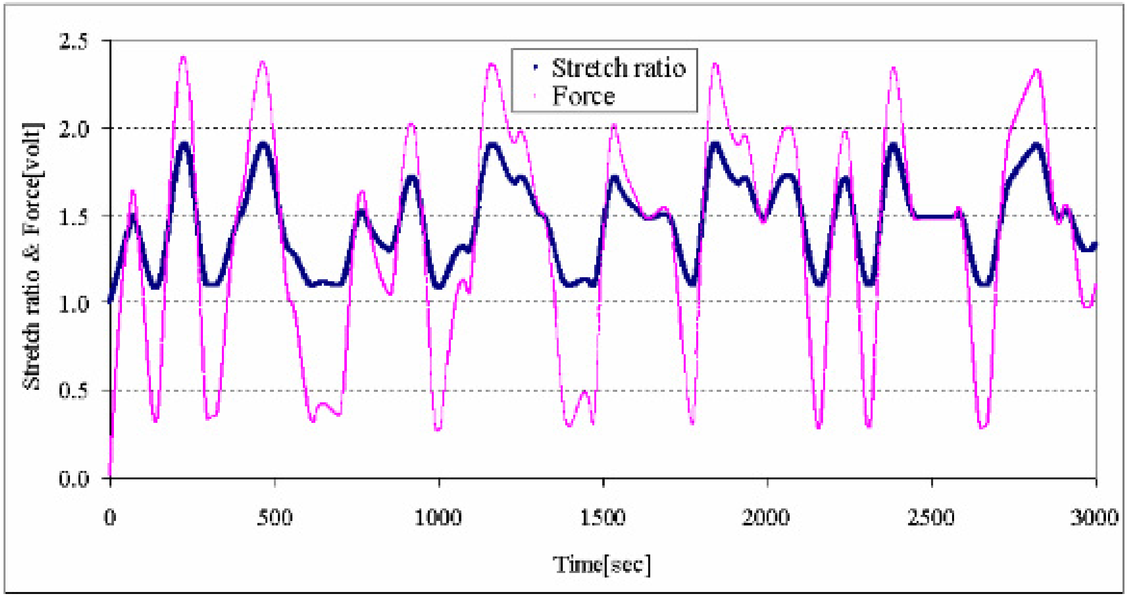

Fig.3.1. The randomized stretch-controlled protocol and corresponding force output profile (0-3000 sec).

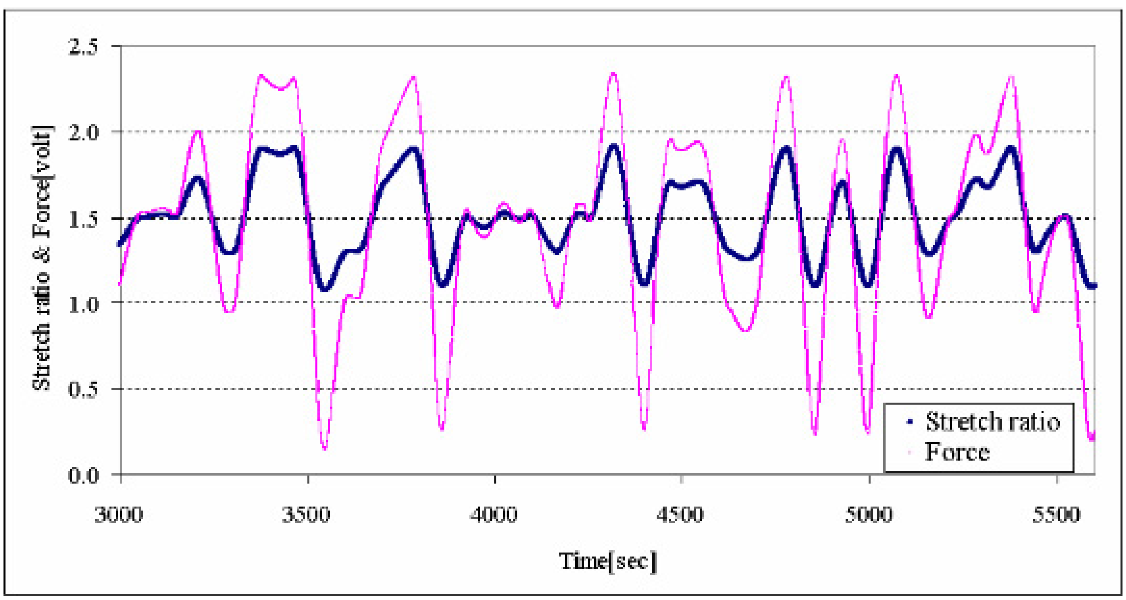

Fig.3.2. The randomized stretch-controlled protocol and corresponding force output profile (3000-5600).

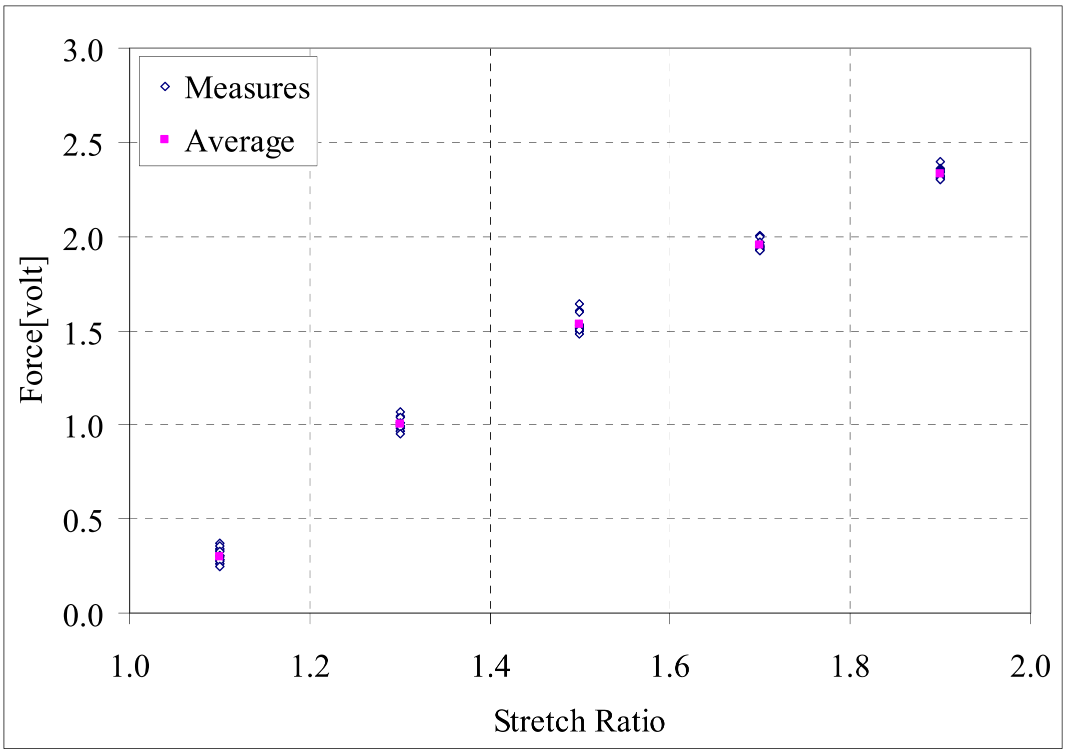

Fig.3.3. Forces obtained by force transducer for each corresponding stretch ratio have been grouped and averaged. It shows the inelastic and nonlinear behaviors.

Table 3.1
A diameter of the rubber fiber, a reference weight obtained by using a paper clip, and a voltage output corresponding to a paper clip's weight (used fixed weight) have been measured for 10 times.

No. of Measure	radius[mm]	ref.weight[gram]	voltage output[volt]
1^{st}	0.1345	0.3938	0.3084
2^{nd}	0.1375	0.3940	0.3067
3^{rd}	0.1360	0.3940	0.3058
4^{th}	0.1345	0.3940	0.3061
5^{th}	0.1345	0.3938	0.3085
6^{th}	0.1415	0.3940	0.3054
7^{th}	0.1400	0.3940	0.3063
8^{th}	0.1415	0.3940	0.3046
9^{th}	0.1345	0.3939	0.3042
10^{th}	0.1415	0.3939	0.3054
Average	0.1376	0.3939	0.3061
Standard deviation	0.0032	0.0001	0.0014
Fractional-Uncertainty[%]	2.3286	0.0214	0.4666
$M = M_{avg} \pm \Delta M$	$r = 0.1376 \pm 0.0032$	$w = 0.3939 \pm 0.0001$	$p = 0.3061 \pm 0.0014$

Stretch Ratio 1.1: The rubber data corresponding to the stretch ratio 1.1 is shown in the table 3.2. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the *Deviation* versus *Time*, H_{t_2} and H_{t_1}. Again, the H_{t_2} is a rate-related stretch history function scanned by a history cut-off t_h = 10 sec, and H_{t_1} is the long-time stretch history function scanned by a history cut-off t_h = 400 sec.

Table 3.2
The rubber data corresponding to the stretch ratio 1.1. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the *Deviation* versus *Time*, H_{t_2} and H_{t_1}.

Time	λ	v_M	v_{avg}	*Deviation*	$d\lambda/dt$	$H_{t_2}(t)$	$H_{t_1}(t)$
[sec]		[volt]	[volt]	v_M - v_{avg}[volt]	[/sec]	t_h=10 sec	t_h=400 sec
130.8	1.1	0.36812	0.30105	0.06707	-0.00485	1.13068	1.07313
299.8		0.33656		0.03551	-0.00004	1.11956	1.21757
622.5		0.33844		0.03739	0.00130	1.09920	1.42234
700.4		0.35875		0.05770	-0.00004	1.10070	1.39949
989.8		0.30500		0.00395	-0.00552	1.14034	1.39676
1391.2		0.30125		0.00020	-0.00138	1.11561	1.47724
1466.3		0.32813		0.02708	-0.00460	1.11815	1.43617
1773.4		0.30406		0.00301	-0.00086	1.11279	1.42072
2157.1		0.28281		-0.01824	-0.00181	1.11982	1.53323
2307.9		0.28625		-0.01480	-0.00248	1.12653	1.51722
2651.5		0.28313		-0.01792	-0.00048	1.11603	1.44743
3561.0		0.30500		0.00395	0.00618	1.08060	1.55591
3858.4		0.26281		-0.03824	-0.00106	1.12729	1.56106
4401.7		0.26281		-0.03824	0.00213	1.11483	1.46314
4858.6		0.27312		-0.02793	0.00472	1.10172	1.51438
4998.9		0.27469		-0.02636	0.00477	1.10422	1.48922
5584.5		0.24688		-0.05417	-0.00472	1.13489	1.54672

It is important to note that, instead of using the stretch-rate calculated by simply dividing the change in stretch ratio by change in time spent testing, we, for the error analysis of stretch history parts, have used the stretch history function H_{t_2} obtained by using the stress relaxation spectrum t_2 corresponding to the early (or fast) exponential decay. It is because the *stretch-rate* is not our control variable for the experiments. The stretch history function H_{t_1} is obtained by using the stress relaxation spectrum t_1 corresponding to the late (or slow) exponential decay. We defined the H_{t_2} and H_{t_1} as rate-related stretch history function and long-time stretch history function, respectively.

The table 3.3 shows the average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.1. They have also been used to calculate the uncertainty in resulting Cauchy stresses. We originally expected that the more *stretch-rate* we have the bigger deviation from the hyperelasticity we get. It's been revealed as true from our other experiments that the *stretch-rate* and the *deviation* from the hyperelasticity have positive relationship.

Table 3.3
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.1. They have also been used to calculate the uncertainty in resulting Cauchy stresses.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\Delta\lambda$	Δv
1.10002	0.30105	0.00008	0.03488

In fact, for the stretch ratio 1.1, it has been discovered by the multivariable linear regression analysis that the rate-related stretch history function H_{t_2} is the most significant factor that causes the deviation from hyperelasticity. The long-time stretch history H_{t_1} has been revealed that it is not significant parameter that causes the deviation from hyperelasticity. These founds are explained in detail in the part of the multivariable linear regression analysis that is in later section.

To get the uncertainty in the Cauchy stress due to $\delta\lambda$, δv, δw, δr, and δp the for the stretch ratio 1.1, the equations (2.26) – (2.32) were employed as

$$\delta t_{\lambda} = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg}w_{avg}}{\pi(r_{avg})^2 p_{avg}}\right|\delta\lambda = \left|\frac{(0.30105)(0.3939)}{\pi(0.1376)^2(0.3061)}\right|(0.00008) \\ = 5.21\times10^{-4}[g/mm^2] \tag{3.1}$$

$$\delta t_{v} = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg}w_{avg}}{\pi(r_{avg})^2 p_{avg}}\right|\delta v = \left|\frac{(1.10002)(0.3939)}{\pi(0.1376)^2(0.3061)}\right|(0.03488) \\ = 8300.67\times10^{-4}[g/mm^2] \tag{3.2}$$

$$\delta t_{w} = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg}v_{avg}}{\pi(r_{avg})^2 p_{avg}}\right|\delta w = \left|\frac{(1.10002)(0.30105)}{\pi(0.1376)^2(0.3061)}\right|(0.0001) \\ = 18.19\times10^{-4}[g/mm^2] \tag{3.3}$$

$$\delta t_{r} = \left|\frac{\partial t}{\partial r}\right|\delta r = \left|\frac{-2\lambda_{avg}v_{avg}w_{avg}}{\pi(r_{avg})^3 p_{avg}}\right|\delta r = \left|\frac{-2(1.10002)(0.30105)(0.3939)}{\pi(0.1376)^3(0.3061)}\right|(0.0032) \\ = 3332.24\times10^{-4}[g/mm^2] \tag{3.4}$$

$$\delta t_{p} = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg}v_{avg}w_{avg}}{\pi(r_{avg})^2 (p_{avg})^2}\right|\delta p = \left|\frac{-(1.10002)(0.30105)(0.3939)}{\pi(0.1376)^2(0.3061)^2}\right|(0.0014) \\ = 327.67\times10^{-4}[g/mm^2] \tag{3.5}$$

Thus, the total uncertainty in t for the stretch ratio 1.1 is

$$
\begin{aligned}
\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial r}\delta r\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\
&= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_r)^2 + (\delta t_p)^2} \\
&= \sqrt{\left\{(5.21)^2 + (8300.67)^2 + (18.19)^2 + (3332.24)^2 + (327.67)^2\right\} \times \left(10^{-4}\right)^2} \qquad (3.6) \\
&= 8950.57 \times 10^{-4} \\
&= 0.90[g/mm^2]
\end{aligned}
$$

We have seen that the uncertainty in the Cauchy stress t due to the $\delta\lambda$ (average deviation or standard deviation of λ) has the lowest uncertainty among the other uncertainties. It is because we used the actuators (CMA-25CCCL, Newport) that provide quite precise motions (Resolution = 0.048828 μm, Speed = 50 – 400 μm/sec). In addition, since the weight of a paper clip has been measured by the scale (Ainsworth, Inc) that has very high resolution (Resolution = 0.0001g), the uncertainty in the Cauchy stress t due to the δw has also a quite low uncertainty. It is important to note that the uncertainty in the Cauchy stress t due to the δv which is standard deviation of forces has the highest uncertainty of the other uncertainties. Thus we can tell that there are huge amount of inelastic behavior in the rubber at a stretch level 1.1. Although the uncertainty in the Cauchy stress t due to the δr alone seems to be relatively high, again, it can be reduced as much as possible to the maximum resolution range that the measuring devices would have. These absolute uncertainties are not our interests because the measured values r, w, and p are static variables and the uncertainties due to them are always constants for both the absolute uncertainty and fractional uncertainty.

Note that, if a certain variable is a dynamic variable, the fractional uncertainty due to the variable will never be constant. The example can be a dynamic variation of force due to the inelastic behavior of a certain material. The figure 3.4 shows the uncertainties in t due to the $\delta\lambda$, δv, δw, δr, and δp. The uncertainty δt_v in t which is due to the δv alone is approximately 2.5 times bigger than the uncertainty δt_v in t which is due to the δr alone.

Now, if we look at the fractional (or relative) uncertainties defined as

$$F.U = \frac{\delta M}{M_{avg}} \tag{3.7}$$

where M_{avg} is the average of the measurements and δM is the standard deviation of M_i, i=1,…,N, and get the fractional uncertainties for each variables, we have

$$\begin{aligned}
\lambda &= \lambda_{avg} \pm \delta\lambda = 1.10002 \pm 0.00008, & \frac{\delta\lambda}{\lambda_{avg}} \times 100 &= 0.007\ \% \\
v &= v_{avg} \pm \delta v = 0.30105 \pm 0.03488, & \frac{\delta v}{v_{avg}} \times 100 &= 11.586\ \% \\
w &= w_{avg} \pm \delta w = 0.39390 \pm 0.00010, & \frac{\delta w}{w_{avg}} \times 100 &= 0.025\ \% \\
r &= r_{avg} \pm \delta r = 0.13760 \pm 0.00320, & \frac{\delta r}{r_{avg}} \times 100 &= 2.326\ \% \\
p &= p_{avg} \pm \delta p = 0.30610 \pm 0.00140, & \frac{\delta p}{p_{avg}} \times 100 &= 0.457\ \%
\end{aligned} \tag{3.8}$$

Using equation (2.25) for the rubber specimen, the Cauchy stress t for the stretch ratio 1.1 is obtained as

$$t = t_{avg} \pm \delta t = 7.16432 \pm 0.89506, \quad \frac{\delta t}{t_{avg}} \times 100 = 12.493\ \% \tag{3.9}$$

Summarized fractional uncertainties are shown in the table 3.4.

Table 3. 4
Fractional uncertainties for the measured data r, w, and p, and obtained data λ and v by the motion controller and the Cauchy stress t for the stretch level 1.1.

Data	$\lambda_{1.1}$	v[volt]	w[gram]	r[mm]	p[volt]	t[g/mm^2]
Average	1.10002	0.30105	0.39390	0.13760	0.30610	7.164322
Standard deviation	0.00008	0.03488	0.00010	0.00320	0.00140	0.895057
Fractional uncertainty[%]	0.007	11.586	0.025	2.326	0.457	12.493

For calculating the (absolute) Cauchy stress, although the uncertainty δt_v in t which is due to the δv alone is approximately only 2.5 times bigger than the uncertainty δt_v in t which is due to the δr alone, the fractional uncertainty for the force v is almost 5 times bigger than the fractional uncertainty for the radius measure r. In fact, the calculation of the uncertainty in the (absolute) Cauchy stress is affected by the formula, i.e., if the scale gets bigger, the uncertainty also gets bigger. But since the fractional uncertainty is ratio-related value, it is not affected by the scale. In addition, if we look at the fractional uncertainties due to the manual measurements such as the measurements of w, r, and p, they are the same for all stretch levels. But, even though we got the force data for the same stretch ratios over time, the fractional uncertainties are changing in deceasing way. The error analysis, therefore, has been devoted to and focused on the uncertainty due to the inelastic behavior of the rubber. The fractional uncertainties in the Cauchy stress t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p is shown in the figure 3.5. The fractional uncertainty for the force v is almost 5 times bigger than the fractional uncertainty for the radius measure r.

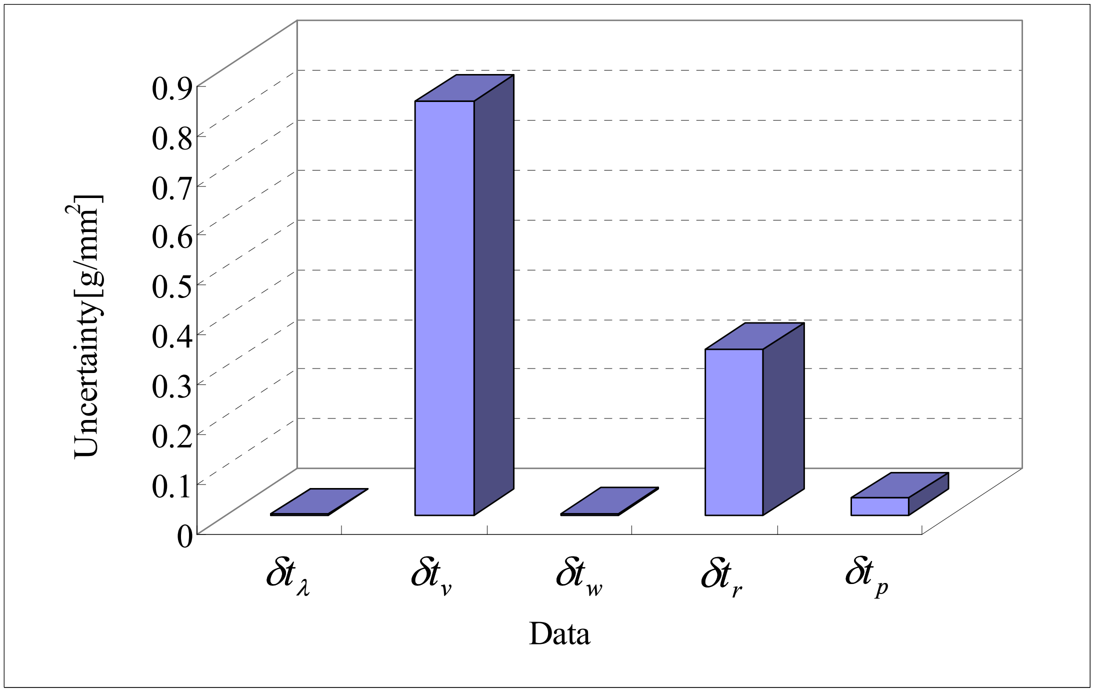

Fig.3.4. Uncertainties in the Cauchy stress t due to the $\delta\lambda$, δv, δw, δr, and δp. The uncertainty δt_v in t which is due to the δv alone is approximately 2.5 times bigger than the uncertainty δt_v in t which is due to the δr alone.

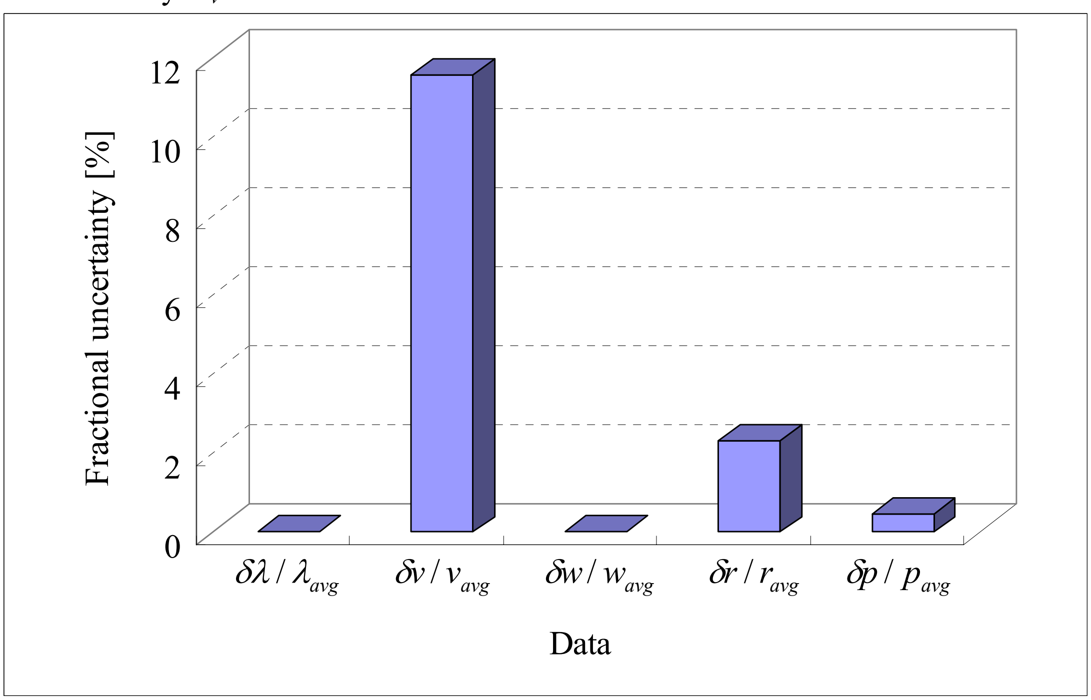

Fig.3.5. Fractional uncertainties in the Cauchy stress t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p. The fractional uncertainty for the force v is almost 5 times bigger than the fractional uncertainty for the radius measure r.

Stretch Ratio 1.3: The rubber data corresponding to the stretch ratio 1.3 is shown in the table 3.5. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the *Deviation* versus *Time*, H_{t_2} and H_{t_1}. Again, the H_{t_2} is a rate-related stretch history function scanned by a history cut-off t_h = 10 sec, and H_{t_1} is the long-time stretch history function scanned by a history cut-off t_h = 400 sec.

Table 3.5
The rubber data corresponding to the stretch ratio 1.3. It shows the times and stretch ratios obtained by the motion controller, forces obtained by the force transducer and calculated deviation, stretch-rate, and stretch histories.

Time	λ	v_M	v_{avg}	*Deviation*	$d\lambda/dt$	$H_{t_2}(t)$	$H_{t_1}(t)$
[sec]		[volt]	[volt]	v_M - v_{avg}[volt]	[/sec]	t_h=10 sec	t_h=400 sec
548.6	1.3	1.01281	1.00094	0.01187	-0.00087	1.31439	1.4341
852.0		1.04281		0.04187	-0.00067	1.30578	1.35408
1086.2		1.06500		0.06406	-0.00331	1.31344	1.30078
2987.1		0.99156		-0.00938	0.00130	1.29646	1.49394
3278.9		0.96969		-0.03125	-0.00413	1.32786	1.55918
3605.5		1.03781		0.03687	0.00074	1.28483	1.53925
4160.5		0.97844		-0.02250	-0.00252	1.31652	1.48771
4626.5		0.96688		-0.03406	-0.00071	1.31578	1.51952
4700.6		1.00719		0.00625	0.00535	1.27817	1.49356
5171.8		0.98875		-0.01219	0.00496	1.28458	1.45993
5439.5		0.94937		-0.05157	-0.00433	1.33973	1.55221

Again for the multivariable linear regression analysis, we have used the stretch history function H_{t_2} obtained by using the stress relaxation spectrum t_2 corresponding to the early (or fast) exponential decay instead of using the stretch-rate $d\lambda/dt$ calculated by simply dividing the change in stretch ratio by change in time spent testing because the *stretch-rate* is not our control variable for the experiments. The table 3.6 shows the average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.3. They have also been used to calculate the uncertainty in resulting Cauchy stresses.

Table 3.6
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.3. They have also been used to calculate the uncertainty in resulting Cauchy stresses.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\Delta\lambda$	Δv
1.30000	1.00094	0.00005	0.03597

To get the Cauchy stress for the stretch ratio 1.3, we again used the equations (2.26) – (2.32) as

$$\delta t_{\lambda} = \left|\frac{\partial t}{\partial \lambda}\right| \delta\lambda = \left|\frac{v_{avg} w_{avg}}{\pi\left(r_{avg}\right)^2 p_{avg}}\right| \delta\lambda = \left|\frac{(1.00094)(0.3939)}{\pi(0.1376)^2(0.3061)}\right| (0.00005) \tag{3.10}$$
$$= 10.83 \times 10^{-4} [g/mm^2]$$

$$\delta t_{v} = \left|\frac{\partial t}{\partial v}\right| \delta v = \left|\frac{\lambda_{avg} w_{avg}}{\pi\left(r_{avg}\right)^2 p_{avg}}\right| \delta v = \left|\frac{(1.30000)(0.3939)}{\pi(0.1376)^2(0.3061)}\right| (0.03597) \tag{3.11}$$
$$= 10116.25 \times 10^{-4} [g/mm^2]$$

$$\delta t_{w} = \left|\frac{\partial t}{\partial w}\right| \delta w = \left|\frac{\lambda_{avg} v_{avg}}{\pi\left(r_{avg}\right)^2 p_{avg}}\right| \delta w = \left|\frac{(1.30000)(1.00094)}{\pi(0.1376)^2(0.3061)}\right| (0.0001) \tag{3.12}$$
$$= 71.47 \times 10^{-4} [g/mm^2]$$

$$\delta t_r = \left|\frac{\partial t}{\partial r}\right|\delta r = \left|\frac{-2\lambda_{avg} v_{avg} w_{avg}}{\pi (r_{avg})^3 p_{avg}}\right|\delta r = \left|\frac{-2(1.30000)(1.00094)(0.3939)}{\pi (0.1376)^3 (0.3061)}\right|(0.0032) \quad (3.13)$$
$$= 13093.29 \times 10^{-4} [g/mm^2]$$

$$\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{\pi (r_{avg})^2 (p_{avg})^2}\right|\delta p = \left|\frac{-(1.30000)(1.00094)(0.3939)}{\pi (0.1376)^2 (0.3061)^2}\right|(0.0014) \quad (3.14)$$
$$= 1287.51 \times 10^{-4} [g/mm^2]$$

Thus, the total uncertainty in t for the stretch ratio 1.3 is

$$\begin{aligned}
\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial r}\delta r\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\
&= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_r)^2 + (\delta t_p)^2} \\
&= \sqrt{\{(10.83)^2 + (10116.25)^2 + (71.47)^2 + (13093.29)^2 + (1287.51)^2\} \times (10^{-4})^2} \quad (3.15) \\
&= 16596.25 \times 10^{-4} \\
&= 1.66 [g/mm^2]
\end{aligned}$$

The uncertainty in the Cauchy stress t due to the $\delta\lambda$ (average deviation or standard deviation of λ) has again the lowest uncertainty among the other uncertainties by the same reason for the case of stretch ratio 1.1. The uncertainty in the Cauchy stress t due to the δw has also a quite low uncertainty. Although the uncertainty in the Cauchy stress t due to δr is now higher than the uncertainty due to the δv, as mentioned above, the uncertainties due to the manual measurements can be easily reduced as much as possible to the maximum resolution of the range of the measuring device (if it is extremely accurate). Additionally, if we look at the fractional uncertainties below, the fractional uncertainty for the force v is still higher than the fractional uncertainty for the r. It is still important to note that the uncertainty in the Cauchy t due to the δv which is standard deviation of forces can not be reduced no matter what we do.

If we look at the fractional uncertainties for each variable, we have

$$
\begin{aligned}
&\lambda = \lambda_{avg} \pm \delta\lambda = 1.30000 \pm 0.00005, && \frac{\delta\lambda}{\lambda_{avg}} \times 100 = 0.004\ \% \\
&v = v_{avg} \pm \delta v = 1.00094 \pm 0.03597, && \frac{\delta v}{v_{avg}} \times 100 = 3.594\ \% \\
&w = w_{avg} \pm \delta w = 0.39390 \pm 0.00010, && \frac{\delta w}{w_{avg}} \times 100 = 0.025\ \% \\
&r = r_{avg} \pm \delta r = 0.13760 \pm 0.00320, && \frac{\delta r}{r_{avg}} \times 100 = 2.326\ \% \\
&p = p_{avg} \pm \delta p = 0.30610 \pm 0.00140, && \frac{\delta p}{p_{avg}} \times 100 = 0.457\ \% \\
&t = t_{avg} \pm \delta t = 28.15058 \pm 1.65963, && \frac{\delta t}{t_{avg}} \times 100 = 5.896\ \%
\end{aligned}
\tag{3.16}
$$

Summarized fractional uncertainties are shown in the table 3.7.

Table 3.7
Fractional uncertainties for the measured data *r*, *w*, and *p*, and obtained data λ and *v* by the motion controller and the Cauchy stress *t* for the stretch level 1.3.

Data	$\lambda_{1.3}$	v[volt]	r[mm]	w[gram]	p[volt]	t[g/mm^2]
Average	1.30000	1.00094	0.13760	0.39390	0.30610	28.15058
Standard deviation	0.00005	0.03597	0.00320	0.00010	0.00140	1.65963
Fractional uncertainty[%]	0.004	3.594	2.326	0.025	0.457	5.896

For the calculation of the absolute Cauchy stress, the uncertainty δt has included all the measured variables such as a reference-weight measure *w*, voltage output for the same weight measure *p*, and the radius measure *r* for the rubber specimen. But the uncertainty arise from the Error-of-definition has nothing to do with those variables because they are not varying and they are just static measured variables. Thus, if we assume the uncertainties that come from those measurements are ignorable (or let say we use the extremely accurate measuring devices), the only factor that we have to consider for the uncertainty that is causing the inelastic behavior is the force v. In other words, those static variables such as r, w, and p can never cause the deviation from the hyperelasticity of the rubber specimen. The uncertainties in t due to the $\delta\lambda$, δv, δw, δr, and δp.are graphically shown in the figure 3.6. The uncertainty δt_r in t which is due to the δr alone is bigger than the uncertainty δt_v in t which is due to the δv alone. Although the uncertainty in the Cauchy stress t due to δr is now higher than the uncertainty due to the δv, the uncertainty due to the δr can be easily reduced as much as possible to the maximum resolution of the range of the measuring device (even if it is not extremely accurate, it can reasonably be assumed to be ignorable).

If we look at the fractional uncertainties in the figure 3.7 below, the fractional uncertainty for the force v is still higher than the fractional uncertainty for the r. The figure 3.7 shows the fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p. The fractional uncertainty for the force v is still bigger than the fractional uncertainty for the radius measure r. But the uncertainty arise from the Error-of-definition has nothing to do with those variables. Thus, if we assume the uncertainties that come from those measurements are ignorable (or let say we use the extremely accurate measuring devices), the only factor that we have to consider for the uncertainty that is causing the inelastic behavior is the force v.

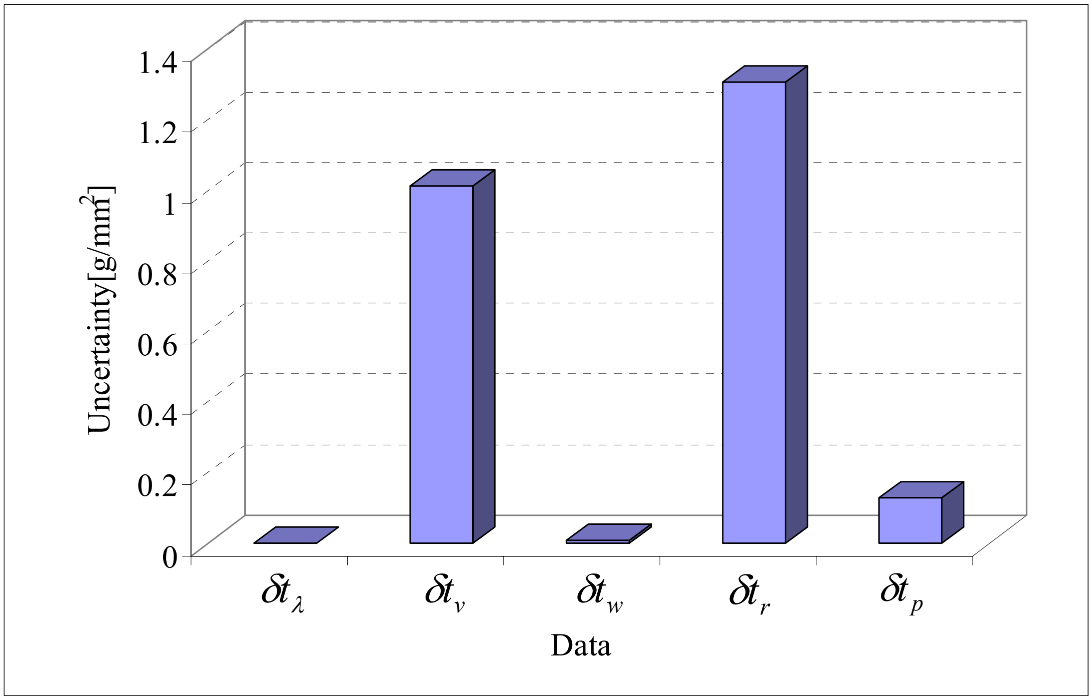

Fig.3.6. Uncertainties in t due to the $\delta\lambda$, δv, δw, δr, and δp. The uncertainty δt_r in t which is due to the δr alone is bigger than the uncertainty δt_v in t which is due to the δv alone. Although the uncertainty in the Cauchy stress t due to δr is now higher than the uncertainty due to the δv, the uncertainty due to the δr can be easily reduced as much as possible to the maximum resolution of the range of the measuring device (even if it is not extremely accurate, it can reasonably be assumed to be ignorable).

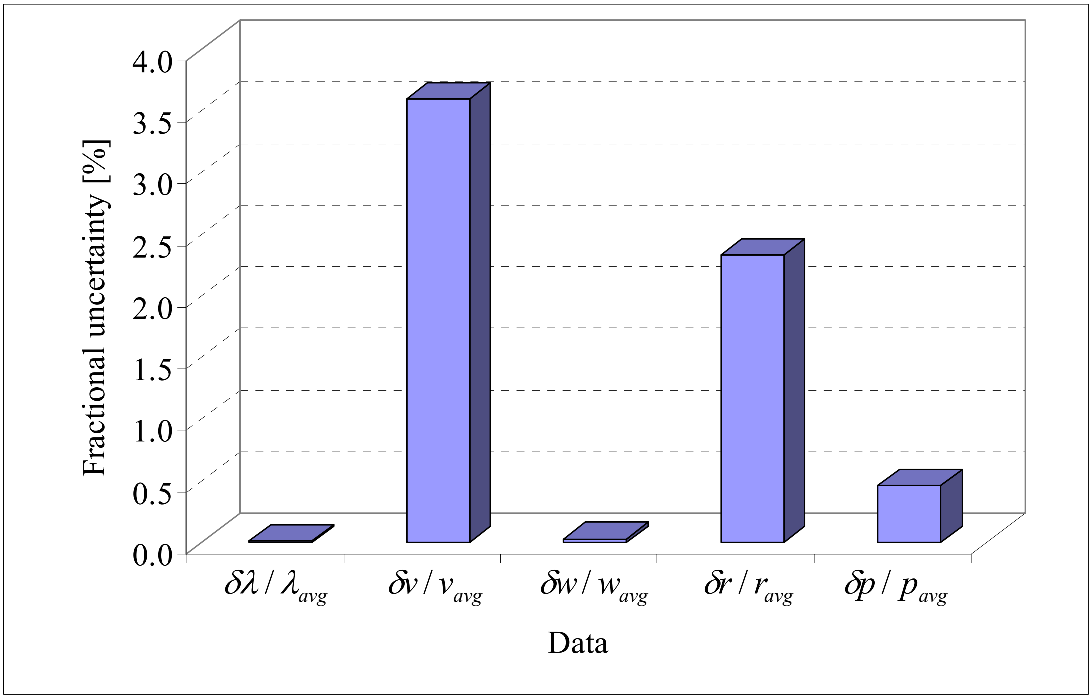

Fig.3.7. Fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p. The fractional uncertainty for the force v is still bigger than the fractional uncertainty for the radius measure r. But the uncertainty arise from the Error-of-definition has nothing to do with those variables.

Stretch Ratio 1.5: The rubber data corresponding to the stretch ratio 1.5 is shown in the table 3.8. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the *Deviation* versus *Time*, H_{t_2} and H_{t_1} where H_{t_2} is a rate-related stretch history function scanned by a history cut-off t_h = 10 sec, and H_{t_1} is the long-time stretch history function scanned by a history cut-off t_h = 400 sec.

Table 3.8
The rubber data corresponding to the stretch ratio 1.5. It shows the times and stretch ratios obtained by the motion controller, forces obtained by the force transducer and calculated deviation, stretch-rate, and stretch histories.

Time [sec]	λ	v_M [volt]	v_{avg} [volt]	*Deviation* v_M - v_{avg}[volt]	$d\lambda/dt$ [/sec]	$H_{t_2}(t)$ t_h=10 sec	$H_{t_1}(t)$ t_h=400 sec
68.1	1.5	1.64500	1.53134	0.11366	0.00638	1.46415	1.04359
398.5		1.60219		0.07085	0.00386	1.47416	1.26212
758.2		1.60156		0.07022	0.00556	1.46095	1.39023
1317.1		1.51031		-0.02103	-0.00031	1.50652	1.49416
1617.8		1.52375		-0.00759	-0.00314	1.51439	1.48207
1698.3		1.52750		-0.00384	-0.00063	1.50438	1.49456
1999.5		1.50812		-0.02322	0.00481	1.48339	1.50564
2445.7		1.49531		-0.03603	-0.00497	1.53378	1.52627
2878.6		1.48219		-0.04915	-0.00634	1.54106	1.5475
3075.0		1.52344		-0.00790	0.00023	1.49976	1.48492
3138.0		1.51719		-0.01415	-0.00130	1.50624	1.48684
3932.6		1.52937		-0.00197	-0.00315	1.50124	1.52586
4049.0		1.51437		-0.01697	-0.00217	1.51008	1.44632
4082.6		1.51594		-0.01540	0.00260	1.48832	1.44999
4245.6		1.51406		-0.01728	-0.00493	1.51756	1.4993
5223.9		1.51625		-0.01509	0.00051	1.49126	1.47043
5515.4		1.50625		-0.02509	0.00071	1.49273	1.53433

For the multivariable linear regression analysis, we have used the stretch history function H_{t_2} obtained by using the stress relaxation spectrum t_2 corresponding to the early (or fast) exponential decay instead of using the stretch-rate $d\lambda/dt$ calculated by simply dividing the change in stretch ratio by change in time spent testing because the *stretch-rate* is not our control variable for the experiments. The table 3.9 shows the average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.5. They have also been used to calculate the uncertainty in resulting Cauchy stresses.

Table 3.9
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.5. They have also been used to calculate the uncertainty in resulting Cauchy stresses *t*.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\Delta\lambda$	Δv
1.50000	1.53134	0.00005	0.04300

Now, we have uncertainties δt_λ, δt_v, δt_w, δt_r, and δt_p as

$$\delta t_\lambda = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg} w_{avg}}{\pi\left(r_{avg}\right)^2 p_{avg}}\right|\delta\lambda = \left|\frac{(1.53134)(0.3939)}{\pi(0.1376)^2(0.3061)}\right|(0.00005) \tag{3.17}$$
$$= 16.56\times10^{-4}[g/mm^2]$$

$$\delta t_v = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg} w_{avg}}{\pi\left(r_{avg}\right)^2 p_{avg}}\right|\delta v = \left|\frac{(1.50000)(0.3939)}{\pi(0.1376)^2(0.3061)}\right|(0.04300) \tag{3.18}$$
$$= 13953.90\times10^{-4}[g/mm^2]$$

$$\delta t_w = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg} v_{avg}}{\pi\left(r_{avg}\right)^2 p_{avg}}\right|\delta w = \left|\frac{(1.50000)(1.53134)}{\pi(0.1376)^2(0.3061)}\right|(0.0001) \tag{3.19}$$
$$= 126.16\times10^{-4}[g/mm^2]$$

$$\delta t_r = \left|\frac{\partial t}{\partial r}\right|\delta r = \left|\frac{-2\lambda_{avg} v_{avg} w_{avg}}{\pi\left(r_{avg}\right)^3 p_{avg}}\right|\delta r = \left|\frac{-2(1.50000)(1.53134)(0.3939)}{\pi(0.1376)^3(0.3061)}\right|(0.0032) \tag{3.20}$$
$$= 23113.21\times10^{-4}[g/mm^2]$$

$$\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{\pi\left(r_{avg}\right)^2\left(p_{avg}\right)^2}\right|\delta p = \left|\frac{-(1.50000)(1.53134)(0.3939)}{\pi(0.1376)^2(0.3061)^2}\right|(0.0014) \tag{3.21}$$
$$= 2272.81\times10^{-4}[g/mm^2]$$

Thus, the total uncertainty in t for the stretch ratio 1.5 is

$$\begin{aligned}
\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial r}\delta r\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\
&= \sqrt{\left(\delta t_\lambda\right)^2 + \left(\delta t_v\right)^2 + \left(\delta t_w\right)^2 + \left(\delta t_r\right)^2 + \left(\delta t_p\right)^2} \\
&= \sqrt{\left\{(16.56)^2 + (13953.90)^2 + (126.16)^2 + (23113.21)^2 + (2272.81)^2\right\}\times\left(10^{-4}\right)^2} \qquad (3.22) \\
&= 27094.54\times10^{-4} \\
&= 2.71\,[g/mm^2]
\end{aligned}$$

We can see the graphical result in the figure.3.8 which is showing the uncertainties in t due to the $\delta\lambda$, δv, δw, δr, and δp for the stretch ratio 1.5. The fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p for the stretch ratio 1.5 have been calculated by following equations and are shown in the figure.3.9.

If we calculate the fractional uncertainties, we have

$$\lambda = \lambda_{avg} \pm \delta\lambda = 1.50000 \pm 0.00005, \quad \frac{\delta\lambda}{\lambda_{avg}} \times 100 = 0.003\ \%$$
$$v = v_{avg} \pm \delta v = 1.53134 \pm 0.04300, \quad \frac{\delta v}{v_{avg}} \times 100 = 2.808\ \%$$
$$w = w_{avg} \pm \delta w = 0.39390 \pm 0.00010, \quad \frac{\delta w}{w_{avg}} \times 100 = 0.025\ \%$$
$$r = r_{avg} \pm \delta r = 0.13760 \pm 0.00320, \quad \frac{\delta r}{r_{avg}} \times 100 = 2.326\ \% \tag{3.23}$$
$$p = p_{avg} \pm \delta p = 0.30610 \pm 0.00140, \quad \frac{\delta p}{p_{avg}} \times 100 = 0.457\ \%$$
$$t = t_{avg} \pm \delta t = 49.69341 \pm 2.70945, \quad \frac{\delta t}{t_{avg}} \times 100 = 5.452\ \%$$

Although the uncertainty in the Cauchy stress t due to δr is higher than the uncertainty due to the δv, the fractional uncertainty for the force v is still higher than the fractional uncertainty for the r. The fractional uncertainties for the radius measure r as well as other manually measured values w and p for all stretch levels are constant. The uncertainty causing the Error-of-definition has nothing to do with those variables. The fractional uncertainties for the force v continuously decreased. The table 3.10 shows the summary of the averages, standard deviations, and fractional uncertainties of the variables we have for stretch ratio 1.5.

Table 3.10
Fractional uncertainties for the measured data r, w, and p, and obtained data λ and v by the motion controller and the Cauchy stress t for the stretch level 1.5.

Data	$\lambda_{1.5}$	v[volt]	r[mm]	w[gram]	p[volt]	t[g/mm^2]
Average	1.50000	1.53134	0.13760	0.39390	0.30610	49.69341
Standard deviation	0.00005	0.04300	0.00320	0.00010	0.00140	2.70945
Fractional uncertainty[%]	0.003	2.808	2.326	0.025	0.457	5.452

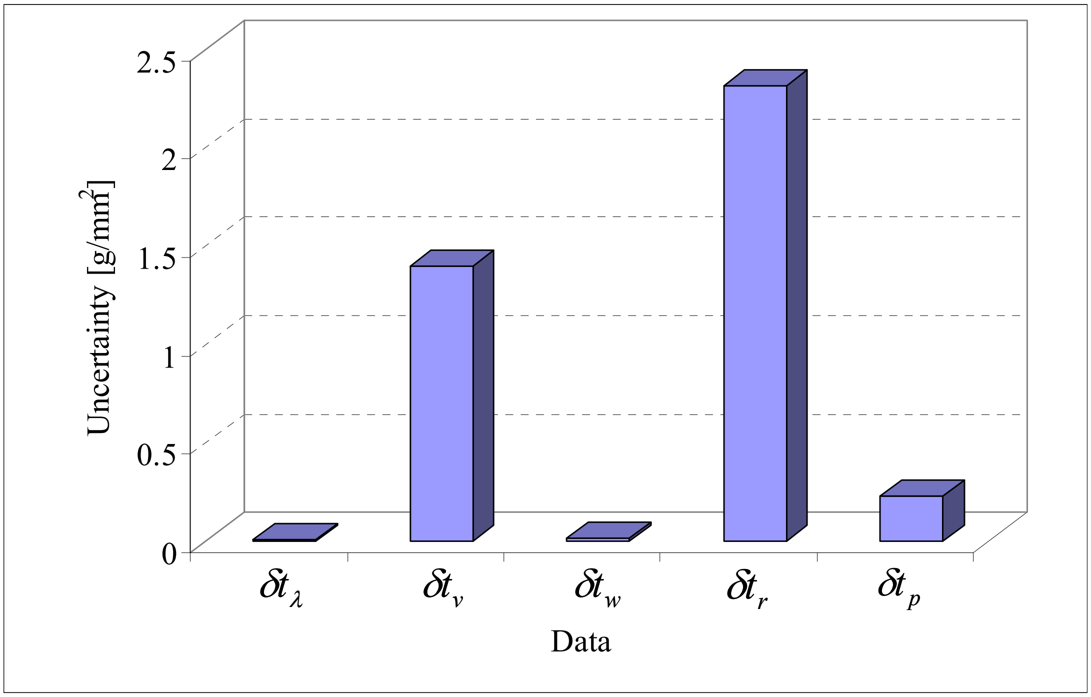

Fig.3.8. Uncertainties in t due to the $\delta\lambda$, δv, δw, δr, and δp for the stretch ratio 1.5.

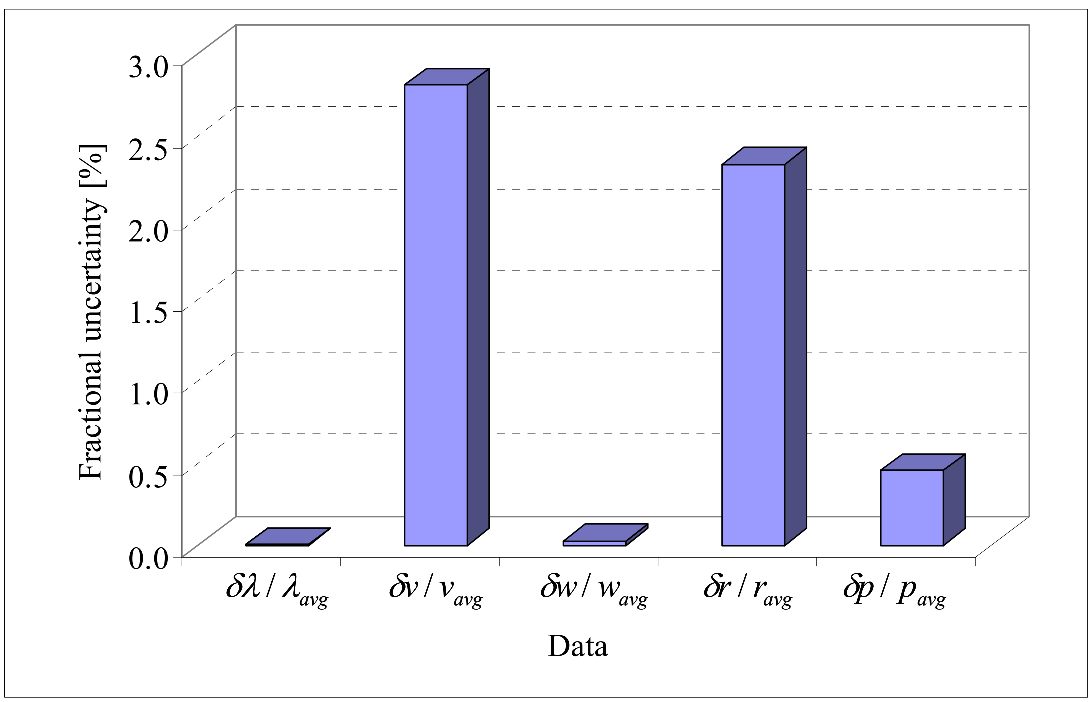

Fig.3.9. Fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p for the stretch ratio 1.5.

Stretch Ratio 1.7: The rubber data corresponding to the stretch ratio 1.7 is shown in the table 3.11. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the *Deviation* versus *Time*, H_{t_2} and H_{t_1} where H_{t_2} is a rate-related stretch history function scanned by a history cut-off t_h = 10 sec, and H_{t_1} is the long-time stretch history function scanned by a history cut-off t_h = 400 sec. The table 3.12 shows the average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.7. They have also been used to calculate the uncertainty in resulting Cauchy stresses *t*.

Table 3.11
The rubber data corresponding to the stretch ratio 1.7. It shows the times and stretch ratios obtained by the motion controller, forces obtained by the force transducer and calculated deviation, stretch-rate, and stretch histories.

Time	λ	v_M	v_{avg}	*Deviation*	$d\lambda/dt$	$H_{t_2}(t)$	$H_{t_1}(t)$
[sec]		[volt]	[volt]	v_M - v_{avg}[volt]	[/sec]	t_h=10 sec	t_h=400 sec
910.9	1.7	2.00281	1.95351	0.04930	0.00083	1.66228	1.39245
1240.7		1.94781		-0.00570	0.00284	1.68818	1.44272
1529.8		1.99750		0.04399	0.00602	1.65074	1.41311
1923.4		1.93813		-0.01538	0.00299	1.68751	1.44299
2082.0		1.93562		-0.01789	-0.00500	1.71875	1.56733
2232.2		1.96875		0.01524	0.00284	1.66541	1.51749
2740.2		1.96906		0.01555	0.00268	1.66506	1.45485
3219.2		1.93781		-0.01570	-0.00594	1.72022	1.53906
3707.4		1.95563		0.00212	0.00126	1.67526	1.52407
4474.6		1.94344		-0.01007	-0.00331	1.69728	1.48998
4551.8		1.92719		-0.02632	-0.00023	1.70312	1.52069
4924.9		1.94562		-0.00789	0.00197	1.6724	1.46919
5297.4		1.92625		-0.02726	-0.00472	1.71694	1.51066

Table 3.12
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.7. They have also been used to calculate the uncertainty in resulting Cauchy stresses *t*.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\Delta\lambda$	Δv
1.70002	1.95351	0.00003	0.02463

We have calculated the uncertainties δt_λ, δt_v, δt_w, δt_r, and δt_p for the stretch ratio 1.7 as

$$\delta t_\lambda = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg}w_{avg}}{\pi(r_{avg})^2 p_{avg}}\right|\delta\lambda = \left|\frac{(1.95351)(0.3939)}{\pi(0.1376)^2(0.3061)}\right|(0.00003) \tag{3.24}$$
$$= 12.68\times 10^{-4}[g/mm^2]$$

$$\delta t_v = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg}w_{avg}}{\pi(r_{avg})^2 p_{avg}}\right|\delta v = \left|\frac{(1.70002)(0.3939)}{\pi(0.1376)^2(0.3061)}\right|(0.02463) \tag{3.25}$$
$$= 9058.46\times 10^{-4}[g/mm^2]$$

$$\delta t_w = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg}v_{avg}}{\pi(r_{avg})^2 p_{avg}}\right|\delta w = \left|\frac{(1.70002)(1.95351)}{\pi(0.1376)^2(0.3061)}\right|(0.0001) \tag{3.26}$$
$$= 182.4\times 10^{-4}[g/mm^2]$$

$$\delta t_r = \left|\frac{\partial t}{\partial r}\right|\delta r = \left|\frac{-2\lambda_{avg}v_{avg}w_{avg}}{\pi(r_{avg})^3 p_{avg}}\right|\delta r = \left|\frac{-2(1.70002)(1.95351)(0.3939)}{\pi(0.1376)^3(0.3061)}\right|(0.0032) \tag{3.27}$$
$$= 33416.97\times 10^{-4}[g/mm^2]$$

$$\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg}v_{avg}w_{avg}}{\pi(r_{avg})^2 (p_{avg})^2}\right|\delta p = \left|\frac{-(1.70002)(1.95351)(0.3939)}{\pi(0.1376)^2(0.3061)^2}\right|(0.0014) \tag{3.28}$$
$$= 3286.02\times 10^{-4}[g/mm^2]$$

Thus, the total uncertainty in t for the stretch ratio 1.7 is

$$\begin{aligned}\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial r}\delta r\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\ &= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_r)^2 + (\delta t_p)^2} \\ &= \sqrt{\left\{(12.68)^2 + (9058.46)^2 + (182.4)^2 + (33416.97)^2 + (3286.02)^2\right\}\times\left(10^{-4}\right)^2} \\ &= 34779.03\times 10^{-4} \\ &= 3.48\ [g/mm^2]\end{aligned} \tag{3.29}$$

, and the fractional uncertainties are

$$\begin{aligned}&\lambda = \lambda_{avg} \pm \delta\lambda = 1.70002 \pm 0.00003, && \frac{\delta\lambda}{\lambda_{avg}}\times 100 = 0.002\ \% \\ &v = v_{avg} \pm \delta v = 1.95351 \pm 0.02463, && \frac{\delta v}{v_{avg}}\times 100 = 1.261\ \% \\ &w = w_{avg} \pm \delta w = 0.39390 \pm 0.00010, && \frac{\delta w}{w_{avg}}\times 100 = 0.025\ \% \\ &r = r_{avg} \pm \delta r = 0.13760 \pm 0.00320, && \frac{\delta r}{r_{avg}}\times 100 = 2.326\ \% \\ &p = p_{avg} \pm \delta p = 0.30610 \pm 0.00140, && \frac{\delta p}{p_{avg}}\times 100 = 0.457\ \% \\ &t = t_{avg} \pm \delta t = 71.84649 \pm 3.47790, && \frac{\delta t}{t_{avg}}\times 100 = 4.841\ \%\end{aligned} \tag{3.30}$$

The absolute uncertainties in in t due to the $\delta\lambda$, δv, δw, δr, and δp for the stretch ratio 1.7 and the fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p for the stretch ratio 1.7 are shown in the figure.3.10 and figure.3.11, respectively. The table 3.13 shows the summary of the averages, standard deviations, and fractional uncertainties of the variables we have for stretch ratio 1.7.

Table 3.13
Fractional uncertainties for the measured data *r*, *w*, and *p*, and obtained data λ and *v* by the motion controller and the Cauchy stress *t* for the stretch level 1.7.

Data	$\lambda_{1.7}$	v[volt]	r[mm]	w[gram]	p[volt]	t[g/mm^2]
Average	1.70002	1.95351	0.13760	0.39390	0.30610	71.84649
Standard deviation	0.00003	0.02463	0.00320	0.00010	0.00140	3.47790
Fractional uncertainty[%]	0.002	1.261	2.326	0.025	0.457	4.841

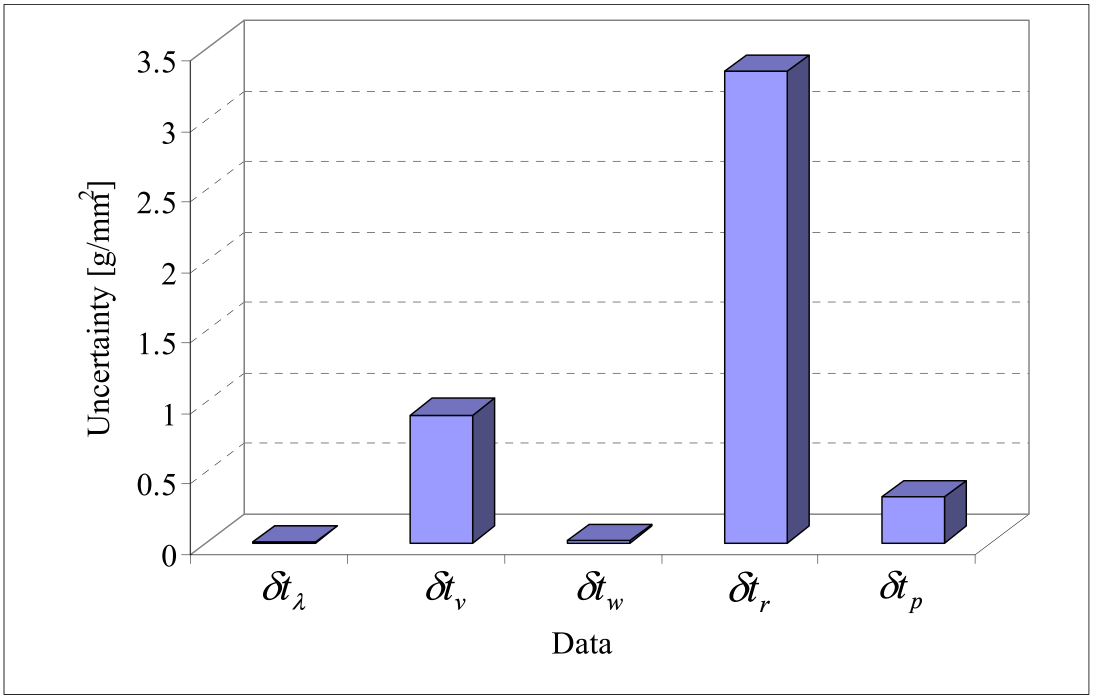

Fig.3.10.Uncertainties in t due to the $\delta\lambda$, δv, δw, δr, and δp for the stretch ratio 1.7.

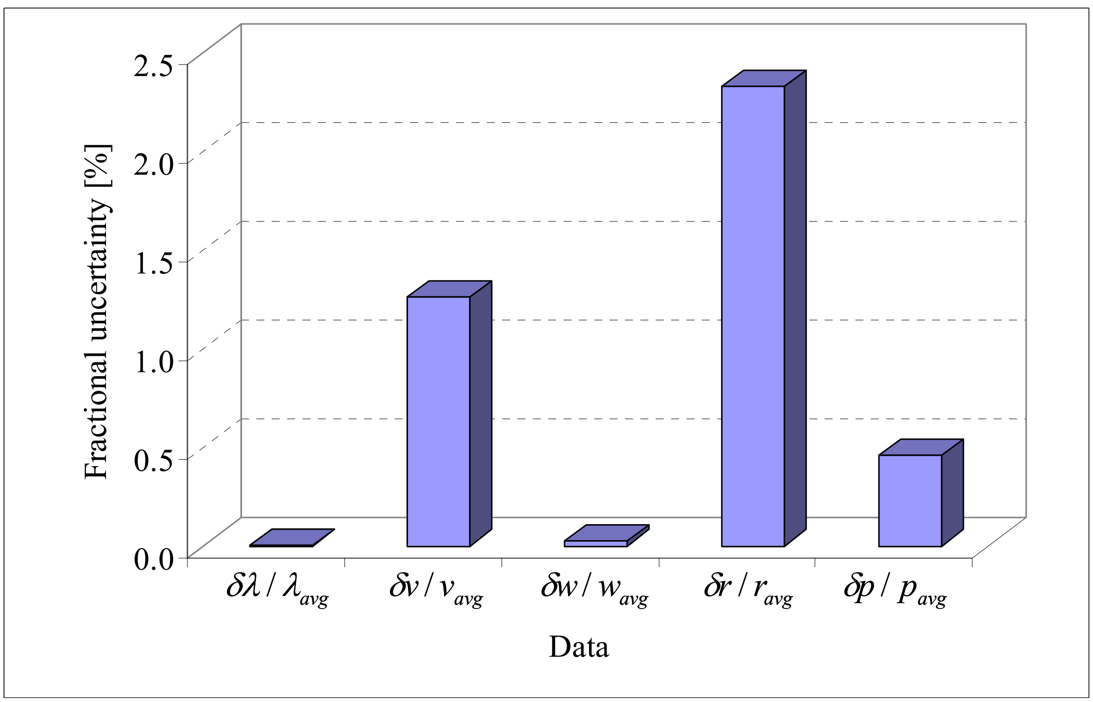

Fig.3.11. Fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p for the stretch ratio 1.7.

Stretch Ratio 1.9: The rubber data corresponding to the stretch ratio 1.9 is shown in the table 3.14. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the *Deviation* versus *Time*, H_{t_2} and H_{t_1} where H_{t_2} is a rate-related stretch history function scanned by a history cut-off t_h = 10 sec, and H_{t_1} is the long-time stretch history function scanned by a history cut-off t_h = 400 sec. The table 3.15 shows the average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.9. They have also been used to calculate the uncertainty in resulting Cauchy stresses *t*.

Table 3.14
The rubber data corresponding to the stretch ratio 1.9. It shows the times and stretch ratios obtained by the motion controller, forces obtained by the force transducer and calculated deviation, stretch-rate, and stretch histories.

Time	λ	v_M	v_{avg}	Deviation	$d\lambda/dt$	$H_{t_2}(t)$	$H_{t_1}(t)$
[sec]		[volt]	[volt]	v_M - v_{avg}[volt]	[/sec]	t_h=10 sec	t_h=400 sec
221.8	1.9	2.39719	2.33031	0.06688	0.00377	1.8644	1.14375
473.0		2.35969		0.02938	-0.00244	1.9009	1.35842
1167.5		2.35500		0.02469	0.00012	1.899	1.35769
1848.3		2.34531		0.01500	-0.00437	1.9031	1.40355
2386.8		2.33687		0.00656	-0.00303	1.8863	1.54094
2817.4		2.32063		-0.00968	0.00106	1.8852	1.49814
3379.4		2.32188		-0.00843	0.00110	1.8817	1.50878
3461.6		2.30063		-0.02968	0.00299	1.8872	1.54695
3782.4		2.30969		-0.02062	0.00126	1.8866	1.54495
4324.1		2.31219		-0.01812	-0.00575	1.9082	1.49347
4775.4		2.31562		-0.01469	0.00236	1.8742	1.52669
5073.6		2.31594		-0.01437	-0.00224	1.8873	1.47706
5371.8		2.30344		-0.02687	0.00398	1.8714	1.50703

Table 3.15
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.9. They have also been used to calculate the uncertainty in resulting Cauchy stresses *t*.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\Delta\lambda$	Δv
1.90001	2.33031	0.00003	0.02762

The uncertainties for the stretch level 1.9 are calculated as

$$\delta t_{\lambda} = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg} w_{avg}}{\pi\left(r_{avg}\right)^2 p_{avg}}\right|\delta\lambda = \left|\frac{(2.33031)(0.3939)}{\pi(0.1376)^2(0.3061)}\right|(0.00003) \tag{3.31}$$
$$= 15.12\times10^{-4}[g/mm^2]$$

$$\delta t_{v} = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg} w_{avg}}{\pi\left(r_{avg}\right)^2 p_{avg}}\right|\delta v = \left|\frac{(1.90001)(0.3939)}{\pi(0.1376)^2(0.3061)}\right|(0.02762) \tag{3.32}$$
$$= 11353.13\times10^{-4}[g/mm^2]$$

$$\delta t_{w} = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg} v_{avg}}{\pi\left(r_{avg}\right)^2 p_{avg}}\right|\delta w = \left|\frac{(1.90001)(2.33031)}{\pi(0.1376)^2(0.3061)}\right|(0.0001) \tag{3.33}$$
$$= 243.18\times10^{-4}[g/mm^2]$$

$$\delta t_{r} = \left|\frac{\partial t}{\partial r}\right|\delta r = \left|\frac{-2\lambda_{avg} v_{avg} w_{avg}}{\pi\left(r_{avg}\right)^3 p_{avg}}\right|\delta r = \left|\frac{-2(1.90001)(2.33031)(0.3939)}{\pi(0.1376)^3(0.3061)}\right|(0.0032) \tag{3.34}$$
$$= 44551.98\times10^{-4}[g/mm^2]$$

$$\delta t_{p} = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{\pi\left(r_{avg}\right)^2 \left(p_{avg}\right)^2}\right|\delta p = \left|\frac{-(1.90001)(2.33031)(0.3939)}{\pi(0.1376)^2(0.3061)^2}\right|(0.0014) \tag{3.35}$$
$$= 4380.97\times10^{-4}[g/mm^2]$$

Thus, the total uncertainty in t for the stretch ratio 1.5 is

$$
\begin{aligned}
\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial r}\delta r\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\
&= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_r)^2 + (\delta t_p)^2} \\
&= \sqrt{\left\{(15.12)^2 + (11353.13)^2 + (243.18)^2 + (44551.98)^2 + (4380.97)^2\right\} \times \left(10^{-4}\right)^2} \\
&= 46184.68 \times 10^{-4} \\
&= 4.62\ [g/mm^2]
\end{aligned}
\tag{3.36}
$$

, and the fractional uncertainties are

$$
\begin{aligned}
&\lambda = \lambda_{avg} \pm \delta\lambda = 1.90001 \pm 0.00003, && \frac{\delta\lambda}{\lambda_{avg}} \times 100 = 0.002\ \% \\
&v = v_{avg} \pm \delta v = 2.33031 \pm 0.02762, && \frac{\delta v}{v_{avg}} \times 100 = 1.185\ \% \\
&w = w_{avg} \pm \delta w = 0.39390 \pm 0.00010, && \frac{\delta w}{w_{avg}} \times 100 = 0.025\ \% \\
&r = r_{avg} \pm \delta r = 0.13760 \pm 0.00320, && \frac{\delta r}{r_{avg}} \times 100 = 2.326\ \% \\
&p = p_{avg} \pm \delta p = 0.30610 \pm 0.00140, && \frac{\delta p}{p_{avg}} \times 100 = 0.457\ \% \\
&t = t_{avg} \pm \delta t = 95.78676 \pm 4.61847, && \frac{\delta t}{t_{avg}} \times 100 = 4.822\ \%
\end{aligned}
\tag{3.37}
$$

The absolute uncertainties in in t due to the $\delta\lambda$, δv, δw, δr, and δp for the stretch ratio 1.9 and the fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p for the stretch ratio 1.9 are shown in the figure.3.12 and figure.3.13, respectively. The table 3.16 shows the summary of the averages, standard deviations, and fractional uncertainties of the variables we have for stretch ratio 1.9.

Table 3.16
Fractional uncertainties for the measured data r, w, and p, and obtained data λ and v by the motion controller and the Cauchy stress t for the stretch level 1.9.

Data	$\lambda_{1.9}$	v[volt]	r[mm]	w[gram]	p[volt]	t[g/mm^2]
Average	1.90001	2.33031	0.13760	0.39390	0.30610	95.78676
Standard deviation	0.00003	0.02762	0.00320	0.00010	0.00140	4.618468
Fractional uncertainty[%]	0.002	1.185	2.326	0.025	0.457	4.822

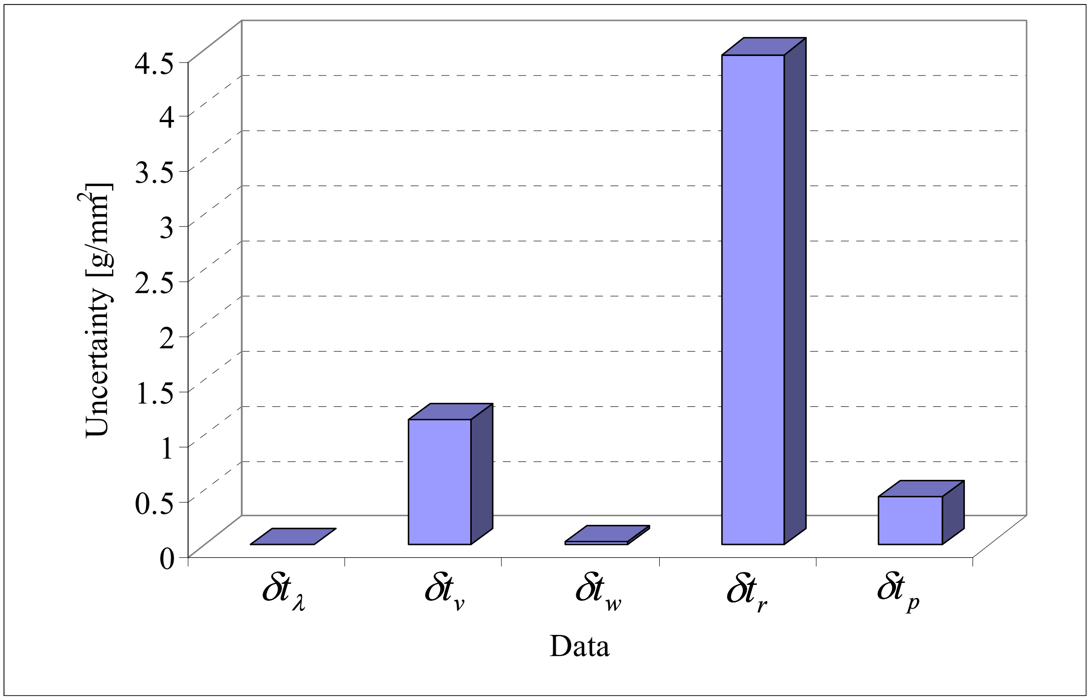

Fig.3.12.Uncertainties in t due to the $\delta\lambda$, δv, δw, δr, and δp for the stretch ratio 1.9.

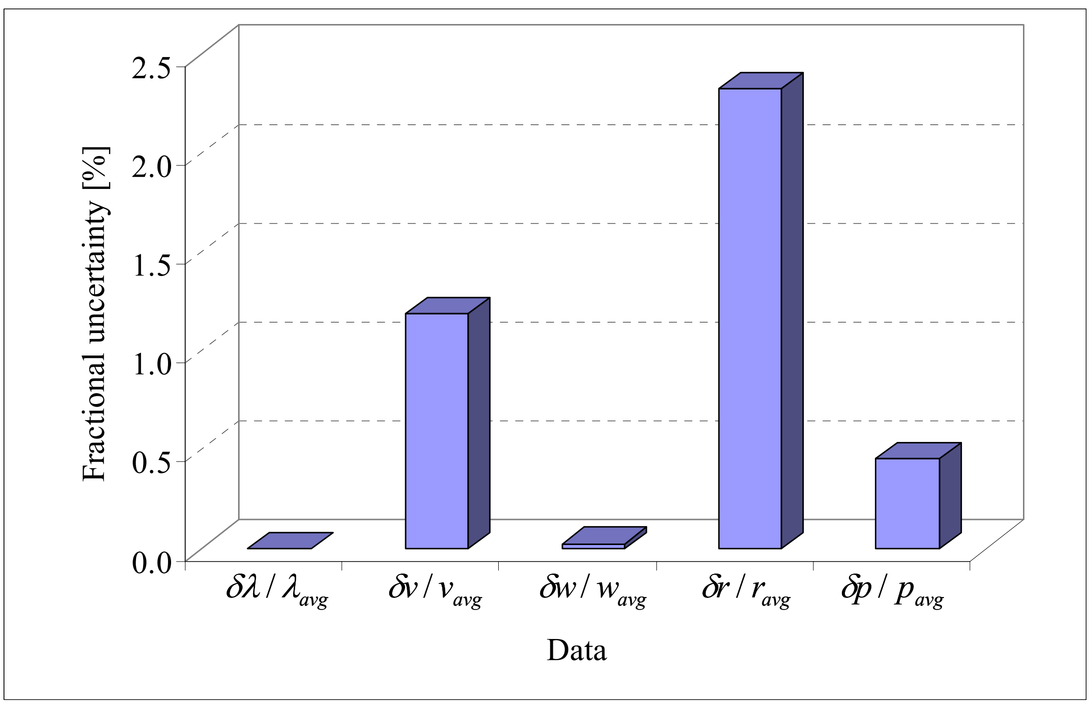

Fig.3.13. Fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, r, and p for the stretch ratio 1.9.

For rubber fiber specimen, we have summarized the values of the absolute uncertainties and fractional uncertainties for each stretch ratio in the table 3.17. Figure 3.14 shows the summarized absolute uncertainties in the Cauchy stress *t* due to the reference-weight measures *w*, the voltage measures *p* to check the resolution of the force transducer, and the radius measures *r*. These values come from the manual measurements. Figure 3.15 shows the summarized absolute uncertainties in the Cauchy stress t due to the stretch ratio data λ, and the force data *v*. These values are obtained by the motion controller/driver (ESP7000, Newport, Inc).

The fractional uncertainties related to the manual measurements are same for entire stretch levels. Note that the fractional uncertainty comes from inelastic behavior noticed from the force data is significant but the fractional uncertainty comes from the data of stretch ratio λ is ignorable and the fractional uncertainties for the static measured variables *r*, *w*, and *p* are constants for all selected stretch levels.

Table 3.17
The summary of absolute uncertainties and fractional uncertainties for each stretch ratio.

		1.1	1.3	1.5	1.7	1.9
λ	λ_{avg}	1.10002	1.30000	1.50000	1.70002	1.90001
	$\Delta\lambda$	0.00008	0.00005	0.00005	0.00003	0.00003
	$\Delta\lambda/\lambda_{avg}$[%]	0.007	0.004	0.003	0.002	0.002
	δt_{λ} [g/mm^2]	0.00052	0.00108	0.00166	0.00127	0.00151
f [volt]	v_{avg}	0.30105	1.00094	1.53134	1.95351	2.33031
	Δv	0.03488	0.03597	0.04300	0.02463	0.02762
	$\Delta v/ v_{avg}$ [%]	11.586	3.594	2.808	1.261	1.185
	δt_v [g/mm^2]	0.83007	1.01163	1.39539	0.90585	1.13531
r [mm]	r_{avg}	0.13760				
	Δr	0.00320				
	$\Delta r/ r_{avg}$ [%]	2.326				
	δt_r [g/mm^2]	0.33322	1.30933	2.31132	3.34170	4.45520
w [g]	w_{avg}	0.39390				
	Δw	0.00010				
	$\Delta w/ w_{avg}$ [%]	0.025				
	δt_w [g/mm^2]	0.00182	0.00715	0.01262	0.01824	0.02432
p [volt]	p_{avg}	0.30610				
	Δp	0.00140				
	$\Delta p/ p_{avg}$ [%]	0.457				
	δt_p [g/mm^2]	0.03277	0.12875	0.22728	0.32860	0.43810
t [g/mm2]	t_{avg}	7.16432	28.1506	49.6934	71.8465	95.7868
	Δt	0.89506	1.65963	2.70945	3.4779	4.61847
	$\Delta t/ t_{avg}$ [%]	12.493	5.896	5.452	4.841	4.822

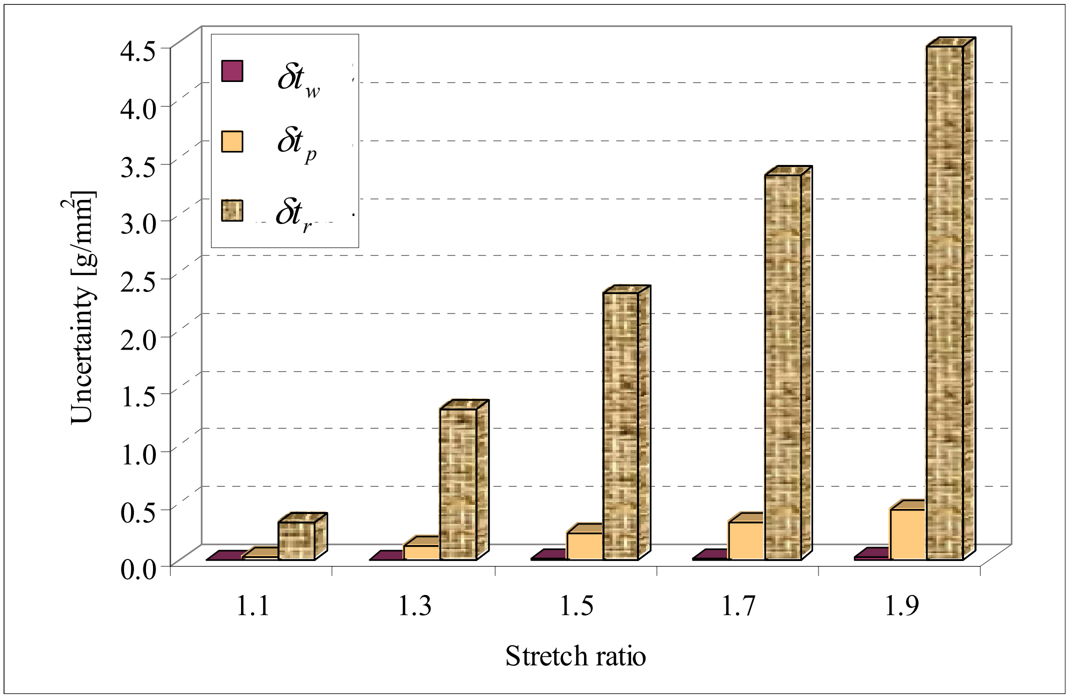

Fig.3.14. The (absolute) uncertainties in the Cauchy stress t due to the reference-weight measures w, the voltage measures p to check the resolution of the force transducer, and the radius measures r. These values come from the manual measurements.

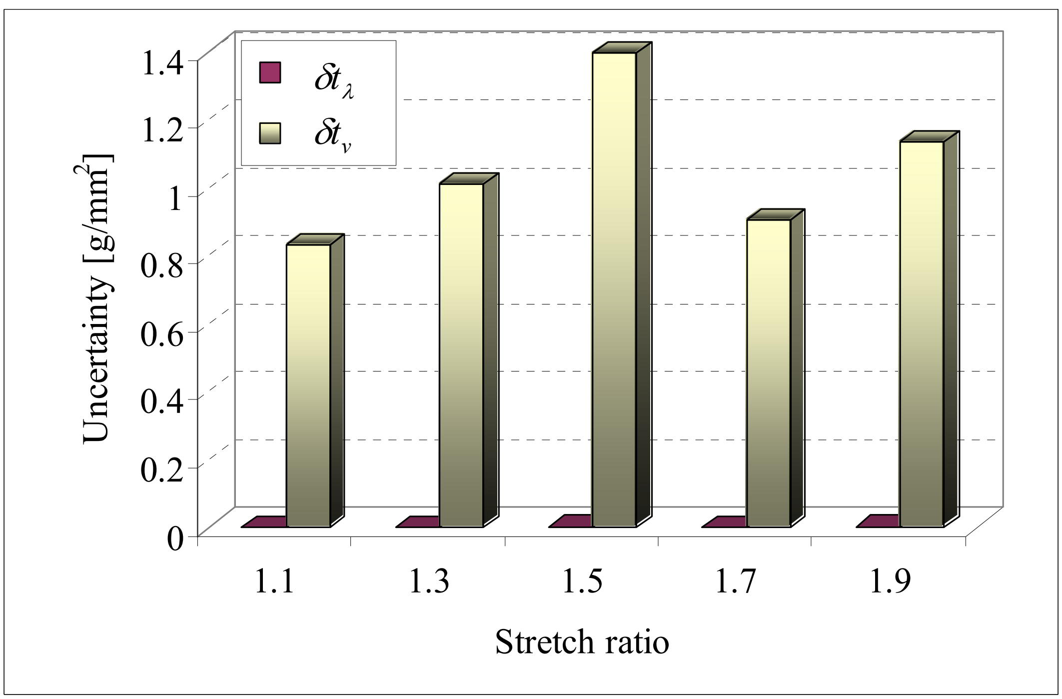

Fig.3.15. The (absolute) uncertainties in the Cauchy stress t due to the stretch ratio data λ, and the force data v. These values are obtained by the motion controller/driver (ESP7000, Newport, Inc).

The reason that the uncertainty in the Cauchy stress t due to δt_r which is the standard deviation of the radius measurements shown in figure 3.14 seems to increase as the stretch ratio gets bigger is because of the calculation process to get the absolute Cauchy stress. Unlikely to the fractional uncertainty, it is affected by the scale and nothing to do with the deviation from the hyperelasticity. Again, any uncertainty due to the physical (or manual) measurements can be reduced as much as to the negligible levels, i.e., with proper assumption, it can be ignored for the error analysis. If we look at the fractional uncertainties shown in figure 3.16, the fractional uncertainties related to the manual measurements such as the reference-weight measures w, the voltage measures p to check the resolution of the force transducer, and the radius measures r are constant for all stretch levels. But the fractional uncertainty due to the force measurements done by the experimental system decreased as the stretch level got bigger. The fractional uncertainty due to the resolution of the motor was inconsequential because we have used very high precision actuators (CMA 25CCCL DC servo motor, Newport, Inc) for the experiments. It is important to note that, for the force measurements, the fractional uncertainty for the stretch ratio 1.1 is approximately 3.2 times bigger than that of the stretch ratio 1.3 and almost 10 times bigger than that of the stretch ratio 1.9. This reveals that the smaller amount of deformation we have the more deviation from the hyperelasticity we get.

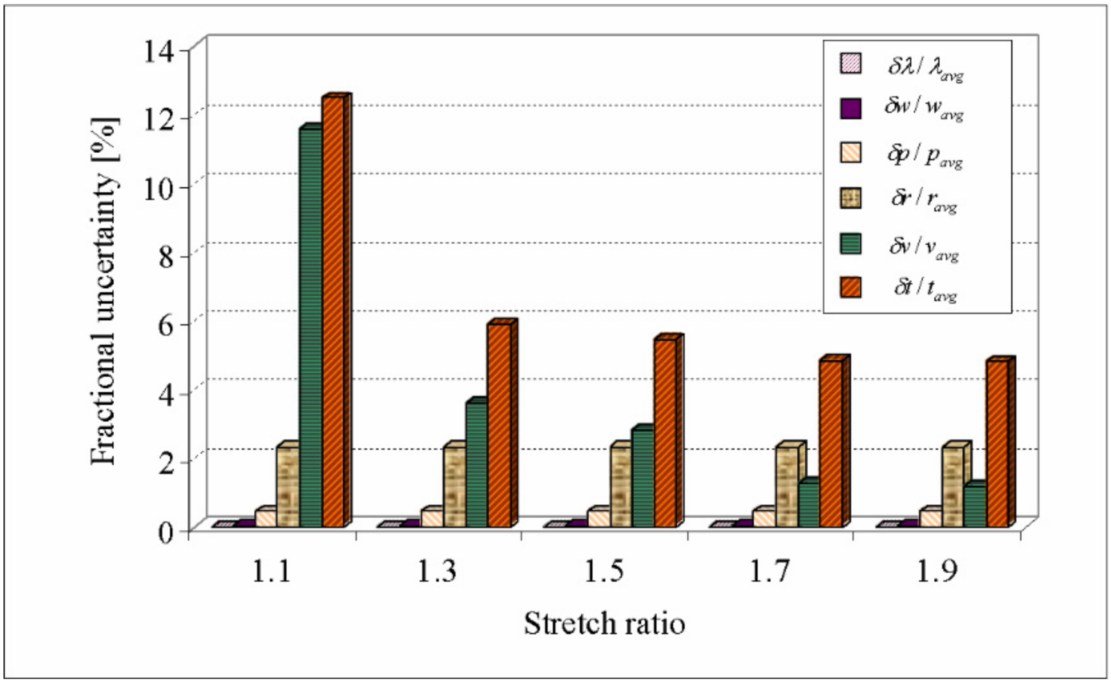

Fig.3. 16. Fraction uncertainties in the data of the λ, w, p, r, v, and t.

Error-of-Definition

The error-of-definition is the uncertainty that comes from the assumption. In this case, the assumption is that the rubber is hyperelastic which is, although it's useful, not truly correct. But many people use the assumption of hyperelasticity for the rubber-like materials and biotissues without knowing the uncertainty due to the assumption. Although the rubber-like materials and biotissues are not perfectly hyperelastic, they predominantly behave like hyperelastic. Thus, the assumption would be useful as long as we find the uncertainty in it. Since we initially assumed that the uncertainty caused by the inelastic behavior of the rubber-like materials and biotissues is due to the *Time*, *Stretch-rate*, and *Stretch history*, we have found their effects on the uncertainty by one of the fundamental statistical theory called multivariable linear regression analysis. This has included the *goodness-of-fit test* and *test of significance* called *t*-test as shown in below. As we mentioned in prior chapter, we used the rate-related stretch history H_{t_2} instead of using the stretch-rate values. Because the motor didn't allow us to control the velocity, the only controllable variable (input) is the stretch ratio. Note again that the stretch-rate and rate-related stretch history are inversely related. Thus, the partial regression coefficient corresponding to the rate-related stretch history should be understood as opposite when we figure out the deviation due to the stretch rate, i.e., if we have a negative sign for the partial regression coefficient for the rate-related stretch history H_{t_2}, the sign for the partial regression coefficient for the stretch-rate should be positive. To do the multivariable linear regression analysis, we have to normalize or nondimensionalize both the dependent variable that is the *Deviation D* and the independent variables that are the *Time T*, rate-related *Stretch History* H_{t_2}, and the long-time *Stretch History* H_{t_1}.

Multivariable Linear Regression Analysis

The existence of several variables forces to use the multivariable regression model. The first order functional relationship of the independent and dependent variables under several assumptions for the residuals was examined by the error analysis. Note that we assumed that there are linear relationships between the dependent variable D and independent variables T, H_{t_2}, and H_{t_1}. We found out how strongly/weakly they are related and which variable(s) can be disregarded for the final regression model. We had to follow some steps to get the best-fitting regression model. This is for getting the maximum effect with the minimum number of independent variables. First, for the regression analysis, we used the whole data T, H_{t_2}, and H_{t_1} as independent variables. Second, we checked the $\overline{R}^2$ as well as calculated ***t***-values that are corresponding to the data T, H_{t_2}, and H_{t_1}. Third, if there was an insignificant variable, we did the regression analysis without having the insignificant variable. It is important to know that the excluding the insignificant independent variable doesn't mean that the excluded independent variable doesn't affect the dependent variable. There could be a reasonably high correlation coefficient between the independent variables. If we include some independent variables that have high correlation coefficient between them, we will have multicollinearity which causes the increasing of standard errors and lower the goodness-of-fit. There is no loss of information even though one of the independent variable of the two that has high correlation coefficient is vanished. Note again that the ideal regression model is the model that has the maximum efficiency with the minimum number of independent variables. This means the good model can predict the future values with the minimum number of the independent variables.

We have set the level of significance α as 0.1 so that the confidence interval (C.I) was

$$\text{C.I} = 100(1 - 0.1) = 90\ \% \tag{3.38}$$

This represents the probability (90 % of probability) that the partial regression coefficient $\hat{\beta}_i$ is likely to be contained within that interval. It physically means that, for the unknown value β, the true value $\hat{\beta}$ will fall into the range of $\hat{\beta} \pm t_c S_{\hat{\beta}}$ for 90 times out of the 100 where $S_{\hat{\beta}}$ is expressed as (for independent variable T)

$$S_{\hat{\beta}} = \frac{S_e}{\sqrt{\sum\left(T_i - \bar{T}\right)^2}} = \frac{\sqrt{\dfrac{\sum\left(D_i - \hat{D}_i\right)^2}{n-k-1}}}{\sqrt{\sum\left(T_i - \bar{T}\right)^2}} \tag{3.39}$$

To carry out the multivariable linear regression analysis, we have nondimensionalized the data to avoid the scale-dependency which cause the partial regression coefficients to be nonsense. We have used $L2$-norm for nondimensionalization as

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \tag{3.40}$$

Stretch Ratio 1.1: The nondimensionalized data for the stretch ratio 1.1 of the rubber specimen is shown in the table 3.18. The dependent variable is Deviation D and the independent variables are Time *T*, rate-related stretch history function H_{t_2}, and long-time stretch history function H_{t_1}.

Table 3.18
Nondimensionalized data using *L*2-norm for the stretch ratio 1.1 of the rubber specimen.

λ	Dependent variable	Independent variables		
	D	***T***	H_{t_2}	H_{t_1}
	$v_M - v_{avg}$		***t_h=10***	***t_h=400***
1.1	0.48072	0.01058	0.24582	0.18016
	0.25452	0.02424	0.24340	0.20441
	0.26800	0.05034	0.23898	0.23879
	0.41356	0.05664	0.23930	0.23495
	0.02833	0.08004	0.24792	0.23449
	0.00145	0.11250	0.24255	0.24801
	0.19410	0.11857	0.24310	0.24111
	0.02159	0.14340	0.24193	0.23852
	-0.13071	0.17443	0.24346	0.25741
	-0.10606	0.18663	0.24492	0.25472
	-0.12842	0.21441	0.24264	0.24300
	0.02833	0.28796	0.23493	0.26121
	-0.27406	0.31200	0.24509	0.26208
	-0.27406	0.35594	0.24238	0.24564
	-0.20016	0.39288	0.23953	0.25424
	-0.18891	0.40423	0.24007	0.25002
	-0.38823	0.45158	0.24674	0.25967
Correlation Coefficients		-0.87	-0.16	-0.76

Since the number of sample for the stretch ratio1.1 is 17, the degree-of-freedom (DOF) for *t* is *n*-*k*-1 = 17-3-1 = 13. Thus, the critical *t* value for the stretch ratio 1.1 for this case is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;13)} = 1.771$ for two-tailed test.

Thus if we have the condition as

$$\left|t_{n-k-1}\right| > 1.771 \tag{3.41}$$

we reject the null hypothesis and accept the alternative hypothesis which means that the specific independent variable(s) affect the dependent variable. The result of multivariable linear regression analysis using entire independent variables *T*, H_{t_2} and H_{t_1} is shown in the table 3.19.

Table 3.19
Result of multivariable linear regression analysis for stretch ratio 1.1 of rubber specimen with entire independent variables *T*, H_{t_2} and H_{t_1}.

	$\hat{\beta}_i$	s_e	*t*-value	$\overline{R}^2$
$\hat{\alpha}$	7.55	1.64	4.62	0.88
T	-1.19	0.20	-5.91	
H_{t_2}	-27.14	6.34	-4.28	
H_{t_1}	-3.08	1.96	-1.58	

Although the $\overline{R}^2$ is high, the *t*-value of the long-time stretch history function H_{t_1} which is -1.58 didn't satisfy the condition above. This means the long-time stretch history H_{t_1} is insignificant. Thus we have eliminated the H_{t_1} and executed the multivariable linear regression analysis again. Since the number of independent variables are reduced to 2, $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;14)} = 1.761$, we should have a new condition as

$$\left|t_{n-k-1}\right| > 1.761 \tag{3.42}$$

to reject the null hypothesis and accept the alternative hypothesis. The result of multivariable linear regression analysis for stretch ratio 1.1 of rubber specimen with independent variables *T*, and H_{t_2} is shown in the table 3.20.

Table 3.20
Result of multivariable linear regression analysis for stretch ratio 1.1 of rubber specimen with independent variables T and H_{t_2} .

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	6.66	1.62	4.11	0.86
T	-1.40	0.15	-9.08	
H_{t_2}	-26.37	6.68	-3.95	

Thus the final regression model for the stretch ratio 1.1 of the rubber specimen can be expressed as

$$\hat{D}_{1.1} = 6.66 - 1.40T - 26.37H_{t_2} \tag{3.43}$$

The optimal independent variables that can predict the deviation D is T and H_{t_2}. In addition, their ability to explain the deviation is 86 %. The rest 14 % can't be explained by this model whatsoever. From the model, we have noticed that the time T and the rate-related stretch history H_{t_2} are the substantial factors that cause the deviation from the hyperelasticity and the rate-related stretch history H_{t_2} is more substantial than the time T. Note that the sign of the partial regression coefficient for the H_{t_2} is exactly the opposite of the stretch-rate.

Stretch Ratio 1.3: The nondimensionalized data for the stretch ratio 1.3 of the rubber specimen is shown in the table 3.21. The dependent variable is deviation D and the independent variables are time T, rate-related stretch history function H_{t_2} , and long-time stretch history function H_{t_1} .

Table 3.21
Nondimensionalized data using $L2$-norm for the stretch ratio 1.3 of the rubber specimen.

λ	Dependent variable	Independent variables		
	D	***T***	H_{t_2}	H_{t_1}
	V_M - V_{avg}		t_h=10	t_h=400
1.3	0.10439	0.04449	0.30317	0.29330
	0.36818	0.06910	0.30119	0.27693
	0.56329	0.08810	0.30296	0.26603
	-0.08245	0.24227	0.29904	0.30553
	-0.27475	0.26593	0.30628	0.31888
	0.32421	0.29242	0.29636	0.31480
	-0.19781	0.33743	0.30367	0.30426
	-0.29946	0.37523	0.30350	0.31077
	0.05498	0.38123	0.29482	0.30546
	-0.10716	0.41945	0.29630	0.29858
	-0.45342	0.44116	0.30902	0.31745
Correlation Coefficients		-0.71	-0.44	-0.77

Since the number of sample for the stretch ratio1.3 is 11, the degree-of-freedom (DOF) for t is n-k-1 = 11-3-1 = 7. For this case, the critical t value for the stretch ratio 1.3 is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;7)} = 1.895$ for two-tailed test.

Thus if we have following constraint as

$$|t_{n-k-1}| > 1.895 \tag{3.44}$$

we reject the null hypothesis and accept the alternative hypothesis which means that the independent variable(s) is significant to predict the value of the dependent variable. The result of multivariable linear regression analysis for the stretch ratio 1.3 of the rubber specimen using entire independent variables T, H_{t_2} and H_{t_1} is shown in the table 3.22.

Table 3.22
Result of multivariable linear regression analysis for stretch ratio 1.3 of rubber specimen with entire independent variables T, H_{t_2} and H_{t_1} .

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	11.33	3.47	3.27	0.77
T	-1.44	0.64	-2.26	
H_{t_2}	-30.79	11.65	-2.64	
H_{t_1}	-5.43	4.76	-1.14	

Although the $\overline{R}^2$ is reasonably high as 77 %, the t-value of the long-time stretch history H_{t_1} is -1.14 which means it is insignificant. So we have executed a regression analysis without H_{t_1} . Since the number of independent variables are reduced to 2 for that case, $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;8)} = 1.860$, we should have a new constraint as

$$|t_{n-k-1}| > 1.860 \tag{3.45}$$

to reject the null hypothesis and accept the alternative hypothesis. The result of multivariable linear regression analysis for stretch ratio 1.3 of rubber specimen with independent variables T, and H_{t_2} is shown in the table 3.23.

Table 3.23
Result of multivariable linear regression analysis for stretch ratio 1.3 of rubber specimen with independent variables T and H_{t_2} .

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	10.74	3.50	3.07	0.76
T	-2.00	0.42	-4.79	
H_{t_2}	-33.74	11.60	-2.91	

Thus the final regression model for the stretch ratio 1.3 of the rubber specimen can be expressed as

$$\hat{D}_{1.3} = 10.74 - 2.00T - 33.74H_{t_2} \tag{3.46}$$

Again, the optimal independent variables that can anticipate the dependent variable is T and H_{t_2}. This model can explain the deviation by 76% using the independent variables T and H_{t_2}. The rest 24 % can't be explained by this model no matter what we do. From the model, for the stretch ratio 1.3, we have noticed again that the time T and the rate-related stretch history H_{t_2} are the substantial factors that cause the deviation from the hyperelasticity and the rate-related stretch history H_{t_2} is more substantial than the time T. Note that the sign of the partial regression coefficient for the H_{t_2} is exactly the opposite of the stretch-rate.

Stretch Ratio 1.5: The nondimensionalized data for the stretch ratio 1.5 of the rubber specimen is shown in the table 3.24. The dependent variable is deviation D and the independent variables are time T, rate-related stretch history function H_{t_2}, and long-time stretch history function H_{t_1}.

Table 3.24
Nondimensionalized data using *L*2-norm for the stretch ratio 1.5 of the rubber specimen.

λ	Dependent variable	Independent variables		
	Deviation	***Time***	***Stretch History H(t)***	
	V_M - V_{avg}		t_h=10	t_h=400
1.5	0.66080	0.00522	0.23681	0.17400
	0.41191	0.03058	0.23843	0.21044
	0.40825	0.05819	0.23629	0.23180
	-0.12227	0.10107	0.24366	0.24913
	-0.04413	0.12415	0.24493	0.24711
	-0.02233	0.13033	0.24332	0.24919
	-0.13501	0.15344	0.23992	0.25104
	-0.20948	0.18769	0.24807	0.25448
	-0.28576	0.22091	0.24925	0.25802
	-0.04594	0.23598	0.24257	0.24759
	-0.08227	0.24081	0.24362	0.24791
	-0.01146	0.30179	0.24281	0.25441
	-0.09867	0.31072	0.24424	0.24115
	-0.08954	0.31330	0.24072	0.24176
	-0.10047	0.32581	0.24545	0.24998
	-0.08774	0.40089	0.24119	0.24517
	-0.14588	0.42325	0.24143	0.25582
Correlation Coefficients		-0.65	-0.79	-0.92

Since the number of sample for the stretch ratio1.5 is 17, the degree-of-freedom (DOF) for *t* is *n*-*k*-1 = 17-3-1 = 13. In this case, the critical *t* value for the stretch ratio 1.5 of the rubber specimen is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;13)} = 1.771$ for two-tailed test.

Thus if we have the following constraint as

$$\left| t_{n-k-1} \right| > 1.771 \quad (3.47)$$

we reject the null hypothesis and accept the alternative hypothesis. This implies that the independent variable(s) affect the dependent variable. The result of multivariable linear regression analysis for the stretch ratio 1.5 of the rubber specimen using entire independent variables *T*, H_{t_2} and H_{t_1} is shown in the table 3.25.

Table 3.25
Result of multivariable linear regression analysis for stretch ratio 1.5 of rubber specimen with entire independent variables T, H_{t_2} and H_{t_1}.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	7.50	1.67	4.49	0.82
T	-0.35	0.20	-1.72	
H_{t_2}	-22.13	8.26	-2.68	
H_{t_1}	-8.48	2.67	-3.17	

Although the modified coefficient of determination $\overline{R}^2$ is high as 82%, the t-value of the time T didn't satisfy the above constraint which means the time T is inconsequential to cause the deviation from the hyperelasticity. So we have reperformed a regression analysis without T. Since the number of independent variables are reduced to 2, $t_{c(\alpha/2;\,n-k-1)} = t_{c(0.05;\,14)} = 1.761$, we should have a new constraint as

$$|t_{n-k-1}| > 1.761 \tag{3.48}$$

to reject the null hypothesis and accept the alternative hypothesis. The result of multivariable linear regression analysis for stretch ratio 1.5 of the rubber specimen with independent variables H_{t_2} and H_{t_1} is shown in the table 3.26.

Table 3.26
Result of multivariable linear regression analysis for stretch ratio 1.5 of rubber specimen with independent variables H_{t_2} and H_{t_1}.

	$\hat{\beta}_i$	s_e	*t*-value	$\overline{R}^2$
$\hat{\alpha}$	7.41	1.79	4.14	0.80
H_{t_2}	-19.89	8.75	-2.27	
H_{t_1}	-10.65	2.52	-4.22	

Thus, the final regression model for the stretch ratio 1.5 of the rubber specimen can be expressed as

$$\hat{D}_{1.5} = 7.41 - 19.89H_{t_2} - 10.65H_{t_1} \tag{3.49}$$

The optimal independent variables that can expect the dependent variable D is H_{t_2} and H_{t_1}. Their ability of the regression model to explain the deviation is 80 %. The rest 20 % can't be explained by this model. From the model for the stretch ratio 1.5 of the rubber specimen, the rate-related stretch history H_{t_2} and the long-time stretch history H_{t_1} are the substantial factors that cause the inelastic deviation. The rate-related stretch history H_{t_2} is more substantial than the long-time stretch history H_{t_1}. The sign of the partial regression coefficient for the H_{t_2} is exactly the opposite of the stretch-rate.

Stretch Ratio 1.7: The nondimensionalized data for the stretch ratio 1.7 of the rubber specimen is shown in the table 3.27. The dependent variable is deviation D and the independent variables are time T, rate-related stretch history function H_{t_2}, and long-time stretch history function H_{t_1}.

Table 3.27
Nondimensionalized data using $L2$-norm for the stretch ratio 1.7 of the rubber specimen.

λ	Dependent variable	Independent variables		
	Deviation	***Time***	***Stretch History H(t)***	
	V_M - V_{avg}		t_h=10	t_h=400
1.7	0.57784	0.07633	0.27336	0.26019
	-0.06680	0.10397	0.27762	0.26958
	0.51561	0.12819	0.27146	0.26405
	-0.18026	0.16118	0.27751	0.26964
	-0.20968	0.17447	0.28265	0.29287
	0.17863	0.18705	0.27387	0.28356
	0.18227	0.22963	0.27382	0.27185
	-0.18401	0.26977	0.28289	0.28759
	0.02486	0.31068	0.27549	0.28479
	-0.11802	0.37497	0.27912	0.27842
	-0.30848	0.38144	0.28008	0.28415
	-0.09247	0.41271	0.27502	0.27453
	-0.31950	0.44392	0.28235	0.28228
Correlation Coefficients		-0.63	-0.84	-0.69

Since the number of sample for the stretch ratio1.7 is 13, the degree-of-freedom (DOF) for t is n-k-1 = 13-3-1 = 9. Thus, the critical t value for this case for the stretch ratio 1.7 is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;9)} = 1.833$ for two-tailed test.

Thus if we have the following constraint as

$$\left|t_{n-k-1}\right| > 1.833 \tag{3.50}$$

we reject the null hypothesis and accept the alternative hypothesis which means that the specific independent variable affect the dependent variable. The result of multivariable linear regression analysis for the stretch ratio 1.7 of the rubber specimen using entire independent variables T, H_{t_2} and H_{t_1} is shown in the table 3.28.

Table 3.28
Result of multivariable linear regression analysis for stretch ratio 1.7 of rubber specimen with entire independent variables T, H_{t_2} and H_{t_1}.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	14.07	2.87	4.90	0.75
T	-0.56	0.34	-1.67	
H_{t_2}	-53.66	13.04	-4.12	
H_{t_1}	3.33	5.79	0.57	

Although the $\overline{R}^2$ is convincingly high as 75%, the t-values of the time T and the long-time stretch history H_{t_1} didn't satisfy the above constraint. This means that the time T and long-time stretch history H_{t_1} don't affect the deviation from the hyperelasticity. The only independent variable that is significant is the rate-related stretch history H_{t_2}. So we have carried out a regression analysis only with H_{t_2}. Since the number of independent variables are reduced to 1, $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;11)} = 1.796$, we should have a new constraint as

$$\left|t_{n-k-1}\right| > 1.796 \tag{3.51}$$

to reject the null hypothesis and accept the alternative hypothesis. The result of multivariable linear regression analysis for stretch ratio 1.7 of the rubber specimen with only independent variable H_{t_2} is shown in the table 3.29.

Table 3.29
Result of multivariable linear regression analysis for stretch ratio 1.7 of rubber specimen with independent variable H_{t_2} .

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	15.06	2.77	5.43	0.72
H_{t_2}	-54.41	9.98	-5.45	

Thus the final regression model for the stretch ratio 1.7 of the rubber specimen can be expressed as

$$\hat{D}_{1.7} = 15.06 - 54.41 H_{t_2} \tag{3.52}$$

and we found that the optimal independent variable that can expect the dependent variable D is rate-related stretch history H_{t_2} only. The ability of the acquired regression model with independent variable H_{t_2} to explain the deviation is 72 %. The rest 28 % can't be explained by this model. From the regression model for the stretch ratio 1.7 of the rubber specimen, we have noticed that the rate-related stretch history H_{t_2} is the only factor that causes the substantial deviation from the hyperelasticity. Note that this doesn't mean that the other two independent variables are not causing the deviation from the hyperelasticity at all; they are just insignificant relative to the stretch-rate. The deviation D can be well-explained only by the rate-related stretch history function H_{t_2} . The sign of the partial regression coefficient for the H_{t_2} is exactly the opposite of the stretch-rate.

Stretch Ratio 1.9: The nondimensionalized data for the stretch ratio 1.9 of the rubber specimen is shown in the table 3.30. The dependent variable is deviation D and the independent variables are time T, rate-related stretch history function H_{t_2} , and long-time stretch history function H_{t_1} .

Table 3.30
Nondimensionalized data using $L2$-norm for the stretch ratio 1.9 of the rubber specimen.

λ	Dependent variable	Independent variables		
	Deviation	***Time***	***Stretch History H(t)***	
	V_M - V_{avg}		***t_h=10***	***t_h=400***
1.9	0.69886	0.01796	0.27397	0.21749
	0.30698	0.03830	0.27933	0.25831
	0.25797	0.09454	0.27905	0.25817
	0.15671	0.14967	0.27966	0.26689
	0.06851	0.19328	0.27719	0.29302
	-0.10120	0.22815	0.27702	0.28488
	-0.08813	0.27366	0.27652	0.28690
	-0.31020	0.28032	0.27732	0.29416
	-0.21552	0.30630	0.27724	0.29378
	-0.18939	0.35016	0.28041	0.28399
	-0.15355	0.38671	0.27541	0.29031
	-0.15021	0.41085	0.27734	0.28087
	-0.28083	0.43500	0.27500	0.28657
Correlation Coefficients		-0.89	-0.07	-0.93

Since the number of sample for the stretch ratio1.9 is 13, the degree-of-freedom (DOF) for t is n-k-1 = 13-3-1 = 9. Thus, the critical t value for this case for the stretch ratio 1.9 of rubber specimen is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;9)} = 1.833$ for two-tailed test. Thus, if we have a following constraint as

$$|t_{n-k-1}| > 1.833 \tag{3.53}$$

we reject the null hypothesis and accept the alternative hypothesis. The result of multivariable linear regression analysis for the stretch ratio 1.9 of the rubber specimen using entire independent variables T, H_{t_2} and H_{t_1} is shown in the table 3.31.

Table 3.31
Result of multivariable linear regression analysis for stretch ratio 1.9 of rubber specimen with entire independent variables T, H_{t_2} and H_{t_1}.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	5.81	5.78	1.01	0.84
T	-1.00	0.29	-3.48	
H_{t_2}	-12.48	19.54	-0.64	
H_{t_1}	-7.59	2.84	-2.68	

Although the $\overline{R}^2$ is very high, the t-value for the rate-related stretch history H_{t_2} didn't satisfy the above constraint. The significant independent variables are the time T and the long-time stretch history function H_{t_1}. Thus we have carried out a regression analysis again without the rate-related stretch history H_{t_2}. Since the number of independent variables are reduced to 2, $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;10)} = 1.812$, we should have a new constraint as

$$|t_{n-k-1}| > 1.812 \tag{3.54}$$

to reject the null hypothesis and accept the alternative hypothesis. The result of multivariable linear regression analysis for stretch ratio 1.9 of the rubber specimen with independent variables T and H_{t_1} is shown in the table 3.32.

Table 3.32
Result of multivariable linear regression analysis for stretch ratio 1.9 of rubber specimen with independent variables T and H_{t_1}.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	2.14	0.67	3.18	0.85
T	-0.95	0.27	-3.56	
H_{t_1}	-6.94	2.56	-2.71	

Thus the final regression model for the stretch ratio 1.9 of the rubber specimen can be expressed as

$$\hat{D}_{1.9} = 2.14 - 0.95T - 6.94H_{t_1} \tag{3.55}$$

From the results, we have noticed that the optimal independent variables that can expect the dependent variable D are T and H_{t_1}. The ability of the regression model to explain the deviation is 85 %. The rest 15 % can't be explained by this model. In addition, we have realized that the rate-related stretch history H_{t_2} is not a significant factor that causes the deviation from the hyperelasticity. The only factors that cause the deviation from the hyperelasticity are the time T and long-time stretch history H_{t_1}.

Nonsense of Correlation Coefficients

We have checked the correlation coefficients between the deviation D (dependent variable) and independent variables T, H_{t_2}, and H_{t_1}. The summary of the correlation coefficients for each stretch ratio for the rubber is shown in the table 3.33.

Table 3.33
The summary of correlation coefficients between the deviation *D* and the time *T*, rate-related stretch history H_{t_2} , and long-time stretch history H_{t_1} .

λ	Correlation Coefficients: *D* between.		
	T	H_{t_2}	H_{t_1}
		t_h=10	t_h=400
1.1	-0.87	-0.16	-0.76
1.3	-0.71	-0.44	-0.77
1.5	-0.65	-0.79	-0.92
1.7	-0.63	-0.84	-0.69
1.9	-0.89	-0.07	-0.93

Generally, the correlation coefficient can be obtained by using the following equation as

$$c = \frac{\sigma_{mn}}{\sigma_m \sigma_n} = \frac{\dfrac{\sum (m_i - \overline{m})(n_i - \overline{n})}{N}}{\sqrt{\dfrac{(m_i - \overline{m})^2}{N}}\sqrt{\dfrac{(n_i - \overline{n})^2}{N}}} = \frac{\sum (m_i - \overline{m})(n_i - \overline{n})}{\sqrt{\sum (m_i - \overline{m})^2 \sum (n_i - \overline{n})^2}} \qquad (3.56)$$

where σ_{mn} is the covariance which is a degree of covarying of two variables *m* and *n*, σ_m is a standard deviation of m_i and σ_n is a standard deviation of n_i. The correlation coefficient just shows whether there is a relationship between one variable and the other or not. It is a relationship only between two variables. For this study, since we have too many variables and scatterness, the assessment and ordering of the correlation coefficients are not meaningful.

Indisputably, if the correlation coefficient between a certain dependent variable and independent variable is high, then we naturally know there is high relationship between them. But again, it merely tells a relationship between the two variables. Additionally, since the correlation coefficient is a *nonratio-related* value, we cannot compare a certain value of a correlation coefficient to the other. For example, if there are two variables A and B that have the correlation coefficients 0.4 and 0.8, respectively, we cannot say that B is two times more related to a certain variable than A is. Besides, if there are many variables, correlation coefficient never tells which one is significant, how intensively related and which one is more related to the other. This can be told only by the multivariable (linear or nonlinear) regression analysis.

For example, despite the correlation coefficients between the deviation D and rate-related stretch history H_{t_2} are much lower than the others for the stretch ratios 1.1 and 1.3, the results of multivariable linear regression analysis for them showed that the rate of the stretch is highly related to the deviation from the hyperelasticity. Moreover, the rate of the stretch is the most substantial factor that causes the deviation from the hyperelasticity for the stretch ratios 1.5 and 1.7 as well.

CHAPTER IV

RESULTS FOR TISSUES

Error-of-Measurement and Error-of-Definition

For the experiment of the biotissues, we have used the strips of pulmonary artery of the adult swine in this study. Unlikely to the results of the experiment of the rubber specimen, the stretch ratio of 2 of the biotissues was almost the maximum stretch ratio that the biotissues can acquire without damaging or breaking the substances of the tissues such as elastin.

If we look at the figure 4.1, wherever the stretch ratio is 2, there was no spiky peak in the output protocol. This means the randomly-arranged elastin in the biotissues went up to almost the maximum stretch ratio. Approximately a stretch ratio of 2 was the maximum stretch ratio that the biotissues could acquire with no damage.

The figures 4.1 and 4.2 show the results of the randomized stretch-controlled protocol and corresponding force output of the strip of the pulmonary artery of the adult swine. The output profile showed the much higher nonlinearity than the rubber did. The experiment of the randomized stretch-controlled motion of the tissue specimen showed that there is a noticeable inelastic deviation from the hyperelasticity. The figure 4.3 shows that there are behaviors of nonlinearity as well as inelasticity. If the hyperelastic assumption is perfectly satisfied, there would be only one point for each stretch ratio.

To find the uncertainty due to the manual measurements, all measurements (width a and length b of the tissue specimen, reference weight w, and voltage output p

for a fixed weight) have been done for 10 times. Again, it is important to note that the uncertainty due to the measurement can be reduced as much as possible as long as we use the extremely accurate measuring devices or equipments. The width *a* and the length *b* of the tissue specimen has been used to get a reference area of the tissue sample. As for the values of reference weight *w* and voltage output *p*, we have used the same values measured for the rubber experiment.

Since the uncertainty due to the manual measurements is fixed or static for entire stretch levels, it can be ignored for the error analysis which was focusing on the uncertainty due to the inelastic behaviors of the specimen. Note that the uncertainty due to the inelastic behavior varies. The summarized manual measurements are shown in the table 4.1. It shows the width *a* and length *b* of the tissue, a reference weight *w*, a voltage output *p* for a fixed weight have been measured for 10 times. It also shows the standard deviations and fractional uncertainties for each measurement.

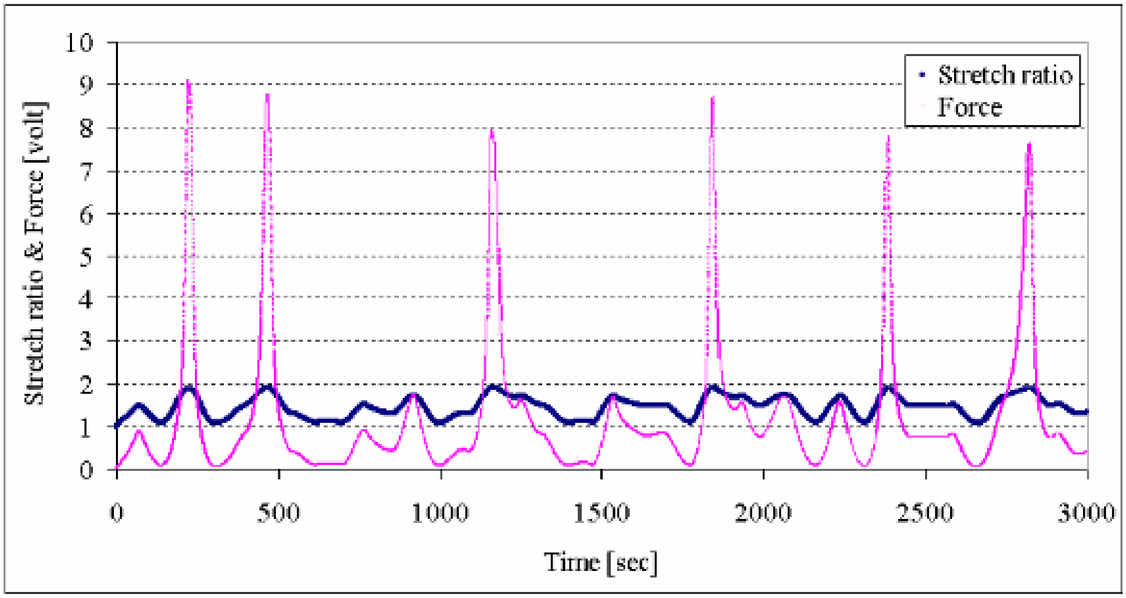

Fig.4.1. The randomized stretch-controlled protocol and corresponding force output profile for the tissue specimen (0-3000 sec).

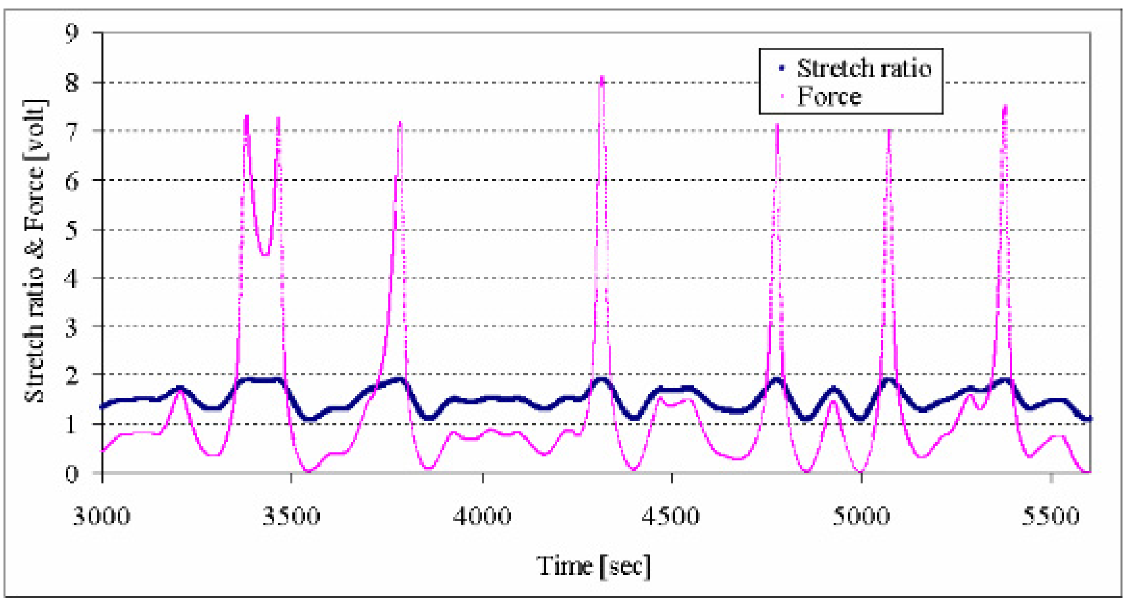

Fig.4.2. The randomized stretch-controlled protocol and corresponding force output profile for the tissue specimen (3000-5600).

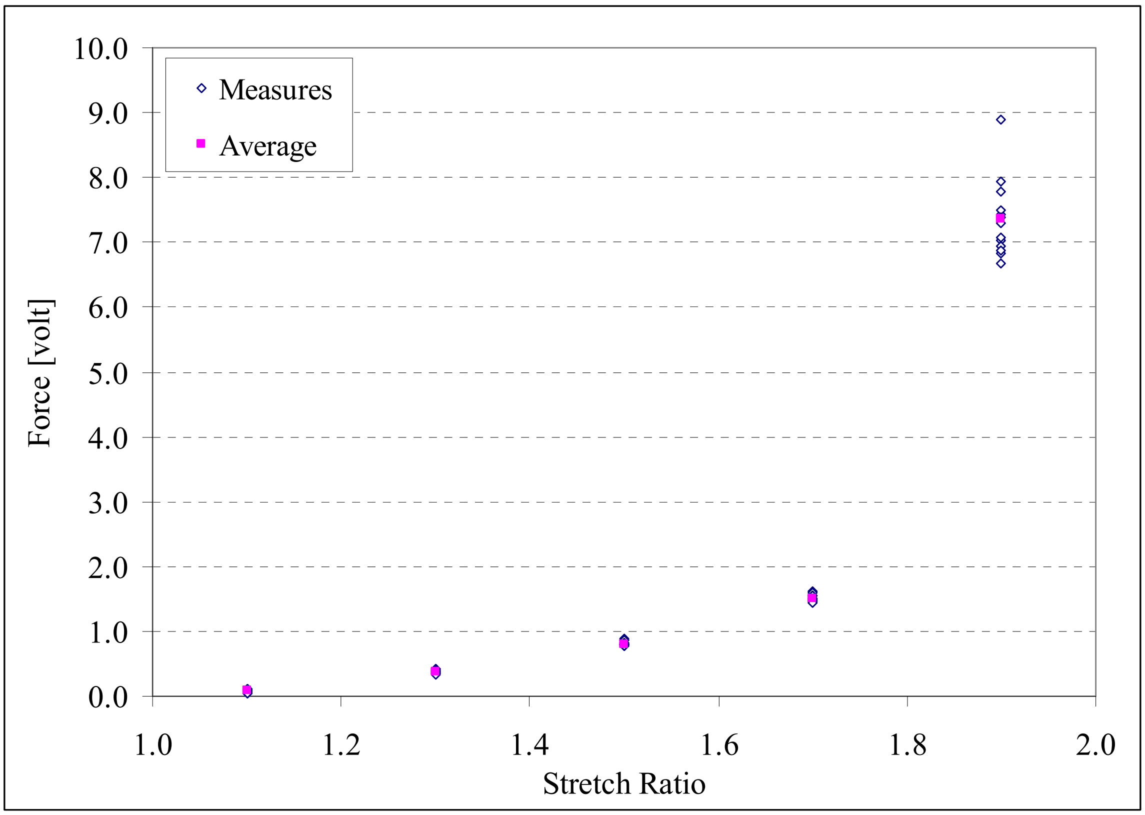

Fig.4.3. Forces obtained by force transducer for each corresponding stretch ratio of the tissue specimen have been grouped and averaged. It shows the inelastic and nonlinear behaviors.

Table 4.1
The width *a* and length *b* of the tissue, a reference weight *w*, a voltage output *p* for a fixed weight have been measured for 10 times.

No. of Measure	width[mm]	length[mm]	ref.weight[gram]	voltage output[volt]
1^{st}	2.4950	2.0210	0.3938	0.3084
2^{nd}	2.6220	2.1480	0.3940	0.3067
3^{rd}	2.7490	1.8940	0.3940	0.3058
4^{th}	2.5966	2.0464	0.3940	0.3061
5^{th}	2.4950	2.1480	0.3938	0.3085
6^{th}	2.6220	2.0210	0.3940	0.3054
7^{th}	2.6220	1.8940	0.3940	0.3063
8^{th}	2.5966	1.9194	0.3940	0.3046
9^{th}	2.7490	1.9194	0.3939	0.3042
10^{th}	2.6220	1.8940	0.3939	0.3054
Average	2.6169	1.9905	0.3939	0.3061
ΔM	0.0853	0.1015	0.0001	0.0014
F.U [%]	3.2611	5.0971	0.0214	0.4666
$M = M_{avg} \pm \Delta M$	$a = 2.6169 \pm 0.0853$	$b = 1.9905 \pm 0.1015$	$w = 0.3939 \pm 0.0001$	$p = 0.3061 \pm 0.0014$

Stretch Ratio 1.1: For the error analysis, we grouped the data according to the same stretch ratios. The table 4.2 shows the grouped data for the stretch ratio 1.1 for the tissue. It shows the times, measured forces by the force transducer and calculated deviation, stretch-rate, and stretch histories. The error analysis has been done on the data of the deviation D versus time T, rate-related stretch history function H_{t_2}, long-time stretch history function H_{t_1}. Although we has found that there is one more stretss relaxation spetrum that made the new stretch history function H_{t_0}, it hasn't been used as one of the independent variable for the multivarible linear regression analysis because there is high correlations between T and H_{t_0}. If there is high correlation between the independent variables, the problem of the multicollinearity will be appeared and it causes the inceasing of the standard error so that it lowers the goodness-of-fit. In the same way to the rubber experiments, H_{t_2} is the rate-related stretch history scanned by a history cut-off t_h = 10 sec, H_{t_1} is the long-time stretch history scanned by a history cut-off t_h = 400 sec, and H_{t_0} is the new stretch history scanned by a history cut-off t_h = 1000 sec.

Table 4.2
The tissue data corresponding to the stretch ratio 1.1. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the *Deviation* versus *Time*, H_{t_2} and H_{t_1} .

Time [sec]	λ	v_M [volt]	v_{avg} [volt]	*D* [volt]	$d\lambda/dt$ [/sec]	$H_{t_2}(t)$ t_h=10	$H_{t_1}(t)$ t_h=400	$H_{t_0}(t)$ t_h=1000
130.8	1.1	0.11438	0.08403	0.03035	-0.00481	1.13068	1.07313	1.02925
299.8		0.10406		0.02003	-0.00008	1.11956	1.21757	1.08703
622.5		0.10719		0.02316	0.00134	1.0992	1.42234	1.18387
700.4		0.11469		0.03066	-0.00004	1.1007	1.39949	1.1899
989.8		0.09469		0.01066	-0.00547	1.14034	1.39676	1.27413
1391.2		0.09844		0.01441	-0.00134	1.11561	1.47724	1.41393
1466.3		0.10625		0.02222	-0.00456	1.11815	1.43617	1.40516
1773.4		0.09156		0.00753	-0.00082	1.11279	1.42072	1.42336
2157.1		0.08812		0.00409	-0.00185	1.11982	1.53323	1.47978
2307.9		0.09063		0.00660	-0.00248	1.12653	1.51722	1.4774
2651.5		0.09406		0.01003	-0.00040	1.11603	1.44743	1.46874
3561.0		0.07531		-0.00872	0.00622	1.0806	1.55591	1.52316
3858.4		0.06844		-0.01559	-0.00102	1.12729	1.56106	1.52006
4401.7		0.06156		-0.02247	0.00213	1.11483	1.46314	1.51433
4858.6		0.04156		-0.04247	0.00468	1.10172	1.51438	1.48701
4998.9		0.04250		-0.04153	0.00472	1.10422	1.48922	1.48863
5584.5		0.03500		-0.04903	-0.00472	1.13489	1.54672	1.51059

Note that, we have used the stretch history function H_{t_2} obtained by using the stress relaxation spectrum t_2 corresponding to the early (or fast) exponential decay instead of using the stretch-rate calculated by simply dividing the change in stretch ratio by change in time spent testing. Those two independent variables are rate-related variables. However, the stretch-rate is not our control variable for the experiments. The stretch history function H_{t_1} is obtained by using the stress relaxation spectrum t_1 corresponding to the late (or slow) exponential decay. We, again, defined the H_{t_2} and H_{t_1} as rate-related stretch history function and long-time stretch history function, respectively.

The table 4.3 shows the average values and standard deviations of stretch ratios and forces. They have been used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.1 of the tissue specimen. They have also been used to calculate the uncertainty in resulting Cauchy stresses. Similarly to the results of rubber experiment, the higher stretch-rate we got the bigger deviation from the hyperelasticity we had. It's been revealed as true from our other experiments that the stretch-rate and the deviation from the hyperelasticity have positive relationship.

Table 4.3
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.1 of tissue. They have also been used to calculate the uncertainty in resulting Cauchy stresses.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\Delta\lambda$	Δv
1.10001	0.08403	0.00008	0.02572

To get the Cauchy stress for the stretch ratio 1.1, the equations (2.33) – (2.40) were employed as

$$\delta t_{\lambda} = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg}w_{avg}}{a_{avg}b_{avg}p_{avg}}\right|\delta\lambda = \left|\frac{(0.08403)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.00008) = 1.66072\times10^{-6}[g/mm^2] \tag{4.1}$$

$$\delta t_{v} = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg}w_{avg}}{a_{avg}b_{avg}p_{avg}}\right|\delta v = \left|\frac{(1.10001)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.02572) = 0.00699[\mathrm{g/mm^2}] \tag{4.2}$$

$$\delta t_w = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg} v_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta w = \left|\frac{(1.10001)(0.08403)}{(2.61690)(1.99050)(0.30610)}\right|(0.00010) \qquad (4.3)$$
$$= 5.7972\times10^{-6}[g/mm^2]$$

$$\delta t_a = \left|\frac{\partial t}{\partial a}\right|\delta a = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{(a_{avg})^2 b_{avg} p_{avg}}\right|\delta a = \left|\frac{-(1.10001)(0.08403)(0.39390)}{(2.61690)^2(1.99050)(0.30610)}\right|(0.08530) \qquad (4.4)$$
$$= 0.00074[\mathrm{g/mm}^2]$$

$$\delta t_b = \left|\frac{\partial t}{\partial b}\right|\delta b = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{a_{avg} (b_{avg})^2 p_{avg}}\right|\delta b = \left|\frac{-(1.10001)(0.08403)(0.39390)}{(2.61690)(1.99050)^2(0.30610)}\right|(0.10150) \qquad (4.5)$$
$$= 0.00116[\mathrm{g/mm}^2]$$

$$\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{a_{avg} b_{avg} (p_{avg})^2}\right|\delta p = \left|\frac{-(1.10001)(0.08403)(0.39390)}{(2.61690)(1.99050)(0.30610)^2}\right|(0.00140) \qquad (4.6)$$
$$= 0.00010[\mathrm{g/mm}^2]$$

Thus, the total uncertainty in t for the stretch ratio 1.1 is

$$\delta t = \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial a}\delta a\right)^2 + \left(\frac{\partial t}{\partial b}\delta b\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2}$$
$$= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_a)^2 + (\delta t_b)^2 + (\delta t_p)^2} \qquad (4.7)$$
$$= 0.00139[\mathrm{g/mm}^2]$$

Since the actuators (CMA-25CCCL, Newport) that we have used provide quite precise motions (Resolution = 0.048828 μm, Speed = 50 – 400 μm/sec), the uncertainty in the Cauchy stress t due to the $\delta\lambda$ (average deviation or standard deviation of λ) has the lowest uncertainty among the other uncertainties. The uncertainty in the Cauchy stress t due to the δw is also quite low. Since the uncertainty in the Cauchy stress t due to the δv which is standard deviation of the forces has the highest uncertainty of the other uncertainties, we can tell that there are huge amount of inelastic behavior in the tissue at a stretch level 1.1. Although the uncertainty in the Cauchy stress t due to the δa and δb seem to be relatively high, it can be reduced as much as possible to the maximum resolution range that the measuring devices would have. The surface roughness of the specimen also affects the increasing of uncertainty. Now, if we look at the fractional uncertainties, we have

$$\begin{aligned}
\lambda &= \lambda_{avg} \pm \delta\lambda = 1.10001 \pm 0.00008, & \frac{\delta\lambda}{\lambda_{avg}} \times 100 &= 0.007\ \% \\
v &= v_{avg} \pm \delta v = 0.08403 \pm 0.02572, & \frac{\delta v}{v_{avg}} \times 100 &= 30.608\ \% \\
w &= w_{avg} \pm \delta w = 0.39390 \pm 0.00010, & \frac{\delta w}{w_{avg}} \times 100 &= 0.025\ \% \\
a &= a_{avg} \pm \delta a = 2.61690 \pm 0.08530, & \frac{\delta a}{a_{avg}} \times 100 &= 3.260\ \% \\
b &= b_{avg} \pm \delta b = 1.99050 \pm 0.10150, & \frac{\delta b}{b_{avg}} \times 100 &= 5.099\ \% \\
p &= p_{avg} \pm \delta p = 0.30610 \pm 0.00140, & \frac{\delta p}{p_{avg}} \times 100 &= 0.457\ \%
\end{aligned} \tag{4.8}$$

The Cauchy stress t for the stretch ratio 1.1 of the tissue, using equation (2.25) for the tissue, we got

$$t_{1.1} = t_{avg} \pm \delta t = 0.02284 \pm 0.00139, \quad \frac{\delta t}{t_{avg}} \times 100 = 6.069\ \% \tag{4.9}$$

Summarized fractional uncertainties are shown in the table 4.4.

Table 4.4
Fractional uncertainties for the measured data a, b, w, and p, and obtained data λ and v by the motion controller and the Cauchy stress t for the stretch level 1.1 of the tissue.

Data	$\lambda_{1.1}$	v[volt]	w[gram]	a[mm]	b[mm]	p[volt]	t[g/mm^2]
Avg	1.10001	0.08403	0.39390	2.61690	1.99050	0.30610	0.02284
S.D	0.00008	0.02572	0.00010	0.08530	0.10150	0.00140	0.00139
F.U [%]	0.007	30.608	0.025	3.260	5.099	0.457	6.069

*S.D = standard deviation, F.U = Avg/S.D

For calculating the absolute Cauchy stress, the uncertainty δt_v in t which is due to the δv alone is approximately 9 times bigger than the uncertainty δt_a in t which is due to the δa alone and 6 times bigger than the uncertainty δt_b in t which is due to the δb alone. The figure 4.4 shows the uncertainties in t due to the $\delta\lambda$, δv, δw, δa, δb and δp. Since the uncertainties δt_w δt_a δt_b , and δt_p are due to the measuring process and the resolution of the measuring devices, those can be reduced to the insignificant level of uncertainties as long as we use the extremely accurate devices. In addition, they can be assumed to be ignorable because the fractional uncertainties corresponding to them are never changed for the whole stretch ranges. This means if we initially measured them very accurately by using extremely accurate measuring devices, they would remain as low amount of uncertainties. The error analysis, therefore, has been devoted to and focused on the uncertainty due to the inelastic behavior of the tissue.

The fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, a, b, and p are shown in the figure 4.5. Unlikely to the rubber case, the reason that the ratio of the absolute uncertainty to the fractional uncertainty is same is that the result of the $\delta t_v/\delta t_a$ is same as $(\Delta v/v_{avg})/\ (\Delta a/a_{avg})$. It can be easily shown as following.

$$\frac{\delta t_v}{\delta t_a} = \frac{\left|\dfrac{\lambda_{avg} w_{avg}}{a_{avg} b_{avg} p_{avg}}\right| \delta v}{\left|\dfrac{-\lambda_{avg} v_{avg} w_{avg}}{(a_{avg})^2 b_{avg} p_{avg}}\right| \delta a} = \frac{a_{avg} \cdot \delta v}{\delta a \cdot v_{avg}} = \frac{\delta v / v_{avg}}{\delta a / a_{avg}} \tag{4.10}$$

Likewise, it can be understood that $\delta t_v/\delta t_b$ is same as $(\Delta v/v_{avg})/\ (\Delta b/b_{avg})$.

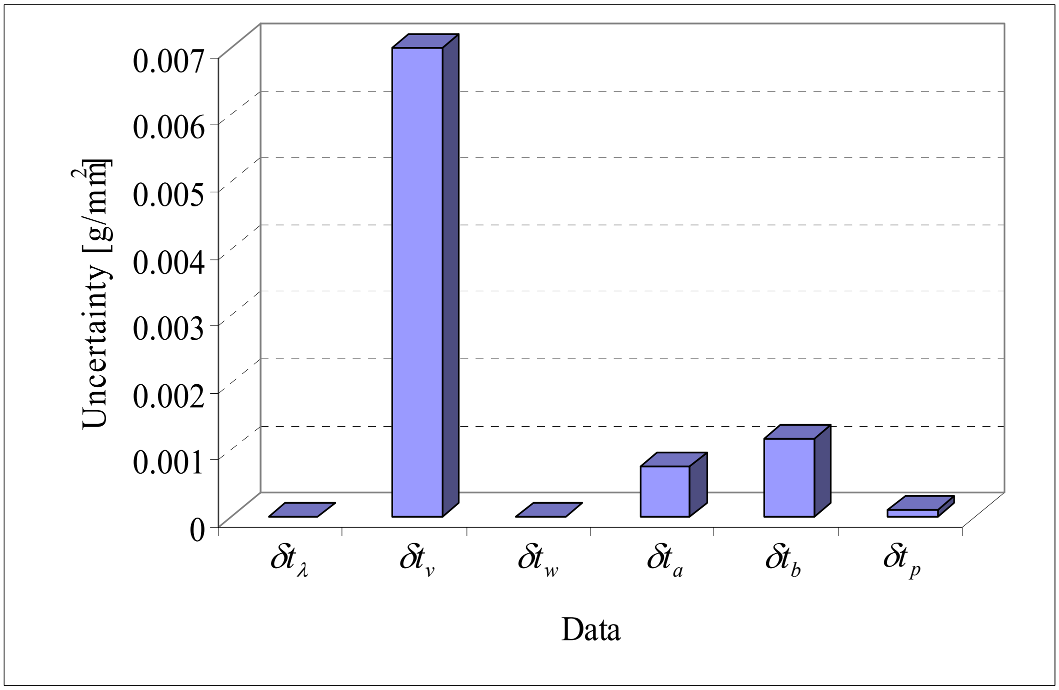

Fig.4.4. Uncertainties in t due to the $\delta\lambda$, δv, δw, δa, δb and δp for the stretch ratio 1.1 of the tissue specimen.

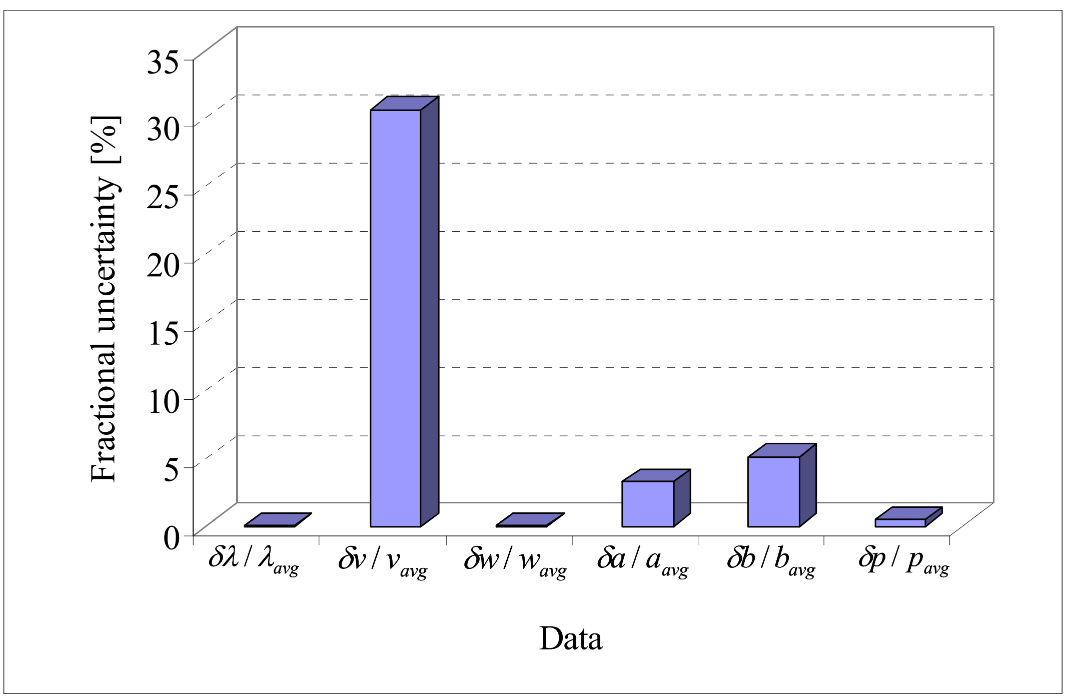

Fig.4.5. Fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, a, b, and p for the stretch ratio 1.1 of the tissue specimen.

Stretch Ratio 1.3: The tissue data corresponding to the stretch ratio 1.3 is shown in the table 4.5. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the D versus T, H_{t_2} and H_{t_1}. Again, the H_{t_2} is a rate-related stretch history function scanned by a history cut-off t_h = 10 sec, and H_{t_1} is the long-time stretch history function scanned by a history cut-off t_h = 400 sec.

Table 4.5
The tissue data corresponding to the stretch ratio 1.3. It shows the times and stretch ratios obtained by the motion controller, forces obtained by the force transducer and calculated deviation, stretch-rate, and stretch histories.

Time	λ	v_M	v_{avg}	D	$d\lambda/dt$	$H_{t_2}(t)$	$H_{t_1}(t)$	$H_{t_0}(t)$
[sec]		[volt]	[volt]	[volt]	[/sec]	t_h=10	t_h=400	t_h=1000
548.6	1.3	0.40250	0.37761	0.02489	-0.00083	1.31439	1.4341	1.1742
852.0		0.41969		0.04208	-0.00067	1.30578	1.35408	1.22832
1086.2		0.41531		0.03770	-0.00335	1.31344	1.30078	1.28872
2987.1		0.36937		-0.00824	0.00130	1.29646	1.49394	1.50735
3278.9		0.36719		-0.01042	-0.00417	1.32786	1.55918	1.50382
3605.5		0.38906		0.01145	0.00074	1.28483	1.53925	1.50908
4160.5		0.36687		-0.01074	-0.00248	1.31652	1.48771	1.52161
4626.5		0.36531		-0.01230	-0.00071	1.31578	1.51952	1.51478
4700.6		0.35844		-0.01917	0.00535	1.27817	1.49356	1.50886
5171.8		0.35656		-0.02105	0.00496	1.28458	1.45993	1.4892
5439.5		0.34344		-0.03417	-0.00429	1.33973	1.55221	1.51785

Again for the multivariable linear regression analysis, we have used the stretch history function H_{t_2} obtained by using the stress relaxation spectrum t_2 corresponding to the early (or fast) exponential decay instead of using the stretch-rate $d\lambda/dt$ calculated by simply dividing the change in stretch ratio by change in time spent testing because the stretch-rate is not our control variable for the experiments. The table 4.6 shows the average values and standard deviations of stretch ratios and forces. They were used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.3 of the tissue. They have also been used to calculate the uncertainty in resulting Cauchy stresses.

Table 4.6
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.3 of the tissue. They have also been used to calculate the uncertainty in resulting Cauchy stresses.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\Delta\lambda$	Δv
1.30000	0.37761	0.00005	0.02523

To get the Cauchy stress for the stretch ratio 1.3, we again have used the equations (2.33) – (2.40) as

$$\delta t_{\lambda} = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg} w_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta\lambda = \left|\frac{(0.37761)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.00005)$$
$$= 4.6643 \times 10^{-6} [g/mm^2] \tag{4.11}$$

$$\delta t_{v} = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg} w_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta v = \left|\frac{(1.30000)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.02523)$$
$$= 0.00810 [\mathrm{g/mm}^2] \tag{4.12}$$

$$\delta t_{w} = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg} v_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta w = \left|\frac{(1.30000)(0.37761)}{(2.61690)(1.99050)(0.30610)}\right|(0.00010)$$
$$= 3.07875 \times 10^{-5} [g/mm^2] \tag{4.13}$$

$$\delta t_a = \left|\frac{\partial t}{\partial a}\right|\delta a = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{(a_{avg})^2 b_{avg} p_{avg}}\right|\delta a = \left|\frac{-(1.30000)(0.37761)(0.39390)}{(2.61690)^2(1.99050)(0.30610)}\right|(0.08530) \qquad (4.14)$$
$$= 0.00395[\mathrm{g/mm^2}]$$

$$\delta t_b = \left|\frac{\partial t}{\partial b}\right|\delta b = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{a_{avg} (b_{avg})^2 p_{avg}}\right|\delta b = \left|\frac{-(1.30000)(0.37761)(0.39390)}{(2.61690)(1.99050)^2(0.30610)}\right|(0.10150) \qquad (4.15)$$
$$= 0.00618[\mathrm{g/mm^2}]$$

$$\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{a_{avg} b_{avg} (p_{avg})^2}\right|\delta p = \left|\frac{-(1.30000)(0.37761)(0.39390)}{(2.61690)(1.99050)(0.30610)^2}\right|(0.00140)$$
$$= 0.00055[\mathrm{g/mm^2}]$$

(4.16)

Thus, the total uncertainty in t for the stretch ratio 1.3 is

$$\begin{aligned}\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial a}\delta a\right)^2 + \left(\frac{\partial t}{\partial b}\delta b\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\ &= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_a)^2 + (\delta t_b)^2 + (\delta t_p)^2} \\ &= 0.00736[\mathrm{g/mm^2}]\end{aligned} \qquad (4.17)$$

The uncertainty in the Cauchy stress t due to the $\delta\lambda$ (average deviation or standard deviation of λ) has again the lowest uncertainty among the other uncertainties by the same reason to the case of stretch ratio 1.1. The uncertainty in the Cauchy stress t due to the δw is also quite low. The uncertainty δt_v due to the inelastic behavior is approximately 2 times bigger than the uncertainty δt_a and 1.3 times bigger than the uncertainty δt_b. The absolute uncertainties in t due to the $\delta\lambda$, δv, δw, δa, δb and δp are shown in the figure 4.6. The uncertainty in the Cauchy stress t due to the δv which is standard deviation of forces can not be reduced no matter what to do. For the fractional uncertainties for each variable for the stretch ratio 1.3, we have

$$
\begin{aligned}
\lambda &= \lambda_{avg} \pm \delta\lambda = 1.30000 \pm 0.00005, && \frac{\delta\lambda}{\lambda_{avg}} \times 100 = 0.004\ \% \\
v &= v_{avg} \pm \delta v = 0.37761 \pm 0.02523, && \frac{\delta v}{v_{avg}} \times 100 = 6.681\ \% \\
w &= w_{avg} \pm \delta w = 0.39390 \pm 0.00010, && \frac{\delta w}{w_{avg}} \times 100 = 0.025\ \% \\
a &= a_{avg} \pm \delta a = 2.61690 \pm 0.08530, && \frac{\delta a}{a_{avg}} \times 100 = 3.260\ \% \\
b &= b_{avg} \pm \delta b = 1.99050 \pm 0.10150, && \frac{\delta b}{b_{avg}} \times 100 = 5.099\ \% \\
p &= p_{avg} \pm \delta p = 0.30610 \pm 0.00140, && \frac{\delta p}{p_{avg}} \times 100 = 0.457\ \%
\end{aligned}
\tag{4.18}
$$

The Cauchy stress t for the stretch ratio 1.3, using equation (2.25) for the tissue, we get

$$
t = t_{avg} \pm \delta t = 0.12127 \pm 0.00736, \quad \frac{\delta t}{t_{avg}} \times 100 = 6.069\ \% \tag{4.19}
$$

Summarized fractional uncertainties are shown in the table 4.7.

Table 4.7
Fractional uncertainties for the measured data a, b, w, and p, and obtained data λ and v by the motion controller and the Cauchy stress t for the stretch level 1.3.

Data	$\lambda_{1.3}$	v[volt]	w[gram]	a[mm]	b[mm]	p[volt]	t[g/mm^2]
Avg	1.30000	0.37761	0.39390	2.61690	1.99050	0.30610	0.12127
S.D	0.00005	0.02523	0.00010	0.08530	0.10150	0.00140	0.00736
F.U [%]	0.004	6.681	0.025	3.260	5.099	0.457	6.069

The fractional uncertainty due to the force is still the highest and the fractional uncertainties measured by manually are constant. The fractional uncertainties in t due to the obtained data λ and v through the motion controller and measured data w, a, b and p is shown in the figure 4.7.

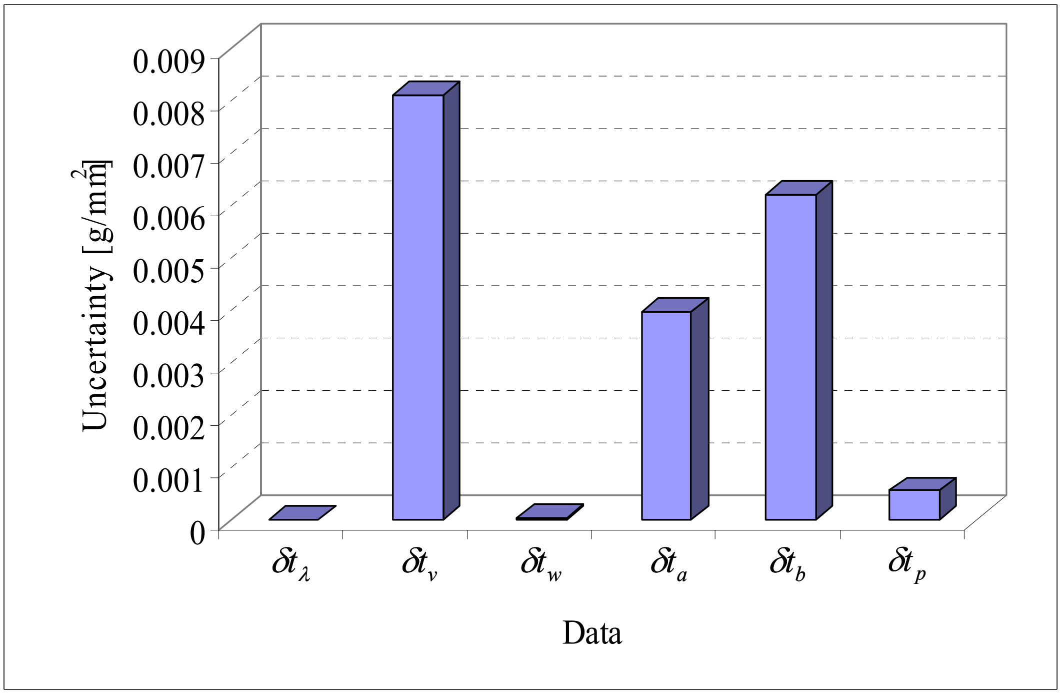

Fig.4.6. Uncertainties in t due to the $\delta\lambda$, δv, δw, δa, δb and δp for the stretch ratio 1.3 of the tissue specimen.

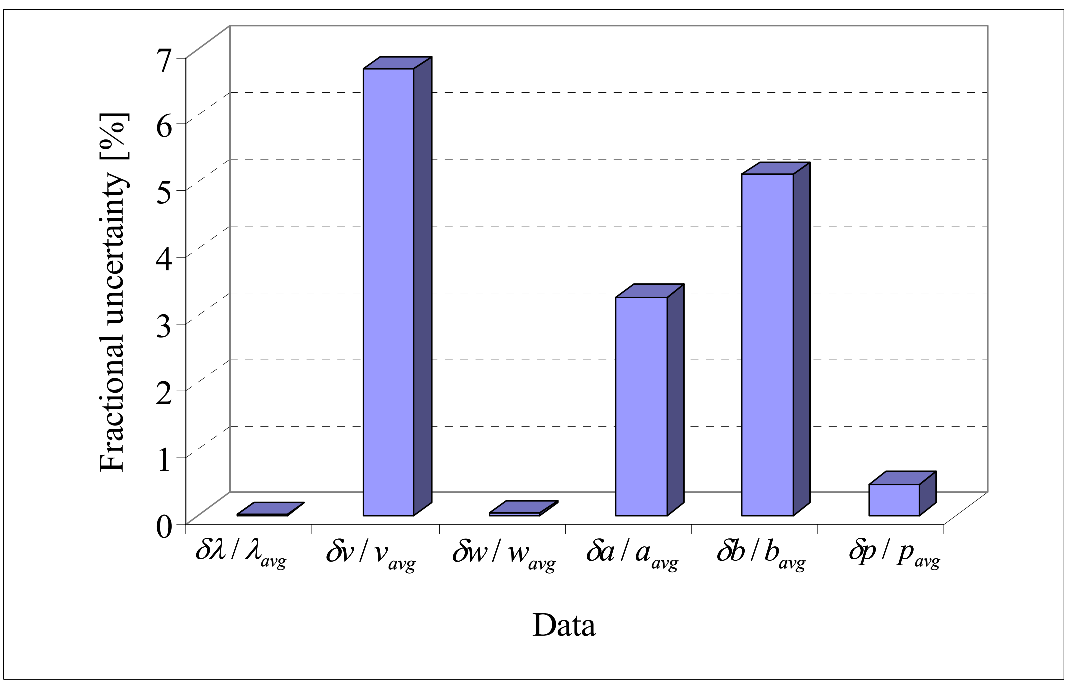

Fig.4.7. Fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, a, b and p for the stretch ratio 1.3 of the tissue specimen.

Stretch Ratio 1.5: The tissue data corresponding to the stretch ratio 1.5 is shown in the table 4.8. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the D versus T, H_{t_2} and H_{t_1} where H_{t_2} is a rate-related stretch history function scanned by a history cut-off t_h = 10 sec, and H_{t_1} is the long-time stretch history function scanned by a history cut-off t_h = 400 sec.

Table 4.8
The tissue data corresponding to the stretch ratio 1.5. It shows the times and stretch ratios obtained by the motion controller, forces obtained by the force transducer and calculated deviation, stretch-rate, and stretch histories.

Time [sec]	λ	v_M [volt]	v_{avg} [volt]	D [volt]	$d\lambda/dt$ [/sec]	$H_{t_2}(t)$ t_h=10	$H_{t_1}(t)$ t_h=400	$H_{t_0}(t)$ t_h=1000
68.1	1.5	0.89219	0.80873	0.08346	0.00634	1.46415	1.04359	1.01744
398.5		0.86656		0.05783	0.00386	1.47416	1.26212	1.10485
758.2		0.86844		0.05971	0.00552	1.46095	1.39023	1.20151
1317.1		0.80781		-0.00092	-0.00031	1.50652	1.49416	1.39912
1617.8		0.81937		0.01064	-0.00314	1.51439	1.48207	1.4188
1698.3		0.81937		0.01064	-0.00059	1.50438	1.49456	1.41631
1999.5		0.79906		-0.00967	0.00477	1.48339	1.50564	1.43855
2445.7		0.79656		-0.01217	-0.00496	1.53378	1.52627	1.50022
2878.6		0.76656		-0.04217	-0.00634	1.54106	1.5475	1.52188
3075.0		0.79562		-0.01311	0.00023	1.49976	1.48492	1.50369
3138.0		0.79156		-0.01717	-0.00130	1.50624	1.48684	1.51689
3932.6		0.80437		-0.00436	-0.00311	1.50124	1.52586	1.51113
4049.0		0.79125		-0.01748	-0.00213	1.51008	1.44632	1.52244
4082.6		0.79000		-0.01873	0.00260	1.48832	1.44999	1.53089
4245.6		0.78844		-0.02029	-0.00497	1.51756	1.4993	1.49661
5223.9		0.78094		-0.02779	0.00048	1.49126	1.47043	1.47368
5515.4		0.77031		-0.03842	0.00075	1.49273	1.53433	1.51271

For the multivariable linear regression analysis, we have used the stretch history function H_{t_2} obtained by using the stress relaxation spectrum t_2 corresponding to the early (or fast) exponential decay instead of using the stretch-rate $d\lambda/dt$ calculated by simply dividing the change in stretch ratio by change in time spent testing because the stretch-rate is not our control variable for the experiments. The table 4.9 shows the average values and standard deviations of stretch ratios and forces. They were used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.5 of the tissue specimen. They have also been used to calculate the uncertainty in resulting Cauchy stresses.

Table 4.9
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.5 of the tissue. They have also been used to calculate the uncertainty in resulting Cauchy stresses *t*.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\Delta\lambda$	Δv
1.50000	0.80873	0.00005	0.03529

Now, we have uncertainties δt_λ, δt_v, δt_w, δt_a, δt_b and δt_p as

$$\delta t_\lambda = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg} w_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta\lambda = \left|\frac{(0.80873)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.00005) \tag{4.20}$$
$$= 9.99\times10^{-6}[g/mm^2]$$

$$\delta t_v = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg} w_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta v = \left|\frac{(1.50000)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.03529) \tag{4.21}$$
$$= 0.01308[\mathrm{g/mm}^2]$$

$$\delta t_w = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg} v_{avg}}{a_{avg} b_{avg} p_{avg}}\right|\delta w = \left|\frac{(1.30000)(0.80873)}{(2.61690)(1.99050)(0.30610)}\right|(0.00010) \tag{4.22}$$
$$= 7.61\times10^{-5}[g/mm^2]$$

$$\delta t_a = \left|\frac{\partial t}{\partial a}\right|\delta a = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{(a_{avg})^2 b_{avg} p_{avg}}\right|\delta a = \left|\frac{-(1.50000)(0.80873)(0.39390)}{(2.61690)^2(1.99050)(0.30610)}\right|(0.08530) \tag{4.23}$$
$$= 0.00977[\mathrm{g/mm^2}]$$

$$\delta t_b = \left|\frac{\partial t}{\partial b}\right|\delta b = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{a_{avg} (b_{avg})^2 p_{avg}}\right|\delta b = \left|\frac{-(1.50000)(0.80873)(0.39390)}{(2.61690)(1.99050)^2(0.30610)}\right|(0.10150) \tag{4.24}$$
$$= 0.01528[\mathrm{g/mm^2}]$$

$$\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{a_{avg} b_{avg} (p_{avg})^2}\right|\delta p = \left|\frac{-(1.50000)(0.80873)(0.39390)}{(2.61690)(1.99050)(0.30610)^2}\right|(0.00140)$$
$$= 0.00137[\mathrm{g/mm^2}]$$

(4.25)

Thus, the total uncertainty in t for the stretch ratio 1.5 is

$$\begin{aligned}\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial a}\delta a\right)^2 + \left(\frac{\partial t}{\partial b}\delta b\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\ &= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_a)^2 + (\delta t_b)^2 + (\delta t_p)^2} \\ &= 0.01819[\mathrm{g/mm^2}]\end{aligned} \tag{4.26}$$

The uncertainties in t due to the $\delta\lambda$, δv, δw, δa, δb and δp for the stretch ratio 1.5 of the tissue is graphically shown in the figure 4.8. Although the uncertainty in the Cauchy stress t due to δb is now higher than the uncertainty due to the δv, the uncertainty due to the δb can be easily reduced as much as possible to the maximum resolution of the range of the measuring device. Again, the uncertainty causing the Error-of-definition has nothing to do with those measuring variables. Thus, if we assume the uncertainties that come from those measurements are ignorable, the only factor that causes the deviation from the hyperelasticity is the inelastic behavior of the tissue shown in the data of force v. The fractional uncertainties have been obtained as

$$
\begin{aligned}
&\lambda = \lambda_{avg} \pm \delta\lambda = 1.50000 \pm 0.00005, && \frac{\delta\lambda}{\lambda_{avg}} \times 100 = 0.003\ \% \\
&v = v_{avg} \pm \delta v = 0.80873 \pm 0.03529, && \frac{\delta v}{v_{avg}} \times 100 = 4.364\ \% \\
&w = w_{avg} \pm \delta w = 0.39390 \pm 0.00010, && \frac{\delta w}{w_{avg}} \times 100 = 0.025\ \% \\
&a = a_{avg} \pm \delta a = 2.61690 \pm 0.08530, && \frac{\delta a}{a_{avg}} \times 100 = 3.260\ \% \\
&b = b_{avg} \pm \delta b = 1.99050 \pm 0.10150, && \frac{\delta b}{b_{avg}} \times 100 = 5.099\ \% \\
&p = p_{avg} \pm \delta p = 0.30610 \pm 0.00140, && \frac{\delta p}{p_{avg}} \times 100 = 0.457\ \%
\end{aligned}
\tag{4.27}
$$

The Cauchy stress t for the stretch ratio 1.5, using equation (2.25) for the tissue, we got

$$
t = t_{avg} \pm \delta t = 0.29969 \pm 0.01819, \quad \frac{\delta t}{t_{avg}} \times 100 = 6.069\ \% \tag{4.28}
$$

The results of the fractional uncertainties for the stretch ratio 1.5 of the tissue is shown in the figure 4.9. The table 4.10 shows the summary of the averages, standard deviations, and fractional uncertainties of the variables we have for stretch ratio 1.5.

Table 4.10
Fractional uncertainties for the measured data w, a, b, and p and obtained data λ and v by the motion controller and the Cauchy stress t for the stretch level 1.5 of the tissue.

Data	$\lambda_{1.5}$	v[volt]	w[gram]	a[mm]	b[mm]	p[volt]	t[g/mm^2]
Avg	1.50000	0.80873	0.39390	2.61690	1.99050	0.30610	0.29969
S.D	0.00005	0.03529	0.00010	0.08530	0.10150	0.00140	0.01819
F.U [%]	0.003	4.364	0.025	3.260	5.099	0.457	6.069

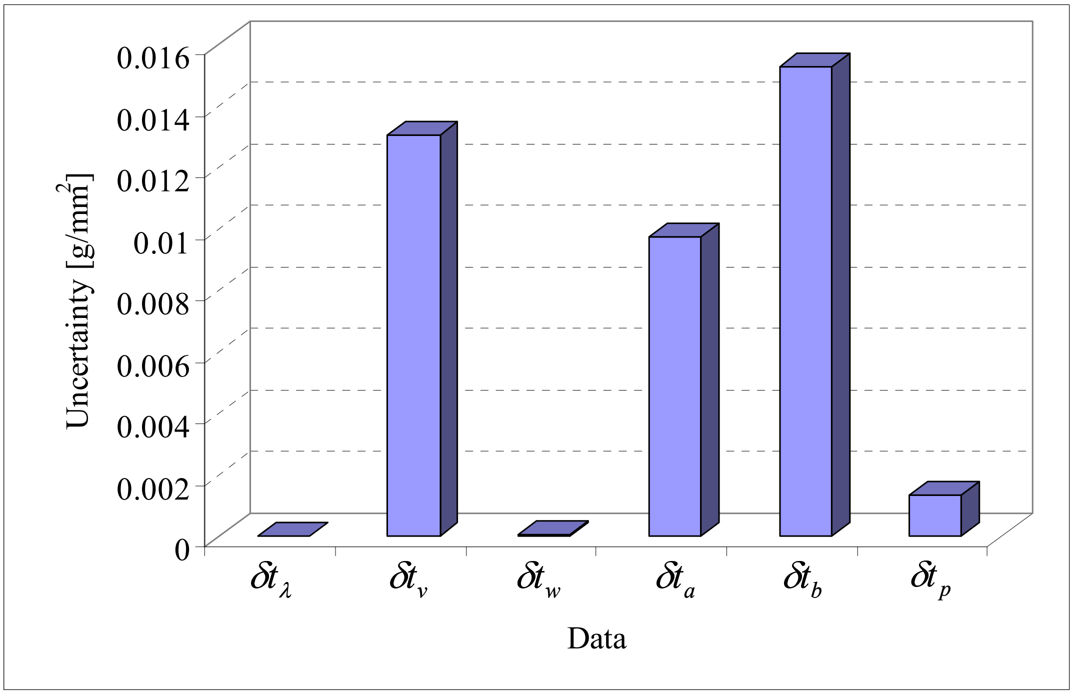

Fig.4.8.Uncertainties in t due to the $\delta\lambda$, δv, δw, δa, δb and δp for the stretch ratio 1.5 of the tissue specimen.

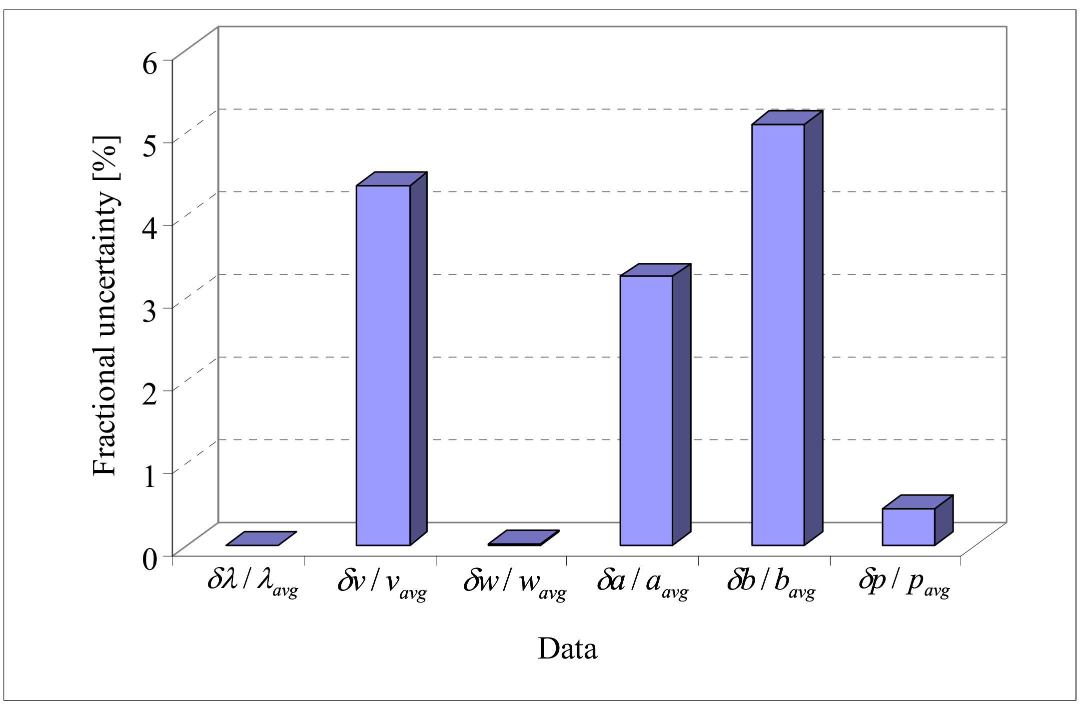

Fig.4.9. Fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, a, b and p for the stretch ratio of the tissue specimen.

Stretch Ratio 1.7: The tissue data corresponding to the stretch ratio 1.7 is shown in the table 4.11. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the D versus T, H_{t_2} and H_{t_1} where H_{t_2} is a rate-related stretch history function scanned by a history cut-off t_h = 10 sec, and H_{t_1} is the long-time stretch history function scanned by a history cut-off t_h = 400 sec. The table 4.12 shows the average values and standard deviations of stretch ratios and forces. They were used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.7 of the tissue. They have also been used to calculate the uncertainty in resulting Cauchy stresses t.

Table 4.11
The tissue data corresponding to the stretch ratio 1.7. It shows the times and stretch ratios obtained by the motion controller, forces obtained by the force transducer and calculated deviation, stretch-rate, and stretch histories.

Time	□	v_M	v_{avg}	D	$d\lambda/dt$	$H_{t_2}(t)$	$H_{t_1}(t)$	$H_{t_0}(t)$
[sec]		[volt]	[volt]	[volt]	[/sec]	t_h=10	t_h=400	t_h=1000
910.9		1.62344		0.11019	0.00083	1.66228	1.39245	1.24823
1240.7		1.50469		-0.00856	0.00280	1.68818	1.44272	1.36554
1529.8		1.59719		0.08394	0.00602	1.65074	1.41311	1.40768
1923.4		1.48406		-0.02919	0.00299	1.68751	1.44299	1.42215
2082.0		1.49719		-0.01606	-0.00496	1.71875	1.56733	1.47176
2232.2		1.55531		0.04206	0.00284	1.66541	1.51749	1.47697
2740.2	1.7	1.55938	1.51325	0.04613	0.00264	1.66506	1.45485	1.46351
3219.2		1.48469		-0.02856	-0.00591	1.72022	1.53906	1.51388
3707.4		1.50906		-0.00419	0.00126	1.67526	1.52407	1.51703
4474.6		1.48344		-0.02981	-0.00331	1.69728	1.48998	1.51296
4551.8		1.46656		-0.04669	-0.00023	1.70312	1.52069	1.52312
4924.9		1.46312		-0.05013	0.00197	1.6724	1.46919	1.47641
5297.4		1.44406		-0.06919	-0.00472	1.71694	1.51066	1.49391

Table 4.12
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.7 of the tissue. They have also been used to calculate the uncertainty in resulting Cauchy stresses *t*.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\Delta\lambda$	Δv
1.70002	1.51325	0.00003	0.05437

The uncertainties δt_λ, δt_v, δt_w, δt_a, δt_b and δt_p for the stretch ratio 1.7 of the tissue have been obtained as

$$\delta t_\lambda = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg}w_{avg}}{a_{avg}b_{avg}p_{avg}}\right|\delta\lambda = \left|\frac{(1.51325)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.00003) \tag{4.29}$$
$$= 1.12\times 10^{-5}[g/mm^2]$$

$$\delta t_v = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg}w_{avg}}{a_{avg}b_{avg}p_{avg}}\right|\delta v = \left|\frac{(1.70002)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.05437) \tag{4.30}$$
$$= 0.02283[\mathrm{g/mm^2}]$$

$$\delta t_w = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg}v_{avg}}{a_{avg}b_{avg}p_{avg}}\right|\delta w = \left|\frac{(1.70002)(1.51325)}{(2.61690)(1.99050)(0.30610)}\right|(0.00010) \tag{4.31}$$
$$= 0.00016[g/mm^2]$$

$$\delta t_a = \left|\frac{\partial t}{\partial a}\right|\delta a = \left|\frac{-\lambda_{avg}v_{avg}w_{avg}}{(a_{avg})^2 b_{avg}p_{avg}}\right|\delta a = \left|\frac{-(1.70002)(1.51325)(0.39390)}{(2.61690)^2(1.99050)(0.30610)}\right|(0.08530) \tag{4.32}$$
$$= 0.02072[\mathrm{g/mm^2}]$$

$$\delta t_b = \left|\frac{\partial t}{\partial b}\right|\delta b = \left|\frac{-\lambda_{avg}v_{avg}w_{avg}}{a_{avg}(b_{avg})^2 p_{avg}}\right|\delta b = \left|\frac{-(1.70002)(1.51325)(0.39390)}{(2.61690)(1.99050)^2(0.30610)}\right|(0.10150) \tag{4.33}$$
$$= 0.03241[\mathrm{g/mm^2}]$$

$$\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{a_{avg} b_{avg} \left(p_{avg}\right)^2}\right|\delta p = \left|\frac{-(1.70002)(1.51325)(0.39390)}{(2.61690)(1.99050)(0.30610)^2}\right|(0.00140)$$
$$= 0.00291[\mathrm{g/mm^2}] \tag{4.34}$$

Thus, the total uncertainty in t for the stretch ratio 1.7 is

$$\begin{aligned}\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial a}\delta a\right)^2 + \left(\frac{\partial t}{\partial b}\delta b\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\ &= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_a)^2 + (\delta t_b)^2 + (\delta t_p)^2} \\ &= 0.03857[\mathrm{g/mm^2}]\end{aligned} \tag{4.35}$$

, and the fractional uncertainties are

$$\begin{aligned}
&\lambda = \lambda_{avg} \pm \delta\lambda = 1.70002 \pm 0.00003, && \frac{\delta\lambda}{\lambda_{avg}} \times 100 = 0.002\ \% \\
&v = v_{avg} \pm \delta v = 1.51325 \pm 0.05437, && \frac{\delta v}{v_{avg}} \times 100 = 3.593\ \% \\
&w = w_{avg} \pm \delta w = 0.39390 \pm 0.00010, && \frac{\delta w}{w_{avg}} \times 100 = 0.025\ \% \\
&a = a_{avg} \pm \delta a = 2.61690 \pm 0.08530, && \frac{\delta a}{a_{avg}} \times 100 = 3.260\ \% \\
&b = b_{avg} \pm \delta b = 1.99050 \pm 0.10150, && \frac{\delta b}{b_{avg}} \times 100 = 5.099\ \% \\
&p = p_{avg} \pm \delta p = 0.30610 \pm 0.00140, && \frac{\delta p}{p_{avg}} \times 100 = 0.457\ \% \\
&t = t_{avg} \pm \delta t = 0.63553 \pm 0.03857, && \frac{\delta t}{t_{avg}} \times 100 = 6.069\ \%
\end{aligned} \tag{4.36}$$

The absolute uncertainties in in t due to the $\delta\lambda$, δv, δw, δa, δb, and δp for the stretch ratio 1.7 and the fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, a, b, and p for the stretch ratio 1.7 are shown in the figure 4.10 and figure 4.11, respectively. The table 4.13 shows the summary of the averages, standard deviations, and fractional uncertainties of the variables for the stretch ratio 1.7 of the tissue.

Table 4.13
Fractional uncertainties for the measured data *w*, *a*, *b*, and *p* and obtained data λ and *v* by the motion controller and the Cauchy stress *t* for the stretch level 1.7 of the tissue.

Data	$\lambda_{1.7}$	*v*[volt]	*w*[gram]	*a*[mm]	*b*[mm]	*p*[volt]	*t*[g/mm^2]
Avg	1.70002	1.51325	0.39390	2.61690	1.99050	0.30610	0.63553
S.D	0.00003	0.05437	0.00010	0.08530	0.10150	0.00140	0.03857
F.U [%]	0.002	3.593	0.025	3.260	5.099	0.457	6.069

Although the fractional uncertainties due to the manually measuring process were constant, the fractional uncertainty due to the force *v* continuously decreased even though we got the data for the same displacements (stretch ratios). This tells that there is inelastic behavior on the tissue. Again, we don't need to consider the uncertainties due to the manual measurements such as *w*, *a*, *b*, and *p*.

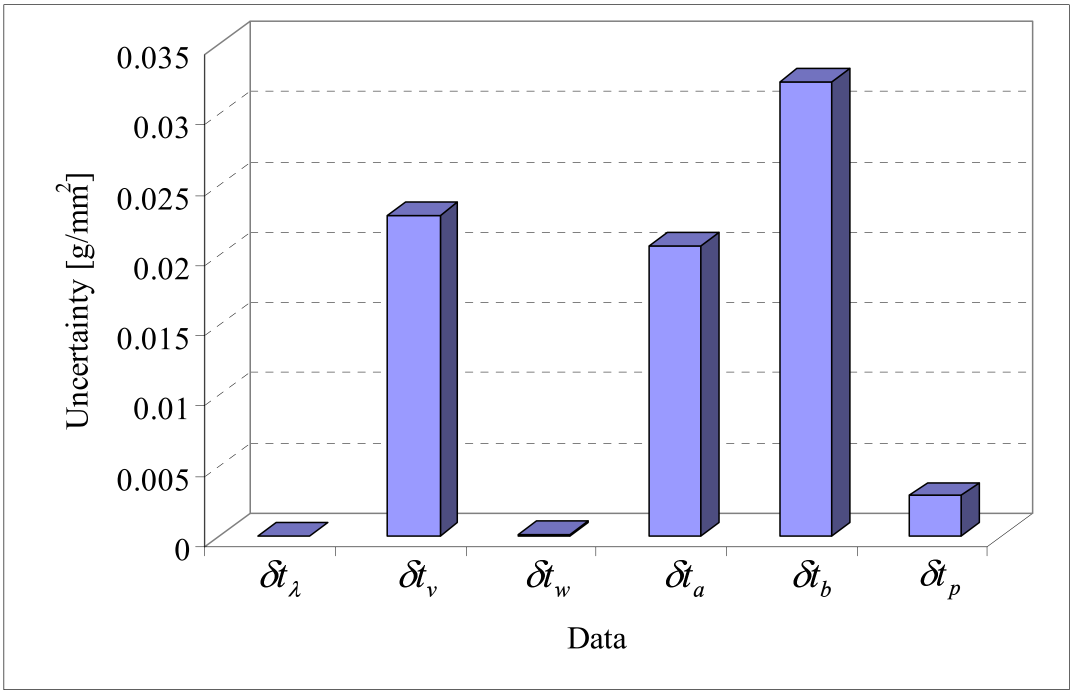

Fig.4.10.Uncertainties in t due to the $\delta\lambda$, δv, δw, δa, δb and δp for the stretch ratio 1.7 of the tissue specimen.

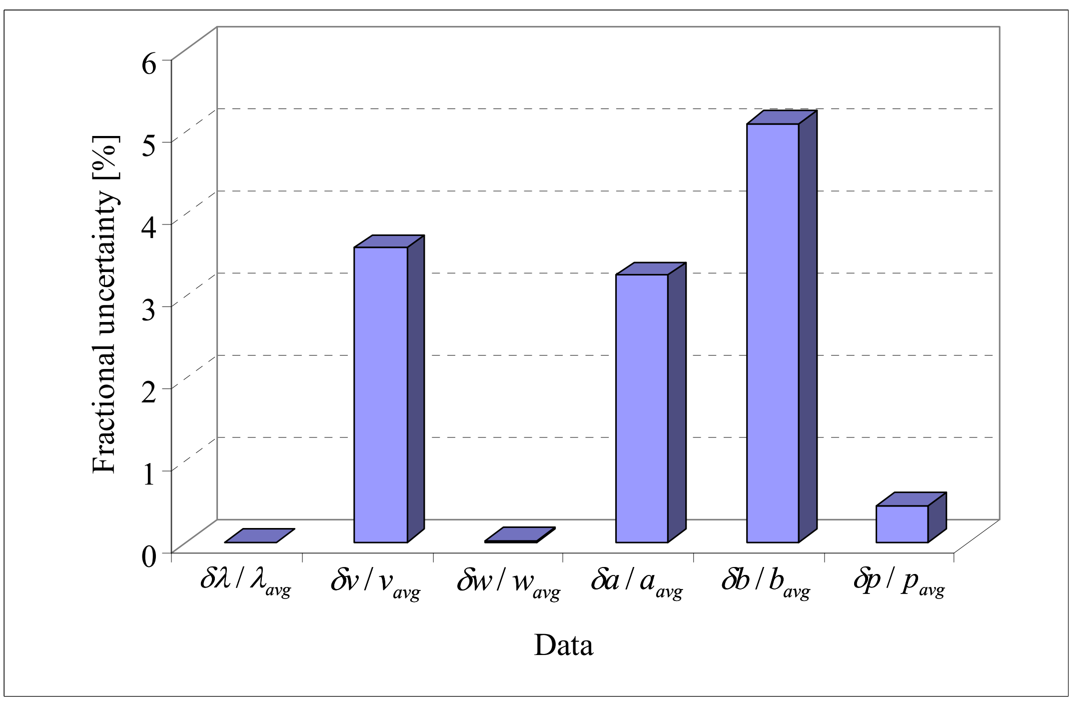

Fig.4.11. Fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, a, b, and p for the stretch ratio 1.7 of the tissue specimen.

Stretch Ratio 1.9: The tissue data corresponding to the stretch ratio 1.9 is shown in the table 4.14. It shows the *times* and stretch ratios obtained by the motion controller, forces v_M obtained by the force transducer and calculated *deviation*, *stretch-rate*, and *stretch histories*. The error analysis has been done on the data of the D versus T, H_{t_2} and H_{t_1} where H_{t_2} is a rate-related stretch history function scanned by a history cut-off t_h = 10 sec, and H_{t_1} is the long-time stretch history function scanned by a history cut-off t_h = 400 sec. The table 4.15 shows the average values and standard deviations of stretch ratios and forces. They were used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.9. They have also been used to calculate the uncertainty in resulting Cauchy stresses t.

Table 4.14
The tissue data corresponding to the stretch ratio 1.9. It shows the times and stretch ratios obtained by the motion controller, forces obtained by the force transducer and calculated deviation, stretch-rate, and stretch histories.

Time	λ	v_M	v_{avg}	D	$d\lambda/dt$	$H_{t_2}(t)$	$H_{t_1}(t)$	$H_{t_0}(t)$
[sec]		[volt]	[volt]	[volt]	[/sec]	t_h=10	t_h=400	t_h=1000
221.8	1.9	8.89437	7.35363	1.54074	0.00377	1.86438	1.14375	1.0575
473.0		7.92938		0.57575	-0.00244	1.90088	1.35842	1.14337
1167.5		7.79125		0.43762	0.00012	1.89895	1.35769	1.326
1848.3		7.43750		0.08387	-0.00437	1.90309	1.40355	1.42373
2386.8		7.49844		0.14481	-0.00303	1.8863	1.54094	1.47655
2817.4		7.38219		0.02856	0.00106	1.88516	1.49814	1.50159
3379.4		7.28531		-0.06832	0.00110	1.88173	1.50878	1.49897
3461.6		7.01875		-0.33488	0.00303	1.88716	1.54695	1.51145
3782.4		7.06625		-0.28738	0.00126	1.8866	1.54495	1.51905
4324.1		6.67469		-0.67894	-0.00579	1.90822	1.49347	1.50724
4775.4		6.93156		-0.42207	0.00240	1.87418	1.52669	1.51081
5073.6		6.82406		-0.52957	-0.00224	1.88732	1.47706	1.50052
5371.8		6.86344		-0.49019	0.00398	1.87136	1.50703	1.51033

Table 4.15
The average values and standard deviations of stretch ratios and forces are used for calculating the absolute uncertainty and relative (or fractional) uncertainty for the stretch ratio 1.9. They have also been used to calculate the uncertainty in resulting Cauchy stresses *t*.

Average		Standard deviation	
λ_{avg}	v_{avg}	$\lambda\Delta$	Δv
1.90001	7.35363	0.00003	0.59756

The uncertainties for the stretch level 1.9 have been calculated as

$$\delta t_\lambda = \left|\frac{\partial t}{\partial \lambda}\right|\delta\lambda = \left|\frac{v_{avg}w_{avg}}{a_{avg}b_{avg}p_{avg}}\right|\delta\lambda = \left|\frac{(7.35363)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.00003)$$
$$= 5.45\times10^{-5}[g/mm^2] \tag{4.37}$$

$$\delta t_v = \left|\frac{\partial t}{\partial v}\right|\delta v = \left|\frac{\lambda_{avg}w_{avg}}{a_{avg}b_{avg}p_{avg}}\right|\delta v = \left|\frac{(1.90001)(0.39390)}{(2.61690)(1.99050)(0.30610)}\right|(0.59756)$$
$$= 0.28049[\mathrm{g/mm}^2] \tag{4.38}$$

$$\delta t_w = \left|\frac{\partial t}{\partial w}\right|\delta w = \left|\frac{\lambda_{avg}v_{avg}}{a_{avg}b_{avg}p_{avg}}\right|\delta w = \left|\frac{(1.90001)(7.35363)}{(2.61690)(1.99050)(0.30610)}\right|(0.00010)$$
$$= 0.00088[g/mm^2] \tag{4.39}$$

$$\delta t_a = \left|\frac{\partial t}{\partial a}\right|\delta a = \left|\frac{-\lambda_{avg}v_{avg}w_{avg}}{(a_{avg})^2 b_{avg}p_{avg}}\right|\delta a = \left|\frac{-(1.90001)(7.35363)(0.39390)}{(2.61690)^2(1.99050)(0.30610)}\right|(0.08530)$$
$$= 0.11251[\mathrm{g/mm}^2] \tag{4.40}$$

$$\delta t_b = \left|\frac{\partial t}{\partial b}\right|\delta b = \left|\frac{-\lambda_{avg}v_{avg}w_{avg}}{a_{avg}(b_{avg})^2 p_{avg}}\right|\delta b = \left|\frac{-(1.90001)(7.35363)(0.39390)}{(2.61690)(1.99050)^2(0.30610)}\right|(0.10150)$$
$$= 0.17601[\mathrm{g/mm}^2] \tag{4.41}$$

$$\delta t_p = \left|\frac{\partial t}{\partial p}\right|\delta p = \left|\frac{-\lambda_{avg} v_{avg} w_{avg}}{a_{avg} b_{avg} \left(p_{avg}\right)^2}\right|\delta p = \left|\frac{-(1.90001)(7.35363)(0.39390)}{(2.61690)(1.99050)(0.30610)^2}\right|(0.00140)$$
$$= 0.01579[\text{g/mm}^2]$$

(4.42)

Thus, the total uncertainty in t for the stretch ratio 1.9 is

$$\begin{aligned}\delta t &= \sqrt{\left(\frac{\partial t}{\partial \lambda}\delta\lambda\right)^2 + \left(\frac{\partial t}{\partial v}\delta v\right)^2 + \left(\frac{\partial t}{\partial w}\delta w\right)^2 + \left(\frac{\partial t}{\partial a}\delta a\right)^2 + \left(\frac{\partial t}{\partial b}\delta b\right)^2 + \left(\frac{\partial t}{\partial p}\delta p\right)^2} \\ &= \sqrt{(\delta t_\lambda)^2 + (\delta t_v)^2 + (\delta t_w)^2 + (\delta t_a)^2 + (\delta t_b)^2 + (\delta t_p)^2} \\ &= 0.209494[\text{g/mm}^2]\end{aligned} \tag{4.43}$$

, and the fractional uncertainties are

$$\begin{aligned}
&\lambda = \lambda_{avg} \pm \delta\lambda = 1.90001 \pm 0.00003, && \frac{\delta\lambda}{\lambda_{avg}} \times 100 = 0.002\ \% \\
&v = v_{avg} \pm \delta v = 7.35363 \pm 0.59756, && \frac{\delta v}{v_{avg}} \times 100 = 8.126\ \% \\
&w = w_{avg} \pm \delta w = 0.39390 \pm 0.00010, && \frac{\delta w}{w_{avg}} \times 100 = 0.025\ \% \\
&a = a_{avg} \pm \delta a = 2.61690 \pm 0.08530, && \frac{\delta a}{a_{avg}} \times 100 = 3.260\ \% \\
&b = b_{avg} \pm \delta b = 1.99050 \pm 0.10150, && \frac{\delta b}{b_{avg}} \times 100 = 5.099\ \% \\
&p = p_{avg} \pm \delta p = 0.30610 \pm 0.00140, && \frac{\delta p}{p_{avg}} \times 100 = 0.457\ \% \\
&t = t_{avg} \pm \delta t = 3.45168 \pm 0.20949, && \frac{\delta t}{t_{avg}} \times 100 = 6.069\ \%
\end{aligned} \tag{4.44}$$

The absolute uncertainties in in t due to the $\delta\lambda$, δv, δw, δa, δb, and δp for the stretch ratio 1.9 and the fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, a, b, and p for the stretch ratio 1.9 are shown in the figure 4.12 and figure 4.13, respectively. The table 4.16 shows the summary of the averages, standard deviations, and fractional uncertainties of the variables for the stretch ratio 1.9 of the tissue.

Note that the fractional uncertainty due to the force *v* has increased. Although the fractional uncertainty due to the force *v* has been decrease as stretch ratio got bigger, the fractional uncertain due to the force *v* for the stretch ratio 1.9 has increased. This is due to the tremendous nonlinear behavior of the tissue.

Table 4.16
Fractional uncertainties for the measured data *w*, *a*, *b*, and *p*, and obtained data λ and *v* by the motion controller and the Cauchy stress *t* for the stretch level 1.9 of the tissue.

Data	$\lambda_{1.9}$	*v*[volt]	*w*[gram]	*a*[mm]	*b*[mm]	*p*[volt]	*t*[g/mm^2]
Avg	1.90001	7.35363	0.39390	2.61690	1.99050	0.30610	3.45168
S.D	0.00003	0.59756	0.00010	0.08530	0.10150	0.00140	0.20949
F.U [%]	0.002	8.126	0.025	3.260	5.099	0.457	6.069

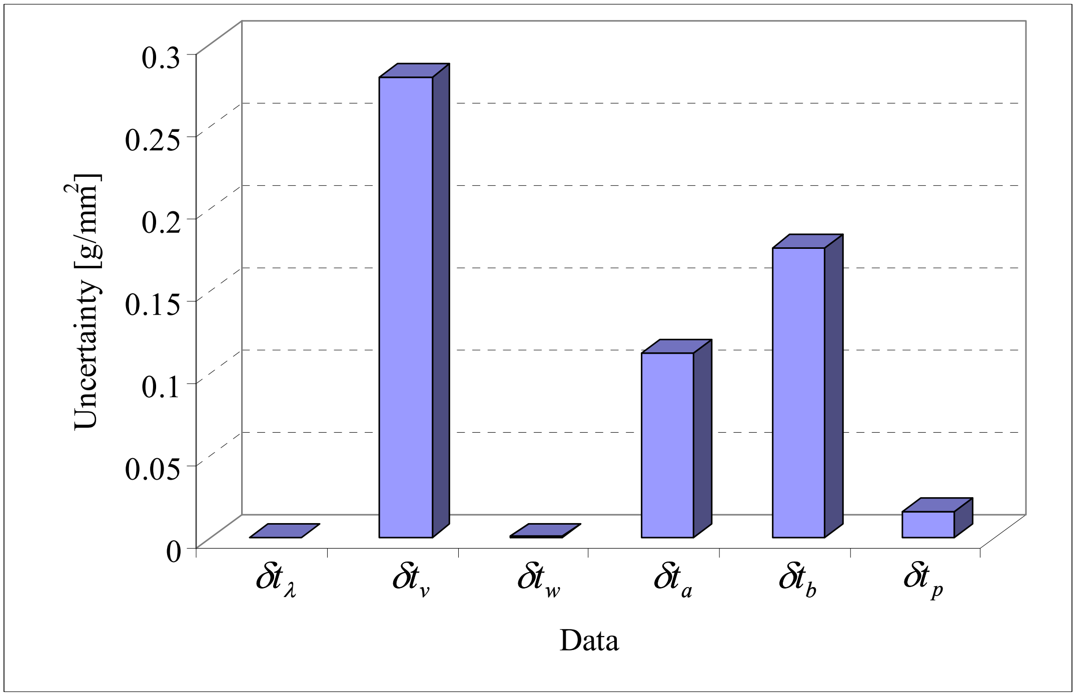

Fig.4.12.Uncertainties in t due to the $\delta\lambda$, δv, δw, δa, δb and δp for the stretch ratio 1.9 of the tissue specimen.

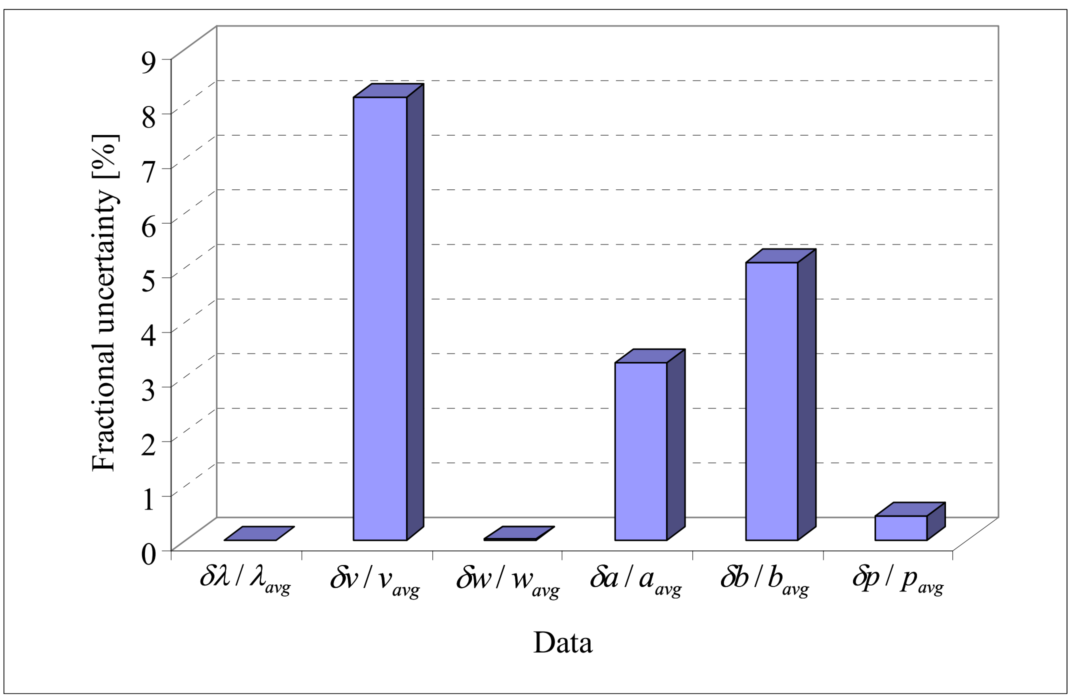

Fig.4.13. Fractional uncertainties in t due to the obtained data λ and v through the motion controller (ESP 7000 Motion Controller/Driver, Newport, Inc) and measured data w, a, b, and p for the stretch ratio 1.9 of the tissue specimen.

For tissue specimen, we have summarized the values of the absolute uncertainties and fractional uncertainties for each stretch ratio in the table 4.17. The absolute uncertainties are graphically shown in the figure 4.14 and figure 4.15. The fractional uncertainties are graphically shown in the figure 4.16. Note that the fractional uncertainty comes from inelastic behavior noticed from the force data is significant but the fractional uncertainty comes from the data of stretch ratio λ is ignorable. The fractional uncertainties for the static measured variables *w*, *a*, *b*, and *p* are constants for all selected stretch levels.

Because of the calculation process to get the absolute Cauchy stress, the uncertainties in the Cauchy stress *t* due to δt_{λ} , δt_{v} , δt_{w} , δt_{a} , δt_{b} , and δt_{p} increase as the stretch ratio gets bigger. If we look at the fractional uncertainties shown in figure 4.16, the fractional uncertainties related to the manual measurements such as the reference-weight measures *w*, the voltage measures *p* to check the resolution of the force transducer, and the width and the length measures a and *b*, respectively are constant for all stretch levels. The fractional uncertainty due to the measurement of the force measured by force transducer is decreasing until the stretch level 1.7 even though we got the data at the same amount of the displacements. Thus, we can conclude that the uncertainty is due to the inelastic behavior. If the uncertainty is totally due to the uncertainty due to the noise or resolution of the force transducer, it won't have any systematic trend, that is, there would be no decreasing or increasing trends of the fractional uncertainty due to the force *v*. Note, again, that any uncertainty due to the physical measurements can be reduced as much as the negligible levels, i.e., with proper assumption, it can be ignored for the error analysis.

Table 4.17
The summary of absolute uncertainties and fractional uncertainties for each stretch ratio of the tissue specimen.

		1.1	1.3	1.5	1.7	1.9
λ	λ_{avg}	1.10001	1.30000	1.50000	1.70002	1.90001
	$\Delta\lambda$	0.00008	0.00005	0.00005	0.00003	0.00003
	$\Delta\lambda/\lambda_{avg}$	0.007	0.004	0.003	0.002	0.002
	δt_λ [g/mm^2]	0.000002	0.000005	0.000010	0.000011	0.000054
f [volt]	v_{avg}	0.08403	0.37761	0.80873	1.51325	7.35363
	Δv	0.02572	0.02523	0.03529	0.05437	0.59756
	$\Delta v/ v_{avg}$ [%]	30.608	6.681	4.364	3.593	8.126
	δt_v [g/mm^2]	0.00699	0.00810	0.01308	0.02283	0.28049
w [g]	w_{avg}	0.3939				
	Δw	0.0001				
	$\Delta w/ w_{avg}$ [%]	0.025				
	δt_w [g/mm^2]	0.000006	0.00003	0.00008	0.00016	0.00088
a [mm]	a_{avg}	2.6169				
	Δa	0.0853				
	$\Delta a /a_{avg}$ [%]	3.260				
	δt_a [g/mm^2]	0.00074	1.30933	2.31132	3.34170	4.45520
b [mm]	b_{avg}	1.9905				
	Δb	0.1015				
	$\Delta b /b_{avg}$ [%]	5.099				
	δt_b [g/mm^2]	0.00116	0.00618	0.01528	0.03241	0.17601
p [volt]	p_{avg}	0.30610				
	Δp	0.00140				
	$\Delta p/ p_{avg}$ [%]	0.457				
	δt_p [g/mm^2]	0.00010	0.00055	0.00137	0.00291	0.01579
t [g/mm^2]	t_{avg}	0.02284	0.12127	0.29969	0.63553	3.45168
	Δt	0.00139	0.00736	0.01819	0.03857	0.20949
	$\Delta t/ t_{avg}$ [%]	6.069	6.069	6.069	6.069	6.069

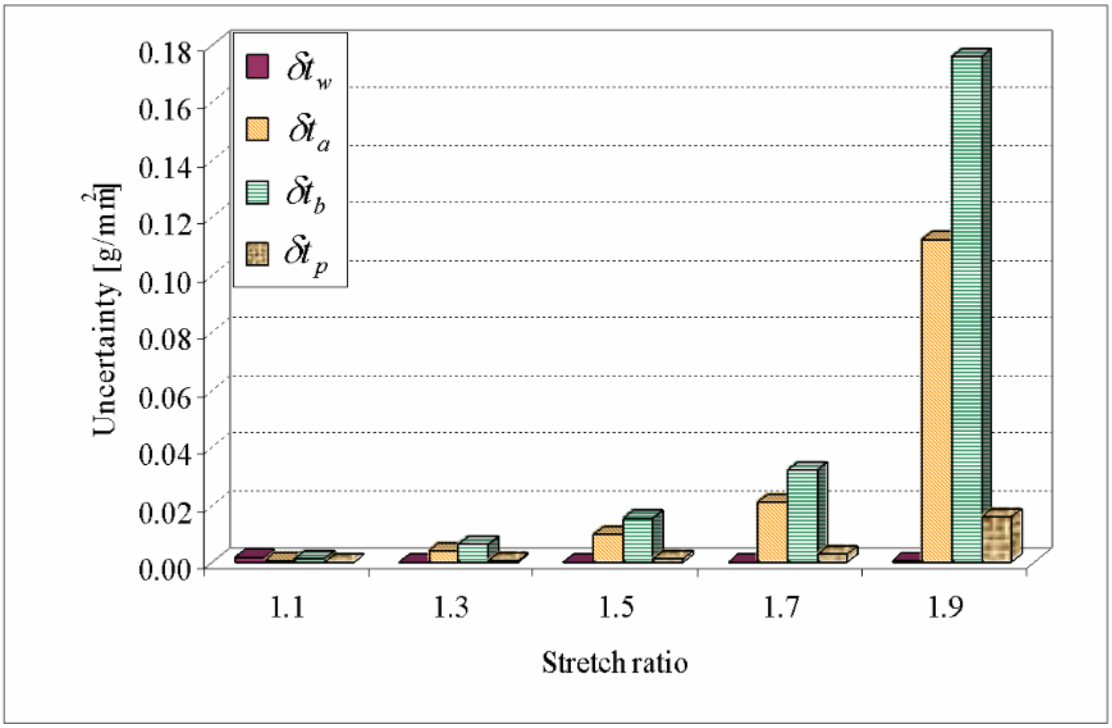

Fig.4.14. The (absolute) uncertainties in the Cauchy stress t due to the reference-weight measures w, the width a, the length b, and the voltage measures p to check the resolution of the force transducer.

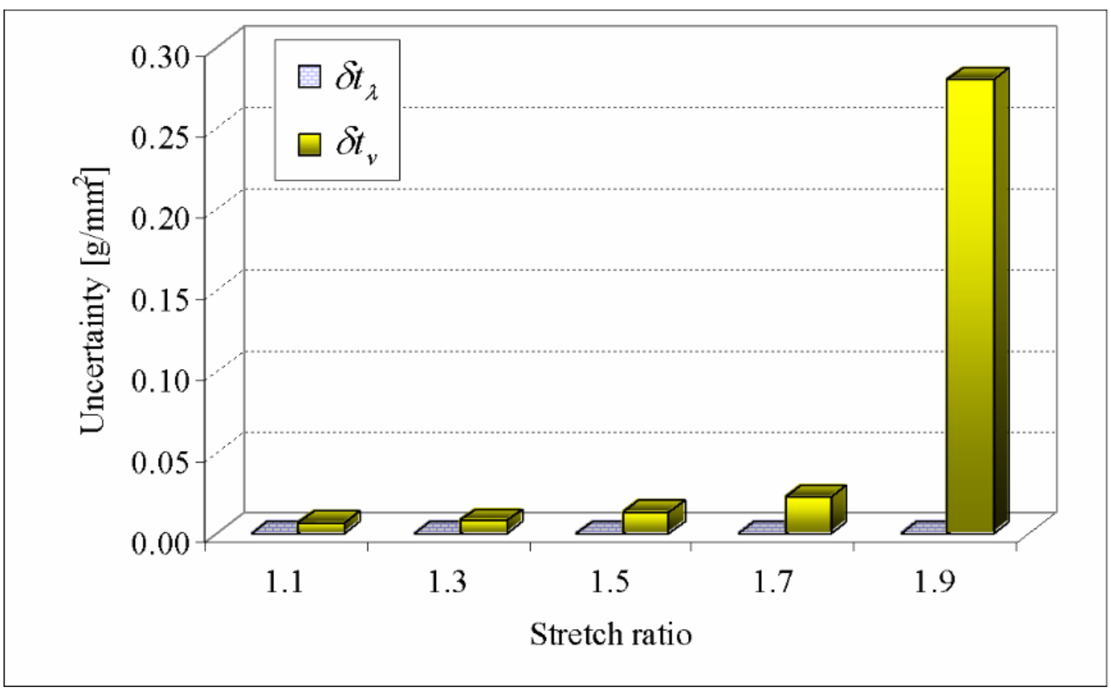

Fig.4.15. The (absolute) uncertainties in the Cauchy stress t due to the stretch ratio data λ, and the force data v of the tissue specimen. These values are obtained by the motion controller/driver (ESP7000, Newport, Inc).

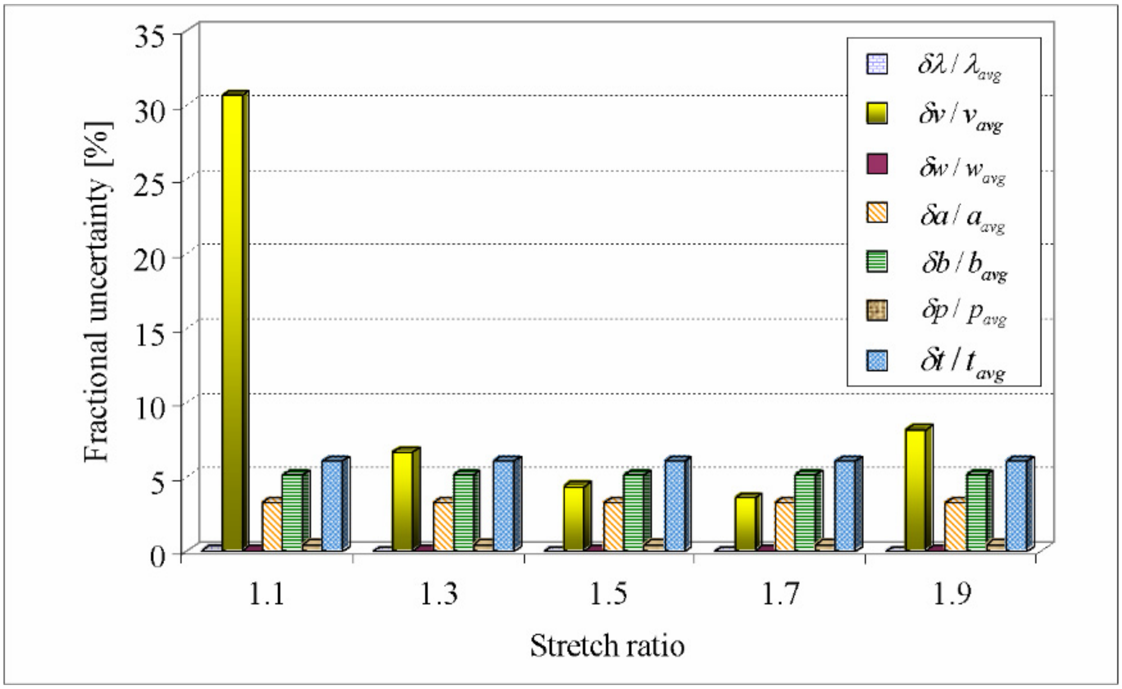

Fig.4.16. Fractional uncertainties in the data of the λ, v, w, a, b, p, and t of the tissue specimen.

As mentioned in the prior chapter, like the rubber-like materials, biotissues have randomly arranged long chain molecules called elastin. Thus, whenever the tissue is stretched, there must be rearrangements of elastin, i.e., the randomly arranged elastin gets close to the ordered configuration so that it causes the decrease of entropy. The reason that the uncertainties especially in the low stretch ranges are important for the rubber-like materials and biotissues, although the stretch ratio λ = 1.3 is not a low strain range for none rubber-like materials, is that their behaviors in the low strain range are primarily related to the entropy. The energy storage at the low range of stain is mostly due to the entropy alone. Thus, in the low stretch range, the strain energy can hardly be determined by the experimentation. That's why there has been a deficiency in understanding of the exclusive nature of the stain energy function for the low strain ranges of those materials. At the higher strain ranges, the energy storage is due to both the entropy and the molecular chemical bonds. That is, if we stretch to the higher strain ranges, we make the long chain molecules rearranged in more ordered manner as well as we stretch the chemical bonds, too. Thus, the energy storage is due to both the entropy and the molecular chemical bonds. (John C. Criscione, 2003). Unlikely to the rubber case, the fractional uncertainties in the total Cauchy stress t for stretch ratio from 1.1 to 1.9 are the same as 6.07 %. It is also because of the difference of the calculation process of the Cauchy stress t.

Error-of-Definition

The approximation of hyperelasticity for modeling the biotissues or rubber-like materials causes the uncertainty. It is called the *Error-of-Definition*. Because the biotissues mostly behave like hyperelastic, most people use the hyperelastic assumption even though it's not truly hyperelastic. Although there is no perfect elastic or hyperelastic material in the world, the assumption would be useful as long as we quantify the uncertainty in it. Again, similarly to rubber, we have assumed that the uncertainty due to the inelastic behavior of the tissue is due to the *Time*, *Stretch-Rate*, and *Stretch- History*. These have been checked by the multivariable linear regression analysis. The multivariable regression analysis, goodness-of-fit test and test of significance called *t*-test have been used for the analysis. As we mentioned in prior chapter, we have used the rate-related stretch history H_{t_2} instead of using the stretch-rate obtained by simple calculation. Note again that the stretch rate and rate-related stretch history are inversely related which means the sign of the rate-related stretch history is exactly the opposite of the sign of the stretch-rate. To do the multivariable linear regression analysis, we have to normalize or nondimensionalize both the dependent variable that is the deviation D and the independent variables that are the time T, rate-related stretch history H_{t_2}, and long-time stretch history H_{t_1}. Although the new stretch history H_{t_0} data has been also normalized, it hasn't been used for the multivariable linear regression analysis because there is high correlation between the time T and the new stretch history H_{t_0} that can give rise to the multicollinearity.

Multivariable Linear Regression Analysis

The multivariable linear regression analysis enables us to employ the multiple variables. Since it is a linear analysis, the first order functional relationship of the independent and dependent variables was investigated. Note that we assumed that there are linear relationships between them. Actually, one of the purposes of the regression model is to get the maximum effect with the minimum number of independent variables. Thus, the ideal regression model should predict the future values or unknown values between the known values with the minimum errors and the minimum number of independent variables.

For the regression analysis, there were several steps we should follow; first, we used the whole independent variables *T*, H_{t_2}, and H_{t_1}. Secondly, we checked the $\overline{R}^2$ as well as calculated *t*-values that are corresponding to the T, H_{t_2}, and H_{t_1}. Lastly, if there was an insignificant variable, we carriedout regression analysis again without the insignificant variable. It is important to know that the excluding the insignificant independent variable doesn't mean that the excluded independent variable doesn't affect the dependent variable. There would be a moderate correlation coefficient between the independent variables. To avoid the multicollinearity which causes the increasing of standard errors and lower the goodness-of-fit, we have checked the correlation coefficients between the independent variables.

Similarly, we have set the level of significance α as 0.1 so that the confidence interval (C.I) was

$$C.I = 100(1- 0.1) = 90\ \% \tag{4.45}$$

This represents the probability (90 % of probability) that the partial regression coefficient $\hat{\beta}_i$ is likely to be contained within that interval. It physically means that, for the unknown value β, the true value $\hat{\beta}$ will fall into the range of $\hat{\beta} \pm t_c S_{\hat{\beta}}$ 90 times out of the 100 where $S_{\hat{\beta}}$ is expressed as (for example, independent variable *T*)

$$S_{\hat{\beta}} = \frac{S_e}{\sqrt{\sum \left(T_i - \overline{T}\right)^2}} = \frac{\sqrt{\frac{\sum \left(D_i - \hat{D}_i\right)^2}{n-k-1}}}{\sqrt{\sum \left(T_i - \overline{T}\right)^2}} \tag{4.46}$$

To carry out the multivariable linear regression analysis, we have nondimensionalized the data to avoid the scale-dependency which cause the partial regression coefficients to be nonsense. We have used *L*2-norm for nondimensionalization as

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \tag{4.47}$$

Stretch Ratio 1.1: The nondimensionalized data for the stretch ratio 1.1 of the rubber specimen is shown in the table 4.18. The dependent variable is deviation *D* and the independent variables are time *T*, rate-related stretch history function H_{t_2}, and long-time stretch history function H_{t_1}. Since the correlation coefficients between the *T* and H_{t_0} is very high as 0.81 and the correlation coefficient between the H_{t_1} and H_{t_0} is also very high as 0.89, although the correlation coefficient between the *D* and H_{t_0} is fairly high, we only use the *T*, H_{t_2}, and H_{t_1} as independent variables for the multiple linear regression analysis.

Table 4.18
Nondimensionalized data using $L2$-norm for the stretch ratio 1.1 of the tissue specimen.

λ	Dependent variable	Independent variables			
	D	***T***	H_{t_2}	H_{t_1}	H_{t_0}
	v_M - v_{avg}		t_h=10	t_h=400	t_h=1000
1.1	0.29500	0.01058	0.24582	0.18016	0.17959
	0.19470	0.02424	0.24340	0.20441	0.18968
	0.22512	0.05034	0.23898	0.23879	0.20657
	0.29801	0.05664	0.23930	0.23495	0.20763
	0.10364	0.08004	0.24792	0.23449	0.22232
	0.14008	0.11250	0.24255	0.24801	0.24672
	0.21598	0.11857	0.24310	0.24111	0.24519
	0.07322	0.14340	0.24193	0.23852	0.24836
	0.03979	0.17443	0.24346	0.25741	0.25821
	0.06418	0.18663	0.24492	0.25472	0.25779
	0.09752	0.21441	0.24264	0.24300	0.25628
	-0.08471	0.28796	0.23493	0.26121	0.26578
	-0.15147	0.31200	0.24509	0.26208	0.26524
	-0.21833	0.35594	0.24238	0.24564	0.26424
	-0.41270	0.39288	0.23953	0.25424	0.25947
	-0.40357	0.40423	0.24007	0.25002	0.25975
	-0.47646	0.45158	0.24674	0.25967	0.26358
Correlation Coefficients		-0.96	0.06	-0.59	-0.70

Since the number of sample for the stretch ratio1.1 is 17, the degree-of-freedom (DOF) for t is n-k-1 = 17-3-1 = 13. Thus, the critical t value for the stretch ratio 1.1 is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;13)} = 1.771$ (for two-tailed test).

Thus if we have the following condition as

$$|t_{n-k-1}| > 1.771 \tag{4.48}$$

we reject the null hypothesis and accept the alternative hypothesis which means that the specific independent variable affect the dependent variable. The result of multivariable linear regression analysis using entire independent variables T, H_{t_2} and H_{t_1} is shown in the table 4.19.

Table 4.19
Result of multivariable linear regression analysis for stretch ratio 1.1 of tissue specimen with entire independent variables T, H_{t_2} and H_{t_1}.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	0.70	1.37	0.51	0.93
T	-1.89	0.17	-11.30	
H_{t_2}	-4.11	5.30	-0.78	
H_{t_1}	2.78	1.63	1.70	

Although the $\overline{R}^2$ is high, the t-values of H_{t_2} and H_{t_1} didn't satisfy the above condition, that is they are insignificant to cause the deviation. In addition, the sign of the partial regression coefficient of the H_{t_1} is positive which is unexpected result; we already know from the results for whole stretch levels that as the stretch history function gets bigger, the deviation gets smaller. It should be negative. So we execute a regression analysis without H_{t_2} and H_{t_1}. Since the number of independent variables are reduced to 1, we have a new critical t-value as $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;15)} = 1.753$ and we should have a following new constraint as

$$|t_{n-k-1}| > 1.753 \tag{4.49}$$

to reject the null hypothesis and accept the alternative hypothesis. The result of multivariable linear regression analysis for stretch ratio 1.1 of tissue specimen with independent variable T is shown in the table 4.20.

Table 4.20
Result of multivariable linear regression analysis for stretch ratio 1.1 of tissue specimen with independent variable *T*.

	$\hat{\beta}_i$	s_e	*t*-value	$\overline{R}^2$
$\hat{\alpha}$	0.34	0.03	10.44	0.92
T	-1.70	0.13	-13.09	

Thus the final regression model for the stretch ratio 1.1 can be expressed as

$$\hat{D}_{1.1} = 0.34 - 1.70T \tag{4.50}$$

The optimal independent variables that can expect the dependent variable is *T* and the ability to explain the deviation is 92 %. The rest 8 % can't be explained by this model. From the model, we have noticed that the time *T* is the substantial factor that causes the deviation from the hyperelasticity. Note that this doesn't mean that the time is the only factor that causes the deviation.

Stretch Ratio 1.3: The nondimensionalized data for the stretch ratio 1.3 of the tissue specimen is shown in the table 4.21. The dependent variable is deviation *D* and the independent variables are time *T*, rate-related stretch history function H_{t_2}, and long-time stretch history function H_{t_1}. Since the correlation coefficient between *T* and H_{t_0} is very high as 0.9, we excluded the H_{t_0} for the multiple linear regression analysis.

Table 4.21
Nondimensionalized data using $L2$-norm for the stretch ratio 1.3 of the tissue specimen.

λ	Dependent variable	Independent variables			
	$\boldsymbol{D}$	$\boldsymbol{T}$	H_{t_2}	H_{t_1}	H_{t_0}
	$v_M - v_{avg}$		$\boldsymbol{t_h=10}$	$\boldsymbol{t_h=400}$	$\boldsymbol{t_h=1000}$
1.3	0.31197	0.04449	0.30317	0.29330	0.24609
	0.52745	0.06910	0.30119	0.27693	0.25743
	0.47255	0.08810	0.30296	0.26603	0.27009
	-0.10333	0.24227	0.29904	0.30553	0.31591
	-0.13065	0.26593	0.30628	0.31888	0.31517
	0.14350	0.29242	0.29636	0.31480	0.31627
	-0.13466	0.33743	0.30367	0.30426	0.31889
	-0.15422	0.37523	0.30350	0.31077	0.31746
	-0.24034	0.38123	0.29482	0.30546	0.31622
	-0.26390	0.41945	0.29630	0.29858	0.31210
	-0.42837	0.44116	0.30902	0.31745	0.31811
Correlation Coefficients		-0.93	-0.06	-0.79	-0.85

In this case, the number of sample for the stretch ratio1.3 is 11 and the degree-of-freedom (DOF) for t is n-k-1 = 11-3-1 = 7. Thus, the critical t value for the stretch ratio 1.3 is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;7)} = 1.895$ (for two-tailed test).

Thus if we have a t-value that satisfies the following constraint as

$$|t_{n-k-1}| > 1.895 \tag{4.51}$$

we reject the null hypothesis and accept the alternative hypothesis which means that the specific independent variable affect the dependent variable. The result of multivariable linear regression analysis for the stretch ratio 1.3 of the tissue specimen using entire independent variables T, H_{t_2} and H_{t_1} is shown in the table 4.22.

Table 4.22
Result of multivariable linear regression analysis for stretch ratio 1.3 of tissue specimen with entire independent variables T, H_{t_2} and H_{t_1}.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	3.63	2.55	1.42	0.86
T	-2.01	0.47	-4.28	
H_{t_2}	-7.36	8.57	-0.86	
H_{t_1}	-2.85	3.50	-0.81	

Although the $\overline{R}^2$ is reasonably high, the t-values of H_{t_2} and H_{t_1} didn't satisfy the above constraint. So we execute a regression analysis only with the T. Since the number of independent variables are reduced to 1, $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;9)} = 1.833$, we should have a new constraint as

$$\left|t_{n-k-1}\right| > 1.833 \tag{4.52}$$

to reject the null hypothesis and accept the alternative hypothesis. The result of multivariable linear regression analysis for stretch ratio 1.3 of tisse specimen with independent variable T is shown in the table 4.23.

Table 4.23
Result of multivariable linear regression analysis for stretch ratio 1.3 of tissue specimen with independent variables T.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	0.64	0.09	6.79	0.87
T	-2.29	0.30	-7.72	

Thus the final regression model for the stretch ratio 1.3 can be expressed as

$$\hat{D}_{1.3} = 0.64 - 2.29T \tag{4.53}$$

Again, the optimal independent variable that can expect the dependent variable is T. In addition, its ability to explain the deviation is 87 %. The rest 13 % can't be explained by this model whatsoever. From the regression results and the model, we figured out that the rate-related stretch history and long-time stretch history are not effective factors that cause the deviation from hyperelasticity for the stretch ratio 1.1 and 1.3 of the tissue.

Stretch Ratio 1.5: The nondimensionalized data for the stretch ratio 1.5 of the tissue specimen is shown in the table 4.24. The dependent variable is deviation D and the independent variables are time T, rate-related stretch history function H_{t_2} , and long-time stretch history function H_{t_1} . Since the correlation coefficient between the T and H_{t_0} is high enough as 0.8 to cause the multicollinearity and also the correlation coefficient between the H_{t_1} and H_{t_0} is very high as 0.9, we carried out the multiple linear regression analysis with the T, H_{t_2} , and H_{t_1} . Since the number of sample for the stretch ratio1.5 is 17, the degree-of-freedom (DOF) for t is n-k-1 = 17-3-1 = 13. Thus, the critical t value for the stretch ratio 1.5 is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;13)} = 1.771$ (for two-tailed test).

Thus if t-value corresponding to a certain variable satisfies the following condition as

$$\left|t_{n-k-1}\right| > 1.771 \tag{4.54}$$

we reject the null hypothesis and accept the alternative hypothesis which means that the specific independent variable affect the dependent variable.

Table 4.24
Nondimensionalized data using $L2$-norm for the stretch ratio 1.5 of the tissue specimen.

λ	Dependent variable	Independent variables			
	D	**T**	H_{t_2}	H_{t_1}	H_{t_0}
	$v_M - v_{avg}$		t_h=10	t_h=400	t_h=1000
1.5	0.59127	0.00522	0.23681	0.17400	0.17317
	0.40969	0.03058	0.23843	0.21044	0.18805
	0.42301	0.05819	0.23629	0.23180	0.20450
	-0.00652	0.10107	0.24366	0.24913	0.23814
	0.07538	0.12415	0.24493	0.24711	0.24149
	0.07538	0.13033	0.24332	0.24919	0.24106
	-0.06851	0.15344	0.23992	0.25104	0.24485
	-0.08622	0.18769	0.24807	0.25448	0.25534
	-0.29875	0.22091	0.24925	0.25802	0.25903
	-0.09288	0.23598	0.24257	0.24759	0.25593
	-0.12164	0.24081	0.24362	0.24791	0.25818
	-0.03089	0.30179	0.24281	0.25441	0.25720
	-0.12384	0.31072	0.24424	0.24115	0.25913
	-0.13269	0.31330	0.24072	0.24176	0.26056
	-0.14374	0.32581	0.24545	0.24998	0.25473
	-0.19688	0.40089	0.24119	0.24517	0.25083
	-0.27218	0.42325	0.24143	0.25582	0.25747
Correlation Coefficients		-0.83	-0.69	-0.85	-0.95

The result of multivariable linear regression analysis for the stretch ratio 1.5 of the tissue specimen using entire independent variables T, H_{t_2} and H_{t_1} is shown in the table 4.25.

Table 4.25
Result of multivariable linear regression analysis for stretch ratio 1.5 of tissue specimen with entire independent variables T, H_{t_2} and H_{t_1}.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	5.85	1.65	3.55	0.85
T	-1.00	0.20	-5.02	
H_{t_2}	-17.48	8.15	-2.15	
H_{t_1}	-5.75	2.63	-2.18	

For the stretch ratio 1.5 of the tissue, the deviation is caused by most of the independent variables, say, T, H_{t_2}, and H_{t_1}. The most effective independent variable is H_{t_2}, that is, the deviation is mostly caused by the rate of stretch. The second effective factor that causes the deviation is H_{t_1}, and lastly, the time *T*. Thus, the final regression model for the stretch ratio 1.5 can be expressed as

$$\hat{D}_{1.5} = 5.85 - T - 17.48H_{t_2} - 5.75H_{t_1} \tag{4.55}$$

Their ability to explain the deviation is 85 %. The rest 15 % can't be explained by this model.

Stretch Ratio 1.7: The nondimensionalized data for the stretch ratio 1.7 of the tissue specimen is shown in the table 4.26. The dependent variable is deviation *D* and the independent variables are time *T*, rate-related stretch history function H_{t_2}, and long-time stretch history function H_{t_1}. Since the correlation coefficient between the H_{t_1} and H_{t_0} is high as 0.8 and the correlation coefficient between the T and H_{t_0} is reasonably high as 0.75, we have performed the error analysis based on the data T, H_{t_2}, and H_{t_1}. In this case, since the number of sample for the stretch ratio1.7 is 13, the degree-of-freedom (DOF) for *t* is *n*-*k*-1 = 13-3-1 = 9. Thus, the critical *t* value for the stretch ratio 1.7 is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;9)} = 1.833$ for two-tailed test.

Thus if we have a *t*-value that satisfies the following condition as

$$|t_{n-k-1}| > 1.833 \tag{4.56}$$

we reject the null hypothesis and accept the alternative hypothesis which means that the specific independent variable affect the dependent variable.

Table 4.26
Nondimensionalized data using $L2$-norm for the stretch ratio 1.7 of the tissue specimen.

λ	Dependent variable	Independent variables			
	Deviation	***Time***	***Stretch History H(t)***		
	v_M - v_{avg}		t_h=10	t_h=400	t_h=1000
1.7	0.58510	0.07633	0.27336	0.26019	0.23790
	-0.04543	0.10397	0.27762	0.26958	0.26026
	0.44572	0.12819	0.27146	0.26405	0.26829
	-0.15497	0.16118	0.27751	0.26964	0.27105
	-0.08525	0.17447	0.28265	0.29287	0.28050
	0.22335	0.18705	0.27387	0.28356	0.28149
	0.24496	0.22963	0.27382	0.27185	0.27893
	-0.15162	0.26977	0.28289	0.28759	0.28853
	-0.02222	0.31068	0.27549	0.28479	0.28913
	-0.15826	0.37497	0.27912	0.27842	0.28835
	-0.24788	0.38144	0.28008	0.28415	0.29029
	-0.26615	0.41271	0.27502	0.27453	0.28139
	-0.36735	0.44392	0.28235	0.28228	0.28472
Correlation Coefficients		-0.74	-0.75	-0.60	-0.68

The result of multivariable linear regression analysis for the stretch ratio 1.7 of the tissue specimen using entire independent variables T, H_{t_2} and H_{t_1} is shown in the table 4.27.

Table 4.27
Result of multivariable linear regression analysis for stretch ratio 1.7 of tissue specimen with entire independent variables T, H_{t_2} and H_{t_1} .

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	11.28	2.95	3.82	0.73
T	-1.05	0.35	-3.03	
H_{t_2}	-47.23	13.40	-3.52	
H_{t_1}	7.42	5.95	1.25	

According to the t-value, the effective independent variables that are significant are T and H_{t_2}. So we have carried out a regression analysis again with the T and H_{t_2}. Since the number of independent variables are reduced to 2, $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;10)} = 1.812$, we should have a new constraint as

$$|t_{n-k-1}| > 1.812 \tag{4.57}$$

to reject the null hypothesis and accept the alternative hypothesis. The result of multivariable linear regression analysis for stretch ratio 1.7 of the tissue specimen with independent variables T and H_{t_2} is shown in the table 4.28.

Table 4.28
Result of multivariable linear regression analysis for stretch ratio 1.7 of tissue specimen with independent variables T and H_{t_2}.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	10.42	2.96	3.52	0.71
T	-0.97	0.35	-2.77	
H_{t_2}	-36.77	10.76	-3.42	

Thus the regression model for the stretch ratio 1.7 of the tissue can be expressed as

$$\hat{D}_{1.7} = 10.42 - 0.97T - 36.77H_{t_2} \tag{4.58}$$

The regression result says that the ability of the independent variables T and H_{t_2} to explain the deviation is 71 %. The rest 29 % can't be explained by this model. In addition, we found that the optimal independent variables that can expect the dependent variable D are T and H_{t_2}. Note again that this doesn't mean that the long-time stretch history doesn't give rise to the deviation from the hyperelasticity; it is just insignificant relative to the significant independent variables T and H_{t_2}. Note that the sign of the partial regression coefficient for the H_{t_2} is exactly the opposite of the stretch-rate.

Stretch Ratio 1.9: The nondimensionalized data for the stretch ratio 1.9 of the tissue specimen is shown in the table 4.29. The dependent variable is deviation D and the independent variables are time T, rate-related stretch history function H_{t_2}, and long-time stretch history function H_{t_1}. Because the correlation coefficient between the T and H_{t_0} is very high as 0.84 and also the correlation coefficient between the H_{t_1} and H_{t_0} is quite high as 0.93, we considered the independent variables as T, H_{t_2}, and H_{t_1} only for the multiple linear regression analysis. Since the number of sample for the stretch ratio1.9 is 13, the degree-of-freedom (DOF) for t is n-k-1 = 13-3-1 = 9. Thus, the critical t value for the stretch ratio 1.9 is $t_{c(\alpha/2;n-k-1)} = t_{c(0.05;9)} = 1.833$ (for two-tailed test).

Thus if we have a t-value that satisfies the following condition as

$$\left|t_{n-k-1}\right| > 1.833 \tag{4.59}$$

we reject the null hypothesis and accept the alternative hypothesis which means that the specific independent variable affect the dependent variable. The result of multivariable linear regression analysis for the stretch ratio 1.9 of the tissue specimen using the entire independent variables T, H_{t_2} and H_{t_1} is shown in the table 4.30.

Table 4.29
Nondimensionalized data using $L2$-norm for the stretch ratio 1.9 of the tissue specimen.

λ	Dependent variable	Independent variables			
	Deviation	***Time***	***Stretch History H(t)***		
	v_M - v_{avg}		t_h=10	t_h=400	t_h=1000
1.9	0.74432	0.01796	0.27397	0.21749	0.20515
	0.27814	0.03830	0.27933	0.25831	0.22181
	0.21141	0.09454	0.27905	0.25817	0.25724
	0.04052	0.14967	0.27966	0.26689	0.27620
	0.06996	0.19328	0.27719	0.29302	0.28644
	0.01380	0.22815	0.27702	0.28488	0.29130
	-0.03300	0.27366	0.27652	0.28690	0.29079
	-0.16178	0.28032	0.27732	0.29416	0.29321
	-0.13883	0.30630	0.27724	0.29378	0.29469
	-0.32799	0.35016	0.28041	0.28399	0.29240
	-0.20390	0.38671	0.27541	0.29031	0.29309
	-0.25583	0.41085	0.27734	0.28087	0.29109
	-0.23681	0.43500	0.27500	0.28657	0.29300
Correlation Coefficients		-0.90	-0.23	-0.88	-0.91

Table 4.30
Result of multivariable linear regression analysis for stretch ratio 1.9 of rubber specimen with entire independent variables T, H_{t_2} and H_{t_1}.

	$\hat{\beta}_i$	s_e	t-value	$\overline{R}^2$
$\hat{\alpha}$	13.56	1.91	7.10	0.98
T	-1.58	0.09	-16.72	
H_{t_2}	-44.76	6.46	-6.93	
H_{t_1}	-2.77	0.94	-2.96	

For the stretch level 1.9, the deviation is quite depend on the whole independent variables such as T, H_{t_2}, and H_{t_1}. Since the $\overline{R}^2$ is very high as 98 %, the model is very good to predict the deviation for the stretch level 1.9 and this model can explain most of the deviation as 98 %. The rest 2 % can't be explained by this model. The final regression model for the stretch ratio 1.9 can be expressed as

$$\hat{D}_{1.9} = 13.56 - 1.58T - 44.76H_{t_2} - 2.77H_{t_1} \tag{4.60}$$

The most significant factor that causes the deviation from the hyperelasticity is the rate-related stretch history which is directly related to the rate of stretch. The second effective factor is the long-time stretch history, and lastly the time.

Nonsense of Correlation Coefficients

We have checked the correlation coefficients between the deviation D (dependent variable) and independent variables T, H_{t_2}, H_{t_1}, and H_{t_0}. The summary of the correlation coefficients for each stretch ratio for the tissue is shown in the table 4.31.

Table 4.31
The summary of correlation coefficients between the deviation D and the time T, rate-related stretch history H_{t_2}, long-time stretch history H_{t_1}, and the new stretch history H_{t_0}.

λ	Correlation Coefficients: D between			
	T	H_{t_2}	H_{t_1}	H_{t_0}
		t_h=10	t_h=400	t_h=400
1.1	-0.96	0.06	-0.59	-0.70
1.3	-0.93	-0.06	-0.79	-0.85
1.5	-0.83	-0.69	-0.85	-0.95
1.7	-0.74	-0.75	-0.60	-0.68
1.9	-0.90	-0.23	-0.88	-0.91

Indisputably, if the correlation coefficient between a certain dependent variable and independent variable is high, then we naturally know there is high relationship between them. But again, it merely tells a relationship between the two variables. The correlation coefficient just shows whether there is a relationship between one variable and the other or not. It is a relationship between only two variables. Since we have too many variables and scattering in the data, the assessment and ordering of the correlation coefficients are not meaningful. If we compare the results obtained by the multiple linear regression analysis and the correlation coefficients shown in the table 4.31, they don't have dependency between them. Again, it says that, in this kind of study, the correlation coefficient is meaningless.

CHAPTER V

PREDICTABILITY

Basic Principle

The test of the predictability should be involved in the final step for any regression analysis to evaluate how well the derived regression model forecasts the intermediate or future values. The best way to accomplish this work is to compare the predicted result to the data that were not used for the formulation of the regression models. We have checked the predictive capability by comparing the unused (deviation) data in the randomized stretch-controlled protocol of the rubber and tissue to the predicted deviation obtained by the regression models. The following equation (5.1) has been used for the interval prediction.

$$\hat{D}_{fc} = \hat{D}_f \pm t_{c(\alpha/2;n-k-1)} \cdot s_f \tag{5.1}$$

where $\hat{D}_{fc}$ is an interval forecasting critical value for the $\hat{D}_f$ with a 100(1-α) confidence, $\hat{D}_f$ is the predicted value on a certain point, $t_{c(\alpha/2;n-k-1)}$ is a critical t value with a degree of freedom n-k-1, s_f is a standard deviation which is consist of the variance of e_i and the variance of $\hat{D}_i$. It can be shown as

$$\begin{aligned} s_f &= \sqrt{Var(\hat{e}_f)} \\ &= \sqrt{Var(e_i) + Var(\hat{D}_i)} \\ &= \sqrt{(s_e)^2 + Var(\hat{D}_i)} \\ &= \sqrt{\left(\frac{\sum (D_i - \hat{D}_i)^2}{n-k-1}\right)^2 + Var(\hat{D}_i)} \end{aligned} \tag{5.2}$$

Rubber

Stretch Ratio 1.1: For λ = 1.1 of rubber specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.761, where α = 0.1, n = 17, k = 2 (for T and H_{t_2}). We used the regression model for λ = 1.1 which is $\hat{D}_{1.1} = 6.66 - 1.40T - 26.37H_{t_2}$ to obtain the $\hat{D}_f$ which is a point forecasting value. By using the equation (5.2), we got s_f = 0.02906 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 0.05117 for λ = 1.1 of the rubber specimen. The table 5.1 shows the unused data for the stretch ratio 1.1 of the rubber specimen. It also shows the time T and rate-related stretch history H_{t_2} that mainly affect the deviation from the hyperelasticity and inelastic deviation D_i, point forecasting $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

We have redefined the deviation from the hyperelasticity as *inelastic deviation* which is, as we have used above, the difference between the *measured* (by force transducer) and *averaged* force. If the rubber specimen was truly elastic, all the measurements would be equal to the average so that the inelastic deviation would be zero. To check if we missed any variables for the prediction, we have newly defined the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Although we have found that the inelastic deviation varies systematically and quantified it, the models that we have derived can never be perfect; there could be any missed variables. The question is “Is the prediction deviation random or systematic?”.

Table 5.1
Unused data for the stretch ratio 1.1 of the rubber specimen. It shows the time T and rate-related stretch history H_{t_2} that mainly affect the deviation from the hyperelasticity and inelastic deviation D_i, point forecasting $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

λ_unused	T	H_{t_2}	D_i	$\hat{D}_f$	$D_i - \hat{D}_f$
	[sec]	$t_h=10$	v_M - v_{avg}[volt]	[volt]	[volt]
1.1	13.6	1.06411	0.18645	0.07771	0.10874
	146.9	1.08978	0.09457	0.05507	0.03950
	297.9	1.12984	0.03239	0.02064	0.01175
	300.7	1.11385	0.03739	0.03339	0.00401
	307.1	1.10001	0.04614	0.04436	0.00178
	324.3	1.09918	0.06051	0.04475	0.01576
	612.2	1.11363	0.02458	0.02864	-0.00406
	697.3	1.10144	0.05770	0.03705	0.02065
	989.9	1.13942	0.00207	0.00205	0.00003
	1011.2	1.08977	0.04301	0.04143	0.00158
	1409.8	1.09719	0.02364	0.02920	-0.00556
	1473.8	1.10285	0.03864	0.02366	0.01498
	2161.9	1.1065	-0.01136	0.00987	-0.02123
	2315.9	1.10143	-0.00511	0.01149	-0.01660
	2669.0	1.09958	-0.00199	0.00740	-0.00938
	3536.5	1.15257	-0.05699	-0.04870	-0.00829
	3863.7	1.1079	-0.02761	-0.01813	-0.00948
	4849.7	1.139	-0.05136	-0.05858	0.00723
	4991.4	1.13732	-0.04386	-0.05948	0.01562
	5600.0	1.08945	-0.04417	-0.03079	-0.01337

According to the result shown in the table 5.1 and the figure 5.1, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we have missed is random and we don't perfectly predict every point. Those points are off in a random way. If those points are off in a systematic way, we would have possibly missed a certain variable that gives rise to the inelastic deviation. The prediction deviation is shown in the figure 5.2. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

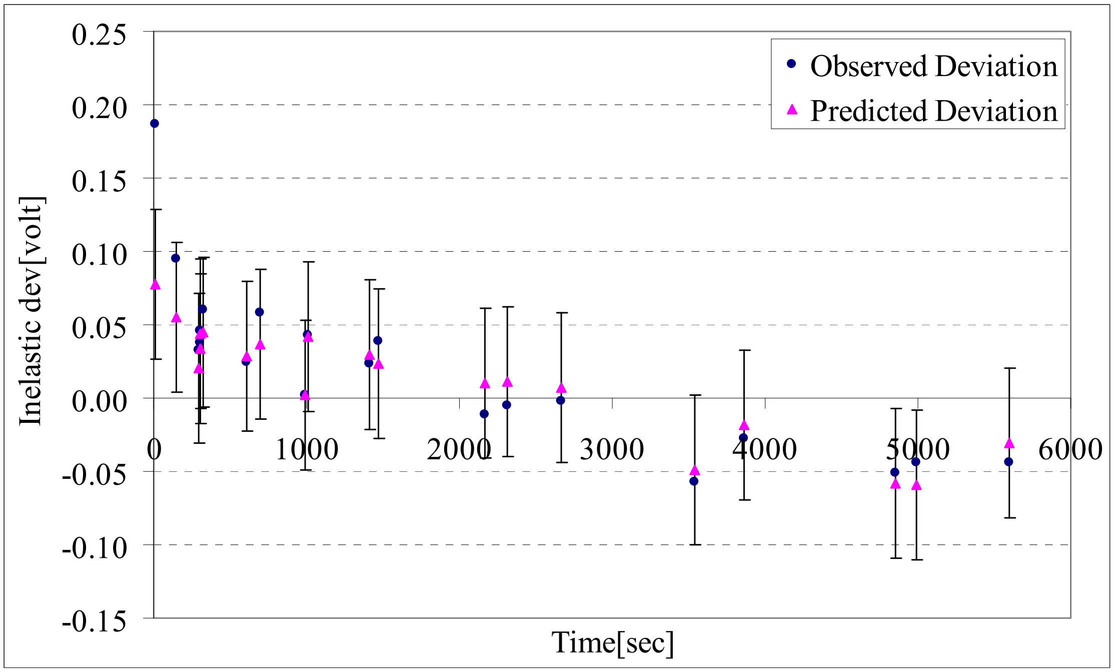

Fig.5.1. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.1 of the rubber specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 0.05117$.

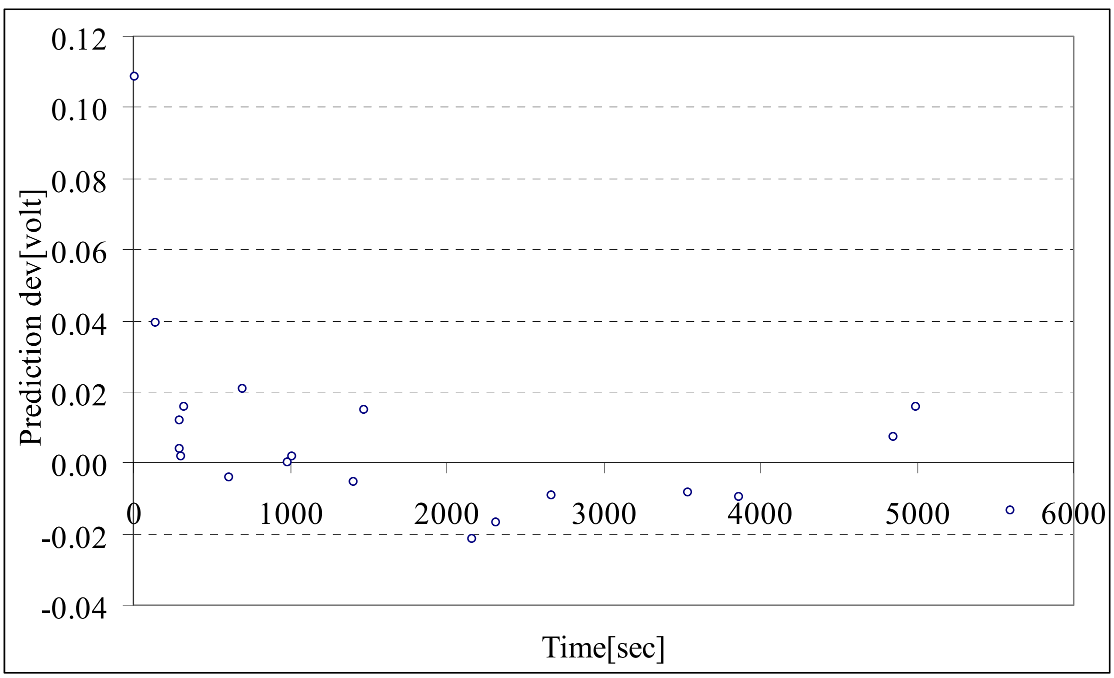

Fig.5.2. It shows the prediction deviation for the stretch ratio 1.1 of the rubber specimen. Note that all points are gathered at near zero but off in a random way.

Stretch Ratio 1.3: For λ = 1.3 of rubber specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.860, where α = 0.1, n = 11, k = 2 (for T and H_{t_2}). We used the regression model for λ = 1.3 which is $\hat{D}_{1.3} = 10.74 - 2.00T - 33.74H_{t_2}$ to obtain the $\hat{D}_f$ which is a point forecasting value. We have s_f = 0.03551 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 0.06605 for λ = 1.3 of the rubber specimen. The table 5.2 shows the unused data for the stretch ratio 1.3 of the rubber specimen. It also shows the time T and rate-related stretch history H_{t_2} that mainly affect the deviation from the hyperelasticity and inelastic deviation D_i , point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$. We have adopted a new terminology as *inelastic deviation* which is, as we have used in previous chapters, the difference between the *measured* (by force transducer) and *averaged* force. To check if we missed any variables for the prediction, we have used the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Since there is no perfect regression model to predict and there could be a certain variable that we would have missed, we had to look at the prediction deviation to see if it is random or systematic. According to the result shown in the table 5.2 and the figure 5.3, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we have missed is random and we don't perfectly predict every point. If those points are off in a systematic way, we would have possibly missed a certain variable. The prediction deviation is shown in the figure 5.4. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

Table 5.2
Unused data for the stretch ratio 1.3 of the rubber specimen. It shows the time T and rate-related stretch history H_{t_2} that mainly affect the deviation from the hyperelasticity and inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

λ_unused	T	H_{t_2}	D_i	$D_i - \hat{D}_f$
	[sec]	t_h=10	v_M - v_{avg}[volt]	[volt]
1.3	40.8	1.26413	0.17312	0.10183
	99.5	1.36032	0.07531	0.01561
	169.0	1.21431	0.14156	0.14356
	275.2	1.41162	0.01437	-0.03303
	362.3	1.24918	0.11062	0.10914
	731.1	1.23324	0.11094	0.11644
	966.0	1.38238	0.00312	-0.01990
	1059.4	1.27895	0.08156	0.06993
	1094.4	1.30153	0.06719	0.04929
	1353.5	1.35634	-0.01438	-0.00400
	1494.7	1.21026	0.09312	0.12269
	1733.8	1.3552	-0.00907	-0.01000
	1794.7	1.20368	0.07250	0.12298
	2125.6	1.37257	-0.03563	-0.03261
	2189.0	1.2253	0.05937	0.09657
	2281.1	1.38594	-0.02250	-0.04731
	2336.3	1.20586	0.06562	0.11106
	2617.7	1.36514	-0.02813	-0.03511
	2696.8	1.22554	0.05968	0.08699
	2965.2	1.31938	-0.02469	-0.00101
	3307.1	1.29023	-0.00813	0.01848
	3630.8	1.30153	0.03000	0.00251
	3638.9	1.2998	0.02750	0.00389
	3834.3	1.40472	-0.05157	-0.09258
	3893.2	1.23404	0.03906	0.05740
	4169.5	1.30037	-0.01563	-0.00641
	4373.2	1.39515	-0.05438	-0.09405
	4426.2	1.21825	0.03625	0.06154
	4626.2	1.31702	-0.03282	-0.02957
	4827.6	1.40265	-0.05531	-0.10907
	4881.8	1.21903	0.03125	0.05245
	4967.9	1.39037	-0.04313	-0.10079
	5019.8	1.20907	0.03281	0.05872
	5147.7	1.33891	-0.03844	-0.05856
	5171.7	1.28437	-0.01531	-0.01073
	5447.4	1.30282	-0.03844	-0.03215
	5555.9	1.36397	-0.04969	-0.08827

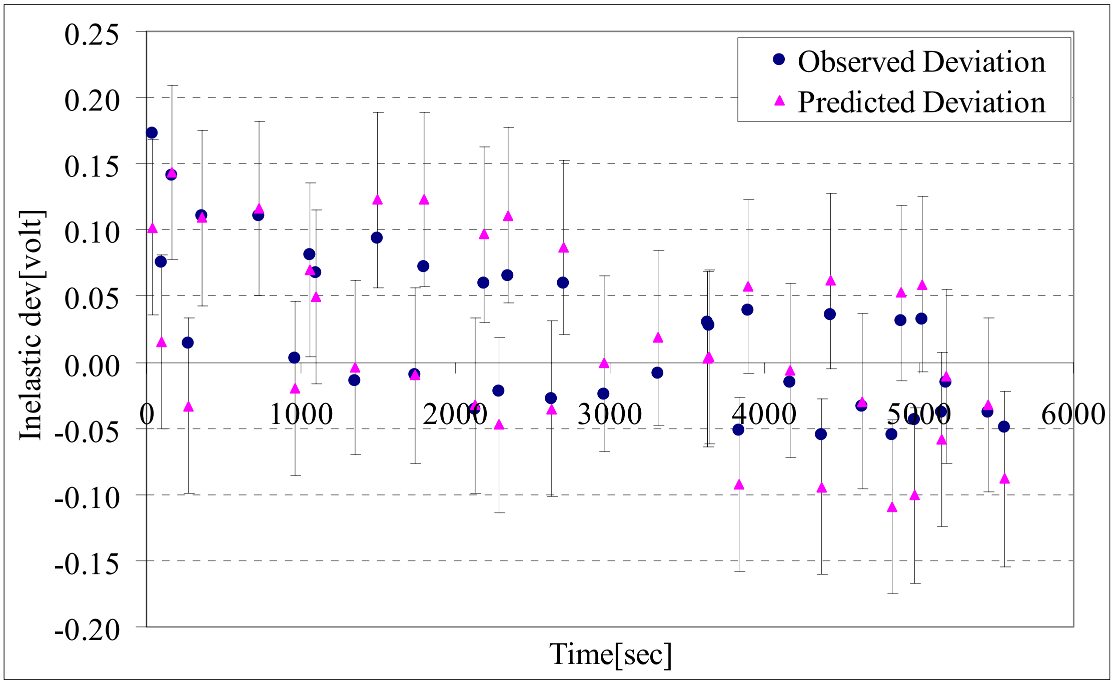

Fig.5.3. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.3 of the rubber specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 0.06605$.

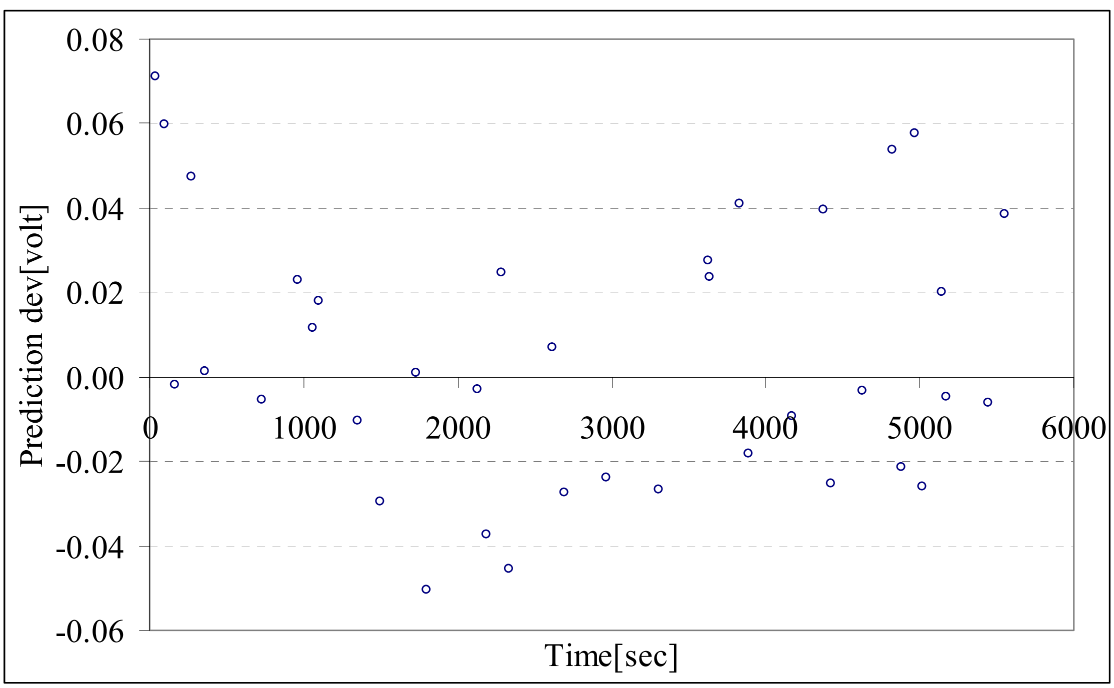

Fig.5.4. It shows the prediction deviation for the stretch ratio 1.3 of the rubber specimen. Note that all points are gathered at near zero but off in a random way.

Stretch Ratio 1.5: For λ = 1.5 of rubber specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.761, where α = 0.1, n = 17, k = 2 (for H_{t_2} and H_{t_1}). We used the regression model for λ = 1.5 which is $\hat{D}_{1.5} = 7.41 - 19.89H_{t_2} - 10.65H_{t_1}$ to obtain the $\hat{D}_f$ which is a point forecasting value. We have s_f = 0.04673 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 0.08230 for λ = 1.5 of the rubber specimen. The table 5.3 shows the unused data for the stretch ratio 1.5 of the rubber specimen. It also shows the time T, rate-related stretch history H_{t_2}, and long-time stretch history H_{t_1}. The major factors for the stretch ratio 1.5 of the rubber that affect the deviation from the hyperelasticity are rate-related stretch history H_{t_2} and long-time stretch history H_{t_1}. The inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$ are also shown in the table. We have adopted a new terminology as *inelastic deviation* which is, as we have used in previous chapters, the difference between the *measured* (by force transducer) and *averaged* force. To check if we missed any variables for the prediction, we have used the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Since there is no perfect regression model to predict and there could be a certain variable that we would have missed, we had to look at the prediction deviation to see if it is random or systematic. According to the result shown in the table 5.3 and the figure 5.5, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we have missed is random and we don't perfectly predict every point. If those points are off in a systematic way, we would have possibly missed a certain variable. The prediction deviation is shown in the figure 5.6. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

Table 5.3
Unused data for the stretch ratio 1.5 of the rubber specimen. It shows the time T, rate-related stretch history H_{t_2} and long-time stretch history H_{t_1}. The inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$ are also shown.

λ_unused	T	H_{t_2}	H_{t_1}	D_i	$\hat{D}_f$	$D_i - \hat{D}_f$
	[sec]	t_h=10	t_h=400	v_M - v_{avg}[volt]	[volt]	[volt]
1.5	73.0	1.46487	1.04371	0.11241	0.14521	-0.03280
	182.6	1.39386	1.09219	0.11772	0.16969	-0.05197
	262.7	1.62443	1.20121	-0.00103	0.00882	-0.00985
	517.2	1.57725	1.41416	-0.01728	-0.03012	0.01284
	775.4	1.51383	1.38047	0.05022	0.01526	0.03496
	883.6	1.43266	1.36893	0.05897	0.06370	-0.00473
	949.4	1.58119	1.41597	0.00179	-0.03285	0.03464
	1117.1	1.41748	1.30227	0.05991	0.09246	-0.03255
	1317.0	1.50673	1.49416	-0.02103	-0.01553	-0.00550
	1509.2	1.40302	1.40458	0.06710	0.06921	-0.00211
	1671.2	1.49463	1.49856	-0.00197	-0.01018	0.00821
	1806.3	1.37534	1.3959	0.05116	0.08718	-0.03602
	1978.6	1.53097	1.49202	-0.03697	-0.02829	-0.00868
	2107.2	1.57193	1.56377	-0.04415	-0.07287	0.02872
	2206.5	1.421	1.51153	0.02928	0.02660	0.00268
	2264.8	1.58104	1.52559	-0.02728	-0.06625	0.03897
	2348.0	1.37787	1.52318	0.03835	0.04691	-0.00856
	2575.9	1.48769	1.48314	-0.01790	-0.00163	-0.01627
	2590.6	1.50916	1.47753	-0.02071	-0.01180	-0.00891
	2714.5	1.4225	1.45058	0.03272	0.04439	-0.01167
	2897.8	1.48558	1.53595	-0.03540	-0.01659	-0.01881
	2922.5	1.51815	1.52018	-0.03571	-0.02980	-0.00591
	3151.0	1.49865	1.48718	-0.01446	-0.00893	-0.00553
	3246.2	1.55972	1.56096	-0.04384	-0.06525	0.02141
	3331.5	1.4243	1.51916	0.00803	0.02244	-0.01441
	3673.3	1.44308	1.51717	0.01616	0.01266	0.00350
	3821.4	1.61723	1.56414	-0.06322	-0.09805	0.03483
	3923.0	1.47094	1.53235	0.00554	-0.00739	0.01293
	4007.1	1.47982	1.47094	-0.00665	0.00645	-0.01310
	4104.9	1.50881	1.46627	-0.01603	-0.00816	-0.00787
	4213.3	1.47016	1.49475	-0.00165	0.00452	-0.00617

Table 5.3 continued

$\lambda_{\_unused}$	T	H_{t_2}	H_{t_1}	D_i	$\hat{D}_f$	$D_i - \hat{D}_f$
	[sec]	$t_h=10$	$t_h=400$	v_M - v_{avg}[volt]	[volt]	[volt]
1.5	4358.4	1.60887	1.48455	-0.06196	-0.06911	0.00715
	4442.1	1.41232	1.47242	0.01116	0.04335	-0.03219
	4588.2	1.55532	1.52805	-0.05447	-0.05277	-0.00170
	4726.3	1.42957	1.4964	0.00303	0.02648	-0.02345
	4814.4	1.61745	1.54292	-0.06728	-0.09169	0.02441
	4898.3	1.41618	1.47224	0.00866	0.04127	-0.03261
	4952.7	1.58455	1.48444	-0.04322	-0.05562	0.01240
	5032.5	1.38449	1.47171	0.01679	0.05896	-0.04217
	5121.2	1.5719	1.47999	-0.06228	-0.04726	-0.01502
	5417.1	1.59402	1.54177	-0.07259	-0.07837	0.00578
	5515.3	1.49255	1.53434	-0.02540	-0.01996	-0.00544
	5524.8	1.50027	1.53476	-0.02697	-0.02436	-0.00261

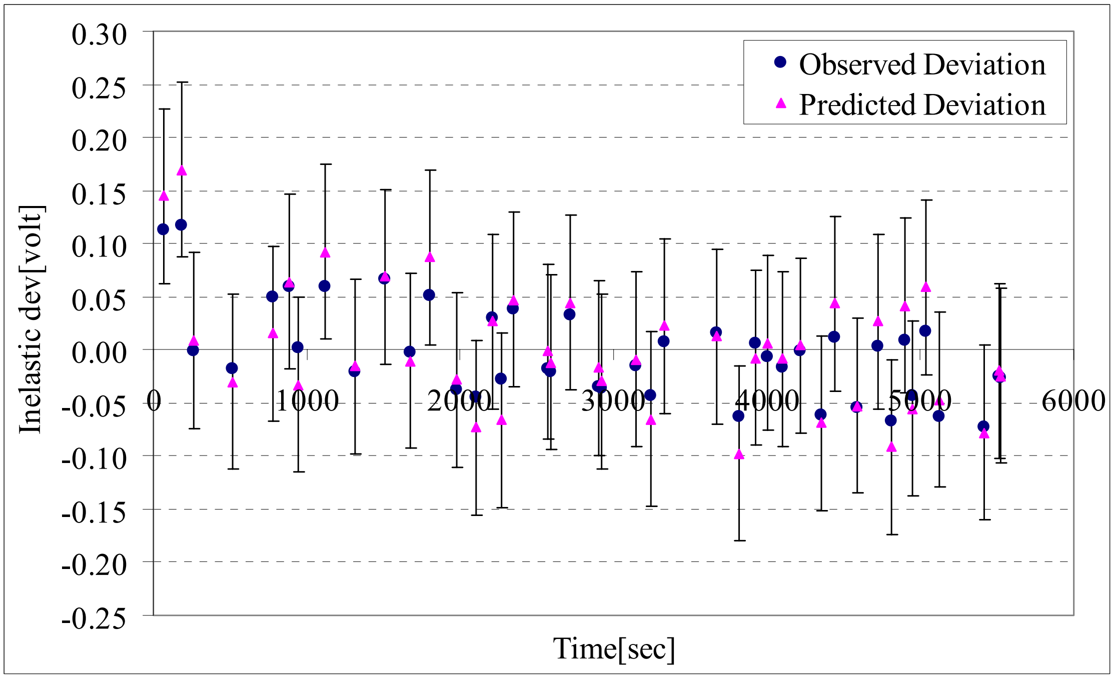

Fig.5.5. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.5 of the rubber specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 0.08230$.

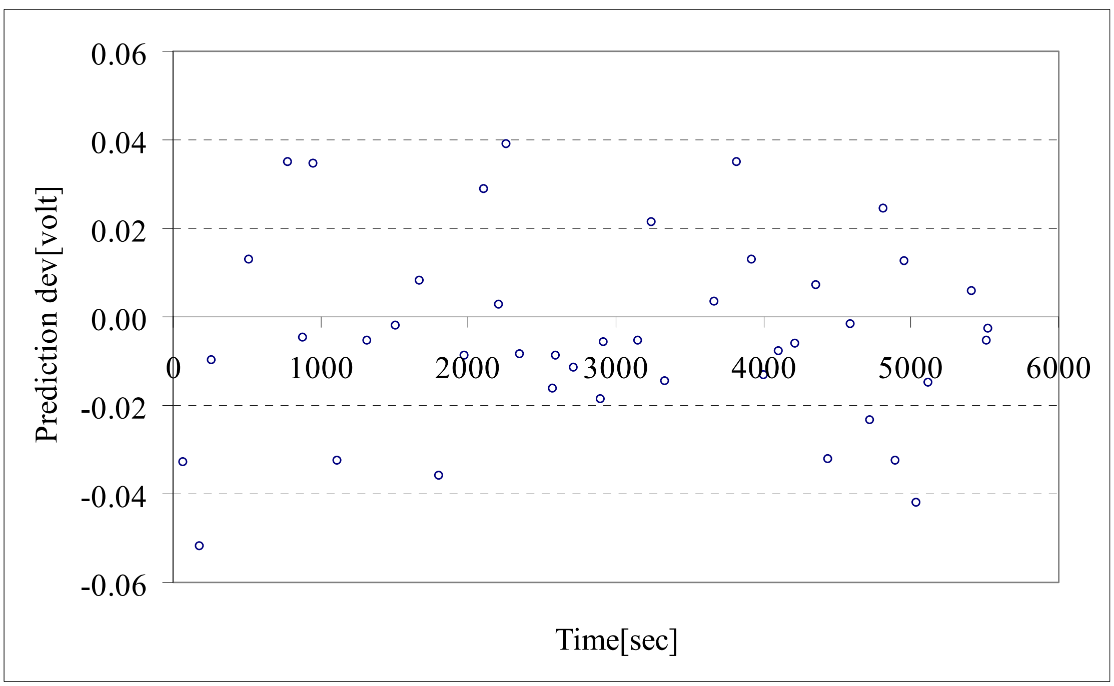

Fig.5.6. It shows the prediction deviation for the stretch ratio 1.5 of the rubber specimen. Note that all points are gathered at near zero but off in a random way.

Stretch Ratio 1.7: For λ = 1.7 of rubber specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.796, where α = 0.1, n = 13, k = 1 (for H_{t_2}). We used the regression model for λ = 1.7 which is $\hat{D}_{1.7} = 15.06 - 54.41H_{t_2}$ to obtain the $\hat{D}_f$ which is a point forecasting value. We have s_f = 0.01790 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 0.03215 for λ = 1.7 of the rubber specimen. The table 5.4 shows the unused data for the stretch ratio 1.7 of the rubber specimen. It also shows the time T and rate-related stretch history H_{t_2} that mainly affect the deviation from the hyperelasticity and inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$. We have adopted a new terminology as *inelastic deviation* which is, as we have used in previous chapters, the difference between the *measured* (by force transducer) and *averaged* force. To check if we missed any variables for the prediction, we have used the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Since there is no perfect regression model to predict and there could be a certain variable that we would have missed, we had to look at the prediction deviation to see if it is random or systematic. According to the result shown in the table 5.4 and the figure 5.7, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we are missing is random and we don't perfectly predict every point. If those points are off in a systematic way, we would have possibly missed a certain variable. The prediction deviation is shown in the figure 5.8. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

Table 5.4
Unused data for the stretch ratio 1.7 of the rubber specimen. It shows the time T and rate-related stretch history H_{t_2} that mainly affect the deviation from the hyperelasticity and inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

$\lambda_{\_unused}$	T	H_{t_2}	D_i	$\hat{D}_f$	$D_i - \hat{D}_f$
	[sec]	t_h=10	v_M - v_{avg}[volt]	[volt]	[volt]
1.7	197.8	1.60838	0.10493	0.05706	0.04787
	251.1	1.79576	0.01430	-0.08599	0.10029
	429.4	1.64134	0.05961	0.03190	0.02771
	499.4	1.77385	-0.00320	-0.06926	0.06606
	910.3	1.66142	0.04899	0.01657	0.03242
	927.8	1.7122	0.03180	-0.02220	0.05400
	1132.6	1.61229	0.05774	0.05408	0.00366
	1221.1	1.72378	-0.01070	-0.03104	0.02034
	1261.4	1.71154	-0.00882	-0.02169	0.01287
	1542.4	1.70843	0.02680	-0.01932	0.04612
	1818.5	1.58761	0.04368	0.07292	-0.02924
	1897.2	1.72034	-0.01695	-0.02841	0.01146
	1897.3	1.7199	-0.01757	-0.02807	0.01051
	1938.2	1.70649	-0.01570	-0.01784	0.00214
	2043.2	1.67517	-0.00320	0.00607	-0.00927
	2241.7	1.70034	0.00805	-0.01314	0.02119
	2361.0	1.59056	0.03305	0.07066	-0.03761
	2416.4	1.76288	-0.02601	-0.06089	0.03488
	2853.4	1.77011	-0.03882	-0.06641	0.02759
	3195.1	1.66976	-0.00132	0.01020	-0.01152
	3349.1	1.6215	0.00461	0.04704	-0.04243
	3489.0	1.7897	-0.05038	-0.08136	0.03098
	3808.8	1.79063	-0.04976	-0.08207	0.03231
	4284.5	1.6278	0.00368	0.04224	-0.03855
	4345.2	1.78685	-0.04663	-0.07919	0.03256
	4467.2	1.6681	-0.00382	0.01147	-0.01529
	4531.8	1.69439	-0.02289	-0.00860	-0.01429
	4745.4	1.62806	-0.00226	0.04204	-0.04430
	4801.8	1.79074	-0.04663	-0.08216	0.03553
	4929.7	1.68925	-0.01038	-0.00468	-0.00570
	5045.6	1.59771	0.00712	0.06521	-0.05809
	5102.1	1.76869	-0.04601	-0.06532	0.01931
	5273.6	1.67274	-0.01476	0.00793	-0.02269
	5328.0	1.68212	-0.02820	0.00077	-0.02897
	5402.4	1.78701	-0.05789	-0.07931	0.02142

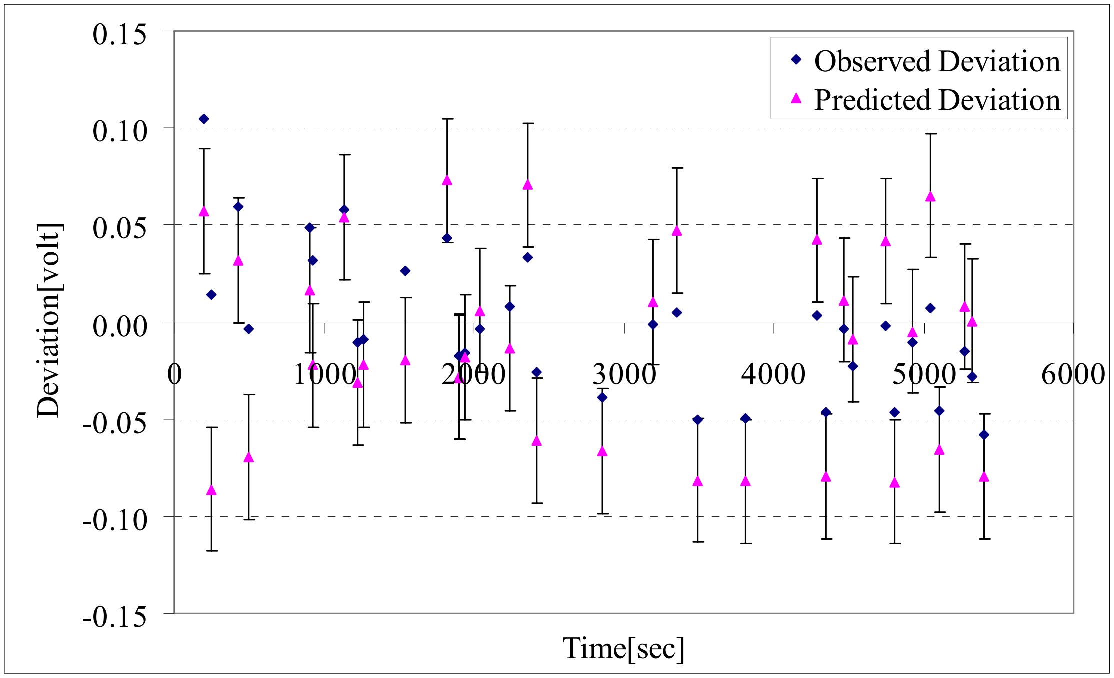

Fig.5.7. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.7 of the rubber specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 0.03215$.

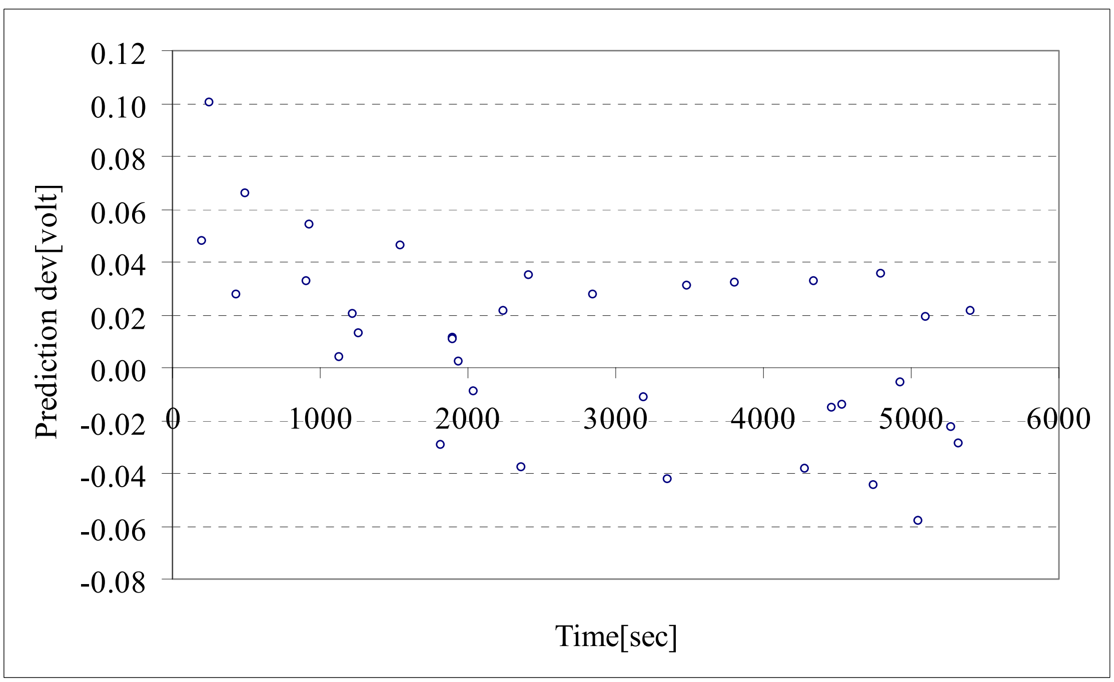

Fig.5.8. It shows the prediction deviation for the stretch ratio 1.7 of the rubber specimen. Note that all points are gathered at near zero but off in a random way.

Stretch Ratio 1.9: For λ = 1.9 of rubber specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.812, where α = 0.1, n = 13, k = 1 (for T and H_{t_1}). We used the regression model for λ = 1.9 which is $\hat{D}_{1.9} = 2.14 - 0.95T - 6.94H_{t_1}$ to obtain the $\hat{D}_f$ which is a point forecasting value. We have s_f = 0.0253 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 0.04584 for λ = 1.9 of the rubber specimen. The table 5.5 shows the unused data for the stretch ratio 1.9 of the rubber specimen. It also shows the time T and long-time stretch history H_{t_1} that mainly affect the deviation from the hyperelasticity and inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$. We have adopted a new terminology as *inelastic deviation* which is, as we have used in previous chapters, the difference between the *measured* (by force transducer) and *averaged* force. To check if we missed any variables for the prediction, we have used the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Since there is no perfect regression model to predict and there could be a certain variable that we would have missed, we had to look at the prediction deviation to see if it is random or systematic. According to the result shown in the table 5.5 and the figure 5.9, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we are missing is random and we don't perfectly predict every point. If those points are off in a systematic way, we would have possibly missed a certain variable. The prediction deviation is shown in the figure 5.10. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

Table 5.5
Unused data for the stretch ratio 1.9 of the rubber specimen. It shows the time T and long-time stretch history H_{t_1}. It also shows the inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

$\lambda_{\_unused}$	T	H_{t_1}	D_i	$\hat{D}_f$	$D_i - \hat{D}_f$
	[sec]	t_h=400	v_M - v_{avg}[volt]	[volt]	[volt]
1.9	221.8	1.14375	0.06688	0.05871	0.00816
	231.5	1.16136	0.05469	0.05642	-0.00173
	463.5	1.34284	0.03750	0.03179	0.00570
	473.0	1.35842	0.02938	0.02976	-0.00038
	1167.4	1.35749	0.02500	0.02476	0.00024
	1838.4	1.39815	0.02625	0.01469	0.01156
	1848.3	1.40355	0.01500	0.01393	0.00106
	2382.4	1.54002	0.01032	-0.00723	0.01755
	2386.8	1.54094	0.00656	-0.00738	0.01394
	2817.4	1.49814	-0.00968	-0.00515	-0.00454
	2826.4	1.5114	-0.01312	-0.00689	-0.00624
	3379.4	1.50878	-0.00843	-0.01063	0.00219
	3383.4	1.50886	-0.01031	-0.01067	0.00035
	3461.6	1.54695	-0.02968	-0.01605	-0.01363
	3467.3	1.55062	-0.02875	-0.01656	-0.01220
	3782.4	1.54495	-0.02062	-0.01816	-0.00246
	4310.6	1.49177	-0.00906	-0.01533	0.00627
	4324.1	1.49347	-0.01812	-0.01565	-0.00248
	4775.4	1.52669	-0.01469	-0.02317	0.00847
	5070.0	1.47555	-0.01125	-0.01888	0.00762
	5371.8	1.50703	-0.02687	-0.02507	-0.00180
	5381.4	1.51279	-0.03187	-0.02587	-0.00600

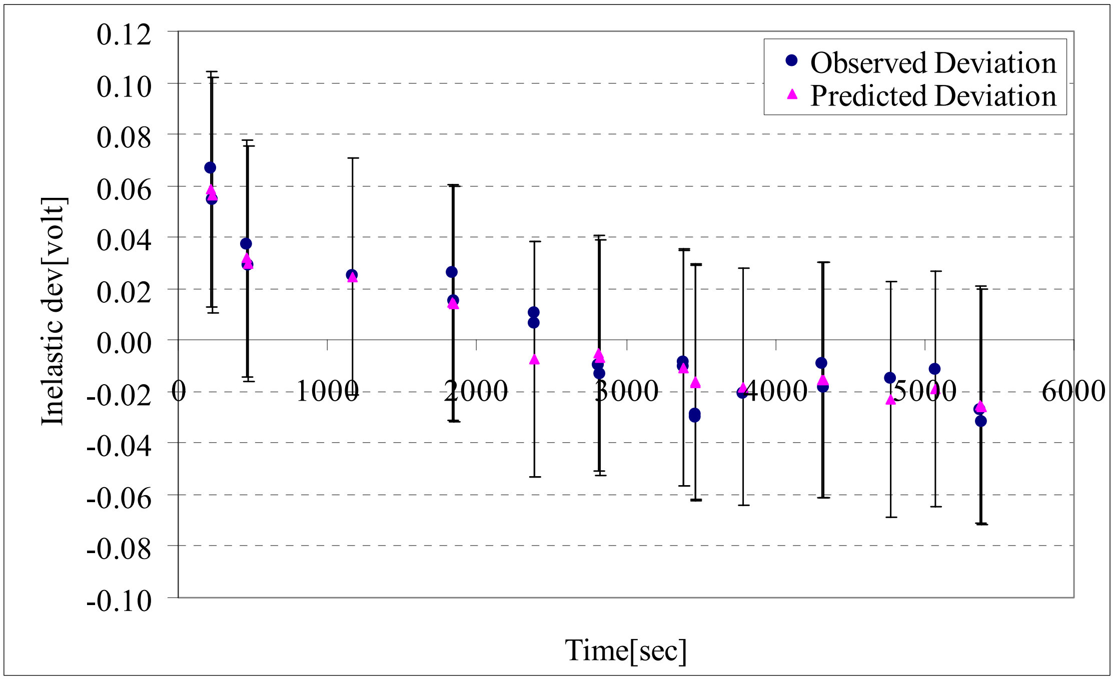

Fig.5.9. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.7 of the rubber specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 0.04584$.

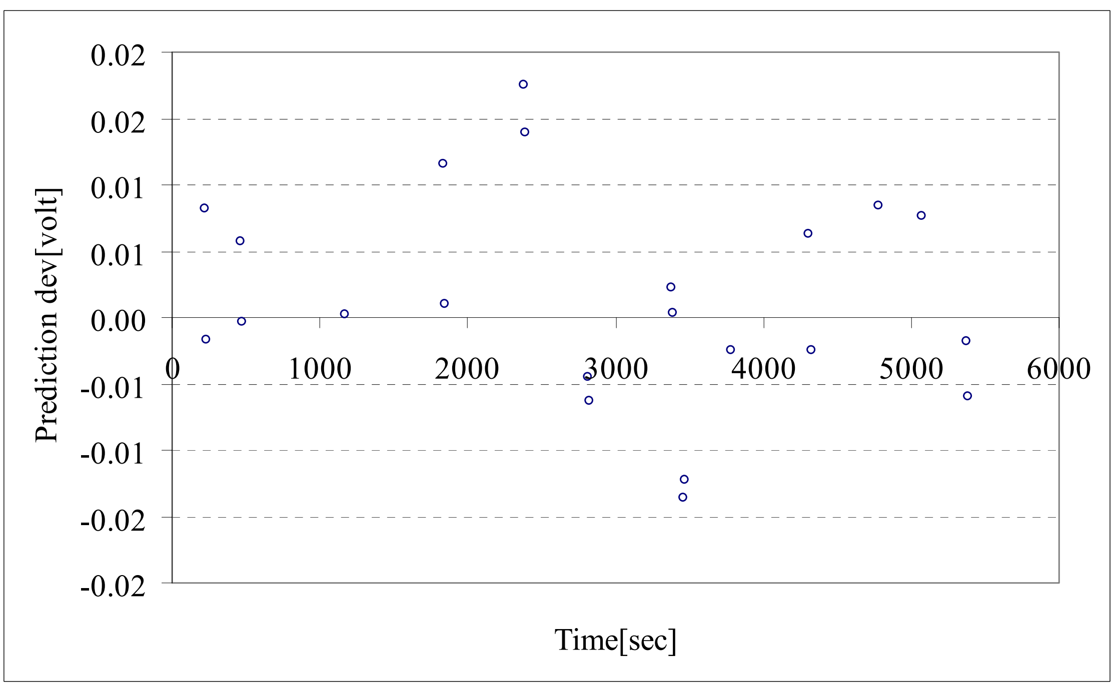

Fig.5.10. It shows the prediction deviation for the stretch ratio 1.9 of the rubber specimen. Note that all points are gathered at near zero but off in a random way.

Tissue

Stretch Ratio 1.1: For λ = 1.1 of tissue specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.753, where α = 0.1, n = 17, and k = 1 (for T). We used the regression model for λ = 1.1 which is $\hat{D}_{1.1} = 0.34 - 1.70T$ to obtain the $\hat{D}_f$ which is a point forecasting value. We have s_f = 0.0251 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 0.04400 for λ = 1.1 of the tissue specimen. The table 5.6 shows the unused data for the stretch ratio 1.1 of the tissue specimen. It also shows the time T that mainly affects the inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$. We have adopted a new terminology as *inelastic deviation* which is, as we have used in previous chapters, the difference between the *measured* (by force transducer) and *averaged* force. To check if we missed any variables for the prediction, we have used the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Since there is no perfect regression model to predict and there could be a certain variable that we would have missed, we had to look at the prediction deviation to see if it is random or systematic. According to the result shown in the table 5.6 and the figure 5.11, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we are missing is random and we don't perfectly predict every point. Those points are off in a random way. If those points are off in a systematic way, we would have possibly missed a certain variable. The prediction deviation is shown in the figure 5.12. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

Table 5.6
Unused data for the stretch ratio 1.1 of the tissue specimen. It shows the time *T* which is the only major factor that gives rise to the inelastic deviation D_i. It also shows the point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

$\lambda_{\_unused}$	T	D_i	$\hat{D}_f$	$D_i - \hat{D}_f$
	[sec]	v_M - v_{avg}[volt]	[volt]	[volt]
1.1	13.6	0.06910	0.03479	0.03431
	146.9	0.03316	0.03291	0.00026
	297.9	0.02003	0.03077	-0.01074
	300.7	0.02034	0.03073	-0.01039
	307.1	0.02222	0.03064	-0.00842
	324.3	0.02441	0.03040	-0.00598
	612.2	0.02097	0.02633	-0.00535
	697.3	0.03035	0.02512	0.00523
	989.9	0.01066	0.02098	-0.01032
	1011.2	0.01878	0.02068	-0.00190
	1409.8	0.01941	0.01504	0.00437
	1473.8	0.02253	0.01414	0.00840
	2161.9	0.00441	0.00440	0.00001
	2315.9	0.00753	0.00223	0.00531
	2669.0	0.01347	-0.00277	0.01624
	3536.5	-0.01997	-0.01504	-0.00493
	3863.7	-0.01403	-0.01967	0.00564
	4849.7	-0.04747	-0.03362	-0.01385
	4991.4	-0.04590	-0.03562	-0.01028
	5600.0	-0.04934	-0.04423	-0.00511

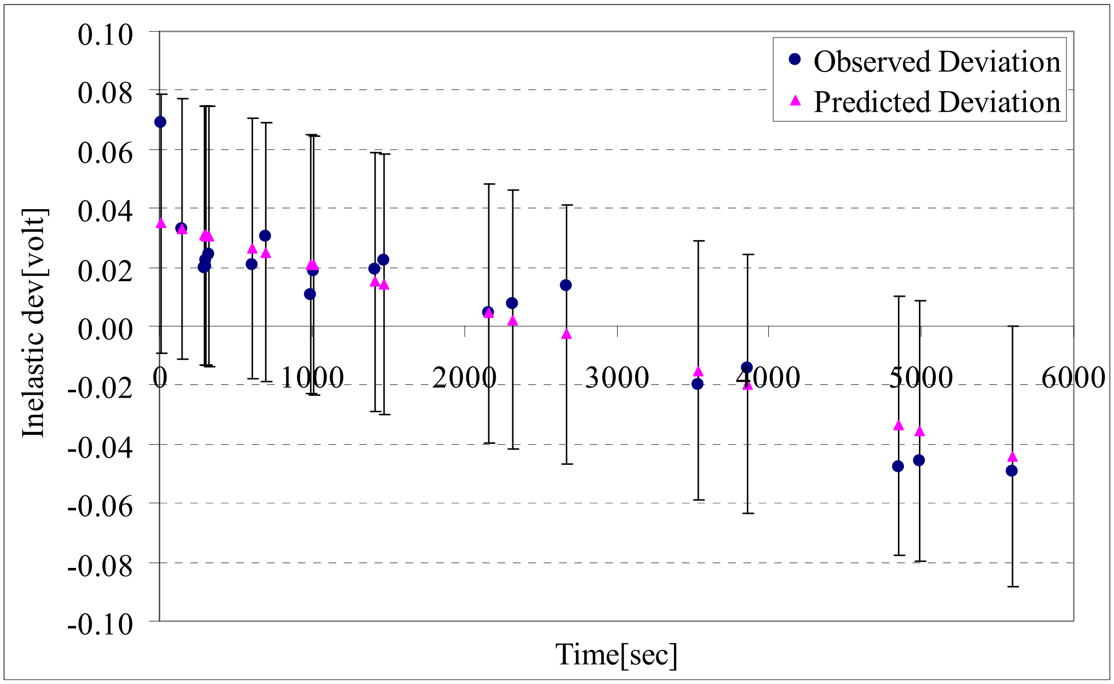

Fig.5.11. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.1 of the tissue specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 0.04400$.

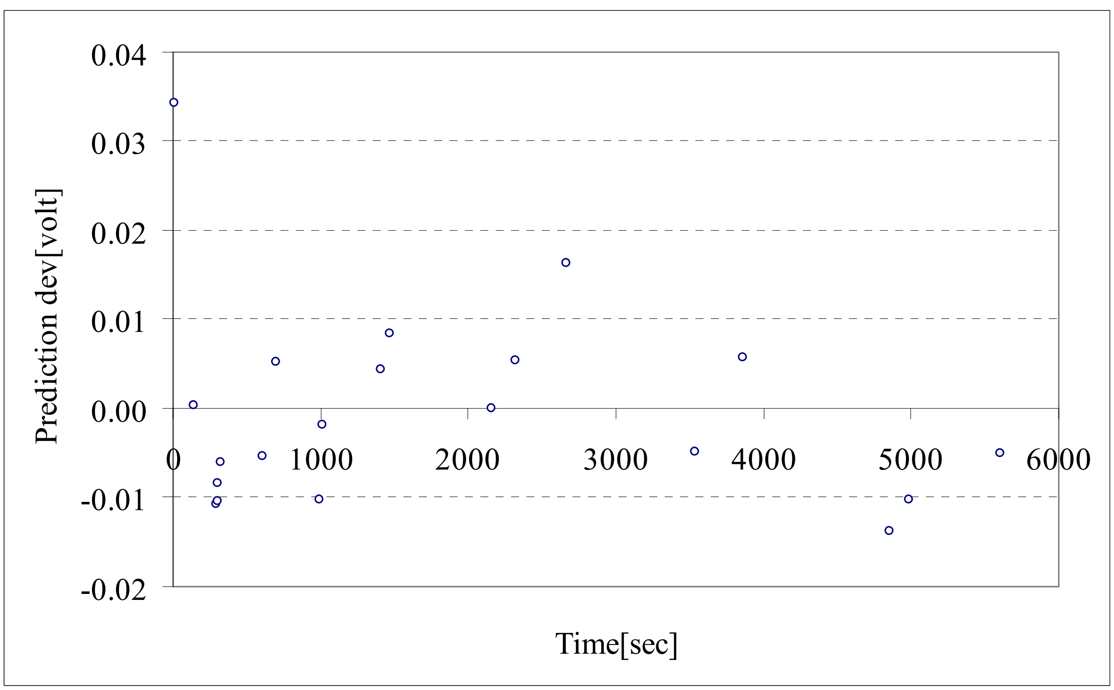

Fig.5.12. It shows the prediction deviation for the stretch ratio 1.1 of the tissue specimen. Note that all points are gathered at near zero but off in a random way.

Stretch Ratio 1.3: For λ = 1.3 of tissue specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.833, where α = 0.1, n = 11, and k = 1 (for T). We used the regression model for λ = 1.3 which is $\hat{D}_{1.3} = 0.64 - 2.29T$ to obtain the $\hat{D}_f$ which is a point forecasting value. We have s_f = 0.02617 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 0.04797 for λ = 1.3 of the tissue specimen. The table 5.7 shows the unused data for the stretch ratio 1.3 of the tissue specimen. It also shows the time T that mainly affects the inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$. We have adopted a new terminology as *inelastic deviation* which is, as we have used in previous chapters, the difference between the *measured* (by force transducer) and *averaged* force. To check if we missed any variables for the prediction, we have used the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Since there is no perfect regression model to predict and there could be a certain variable that we would have missed, we had to look at the prediction deviation to see if it is random or systematic. According to the result shown in the table 5.7 and the figure 5.13, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we are missing is random and we don't perfectly predict every point. If those points are off in a systematic way, we would have possibly missed a certain variable. The prediction deviation is shown in the figure 5.14. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

Table 5.7
Unused data for the stretch ratio 1.3 of the tissue specimen. It shows the time *T* which is the only major factor that gives rise to the inelastic deviation D_i. It also shows the point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

λ_{unused}	T	D_i	$\hat{D}_f$	$D_i - \hat{D}_f$
	[sec]	v_M - v_{avg}[volt]	[volt]	[volt]
1.3	40.8	0.09427	0.05045	0.04382
	99.5	0.04895	0.04958	-0.00063
	169.0	0.06520	0.04855	0.01665
	362.3	0.05301	0.04569	0.00732
	731.1	0.05770	0.04022	0.01748
	1059.4	0.04270	0.03536	0.00734
	1094.4	0.03864	0.03484	0.00380
	1353.5	0.01676	0.03100	-0.01424
	1733.8	0.01333	0.02537	-0.01204
	1794.7	0.03395	0.02446	0.00949
	2125.6	0.00208	0.01956	-0.01748
	2189.0	0.03145	0.01862	0.01283
	2617.7	0.01270	0.01227	0.00043
	2696.8	0.03395	0.01110	0.02285
	2965.2	-0.01324	0.00712	-0.02036
	3307.1	-0.00324	0.00206	-0.00530
	3630.8	0.01208	-0.00274	0.01482
	3638.9	0.01052	-0.00286	0.01338
	3834.3	-0.01449	-0.00575	-0.00874
	3893.2	0.01489	-0.00663	0.02152
	4169.5	-0.00917	-0.01072	0.00155
	4373.2	-0.02198	-0.01374	-0.00824
	4426.2	0.00739	-0.01452	0.02191
	4626.2	-0.01167	-0.01749	0.00582
	4827.6	-0.04199	-0.02047	-0.02152
	4967.9	-0.03824	-0.02255	-0.01569
	5147.7	-0.03074	-0.02521	-0.00553
	5171.7	-0.02105	-0.02557	0.00452
	5447.4	-0.03011	-0.02965	-0.00046

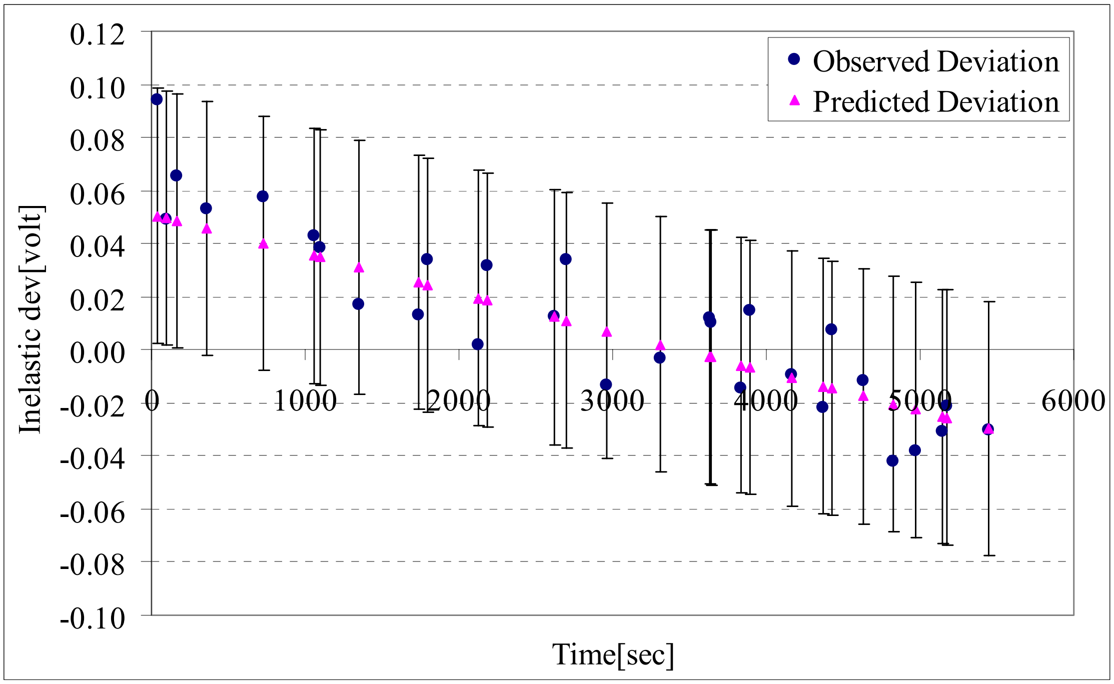

Fig.5.13. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.3 of the tissue specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 0.04797$.

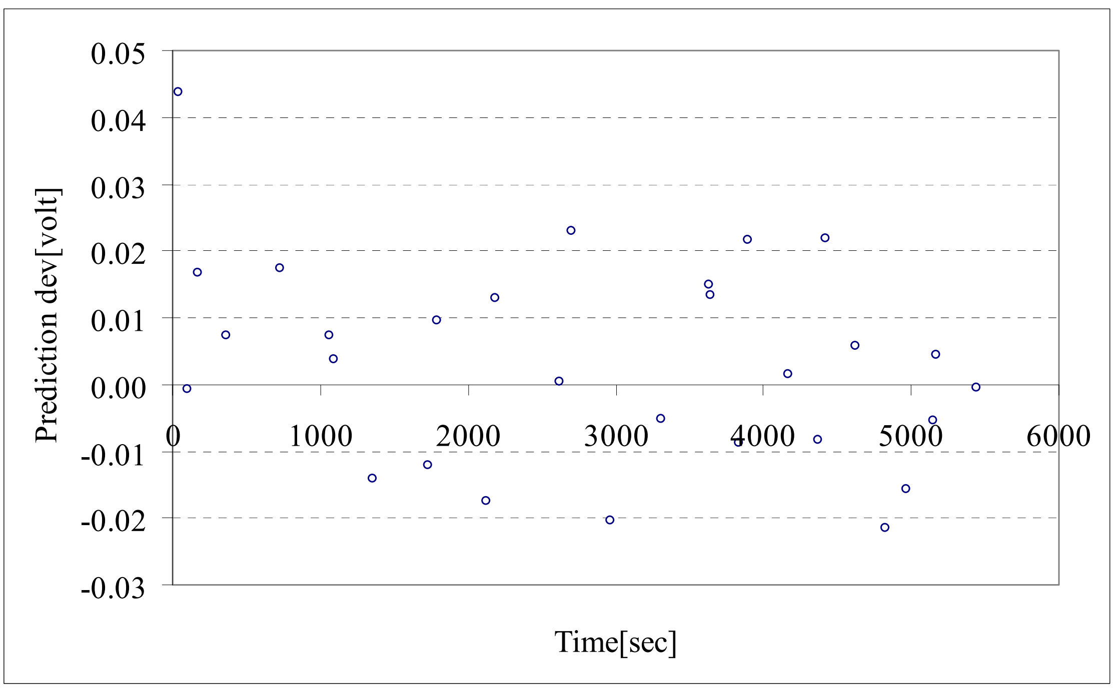

Fig.5.14. It shows the prediction deviation for the stretch ratio 1.3 of the tissue specimen. Note that all points are gathered at near zero but off in a random way.

Stretch Ratio 1.5: For λ = 1.5 of tissue specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.771, where α = 0.1, n = 17, and k = 3 (for T, H_{t_2} and H_{t_1}). We used the regression model for λ = 1.5 which is $\hat{D}_{1.5} = 5.85 - T - 17.48H_{t_2} - 5.75H_{t_1}$ to obtain the $\hat{D}_f$ which is a point forecasting value. We have s_f = 0.03638 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 0.06443 for λ = 1.5 of the tissue specimen. The table 5.8 shows the unused data for the stretch ratio 1.5 of the tissue specimen. It also shows the time T that mainly affects the inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$. We have adopted a new terminology as *inelastic deviation* which is, as we have used in previous chapters, the difference between the *measured* (by force transducer) and *averaged* force. To check if we missed any variables for the prediction, we have used the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Since there is no perfect regression model to predict and there could be a certain variable that we would have missed, we had to look at the prediction deviation to see if it is random or systematic. According to the result shown in the table 5.8 and the figure 5.15, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we are missing is random and we don't perfectly predict every point. If those points are off in a systematic way, we would have possibly missed a certain variable. The prediction deviation is shown in the figure 5.16. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

Table 5.8
Unused data for the stretch ratio 1.5 of the tissue specimen. It shows the time T, rate-related stretch history H_{t_2}, long-time stretch history H_{t_1}, and the inelastic deviation D_i. It also shows the point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

λ_unused	T [sec]	H_{t_2} t_h=10	H_{t_1} t_h=400	D_i v_M - v_{avg}[volt]	$\hat{D}_f$ [volt]	$D_i - \hat{D}_f$ [volt]
1.5	73.0	1.46487	1.04371	0.08346	0.09913	-0.01567
	517.2	1.57725	1.41416	0.00190	-0.00065	0.00255
	775.4	1.51383	1.38047	0.04783	0.02642	0.02141
	883.6	1.43266	1.36893	0.05690	0.05920	-0.00230
	1117.1	1.41748	1.30227	0.04658	0.07175	-0.02517
	1317.0	1.50673	1.49416	-0.00092	0.00800	-0.00892
	1671.2	1.49463	1.49856	0.01190	0.00839	0.00351
	1806.3	1.37534	1.3959	0.03533	0.06843	-0.03310
	1978.6	1.53097	1.49202	-0.01811	-0.00855	-0.00956
	2107.2	1.57193	1.56377	-0.01748	-0.03600	0.01852
	2206.5	1.421	1.51153	0.02439	0.03022	-0.00583
	2575.9	1.48769	1.48314	0.00190	0.00345	-0.00155
	2590.6	1.50916	1.47753	0.00564	-0.00452	0.01016
	2714.5	1.4225	1.45058	0.03065	0.03237	-0.00172
	2897.8	1.48558	1.53595	-0.03279	-0.00634	-0.02645
	2922.5	1.51815	1.52018	-0.03217	-0.01747	-0.01470
	3151.0	1.49865	1.48718	-0.01748	-0.00770	-0.00978
	3246.2	1.55972	1.56096	-0.02904	-0.04309	0.01405
	3331.5	1.4243	1.51916	-0.00154	0.01569	-0.01723
	3673.3	1.44308	1.51717	0.00783	0.00476	0.00307
	3821.4	1.61723	1.56414	-0.04310	-0.07270	0.02960
	3923.0	1.47094	1.53235	-0.00092	-0.01112	0.01020
	4007.1	1.47982	1.47094	-0.00560	-0.00726	0.00166
	4104.9	1.50881	1.46627	-0.01811	-0.01926	0.00115
	4213.3	1.47016	1.49475	-0.01123	-0.00886	-0.00237
	4358.4	1.60887	1.48455	-0.04810	-0.06441	0.01631
	4442.1	1.41232	1.47242	-0.00404	0.01476	-0.01880
	4726.3	1.42957	1.4964	-0.02592	0.00156	-0.02748
	5032.5	1.38449	1.47171	-0.02061	0.01957	-0.04018
	5417.1	1.59402	1.54177	-0.06560	-0.07769	0.01209
	5515.3	1.49255	1.53434	-0.03842	-0.03726	-0.00116
	5524.8	1.50027	1.53476	-0.04092	-0.04050	-0.00042

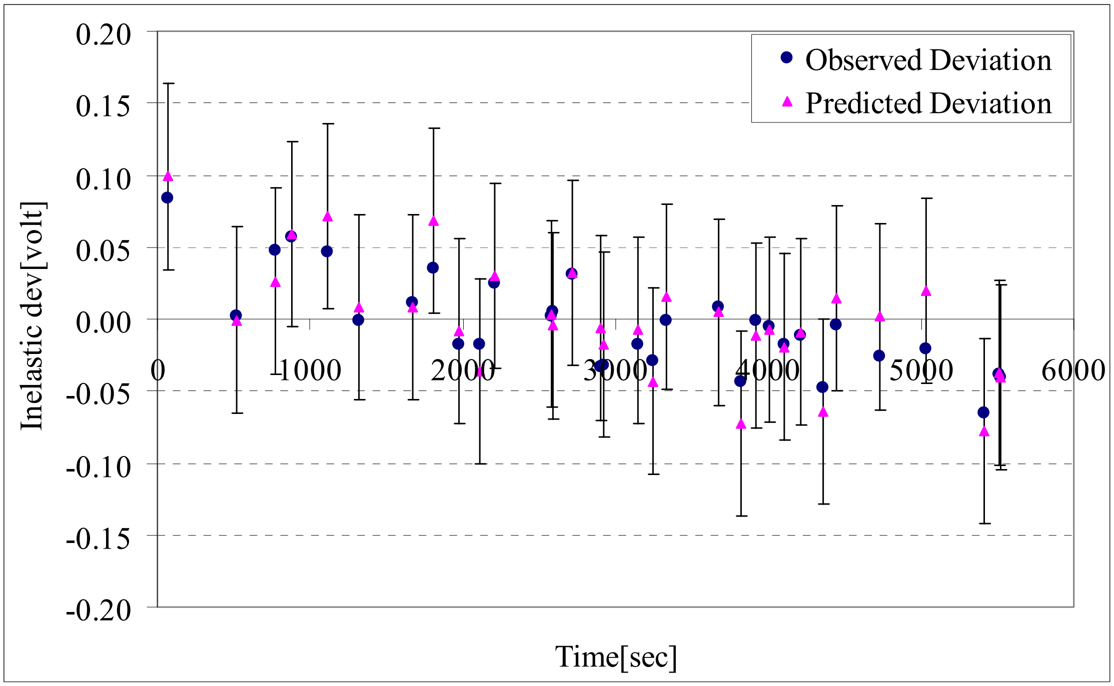

Fig.5.15. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.5 of the tissue specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 0.06443$.

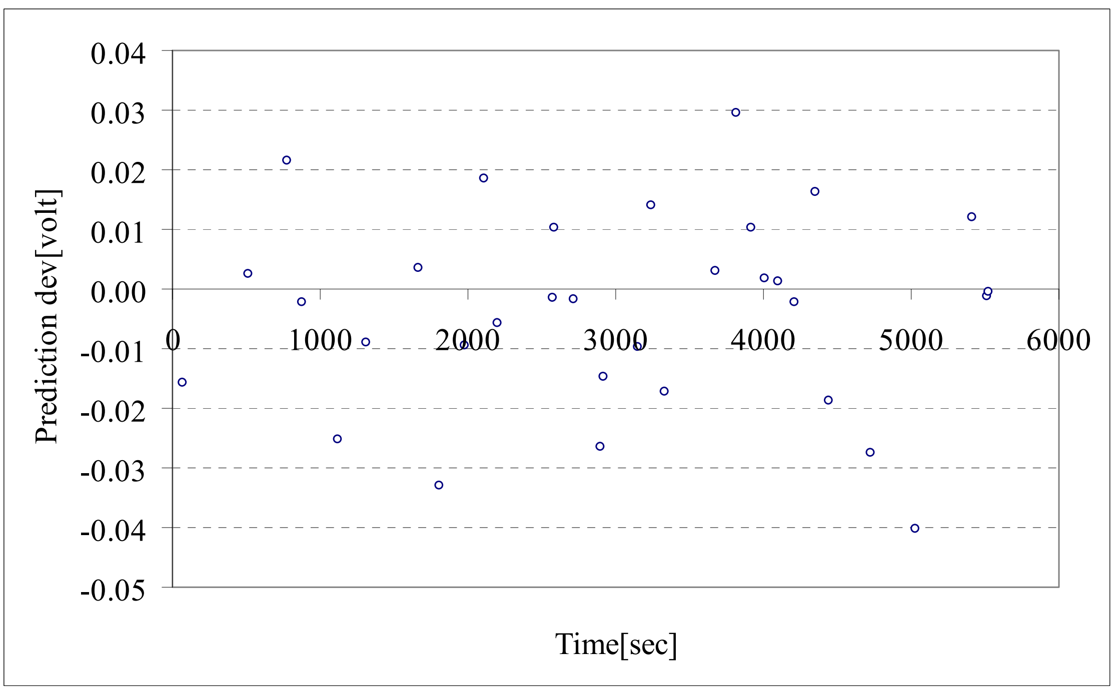

Fig.5.16. It shows the prediction deviation for the stretch ratio 1.5 of the tissue specimen. Note that all points are gathered at near zero but off in a random way.

Stretch Ratio 1.7: For λ = 1.7 of tissue specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.812, where α = 0.1, n = 13, and k = 2 (for T and H_{t_2}). We used the regression model for λ = 1.7 which is $\hat{D}_{1.7} = 10.42 - 0.97T - 36.37H_{t_2}$ to obtain the $\hat{D}_f$ which is a point forecasting value. We have s_f = 0.041690 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 0.07554 for λ = 1.7 of the tissue specimen. The table 5.9 shows the unused data for the stretch ratio 1.7 of the tissue specimen. It also shows the time T that mainly affects the inelastic deviation D_i, point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$. We have adopted a new terminology as *inelastic deviation* which is, as we have used in previous chapters, the difference between the *measured* (by force transducer) and *averaged* force. To check if we missed any variables for the prediction, we have used the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Since there is no perfect regression model to predict and there could be a certain variable that we would have missed, we had to look at the prediction deviation to see if it is random or systematic. According to the result shown in the table 5.9 and the figure 5.17, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we are missing is random and we don't perfectly predict every point. If those points are off in a systematic way, we would have possibly missed a certain variable. The prediction deviation is shown in the figure 5.18. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

Table 5. 9

Unused data for the stretch ratio 1.7 of the tissue specimen. It shows the time T, rate-related stretch history H_{t_2} , and the inelastic deviation D_i. It also shows the point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

λ_{unused}	T	H_{t_2}	D_i	$\hat{D}_f$	$D_i - \hat{D}_f$
	[sec]	t_h=10	v_M - v_{avg}[volt]	[volt]	[volt]
1.7	197.8	1.60838	0.21706	0.12777	0.08929
	251.1	1.79576	0.00769	-0.08644	0.09413
	429.4	1.64134	0.13175	0.08669	0.04506
	499.4	1.77385	-0.01669	-0.06529	0.04860
	910.3	1.66142	0.10987	0.05646	0.05341
	927.8	1.7122	0.07331	-0.00164	0.07495
	1132.6	1.61229	0.12175	0.10901	0.01274
	1221.1	1.72378	-0.02794	-0.01932	-0.00862
	1261.4	1.71154	-0.01481	-0.00599	-0.00882
	1542.4	1.70843	0.05175	-0.00675	0.05850
	1897.2	1.72034	-0.04262	-0.02575	-0.01687
	1897.3	1.7199	-0.04387	-0.02525	-0.01862
	1938.2	1.70649	-0.02669	-0.01060	-0.01609
	2043.2	1.67517	0.01363	0.02346	-0.00983
	2241.7	1.70034	0.03175	-0.00825	0.04000
	2416.4	1.76288	-0.05075	-0.08214	0.03139
	2853.4	1.77011	-0.08512	-0.09707	0.01195
	3349.1	1.6215	0.01300	0.06459	-0.05159
	4345.2	1.78685	-0.10825	-0.13897	0.03072
	4467.2	1.6681	-0.02044	-0.00560	-0.01484
	4531.8	1.69439	-0.04481	-0.03653	-0.00828
	4929.7	1.68925	-0.05356	-0.03677	-0.01679
	5045.6	1.59771	-0.03356	0.06571	-0.09927
	5102.1	1.76869	-0.11981	-0.12988	0.01007
	5273.6	1.67274	-0.04200	-0.02323	-0.01877
	5328.0	1.68212	-0.06169	-0.03475	-0.02694

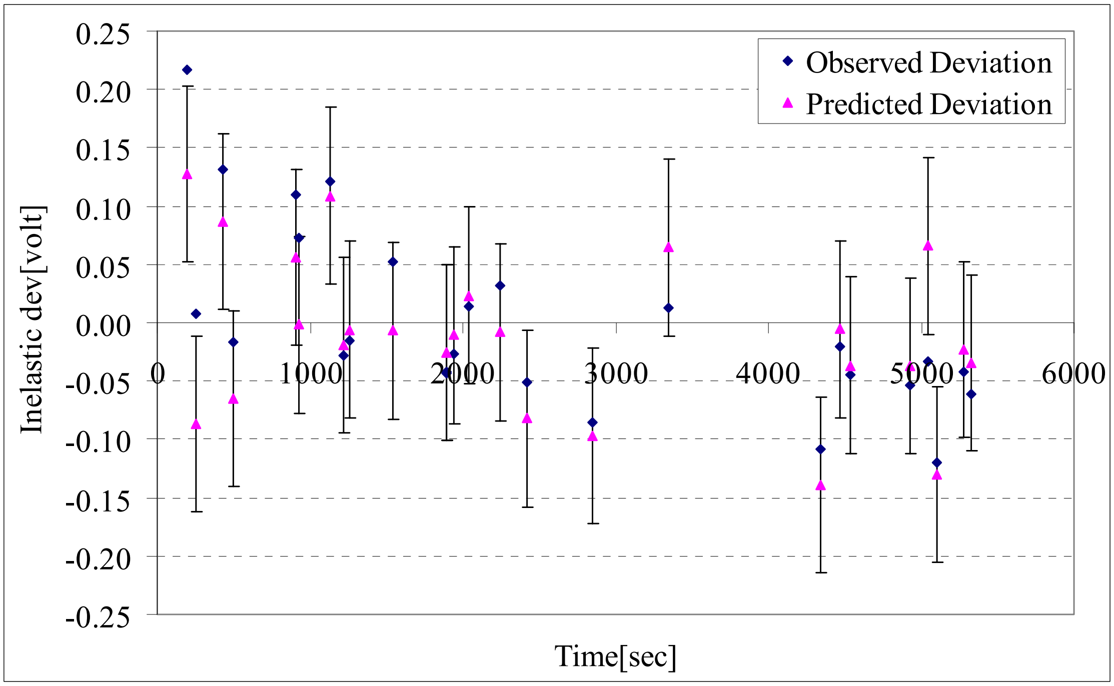

Fig.5.17. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.7 of the tissue specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 0.07554$.

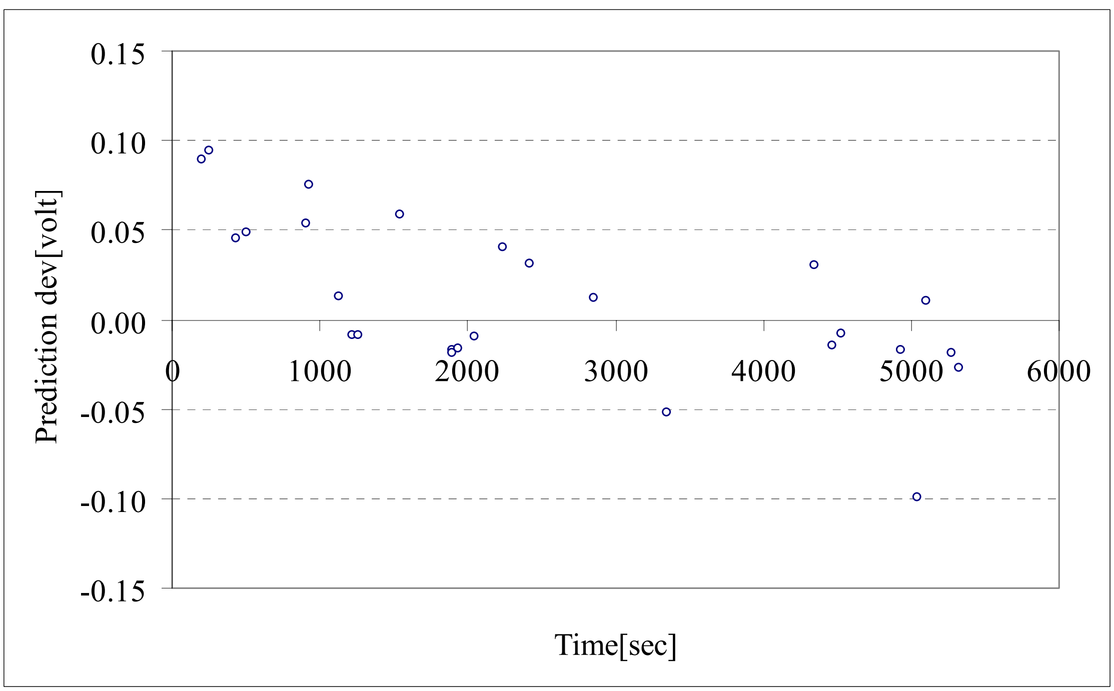

Fig.5.18. It shows the prediction deviation for the stretch ratio 1.7 of the tissue specimen. Note that all points are gathered at near zero but off in a random way.

Stretch Ratio 1.9: For λ = 1.9 of tissue specimen, we have $t_{c(\alpha/2;n-k-1)}$ = 1.833, where α = 0.1, n = 13, and k = 3 (for T, H_{t_2}, and H_{t_1}). We used the regression model for λ = 1.9 which is $\hat{D}_{1.9} = 13.56 - 1.58T - 44.76H_{t_2} - 2.77H_{t_1}$ to obtain the $\hat{D}_f$ which is a point forecasting value. We have s_f = 0.56092 and $t_{c(\alpha/2;n-k-1)} \cdot s_f$ = 1.02816 for λ = 1.9 of the tissue specimen. The table 5.10 shows the unused data for the stretch ratio 1.9 of the tissue specimen. It shows the time T, rate-related stretch history H_{t_2}, long-time stretch history H_{t_1}, and the inelastic deviation D_i. It also shows the point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$. We have adopted a new terminology as *inelastic deviation* which is, as we have used in previous chapters, the difference between the *measured* (by force transducer) and *averaged* force. To check if we missed any variables for the prediction, we have used the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Since there is no perfect regression model to predict and there could be a certain variable that we would have missed, we had to look at the prediction deviation to see if it is random or systematic. According to the result shown in the table 5.10 and the figure 5.19, the prediction deviation looks quite random because the trends of the observed deviation and predicted deviation are almost the same. This tells what we are missing is random and we don't perfectly predict every point. If those points are off in a systematic way, we would have possibly missed a certain variable. The prediction deviation is shown in the figure 5.20. All points for the stretch ratio of the rubber specimen are near zero but they are off in a random way.

Table 5.10
Unused data for the stretch ratio 1.9 of the tissue specimen. It shows the time *T*, rate-related stretch history H_{t_2}, long-time stretch history H_{t_1}, and the inelastic deviation D_i. It also shows the point forecasting value $\hat{D}_f$ and prediction deviation $D_i - \hat{D}_f$.

$\lambda_{\_unused}$	T	H_{t_2}	H_{t_1}	D_i	$\hat{D}_f$	$D_i - \hat{D}_f$
	[sec]	t_h=10	t_h=400	v_M - v_{avg}[volt]	[volt]	[volt]
1.9	221.8	1.86438	1.14375	1.54074	1.37910	0.16164
	231.5	1.90128	1.16136	0.97012	0.85491	0.11521
	463.5	1.87667	1.34284	0.97668	0.93068	0.04600
	473.0	1.90088	1.35842	0.57575	0.58154	-0.00579
	1167.4	1.89889	1.35749	0.43887	0.42573	0.01314
	1838.4	1.85157	1.39815	0.60481	0.84797	-0.24316
	1848.3	1.90309	1.40355	0.08387	0.13800	-0.05413
	2382.4	1.86404	1.54002	0.27699	0.37941	-0.10242
	2386.8	1.8863	1.54094	0.14481	0.07418	0.07063
	2817.4	1.88516	1.49814	0.02856	0.02233	0.00623
	2826.4	1.89999	1.5114	-0.16988	-0.19643	0.02655
	3379.4	1.88173	1.50878	-0.06832	-0.09143	0.02311
	3383.4	1.89349	1.50886	-0.14238	-0.25268	0.11030
	3461.6	1.88716	1.54695	-0.33488	-0.22876	-0.10612
	3467.3	1.89543	1.55062	-0.37426	-0.34686	-0.02740
	3782.4	1.8866	1.54495	-0.28738	-0.30391	0.01653
	4310.6	1.86266	1.49177	-0.25176	-0.05985	-0.19191
	4324.1	1.90822	1.49347	-0.67894	-0.68562	0.00668
	4775.4	1.87418	1.52669	-0.42207	-0.37790	-0.04417
	5070.0	1.86994	1.47555	-0.45644	-0.34242	-0.11402
	5371.8	1.87136	1.50703	-0.49019	-0.47602	-0.01417
	5381.4	1.90098	1.51279	-0.77707	-0.88812	0.11105

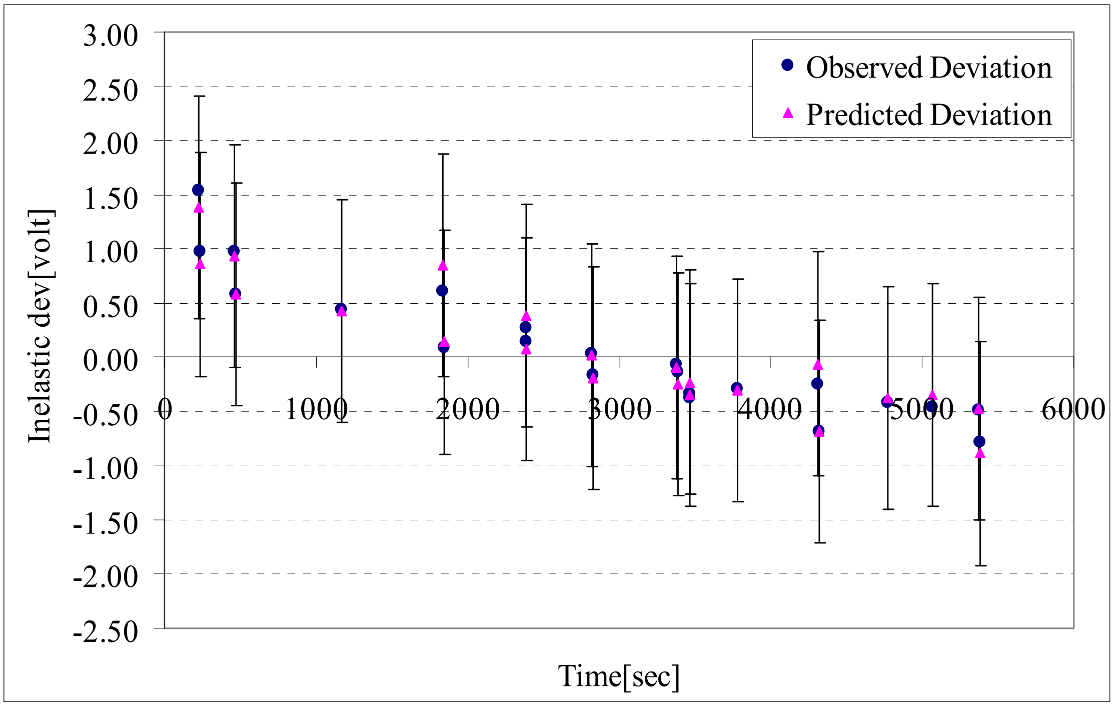

Fig.5.19. It shows the inelastic deviation D_i obtained from the unused data and point forecasting $\hat{D}_f$ using the regression model. For λ = 1.9 of the tissue specimen, $t_{c(\alpha/2;n-k-1)} \cdot s_f = 1.02816$.

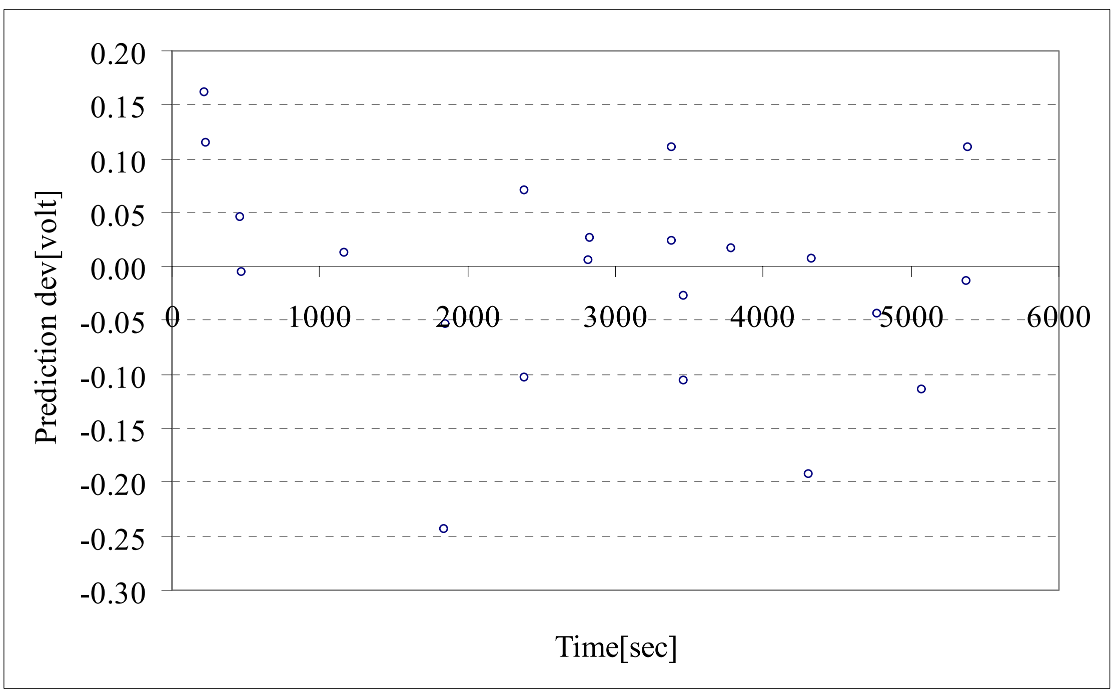

Fig.5.20. It shows the prediction deviation for the stretch ratio 1.9 of the tissue specimen. Note that all points are gathered at near zero but off in a random way.

CHAPTER VI

DISCUSSION

The hyperelastic models for the rubber-like materials and biotissues are never exact but they are useful because they guide our directions. The efficacy of a model is in its providing of insights into the mechanisms underlying the mechanical behavior of the materials. In lieu of the perfect models, hyperelastic models are currently most common choice for the high strain materials and the biotissues. In addition, the usage of those models reduces the trial and error. The reason that we have done the error analysis for the hyperelastic models of high strain materials and biotissues is that, even though they are never perfectly correct, those models are comprehensively being used and regarded as useful in the area of biomechanics and mechanobiology. If we have all infinite number of higher order terms, then the model would be perfect. But a perfect model is unfeasible because we can't have infinite number of data points. Thus, we need to look for the uncertainty due to the approximation.

Although, so far, many different types of the hyperelastic models for the rubber-like materials and biotissues have been enthusiastically developed since the 1940's, it is understood that no model is superior to the other and there is no agreement for those models. There is an ambiguity for developing the hyperelastic models of high strain materials as well as of biotissues. Since the error analysis that should been done in 1940's has not been tried so far, the ambiguity is being appeared by the lack of understanding the uncertainty due to the approximation. The approximation is equivalently the assumption of hyperelasticity of the rubber-like materials and biotissues. A model can be regarded as useful only if the uncertainty in the model is quantified and acceptable for the particular application of the model. If the error analysis is not executed on it, a model cannot be evaluated.

Even the well-known experts in this area such as Mullins, Mooney, Rivlin, Y.C. Fung, Holzapfel, Gasser and Ogden (2000), and so on have published their own models without discussing the uncertainties in them; they didn't show how their models deviated from the hyperelastic assumption. They got the results without finding out what the errors in the experimental data were. It should be noted that no hyperelastic models have been suggested without doing experiments and none of the measurement can be made without having some degree of uncertainties. This implies that it is extremely important to know if there is an error in any model.

The completely randomized stretch controlled protocol is indispensable for the error analysis. By the assumption that a hyperelastic model for the high strain materials and biotissues is a function of three variables such as time T, stretch-rate S and stretch history H, those three variables should be completely independent of each other so that they are not coupled or correlated. The randomized stretch- controlled protocol, therefore, enabled us to look at the three suspicious variables as independent variables.

The error analysis is motivated by the fact that no physical quantity can be measured without having some degree of uncertainties. Although many different types of models for hyperelasticity have been developed since the 1940's, there are very few models that discussed the errors in the data for the constitutive modeling. Since having some degree of uncertainties is inevitable especially in experiments on high strain material and biotissues, validity of the model should be justified by understanding the uncertainty in the data. Every constitutive relation in hyperelasticity, therefore, should be modeled after characterizing the uncertainties in the data. The fundamental statistical theory called multivariable linear regression analysis has been used to find the significant factors that cause the deviation from the hyperelasticity as well as to characterize the uncertainties. We initially suspected that the factors which cause the deviation from the hyperelasticity are the time, stretch-rate, and stretch history. Finally, the error analysis has revealed that they underlie the deviation of rubber and tissues from hyperelasticity as evident in a uniaxial (randomized stretch-controlled protocols) stretch test. We used the dependent parameter as the deviation *D* and independent parameters as the time *T*, rate-related stretch history function H_{t_2} , and stretch history H_{t_1}. Although there was one more independent variable for the tissue which is H_{t_0}, it hasn't been employed for the multiple linear regression analysis because there was high correlation between the time *T* and H_{t_0}. If we use the independent variables which have high correlation, then we will have a trouble with the multicollinearity which causes the increasing of standard error of the partial regression coefficients and decreasing the goodness-of-fit of the model. Excluding one of the variable that has high correlation to another never cause the loss of the predictability of the model.

After obtaining the regression models for each stretch level, we have tested the model with the **goodness-of-fit test** by using $\overline{R}^2$ called coefficient of determination and test of significance called ***t*-test**. The table 6.1 and the table 6.2 show the regression models and *t*-values for each stretch level for rubber and tissue.

Table 6.1
Multiple linear regression models for each stretch ratio for the rubber specimen. The critical values of t for two-tailed test and t-values for each independent variable are also shown.

λ	Regression Model of Rubber	$t_{c(\alpha/2;n\text{-}k\text{-}1)}$, α=0.1	t-value of			$\overline{R}^2$
			T	H_{t_2}	H_{t_1}	
1.1	$\hat{D}=6.66-1.40T-26.37H_{t_2}$	1.761	-9.08	-3.95		0.86
1.3	$\hat{D}=10.74-2.00T-33.74H_{t_2}$	1.860	-4.79	-2.91		0.76
1.5	$\hat{D}=7.41-19.89H_{t_2}-10.65H_{t_1}$	1.761		-2.27	-4.22	0.80
1.7	$\hat{D}=15.06-54.41H_{t_2}$	1.796		-5.45		0.72
1.9	$\hat{D}=2.14-0.95T-6.94H_{t_1}$	1.812	-3.56		-2.71	0.85

Table 6.2
Multiple linear regression models for each stretch ratio for the tissue specimen. The critical values of t for two-tailed test and t-values for each independent variable are also shown.

λ	Regression Model of Tissue	$t_{c(\alpha/2;n\text{-}k\text{-}1)}$, α=0.1	t-value of			$\overline{R}^2$
			T	H_{t_2}	H_{t_1}	
1.1	$\hat{D}=0.34-1.70T$	1.753	-13.09			0.92
1.3	$\hat{D}=0.64-2.29T$	1.833	-7.72			0.87
1.5	$\hat{D}=5.85-T-17.48H_{t_2}-5.75H_{t_1}$	1.771	-5.02	-2.15	-2.18	0.85
1.7	$\hat{D}=10.42-0.97T-36.77H_{t_2}$	1.812	-2.77	-3.42		0.71
1.9	$\hat{D}=13.56-1.58T-44.76H_{t_2}-2.77H_{t_1}$	1.833	-16.72	-6.93	-2.96	0.98

Rubber

If we look at the multiple linear regression models of the rubber summarized in the table 6.1, not all of the independent variables were employed for the final regression models. This doesn't mean that all the unemployed variables were not giving rise to the deviation from the hyperelasticity. Note again that the ideal regression model is the model that has the maximum efficiency with the minimum number of independent variables. The good model should predict the future or intermediate values as close as possible with the minimum number of the independent variables.

According to the results shown in the table 6.1, the independent variable H_{t_2} is mostly related to the deviation *D* for most of the stretch ratios. This tells that the rate of the stretch is highly effective to give rise to the inelastic deviation which is deviation from the hyperelasticity. If we look at the regression model for stretch ratio 1.1 of the rubber, the considerable amount of the deviation *D* is determined by the rate-related stretch history function H_{t_2}. It, again, means the stretch-rate is highly related to the inelastic deviation. For the stretch ratio λ = 1.1, we have found that both the independent variables *T* and H_{t_2} are optimal and significant variables to anticipate the inelastic deviation *D*. Note that the rate-related stretch history function H_{t_2} is almost 19 times more effective than the time *T* to give rise to the inelastic deviation *D*.

For the stretch ratio λ = 1.3, the rate-related stretch history function H_{t_2} is almost 17 times more effective than the time *T* to give rise to the inelastic deviation *D*.

For the stretch ratio λ = 1.5, the rate-related stretch history function H_{t_2} is the most significant factor that induce the inelastic deviation from the hyperelasticity. There is no *T* term here and H_{t_1} has involved instead. This is because there is a certain amount of correlation between the independent variables *T* and H_{t_1}, and the correlation coefficient between *D* and H_{t_1} is higher than the correlation coefficient between *D* and *T*.

For the stretch ratio λ = 1.7, the inelastic deviation *D* can mostly be determined by the rate-related stretch history function H_{t_2}.

The only stretch level where the rate-related stretch history function H_{t_2} has nothing to do with the inelastic deviation *D* is λ = 1.9. For the stretch ratio λ = 1.9, the inelastic deviation *D* is affected by the time T and the long-time stretch history function H_{t_1}. The rate-related stretch history function H_{t_2} is the most significant factor to give rise to the inelastic deviation from the hyperelasticity for the stretch ranges from λ = 1.1, 1.3, 1.5, and 1.7. But it is not inconsequential to cause the inelastic deviation for the stretch ratio 1.9.

Tissue

Again, the multiple linear regression models of the tissue didn't show all of the independent variables in the final regression models; this doesn't imply that all the unemployed variables are not giving rise to the deviation from the hyperelasticity. Note again that the ideal regression model is the model that has the maximum efficiency with the minimum number of independent variables. The good model should predict the future and intermediate values as close as possible with the minimum number of the independent variables. According to the results shown in the table 6.2 for the tissue, the independent variable *T* is entirely related to the inelastic deviation *D* for the all selected stretch ratios. This says that the deviation is progressing with time. So there must be a creep that is a time-dependent deformation for the same amount of stretch.

For the stretch ratios λ = 1.1 and 1.3, we have found that the independent variables *T* only is the optimal and significant variable to anticipate the deviation *D*. Interestingly, it has been found that the rate-related stretch history function H_{t_2} is not significant factor for inducing the inelastic deviation *D* for the stretch ratios λ = 1.1 and 1.3. If we compare the regression models for the tissue and rubber for the stretch ratio λ = 1.1 and 1.3, the effects of the time *T* for the prediction of the *D* are reasonably identical. For example, for the rubber, if the *T* is changing one unit, the *D* is changing 1.4 units, and, for the tissue, if the *T* is changing one unit the *D* is changing 1.7 units. It can be understood likewise for the stretch ratio λ = 1.3. This tells that the time is almost equally effective to give rise to the inelastic deviation for the rubber and tissue.

For the stretch ratio λ = 1.5, the independent variables that induce the inelastic deviation from the hyperelasticity are the time *T*, rate-related stretch history function H_{t_2} and long time stretch history H_{t_1}. The rate-related stretch history function H_{t_2} is the most significant factor, and secondly, the long-time stretch histories function H_{t_1} and lastly, the time *T*.

For the stretch ratio λ = 1.7, the inelastic deviation D can mostly be determined by the rate-related stretch history function H_{t_2} and the time T. In addition, the regression model is dominated by the rate-related stretch history function H_{t_2}.

For the stretch ratio λ = 1.9, the regression model has all the independent variables such as T, H_{t_2}, and H_{t_1}. This implies that the inelastic deviation D is affected by the all independent variables and is well-determined by them.

From the regression models, for the stretch ranges 1.5, 1.7, and 1.9 of the tissue, we figured out that the rate-related stretch history function H_{t_2} is the most significant factor to give rise to the inelastic deviation from the hyperelasticity.

Finally, we can apply those models as

$$f\Big|_{\lambda_0} = f_{avg}\Big|_{\lambda_0} + \hat{D}\Big|_{\lambda_0} \tag{6.1}$$

where, f is the force for a specific stretch ratio λ_0, and f_{avg} is the average of the forces corresponding to the λ_0. Equivalently, if we use the same notation that we have used for this study, the equation (5.1) can be written as

$$v\Big|_{\lambda_0} = v_{avg}\Big|_{\lambda_0} + \hat{D}\Big|_{\lambda_0} \tag{6.2}$$

It is known that the rubber-like materials have randomly arranged long chain molecules. Thus, whenever the materials are stretched, there must be rearrangements of those molecular chains, i.e., the randomly arranged molecular chains get close to the ordered configuration so that it causes the decrease of entropy. In addition, whenever the rubber is deformed, it is likely to go back to its original configuration by the force that is generated by the constant thermal activity of the long chain molecules (L. Mullins, 1947). The reason that the uncertainties especially in the low stretch ranges are important, although the stretch ratio λ = 1.3 is not a low strain range for none rubber-like materials, is that the behavior of the rubber in the low strain range is primarily related to the entropy. The energy storage at the low range of stain is mostly due to the entropy alone. Thus, in the low stretch range, the strain energy can hardly be determined by the experimentation. That's why there has been a deficiency in understanding of the exclusive nature of the stain energy function for the low strain ranges of the rubber. At the higher strain ranges, the energy storage is due to both the entropy and the molecular chemical bonds. That is, if we stretch to the higher strain ranges, we make the long chain molecules rearranged in more ordered manner as well as we stretch the chemical bonds, too. Thus, the energy storage is due to both the entropy and the molecular chemical bonds. (John C. Criscione, 2003).

The test of the predictability has been involved in the final step for any regression analysis to evaluate how well the derived regression model forecasted the intermediate or future values. We have checked the predictive capability by comparing the unused (deviation) data in the randomized stretch-controlled protocol of the rubber and tissue to the predicted deviation obtained by the regression models.

We have redefined the deviation from the hyperelasticity as *inelastic deviation* which is the difference between the *measured* (by force transducer) and *averaged* force. If the testing specimens were truly elastic, all the measurements would be equal to the average so that the inelastic deviation would be zero. To check if we have missed any variables for the prediction, we have newly defined the $D_i - \hat{D}_f$ as *prediction deviation* which is difference between the observed deviation D_i and the point forecasting $\hat{D}_f$. Although we have found that the inelastic deviation varies systematically and we have quantified it, the models that we have derived can never be perfect; there could be any missed variables. According to the results, we have realized that the prediction deviation is random because the trends of the observed deviation and predicted deviation were almost the same. This tells what we have missed is random and we don't perfectly predict every point. Those points are off in a random way. If those points are off in a systematic way, we would have possibly missed a certain variable that gives rise to the inelastic deviation. Therefore we could conclude that we never missed any significant factors that give rise to the inelastic deviation from the hyperelasticity for the fitting.

Future Works and Limitations

To get more strict predictive capability, the non-randomized cyclic loading tests are needed. Although it seemed that there are mostly linear relationships between them, we need a much longer motion protocol to get enough number of data points be to sure of the linearity. We assumed that there are linear relationships between the dependent variable D and the independent variables T, H_{t_2}, and H_{t_1} to get the regression models. But, there could be nonlinear relationships between them, and in such case, a nonlinear regression models should be needed to get more accurate predictability of the deviation from the hyperelasticity.

For the future works, we suggest the biaxial test of the high strain materials and biotissues to get a reliable constitutive model which has a specified uncertainty obtained by multivariable nonlinear regression analysis. It can be much more practically useful because most of the high strain material and biotissues are treated as the membranes and, in fact, they are mostly anisotropic.

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