COMPACTNESS OF THE $\overline{\partial}$-NEUMANN PROBLEM AND STEIN NEIGHBORHOOD BASES

A Dissertation

by

SÖNMEZ ŞAHUTOĞLU

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2006

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Approved by:

Chair of Committee,	Emil J. Straube
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ABSTRACT

Compactness of the $\overline{\partial}$-Neumann Problem

and Stein Neighborhood Bases. (May 2006)

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Chair of Advisory Committee: Dr. Emil J. Straube

This dissertation consists of two parts. In the first part we show that for $1 \leq k \leq n-1$, a complex manifold M of dimension at least k in the boundary of a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$ is an obstruction to compactness of the $\overline{\partial}$-Neumann operator on (p,q)-forms for $0 \leq p \leq k \leq n$, provided that at some point of M, the Levi form of $b\Omega$ has the maximal possible rank $n-1-dim(M)$ (i.e. the boundary is strictly pseudoconvex in the directions transverse to M). In particular, an analytic disc is an obstruction to compactness of the $\overline{\partial}$-Neumann operator on $(p,1)$-forms, provided that at some point of the disc, the Levi form has only one vanishing eigenvalue (i.e. the eigenvalue zero has multiplicity one). We also show that a boundary point where the Levi form has only one vanishing eigenvalue can be picked up by the plurisubharmonic hull of a set only via an analytic disc in the boundary.

In the second part we obtain a weaker and quantified version of McNeal's Property $(\widetilde{P})$ which still implies the existence of a Stein neighborhood basis. Then we give some applications on domains in $\mathbb{C}^2$ with a defining function that is plurisubharmonic on the boundary.

To my family.

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CHAPTER I

INTRODUCTION

This dissertation concerns compactness of the $\overline{\partial}$-Neumann problem and the existence of Stein neighborhood bases for the closure of smooth bounded pseudoconvex domains.

Pseudoconvex domains are central objects in several complex variables as they are natural domains of existence of holomorphic functions. It is known that a domain is pseudoconvex if and only if there exists a holomorphic function that is not extendable through any boundary point of the domain. It turns out that boundaries of domains play a leading role in the theory of several complex variables. In particular, it is important to study the interplay of the complex geometry of $\mathbb{C}^n$ with the geometry of the boundary of a domain. The $\overline{\partial}$-Neumann problem, via its boundary conditions, is very much connected to this interplay. For example, Catlin([12, 13, 14]) showed that a smooth bounded pseudoconvex domain is of finite type in the sense of D'Angelo([20]) if and only if the $\overline{\partial}$-Neumann problem of the domain satisfies a subelliptic estimate.

Global regularity properties of the $\overline{\partial}$-Neumann problem are important from the partial differential equations perspective as well as from that of several complex variables. The reason is that many function theoretic problems reduce to solving the $\overline{\partial}$-Neumann problem with some regularity properties. In the last couple of decades it has become clear that global regularity is very subtle [9, 16]. Compactness, on the other hand, is known to be more robust than global regularity. For example, compactness is a local property whereas global regularity is not. Another motivation for studying the compactness of the $\overline{\partial}$-Neumann problem comes from its connections to

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Toeplitz operators ([31] and its references), semiclassical analysis of Schrödinger operators [17, 32], and kernels solving $\overline{\partial}$ [35]. Our motivation for studying compactness of the $\overline{\partial}$-Neumann problem, however, comes from its connections to the geometry of the boundaries of pseudoconvex domains. To be more precise, we would like to know if it is possible to have an analytic disc (or plurisubharmonic hulls in general) in the boundary of a smooth bounded pseudoconvex domain when the $\overline{\partial}$-Neumann operator is compact. This problem has been solved by Catlin in $\mathbb{C}^2$, but the problem is still open in general.

Let $0 \leq p \leq n, 1 \leq q \leq k \leq n-1$ and Ω be a smooth bounded pseudoconvex domain whose Levi form of has at most k vanishing eigenvalues. (That is, the Levi form is of rank at least $n-k-1$ at every point on $b\Omega$.) We show that when the $\overline{\partial}$-Neumann operator of Ω on (p,q)-forms is compact there cannot be a complex manifold of dimension k in $b\Omega$. In particular, if the Levi form has at most one vanishing eigenvalue and the $\overline{\partial}$-Neumann operator of Ω on $(p,1)$-forms is compact then there cannot be an analytic disc in $b\Omega$. Although we believe that the conclusion of the theorem should not depend on the rank of the Levi form, our methods do not give a more general result. We also show that in case the Levi form has at most one vanishing eigenvalue, absence of analytic discs in $b\Omega$ is equivalent to $\widehat{K} \cap b\Omega = K \cap b\Omega$ for any compact set $K \subset \overline{\Omega}$, where $\widehat{K}$ is the plurisubharmonic hull of K.

The second part of the thesis deals with the existence of a Stein neighborhood basis for the closure of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let Ω be a domain in $\mathbb{C}^n$. Then $\overline{\Omega}$ is said to have a Stein neighborhood basis if for every neighborhood V of $\overline{\Omega}$ there exists another pseudoconvex domain Ω_V such that $\overline{\Omega} \subset \Omega_V \subset V$. That is, $\overline{\Omega}$ has a neighborhood basis that consists of pseudoconvex domains. The problem of whether the closure of every smooth bounded pseudoconvex domain has a Stein neighborhood basis first appeared in the literature in [4]. It is motivated in part by the fact that any pseudoconvex domain can be exhausted by smooth pseudoconvex domains. Approximation from outside, however, is not always possible. For example, neither the closed Hartogs triangle nor the closure of the worm domain of Diederich and Fornæss([21]), a smooth bounded pseudoconvex domain in $\mathbb{C}^2$, have a Stein neighborhood basis. Therefore it is interesting to know when the closure of a domain has a Stein neighborhood basis.

Existence of a Stein neighborhood basis for $\overline{\Omega}$ is known to be connected to global regularity of the $\overline{\partial}$-Neumann problem [55] and the Mergelyan approximation property [19]. In fact, all classes of smooth domains whose closure is known to have a Stein neighborhood basis have the Mergelyan approximation property. On the other hand, it is still open whether some sort of approximation property, like Mergelyan or Runge, implies the existence of a Stein neighborhood basis for the closure of a smooth bounded pseudoconvex domain.

One way to get a Stein neighborhood basis for the closure is through McNeal's property $(\widetilde{P})$. Sibony([53, 54]) showed that if a smooth bounded pseudoconvex domain satisfies property $(\widetilde{P})$ then there exists a plurisubharmonic function f that vanishes on the domain and stays strictly positive outside of the domain. Hence one can get a Stein neighborhood basis out of level sets of f. In fact, $\overline{\Omega}$ is uniformly H-convex. On the other hand, McNeal([46]) showed that property $(\widetilde{P})$ is sufficient for compactness of the $\overline{\partial}$-Neumann problem. This suggests that there might be some connections between compactness of the $\overline{\partial}$-Neumann problem and existence of a Stein neighborhood basis for the closure. Examining the proof of Sibony one can see that property $(\widetilde{P})$ is much stronger than what is needed. We will introduce a weaker and a quantitative version of property $(\widetilde{P})$ that still implies the existence of a Stein neighborhood basis for the closure. Additionally, we will give an application on domains in $\mathbb{C}^2$ with a defining function that is plurisubharmonic on the boundary.

CHAPTER II

BACKGROUND

A domain Ω in $\mathbb{C}^n$ is said to have C^k-smooth boundary, $1 \leq k \leq \infty$, if there exists a neighborhood U of $\overline{\Omega}$ and a real valued C^k-smooth function ρ defined on U such that $\Omega = \{z \in U : \rho(z) < 0\}, b\Omega = \{z \in U : \rho(z) = 0\}, U \setminus \overline{\Omega} = \{z \in U : \rho(z) > 0\}$, and the gradient of ρ does not vanish on the boundary $b\Omega$ of Ω. In this case ρ is called a defining function for Ω. In case ρ is a C^2-smooth defining function we define the complex Hessian of ρ at $z \in b\Omega$ as follows:

$$\mathcal{L}_\rho(z; A, \overline{B}) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k} a_j \bar{b}_k,$$

where A and B are vectors of type $(1,0)$ in $\mathbb{C}^n$ with $A = \sum_{j=1}^n a_j \frac{\partial}{\partial z_j}$, and $B = \sum_{j=1}^n b_j \frac{\partial}{\partial z_j}$. For convenience we will denote $\mathcal{L}_r(z; A, \overline{A})$ by $\mathcal{L}_r(z; A)$ and suppress z when it is not confusing.

A function is said to be plurisubharmonic on an open set V if its restriction on any complex line that passes through V is subharmonic. One can check that a C^2-smooth function f is plurisubharmonic on V if and only if $\mathcal{L}_f(z; W) \geq 0$, for $z \in V$ and any vector W of type $(1,0)$.

Definition 1. *A domain $\Omega \subset \mathbb{C}^n$ is said to be pseudoconvex if there exists a continuous plurisubharmonic function ρ on Ω such that $\{z \in \Omega : \rho(z) < c\}$ is a precompact subset of Ω for any real number $c \geq 0$.*

Notice that in this definition no smoothness is assumed. When the domain has at least twice continuously differentiable boundary it is a well known fact that pseudoconvexity can be defined using the Levi form. We refer the reader to the books [42, 50] for a proof of the following theorem and other equivalent definitions of pseudoconvexity.

Theorem 1. *A domain Ω with C^2-smooth boundary in $\mathbb{C}^n, n \geq 2$, is pseudoconvex if and only if it has a defining function ρ such that $\mathcal{L}_\rho(z;W) \geq 0$ for $z \in b\Omega$ and $W(\rho)(z) = \sum_{j=1}^n \frac{\partial \rho(z)}{\partial z_j} w_j = 0$ where $W = \sum_{j=1}^n w_j \frac{\partial}{\partial z_j}$.*

A vector W of type $(1,0)$ is called complex tangential to $b\Omega$ at q if $W(\rho)(q) = 0$ for a defining function ρ of Ω. One can check that, in the above theorem, the complex Hessian of any defining function is non-negative on complex tangential vectors of type $(1,0)$ on the boundary. Therefore, pseudoconvexity is independent of the defining function. The restriction of the complex Hessian on the space of complex tangential vectors is called the Levi form.

A. The $\overline{\partial}$-Neumann Problem

Studying the $\overline{\partial}$-Neumann problem was proposed by Garabedian and Spencer([33]) in order to study the $\overline{\partial}$-problem. Morrey([47]) proved an a priori estimate on $(0,1)$-forms and later Kohn([40]) generalized the a priori estimates, proved the existence of the $\overline{\partial}$-Neumann operator on (p,q)-forms and showed the boundary regularity on strongly pseudoconvex domains. Hörmander([37]) used weighted L^2-theory to solve the $\overline{\partial}$-problem (and hence the $\overline{\partial}$-Neumann problem) in L^2 (without weights) on bounded pseudoconvex domains.

In this section we sketch the setup of the $\overline{\partial}$-Neumann problem. We refer the reader to the books [15, 24] and a survey [9] for a more detailed treatment of the topic.

Let Ω be a bounded domain in $\mathbb{C}^n, n \geq 2$, and $0 \leq p, q \leq n$. We denote the space of square integrable and smooth (p,q)-forms by $L^2_{(p,q)}(\Omega)$ and $C^\infty_{(p,q)}(\Omega)$, respectively. Let $z = (z_1, \ldots, z_n)$ denote the complex coordinates for $\mathbb{C}^n$. Any square integrable

(p,q)-form f can be written as

$$f = \sum_{I,J}{}' f_{IJ} dz_I \wedge d\bar{z}_J$$

where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multiindices, $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$ and $\sum'$ denotes the summation over strictly increasing multi-indices.

For the sake of simplicity we will drop the indices from $\overline{\partial}_{(p,q)}$ and just write $\overline{\partial}$. $L^2_{(p,q)}(\Omega)$ is a Hilbert space with the inner product coming from the following norm:

$$\|f\|^2 = \sum_{I,J}{}' \int_{\Omega} |f_{IJ}|^2 dV$$

where dV is the volume element on $\mathbb{C}^n$. When f is a smooth (p,q)-form we define the action of $\overline{\partial}$ as follows:

$$\overline{\partial} f = \sum_{I,J}{}' \sum_{k} \frac{\partial f_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J.$$

Then we extend $\overline{\partial}$ to its weak closure and still denote it by $\overline{\partial}$. Hence $u \in Dom(\overline{\partial})$ if $u \in L^2_{(p,q)}(\Omega)$ and $\overline{\partial} u \in L^2_{(p,q+1)}(\Omega)$ where $\overline{\partial} u$ is defined in the distribution sense. One can check that $\overline{\partial}$ is a linear, closed, and densely defined operator. Then the Hilbert space adjoint $\overline{\partial}^* : L^2_{(p,q+1)}(\Omega) \to L^2_{(p,q)}(\Omega)$ is linear, closed and densely defined. A square integrable (p,q)-form f belongs to $Dom(\overline{\partial}^*)$ if there exists $g \in L^2_{(p,q)}(\Omega)$ such that

$$\left\langle f, \overline{\partial} \varphi \right\rangle = \langle g, \varphi \rangle \quad \text{for} \quad \varphi \in Dom(\overline{\partial}) \cap L^2_{(p,q)}(\Omega)$$

where $\langle .,. \rangle$ is the inner product on the corresponding Hilbert spaces. When Ω is a bounded domain one can easily see that $C^\infty_{(p,q)}(\overline{\Omega}) \subset Dom(\overline{\partial})$. But for a (p,q)-form f to be in $Dom(\overline{\partial}^*)$ it must satisfy a boundary condition in the weak sense. In case Ω has C^1 boundary, using integration by parts, one can show that a C^1-smooth

(p,q)-form f is in the domain of $\overline{\partial}^*$ if and only if it satisfies the following:

$$\sum_k f_{I,kK} \frac{\partial \rho}{\partial z_k} = 0 \quad \text{on} \quad b\Omega$$

for all strictly increasing multiindices I, K such that $|I| = p$ and $|K| = q-1$.

Let $\wedge$ denote the exterior product of forms. One can show that $f \in C^1_{(p,q)}(\overline{\Omega}) \cap Dom(\overline{\partial}^*)$ if and only if $\overline{\partial}\rho \vee f = 0$ on $b\Omega$ where the interior product $\vee$ is defined as follows:

$$\langle g \wedge \overline{\partial}\rho, f \rangle = \langle g, \overline{\partial}\rho \vee f \rangle$$

for any (p,q)-form f and a smooth $(p,q-1)$-form g. Now we will define the complex Laplacian $\Box_{(p,q)}$.

Definition 2. $\Box_{(p,q)} = \overline{\partial}_{(p,q-1)}\overline{\partial}^*_{(p,q)} + \overline{\partial}^*_{(p,q+1)}\overline{\partial}_{(p,q)}$ *is a linear operator defined on* $L^2_{(p,q)}(\Omega)$ *such that a square integrable* (p,q)*-form* f *is in* $Dom(\Box_{(p,q)})$ *if and only if* $f \in Dom(\overline{\partial}_{(p,q)}) \cap Dom(\overline{\partial}^*_{(p,q)})$ *and* $\overline{\partial} f \in Dom(\overline{\partial}^*_{(p,q+1)}), \overline{\partial}^* f \in Dom(\overline{\partial}_{(p,q-1)})$.

One can check that $\Box_{(p,q)}$ is a densely defined closed (unbounded) linear operator on $L^2_{(p,q)}(\Omega)$. The $\overline{\partial}$-Neumann problem is defined as finding a solution to $\Box_{(p,q)} f = g$ on D for $f \in Dom(\Box_{(p,q)})$. Existence of a solution for the $\overline{\partial}$-Neumann problem on pseudoconvex domains is guarantied by the following theorem. We refer the reader to [15] for a proof.

Theorem 2 (Hörmander). *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n, n \geq 2$, *and* e *be the base of the natural logarithm. For each* $0 \leq p \leq n, 1 \leq q \leq n$, *there exists a bounded operator, called the* $\overline{\partial}$*-Neumann operator,* $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ *such that*

(1) $Range(N_{(p,q)}) \subset Dom(\Box_{(p,q)})$, *and*
$N_{(p,q)}\Box_{(p,q)} = \Box_{(p,q)}N_{(p,q)} = I$ *on* $Dom(\Box_{(p,q)})$.

(2) For any $f \in L^2_{(p,q)}(\Omega), f = \overline{\partial}\,\overline{\partial}^* N_{(p,q)} f \oplus \overline{\partial}^* \overline{\partial} N_{(p,q)} f$.

(3) $\overline{\partial} N_{(p,q)} = N_{(p,q+1)} \overline{\partial}$ *on* $Dom(\overline{\partial}), 1 \leq q \leq n-1$.

(4) $\overline{\partial}^* N_{(p,q)} = N_{(p,q-1)} \overline{\partial}^*$ *on* $Dom(\overline{\partial}^*), 2 \leq q \leq n$.

(5) Let δ *be the diameter of* Ω*. The following estimates hold for any* $f \in L^2_{(p,q)}(\Omega)$:

$$\|N_{(p,q)} f\| \leq \frac{e\delta^2}{q} \|f\|,$$

$$\|\overline{\partial} N_{(p,q)} f\| \leq \sqrt{\frac{e\delta^2}{q}} \|f\|,$$

$$\|\overline{\partial}^* N_{(p,q)} f\| \leq \sqrt{\frac{e\delta^2}{q}} \|f\|.$$

We note that $N_{(p,0)}$ has a similar existence theorem. The main difference between $N_{(p,0)}$ and $N_{(p,q)}$ for $q \geq 1$ is that $\Box_{(p,0)}$ is not onto. We refer the reader to [15] for more information on this matter. Using the above theorem one can show that when Ω is bounded and pseudoconvex, an L^2 solution to the $\overline{\partial}$-problem exists. In fact, the solution operator with minimal norm in the L^2 sense is $\overline{\partial}^* N_{(p,q)}$, as the following corollary shows.

Corollary 1. *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n, n \geq 2$*. Assume that* $0 \leq p \leq n, 1 \leq q \leq n$, $g \in L^2_{(p,q)}(\Omega)$*, and* $\overline{\partial} g = 0$*. Then* $f = \overline{\partial}^* N_{(p,q)} g$ *satisfies* $\overline{\partial} f = g$ *and*

$$\|f\| \leq \sqrt{\frac{e\delta^2}{q}} \|g\|. \tag{2.1}$$

f *is the unique solution to* $\overline{\partial} u = g$ *that is orthogonal to* $Ker(\overline{\partial})$.

$\overline{\partial}^* N_{(p,q)}$ is called the canonical solution operator for the $\overline{\partial}$-problem.

B. Compactness of the $\overline{\partial}$-Neumann Problem

In this section we introduce compactness of the $\overline{\partial}$-Neumann problem. We refer the reader to [30, 31, 32, 46] for more information.

We will use the notation $W^s_{(p,q)}(\Omega)$ for (p,q)-forms with coefficient functions from the Sobolev space $W^s(\Omega)$. The norm on $W^s(\Omega)$ is denoted by $\|.\|_s$. Compactness of the $\overline{\partial}$-Neumann problem can be formulated in several useful ways:

Lemma 1. *Let Ω be a bounded pseudoconvex domain, $0 \leq p \leq n, 1 \leq q \leq n$. Then the following are equivalent:*

i) The $\overline{\partial}$-Neumann operator, $N_{(p,q)}$, is compact from $L^2_{(p,q)}(\Omega)$ to itself.

ii) The embedding of the space $Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*)$, provided with the graph norm $u \to \|\overline{\partial}u\| + \|\overline{\partial}^* u\|$, into $L^2_{(p,q)}(\Omega)$ is compact.

iii) For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\|u\| \leq \varepsilon(\|\overline{\partial}u\| + \|\overline{\partial}^* u\|) + C_\varepsilon \|u\|_{-1}, \quad for \;\; u \in Dom(\overline{\partial}^*) \cap Dom(\overline{\partial}).$$

iv) The canonical solution operators $\overline{\partial}^* N_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q-1)}(\Omega)$ and $\overline{\partial}^* N_{(p,q+1)} : L^2_{(p,q+1)}(\Omega) \to L^2_{(p,q)}(\Omega)$ are compact.

The statement in (iii) is called a compactness estimate. The equivalence of (ii) and (iii) is a result of Lemma 1.1 in [41]. The general L^2-theory and the fact that $L^2_{(p,q)}(\Omega)$ embeds compactly into $W^{-1}_{(p,q)}(\Omega)$ shows that (i) is equivalent to (ii) and (iii). Finally, the equivalence of (i) and (iv) follows from the formula

$$N_{(p,q)} = (\overline{\partial}^* N_{(p,q)})^* \overline{\partial}^* N_{(p,q)} + \overline{\partial}^* N_{(p,q+1)} (\overline{\partial}^* N_{(p,q+1)})^*$$

(see [24], p.55, [49]). We refer the reader to [46] for similar calculations.

The following lemma is well known but has not appeared in the literature. So we will give a proof for the convenience of the reader.

Lemma 2. *Let* Ω_1 *and* Ω_2 *be two bounded pseudoconvex domains in* $\mathbb{C}^n, n \geq 2$, *and* $F : \Omega_1 \to \Omega_2$ *be a biholomorphism that is smooth up to the boundary. Then the* $\overline{\partial}$*-Neumann problem on* (p,q)*-forms on* Ω_1 *is compact if and only if the* $\overline{\partial}$*-Neumann problem on* (p,q)*-forms on* Ω_2 *is compact for* $0 \leq p \leq n, 1 \leq q \leq n$.

Proof. The proof of this lemma is implicit in [30]. $N_{(p,q)}$ is compact if and only if $\overline{\partial}^* N_{(p,q)}$ and $\overline{\partial}^* N_{(p,q+1)}$ are compact. (See *iv*) in Lemma 1). Therefore it is enough to show that compactness of $\overline{\partial}^* N_{(p,q)}$ and $\overline{\partial}^* N_{(p,q+1)}$ are invariant under biholomorphisms. Let Ω_1 and Ω_2 be two pseudoconvex domains and $F : \Omega_1 \to \Omega_2$ be a biholomorphism. Let $\{f_j\}$ be a sequence of bounded $\overline{\partial}$-closed forms in $L^2_{(p,q)}(\Omega_2)$. Since F is holomorphic, F^*, which pulls back forms, and $\overline{\partial}$ commute. Therefore $\{F^*(f_j)\}$ is a sequence of bounded $\overline{\partial}$-closed forms in $L^2_{(p,q)}(\Omega_1)$. If $\overline{\partial}^* N_{(p,q)} : L^2_{(p,q)}(\Omega_1) \to L^2_{(p,q-1)}(\Omega_1)$ is compact then $\{\overline{\partial}^* N_{(p,q)} F^*(f_j)\}$ has a convergent subsequence in $L^2_{(p,q-1)}(\Omega_1)$. By passing to a subsequence if necessary we may assume that $\{\overline{\partial}^* N_{(p,q)} F^*(f_j)\}$ is convergent. Since F is a biholomorphism $\{(F^{-1})^* \overline{\partial}^* N_{(p,q)} F^*(f_j)\}$ is convergent too. One can check that

$$\overline{\partial}(F^{-1})^* \overline{\partial}^* N_{(p,q)} F^*(f_j) = (F^{-1})^* \overline{\partial}\overline{\partial}^* N_{(p,q)} F^*(f_j) = f_j$$

Therefore $(F^{-1})^* \overline{\partial}^* N_{(p,q)} F^*$ is a compact solution operator for $\overline{\partial}$ on (p,q)-forms on Ω_2. We get $\overline{\partial}^* N_{(p,q)}$ on Ω_2 by applying the projection on the complement of the kernel of $\overline{\partial}$ to this solution operator (see 1). Hence the operator $\overline{\partial}^* N_{(p,q)}$ on Ω_2 is compact. Similarly one can show that compactness of $\overline{\partial}^* N_{(p,q+1)}$ is invariant under biholomorphisms. □

Now we will introduce another characterization for compactness that has not appeared in the literature before.

Let Ω be a smooth bounded domain in $\mathbb{C}^n$, $K \subset b\Omega$ and U be an open neighborhood of K. Define

$$C^\infty_{0,(p,q)}(U) = \left\{ \sum_{|I|=p,|J|=q}{}' f_{IJ} dz_I \wedge d\bar{z}_J : f_{IJ} \in C^\infty_0(U) \right\}$$

for $0 \le q \le n$ where $C^\infty_0(U)$ denotes the space of smooth functions with compact support in U. Define $\lambda_{(p,q)}(U)$ as follows: For $0 \le p \le n,\ 1 \le q \le n$

$$\lambda_{(p,q)}(U) = \inf \left\{ \frac{\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2}{\|f\|^2} : f \in Dom(\overline{\partial}^*) \cap C^\infty_{0,(p,q)}(U),\ f \neq 0 \right\},$$

where $Dom((\overline{\partial}^*)$ refers to the domain of $\overline{\partial}^*$ on Ω, and

$$\lambda_{(p,0)}(U) = \inf \left\{ \frac{\|\overline{\partial} f\|^2}{\|f\|^2} : f \in (Ker\overline{\partial})^\perp \cap C^\infty_{0,(p,0)}(U),\ f \neq 0 \right\}$$

where $(Ker\overline{\partial})^\perp$ is the orthogonal complement of $(Ker\overline{\partial})$ in $L^2_{(p,q)}(\Omega)$. Notice that $\lambda_{(p,q)}(U) \le \lambda_{(p,q)}(V)$ if $V \subset U$.

Theorem 3. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let $0 \le p, q \le n$, be given. Then the following are equivalent:*

(i) The $\overline{\partial}$-Neumann operator $N_{(p,q)}$ of Ω is compact on square integrable (p,q)-forms on Ω

(ii) for all $K \subset b\Omega$ and $M > 0$ there exists an open neighborhood U of K such that $\lambda_{(p,q)}(U) > M$,

(iii) for all $M > 0$ there exists an open neighborhood U of the set of infinite type points in $b\Omega$ such that $\lambda_{(p,q)}(U) > M$.

Proof. We'll show the equivalences for $0 \le p \le n, 1 \le q \le n$. The proof can be mimicked for the case $q = 0$ using the following: Compactness of N_0 is equivalent to the following compactness estimate: $\forall\varepsilon > 0, \exists D_\varepsilon > 0$ such that

$$\|g\|^2 \le \varepsilon\|\overline{\partial}g\|^2 + D_\varepsilon\|g\|_{-1}^2 \text{ for } g \in (Ker\overline{\partial})^\perp \cap Dom(\overline{\partial})$$

$(i) \Rightarrow (ii)$: Assume that the $\overline{\partial}$-Neumann operator of Ω is compact, and there exist $K \subset b\Omega$ and $M > 0$ such that $\lambda_{(p,q)}(U) < M$ for all open neighborhoods U of K. We may assume that there exist a sequence of open neighborhoods $\{U_k\}$ of K, and a sequence of nonzero (p,q)-forms $\{f_k\}$ such that

$$U_{k+1} \subset\subset U_k,\ K \subset \bigcap_{k=1}^{\infty} U_k \subset b\Omega,\ f_k \in Dom(\overline{\partial}^*) \cap C_{0,(p,q)}^{\infty}(U_k),$$

$$\|f_k\| = 1, \text{ and } \|\overline{\partial}f_k\|^2 + \|\overline{\partial}^* f_k\|^2 < M \text{ for } k = 1, 2, 3, \cdots$$

Let's choose $f_{k_1} = f_1$. Since $K \subset \bigcap_{k=1}^{\infty} U_k, U_{k+1} \subset\subset U_k, \|f_k\| = 1$, and $f_k \in C_{0,(p,q)}^{\infty}(U_k)$ there exists k_2 such that $\int_{\Omega\setminus U_{k_2}} |f_{k_1}|^2 > 1/2$. So $\|f_{k_1} - f_{k_2}\|^2 \ge 1/2$. Similarly, we can choose a subsequence $\{f_{k_j}\}$ so that $\|f_{k_s} - f_{k_t}\|^2 \ge 1/2$ for $s \ne t$. We denote this subsequence by $\{f_k\}$. Compactness of the $\overline{\partial}$-Neumann operator is equivalent to the following so called compactness estimate (see Lemma 1): $\forall\varepsilon > 0, \exists D_\varepsilon > 0$ such that

$$\|g\|^2 \le \varepsilon(\|\overline{\partial}g\|^2 + \|\overline{\partial}^* g\|^2) + D_\varepsilon\|g\|_{-1}^2 \text{ for } g \in Dom(\overline{\partial}^*) \cap Dom(\overline{\partial}) \qquad (2.2)$$

Choose $\varepsilon = \frac{1}{16M}$. Since $Dom(\overline{\partial}^*) \cap C_{0,(p,q)}^{\infty}(U_k) \subset Dom(\overline{\partial}^*) \cap Dom(\overline{\partial})$ using (2.2) we get

$$\|f_k - f_l\|_{-1}^2 \ge \frac{1}{4D_\varepsilon} > 0 \text{ for } k \ne l \qquad (2.3)$$

The imbedding from $L^2(\Omega)$ to $W^{-1}(\Omega)$ is compact and $\{f_k\}$ is a bounded sequence in $L_{(p,q)}^2(D)$. Hence $\{f_k\}$ has a convergent subsequence in $W_{(p,q)}^{-1}(\Omega)$. This contradicts (2.3).

$(ii) \Rightarrow (iii)$: This part is obvious because the set of infinite type points is compact.

$(iii) \Rightarrow (i)$: Let K be the set of infinite type points in $b\Omega$ and $u \in Dom(\overline{\partial}^*) \cap C^\infty_{(p,q)}(\overline{\Omega})$. Assume that $\lambda_{(p,q)}(U_k) > k$ where $\{U_k\}$ is a sequence of open neighborhoods of K such that $U_{k+1} \subset\subset U_k,\ K \subset \bigcap_{k=1}^\infty U_k \subset b\Omega$. Let $\varphi_k \in C_0^\infty(U_k)$ such that $0 \leq \varphi_k \leq 1$ and $\varphi_k \equiv 1$ in a neighborhood of K. Define $\psi_k = 1 - \varphi_k$. Notice that ψ_k is supported away from K. We will use general constants in the following estimates. That is the constants we use won't depend on u but they might change at each step. Away from K we have subelliptic estimates as $b\Omega \setminus K$ is the set of finite type points. Hence, there exists $s > 0$ such that $\forall \varepsilon > 0\ \exists D_\varepsilon > 0$ such that

$$\begin{aligned} \|\psi_k u\|^2 &\leq \varepsilon\|\psi_k u\|_s^2 + D_\varepsilon\|\psi_k u\|_{-1}^2 \\ &\leq \varepsilon C_k(\|\overline{\partial}(\psi_k u)\|^2 + \|\overline{\partial}^*(\psi_k u)\|^2) + C_k D_\varepsilon\|u\|_{-1}^2 \\ &\leq \varepsilon C_k(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 + \|u\|^2) + C_k D_\varepsilon\|u\|_{-1}^2 \end{aligned} \tag{2.4}$$

The first inequality follows because L^2 imbedds compactly into W^s for $s > 0$. We used the subelliptic estimate for the second inequality. If we use $\lambda_{(p,q)}(U_k) > k$ we get:

$$\begin{aligned} \|\varphi_k u\|^2 &\leq \frac{1}{k}(\|\overline{\partial}(\varphi_k u)\|^2 + \|\overline{\partial}^*(\varphi_k u)\|^2) \\ &\leq \frac{1}{k}(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2) + \frac{C_k}{k}\|\xi_k u\|^2 \end{aligned} \tag{2.5}$$

where $\xi_k \equiv 0$ in a neighborhood of K and $\xi_k \equiv 1$ in a neighborhood of support of ψ_k. Calculations that are similar to ones in (2.4) show that

$$\|\xi_k u\|^2 \leq \varepsilon'\tilde{C}_k(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 + \|u\|^2) + \tilde{C}_k D_{\varepsilon'}\|u\|_{-1}^2 \tag{2.6}$$

By choosing $\varepsilon, \varepsilon'$ small enough and combining (2.4) and (2.6)we get: For all $k = 1, 2, 3, \cdots$ there exists $D_k > 0$ such that

$$\|u\|^2 \leq \frac{2}{k}(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 + \|u\|^2) + D_k\|u\|_{-1}^2 \text{ for } u \in Dom(\overline{\partial}^*) \cap C^\infty_{(p,q)}(\overline{\Omega}) \tag{2.7}$$

$Dom(\overline{\partial}^*) \cap C^\infty_{(p,q)}(\overline{\Omega})$ is dense in $Dom(\overline{\partial}^*) \cap Dom(\overline{\partial})$. Therefore, the above estimate (2.7) holds on $Dom(\overline{\partial}^*) \cap Dom(\overline{\partial})$. That is, the $\overline{\partial}$-Neumann operator of Ω is compact on (p, q)-forms for $0 \leq p \leq n, 1 \leq q \leq n$. □

One of the nice properties of compactness is that it localizes.

Lemma 3 ([31]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$, and $N_{(p,q)}$ be the $\overline{\partial}$-Neumann operator on $L^2_{(p,q)}(\Omega)$ where $0 \leq p \leq n, 1 \leq q \leq n$.*

1) *If for every boundary point p there exists a pseudoconvex domain U that contains p such that the $\overline{\partial}$-Neumann operator on (the domain) $U \cap \Omega$ is compact, then $N_{(p,q)}$ is compact.*
2) *If U is smooth bounded and strictly pseudoconvex and $U \cap \Omega$ is a domain, then if $N_{(p,q)}$ is compact, so is the corresponding $\overline{\partial}$-Neumann operator on $U \cap \Omega$.*

In the introduction we mentioned that compactness of the $\overline{\partial}$-Neumann problem is weaker than global regularity. Actually more is true:

Theorem 4 ([41]). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. If $N_{(p,q)}$ is compact on $L^2_{(p,q)}(\Omega)$, then N is compact (in particular, continuous) as an operator from $W^s_{(p,q)}(\Omega)$ to itself, for all $s \geq 0$.*

We would like to remark that in the above theorem, the implication in the other direction is valid as well. That is, if $N_{(p,q)}$ is a compact operator on $W^s_{(p,q)}(\Omega)$ for some $s \geq 0$, then $N_{(p,q)}$ is compact in $L^2_{(p,q)}(\Omega)$. This implication follows from a theorem about compact operators over Banach spaces, symmetric with respect to a scalar product (see, e.g., [43], Corollary 2).

In case of convex domains, compactness of the $\overline{\partial}$-Neumann problem is completely understood:

Theorem 5 ([30]). *Let* Ω *be a bounded convex domain in* $\mathbb{C}^n$. *Let* $1 \leq q \leq n$. *The following are equivalent:*

1) *There exists a compact solution operator for* $\overline{\partial}$ *on* (p, q)*-forms.*
2) *The boundary of* Ω *does not contain any affine variety of dimension greater than or equal to* q.
3) *The boundary of* Ω *does not contain any analytic variety of dimension greater than or equal to* q.
4) *The* $\overline{\partial}$*-Neumann operator* $N_{(p.q)}$ *is compact.*

C. Stein Neighborhood Bases

In this section we will discuss existence of a Stein neighborhood basis for the closure of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. There are two sources of existence of Stein neighborhood bases for the closure. The first is through existence of transversal holomorphic vector fields on a neighborhood of the weakly pseudoconvex points on the boundary. This method has been developed by Fornæss and Nagel in [26] (see also [2]).

Theorem 6 ([26]). *Let* Ω *be a* C^3*-smooth bounded pseudoconvex domain in* $\mathbb{C}^n$ *with a defining function* ρ, *and let* K *be the set of weakly pseudoconvex points in* $b\Omega$. *Assume that there exist an open neighborhood* U *of* K *and a vector field* $F = (F_1, \ldots, F_n)$ *on* U *such that* F_j *is holomorphic for* $j = 1, \ldots, n$ *and* $Re\left(\sum_{j=1}^n F_j \frac{\partial \rho}{\partial z_j}\right) > 0$ *on* $U \cap b\Omega$. *Then* $\overline{\Omega}$ *has a Stein neighborhood basis.*

The condition $Re\left(\sum_{j=1}^n F_j \frac{\partial \rho}{\partial z_j}\right) > 0$ enables us to push the boundary out. Moreover, since F has holomorphic components the pseudoconvexity of the boundary is preserved. We give a proof simpler than the one in [26] for the convenience of the reader.

Proof. Choose open sets V_1, V_2 in $\mathbb{C}^n$ such that $K \subset\subset V_1 \subset\subset V_2 \subset\subset U$ and a smooth compactly supported function ϕ on V_2 such that $0 \leq \phi \leq 1$, and $\phi \equiv 1$ on V_1. We may choose a neighborhood U_1 of $b\Omega$ and a defining function r of Ω such that $\mathcal{L}_r(z;\tau) > 0$ for $z \in \overline{U}_1 \setminus V_1, \tau \in \mathbb{C}^n \setminus \{0\}$. Let

$$H(z) = \phi(z)F(z) + (1-\phi(z))\left(\frac{\partial r}{\partial \bar{z}_1}, \cdots, \frac{\partial r}{\partial \bar{z}_n}\right).$$

Now we define r_ε such that $r(z) = r_\varepsilon(A^\varepsilon(z))$ for $A^\varepsilon(z) = z + \varepsilon H(z)$. Notice that H pushes the boundary out and so r_ε is a defining function for a domain that includes $\overline{\Omega}$. We have

$$\begin{aligned}
\frac{\partial r}{\partial z_k} &= \sum_{s=1}^n \frac{\partial r_\varepsilon}{\partial w_s}\frac{\partial A_s^\varepsilon}{\partial z_k} + \sum_{s=1}^n \frac{\partial r_\varepsilon}{\partial \bar{w}_s}\frac{\partial \bar{A}_s^\varepsilon}{\partial z_k} \text{ and} \\
\frac{\partial^2 r}{\partial \bar{z}_l \partial z_k} &= \sum_{s,t=1}^n \frac{\partial^2 r_\varepsilon}{\partial w_t \partial w_s}\frac{\partial A_t^\varepsilon}{\partial \bar{z}_l}\frac{\partial A_s^\varepsilon}{\partial z_k} + \sum_{s,t=1}^n \frac{\partial^2 r_\varepsilon}{\partial \bar{w}_t \partial w_s}\frac{\partial \bar{A}_t^\varepsilon}{\partial \bar{z}_l}\frac{\partial A_s^\varepsilon}{\partial z_k} \\
&+ \sum_{s,t=1}^n \frac{\partial^2 r_\varepsilon}{\partial w_t \partial \bar{w}_s}\frac{\partial A_t^\varepsilon}{\partial \bar{z}_l}\frac{\partial \bar{A}_s^\varepsilon}{\partial z_k} + \sum_{s,t=1}^n \frac{\partial^2 r_\varepsilon}{\partial \bar{w}_t \partial \bar{w}_s}\frac{\partial \bar{A}_t^\varepsilon}{\partial \bar{z}_l}\frac{\partial \bar{A}_s^\varepsilon}{\partial z_k} \\
&+ \sum_{s=1}^n \frac{\partial r_\varepsilon}{\partial w_s}\frac{\partial^2 A_s^\varepsilon}{\partial \bar{z}_l \partial z_k} + \sum_{s=1}^n \frac{\partial r_\varepsilon}{\partial \bar{w}_s}\frac{\partial^2 \bar{A}_s^\varepsilon}{\partial \bar{z}_l \partial z_k}
\end{aligned}$$

Let f be a differentiable function from an open set in $\mathbb{C}^n$ into $\mathbb{C}^n$ and $J_\partial(f)$ denote the matrix $\left\{\frac{\partial f_j}{\partial z_k}\right\}_{jk}$. Therefore, we have

$$\begin{aligned}
\mathcal{L}_r(z;\tau) &= \sum_{s,t=1}^n \frac{\partial^2 r_\varepsilon(A^\varepsilon(z))}{\partial w_t \partial w_s}(J_{\bar{\partial}}(A^\varepsilon)\bar{\tau})_t (J_\partial(A^\varepsilon)\tau)_s + \mathcal{L}_{r_\varepsilon}(A^\varepsilon(z); J_\partial(A^\varepsilon)\tau) \\
&+ \mathcal{L}_{r_\varepsilon}(A^\varepsilon(z); J_{\bar{\partial}}(A^\varepsilon)\bar{\tau}) + \sum_{s,t=1}^n \frac{\partial^2 r_\varepsilon(A^\varepsilon(z))}{\partial \bar{w}_t \partial \bar{w}_s}(J_{\bar{\partial}}(\bar{A}^\varepsilon)\bar{\tau})_t (J_\partial(\bar{A}^\varepsilon)\tau)_s \\
&+ \sum_{s,l,k=1}^n \frac{\partial r_\varepsilon(A^\varepsilon(z))}{\partial w_s}\frac{\partial^2 A_s^\varepsilon(z)}{\partial \bar{z}_l \partial z_k}\bar{\tau}_l \tau_k + \sum_{s,l,k=1}^n \frac{\partial r_\varepsilon(A^\varepsilon(z))}{\partial \bar{w}_s}\frac{\partial^2 \bar{A}_s^\varepsilon(z)}{\partial \bar{z}_l \partial z_k}\bar{\tau}_l \tau_k
\end{aligned}$$

Notice that $\frac{\partial A^{\varepsilon}_{s}}{\partial \bar{z}_k}$, $\frac{\partial^2 A^{\varepsilon}_{s}(z)}{\partial \bar{z}_l \partial z_k}$, and $\frac{\partial^2 \bar{A}^{\varepsilon}_{s}(z)}{\partial \bar{z}_l \partial z_k}$ are of order ε for $1 \leq s, l, k \leq n$. Therefore, every term on the right hand side of the equality above is of order ε except the second one. Namely we get the following formulas:

$$\begin{aligned} \mathcal{L}_r(z;\tau) &= \mathcal{L}_{r_\varepsilon}(z + \varepsilon H(z); \tau + \varepsilon J_\partial(H)\tau) + O(\varepsilon) \quad \text{for } z \in U_1 \setminus V_1 \\ \mathcal{L}_r(z;\tau) &= \mathcal{L}_{r_\varepsilon}(z + \varepsilon H(z); \tau + \varepsilon J_\partial(H)\tau) \quad \text{for } z \in V_1 \end{aligned}$$

Using Sard's theorem, one can choose a positive decreasing sequence $\{a_n\}$ that converges to 0 such that the domains $U_n = \{z \in \mathbb{C}^n : r_{a_n}(z) < 0\}$ have smooth boundary. Moreover, $\overline{\Omega} \subset\subset U_n$, and

$$\mathcal{L}_{r_{a_n}}(z;\tau) \geq 0 \text{ for } z \in bU_n, \text{ and } \sum_{j=1}^{n} \frac{\partial r_{a_n}(z)}{\partial z_j}\tau_j = 0.$$

Therefore $\overline{\Omega}$ has a Stein neighborhood basis. □

The second source of existence of a Stein neighborhood basis for the closure is property $(\widetilde{P})$.

Definition 3 ([13, 46]). *A domain $\Omega \subset \mathbb{C}^n$ is said to satisfy property (P) $\Big($property $(\widetilde{P})\Big)$ if for every $M > 0$ there exists $\phi_M \in C^2(\overline{\Omega})$ such that for every vector $W = \sum_{j=1}^{n} w_j \frac{\partial}{\partial z_j}$ of type $(1,0)$ the following inequalities are satisfied:*

i) $0 \leq \phi_M \leq 1$ *on* $\overline{\Omega}$ $\Big(|W(\phi_M)(z)|^2 \leq \mathcal{L}_{\phi_M}(z;W)$ *for* $z \in \overline{\Omega}\Big)$

ii) $\mathcal{L}_{\phi_M}(z;W) \geq M \sum_{j=1}^{n} |w_j|^2$ *for* $z \in b\Omega$.

By exponentiating and scaling the functions one can show that property (P) implies property $(\widetilde{P})$. As mentioned in the introduction, Sibony showed that for a smooth bounded pseudoconvex domain, property $(\widetilde{P})$ implies the existence of a Stein neighborhood basis for the closure. Although he stated the theorem for property (P) his proof shows that only property $(\widetilde{P})$ is needed:

Theorem 7 ([53]). *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that Ω satisfies property $(\widetilde{P})$. Then $\overline{\Omega}$ has a Stein neighborhood basis.*

When a pseudoconvex domain has a "well behaving" vector field it does not necessarily satisfy property $(\widetilde{P})$. For example, any convex domain with an analytic disc in its boundary has a holomorphic vector field that is transversal to the boundary. However, because of the disc in the boundary it does not satisfy property $(\widetilde{P})$ (see [57], section 2.2). It will be interesting to know if and how these two methods are connected.

CHAPTER III

COMPACTNESS OF THE $\overline{\partial}$-NEUMANN PROBLEM*

Basic definitions and results about compactness of the $\overline{\partial}$-Neumann problem have already been discussed in the previous chapter. In this chapter we will state and prove our results.

Throughout this chapter by the *$\overline{\partial}$-Neumann problem* we mean the $\overline{\partial}$-Neumann problem on $(0, 1)$-forms unless otherwise specified. There have been two different approaches for compactness of the $\overline{\partial}$-Neumann problem. The first is a potential theoretic approach. Catlin in [13] introduced property (P) and showed that it implies the compactness of the $\overline{\partial}$-Neumann problem. Later, Sibony([53]) undertook a systematic study of property (P) under the name B-regularity. He showed that it is equivalent to the property that any continuous function on the boundary of the domain can be extended to a function that is continuous on the closure and plurisubharmonic on the domain. Recently, McNeal introduced property $(\widetilde{P})$, and showed that it still implies compactness of the $\overline{\partial}$-Neumann problem. While it is easy to see that property (P) implies property $(\widetilde{P})$, the precise relations between the two properties is not understood. The second approach is geometric in nature. Recently, Straube([56]) introduced a geometric condition that implies compactness of the $\overline{\partial}$-Neumann operator on domains in $\mathbb{C}^2$. We still do not know how property (P) or $(\widetilde{P})$ interact with Straube's geometric condition.

It is well known that analytic discs in the boundary of a smooth bounded pseudoconvex domain are a violation of all sufficient conditions for compactness discussed

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in the previous paragraph. So it will be interesting to know whether analytic discs in the boundary cause an obstruction for compactness of the $\overline{\partial}$-Neumann problem as well. Catlin, in his unpublished result, showed that presence of an analytic disc in the boundary is an obstruction for compactness in case the domain is in $\mathbb{C}^2$. A proof for the case of domains with Lipschitz boundary is in [31]. It is folklore that the methods that work in $\mathbb{C}^2$ also show that, in any dimension n, if the $\overline{\partial}$-Neumann operator is compact, then the boundary cannot contain an $(n-1)$-dimensional complex manifold. Matheos, on the other hand, showed that absence of analytic discs in the boundary is not sufficient for compactness of the $\overline{\partial}$-Neumann operator. He constructed a smooth bounded complete Hartogs domain without an analytic disc in the boundary, with a noncompact $\overline{\partial}$-Neumann operator [31, 45]. The problem is still open in general. That is, it is still open whether presence of an analytic disc in the boundary is a violation of compactness of the $\overline{\partial}$-Neumann operator in $\mathbb{C}^n, n > 2$. However, we show that the answer is still yes if the disc contains a point at which the Levi form is of rank $n-2$. That is, the boundary is strictly pseudoconvex in the directions transversal to the disc. More generally, we show that it is enough to have a complex manifold of dimension equal to the number of vanishing eigenvalues of the Levi form.

Theorem 8. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$; and $0 \leq p \leq n, 1 \leq q \leq k \leq n-1$. Let $z_0 \in b\Omega$ and assume that the Levi form of $b\Omega$ at z_0 has at most k vanishing eigenvalues. (That is, the Levi form is of rank at least $n-k-1$ at z_0). If there exists a compact solution operator for $\overline{\partial}$ on (p,q)-forms (in particular, if the $\overline{\partial}$-Neumann operator on (p,q)-forms is compact), then $b\Omega$ does not contain a k-dimensional complex manifold through z_0.*

We would like to note that the above theorem generalizes Theorem 1.1 in [39], where the domains are fibered over a Reinhardt domain in $\mathbb{C}^2$. In case the Levi form has at most one vanishing eigenvalue on the boundary, compactness of the $\overline{\partial}$-Neumann operator implies that there isn't any analytic disc in the boundary. It is a well known result that if the set of *Levi flat* boundary points (where all the eigenvalues are zero) has nonempty interior with respect to relative topology on the boundary, then it is foliated by $(n-1)$-dimensional complex manifolds. So, this is an obstruction for compactness of the $\overline{\partial}$-Neumann operator. It is an easy consequence of Theorem 8 that this result holds for the (in general much bigger) set of *weakly pseudoconvex* points.

Corollary 2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$. If there exists a compact solution operator for $\overline{\partial}$ on $(0,1)$-forms (in particular, if the $\overline{\partial}$-Neumann operator on Ω is compact), then the set of weakly pseudoconvex points in $b\Omega$ has empty (relative) interior.*

We would like to remark that the higher the number of vanishing eigenvalues the "flatter" the boundary is. In general, one would expect this situation to be even more favorable to noncompactness of the $\overline{\partial}$-Neumann operator. However, our present methods do not give this. The reason is that having a zero eigenvalue in the transversal direction to the complex manifold allows for complications in the geometry of the boundary.

Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$, and $P(\overline{\Omega})$ denote the plurisubharmonic functions on Ω that are continuous up to the boundary of Ω. Define the plurisubharmonic hull, $\widehat{K}$, of a compact set $K \subset \overline{\Omega}$ as follows:

$$\widehat{K} = \left\{ z \in \overline{\Omega} : f(z) \leq \sup_{w \in K} f(w) \text{ for all } f \in P(\overline{\Omega}) \right\}$$

We get exactly the same hull as $\widehat{K}$ if we define it with respect to any of the following: plurisubharmonic functions that are smooth up to the boundary, the absolute value of holomorphic functions that are smooth up to the boundary, or the absolute value of holomorphic functions that are continuous up to the boundary (see for example [10, 11, 34]). We say that Ω satisfies the hull condition if $\widehat{K} \cap b\Omega = K \cap b\Omega$ for every compact set $K \subset \overline{\Omega}$. Catlin in his thesis [10] showed that in $\mathbb{C}^2$ a smooth bounded pseudoconvex domain Ω satisfies the hull condition if and only if there is no analytic disc in the boundary of Ω. He also showed that this is not true in higher dimensions. So for smooth bounded pseudoconvex domains in $\mathbb{C}^2$ compactness of the $\overline{\partial}$-Neumann problem implies the hull condition. We will generalize this result to $\mathbb{C}^n$ under the restriction that the Levi form has at most one vanishing eigenvalue on the boundary.

Theorem 9. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$. Let p be a boundary point where the Levi form of $b\Omega$ has at most one vanishing eigenvalue. Then there is a compact subset K of $\overline{\Omega}$ with $p \in \widehat{K} \setminus K$ if and only if $b\Omega$ contains an analytic disc through p.*

The proof of Theorem 9 will be along the lines of the proof for the $\mathbb{C}^2$ case given in [10], but some additional work is needed. The details will be given in Section B of this chapter.

Sibony([52]) showed that in $\mathbb{C}^n$ the hull condition is equivalent to the L^2_{loc}-hypoellipticity of the $\overline{\partial}$-problem. That is, the $\overline{\partial}$-problem can be solved in L^2_{loc} topology preserving the singular support on the closure of the domain. Therefore, L^2_{loc}-hypoellipticity of the $\overline{\partial}$-problem follows from compactness of the $\overline{\partial}$-Neumann problem under the restriction that the Levi form has at most one vanishing eigenvalue. At present there is no direct proof, even in $\mathbb{C}^2$. Such a direct proof would be very interesting to have.

A. Proofs of Theorem 8 and Corollary 2

Using the following lemma we will be able to simplify the boundary locally near a point of a complex manifold in the boundary.

Lemma 4. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, M a complex manifold of dimension k in $b\Omega$, and $p \in M$. Then there is a ball B centered at p, a biholomorphic map $G : B \to G(B)$ such that*

(i) $G(p) = 0$

(ii) $G(M \cap B) = \{w \in G(B) | w_{k+1} = \cdots = w_n = 0\}$

(iii) the real normal to $G(b\Omega \cap B)$ at points of $G(M \cap B)$ is given by the $Re(w_n)$-axis.

In other words, there is a local holomorphic change of coordinates so that in the new coordinates, M is affine and the (real) unit normal to the boundary is constant on M.

Proof of Lemma 4 . Using holomorphic change of coordinates we may assume that $p = 0$, and M is locally given by $\{z | z_{k+1} = \cdots = z_n = 0\}$. In these coordinates, let ρ be a defining function for $b\Omega$ near 0 such that $\partial\rho/\partial z_n \neq 0$. There exists a real-valued C^∞ function h in a neighborhood of 0 such that the normal $e^h \left(0, \cdots, 0, \frac{\partial\rho}{\partial\bar{z}_{k+1}}, \cdots, \frac{\partial\rho}{\partial\bar{z}_n}\right)$ is conjugate holomorphic on M (each component is conjugate holomorphic). This can be shown by adapting the argument in the proof of Lemma 1 in [3] where the same result is shown in case M is an analytic disc. Alternatively, the statement is a special case of the theorem in [58], specifically the equivalence of (ii) and (iv). Note that the form α appearing in (iv) of that theorem is real, and its restriction to M is closed by the lemma in section 2 of [8]. Therefore, there is a real valued function h such that $d(-h) = \alpha$ on M (near 0). The proof of $(ii) \Leftrightarrow (iv)$ in [58] shows that any real-valued C^∞ extension of h to a full neighborhood of 0 will do.

We now define a biholomorphic change of coordinates (near 0) by

$$\widehat{G}(z_1,\cdots,z_n)=\left(z_1,\cdots,z_{n-1},\sum_{j=k+1}^{n} z_j e^{h(z_1,\cdots,z_k,0,\cdots,0)}\frac{\partial\rho}{\partial z_j}(z_1,\cdots,z_k,0,\cdots,0)\right)$$

Then $\widehat{G}(M)\subset\{z_{k+1}=\cdots=z_n=0\}$, and the complex tangent space to $b\Omega$ is mapped onto the complex hypersurface $\{z_n=0\}$ by the the complex derivative of $\widehat{G}$ at points of M. Notice that the image of the complex normal is not necessarily parallel to the vector $(0,\cdots,0,1)$. In these new coordinates (which we again denote by $(z_1,\cdots,z_n)$), M is of the same form as before, but the complex tangent space to $b\Omega$ is constant, namely it is the hyperplane $\{z_n=0\}$. Consider now the real unit normal to $b\Omega$. Since it is perpendicular to complex tangential space its restriction to M is of the form $(0,\cdots,0,e^{i\theta})$. Using Lemma 1 from [3] once more shows that the function θ is harmonic on each disc in M, i.e. it is pluriharmonic on M. Denote by h_1 a pluriharmonic conjugate. The final coordinate change

$$(z_1,\cdots,z_n)\to\left(z_1,\cdots,z_{n-1},z_n e^{-h_1(z_1,\cdots,z_k,0,\cdots,0)-i\theta(z_1,\cdots,z_k,0,\cdots,0)}\right)$$

keeps the complex tangential space the same and rotates the real normal so that the unit normal becomes constant on M. Combining the three local biholomorphic coordinate changes gives the map G with the properties required in Lemma 4. □

The following lemma will be essential in the proof of Theorem 8. Let $A(D)$ denote the Hilbert space of square integrable holomorphic functions on D.

Lemma 5. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, smooth near the strictly pseudoconvex boundary point p. Assume the pseudoconvex domain Ω_1 is contained in Ω, shares the boundary point p, and is smooth near p. Then the restriction operator from $A(\Omega)$ to $A(\Omega_1)$ is not compact.*

Proof of Lemma 5. The proof of the lemma uses ideas from [30], p. 637. First we construct a sequence of norm 1 functions in $A(\Omega)$ which has no convergent subsequence in $A(\Omega_1)$. Let $\{p_j\}_{j=1}^{\infty}$ be a sequence of points on the common interior normal to $b\Omega$ and $b\Omega_1$ at p such that $\lim_{j\to\infty} p_j = p$. Let $f_j(z) = K_\Omega(z, p_j)/K_\Omega(p_j, p_j)^{1/2}$, where K_Ω denotes the Bergman kernel of Ω. Notice that $\| f_j \|_{L^2(\Omega)} = 1$ for all j. For $z \in \Omega$ fixed, the function $K_\Omega(z, \cdot)$ is smooth up to the boundary near p (in fact, the subelliptic estimates for the $\overline{\partial}$-Neumann problem near p have considerably stronger consequences for the kernel function [5, 6]), and $K_\Omega(p_j, p_j) \to \infty$ as $j \to \infty$ (see e.g. [37], Theorem 3.5.1). Consequently, $f_j(z) \to 0$ for all $z \in \Omega$. On the other hand,

$$\| f_j \|^2_{L^2(\Omega_1)} = \frac{\| K_\Omega(\cdot, p_j) \|^2_{L^2(\Omega_1)}}{K_\Omega(p_j, p_j)} \geq \frac{K_\Omega(p_j, p_j)}{K_{\Omega_1}(p_j, p_j)} \geq C > 0 \ .$$

The first inequality follows by applying the reproducing property of K_{Ω_1} to the function $K_\Omega(\cdot, p_j)$ viewed as an element of $A(\Omega_1)$ together with an argument using Hölder's inequality. The second inequality follows from the fact that both Ω and Ω_1 are strictly pseudoconvex at p, and hence the kernel asymptotics at p are the same ([37], Theorem 3.5.1), and the second inequality follows. We conclude that $\{f_j\}_{j=1}^{\infty}$ has no subsequence that converges in $A(\Omega_1)$, and the proof of Lemma 5 is complete. □

Proof of Theorem 8. Since the theory of (p, q)-forms and $(0, q)$-forms is the same for the $\overline{\partial}$-Neumann problem we will prove the theorem for $p = 0$. If $\overline{\partial}$ has a compact solution operator on $(0, q)$-forms then $\overline{\partial}^* N_{(0,q)}$ is compact. We will show that if the boundary contains a k-dimensional complex manifold through z_0 and the rank of the Levi form at z_0 is $n - k - 1$, then the $\overline{\partial}$-Neumann operator on $(0, q)$-forms on Ω is not compact for $1 \leq q \leq k$. The following facts will be useful in simplifying the proof: Compactness of the $\overline{\partial}$-Neumann problem on $(0, q)$-forms is invariant under biholomorphisms (Lemma 1); and it is a local property (Lemma 3). First we will use Lemma 4 to pass to a coordinate system where the boundary geometry is simple around z_0. Then we will construct a sequence of $\overline{\partial}$-closed $(0,q)$-forms that will lead to a contradiction when combined with compactness of the $\overline{\partial}^* N_{(0,q)}$ on Ω. A very similar argument appeared in [30], section 4, and [31], proof of Proposition 4.1. In turn, those arguments draw substantially on ideas from [12] and [23]. Using Lemma 1, Lemma 3, Lemma 4, and scaling if necessary we may assume that $M = \{(z', z'', 0) : |z'| + |z''| < 2\} \subset b\Omega$, where $z' = (z_1, \dots, z_q), z'' = (z_{q+1}, \dots, z_k)$, and the unit (real) normal to the boundary is $(0, \dots, 0, 1)$ along M. Let $z''' = (z_{k+1}, \dots, z_n)$ and Ω_1 be the $(n-k)$-dimensional slice of Ω in z''' variables. We note that when $k = q$ the variable z'' is not defined and $z = (z', z''')$. Since the normal to $b\Omega$ is constant along M we can choose a ball Ω_2 in $\mathbb{C}^{n-k}$ (in z''' variables) centered at $(0, \dots, 0, -\delta)$ with radius $\delta > 0$ such that $\{(z', z'') \in \mathbb{C}^k : |z'| < 1/2, |z''| < 1/2\} \times \Omega_2 \subset \Omega$. Since the rank of the Levi form is preserved under biholomorphisms Ω_1 is strongly pseudoconvex at 0. Therefore Lemma 5 shows that we can choose a sequence of holomorphic functions $\{f_j\}$ in $A(\Omega_1)$ without any convergent subsequence in $A(\Omega_2)$. The rest of the proof is essentially the same as the proof of (1) $\Rightarrow$ (2) of Theorem 1.1 in [30]. We put the details here for the convenience of the reader.

We use the Ohsawa-Takegoshi extension theorem [48] to extend f_j to get a bounded sequence $\{F_j\}$ in $A(\Omega)$. Let $\alpha_j = F_j(z', z'', z''')d\bar{z}_1 \wedge \dots \wedge d\bar{z}_q$. Then $\{\alpha_j\}$ is a $\overline{\partial}$-closed bounded sequence in $L^2_{(0,q)}(\Omega)$. Let $g_j = \overline{\partial}^* N_{(0,q)}\alpha_j$ and $\hat{g}_j$ denote the form obtained from g_j by omitting the terms containing $d\bar{z}_j$ with $q+1 \leq j \leq n$. We still have $\overline{\partial}_{z'}\hat{g}_j = \alpha_j$ for fixed (z'', z''') where $\overline{\partial}_{z'}$ is the $\overline{\partial}$ in z' variables. Let $\langle ., . \rangle$ denote the standard pointwise inner product in z' variables and $\chi \in C^\infty_0(-\infty, \infty)$ be a cut-off function such that $0 \leq \chi \leq 1, \chi(t) = 1$ when $|t| \leq 1/4$ and $\chi(t) = 0$ when $|t| \geq 2/5$. Let $\beta(z) = \chi(|z'|)d\bar{z}_1 \wedge \dots \wedge d\bar{z}_q$. In the following equalities we will use the mean value properties of holomorphic functions. There exists $C, D > 0$ such that for

$|z''| \leq 1/2$ and $z''' \in \Omega_2$ fixed we have

$$\begin{aligned}
|F_j(0,z'',z''') - F_k(0,z'',z''')| &= C\left|\int_{|z'|<1/2} \langle \alpha_j - \alpha_k, \beta\rangle dV(z')\right| \\
&= C\left|\int_{|z'|<1/2} \langle \hat{g}_j - \hat{g}_k, \vartheta_{z'}\beta\rangle dV(z')\right| \\
&\leq CD\left\{\int_{|z'|<1/2} |\hat{g}_j - \hat{g}_k|^2 dV(z')\right\}^{1/2}.
\end{aligned}$$

We apply the submean value property for the plurisubharmonic functions $|F_j(0,z'',z''')-F_k(0,z'',z''')|^2$. So there exists $E > 0$ such that

$$|f_j(z''') - f_k(z''')|^2 \leq E^2 \int_{|z''|\leq 1/2} |F_j(0,z'',z''') - F_k(0,z'',z''')|^2 dV(z'')$$

If we integrate out z''' and combine the previous estimates we get:

$$\|f_j - f_k\|_{L^2(\Omega_2)} \leq CDE\|\hat{g}_j - \hat{g}_k\|_{L^2_{(0,q-1)}(\Omega)} \leq CDE\|g_j - g_k\|_{L^2_{(0,q-1)}(\Omega)}$$

$\{f_j\}$ has no convergent subsequence therefore neither does $\{g_j\}$. This is a contradiction to compactness of $\overline{\partial}^* N_{(0,q)}$. □

The idea of slicing a domain arises in other contexts too. For example, it can be used to analyze the asymptotic behavior of the Bergman kernel in terms of the distance to the boundary.

Let $d_{b\Omega}(z)$ be the function defined on Ω that measures the (minimal) distance from $z \in \Omega$ to $b\Omega$. The Bergman kernel function of Ω on the diagonal can be defined by

$$K_\Omega(z,z) = \sup\{|f(z)|^2 : f \in L^2_h(\Omega), \|f\|_{L^2_h(\Omega)} \leq 1\}$$

For more information on the Bergman kernel function we refer the reader to [38].

Proposition 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary near $p \in b\Omega$. If the Levi form has rank k at p then there exist a constant $C > 0$ and* *a neighborhood* U *of* p *such that*

$$K_\Omega(z,z) \geq \frac{C}{d_{b\Omega}^{k+2}(z)} \;\; for \;\; z \in U \cap \Omega.$$

Fu ([29]) proved the above proposition when Ω is a weakly pseudoconvex domain in $\mathbb{C}^n$ (i.e., $k = 0$). One can give a proof for this case by following the same proof except that at the last step, one should use the asymtotics of the Bergman kernel for strongly pseudoconvex domains [37], (rather than for planar domains, as in [29]). We next prove Corollary 2.

Proof of Corollary 2. Let V be a nonempty open subset of $b\Omega$ contained in the set of weakly pseudoconvex points. Define m to be the maximum rank of the Levi form on V and let $p \in V$ be a point where the Levi form has rank m (such a point exists, since the rank assumes only finitely many values on V). Then near p, the rank is at least m, hence equal to m. Therefore, $b\Omega$ is foliated, near p, by complex manifolds of dimension $n-1-m$ (see for example [28]). Theorem 8 (for $k = n-1-m$) implies that the $\overline{\partial}$-Neumann operator on Ω is not compact. This contradicts the assumption in Corollary 2. □

B. Proof of Theorem 9

The following simple lemma appears in [44] as Lemma 2.1. We give a proof of it for the convenience of the reader.

Lemma 6. *Let* Ω *be a smooth bounded pseudoconvex domain in* $\mathbb{C}^n, n \geq 3$ *and* ρ *be a defining function. Assume that the Levi form* $\mathcal{L}_\rho(.,.)$ *has at least* $n-2$ *positive eigenvalues at* $p \in b\Omega$*. Then there exist a neighborhood* U *of* p *and complex tangent vector fields* $X_1, \cdots, X_{n-1}$ *of type* $(1,0)$ *defined on* U *such that* $\mathcal{L}_\rho(X_j, X_k) = 0$ *if* $j \neq k$ *and* $j,k = 1,2,\cdots,n-1,$ *and* $\mathcal{L}_\rho(X_j) = 1$ *for* $j = 2,3,\cdots,n-1.$

Proof. Without loss of generality we may assume that $\frac{\partial r}{\partial z_n} = 1$ at $p = 0$, $|\nabla(r)| = 1$ on $b\Omega$, $\mathcal{L}_\rho(\frac{\partial}{\partial z_j}) > 0$ for $j = 2, 3, \cdots, n-1$, and $\mathcal{L}_\rho(\frac{\partial}{\partial z_1}) = 0$. Let

$$Y_j = \frac{\partial}{\partial z_j} - \left(\frac{\partial r}{\partial z_n}\right)^{-1} \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_n} \text{ for } j = 1, 2, \cdots, n-1.$$

Then the Levi form is positive definite on the space spanned by $\{Y_2, Y_3, \cdots, Y_{n-1}\}$ at p. By continuity there exists a neighborhood U of p such that the Levi form $\mathcal{L}_\rho(.,.)$ is positive definite on the space spanned by $\{Y_2, Y_3, \cdots, Y_{n-1}\}$ on U. Using Gram-Schmidt orthogonalization process with the inner product coming from the Levi form we get an orthonormal frame consisting of vector fields $\{X_2, X_3, \cdots, X_{n-1}\}$ on U such that $\mathcal{L}_\rho(X_j, X_k) = \delta_{jk}$ for $2 \leq j, k \leq n-1$ where δ_{jk} is the Kronecker delta function. Define $X_1 = Y_1 - L_\rho(Y_1, X_2)X_2 - \cdots - L_\rho(Y_1, X_{n-1})X_{n-1}$. Then $L_\rho(X_1, X_j) = 0$ for $j \neq 1$. □

Let J denote the complex structure map of (the tangent bundle of) $\mathbb{C}^n$ and $A^\theta = \cos(\theta)A + \sin(\theta)J(A)$. We will denote the flow generated by the (real) vector field A by $F^t_A(p)$. Namely, $F^t_A(p)$ is the point that is reached by flowing along A for a time t.

Proof of Theorem 9. Assume that r is a defining for Ω. If there is an analytic disc in the boundary one can choose K to be the boundary of the disc and p to be the center. Then $\widehat{K}$ is the disc, and $\widehat{K} \cap b\Omega \neq K \cap b\Omega$. For the other direction we may assume that there exists a weakly pseudoconvex point $p \in b\Omega$ and a compact set $K \subset \overline{\Omega}$ such that $p \in \widehat{K} \setminus K$. Using Lemma 6 we can diagonalize the Levi form locally around p. That is, we can choose an open neighborhood V of p and complex tangential smooth vector fields $X_1, X_2, ..., X_{n-1}$ of type $(1, 0)$ that diagonalize the Levi form on $V \cap b\Omega$. Furthermore, we can choose the vector fields such that the Levi form vanishes only in X_1 direction. Let's define $A = Re(X_1)$ and

$$M_s = \left\{ F^t_{A^\theta}(p) : 0 \le t < s, 0 \le \theta \le 2\pi \right\} \text{ for } 0 < s$$

Then M_s is a two dimensional smooth manifold.

Claim 1. *Let's choose $s_0 > 0$ such that $M_{s_0} \subset\subset V \cap b\Omega$. Then either M_s is a complex manifold for some $0 < s < s_0$ or there exist sequences $\{s_j\}_{j=1}^\infty$ and $\{\theta_j\}_{j=1}^\infty$ of positive numbers such that $\lim_{j\to\infty} s_j = 0$, $\theta_j \in [0, 2\pi]$ and the Levi form of r is positive at $F^{s_j}_{A^{\theta_j}}(p)$ in the direction $X_1 = A - iJ(A)$.*

Proof of the Claim 1. Assume that there exists $s > 0$ such that the Levi form of r is zero at $F^t_{A^\theta}(p)$ in the direction $X_1 = A - iJ(A)$ for $0 \le t \le s$ and $\theta \in [0, 2\pi]$. We will show that the real two-dimensional manifold M_s is a complex manifold. We follow [10] and indicate the necessary modifications. It suffices to show that the tangent space of M_s is spanned by A and $J(A)$ because an even dimensional smooth manifold is a complex manifold if and only if at each point the (real) tangent space is J invariant(see for example Proposition 1.3.19 in [1]). Let $f(t, \theta) = F^t_{A^\theta}(p)$. Using the chain rule we get the following ordinary differential equation in t with parameter θ:

$$\begin{cases} \left(\frac{\partial f}{\partial \theta}\right)' & = & \frac{\partial A^\theta}{\partial x}\frac{\partial f}{\partial \theta} + \frac{\partial A^\theta}{\partial \theta} \\ \frac{\partial f}{\partial \theta}(0) & = & 0 \end{cases} \tag{3.1}$$

where $\frac{\partial A^\theta}{\partial x}$ is the Jacobian of A^θ. One can check that (3.1) is equivalent to the following system:

$$\begin{cases} \left(\frac{\partial f}{\partial \theta}\right)' & = & D_{\frac{\partial f}{\partial \theta}} A^\theta + \frac{\partial A^\theta}{\partial \theta} \\ \frac{\partial f}{\partial \theta}(0) & = & 0 \end{cases} \tag{3.2}$$

where $D_{\frac{\partial f}{\partial \theta}} A^\theta$ is the Lie derivative of A^θ by $\frac{\partial f}{\partial \theta}$. We will find a solution to (3.2) in the form $a(t)A + b(t)J(A)$. Then $\frac{\partial f}{\partial \theta}$, the unique solution of (3.2), is of this form.

Therefore, both $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial \theta}$ are linear combinations of A and $J(A)$. This will complete the proof of Claim 1. If we substitute $a(t)A + b(t)J(A)$ for $\frac{\partial f}{\partial \theta}$ into (3.2) use the fact that $D_V W - D_W V = [V, W]$ we get:

$$\begin{aligned} 0 &= a'(t)A + (a(t)\sin(\theta) - b(t)\cos(\theta))[J(A), A] \\ &\quad + b'(t)J(A) + \sin(\theta)A - \cos(\theta)J(A) \\ 0 &= a(0) = b(0) \end{aligned} \tag{3.3}$$

Suppose that we can show that $[J(A), A]$ is a linear combination of A and $J(A)$ at points of M_s with smooth coefficient functions. Then after collecting terms containing A and $J(A)$, respectively, (3.3) becomes an initial value problem for $(a(t), b(t))$ that has a (unique) solution, and we are done. Cartan's formula shows that if X is complex tangential vector field of type $(1, 0)$ then $\mathcal{L}_\rho(X)$ gives the component of $[X, \overline{X}]$ that is not complex tangential. Therefore, $[J(A), A]$ is complex tangential at points of M_s because $A - iJ(A)$ is a Levi null direction. In $\mathbb{C}^2$, this means that it is a (real) linear combination of A and $J(A)$. The argument so far comes entirely from [10]. In higher dimensions, some additional work is needed.

Since $X_1 = A - iJ(A)$ the commutator we are interested in is $\left[X_1, \overline{X_1}\right] = 2i[A, J(A)]$. If X_1 is a Levi null field in a *full* neighborhood of p, then $[X_1, \overline{X_1}]$ would likewise be ([28]). However, we only know that $\mathcal{L}_\rho(X_1, \overline{X_1}) = 0$ on M_s. What we can assert is that $\left[X_1, \overline{X_1}\right] = Y - \overline{Y} + \varphi(L_n - \overline{L_n})$, where Y is a smooth complex tangential field of type $(1, 0)$, L_n is the complex normal to the boundary, and φ is a smooth, nonnegative function that vanishes on M_s. The nonnegativity of φ is a consequence of the pseudoconvexity of Ω. What we need is that on M_s, Y is a multiple of X_1. The Jacobi identity for X_1, $\overline{X_1}$, and X_k gives

$$\left[Y - \overline{Y} + \varphi(L_n - \overline{L_n}), X_k\right] + \left[[\overline{X}_1, X_k], X_1\right] + \left[[X_k, X_1], \overline{X}_1\right] = 0, \tag{3.4}$$

for $k = 1, \cdots, n-1$. We have replaced $\left[X_1, \overline{X_1}\right]$ in the first term by $Y - \overline{Y} + \varphi(L_n - \overline{L_n})$. We will use the the following fact : let X and Y be two complex tangential vector fields of type $(1,0)$. Then $[X, \overline{Y}]$ is complex tangential at q if X or Y is in Levi null direction at q. The second and third commutators in (3.4) are complex tangential at points of M_s if $k \geq 2$. To see this, note that X_1 is in the nullspace of the Levi form at points of M_s and that both the commutators $[X_k, X_1]$ and $\left[\overline{X_1}, X_k\right]$ are complex tangential in a *full* neighborhood of p in $b\Omega$ (the latter because $X_1, \cdots, X_{n-1}$ diagonalize the Levi form near p). Consequently, the first commutator is complex tangential at points of M_s as well. Since φ is nonnegative in a neighborhood of M_s in $b\Omega$ and vanishes on M_s its first order tangential derivatives vanish at points of M_s. Therefore, $\left[\varphi(L_n - \overline{L_n}), X_k\right]$ is zero, and $\left[Y - \overline{Y}, X_k\right]$ and $\left[\overline{Y}, X_k\right]$ are complex tangential at points of M_s. It follows that at points of M_s, Y is in the nullspace of the Levi form and thus is a multiple of X_1. This completes the proof of Claim 1. We remark that a similar use of the Jacobi identity occurs in [28].

Once Lemma 6 and Claim 1 are in hand, the proof of Theorem 9 can be completed as in [10], with only small modifications. Assume that there is a compact subset K of $\overline{\Omega}$ and a boundary point $p \in \widehat{K} \setminus K$ and there is no analytic disc in the boundary through p. Then p is a weakly pseudoconvex point and by assumption zero is an eigenvalue (with multiplicity one) of the Levi form at p. By Lemma 6 take V to be a neighborhood of p small enough so that $V \cap K = \emptyset$ and so that there are complex tangential fields $X_1, \cdots, X_{n-1}$ of type $(1,0)$ which diagonalize the Levi form in a neighborhood of $\overline{b\Omega \cap V}$, with $\mathcal{L}_r(X_1, \overline{X_1})(p) = 0$. By Claim 1, there exist θ and $t_0 > 0$ such that $F^{t_0}_{A^\theta}(p) \in V$ and $F^{t_0}_{A^\theta}(p)$ is a strongly pseudoconvex point, where $A = X_1 + \overline{X_1}$. Near p, choose a boundary coordinate system $(t_1, t_2, \cdots, t_{2n-1}, r)$ such that $A^\theta = \frac{\partial}{\partial t_1}$. Without loss of generality we may assume that V is contained in this coordinate patch. We will follow [10] to show that the integral curve of A^θ

from $t_1 = 0$ to $t_1 = t_0$, and hence p, can be separated by the level set of a strictly plurisubharmonic function from any compact subset of $\overline{\Omega} \setminus V$. This contradiction (to $p \in \widehat{K} \setminus K$ for some compact K) will complete the proof. As in [10], consider the auxiliary function

$$g(t_1, t_2, \cdots, t_{2n-1}, r) = \frac{t_1 + \frac{1}{m}}{1 + m^2(t_2^2 + \cdots + t_{2n-1}^2)}$$

and the sets $S_c = \{(t_1, t_2, \cdots, t_{2n-1}, 0) : g(t_1, t_2, \cdots, t_{2n-1}, 0) = c, 0 \leq t_1 \leq t_0\}$. Choose m sufficiently large such that $S_c \subset\subset V \cap b\Omega$ for all c with $1/m \leq c \leq t_0 + 1/m$. By shrinking V if necessary we may assume that any point in the set $\{z \in b\Omega \cap V : t_1(z) \geq t_0\}$ is strongly pseudoconvex. In [10], g is modified, but the modification is specified to be of order r^2. Here, it is important to also have a term that is of order r. Specifically, set $h = g + \mu r + \nu r^2$ for μ, ν positive numbers to be determined. Now we want to calculate the Levi form of the hypersurface $\{z \in V : h(z) = \zeta\}$ on $b\Omega$.

$$\begin{aligned} \mathcal{L}_h(M) &= \mathcal{L}_g(M) + 2\nu|\alpha_n|^2 + \mu \sum_{j=1}^{n-1} |\alpha_j|^2 \mathcal{L}_r(X_j) \\ &+ 2\mu \mathrm{Re}\left(\sum_{j=1}^{n-1} \alpha_n \overline{\alpha_j} \mathcal{L}_r(X_n, X_j)\right) + \mu|\alpha_n|^2 \mathcal{L}_r(X_n) \end{aligned}$$

where $M = \sum_{j=1}^n \alpha_j X_j$ and $M(h) = \sum_{j=1}^n \alpha_j X_j(g) + \mu\alpha_n = 0, \sum_{j=1}^n |\alpha_j|^2 = 1$. Using the inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ for $\varepsilon > 0$ and $M(h) = 0$ we get the following: There exists $C > 0$ such that $|\mathcal{L}_g(M)|, |\mathcal{L}_r(X_j, X_k)| < C$ for $j, k = 1, \cdots, n$. Hence we have

$$\begin{aligned} \mathcal{L}_h(M) \geq& \mathcal{L}_g(M) + (2\nu - \mu C)|\alpha_n|^2 + \mu \sum_{j=1}^{n-1} (|\alpha_j|^2 \mathcal{L}_r(X_j) - 2C|\alpha_n \overline{\alpha_j}|) \\ \geq& -C + \left(2\nu - \mu C - \frac{(n-2)\mu C^2}{\varepsilon}\right)|\alpha_n|^2 + \mu \sum_{j=2}^{n-1} |\alpha_j|^2 (\mathcal{L}_r(X_j) - \varepsilon) \\ & -2\mu C|\alpha_n \overline{\alpha_1}| \end{aligned}$$

Using $M(h) = \sum_{j=1}^{n} \alpha_j X_j(g) + \mu\alpha_n = 0$ we get

$$\begin{aligned}
\mathcal{L}_h(M) \geq & -C + \left(2\nu - \mu C - \frac{(n-2)\mu C^2}{\varepsilon}\right)|\alpha_n|^2 + \mu \sum_{j=2}^{n-1} |\alpha_j|^2 (\mathcal{L}_r(X_j) - \varepsilon) \\
& - \frac{2\mu C}{|X_1(g)|} \sum_{j=2}^{n-1} |\alpha_n|.|\alpha_j|.|X_j(g)| - \left(\frac{2\mu^2 C}{|X_1(g)|} + \frac{2\mu C|X_n(g)|}{|X_1(g)|}\right)|\alpha_n|^2 \\
\geq & -C + 2\nu|\alpha_n|^2 + \mu \sum_{j=2}^{n-1} \left(\mathcal{L}_r(X_j) - \varepsilon - \frac{\varepsilon C|X_j(g)|^2}{|X_1(g)|}\right)|\alpha_j|^2 \\
& - \left(\mu C + \frac{(n-2)\mu C^2}{\varepsilon} + \frac{2\mu^2 C}{|X_1(g)|} + \frac{2\mu C|X_n(g)|}{|X_1(g)|} + \frac{(n-2)\mu C}{\varepsilon|X_1(g)|}\right)|\alpha_n|^2
\end{aligned}$$

Notice that $|X_1(g)| > 0$ on U and $\mathcal{L}_r(X_j) = 1$ for $j = 2, \cdots, n-1$. So by choosing sufficiently small $\varepsilon > 0$ and sufficiently large μ, ν we get

$$\mathcal{L}_h(M) \geq -C + \frac{\mu}{2} \sum_{j=2}^{n} |\alpha_j|^2$$

There exists $0 < \delta < 1$ such that $|\alpha_1| < 1 - \delta$. Hence If we choose μ sufficiently large we get a smooth function whose level surfaces are strongly pseudoconvex on $V \cap b\Omega$. Let ψ_s be a smooth real valued function such that $\psi_s(t_1) = 1$ for $t_1 \leq t_0, \psi_s(t_1) = 0$ for $t_1 \geq s$. Choose $s > 0$ such that

$$\{(t_1, t_2, \cdots, t_{2n-1}, 0) : \psi_s(t_1) g(t_1, t_2, \cdots, t_{2n-1}, 0) \geq 1/m\} \subset\subset V \cap b\Omega.$$

Let $\lambda(z) = e^{\tau h(z)}$ where τ is a real number. For sufficiently large τ the function $\lambda(z)$ is strongly plurisubharmonic in a neighborhood of $V \cap b\Omega$. The function $\psi_s(t_1(z))\lambda(z)$ is strongly plurisubharmonic at all points $z \in V \cap b\Omega$ where $t_1(z) \leq t_0$.

From here on, the argument is exactly as in [10], p. 54-55; we only sketch it. To deal with the direction transverse to the boundary, Catlin applies his construction in [10], Theorem 3.1.6 (see also [11], Proposition 3.1.6). In our situation, this construction yields a strictly plurisubharmonic function on a neighborhood V_1 of p

with $\{F^t_{A^\theta}(p) : 0 \leq t \leq t_0\} \subset\subset V_1 \subset\subset V$, whose superlevel set determined by p is a compact subset of V_1. Composition with a suitable convex increasing function finally results in a plurisubharmonic function defined on all of $\overline{\Omega}$ (by extension by 0) that separates p from any compact subset of $\overline{\Omega} \setminus V$. This completes the proof of (the nontrivial direction of) Theorem 9. $\square$

CHAPTER IV

STEIN NEIGHBORHOOD BASES

In this chapter we state and prove our main result about the existence of a Stein neighborhood basis for the closure. We finish the chapter with some applications.

We first introduce some notation. We identify $\mathbb{C}^n$ with the $(1,0)$ tangent bundle of $\mathbb{C}^n$. Namely, $(a_1,\ldots,a_n)$ is identified with $\sum_{j=1}^n a_j \frac{\partial}{\partial z_j}$. We denote $\sum_{j=1}^n |w_j|^2$ by $\|W\|^2$, where $W = (w_1,\ldots,w_n) \in \mathbb{C}^n$. Let $\vec{n}(z) \in \mathbb{R}^{2n}$ ($\cong \mathbb{C}^n$ by the standard identification) be the unit outward normal vector of $b\Omega$ at z. We denote the directional derivative in the direction $\vec{n}(z)$ at the point z by $\frac{d}{d\vec{n}(z)}$. We define

$$\frac{d}{dr(z)} = \frac{1}{\|\nabla r(z)\|}\frac{d}{d\vec{n}(z)}.$$

Here, r is a defining function for the domain. Therefore, the gradient of r, $\nabla r(z)$, does not vanish on the boundary. We define

$$\Gamma_\Omega = \{(z,W) \in b\Omega \times \mathbb{C}^n : W(r)(z) = 0, \|W\| = 1, \mathcal{L}_r(z;W) = 0\}.$$

For a vector (of type $(1,0)$) $A \in \mathbb{C}^n$ and $z \in b\Omega$ we will denote

$$\begin{aligned} C_r(z;A) &= \frac{d\mathcal{L}_r(z;A)}{dr(z)}, \\ D_r(z;A) &= |A(\ln\|\nabla r\|)(z)|, \\ N_r &= \sum_{j=1}^n \frac{\partial r}{\partial \bar{z}_j}\frac{\partial}{\partial z_j}, \quad \text{and} \\ E_r(z;A) &= \frac{4}{\|\nabla r(z)\|^2}|\mathcal{L}_r(z;A,\overline{N}_r)|, \end{aligned}$$

where r is a defining function for Ω. We note that C_r, E_r and D_r are not independent of the defining function r. However, $D_r \equiv 0$ for a defining function r with the property that $\|\nabla r(z)\| = 1$ for $z \in b\Omega$. Now we can state the main theorem of the chapter.

Theorem 10. *Let Ω be a C^3-smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$. Assume that there exist $\varepsilon > 0, h \in C^2(\overline{\Omega})$ and a defining function r of Ω with the property that:*

i) $|W(h)(z)|^2 \leq \mathcal{L}_h(z;W)$

ii) $\mathcal{L}_h(z;W) > 4(1+\varepsilon)\left(2E_r(z;W)D_r(z;W) + \frac{(E_r(z;W))^2}{\varepsilon} - C_r(z;W)\right)$

for $(z,W) \in \Gamma_\Omega$. Then $\overline{\Omega}$ has a Stein neighborhood basis.

The conditions we have in Theorem 10 may be viewed as a quantified version of property $(\widetilde{P})$ in that the eigenvalues of the complex Hessians are only required to have a lower bound depending on the defining function of the domain. In the literature, the inequality i) in Theorem 10 is referred to as having self-bounded complex gradient.

We will construct a defining function for the domain such that the level sets that are sufficiently close to the closure will be boundaries of pseudoconvex domains.

Proof of Theorem 10. Assume that Ω satisfies the conditions in Theorem 10. Then Ω has a defining function, r, and there exists a function $h \in C^2(\overline{\Omega})$ and $\varepsilon > 0$ such that i) and ii) in Theorem 10 are satisfied. We extend h to $\mathbb{C}^n$ as a C^2 function and call the extension h. We scale h, if necessary, so that there is a neighborhood of $\overline{\Omega}$ on which the conditions of the theorem are still satisfied. We define $\rho(z) = r(z)e^{\lambda h(z)}$, where $2\lambda(1+\varepsilon) = 1$. We will build a Stein neighborhood basis as follows: we will show that there exists a neighborhood V of $\overline{\Omega}$ such that $\nabla\rho$ is nonvanishing on $V \setminus \Omega$, and the Levi form of ρ is nonnegative on vectors complex tangential to the level sets of ρ in $V \setminus \Omega$. Since Ω is bounded and $\|\nabla\rho\|$ is continuous and strictly positive on $b\Omega$, the first part of the above argument follows immediately. Since Ω is bounded it suffices to argue near a boundary point q.

Using translation and rotation we can move any point, $q \in b\Omega$, to the origin such that the y_n-axis is the outward normal direction at 0. There exists a neighborhood

$\widetilde{U}$ of 0 on which $\frac{\partial\rho}{\partial z_n}$ does not vanish. If $W = (w_1, \dots, w_n)$ is a complex tangential vector to the level set of ρ at $z \in \widetilde{U} \setminus \Omega$ (i.e. $W(\rho)(z) = 0$) then

$$w_n = -\left(\left(\frac{\partial\rho(z)}{\partial z_n}\right)^{-1}\right)\sum_{j=1}^{n-1}\frac{\partial\rho(z)}{\partial z_j}w_j. \tag{4.1}$$

We introduce an auxiliary real valued function f to show the level sets of ρ are pseudoconvex locally. We define f on $(\widetilde{U} \setminus \Omega) \times \mathbb{C}^{n-1}$ as follows:

$$f(z, W') = e^{-\lambda h(z)}\mathcal{L}_\rho(z; W),$$

where $W = (w_1, \dots, w_n)$ and $W' = (w_1, \dots, w_{n-1})$, with w_n given by (4.1).

Claim 2. *To satisfy the local property above it is enough to have*

$$\frac{df(z, W')}{d\vec{n}(p)}\bigg|_{z=p} > 0 \tag{4.2}$$

for any $(p, W) \in ((\widetilde{U} \cap b\Omega) \times \mathbb{C}^n) \cap \Gamma_\Omega$.

Proof of Claim 2. The localization can be phrased in terms of f as follows: for every $p \in \widetilde{U} \cap b\Omega$ there exists a neighborhood, U_p, of p such that $f(z, W') \geq 0$ for $z \in U_p \setminus \Omega$ and $W' \in \mathbb{C}^{n-1}$. Due to the continuity of the Levi form this is always true for a strongly pseudoconvex point $p \in \widetilde{U} \cap b\Omega$. So we only need to consider weakly pseudoconvex points in $p \in \widetilde{U} \cap b\Omega$. We choose an open neighborhood U of 0 such that $U \subset\subset \widetilde{U}$. Let $S = \{W' \in \mathbb{C}^{n-1} : \|W'\| = 1\}$. Notice that $(\overline{U \cap b\Omega}) \times S$ is compact; and it is enough to show that $f(z, W') \geq 0$ for $z \in U \setminus \Omega$ and $W' \in S$. Assume that the conditions in Claim 2 are satisfied for $(p, W_p) \in ((\widetilde{U} \cap b\Omega) \times \mathbb{C}^n) \cap \Gamma_\Omega$. Then there exist a neighborhood U_p of p and an $\varepsilon_p > 0$ such that $f(z, W') \geq 0$ for $z \in U_p \setminus \Omega$ and $\|W'_p - W'\| < \varepsilon_p$. This follows because f is nonnegative on $(\widetilde{U} \cap b\Omega) \times \mathbb{C}^{n-1}$, the variable W' is independent of the variable of differentiation z and the left hand side of (4.2) is continuous. Then we can cover the compact set $(\overline{U \cap b\Omega}) \times S$ by finitely many sets open in $\mathbb{C}^n \times \mathbb{C}^{n-1}$. We now choose an open set $U_1 \supset\supset \overline{U \cap b\Omega}$ such that $f(z, W') \geq 0$ for $(z, W') \in U_1 \times S$. $\square$

Let $z \in \widetilde{U} \setminus \Omega$, and $W \in \mathbb{C}^n$, be a complex tangential vector to the level set of ρ at z. Namely,

$$W(\rho)(z) = e^{\lambda h(z)}\Big(W(r)(z) + \lambda r(z) W(h)(z)\Big) = 0. \tag{4.3}$$

Now we will calculate the Levi form of ρ at z in the direction W. So first we differentiate ρ with respect to z_j to get

$$\frac{\partial \rho}{\partial z_j}(z) = e^{\lambda h(z)} \frac{\partial r}{\partial z_j}(z) + \lambda r(z) e^{\lambda h(z)} \frac{\partial h}{\partial z_j}(z) \tag{4.4}$$

and if we differentiate (4.4) with respect to $\bar{z}_k$ we get

$$\begin{aligned} \frac{\partial^2 \rho}{\partial \bar{z}_k \partial z_j}(z) &= e^{\lambda h(z)} \frac{\partial^2 r}{\partial \bar{z}_k \partial z_j}(z) + \lambda e^{\lambda h(z)} \frac{\partial h}{\partial \bar{z}_k}(z) \frac{\partial r}{\partial z_j}(z) + \lambda e^{\lambda h(z)} \frac{\partial r}{\partial \bar{z}_k}(z) \frac{\partial h}{\partial z_j}(z) \\ &+ \lambda^2 r(z) e^{\lambda h(z)} \frac{\partial h}{\partial \bar{z}_k}(z) \frac{\partial h}{\partial z_j}(z) + \lambda r(z) e^{\lambda h(z)} \frac{\partial^2 h}{\partial \bar{z}_k \partial z_j}(z) \end{aligned}$$

Using (4.3) in the last equality we get

$$\mathcal{L}_\rho(z; W) = e^{\lambda h(z)}\Big(\lambda r(z) \mathcal{L}_h(z; W) + \mathcal{L}_r(z; W) - \lambda^2 r(z) |W(h)(z)|^2\Big).$$

Therefore

$$f(z, W') = \lambda r(z) \mathcal{L}_h(z; W) + \mathcal{L}_r(z; W) - \lambda^2 r(z) |W(h)(z)|^2 \tag{4.5}$$

for $z \in \widetilde{U} \setminus \Omega$ and $W = (W', w_n)$ where w_n is defined by (4.1). We will examine the vector field $W = (W', w_n)$ with $W' \in \mathbb{C}^{n-1}$ fixed and w_n given by (4.1). Recall that $(p, W(p)) \in \Gamma_\Omega$. Differentiate $f(z, W')$ with respect to $r(z)$ at $p \in b\Omega$. Using (4.5) we get:

$$\lambda \mathcal{L}_h(p; W) + C_r(p; W) + 2Re\Big(\mathcal{L}_r(p; W, d\overline{W}/dr)\Big) - \lambda^2 |W(h)(p)|^2. \tag{4.6}$$

Since W is a weakly pseudoconvex direction we only need to compute the complex normal component of $\frac{dW}{dr(p)}$ at p to estimate the third term of the above expression. Hence, we need to compute $\frac{dW}{dr(p)}(r)(p)$ which represents the following: first we differentiate W by $\frac{d}{dr(p)}$ at p then apply the result to r and evaluate at p. If we differentiate the left hand side of

$$W(r)(z) + \lambda r(z)W(h)(z) = 0$$

we get:

$$\begin{aligned}
\frac{d\{W(r)(z) + \lambda r(z)W(h)(z)\}}{dr(p)}\Big|_{z=p} &= \frac{dW}{dr(p)}(r)(p) + \sum_{j=1}^{n} w_j \frac{\nabla r(p)}{\|\nabla r(p)\|^2} \cdot \nabla\left(\frac{\partial r}{\partial z_j}\right)(p) \\
&\quad + \lambda W(h)(p) \\
&= \frac{dW}{dr(p)}(r)(p) + \frac{1}{2}W\left(\ln \|\nabla r\|^2\right)(p) + \lambda W(h)(p) \\
&= \frac{dW}{dr(p)}(r)(p) + W(\ln \|\nabla r\|)(p) + \lambda W(h)(p).
\end{aligned}$$

We note that the second term in the first equality consists of a summation of dot product of vectors. Thus we have:

$$\frac{dW}{dr(p)}(r)(p) + W(\ln \|\nabla r\|)(p) + \lambda W(h)(p) = 0. \tag{4.7}$$

If $Y = \tau N_r + \xi$ where ξ is the complex tangential component of Y then

$$\tau = \frac{Y(r)(p)}{N_r(r)(p)} = \frac{4Y(r)(p)}{\|\nabla r(p)\|^2}.$$

Then using the above observation with (4.7) we conclude that the absolute value of the third term in (4.6) is bounded from above by

$$2E_r(p;W)\Big(D_r(p;W) + \lambda |W(h)(p)|\Big)$$

Hence we want

$$\lambda\mathcal{L}_h(p;W) - \lambda^2|W(h)(p)|^2 - 2E_r(p;W)\Big(\lambda|W(h)(p)| + D_r(z;W)\Big) + C_r(p;W) \quad (4.8)$$

to be positive for $(p,W) \in \Gamma_\Omega$. Using the following inequalities

$$2\lambda E_r(p;W)|W(h)(p)| \leq \varepsilon\lambda^2|W(h)(p)|^2 + \frac{(E_r(p;W))^2}{\varepsilon},$$

$$2\lambda(1+\varepsilon)|W(h)(p)|^2 \leq \mathcal{L}_h(p;W),$$

and

$$\lambda\mathcal{L}_h(p;W) > 2\Big(2E_r(z;W)D_r(z;W) + \frac{(E_r(z;W))^2}{\varepsilon} - C_r(p;W)\Big)$$

with $2\lambda(1+\varepsilon) = 1$ one obtains that (4.8) is positive for $(p,W) \in \Gamma_\Omega$. □

Remark 1. We cannot expect to obtain a plurisubharmonic defining function because even a bounded pseudoconvex domain with real analytic boundary need not have a plurisubharmonic defining function (see for example [25]). Regardless, when we have a function h satisfying the conditions of Theorem 10 we can produce a defining function whose level sets give a Stein neighborhood basis. In that case we get a uniformly H-convex domain, which also has the Mergelyan approximation property [18]. Indeed, all examples of smooth domains whose closures are known to have a Stein neighborhood basis are uniformly H-convex. It would be interesting to know if it is possible for a closure of a smooth domain to have a Stein neighborhood basis without being uniformly H-convex or without satisfying the Mergelyan approximation property.

We denote the diameter of a set S by τ_S. For a defining function r we define

$$\begin{aligned} C_r &= \inf\{C_r(z;W) : (z;W) \in \Gamma_\Omega\}, \\ D_r &= \sup\{D_r(z;W) : (z;W) \in \Gamma_\Omega\}, \end{aligned}$$

$$E_r = \sup\{E_r(z;W) : (z;W) \in \Gamma_\Omega\}$$

Corollary 3. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$, with defining function r and K be the set of weakly pseudoconvex points in $b\Omega$. Assume that one of the following properties is satisfied:*

$$E_r = 0 \quad \text{and} \quad C_r > -\frac{1}{4\tau_K^2}, \tag{4.9}$$

$$E_r > 0 \quad \text{and} \quad 2E_rD_r - C_r < 0, \quad \text{or} \tag{4.10}$$

$$E_r > 0,\ 2E_rD_r - C_r \geq 0 \quad \text{and} \quad \left(E_r + \sqrt{2E_rD_r - C_r}\right)^2 < \frac{1}{4\tau_K^2}. \tag{4.11}$$

Then $\overline{\Omega}$ has a Stein neighborhood basis.

Proof. Without loss of generality we may assume that K is contained in a ball centered at the origin with radius of τ_K. Let $h(z) = \frac{1}{\tau_K^2}\|z\|^2$. Then

$$|W(h)(z)|^2 \leq \frac{1}{\tau_K^4}\left|\sum_{j=1}^n \bar{z}_j w_j\right|^2 \leq \frac{\|W\|^2}{\tau_K^2} = \mathcal{L}_h(z;W)$$

for $(z,W) \in \overline{\Omega} \times \mathbb{C}^n$. So the first condition in Theorem 10 is satisfied. Additionally,

$$\sup_{(z;W)\in\Gamma_\Omega}\left\{2E_r(z;W)D_r(z;W) + \frac{(E_r(z;W))^2}{\varepsilon} - C_r(z;W)\right\} \leq 2E_rD_r + \frac{(E_r)^2}{\varepsilon} - C_r.$$

We will show that if one of (4.9), (4.10), or (4.11) is satisfied, then there exists $\varepsilon > 0$ such that

$$(1+\varepsilon)\left(2E_rD_r + \frac{(E_r)^2}{\varepsilon} - C_r\right) < \frac{1}{4\tau_K^2} \tag{4.12}$$

That is, the second condition in Theorem 10 is satisfied. It is easy to see that (4.12) is satisfied for small enough $\varepsilon > 0$ if (4.9) holds. If (4.10) is satisfied then one can choose $\varepsilon > 0$ large enough so that the left hand side of (4.12) is negative. If (4.11) is satisfied with $2E_rD_r - C_r = 0$, one can choose $\varepsilon > 0$ large enough so that $(1+1/\varepsilon)(E_r)^2 < \frac{1}{4\tau_K^2}$. On the other hand, if (4.11) is satisfied with $2E_rD_r - C_r > 0$, the minimum of the left hand side of (4.12) is $\left(E_r + \sqrt{2E_rD_r - C_r}\right)^2$ and it is attained when

$$\varepsilon = \frac{E_r}{\sqrt{2E_rD_r - C_r}} > 0.$$

Thus one can choose $\varepsilon > 0$ so that (4.12) is satisfied. $\square$

Remark 2. We would like to note that when the defining function r is plurisubharmonic at z the Levi form (complex Hessian) is a positive semi-definite, self-adjoint matrix. Then we can use Cauchy-Schwarz inequality to get $E_r = 0$ when the domain has a defining function r that is plurisubharmonic on the boundary. This class of domains has recently been studied in connection with global regularity of the $\overline{\partial}$-Neumann problem and the Bergman projection. See for example [7, 9, 36].

A. Applications

Let $\frac{4}{\|\nabla\rho\|^2}N_\rho = n + iT$ where T is the "bad" (real tangential) direction, n is the unit outward real normal vector, and ρ is a defining function. For $X = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j}$ and a vector field Y we define: $D_Y X = \sum_{j=1}^n Y(\xi_j)\frac{\partial}{\partial z_j}$ and $D_Y\overline{X} = \sum_{j=1}^n Y(\bar{\xi}_j)\frac{\partial}{\partial \bar{z}_j}$.

Theorem 11. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$. Assume that there exists a defining function ρ that is plurisubharmonic on K and $K = \overline{K}_1$ such that for any $p \in K_1$ there exist two sequences $\{x_j\}$ and $\{y_j\}$ of K that converge to p and two smooth one real dimensional curves γ and $\tilde{\gamma}$ (in $\mathbb{C}^n$) with linearly independent complex tangential (real) vectors at p such that $x_j \in \gamma, y_j \in \tilde{\gamma}$. Then $\overline{\Omega}$ has a Stein neighborhood basis.*

Proof. Since Ω is pseudoconvex we have $T(\mathcal{L}_\rho(z;X)) = 0$ for a complex tangential vector field X of type $(1,0)$ and $z \in K$. Therefore

$$\frac{\|\nabla\rho\|^2}{4}C_\rho(z,X) = N_\rho(\mathcal{L}_\rho(z;X)) = \overline{N}_\rho(\mathcal{L}_\rho(z;X)) = n_\rho(\mathcal{L}_\rho(z;X)) \in \mathbb{R}$$

for $z \in K$. For a vector field W of type $(1,0)$ one can easily check that:

$$\begin{aligned}\overline{X}(\mathcal{L}_\rho(z;X,\overline{W})) &= \overline{W}(\mathcal{L}_\rho(z;X,\overline{X})) + \mathcal{L}_\rho(z;D_{\overline{X}}X,\overline{W}) + \mathcal{L}_\rho(z;X,D_{\overline{X}}\overline{W}) \\ &\quad -\mathcal{L}_\rho(z;D_{\overline{W}}X,\overline{X}) - \mathcal{L}_\rho(z;X,D_{\overline{W}}\overline{X}).\end{aligned}$$

Also $D_{\overline{X}}X(\rho)(z) = \overline{X}(X(\rho))(z) - \mathcal{L}_\rho(z;X)$. Since X is complex tangential ($X(\rho) = 0$ on $b\Omega$) we have $D_{\overline{X}}X(\rho)(z) = -\mathcal{L}_\rho(z;X)$. Let's take $W_\rho = \frac{4}{\|\nabla\rho\|^2}N_\rho$. Therefore $D_{\overline{X}}X$ is complex tangential at weakly pseudoconvex points and $\mathcal{L}_\rho(z;X,\overline{N}_\rho) = 0$ implies that

$$C_\rho(z,X) = \overline{X}(\mathcal{L}_\rho(z;X,\overline{W}_\rho)) \in \mathbb{R} \tag{4.13}$$

for $z \in K$. We will show that $C_\rho(z,X) > \frac{-1}{4\tau_K^2}$ for $z \in K$. By Corollary 3 that is sufficient to get a Stein neighborhood basis (recall that $E_\rho = 0$, since ρ is plurisubharmonic on K_1). Let $X = A - iJ(A), \tilde{X} = B - iJ(B)$ for $A = \gamma'$ and $B = \tilde{\gamma}'$ and $\mathcal{L}_\rho(z;X,\overline{W}_\rho) = \alpha + i\beta$. By assumption $\alpha = \beta = 0$ on $\{x_j\} \cup \{y_j\}$. Then by smoothness of ρ and the mean value theorem applied to α and β on γ and $\tilde{\gamma}$ respectively we get $A(\alpha) = A(\beta) = B(\alpha) = B(\beta) = 0$ at p. Since A and B are linearly independent and complex tangential at p we can write $J(A)$ as a linear combination of A and B. Therefore using (4.13) we conclude that $C_\rho(p,X) = A(\alpha) + J(A)(\beta) = 0$ on K_2. □

If a domain has a plurisubharmonic defining function and the set of weakly pseudoconvex points is the closure of a set foliated by complex manifolds then it satisfies the conditions in the above theorem. And hence has a Stein neighborhood basis. Therefore, we have the following corollary.

Corollary 4. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ and K be a subset of $b\Omega$ such that for any $p \in K$ there exists an analytic disc $D_p \subset b\Omega$ that passes through p. Assume that ρ is a defining function of Ω that is plurisubharmonic on $b\Omega$ and every point in $b\Omega \setminus \overline{K}$ is strongly pseudoconvex. Then $\overline{\Omega}$ has a Stein neighborhood* *basis.*

Remark 3. When the set of weakly pseudoconvex points $\overline{K}$ is very thin (for example if $K = \emptyset$) or very thick (if K is an open set in the boundary foliated by complex manifolds) existence of a defining function that is plurisubharmonic on the boundary is sufficient for the existence of a Stein neighborhood basis for the closure. Therefore we expect that existence of a Stein neighborhod basis should be independent of the size and geometry of the set of weakly pseudoconvex points under the assumption that the defining function is plurisubharmonic on the boundary.

Remark 4. Existence of a Stein neighborhood basis for the closure when K_1 is foliated by complex manifolds has been studied by many authors including Bedford and Fornæss([2]), Straube and Sucheston([58]), and Forstneric and Laurent-Thiebaut([27]). They all use the method of "transversal holomorphic vector field" developed by Fornæss and Nagel in [26] to get a Stein neighborhood basis for the closure (See Theorem 6). In all those results to guarantee the existence of the vector field some restrictions on the "geometry" of the foliation are required. However, in our case we have no condition on the foliation whatsoever. In fact, Theorem 11 does not require any foliation at all. In case K_1 is foliated by complex manifolds but the domain does not have a defining function that is plurisubharmonic on the boundary there may not be any Stein neighborhood basis. For example, using a similar construction as in [21], Bedford and Fornæss([2, page 21]) constructed a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ with the following properties: weakly pseudoconvex points constitute a Levi flat hypersurface in $\mathbb{C}^2$ foliated by complex manifolds, and the closure of the domain does not have a Stein neighborhood basis.

Corollary 5. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2$, with a plurisubharmonic (on $\overline{\Omega}$) defining function ρ. Assume that for any weakly pseudo-*

convex direction on the boundary (z, W) *(i.e.* $(z, W) \in \Gamma_{\Omega}$*) there exists a sequence* $\{(z_j, W_j)\}$ *in* $\Omega \times \mathbb{C}^n$ *that converges to* (z, W) *such that* $\mathcal{L}_{\rho}(z_j; W_j) = 0$ *for all* j*. Then* $\overline{\Omega}$ *has a Stein neighborhood basis.*

Proof. Using plurisubharmonicity of ρ on $b\Omega$ one can check that $E_{\rho} = 0$. To satisfy the first condition in Corollary 3 we only need to show that $C_{\rho} = 0$. Let $f(z, W) = \mathcal{L}_{\rho}(z; W)$. Then f is a nonnegative real valued function on $\overline{\Omega} \times \mathbb{C}^n$ and hence any first derivative of it vanishes on (z_j, W_j) for all j. Since f is smooth on $\overline{\Omega} \times \mathbb{C}^n$ we conclude that $C_{\rho} = 0$. □

The following proposition is useful in producing examples of domains whose closures have Stein neighborhod bases. In theory, it could happen that the set of weakly pseudoconvex points is the finite union of disjoint compact sets such that each is "good" for the purpose of having a pseudoconvex surface sufficiently close to the closure. In that case we show that the closure of the domain still has a Stein neighborhood basis. The idea is to "glue" the defining functions away from the set of weakly pseudoconvex points.

Proposition 2. *Let* Ω *be a smooth bounded pseudoconvex domain in* $\mathbb{C}^n, n \geq 2$*. Assume that the set of weakly pseudoconvex points in the boundary is the finite union of pairwise disjoint compact sets* $\{K_1, \ldots, K_k\}$*. Assume further that for every* $j \in \{1, \ldots, k\}$ *there exist a neighborhood* U_j *of* K_j *and* $\rho_j \in C^2(U_j)$ *such that*

i) $U_j \cap \Omega = \{z \in \mathbb{C}^n : \rho_j(z) < 0\}, U_j \cap b\Omega = \{z \in \mathbb{C}^n : \rho_j(z) = 0\}, U_j \setminus \overline{\Omega} = \{z \in \mathbb{C}^n : \rho_j(z) > 0\}$ *and the gradient of* ρ_j *does not vanish on* $U_j \cap b\Omega$*.*

ii) $\mathcal{L}_{\rho_j}(z; W) \geq 0$ *for* $z \in U_j \setminus \Omega$ *and* $W(\rho_j)(z) = 0$*.*

Then $\overline{\Omega}$ *has a Stein neighborhood basis.*

Proof. Since there are only finitely many of them, we shrink each U_j, if necessary, to get a finite collection of pairwise disjoint open sets $\{U_1, \ldots, U_k\}$. For every $j \in \{1, \ldots, k\}$ we choose open neighborhoods $\widetilde{U}_j, \widetilde{\widetilde{U}}_j$ of K_j and $\phi_j \in C_0^\infty(U_j)$ such that

i) $K_j \subset\subset \widetilde{\widetilde{U}}_j \subset\subset \widetilde{U}_j \subset\subset U_j$ and,

ii) $0 \leq \phi_j \leq 1$, $\phi_j \equiv 1$ on $\widetilde{\widetilde{U}}_j$ and $\phi_j = 0$ out of $\widetilde{U}_j$

We also choose a defining function ρ for Ω and define $\phi = 1 - \sum_{j=1}^k \phi_j$, and

$$r = \phi\rho + \sum_{j=1}^{n} \phi_j \rho_j.$$

One can easily check that r is a defining function for Ω. Since $r = \rho_j$ on $\widetilde{\widetilde{U}}_j$ we have $W(r)(z) = W(\rho_j)(z)$ and $\mathcal{L}_r(z; W) = \mathcal{L}_{\rho_j}(z; W)$ on $\widetilde{\widetilde{U}}_j$. Therefore, by assumption the level sets of r give pseudoconvex surfaces in $\left(\cup_{j=1}^k \widetilde{\widetilde{U}}_j\right) \setminus \Omega$. Since

$$S = b\Omega \setminus \left(\bigcup_{j=1}^{k} \widetilde{\widetilde{U}}_j\right)$$

is compact and any $z \in S$ is strongly pseudoconvex, by continuity of the Levi form there exists a neighborhood V of S such that $\mathcal{L}_r(z; W) > 0$ for $z \in V$ and $W(r)(z) = 0$. Therefore

$$b\Omega \subset U = V \cup \left(\bigcup_{j=1}^{k} \widetilde{\widetilde{U}}_j\right)$$

and for any $z \in U$ and $W(r)(z) = 0$ we have $\mathcal{L}_r(z; W) \geq 0$. By Sard's theorem we can choose a Stein neighborhood basis for $\overline{\Omega}$ out of the level sets of r. $\square$

CHAPTER V

SUMMARY

In chapter I we gave some motivation for studying the compactness of the $\overline{\partial}$-Neumann problem and Stein neighborhood bases for the closure of a smooth domain in $\mathbb{C}^n$.

In the first part of chapter II we gave the set-up of the $\overline{\partial}$-Neumann problem. Then we defined the compactness property and gave basic properties of the compactness of the $\overline{\partial}$-Neumann problem. We also we gave a "potential theoretic" characterization of compactness. In the second part we gave some background as well as sufficient conditions for Stein neighborhood bases for the closure of a smooth domain in $\mathbb{C}^n$.

In chapter III we stated the main theorems about compactness and gave the proofs. We showed that a complex manifold M of dimension at least q for $1 \leq q \leq n-1$ in the boundary of a smooth bounded pseudoconvex domain Ω in $\mathbb{C}^n$ is an obstruction to compactness of the $\overline{\partial}$-Neumann operator on (p, q)-forms for $0 \leq p \leq n$, on Ω, provided that at some point of M, the boundary is strictly pseudoconvex in the directions transverse to M. Although we believe that the conclusion of the theorem should not depend on the rank of the Levi form our methods do not give a more general result. We also showed that a boundary point where the Levi form has only one vanishing eigenvalue can be picked up by the plurisubharmonic hull of a set only via an analytic disc in the boundary.

In chapter IV we obtained a weaker and quantified version of McNeal's Property $(\widetilde{P})$ which still implies the existence of a Stein neighborhood basis. Then we give some applications on domains in $\mathbb{C}^2$ with a defining function that is plurisubharmonic on the boundary.

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