GLOBAL EXISTENCE OF REACTION-DIFFUSION EQUATIONS OVER MULTIPLE DOMAINS

A Dissertation

by

JOHN MAURICE-CAR RYAN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2004

Major Subject: Mathematics

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Approved as to style and content by:

Jay Walton
(Chair of Committee)

Joseph Pasciak
(Member)

Bruce Lowe
(Member)

D. Scott Lee
(Member)

Albert Boggess
(Head of Department)

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ABSTRACT

Global Existence of Reaction-Diffusion Equations

over Multiple Domains. (December 2004)

John Maurice-Car Ryan, B.A., University of South Florida;

M.S., University of Florida;

M.S., Texas A&M University

Chair of Advisory Committee: Dr. Jay Walton

Systems of semilinear parabolic differential equations arise in the modelling of many chemical and biological systems. We consider m component systems of the form

$$u_t = D\Delta u + f(t, x, u)$$

$$\partial u_k/\partial \eta = 0 \quad k = 1, ...m$$

where $u(t, x) = (u_k(t, x))_{k=1}^m$ is an unknown vector valued function and each u_{0_k} is zero outside $\Omega_{\sigma(k)}$, $D = diag(d_k)$ is an $m \times m$ positive definite diagonal matrix, $f : R \times R^n \times R^m \to R^m$, u_0 is a componentwise nonnegative function, and each Ω_i is a bounded domain in R^n where $\partial\Omega_i$ is a $C^{2+\alpha}$ manifold such that Ω_i lies locally on one side of $\partial\Omega_i$ and has unit outward normal η. Most physical processes give rise to systems for which $f = (f_k)$ is locally Lipschitz in u uniformly for $(x, t) \in \overline{\Omega} \times [0, T]$ and $f(\cdot, \cdot, \cdot) \in L^\infty(\Omega \times [0, T) \times U)$ for bounded U and the initial data u_0 is continuous and nonnegative on $\overline{\Omega}$.

The primary results of this dissertation are three-fold. The work began with a proof of the well posedness for the system . Then we obtained a global existence result if f is polynomially bounded, quaipositive and satisfies a linearly intermediate sums condition. Finally, we show that systems of reaction-diffusion equations with large diffusion coefficients exist globally with relatively weak assumptions on the vector field f.

To Maguelonne, John, M and Jack.

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CHAPTER I

INTRODUCTION

Systems of semilinear parabolic differential equations arise in the modelling of many chemical and biological systems [6, 9]. In this setting, the systems are often referred to as reaction-diffusion systems, and in their simplest form they can be written as

$$\begin{cases} u_t = D\Delta u + f(u) & t > 0, x \in \Omega \\ \partial u_k/\partial \eta = 0 & t > 0, x \in \partial\Omega \quad k = 1, ...m \\ u_k(0, \cdot) = u_{0_k(\cdot)} & t = 0, x \in \overline{\Omega} \end{cases} \tag{1.1}$$

where $u(t, x) = (u_k(t, x))_{k=1}^m$ is an unknown vector valued function, $D = diag(d_k)$ is an $m \times m$ positive definite diagonal matrix, $f : R^m \to R^m$, u_0 is a componentwise nonnegative function, and Ω is a bounded domain in R^n where $\partial\Omega$ is a $C^{2+\alpha}$ manifold such that Ω lies locally on one side of $\partial\Omega$ and has unit outward normal η. Most physical processes give rise to systems for which $f = (f_k)$ is locally Lipschitz and the initial data u_0 is continuous on $\overline{\Omega}$. These conditions guarantee the following well known result [12, 26].

Theorem 1.1 *There exists a $T_{max} \in (0, \infty]$ such that (1.1) has a unique, classical, noncontinuable solution on $[0, T_{max}) \times \overline{\Omega}$. Furthermore, if $T_{max} < \infty$, then*

$$\lim_{t \to T_{max}^-} \|u(t, \cdot)\|_{\infty,\Omega} = \infty \tag{1.2}$$

The journal model is *Journal of Differential Equations.*

A consequence of this result is that solutions of (1.1) are guaranteed to exist globally (i.e. for all $t > 0$) provided that they do not blow up in the sup-norm in finite time. Over the past twenty-five years a great deal of research has been directed towards answering questions of global existence and large time behavior of solutions to (1.1) [4, 7, 13, 15, 16, 17, 22, 24]. Of course, the results which have been obtained are a consequence of the hypotheses that have been placed on the systems.

Two of the most fundamental assumptions associated with (1.1) are preservation of positivity and conservation or reduction of total mass. Both of these assumptions can be translated into very simple mathematical terms.

Definition 1.2 *A function $f : R^m \to R^m$ is said to be quasipositive if and only if for all $k = 1, ..., m$*

$$f_k(v) \geq 0 \text{ for all } v \in R^m_+ \text{ with } v_k = 0 \tag{1.3}$$

Solutions of system (1.1) will be componentwise nonnegative for all choices of nonnegative initial data if f is quasipositive. This can be seen by considering the system (1.1) with $f(u)$ replaced by $f(u^+)$

$$\begin{cases} u_t = D\Delta u + f(u^+) & t > 0, x \in \Omega \\ \frac{\partial u_k}{\partial \eta} = 0 & t > 0, x \in \partial\Omega \quad k = 1, ..., m \\ u_k(0, \cdot) = u_{0_k} & t = 0, x \in \overline{\Omega} \end{cases} \tag{1.4}$$

where $u^+ = max\{u, 0\}$ and $u^- = -min\{u, 0\}$. $f(u^+)$ is locally Lipschitz since f is locally Lipschitz. Theorem 1.1 guarantees there exists a unique solution of (1.4). Multiplying the k^{th} component equation of (1.4) by u_k^- and integrating over $(0, t) \times \Omega$ we obtain

$$\int_0^t \int_\Omega u_k^- \frac{\partial}{\partial t}(u_k)dxdt \quad = \quad \int_0^t \int_\Omega d_k u_k^- \Delta u_k dxdt + \int_0^t \int_\Omega u_k^- f_k(u^+)dxdt$$

Note that

$$(u_k)_t = -(u_k^-)_t \text{ and } \Delta u_k = -\Delta u_k^- \text{ whenever } u_k^- > 0.$$

Making this substitution and integrating the equation above by parts yields

$$-\tfrac{1}{2}\int_\Omega (u_k^-)^2 dx = d_k \int_0^t \int_\Omega \left|\nabla u_k^-\right|^2 dxdt + \int_0^t \int_\Omega u_k^- f_k(u^+)dxdt$$

Also,

$$u_k^- f_k(u^+) = \begin{cases} 0 & \text{if } u \geq 0 \\ \geq 0 & \text{if } u \leq 0 \end{cases}$$

since f_k is quasipositive.

This gives

$$-\tfrac{1}{2}\int_\Omega (u_k^-)^2 dx \geq d_k \int_0^t \int_\Omega \left|\nabla u_k^-\right|^2 dxdt$$

implying $u_k^- = 0$. As a result, $u = u^+$, and u solves (1.1). Therefore, by uniqueness the solution to (1.1) is componentwise nonnegative.

It is a simple matter to determine an assumption that leads to conservation or reduction of total mass. The total mass of the system at time t is given by

$$\sum_{k=1}^m \int_\Omega u_k(t,x)dx \tag{1.5}$$

Consequently, we can state mathematically that total mass does not increase by requiring

$$\int_{\Omega} \sum_{k=1}^{m} u_k(t,x)dx \leq \int_{\Omega} \sum_{k=1}^{m} u_k(0,x)dx \text{ for all } t \geq 0 \tag{1.6}$$

We can see how this condition effects (1.1) by integrating the k-th component of u over $(0,t) \times \Omega$. This yields

$$\int_{\Omega} u_k(t,x)dx - \int_{\Omega} u_k(0,x)dx = \int_0^t \int_{\Omega} d_k \Delta u_k(s,x)dxds + \int_0^t \int_{\Omega} f_k(s,x)dxds \tag{1.7}$$

Integration by parts and the boundary conditions in (1.1) imply

$$\int_0^t \int_{\Omega} d_k \Delta u_k(s,x)dxds = 0 \tag{1.8}$$

Substituting this information above and summing over k yields

$$\int_{\Omega} \sum_{k=1}^{m} u_k(t,x)dx = \int_{\Omega} \sum_{k=1}^{m} u_k(0,x)dx + \int_0^t \int_{\Omega} \sum_{k=1}^{m} f_k(u(s,x))dxds \tag{1.9}$$

Consequently, we can only expect (1.6) to hold for all choices of initial data if

$$\sum_{k=1}^{m} f_k(v) \leq 0 \qquad \forall v \in R_+^m \tag{1.10}$$

This motivates the following well known definition [5, 7, 15, 16, 17, 18, 20, 21].

Definition 1.3 *A function f is said to be balanced if there exist constants $c_k > 0$, such that*

$$\sum_{k=1}^{m} c_k f_k(v) \leq 0 \qquad \forall v \in R_+^m. \tag{1.11}$$

If the diffusion coefficients d_i are all equal, and f is quasipositive and balanced, then solutions to (1.1) exist globally. We can see this as follows. First, u is componentwise nonnegative since f is quasipositive. Setting $w = \sum_{k=1}^{m} c_k u_k$, and incorporating (1.1) and (1.11) implies

$$\begin{cases} w_t \leq d\Delta w & 0 < t < T_{max}, x \in \Omega \\ \partial w/\partial \eta = 0 & t > 0, x \in \partial\Omega \\ w = w_0 & t = 0, x \in \overline{\Omega} \end{cases} \tag{1.12}$$

where $w_0 = \sum_{k=1}^{m} c_k u_{0_k}$ and $d = d_k$ for all k. The maximum principle gives us $\|w(t,\cdot)\|_{\infty,\Omega} \leq \|w_0\|_{\infty}$ for all $0 < t < T_{max}$. Consequently, $u(t,x)$ is uniformly bounded, and from Theorem 1.1 we have $T_{max} = \infty$.

The quasipositivity assumption is made to guarantee that solutions that begin in R_+^m remain in R_+^m. A more general assumption is that of an invariant region. The idea is to find a set I in the state space from which solutions can not escape.

Definition 1.4 *A set $I \subseteq R^m$ is invariant with respect to the system (1.1) iff $u(t,x) \in I$ for all $t \in [0, T_{\max}), x \in \Omega$ whenever $u_0 \in C(\overline{\Omega}, I)$.*

Definition 1.5 *f does not point out of I iff for every $u \in \partial I$ and $\eta \in R^m$ normal to I at u we have $\eta \cdot f(u) \leq 0$.*

The general condition [3] demands that all outer normals of an invariant region are left eigenvectors of the diffusion matrix, as shown below.

Suppose I is an invariant region. Choose v_0 to be on the boundary of I. Let ψ be the outward normal of I at v_0. We will show that ψ must be a left eigenvector of D. By way of contradiction, suppose ψ is not a left eigenvector. First pick a vector ξ such that $\psi \cdot \xi < 0$ and $\psi \cdot (D\xi) > 0$. We are able to find ξ because D is symmetric positive definite and ψ is not a left eigenvector of D. Choose λ such that

$$\lambda\psi \cdot (D\xi) + \psi \cdot f(v_0) > 0.$$

Let $\hat{u}(x) = v_0 + \frac{1}{2}\lambda \mid x \mid^2 \xi$. We know that there exists $\delta > 0$ such that $\hat{u}(x) \in I$ if $|x| < \delta$. let $u_0(x)$ be a smooth function contained in I such that $u_0(x) = \hat{u}(x)$ if $|x| < \delta$. Let u be the solution to (1.1) where $u_0(x)$ is our initial data.

$$\psi \cdot u_t = \psi \cdot D\Delta u + \psi \cdot f(u)$$

$$(\psi \cdot u)_t = \psi \cdot D(\lambda\xi) + \psi \cdot f(u)$$

$$\psi \cdot u_t|_{t=0} = (\psi u)_t|_{t=0} = \lambda\psi \cdot D(\xi) + \psi f(v_0) > 0$$

by choice of λ. Therefore, v is pointing out of I. Which contradicts I being an invariant region. Hence ψ must be a left eigenvector of D.

One consequence of the result above is that if the diffusion coefficients are all distinct then the only invariant regions, I, are m-hypercubes whose "sides" are parallel to the coordinate hyperplanes such that f does not point out of I.

Definition 1.6 *Let I be invariant with respect to (1.1). A function $H : I \to [0, \infty)$ is a convex seperable Lyapunov function for (1.1) iff H is a convex function that has a unique zero and can be written in the form $H(u) = \Sigma h_i(u)$ for nonnegative functions $h_i \in C^2$ and $\nabla_u H(u) \cdot f(u) \leq 0$ for every $u \in I$.*

If the diffusion coefficients are all equal, and (1.1) has an invariant region I and a convex seperable Lyapunov function, then solutions to (1.1) exist globally. We can see this as follows.

Let H be the convex seperable Lyapunov function and z be the zero of H. Note: $h_i''(x) \geq 0$ for every $x \in I$ and $\nabla H(x) \neq 0$ if $x \neq z$. Consider the case of equal diffusion coefficients. Then

$$u_t = D\Delta u + f(u) = dI\Delta u + f(u)$$

for some $d > 0$. Also

$$\frac{d}{dt}H(u) = \nabla_u H(u) \cdot u_t$$

and

$$\Delta_x H(u) = \nabla_x \cdot \nabla_u H(u) \nabla_x u = \Delta_u H(u) \mid \nabla_x u \mid^2 + \nabla_u H(u) \cdot \Delta_x u$$

Therefore,

$$\frac{d}{dt}H(u) - d\Delta_x H(u) = \nabla_u H(u) \cdot u_t - d\nabla_u H(u) \cdot \Delta_x u - d\Delta_u H(u) \mid \nabla_x u \mid^2 \leq \nabla H(u) \cdot f(u)$$

Thus

$$\frac{d}{dt}H(u) \leq d\Delta_x H(u) + \Delta H(u) \cdot f(u)$$

So, if

$$\nabla_u H(u) \cdot f(u) \leq 0,$$

then

$$\frac{d}{dt}H(u) \leq d\Delta_x H(u)$$

$$\frac{\partial}{\partial \eta}H(u) = \nabla H(u) \cdot \frac{\partial u}{\partial \eta} = 0.$$

So, by the maximum principle, $H(u)$ is bounded.

An additional assumption that follows from the physical properties of many models is that

$$|f(v)| \text{ is bounded above by a polynomial} \tag{1.13}$$

In the case where the diffusion coefficients are distinct, an example of Pierre and Schmidt [24] shows that (1.3), (1.11) and (1.13) are not enough to guarantee global existence of solutions.

For the past twenty-five years many authors have struggled with this global existence question. Alikakos [1] worked on systems with $n = 2$, homogeneous Neumann boundary conditions (1.1), and f having the form

$$\begin{aligned} f_1(u) &= -u_1 g(u_2) \\ f_2(u) &= u_1 g(u_2) \end{aligned} \tag{1.14}$$

with g nonnegative and polynomially bounded. Notice that f satisfies both with (1.3) and (1.11) with $c_1 = c_2 = 1$. This system was originally proposed by Martin, and in [14], Hollis, Martin and Pierre analyzed this system and others of the form (1.1) under assumptions (1.3), (1.11) and (1.13) with $m = 2$, and proved global existence in any spatial dimension provided that an *a priori* L^∞ bound is available for one component. Morgan [20, 21] extended these results to handle arbitrary m component systems of the form (1.1) under conditions (1.3), (1.11) and (1.13) along with an intermediate sums condition.

Kanel' [17, 18] obtained some results on related problems without the intermediate sums condition by placing stricter requirements on the polynomial bounds on the components of f. He has shown that solutions of (1.1) with $\Omega = R^n$ exist globally provided that in addition to (1.3) and (1.11), each f_k is at most quadratic if $n \geq 2$ and at most cubic if $n = 1$. Kanel' also obtained the last result for cubic f_k's and

$n = 1$ on the bounded domain $\Omega = (0, L)$ with u_k satisfying homogeneous Neumann conditions at the endpoints.

Redheffer, Redlinger and Walter [25] showed that in the case of equal diffusion coefficients, the existence of a convex Lyapunov function V guarantees global existence for solutions to (1.1). Moreover, if V is strictly convex, then the omega limit set of (1.1) is the same as the omega limit set of the associated system of ordinary differential equations given by

$$u' = f(u).$$

Conway, Hoff and Smoller [4] showed that if (1.1) admits a bounded invariant region and the diffusion coefficients are sufficiently large then

$$\left\| u(t, \cdot) - \overline{u(t)} \right\|_{\infty,\Omega} \to 0 \quad (exponentially) \quad as \quad t \to \infty \tag{1.15}$$

where

$$\overline{u(t)} = \frac{1}{|\Omega|} \int_{\Omega} u(t, x) dx \tag{1.16}$$

Hale [10] showed that if $K \subset R^n$ is a compact attractor for the ordinary differential equation

$$v'(t) = f(v(t)) \tag{1.17}$$

and the diffusion coefficients are sufficiently large then K is a compact attractor for (1.1). These results were also obtained by Cupps [5] for systems of the form (1.1) using only the assumptions (1.3) and (1.11). This is remarkable given that assumptions (1.3) and (1.11) do not guarantee the existence of bounded invariant regions or compact attractors for (1.1).

The focus of this research is to examine the effect of assumptions of the form (1.3) and (1.11) on global existence of solutions to reaction-diffusion systems on multiple domains.

Problems of this type can arise in the modelling of biological systems, and are only recently being studied as mathematical models. For example, one such system which is analyzed by Fitzgibbon, Langlais and Morgan [6] models the interaction of two hosts and a vector population.

We will consider a reation-diffusion system on noncoincident spatial domains to help motivate the types of systems studied in this dissertation. One type of model studied is the so-called "criss-cross" model. Criss-Cross models have been put forth to describe the transmission of vector hosts such as malaria. Typically these models assume two independent populations, each of which are subdividedd into three subclasses: susceptible, S_i for $i = 1$ or 2, infectives, I_i for $i = 1$ or 2 and removed, R_i for $i = 1$ or 2. Susceptibles are individuals capable of contracting the disease, and the infectives are individuals infected with the disease and capable of transmitting it. The removed class are those individuals that have either died or gained permanent immunity from the disease. Basically the disease is transmitted by infectives of one population interacting with the susceptibles of the other population producing infectives of the first population. If there is no loss of immunity or resurrection the removed classes do not affect the dynamics of the process and are not considered.

The following system of differential equations describes a basic process:

$$\begin{aligned} dS_1/dt &= -k_1 S_1 I_2 \\ dS_2/dt &= -k_2 S_2 I_1 \\ dI_1/dt &= k_1 S_1 I_2 - \lambda_1 I_1 \\ dI_2/dt &= k_2 S_2 I_1 - \lambda_2 I_2 \end{aligned} \tag{1.18}$$

In a more complex setting [6], a disease is transmitted in a criss-cross fashion from one host through a vector to another host. It is assumed that the disease is benign for one host and lethal to the other. This dynamic can be described by the following set of ordinary differential equations:

$$\begin{array}{ll} \begin{pmatrix} \phi_t = -k_1\phi\beta + \lambda_1\psi \\ \psi_t = k_1\phi\beta - \lambda_1\psi \end{pmatrix} & \text{host 1} \\ \begin{pmatrix} \alpha_t = -k_2\alpha\psi - k_3\alpha v + \lambda_2\beta \\ \beta_t = k_2\alpha\psi + k_3\alpha v - \lambda_2\beta \end{pmatrix} & \text{vector} \\ \begin{pmatrix} v_t = -k_4 v\beta \\ w_t = k_4 v\beta - \lambda_3 w \end{pmatrix} & \text{host 2} \end{array} \tag{1.19}$$

with positive constants k_i and $\lambda_{j.}$ Here the host where the disease is benign is given by the first set of equations with ϕ representing the susceptibles and ψ representing the infectives. Because the disease is considered benign, the recovery rate is a constant $\lambda_1 > 0$ with no mortality. The third set of equations describes the circulation of the disease through the second host. In this case the disease can be fatal if there is no recovery term. Essentially, this is an SIR model with incidence term $k_4 v\beta$. The susceptible vector and infective vector populations are represented by α and β respectively. Basically, this system is a coupling of two SIS models with an SIR model. Such models can be used to describe the epidemiological dynamics of encephalitis whereby the disease is transferred between avian and human populations via mosquitos.

Now consider a spatially distributed population. The dispersion of the population is modeled by Fickian diffusion. In this model there are three populations confined to seperate habitats which overlap. The possibility of physically separated habitats for the vulnerable and resistant hosts are allowed, each of which intersects with the domain of the vector.

$$\begin{cases} \left(\begin{array}{ll} \phi_t = d_1\Delta\phi - k_1(x)\phi\beta + \lambda_1\psi & \\ & \text{for } x \in \Omega_1, t > 0 \\ \psi_t = d_2\Delta\psi + k_1(x)\phi\beta - \lambda_1\psi & \end{array} \right) & \text{host 1} \\ \left(\begin{array}{ll} \alpha_t = d_3\Delta\alpha - k_2(x)\alpha\psi - k_3(x)\alpha v + \lambda_2\beta & \\ & \text{for } x \in \Omega_2, t > 0 \\ \beta_t = d_4\Delta\beta + k_2(x)\alpha\psi + k_3(x)\alpha v - \lambda_2\beta & \end{array} \right) & \text{vector} \\ \left(\begin{array}{ll} v_t = d_5\Delta v - k_4(x)v\beta & \\ & \text{for } x \in \Omega_3, t > 0 \\ w_t = d_6\Delta w + k_4(x)v\beta - \lambda_3 w & \end{array} \right) & \text{host 2} \end{cases} \tag{1.20}$$

Here k_1, k_2, k_3 and k_4 are nonnegative functions, and λ_1, λ_2 and λ_3 are positive constants. Furthermore, the supports of k_1 and k_2 are contained in the intersection of Ω_1 and Ω_2, the supports of k_3 and k_4 are contained in the intersection of Ω_2 and Ω_3. Finally, the values d_i and λ_j are positive constants for $i = 1, 2, ..., 6$ and $j = 1, 2, 3$. We impose homogeneous Neumann boundary conditions on each domain Ω_1, Ω_2, and Ω_3.

$$\begin{aligned}
\partial\phi/\partial\eta &= \partial\psi/\partial\eta = 0 \quad for x \in \partial\Omega_1, t > 0 \\
\partial\alpha/\partial\eta &= \partial\beta/\partial\eta = 0 \quad for x \in \partial\Omega_2, t > 0 \\
\partial v/\partial\eta &= \partial w/\partial\eta = 0 \quad for x \in \partial\Omega_3, t > 0
\end{aligned} \tag{1.21}$$

and specify continuous nonnegative initial data.

$$\begin{aligned}
&\phi(x,0) = \phi_0(x), \quad \psi(x,0) = \psi_0(x) \quad \text{for every } x \in \overline{\Omega}_1 \\
&\alpha(x,0) = \alpha_0(x), \quad \beta(x,0) = \beta_0(x) \quad \text{for every} x \in \overline{\Omega}_2 \\
&v(x,0) = v_0(x), \quad w(x,0) = w_0(x) \quad \text{for every} x \in \overline{\Omega}_3
\end{aligned} \tag{1.22}$$

Variants of the quasipositivity and balancing assumptions occur on each component domain. The quasipositivity is obvious, and the balancing holds on each component domain. For Ω_1 the vector field

$$\begin{pmatrix} -k_1(x)\phi\beta + \lambda_1\psi \\ +k_1(x)\phi\beta - \lambda_1\psi \end{pmatrix} \tag{1.23}$$

has components that clearly sum to zero. Similarly, on Ω_2 the vector field

$$\begin{pmatrix} -k_2(x)\alpha\psi - k_3(x)\alpha v + \lambda_2\beta \\ k_2(x)\alpha\psi + k_3(x)\alpha v - \lambda_2\beta \end{pmatrix} \tag{1.24}$$

also sums to zero. The same mechanism can be seen on Ω_3 since the function

$$\begin{pmatrix} -k_4(x)v\beta \\ k_4(x)v\beta - \lambda_3 w \end{pmatrix} \tag{1.25}$$

has components that sum to less than zero.

The system described above is an example of the type of systems that will be the focus of this dissertation. Consider domains $\Omega_{1,}\Omega_{2,}..., \Omega_m$ where each Ω_k is a bounded $C^{2+\alpha}$ manifold such that Ω_k lies locally on one side of $\partial\Omega_k$ and has unit outward normal η. Each domain Ω_k will have n_k species.

Notationally each species is associated with the appropriate domain by partitioning the set $\{1, 2, ..., s\}$ into k disjoint sets, $O_1, O_2, ..., O_k$, where $i \in O_j$ can be interpreted as meaning the i^{th} species is defined on Ω_j. Define the mapping $\sigma : \{1, 2, ..., s\} \to \{1, 2, ..., m\}$ via $i \in O_{\sigma(i)}$. The k^{th} species dynamics is governed by the following system:

$$\begin{cases} (u_k)_t = d_k \Delta u_k + f_k(t, x, u) & x \in \Omega_{\sigma(k)}, t > 0 \\ \frac{\partial u_k}{\partial \eta} = 0 & x \in \partial\Omega_{\sigma(k)} \\ u_k(0, x) = 0 & x \notin \Omega_{\sigma(k)} \end{cases} \tag{1.26}$$

where $f_k(t, x, u) = 0$ if $x \notin \Omega_{\sigma(k)}$, $u_k(0, x)$ is continuous and nonnegative.

The quasipositivity and balancing assumptions given in (1.3) and (1.11) do not give rise to bounded invariant regions or compact attractors for (1.1), let alone (1.26). As a result, the results in [4, 10, 25] do not apply. However, it seems possible to use these assumptions and apply the methods in [20] to obtain both $L^1(\Omega)$ and $L^2((0, T) \times \Omega)$ bounds on the unknowns. We will use this to prove two results. First, we will show a result analogous to Theorem 1 for the system we are solving. Next, we will show that if the initial data is sufficiently small then the system (1.26) has a global solution which is uniformly bounded. This result is an extension of the results in [16]. Then we show that solutions to (1.26) exist globally provided the diffusion coefficients are sufficiently large. This analysis is an extension of the techniques employed in [5], and the result extends the results in [4, 5, 10].

The material in this dissertation is organized in the following manner. Conventions on notation and statements of main results are given in Chapter II. Chapter

III contains statements of fundamental results that will be used in proving the main theorems. Chapter IV contains he proofs of the theorems stated in Chapter II, while Chapter V contains some applications and suggestions for further research.

CHAPTER II

NOTATION AND BASIC RESULTS

A. Notation

Let $1 \leq p \leq \infty$ and suppose Ω is a bounded domain of R^n. $L^p(\Omega, R^n)$ will denote the Banach space of measurable functions with norm given by

$$\|f\|_{p,\Omega} = \left(\sum_{i=1}^{m} \int_{\Omega} |f_i(x)|^p \, dx \right)^{1/p} \quad \text{for } 1 \leq p < \infty$$

and

$$\|f\|_{\infty,\Omega} = \sum_{i=1}^{n} |f_i(x)|_{\infty,\Omega}$$

where

$$|f_i(x)|_{\infty,\Omega} = inf\{K : \ |f_i(x)| \leq K \text{ for almost every } x \in \Omega\}.$$

All derivatives are understood to be in the distributional sense. D^α denotes $\partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}$ where $\partial_i = \partial/\partial x_i$ and $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, $|\alpha| = \sum\limits_{i=1}^{n} \alpha_i$.

For $p \geq 1$ and $k > 0$,

$$W^k(\Omega) = \{f : \Omega \to R : D^\alpha f \text{ exists for all } \alpha, \text{with } \ |\alpha| \leq k\}.$$

$$W_p^k(\Omega) = \{f \in W^k(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all } \alpha, \text{with } \ |\alpha| \leq k\}.$$

$W_p^k(\Omega)$ is equipped with the norm

$$\|f\|_{p,\Omega}^{(k)} = (\int\limits_{\Omega} \sum_{|\alpha| \leq k} |D^\alpha f|^p \, dx)^{1/p}.$$

$H^k(\Omega)$ will be used to denote $W_2^k(\Omega)$.

For real numbers τ and t with $0 \leq \tau < t$, $Q_{(\tau,t)}$ will denote the cylinder $(\tau, t) \times \Omega$.

$W_p^{1,2}(Q_{(\tau,t)})$ will denote the Banach space of elements f in $L^p(Q_{(\tau,t)})$ having weak derivatives $\partial_t^r \partial_x^s f$, with $r \le 1$ and $s \le 2$ lying in $L^p(Q_{(\tau,t)})$. This space is equipped with the norm

$$\|f\|_{p,Q_{(\tau,t)}}^{(1,2)} = (\int_\tau^t \int_\Omega \sum_{r\le 1 s\le 2} |\partial_t^r \partial_x^s f|^p \, dx)^{1/p}.$$

Ω_i will denote a bounded domain in R^n that lies locally on one side of its $C^{2+\alpha}$ boundary $\partial\Omega_i$. $\overline{\Omega_i}$ will denote the closure of Ω_i and $|\Omega_i|$ is the measure of Ω_i. The gradient and Laplacian operators will be represented by ∇ and Δ respectively. χ_i will denote the characteristic function on Ω_i. Finally, R_+^m will denote the nonnegative orthant of R^m.

B. Main Results

The primary focus of this dissertation is the reaction-diffusion system described below.

Consider the domains $\Omega_{1,}\Omega_{2,}..., \Omega_m$. We define $\Omega = \cup_{i=1}^m \Omega_i$. Each domain Ω_k will have n_k species. Notationally each species is associated with the appropriate domain by partitioning the set $\{1, 2, ..., s\}$ into m disjoint sets, $O_1, O_2, ..., O_m$, where $i \in O_j$ can be interpreted as meaning the i^{th} species is defined on Ω_j.

Define the mapping $\sigma : \{1, 2, ..., s\} \to \{1, 2, ..., m\}$ via $i \in O_{\sigma(i)}$.

$$\begin{cases} u_t = D\Delta u + f(t, x, u) & t > 0, x \in \Omega \\ \partial u_k/\partial\eta = 0 & t > 0, x \in \partial\Omega_{\sigma(k)} \quad k = 1, ...m \\ u_k(0, \cdot) = u_{0_k(\cdot)} & t = 0, x \in \overline{\Omega} \quad k = 1, ...m \end{cases} \tag{2.1}$$

where $u(t, x) = (u_k(t, x))_{k=1}^m$ is an unknown vector valued function and each u_{0_k} is zero outside $\Omega_{\sigma(k)}$, $D = diag(d_k)$ is an $m \times m$ positive definite diagonal matrix, $f : R \times R^n \times R^m \to R^m$, u_0 is a componentwise nonnegative function, and each Ω_i is a bounded domain in R^n where $\partial\Omega_i$ is a $C^{2+\alpha}$ manifold such that Ω_i lies locally on one side of $\partial\Omega_i$ and has unit outward normal η. Most physical processes give rise to systems for which $f = (f_k)$ is locally Lipschitz in u uniformly for $(x,t) \in \overline{\Omega} \times [0,T]$ and $f(\cdot,\cdot,\cdot) \in L^\infty(\Omega \times [0,T) \times U)$ for bounded U and the initial data u_0 is continuous and nonnegative on $\overline{\Omega}$. We make these assumptions on f and u_0 throughout the remainder of this work.

In some of our results we will assume there exists a $z \in I$ such that

$$f(\cdot,\cdot,z) = 0 \tag{2.2}$$

Definition 2.1 *A set $I \subseteq R^m$ is invariant with respect to the system (2.1) iff $u(t,x) \in I$ for all $t \in [0, T_{\max}), x \in \Omega$ whenever $u_0 \in C(\overline{\Omega}, I)$.*

Before we state our results, we introduce a truncated system associated with (2.1). To this end let $r > max\{\|u_0\|_{\infty,\Omega}, |z|, 1\}$. and define $\Phi_r \in C^\infty(R^m, [0,1])$ via

$$\Phi_r(u) = \begin{cases} 1, & u \in B_r(0) \\ 0, & u \notin B_{2r}(0) \end{cases}$$

where $\left|\frac{\partial \Phi_r(u)}{u_i}\right| \leq 2$ for all $i = 1, ..., m$. Let $\widehat{f}_i(t,x,u) = \Phi_r(u) f_i(t,x,u)$ and consider the so-called truncated system given by

$$\begin{cases} u_t = D\Delta u + \widehat{f}(t,x,u) & t > 0, x \in \Omega \\ \partial u_k/\partial\eta = 0 & t > 0, x \in \partial\Omega_{\sigma(k)} \quad k = 1, ..., m \\ u_k(0,\cdot) = u_{0_k}(\cdot) & t = 0, x \in \overline{\Omega} \end{cases} \tag{2.3}$$

Definition 2.2 *Suppose $I \subseteq R^m$ is invaariant with respect to (2.1). A function $H : I \to [0,\infty)$ is a convex seperable Lyapunov function for (2.1) iff H is a convex* *function that has a unique zero and can be written in the form* $H(u) = \Sigma h_i(u)$ *for nonnegative functions* $h_i \in C^2$ *and* $\nabla_u H(u) \cdot f(u) \leq 0$ *for every* $u \in I$.

Remark: The case $I = R_+^m$ and $H(u) = \Sigma c_i u_i$ corresponds to f being balanced.

Our first result is an extension of Theorem (1.1) to the setting of (2.1).

Theorem 2.3 *There exists a* $T_{max} \in (0, \infty]$ *such that (2.1) has a unique noncontinuable solution on* $[0, T_{max}) \times \overline{\Omega}$. *Furthermore, if* $T_{max} < \infty$, *then*

$$\lim_{t \to T_{max}^-} \|u(t, \cdot)\|_{\infty,\Omega} = \infty \tag{2.4}$$

We continue our development by introducing a variant of the linear intermediate sums condition from Chapter I to the setting of (2.1).

Definition 2.4 *We say that the reaction-diffusion system satisfies the linear intermediate sums condition if for every* k *(associated with domain* Ω_k*) there exists* $M_k, N_k \geq 0$ *and a lower triangular matrix* $A^{(k)}$ *such that* $a_{ii}^{(k)} > 0$, $a_{n_k,i}^{(k)} > 0$, *and* $\sum_{j=1}^{n_k} a_{io_j} f_{io_j}(x, t, u) \leq M_k \sum_{j \in O_k} u_j + N_k$ *for each* i *and* $a_{n_k,j} > 0$ *for every* $u \in R_+^m$.

This extension allows us to generalize some of the results in [20, 21].

Theorem 2.5 *Suppose that* f *is quasipositive and satisfies the linear intermediate sums condition and satisfies (1.13). Then the solution of (2.1) is nonnegative and exists globally.*

We are now in a position to extend some results of [5].

Theorem 2.6 *Suppose f satisfies (1.3) and (2.2). Let $M > 0$ and suppose there exists R, L_M such that if $r \geq R$, $\|u_0 - z\|_{\infty,\Omega} \leq M$ and u solves (2.3) then $\|u(t,\cdot) - z\|_{1,\Omega} \leq L_M$. Then there exists constants $d_M, K_M > 0$ so that if $d_i \geq d_M$ for all i, then the solution u of (2.1) exists globally and*

$$\|u(t,\cdot) - z\|_{\infty,\Omega} \leq K_M \qquad \forall t \geq 0.$$

Moreover, if $L_M \to 0$ as $\|u_0 - z\|_{\infty,\Omega} \to 0$ and the initial data is sufficiently close to z then no additional assumptions on the size of the diffusion coefficients are necessary to guarantee the solution u of (2.1) exists globally and

$$\|u(t,\cdot) - z\|_{\infty,\Omega} \leq K_M \qquad \forall t \geq 0.$$

At first glance, the result above might seem untractable. However, the following result is an immediate consequence.

Theorem 2.7 *Suppose that f, D and u_0 are as in Theorem 2.6 and f is also balanced. If the diffusion coefficients d_i are sufficiently large, then the solution of (2.1) exists globally and is uniformly bounded. Also, if the initial data is sufficiently close to the equilibrium point z then no additional assumptions on the size of the diffusion coefficients are necessary.*

Theorem 2.8 *Suppose there exists an invariant region I and convex separable Lyapunov function associated with (2.1), and f satisfies (2.2). Further suppose that D and u_0 are as in Theorem 2.6 and $u_0 \in I$. If the diffusion coefficients d_i are sufficiently large or u_0 is sufficiently close to the equilibrium point z, then the solution to (2.1) exists globally and is uniformly bouded.*

Theorem 2.9 *Suppose that D and u_0 are as in Theorem 2.6 and $f \in C^1(R_+^m, R^m)$ is balanced and quasipositive. If the d_is are sufficiently large, then the solution of (2.1) exists globally, is uniformly bounded and*

$$\|u_k(t,\cdot) - \overline{u}_k(t)\|_{\infty,\Omega_{\sigma(k)}} \to 0 \text{ as } t \to \infty \text{ for every } k.$$

where

$$\overline{u}_k(t) = \frac{1}{\left|\Omega_{\sigma(k)}\right|} \int_{\Omega_{\sigma(k)}} u_k(t,x)dx.$$

CHAPTER III

PRELIMINARIES

A. Fundamental Results

We will need the following fractional-Sobolev embedding theorem of Amann [2].

Theorem 3.1 *If* $0 \leq s' \leq s \leq 2$ *and* $1 < p, q < \infty$ *then* $W_p^s(\Omega)$ *embeds continuously into* $W_q^{s'}(\Omega)$ *whenever* $1/p \geq 1/q$ *and* $s - n/p \geq s' - n/q$.

The following regularity estimate by Ladyzenskaja et al. [19] will be crucial in obtaining several estimates.

Theorem 3.2 *Suppose* $1 < q < \infty$, $\tau < t < T$, $\theta \in L^q(Q_{(\tau,T)})$, $\phi_0 \in W_q^{2-2/q}(\Omega)$ *and* ϕ *solves the scalar equation*

$$\begin{cases} \phi_t = d\Delta\phi + \theta & t \in (\tau, T), x \in \Omega \\ \frac{\partial \phi}{\partial \eta} = 0 & t \in (\tau, T), x \in \Omega \\ \phi = \phi_0 & t = \tau, x \in \Omega \end{cases} \tag{3.1}$$

Then there exists $C(q, d, \Omega, T - \tau) > 0$ *such that*

$$\|\phi\|_{q,Q_{(\tau,T)}}^{(1,2)} \leq C(q, d, \Omega, T - \tau)(\|\theta\|_{q,Q_{(\tau,T)}} + \|\phi_0\|_{q,\Omega}^{2-2/q}).$$

Theorem 3.3 *(Shauder's fixed point theorem) If* A *is a closed, bounded and convex subset of a normed linear space* X *and* $T : X \to X$ *is a compact, continuous function such that* $T(A) \subseteq A$ *then there exists a* $u \in A$ *such that* $Tu = u$.

Theorem 3.4 *(Gronwall's inequality) Let*

$$\begin{aligned} u &: [a,b] \to [0,\infty) \\ v &: [a,b] \to R \end{aligned}$$

be continuous functions and let C be a constant. Then if

$$v(t) \le C + \int_a^t v(s)u(s)ds \tag{3.2}$$

for $t \in [a,b]$, it follows that

$$v(t) \le C \exp(\int_a^t u(s)ds) \tag{3.3}$$

for $t \in [a,b]$.

We will also need a result from semigroup theory.

Definition 3.5 *Let X be a Banach space. A one parameter family of bounded linear operators $\{T(t)\}_{t \ge 0}$ from X into X is a strongly continuous semigroup of contractions on X if*

i. $T(t+s) = T(t)T(s) \qquad \forall t, s \in R_+$

ii. $T(0) = I$

iii. $T(\cdot)f \in C(R_+, X) \qquad \forall\ f \in X$

iv. $\|T(t)\| \le 1 \qquad \forall t \in R_+$

Theorem 3.6 *The operator $d\Delta$ subject to homogeneous Neumann boundary conditions generates a strongly continuous semigroup of contractions on $C(\overline{\Omega})$.*

B. Preliminary Estimates

Many of the following estimates can be found in [2, 5, 14, 19, 23, 28]. These results are somewhat obscure and consequently are included.

Lemma 3.7 *Let $\{S(t)\}_{t\geq 0}$ be the semigroup generated by $d\Delta$ subject to homogeneous Neumann boundary conditions on $L^p(\Omega)$ with $1 \leq p < \infty$. Then there exist constants $M, \delta > 0$ so that*

$$\|PS(t)w\|_{p,\Omega} \leq Me^{-\delta t} \|w\|_{p,\Omega}$$

for $w \in L^p(\Omega)$, where

$$P\Psi = \Psi - \tfrac{1}{|\Omega|} \int_\Omega \Psi \qquad \text{for } \Psi \in L^p(\Omega).$$

Proof:

$$PS(t)\phi = S(t)\phi - \tfrac{1}{|\Omega|} \int_\Omega S(t)\phi = S(t)[\phi - \tfrac{1}{|\Omega|} \int_\Omega \phi] = S(t)P\phi \qquad \forall \phi \in L^p(\Omega)$$

Setting $v = PS(t)\phi$ we see v satisfies

$$v_t = d\Delta v$$
$$v(0) = P\phi$$

So $\{PS(t)\}_{t\geq 0}$ is the semigroup generated by the restriction of $d\Delta$ to the subspace $\{\phi \in L^p(\Omega) : \int_\Omega \phi = 0\}$. It follows that for any $\phi \in L^p(\Omega)$,

$$\begin{aligned} &\|PS(t)\phi\|_{p,\Omega} = \|P^2 S(t)\phi\|_{p,\Omega} = \|PS(t)P\phi\|_{p,\Omega} \\ &\leq M_1 e^{-\delta t} \|P\phi\|_{p,\Omega} \leq Me^{-\delta t} \|\phi\|_{p,\Omega}. \end{aligned}$$

$\square$

Now consider the scalar equation

$$\begin{cases} \phi_t = d\Delta\phi + \theta & t \in (\tau, T), x \in \Omega \\ \frac{\partial \phi}{\partial \eta} = 0 & t \in (\tau, T), x \in \partial\Omega \\ \phi = 0 & t = \tau, x \in \Omega \end{cases} \tag{3.4}$$

Applying Theorem 3.2 we have

Lemma 3.8 *Suppose $q > 1, \theta \in L^q(Q_{(\tau,T)})$ and that ϕ solves (3.4). Then there is a constant $C(q, d, (T - \tau))$ such that*

$$\|\phi\|_{q,Q_{(\tau,T)}}, \|\Delta\phi\|_{q,Q_{(\tau,T)}} \leq C(q, d, (T - \tau)) \|\theta\|_{q,Q_{(\tau,T)}}$$

Lemma 3.9 *Under the assumptions of Lemma 3.8, there is a constant $\widehat{C}(q, d)$ so that*

$$\|P\phi\|_{q,Q_{(\tau,T)}}, \|\Delta\phi\|_{q,Q_{(\tau,T)}} \leq \widehat{C}(q, d) \|\theta\|_{q,Q_{(\tau,T)}}$$

Proof:

$$\begin{aligned} (P\phi)_t &= \phi_t - \tfrac{1}{|\Omega|} \int_\Omega \phi_t \\ &= d\Delta\phi + \theta - \tfrac{1}{|\Omega|} \int_\Omega (\phi_t + \theta) \\ &= d\Delta\phi + \theta - \tfrac{1}{|\Omega|} \int_\Omega \theta \\ &= d\Delta(\phi - \tfrac{1}{|\Omega|} \int_\Omega \phi) + \theta - \tfrac{1}{|\Omega|} \int_\Omega \theta \\ &= d\Delta P\phi + P\theta \end{aligned}$$

So $P\phi$ solves

$$\begin{cases} P\phi_t = d\Delta P\phi + P\theta & t \in (\tau, T), x \in \Omega \\ \frac{\partial P\phi}{\partial \eta} = 0 & t \in (\tau, T), x \in \partial\Omega \\ P\phi = 0 & t = \tau, x \in \Omega \end{cases} \tag{3.5}$$

By variation of parameters,

$$\phi(t,\cdot) = \int_\tau^T S(t-s)\theta(s,\cdot)ds$$

we find

$$P\phi(t,\cdot) = \phi(t,\cdot) - \overline{\phi}(t)$$
$$= \int_\tau^t S(t-s)\theta(s,\cdot)ds - \frac{1}{|\Omega|}\int_\Omega\int_\tau^t S(t-s)\theta(s,x)dsdx$$
$$= \int_\tau^t PS(t-s)\theta(s,\cdot)ds.$$

This gives us

$$\|P\phi\|_{q,Q_{(\tau,T)}}^q = \int_\tau^T\int_\Omega |P\phi(t,x)|^q\, dxdt$$
$$= \int_\tau^T \|P\phi(t,\cdot)\|_q^q\, dt$$
$$= \int_\tau^T \left\|\int_\tau^t PS(t-s)\theta(s,\cdot)ds\right\|_q^q dt$$
$$\leq \int_\tau^T(\int_\tau^t \|PS(t-s)\theta(s,\cdot)\|_q\, ds)^q dt$$
$$\leq \int_\tau^T(\int_\tau^t (Me^{-\delta(t-s)}\, \|\theta(s,\cdot)\|_q)ds)^q dt$$
$$= M^q\int_\tau^T(\int_\tau^t (e^{-\delta(t-s)}\, \|\theta(s,\cdot)\|_q)ds)^q dt$$

Setting $y(s) = \|\theta(s,\cdot)\|_q$ and applying Holder's inequality we see

$$\int_\tau^t e^{-\delta(t-s)}y(s)ds \leq (\int_\tau^t e^{-\delta(t-s)}y(s)^q ds)^{1/q}(\int_\tau^t e^{-\delta(t-s)}ds)^{1/p}$$

and this gives us

$$\int_\tau^T(\int_\tau^t (e^{-\delta(t-s)}y(s)_q)ds)^q dt \leq \int_\tau^T(\int_\tau^t e^{-\delta(t-s)}y(s)^q ds)(\int_\tau^t e^{-\delta(t-s)}ds)^{q/p}dt$$

$$\|P\phi\|_{q,Q_{(\tau,T)}}^q \leq M^q\int_\tau^T(\int_\tau^t (e^{-\delta(t-s)}y(s)ds)^q dt$$
$$\leq M^q\int_\tau^T(\int_\tau^t e^{-\delta(t-s)}y(s)^q ds)(\int_\tau^t e^{-\delta(t-s)}ds)^{q/p}dt$$
$$= M^q\delta^{-q/p}\int_\tau^T(\int_\tau^t e^{-\delta(t-s)}y(s)^q ds)dt = M^q\delta^{-q/p}\int_\tau^T y(s)^q\int_s^T e^{-\delta(t-s)}dtds$$
$$\leq M^q\delta^{-q/p}\delta^{-1}\|\theta\|_{q,Q_{(\tau,T)}}^q = M^q\delta^{-q}\|\theta\|_{q,Q_{(\tau,T)}}^q.$$

We are now in a position to get an estimate for $\Delta\phi$. Let ϕ and $\Delta\phi$ be the solutions of (3.4) and (3.5) , respectively. Define $g \in C^\infty(R, [0,1])$ via

$$g(t) = \begin{cases} 1 & if\ t \in [-1,1] \\ 0 & if\ t \notin [-2,2] \end{cases}$$

For $t_0 \geq 2$, set $\Psi(t,x) = g(t-t_0)P\phi(t,x)$. Ψ satisfies

$$\begin{cases} \Psi_t = d\Delta\Psi + g(t-t_0)P\theta + g'(t-t_0)P\phi & t > t_0 - 2, x \in \Omega \\ \frac{\partial\Psi}{\partial\eta} = 0 & t > t_0 - 2, x \in \partial\Omega \\ \Psi = 0 & t = t_0 - 2, x \in \Omega \end{cases} \tag{3.6}$$

Setting $C_1(q,d) = C(q,d,4)$ from Lemma 3.8 we have

$$\|\Delta\Psi\|^q_{q,(t_0-2,t_0+2)\times\Omega} \leq C_1(q,d)^q(\|P\theta\|_{q,(t_0-2,t_0+2)\times\Omega} + \|g'\|_\infty \|P\phi\|_{q,(t_0-2,t_0+2)\times\Omega})^q$$

and this gives us

$$\|\Delta\Psi\|^q_{q,(t_0-2,t_0+2)\times\Omega} \leq 2^{q-1}C_1(q,d)^q(\|P\theta\|^q_{q,(t_0-2,t_0+2)\times\Omega} + \|g'\|^q_\infty \|P\phi\|^q_{q,(t_0-2,t_0+2)\times\Omega})$$

By the construction of Ψ we have

$$\begin{aligned} &\|\Delta\phi\|^q_{q,(t_0-1,t_0+1)\times\Omega} \leq \|\Delta P\phi\|^q_{q,(t_0-1,t_0+1)\times\Omega} \\ &= \|\Delta\Psi\|^q_{q,(t_0-1,t_0+1)\times\Omega} \\ &\leq \|\Delta\Psi\|^q_{q,(t_0-2,t_0+2)\times\Omega} \end{aligned}$$

so

$$\|\Delta\phi\|^q_{q,(t_0-1,t_0+1)\times\Omega} \leq 2^{q-1}C_1(q,d)^q(\|P\theta\|^q_{q,(t_0-2,t_0+2)\times\Omega} + \|g'\|^q_\infty \|P\phi\|^q_{q,(t_0-2,t_0+2)\times\Omega})$$

We will now find a bound for $P\theta$ in terms of θ.

$$\begin{aligned}
&\|P\theta\|^q_{q,(t_0-2,t_0+2)\times\Omega} = \int_{t_0-2}^{t_0+2}\int_\Omega \left|\theta(t,x)-\overline{\theta}(t,\cdot)\right|^q dxdt \\
&\leq \int_{t_0-2}^{t_0+2}\int_\Omega (|\theta(t,x)| + |\Omega|^{-1}\int_\Omega |\theta(t,y)|\,dy)^q dxdt \\
&\leq 2^{q-1}(\|\theta\|^q_{q,(t_0-2,t_0+2)\times\Omega} + \int_{t_0-2}^{t_0+2}\int_\Omega (|\Omega|^{-1}\int_\Omega |\theta(t,y)|\,dy)^q dxdt) \\
&= 2^{q-1}(\|\theta\|^q_{q,(t_0-2,t_0+2)\times\Omega} + |\Omega|^{1-q}\int_{t_0-2}^{t_0+2}(\int_\Omega |\theta(t,y)|\,dy)^q dt)
\end{aligned}$$

From Holder's inequality,

$$\int_\Omega |\theta(t,y)|\,dy \leq |\Omega|^{1/p}\,(\int_\Omega |\theta(t,y)|^q\,dy)^{1/q}$$

and thus

$$\|P\theta\|^q_{q,(t_0-2,t_0+2)\times\Omega} \leq 2^q\,\|\theta\|^q_{q,(t_0-2,t_0+2)\times\Omega}\,.$$

It then follows that

$$\|\Delta\phi\|^q_{q,(t_0-1,t_0+1)\times\Omega} \leq 2^{2q-1}C_1(q,d)^q(\|\theta\|^q_{q,(t_0-2,t_0+2)\times\Omega} + \|g'\|^q_\infty\,\|P\phi\|^q_{q,(t_0-2,t_0+2)\times\Omega}).$$

Applying these inequalities with $t_0 = 2, 4, ..., 2k, ...$ and summing over k, we obtain

$$\begin{aligned}
&\|\Delta\phi\|^q_{q,(1,\infty)\times\Omega} \\
&\leq 2^{2q-1}C_1(q,d)^q(\|\theta\|^q_{q,(0,2)\times\Omega} + \|g'\|^q_\infty\,\|P\phi\|^q_{q,(0,2)\times\Omega} + 2\,\|\theta\|^q_{q,(2,\infty)\times\Omega} + 2\,\|g'\|^q_\infty\,\|P\phi\|^q_{q,(2,\infty)\times\Omega}).
\end{aligned}$$

Appealing to Lemma 3.8 on the time interval (0,1) and noting that $C(q,d,1)$ can be chosen so that $C(q,d,1) \leq C(q,d)$ we have

$$\begin{gathered}
\|\Delta\phi\|^q_{q,(0,1)\times\Omega} \leq C_1(q,d)^q\,\|\theta\|^q_{q,(0,1)\times\Omega} \leq \\
2^{2q-1}C_1(q,d)^q(\|\theta\|^q_{q,(0,2)\times\Omega} + \|g'\|^q_\infty\,\|P\phi\|^q_{q,(0,2)\times\Omega})
\end{gathered}$$

and thus

$$\|\Delta\phi\|^q_{q,(0,\infty)\times\Omega} \leq 2^{2q}C_1(q,d)^q(\|\theta\|^q_{q,(0,\infty)\times\Omega} + \|g'\|^q_\infty\,\|P\phi\|^q_{q,(0,\infty)\times\Omega})$$

By virtue of (3.6)

$$\|\Delta\phi\|_{q,(0,\infty)\times\Omega}^{q} \leq (2^{2q}C_1(q,d)^q(1+\|g'\|_\infty^q M^q\delta^{-q}))^{1/q}\,\|\theta\|_{q,(0,\infty)\times\Omega}\,.$$

Applying this estimate with $\theta = 0$ outside of (τ, T) yields the required result. □

Lemma 3.10 *If $d \geq 1$ and ϕ solves (3.4) , then there exists a constant, $\overline{C}(q)$, depending only on q such that*

$$\|P\phi\|_{q,(\tau,T)\times\Omega}\,, \|\Delta\phi\|_{q,(\tau,T)\times\Omega} \leq \tfrac{\overline{C}(q)}{d}\,\|\theta\|_{q,(\tau,T)\times\Omega}$$

and

$$\|\phi_t\|_{q,(\tau,T)\times\Omega} \leq \overline{C}(q)\,\|\theta\|_{q,(\tau,T)\times\Omega}$$

Proof:

Define $w(t,x) = \phi(\frac{t}{d},x)$. Then w satisfies

$$\begin{array}{ll} w_t = \Delta w + \frac{1}{d}\widehat{\theta} & t\in(d\tau,dT), x\in\Omega \\ \frac{\partial w}{\partial\eta} = 0 & t\in(d\tau,dT), x\in\partial\Omega \\ w = 0 & t = d\tau, x\in\Omega \end{array} \tag{3.7}$$

where $\widehat{\theta}(t,x) = \theta(\frac{t}{d},x)$. From Lemma 3.9, there exists a $\widetilde{C}(q)$ so that

$$\|\Delta w\|_{q,(d\tau,dT)\times\Omega} \leq \widetilde{C}(q)\left\|\tfrac{1}{d}\widehat{\theta}\right\|_{q,(d\tau,dT)\times\Omega}.$$

This implies

$$\int_{d\tau}^{dT}\int_\Omega |\Delta w(t,x)|^q\,dxdt \leq \left(\tfrac{\widetilde{C}(q)}{d}\right)^q \int_{d\tau}^{dT}\int_\Omega \left|\widehat{\theta}(t,x)\right|^q dxdt$$

or equivalently

$$\int_{d\tau}^{dT}\int_\Omega |\Delta\phi(t/d,x)|^q\,dxdt \leq \left(\tfrac{\widetilde{C}(q)}{d}\right)^q \int_{d\tau}^{dT}\int_\Omega |\theta(t/d,x)|^q\,dxdt.$$

Making a change of variables gives us

$$d\int_{d\tau}^{dT}\int_{\Omega}|\Delta\phi(s,x)|^q\,dxds \le \frac{\widetilde{C}(q)^q}{d^{q-1}}\int_{d\tau}^{dT}\int_{\Omega}|\theta(s,x)|^q\,dxds$$

and we have

$$\int_{d\tau}^{dT}\int_{\Omega}|\Delta\phi(s,x)|^q\,dxds \le \left(\frac{\widetilde{C}(q)}{d}\right)^q\int_{d\tau}^{dT}\int_{\Omega}|\theta(s,x)|^q\,dxds.$$

$$\|\Delta\phi\|_{q,(\tau,T)\times\Omega} \le \frac{\widetilde{C}(q)}{d}\|\theta\|_{q,(\tau,T)\times\Omega}$$

The same argument gives the estimate for $P\phi$. To obtain the estimate on the time derivative, note that

$$\begin{aligned}&\|\phi_t\|_{q,(\tau,T)\times\Omega} = \|d\Delta\phi+\theta\|_{q,(\tau,T)\times\Omega}\\ &\le \|d\Delta\phi\|_{q,(\tau,T)\times\Omega} + \|\theta\|_{q,(\tau,T)\times\Omega} \le (\widetilde{C}(q)+1)\,\|\theta\|_{q,(\tau,T)\times\Omega}\end{aligned}$$

Setting $\overline{C}(q) = \widetilde{C}(q)+1$ gives the desired result. □

We also need an estimate for the scalar equation

$$\begin{cases} \chi_t = d\Delta\chi & t\in(d\tau,dT), x\in\Omega\\ \frac{\partial\chi}{\partial\eta} = 0 & t\in(d\tau,dT), x\in\partial\Omega\\ \chi = \widehat{\chi} & t = d\tau, x\in\Omega. \end{cases} \tag{3.8}$$

Lemma 3.11 *Let $d \ge 1, q\in(1,\infty)$ and $\widehat{\chi}\in W_q^{2-2/q}(\Omega)$. Suppose χ solves (3.8). Then there exists a constant $\overline{K}(q)$ so that*

$$\|P\chi\|_{q,(\tau,T)\times\Omega}, \|\Delta\chi\|_{q,(\tau,T)\times\Omega}, \|\chi_t\|_{q,(\tau,T)\times\Omega} \le \overline{K}(q)\,\|\widehat{\chi}\|_{q,\Omega}^{2-2/q}.$$

Proof:

Define $\mu(t,x) = \chi(\frac{t}{d},x)$. Then μ satisfies

$$\begin{cases} \mu_t = \Delta\mu & t \in (d\tau, dT), x \in \Omega \\ \frac{\partial\mu}{\partial\eta} = 0 & t \in (d\tau, dT), x \in \partial\Omega \\ \mu = \widehat{\chi} & t = d\tau, x \in \Omega. \end{cases} \tag{3.9}$$

We assume without loss of generality that d is sufficiently large so that $dT \geq d\tau + 1$. We know from Theorem 3.1 that there exists a constant $C_1(q)$, depending only on q so that

$$\|\mu\|_{q,(d\tau,d\tau+1)\times\Omega}^{(1,2)} \leq C_1(q)\, \|\widehat{\chi}\|_{q,\Omega}^{2-2/q} \tag{3.10}$$

Now let $\phi \in C^1([d\tau, dT], [0,1])$ be such that

$$\phi(t) = \begin{cases} 0 & t = d\tau \\ 1 & t \geq d\tau + 1 \end{cases}$$

Note that there exists an $M > 0$ so that

$$|\phi'(t)| \leq M \qquad \text{for all } t \in [d\tau, dT].$$

Define $v(t,x) = \phi(t)\mu(t,x)$ and $w(t,x) = (1 - \phi(t))\mu(t,x)$. Then v satisfies

$$\begin{cases} v_t = \Delta v + \phi'(t)\mu & t \in (d\tau, dT), x \in \Omega \\ \frac{\partial v}{\partial\eta} = 0 & t \in (d\tau, dT), x \in \partial\Omega \\ v = 0 & t = d\tau, x \in \Omega. \end{cases} \tag{3.11}$$

and w satisfies

$$\begin{cases} w_t = \Delta w - \phi'(t)\mu & t \in (d\tau, d\tau + 1), x \in \Omega \\ \frac{\partial w}{\partial\eta} = 0 & t \in (d\tau, d\tau + 1), x \in \partial\Omega \\ w = \widehat{\chi} & t = d\tau, x \in \Omega. \end{cases} \tag{3.12}$$

Moreover, note that

$i.\ \mu = v + w$ for $t \in [d\tau, d\tau + 1]$

$ii.\ \mu = v$ for $t \in [d\tau, dT]$

$iii. \phi' = 0$ for $t \in [d\tau + 1, dT]$.

By Lemma 3.10 , there exists a constant $C_2(q)$ so that

$$\|Pv\|_{q,(d\tau,dT)\times\Omega}, \|v_t\|_{q,(d\tau,dT)\times\Omega}, \|\Delta v\|_{q,(d\tau,dT)\times\Omega} \leq C_2(q) \|\phi'\mu\|_{q,(d\tau,dT)\times\Omega}$$
$$\leq MC_2(q) \|\mu\|_{q,(d\tau,d\tau+1)\times\Omega}$$

Appealing to (3.10), we have

$$\|Pv\|_{q,(d\tau,dT)\times\Omega}, \|v_t\|_{q,(d\tau,dT)\times\Omega}, \|\Delta v\|_{q,(d\tau,dT)\times\Omega} \leq MC_1(q)C_2(q) \|\widehat{\chi}\|_{q,\Omega}^{2-2/q}$$

Again, by virtue of Theorem 3.1, there exists $C_3(q)$ so that

$$\|w\|^{(1,2)}_{q,(d\tau,dT)\times\Omega} \leq C_3(q)(\|\phi'\mu\|_{q,(d\tau,d\tau+1)\times\Omega} + \|\widehat{\chi}\|_{q,\Omega}^{2-2/q})$$

which implies

$$\|w\|^{(1,2)}_{q,(d\tau,dT)\times\Omega} \leq C_3(q)(MC_1(q) + 1) \|\widehat{\chi}\|_{q,\Omega}^{2-2/q}$$

and hence

$$\|Pw\|^{(1,2)}_{q,(d\tau,d\tau+1)\times\Omega} \leq 2C_3(q)(MC_1(q) + 1) \|\widehat{\chi}\|_{q,\Omega}^{2-2/q}$$

It then follows that

$$\|\Delta v\|_{q,(d\tau,dT)\times\Omega} = \left(\int\limits_{d\tau}^{d\tau+1} \int\limits_{\Omega} |\Delta v + \Delta w|^q \, dxdt + \int\limits_{d\tau+1}^{dT} \int\limits_{\Omega} |\Delta v|^q \, dxdt \right)^{1/q}$$
$$\leq \|\Delta v + \Delta w\|_{q,(d\tau,d\tau+1)\times\Omega} + \|\Delta v\|_{q,(d\tau+1,dT)\times\Omega}$$
$$\leq \|\Delta w\|_{q,(d\tau,d\tau+1)\times\Omega} + 2 \|\Delta v\|_{q,(d\tau,dT)\times\Omega}$$
$$\leq (C_3(q)(MC_1(q) + 1) + 2MC_1(q)C_2(q)) \|\widehat{\chi}\|_{q,\Omega}^{2-2/q}$$

Setting $\overline{K}(q) = 2C_3(q)(MC_1(q)+1) + 2MC_1(q)C_2(q)$, we have

$$\|\Delta\mu\|_{q,(d\tau,dT)\times\Omega} \leq \overline{K}(q)\, \|\widehat{\chi}\|_{q,\Omega}^{2-2/q} \tag{3.13}$$

In terms of χ, this is equivalent to

$$\left(\int_{d\tau}^{dT}\int_{\Omega} |\Delta\chi(t/d,x)|^q \, dxdt\right)^{1/q} \leq \overline{K}(q)\, \|\widehat{\chi}\|_{q,\Omega}^{2-2/q}$$

and hence

$$\|\Delta\chi\|_{q,(\tau,T)\times\Omega} \leq \overline{K}(q)\, \|\widehat{\chi}\|_{q,\Omega}^{2-2/q} \tag{3.14}$$

since $d \geq 1$. The same analysis provides the estimate for $\|P\chi\|_{q,(\tau,T)\times\Omega}$ and $\|\chi_t\|_{q,(\tau,T)\times\Omega}$. □

We now provide some results regarding equivalent norms on some Sobolev spaces on which we will be working.

Lemma 3.12 *Let $X = \{u \in W_p^2(\Omega) : \frac{\partial u}{\partial \eta} = 0$ on $\partial\Omega\}$. Then $\|\Delta u\|_{p,\Omega} + \|u\|_{p,\Omega}$ defines a norm equivalent to the standard $\|\cdot\|_{p,\Omega}^{(2)}$ norm on X.*

Proof:

Clearly, there exists a C_1 such that

$\|\Delta u\|_{p,\Omega} + \|u\|_{p,\Omega} \leq C_1 \|u\|_{p,\Omega}^{(2)}$ for all $u \in X$. On the other hand, for $u \in X$ we have that u solves

$$\begin{aligned} -\Delta u + u = f \quad & x \in \Omega \\ \frac{\partial u}{\partial \eta} = 0 \quad & x \in \partial\Omega \end{aligned}$$

for some $f \in L^p(\Omega)$. From standard elliptic regularity results , we have

$$\|u\|_{p,\Omega}^{(2)} \leq C_2 \|f\|_{p,\Omega} = C_2 \|-\Delta u + u\|_{p,\Omega} \leq C_2(\|\Delta u\|_{p,\Omega} + \|u\|_{p,\Omega}).$$

$\square$

Lemma 3.13 *Let* $X = \{u \in W_p^2(\Omega) : \frac{\partial u}{\partial \eta} = 0$ *on* $\partial\Omega\}$. *Then* $\|\Delta u\|_{p,\Omega} + \|u\|_{1,\Omega}$ *defines a norm equivalent to the standard norm on* X.

Proof:

By virtue of Lemma 3.12, it suffices to show that $\|\Delta u\|_{p,\Omega} + \|u\|_{1,\Omega}$ gives a norm equivalent to the norm defined therein. Clearly, since Ω is bounded, there exists a k_1 so that

$$\|\Delta u\|_{p,\Omega} + \|u\|_{1,\Omega} \leq k_1(\|\Delta u\|_{p,\Omega} + \|u\|_{p,\Omega}). \tag{3.15}$$

To obtain the inequality in the other direction, it suffices to show that there exists a k_2 such that

$$\|u\|_{p,\Omega} \leq k_2(\|\Delta u\|_{p,\Omega} + \|u\|_{1,\Omega})$$

Suppose by way of contradiction that this does not hold. Then there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset X$ so that

$\|u\|_{p,\Omega} > n(\|\Delta u_n\|_{p,\Omega} + \|u_n\|_{1,\Omega})$ for all $n \in N$.

Define a new sequence $\{v_n\}$, by $v_n = \frac{u_n}{\|u_n\|}$ and note that

$1 = \|v_n\|_{p,\Omega} > n(\|\Delta v_n\|_{p,\Omega} + \|v_n\|_{1,\Omega})$ for all $n \in N$. Clearly this implies that $\|\Delta v_n\|_{p,\Omega}, \|v_n\|_{1,\Omega} \to 0$ as $n \to \infty$. In particular, $\{v_n\}_{n=1}^{\infty}$ is a bounded sequence in X. Since $W_p^2(\Omega)$ is compactly imbedded in $L^p(\Omega)$ [2], there must be a subsequence $\{v_{n_j}\}$ which converges in $L^p(\Omega)$. Let $v \in L^p(\Omega)$ be such that $v_{n_j} \to v$. Then v satisfies $\|v\|_{p,\Omega} = 1$ and $\|v\|_{p,\Omega} = 0$, which is a contradiction. Therefore the result follows. $\square$

Lemma 3.14 *Let* $\widehat{X} = \{u \in W_p^{1,2}(\Omega) : \frac{\partial u}{\partial \eta} = 0$ *on* $(\tau, T) \times \partial\Omega\}$*. Then* $\|u_t\|_{p,\Omega} + \|u\|_{p,\Omega} + \|\Delta u\|_{p,\Omega}$ *defines a norm equivalent to the standard norm on* $\widehat{X}$*.*

Proof:

Easily, there exists a k_1 so that

$$\|u_t\|_{p,\Omega} + \|u\|_{p,\Omega} + \|\Delta u\|_{p,\Omega} \leq k_1 \|u\|_{p,\Omega}^{(1,2)}.$$

In order to obtain the inequality in the other direction, fix $t \in (\tau, T)$. Then $u(t, \cdot) \in W_p^2(\Omega)$ and satisfies $\frac{\partial u}{\partial \eta} = 0$ for $x \in \partial\Omega$. Consequently, from Lemma 3.12, we have

$$\|u(t,\cdot)\|_{p,\Omega}^{(2)} \leq k_2(\|\Delta u(t,\cdot)\|_{p,\Omega} + \|u(t,\cdot)\|_{p,\Omega})$$

with k_2 independent of u. This yields

$$\int_\tau^T \int_\Omega \sum_{|\alpha| \leq 2} \left| D^{(0,\alpha)} u \right|^p dxdt \leq \widehat{k}_2 \left(\int_\tau^T \int_\Omega (|\Delta u|^p + |u|^p) dxdt \right)$$

and it follows that

$$\int_\tau^T \int_\Omega |u_t|^p \, dxdt + \int_\tau^T \int_\Omega \sum_{|\alpha| \leq 2} \left| D^{(0,\alpha)} u \right|^p dxdt \leq \widetilde{k}_2 \left(\int_\tau^T \int_\Omega (|\Delta u|^p + |u|^p + |u_t|^p) dxdt \right).$$

Thus giving the desired result. □

We close this chapter with some estimates that we will use directly in the proofs of Theorems 2.6 and 2.9.

Lemma 3.15 *Suppose that* $d \geq 1$ *and that* ϕ *solves (3.1) with* $T - \tau > 1/2$*. Then there exists a* $t \in (\tau, T)$ *and a constant* $C(q)$ *depending only on* q *so that*

$$\|\Delta\phi(t,\cdot)\|_{q,\Omega} \leq C(q) \left(\|\theta\|_{q,(\tau,T)\times\Omega} + \|\phi_0\|_{q,\Omega}^{2-2/q} \right).$$

Proof:

Note that $\phi = v + w$, where v solves

$$\begin{cases} v_t = d\Delta v + \theta & t \in (\tau, T), x \in \Omega \\ \frac{\partial v}{\partial \eta} = 0 & t \in (\tau, T), x \in \partial\Omega \\ v = 0 & t = \tau, x \in \Omega. \end{cases} \tag{3.16}$$

and w solves

$$\begin{cases} w_t = d\Delta w & t \in (\tau, T), x \in \Omega \\ \frac{\partial w}{\partial \eta} = 0 & t \in (\tau, T), x \in \partial\Omega \\ w = \phi_0 & t = \tau, x \in \Omega. \end{cases} \tag{3.17}$$

Lemmas 3.10 and 3.11 imply that

$$\begin{aligned} &\|\Delta\phi\|_{q,(\tau,T)\times\Omega} = \|\Delta v + \Delta w\|_{q,(\tau,T)\times\Omega} \leq \|\Delta v\|_{q,(\tau,T)\times\Omega} + \|\Delta w\|_{q,(\tau,T)\times\Omega} \\ &\leq \overline{C}(q)\, \|\theta\|_{q,(\tau,T)\times\Omega} + \overline{K}(q)\, \|\phi_0\|_{q,\Omega}^{2-2/q} \leq \widehat{C}(q) \left(\|\theta\|_{q,(\tau,T)\times\Omega} + \|\phi_0\|_{q,\Omega}^{2-2/q} \right) \end{aligned}$$

where $\widehat{C}(q) = max\{\overline{C}(q), \overline{K}(q)\}$. The Mean Value Theorem together with the fact the $T - \tau > 1/2$ implies that there exists a $t \in (\tau, T)$ so that

$$\|\Delta\phi(t,\cdot)\|_{q,\Omega} \leq 2\widehat{C}(q) \left(\|\theta\|_{q,(\tau,T)\times\Omega} + \|\phi_0\|_{q,\Omega}^{2-2/q} \right).$$

Setting $C(q) = 2\widehat{C}(q)$ gives the desired result. □

Lemma 3.16 *Suppose that $d \geq 1$ and that ϕ solves (3.1). Then there exists a constant $K(q)$ depending only on q so that*

$$\|P\phi\|_{q,(\tau,T)\times\Omega}^{(1,2)} \leq K(q) \left(\|\theta\|_{q,(\tau,T)\times\Omega} + \|\phi_0\|_{q,\Omega}^{2-2/q} \right)$$

Proof:

Proceeding as in Lemma 3.15, we have that $\phi = v + w$ where v and w solve (3.9) and (3.10), respectively. By virtue of Lemmas 3.10 and 3.11, we have

$$\begin{aligned}\|P\phi\|_{q,(\tau,T)\times\Omega} &= \|Pv+Pw\|_{q,(\tau,T)\times\Omega} \le \|Pv\|_{q,(\tau,T)\times\Omega} + \|Pw\|_{q,(\tau,T)\times\Omega}\\ &\le \overline{C}(q)\,\|\theta\|_{q,(\tau,T)\times\Omega} + K(q)\,\|\phi_0\|_{q,\Omega}^{2-2/q} \le K_1(q)(\|\theta\|_{q,(\tau,T)\times\Omega} + \|\phi_0\|_{q,\Omega}^{2-2/q})\end{aligned}$$

where $K_1(q) = max\{\overline{C}(q), \overline{K}(q)\}$. Similarly, we obtain

$$\|\Delta\phi\|_{q,(\tau,T)\times\Omega}\,, \|\phi_t\|_{q,(\tau,T)\times\Omega} \le K_1(q)(\|\theta\|_{q,(\tau,T)\times\Omega} + \|\phi_0\|_{q,\Omega}^{2-2/q}).$$

Noting that $\Delta\phi = \Delta(P\phi)$ and that $\|(P\phi)_t\|_{q,(\tau,T)\times\Omega} \le 2\,\|\phi_t\|_{q,(\tau,T)\times\Omega}$, we have that

$$\|P\phi_t\|_{q,(\tau,T)\times\Omega} + \|P\phi\|_{q,(\tau,T)\times\Omega} + \|\Delta(P\phi)\|_{q,(\tau,T)\times\Omega} \le 4K_1(q)(\|\theta\|_{q,(\tau,T)\times\Omega} + \|\phi_0\|_{q,\Omega}^{2-2/q}).$$

From Lemma 3.14, there exists a $\tilde{c}$ so that

$$\|P\phi\|_{q,(\tau,T)\times\Omega}^{(1,2)} \le \tilde{c}\left(\|P\phi_t\|_{q,(\tau,T)\times\Omega} + \|P\phi\|_{q,(\tau,T)\times\Omega} + \|\Delta(P\phi)\|_{q,(\tau,T)\times\Omega}\right)$$

and the result follows with $K(q) = 4\tilde{c}K_1(q)$. $\square$

We will also need the following standard algebraic estimate.

Lemma 3.17 *Suppose that $K, L, y \ge 0$ and $0 < \epsilon < 1$. If $y \le K + Ly^\epsilon$, then*

$$y < \tfrac{K}{1-\epsilon} + L^{\frac{1}{1-\epsilon}}.$$

Proof:

The result is clear if $L = 0$. Suppose $L > 0$, and for each $K \ge 0$, define $u(K)$ to be the unique positive solution of $u = K + Lu^\epsilon$. Then $y \le u(K)$ and $u'(K) = \frac{u(K)}{(1-\epsilon)u(K)+\epsilon K} < \frac{1}{1-\epsilon}$. Consequently,

$$\begin{aligned} y &\le u(K)\\ &< \tfrac{K}{1-\epsilon} + u(0)\\ &= \tfrac{K}{1-\epsilon} + L^{\frac{1}{1-\epsilon}}.\end{aligned}$$

□

The following two lemmas will be used in the proof of Theorem 2.9. These proofs are well known [4], but will be included for the sake of completeness.

Lemma 3.18 *Let $w \in H^2(\Omega)$ with $\frac{\partial w}{\partial \eta} = 0$ on $\partial\Omega$. Then*

$$\|\Delta w\|_{2,\Omega}^2 \geq \lambda_1 \|\nabla w\|_{2,\Omega}^2$$

where λ_1 is the smallest positive eigenvalue of $-\Delta$ subject to Neumann boundary conditions.

Proof:

Let $\{\phi_k\}_{k=0}^{\infty}$ be a complete set of orthonormal eigenfunctions in $L^2(\Omega)$ of $-\Delta$ with homogeneous Neumann boundary conditions. Let $\{\lambda_k\}_{k=0}^{\infty}$ be the corresponding set of eigenvalues listed in increasing order. We can write w as

$$w = \sum_{k=0}^{\infty} w_k \phi_k$$

and hence

$$\Delta w = \sum_{k=0}^{\infty} -\lambda_k w_k \phi_k.$$

Using integration by parts, we have

$$\|\nabla w\|_{2,\Omega}^2 = \int_{\Omega} |\nabla w|^2 \, dx = \int_{\partial\Omega} w \frac{\partial w}{\partial \eta} d\sigma - \int_{\Omega} w \Delta w dx = -\int_{\Omega} w \Delta w dx = \sum_{k=1}^{\infty} \lambda_k w_k^2$$

It now follows that

$$\|\Delta w\|_{2,\Omega}^2 = \sum_{k=1}^{\infty} \lambda_k^2 w_k^2 \geq \lambda_1 \sum_{k=1}^{\infty} \lambda_k w_k^2 \geq \lambda_1 \|\nabla w\|_{2,\Omega}^2.$$

□

Lemma 3.19 *Let $w \in H^2(\Omega)$ with $\frac{\partial w}{\partial \eta} = 0$ on $\partial\Omega$. Then*

$$\|\nabla w\|_{2,\Omega}^2 \geq \lambda_1 \|w - \overline{w}\|_{2,\Omega}^2$$

where $\overline{w} = |\Omega|^{-1} \int\limits_{\Omega} w dx$.

Proof:

Let $\{\phi_k\}_{k=0}^{\infty}$ be a complete set of orthonormal eigenfunctions in $L^2(\Omega)$ of $-\Delta$ with homogeneous Neumann boundary conditions. Let $\{\lambda_k\}_{k=0}^{\infty}$ be the corresponding set of eigenvalues listed in increasing order. We can write w as

$$w = \sum_{k=0}^{\infty} w_k \phi_k$$

and hence

$$\|\nabla w\| = \sum_{k=0}^{\infty} \lambda_k w_k^2.$$

ϕ_0 is a constant function associated with λ_0. Hence it follows that

$$\overline{w} = |\Omega|\,.^{-1} \int\limits_{\Omega} w dx = w_0 \phi_0$$

We now have that

$$w - \overline{w} = \sum_{k=1}^{\infty} \lambda_k w_k$$

and

$$\|w - \overline{w}\|_{2,\Omega}^2 = \sum_{k=1}^{\infty} w_k^2.$$

It then follows that

$$\|\nabla w\| = \sum_{k=0}^{\infty} \lambda_k w_k^2 \geq \lambda_1 \sum_{k=1}^{\infty} w_k^2 = \lambda_1 \|w - \overline{w}\|_{2,\Omega}^2 .$$

$\square$

CHAPTER IV

MAIN RESULTS

A. Proof of Theorem 2.3

We begin by restating (2.1) for convenience as

$$\begin{cases} u_t = D\Delta u + f(t,x,u) & t > 0, x \in \Omega \\ \partial u_k/\partial \eta = 0 & t > 0, x \in \partial\Omega_{\sigma(k)} \quad k = 1, ...m \\ u_k(0,\cdot) = u_{0_k(\cdot)} & t = 0, x \in \overline{\Omega}_{\sigma(k)} \quad k = 1, ...m \end{cases} \tag{4.1}$$

Recall the truncated system associated with (4.1). Let $r > 0$ and define $\Phi_r \in C^\infty(R^m, [0,1])$ via

$$\Phi_r(u) = \begin{cases} 1, & u \in B_r(0) \\ 0, & u \notin B_{2r}(0) \end{cases} \tag{4.2}$$

The truncated system is given by

$$\begin{cases} u_t = D\Delta u + \widehat{f}(t,x,u) & t > 0, x \in \Omega_{\sigma(k)} \quad k = 1, ...m \\ \partial u_k/\partial \eta = 0 & t > 0, x \in \partial\Omega_{\sigma(k)} \quad k = 1, ...m \\ u_k(0,\cdot) = u_{0_k(\cdot)} & t = 0, x \in \overline{\Omega}_{\sigma(k)} \quad k = 1, ...m \end{cases} \tag{4.3}$$

where $\widehat{f_k}(t,x,u) = \Phi_r(u) f_k(t,x,u)$

We establish that this system has at least one solution. To this end we recast the system as a fixed point problem and apply Shauder's fixed point theorem. Define $T_{r,\tau} : C(\overline{Q}_\tau) \to C(\overline{Q}_\tau)$ via $T_{r,\tau}(v) = u$ where u solves

$$\begin{cases} u_t = D\Delta u + \widehat{f}(t,x,v) & t > 0, x \in \Omega \\ \partial u_k/\partial \eta = 0 & t > 0, x \in \partial\Omega_{\sigma(k)} \quad k = 1, ...m \\ u_k(0,\cdot) = u_{0_k(\cdot)} & t = 0, x \in \overline{\Omega}_{\sigma(k)} \quad k = 1, ...m \end{cases} \tag{4.4}$$

We show that this mapping is well-defined. Note that $v \in C(\overline{\Omega}, R^m)$ implies there exists a constant $K_{r,\tau} > 0$ such that $\left|\widehat{f}(t,x,v)\right| \leq K_{r,\tau}$. This gives us that $\widehat{f}(t,x,v) \in L^p(\overline{Q}_\tau)$ for all $1 \leq p < \infty$.

We get the existence of $u \in W_p^{1,2}(\overline{Q}_\tau)$ for all $1 \leq p < \infty$ by [19]. Note that if p is sufficiently large $W_p^{1,2}(\overline{Q}_\tau)$ imbeds compactly into $C(\overline{Q}_\tau)$. Therefore, $T(v)_{r,\tau} \in C(\overline{Q}_\tau)$, and we have $T_{r,\tau}$ is well defined.

We will now show that $T_{r,\tau}$ is continuous. To this end, let $z = T_{r,\tau}(v) - T_{r,\tau}(w)$ for $v, w \in C(\overline{Q_\tau})$. Then z solves

$$\begin{cases} (z_k)_t = d_k\Delta z_k + \widehat{f}_k(t,x,v) - \widehat{f}_k(t,x,w) & t > 0, x \in \Omega_{\sigma(k)} \quad k = 1, ...m \\ \partial z_k/\partial \eta = 0 & t > 0, x \in \partial\Omega_{\sigma(k)} \quad k = 1, ...m \\ z_k(0,x) = 0 & t = 0, x \in \overline{\Omega}_{\sigma(k)} \quad k = 1, ...m \end{cases} \tag{4.5}$$

From [19] there exists a constant $L_{p,\tau}$ such that

$$\|z_k\|_{W_p^{1,2}(Q_\tau)} \leq L_{p,\tau} \left\|\widehat{f}_k(\cdot,\cdot,v) - \widehat{f}_k(\cdot,\cdot,w)\right\|_{p,Q_\tau} \leq a_{\max} L_{p,\tau} \|v - w\|_{p,Q_\tau} \tag{4.6}$$

So,

$$\|z_k\|_{W_p^{1,2}(Q_\tau)} \leq a_{\max} L_{p,\tau} \left\|\Omega_{\sigma(k)}\right\|^{1/p} \tau^{1/p} \|v - w\|_{\infty,Q_\tau} \tag{4.7}$$

Applying the Sobolev embedding theorem for p large enough [19] there exists $C_{p,\tau,\Omega}$ such that

$$\|z_k\|_\infty \le C_{p,\tau,\Omega} \|z_k\|_{W_p^{1,2}(Q_\tau)} \tag{4.8}$$

This gives us

$$\|T_{r,\tau}(v) - T_{r,\tau}(w)\|_\infty = \|z_k\|_\infty \le \widetilde{C} \|v - w\|_{\infty, Q_\tau} \tag{4.9}$$

So, $T_{r,\tau}$ is continuous. Furthermore, the $W_p^{1,2}(\overline{Q_\tau})$ estimate for u and (4.6) imply $T_{r,\tau}$ is compact. Finally, note that the truncation implies that $[-2r, 2r]^m = B_{2r}$ is an invariant m-rectange for (4.3). Therefore, $T_{r,\tau} : C(\overline{Q_\tau}, B_{2r}) \to C(\overline{Q_\tau}, B_{2r})$ and the Shauder fixed point theorem gives us that there exists a $u \in C(\overline{Q_\tau}, B_{2r})$ such that $T_{r,\tau}(u) = u$.

We denote that solution u to (4.3) by $u^{(r)}$. Now choose $r > \|u_0\|_\infty$. By continuity there exists an $\epsilon_r > 0$ such that $\|u(t,\cdot)\|_\infty \le r$ for every $t \in [0, \epsilon_r]$. Note that $u^{(r)}$ solves (4.1) for $[0, \epsilon_r]$. As a result, we have a local solution to (4.1).

We must now show that the solution to (4.1) on Q_{ϵ_r} is unique. Suppose that $w \in C(\overline{Q_\tau}, R^m)$ solves (4.1) on Q_{ϵ_r}.
Let $\phi = u^{(r)} - w$ and $\theta = f(\cdot,\cdot,u^{(r)}) - f(\cdot,\cdot,w)$. Then ϕ solves

$$\begin{cases} (\phi_k)_t = d_k \Delta \phi_k + \theta & t > 0, x \in \Omega_{\sigma(k)} \quad k = 1, ...m \\ \partial\phi_k / \partial\eta = 0 & t > 0, x \in \partial\Omega_{\sigma(k)} \quad k = 1, ...m \\ \phi_k(0, x) = 0 & t = 0, x \in \overline{\Omega}_{\sigma(k)} \quad k = 1, ...m \end{cases} \tag{4.10}$$

Multiplying both sides by ϕ_k gives

$$\phi_k (\phi_k)_t = d_k \phi_k \Delta \phi_k + \phi_k \theta_k \tag{4.11}$$

Integrating both sides over $Q_{\tau,\Omega_{\sigma(k)}}$ gives

$$\int_0^t \int_{\Omega_{\sigma(k)}} \phi_k(\phi_k)_t dxdt = \int_0^t \int_{\Omega_{\sigma(k)}} d_k\phi_k\Delta\phi_k dxdt + \int_0^t \int_{\Omega_{\sigma(k)}} \phi_k\theta_k dxdt \tag{4.12}$$

As a result,

$$\frac{1}{2}\int_{\Omega_{\sigma(k)}} \phi_k^2 dx = -d_k \int_0^t |\nabla\phi_k|^2 \, dt + \int_0^t \int_{\Omega_{\sigma(k)}} \phi_k\theta_k dxdt \le \widetilde{K} \int_0^t \int_{\Omega_{\sigma(k)}} \phi_k^2 dxdt \tag{4.13}$$

Now define $y = \frac{1}{2} \int\limits_{\Omega_{\sigma(k)}} \phi_k^2 dx$.

$$y \le 2\widetilde{K} \int_0^t ydt$$

So, Gronwall's inequality gives us $y = 0$. Therefore, $u^{(r)} = w$. So, to finish we define

$$\epsilon_r = \sup\{\epsilon \, | \|u(t,\cdot)\|_\infty \le r \text{ for every } t \in [0,\epsilon]\}$$

and let

$$T_{\max} = \lim_{r\to\infty} \epsilon_r$$

The analysis above implies if $T_{\max} < \infty$ then solution to (4.1) blows up in finite time.

B. Invariance of R_+^m

In the introduction we mentioned that if the $f_i s$ are quasipositive, then R_+^m is invariant for (4.1). We demonstrate this below.

Consider the system (4.1) with $f(t,x,u)$ replaced by $f(t,x,u^+)$

$$\begin{cases} u_t = D\Delta u + f(t,x,u^+) & t > 0, x \in \Omega \\ \frac{\partial u_k}{\partial \eta} = 0 & t > 0, x \in \partial\Omega_{\sigma(k)} \quad k = 1,...,m \\ u_k(0,\cdot) = u_{0_k} & t = 0, x \in \overline{\Omega}_{\sigma(k)} \quad k = 1,...m \end{cases} \tag{4.14}$$

where $u^+ = max\{u, 0\}$, $u^- = -min\{u, 0\}$ and $f(t, x, u^+)$ is continuous and is locally Lipschitz in its third argument uniformly for bounded x and t. Theorem 2.3 guarantees there exists a unique solution of (4.14). Multiplying the k^{th} component of (4.14) and integrating over $(0, t) \times \Omega_{\sigma(k)}$ we obtain

$$\int_0^t \int_{\Omega_{\sigma(k)}} u_k^-(u_k)_t dxdt = \int_0^t \int_{\Omega_{\sigma(k)}} d_k u_k^- \Delta u_k dxdt + \int_0^t \int_{\Omega_{\sigma(k)}} u_k^- f_k(u^+) dxdt$$

Note that

$$(u_k)_t = -(u_k^-)_t$$

and

$$\Delta u_k = -\Delta u_k^-$$

whenever $u_i^- > 0$.

Integrating the equation above by parts yields

$$-\frac{1}{2} \int_{\Omega_{\sigma(k)}} (u_k^-)^2 dx = d_k \int_0^t \int_{\Omega_{\sigma(k)}} \left|\nabla u_k^-\right|^2 dxdt + \int_0^t \int_{\Omega_{\sigma(k)}} u_k^- f_k(u^+) dxdt$$

Since f_k is quasipositive,

$$u_k^- f_k(u^+) = \begin{cases} 0 & \text{if } u \geq 0 \\ \geq 0 & \text{if } u < 0 \end{cases}$$

This gives us

$$-\frac{1}{2} \int_{\Omega_{\sigma(k)}} (u_k^-)^2 dx \geq d_k \int_0^t \int_{\Omega_{\sigma(k)}} \left|\nabla u_k^-\right|^2 dxdt$$

Hence $u_k^- = 0$, the desired result.

C. Proof of Theorem 2.5

We must first establish an a priori estimate for

$$\sum_{i=1}^{n_k} \int_0^{T_{\max}} u_{o_{k(i)}}(s,x)ds \tag{4.15}$$

We have $(u_i)_t = d_i \triangle u_i + f_i(t,x,u)$. Multiplting both sides by $a_{n_k,i}$ and summing we get

$$\sum_{i=1}^{n_k} a_{n_k,i}(u_{o_{k(i)}})_t \leq \triangle(\sum_{i=1}^{n_k} a_{n_k,i} d_i u_{o_{k(i)}}) + M_k(\sum_{i=1}^{n_k} u_{o_{k(i)}}) + N_k \tag{4.16}$$

Setting $d_{\max} = \max\{d_i\}$ and integrating both sides we get

$$\sum_{i=1}^{n_k} a_{n_k,i} u_{o_{k(i)}}(x,t) \leq N_k t +$$
$$d_{\max}\triangle(\int_0^t \sum_{i=1}^{n_k} a_{n_k,i} u_{o_{k(i)}}(x,s)ds) + \sum_{i=1}^{n_k} a_{n_k,i} u_{o_{k(i)}}(x,0) + \widetilde{M_k}(\int_0^t \sum_{i=1}^{n_k} a_{n_k,i} u_{o_{k(i)}}(x,s)ds)$$

where $\widetilde{M_k} = M_k \cdot \{\frac{1}{a_{n_k,o_{k(i)}} d_i}\}$.

Define $\varphi = \int_0^t \sum_{i=1}^{n_k} a_{n_k,i} u_{o_{k(i)}}(x,t)$ and we get

$$\varphi_t \leq d_{\max}\triangle\varphi + \widetilde{M_k}\varphi + \sum_{i=1}^{n_k} a_{n_k,i} u_{o_{k(i)}}(x,0) + N_k t \tag{4.17}$$

Suppose that Ψ is a solution to

$$\begin{cases} \Psi'(t) = M_k\Psi + \|v_0\|_\infty + N_k t \\ \Psi(0) = 0. \end{cases}$$

We can see that Ψ is an upperbound for φ and

$$\Psi(t) = e^{M_k t} \int_0^t e^{-M_k s}(\|v_0\|_\infty + N_k s)ds.$$

Therefore, φ is bounded for all bounded t.

Using the first row of $a_{ij}^{(k)}$ we have that and applying the intermediate sums condition we have

$$f_{o_{k(i)}}(x,t,u) \leq M_k \sum_{i=1}^{n_k} u_{o_{k(i)}} + N_k \tag{4.18}$$

Suppose that $v_{k,1}$ solves

$$\begin{cases} (v_{k,1})_t = d_k \Delta v_{k,1} + M_k \sum\limits_{i=1}^{n_k} u_{o_{k(i)}} + N_k - f_{o_{k(1)}} & t>0, x \in \Omega_{\sigma(k)} \\ \partial v_{k,1}/\partial\eta = 0 & t>0, x \in \partial\Omega_{\sigma(k)} \\ v_{k,1}(0,x) = 0 & t=0, x\in \overline{\Omega}_{\sigma(k)} \end{cases} \tag{4.19}$$

Note: $v_{k,1} \geq 0$

Consider $v_{k,1} + u_{k,1} = w_{k,1}$

$$\begin{cases} (w_{k,1})_t = d_k \Delta w_{k,1} + M_k \sum\limits_{i=1}^{n_k} u_{o_{k(i)}} + N_k & t>0, x \in \Omega_{\sigma(k)} \\ \partial w_{k,1}/\partial\eta = 0 & t>0, x \in \partial\Omega_{\sigma(k)} \\ w_{k,1}(0,x) = u_{k,1}(0,x) & t=0, x\in \overline{\Omega}_{\sigma(k)} \end{cases} \tag{4.20}$$

$$w_k(t,x) = d_k \Delta \int_0^t w_k + w_k(0,x) + \int_0^t (M_k \sum_{i=1}^{n_k} u_{o_{k(i)}}(s,x) + N_k) ds \tag{4.21}$$

Note: $\int\limits_0^t (M_k \sum\limits_{i=1}^{n_k} u_{o_{k(i)}}(s,x) + N_k) ds$ is bounded for bounded t.

This implies there exists a constant $L(d_k, t, p)$ such that

$$\|(w_{k,1})_t\|_{p,Q_t} \leq L \left\| w_{k,1}(0,x) + \int_0^t (M_k \sum_{i=1}^{n_k} u_{o_{k(i)}}(s,x) + N_k) ds \right\|_{p,Q_t} \tag{4.22}$$

Thus $\|w_{k,1}\|_{p,Q_t}$ is bounded.

Since $\|u_{k,1}\|_{p,Q_t} \leq \|w_{k,1}\|_{p,Q_t}$ we have that

$\|u_{k,1}\|_{p,Q_t}$ is bounded for every $p \geq 1$.

Recall $a^{(k)}_{2,1} f_{o_{k(1)}}(t,x,u) + a^{(k)}_{2,2} f_{o_{k(2)}}(t,x,u) \leq M_k \sum\limits_{i=1}^{n_k} u_{o_{k(i)}}(s,x) + N_k$

Suppose that $v_{k,2}$ solves

$$\begin{cases} (v_{k,2})_t = d_k \Delta v_{k,2} + M_k \sum\limits_{i=1}^{n_k} u_{o_{k(i)}} + N_k - a^{(k)}_{2,1} f_{o_{k(1)}} - a^{(k)}_{2,2} f_{o_{k(2)}} & t > 0, x \in \Omega_{\sigma(k)} \\ \partial v_{k,2}/\partial \eta = 0 & t > 0, x \in \partial\Omega_{\sigma(k)} \\ v_{k,2}(0,x) = 0 & t = 0, x \in \overline{\Omega}_{\sigma(k)} \end{cases}$$

Note: $v_{k,2} \geq 0$

We have

$$(\mathrm{a}^{(k)}_{2,1} u_{o_{k(1)}}(t,x,u) + a^{(k)}_{2,2} u_{o_{k(2)}}(t,x,u) + v_{k,2})_t =$$
$$\Delta(d_{o_{k(1)}} a^{(k)}_{2,1} u_{o_{k(1)}} + d_{o_{k(2)}} a^{(k)}_{2,2} u_{o_{k(2)}} + d_{o_{k(2)}} v_{k,2}) + M_k \sum\limits_{i=1}^{n_k} u_{o_{k(i)}}(s,x) + N_k$$

Setting

$$T_1 = M_k \sum_{i=1}^{n_k} u_{o_{k(i)}}(s,x) + N_k \tag{4.23}$$
$$T_2 = a^{(k)}_{2,1} u_{o_{k(1)}}(0,x,u) + a^{(k)}_{2,2} u_{o_{k(2)}}(0,x,u) \tag{4.24}$$
$$T_3 = (a^{(k)}_{2,1} \frac{d_{o_{k(1)}}}{d_{o_{k(2)}}} - 1) u_{o_{k(1)}} \tag{4.25}$$

we see that T_1 and T_2 are bounded and that T_3 has an $L^p(Q_{k,t})$ bound.

Choose $\phi = \int\limits_0^t (\frac{d_{o_{k(1)}}}{d_{o_{k(2)}}} a^{(k)}_{2,1} u_{o_{k(1)}} + a^{(k)}_{2,2} u_{o_{k(2)}} + v_{k,2})$. ϕ satisfies

$$\phi_t = d_{o_{k(2)}} \Delta\phi + T_1 + T_2 + T_3. \tag{4.26}$$

This gives us that $\left\| u_{o_{k(2)}} \right\|_{p,Q_{k,t}}$ is bounded for $p \geq 1$.

Recall that

$$\begin{cases} (u_{o_k(i)})_t = d_{o_k(i)}\Delta u_{o_k(i)} + f_{o_k(i)}(t,x,u) & t>0, x\in\Omega_{\sigma(k)} \\ \partial u_{o_k(i)}/\partial\eta = 0 & t>0, x\in\partial\Omega_{\sigma(k)} \quad k=1,...m \\ u_{o_k(i)}(0,\cdot) = u_{o_k(i)}(\cdot) & t=0, x\in\overline{\Omega}_{\sigma(k)} \quad k=1,...m \end{cases} \tag{4.27}$$

and $u_{o_k(i)} \leq \widetilde{u}_{o_k(i)}$ where $\widetilde{u}_{o_k(i)}$ solves

$$\begin{cases} (\widetilde{u}_{o_k(i)})_t = d_{o_k(i)}\Delta \widetilde{u}_{o_k(i)} + f_{o_k(i)}(t,x,\widetilde{u}) & t>0, x\in\Omega_{\sigma(k)} \\ \partial \widetilde{u}_{o_k(i)}/\partial\eta = 0 & t>0, x\in\partial\Omega_{\sigma(k)} \quad k=1,...m \\ \widetilde{u}_{o_k(i)}(0,\cdot) = \left\| u_{o_k(i)}(\cdot)\right\|_\infty & t=0, x\in\overline{\Omega}_{\sigma(k)} \quad k=1,...m \end{cases} \tag{4.28}$$

and $\widetilde{u}_{o_k(i)} = \Phi_{k,i} + \Psi_{k,i}$ where $\Phi_{k,i}$ solves

$$\begin{cases} (\Phi_{k,i})_t = d_{o_k(i)}\Delta \Phi_{k,i} & t>0, x\in\Omega_{\sigma(k)} \\ \partial \Phi_{k,i}/\partial\eta = 0 & t>0, x\in\partial\Omega_{\sigma(k)} \quad k=1,...m \\ \Phi_{k,i}(0,\cdot) = \left\| u_{o_k(i)}(\cdot)\right\|_\infty & t=0, x\in\overline{\Omega}_{\sigma(k)} \quad k=1,...m \end{cases} \tag{4.29}$$

and $\Psi_{k,i}$ solves

$$\begin{cases} (\Psi_{k,i})_t = d_{o_k(i)}\Delta \Psi_{k,i} + f_{o_k(i)}(t,x,\widetilde{u}) & t>0, x\in\Omega_{\sigma(k)} \\ \partial \Psi_{k,i}/\partial\eta = 0 & t>0, x\in\partial\Omega_{\sigma(k)} \quad k=1,...m \\ \Psi_{k,i}(0,\cdot) = 0 & t=0, x\in\overline{\Omega}_{\sigma(k)} \quad k=1,...m \end{cases} \tag{4.30}$$

Note that the maximum principle imples that $\Phi_{k,i}$ is bounded by $\left\| u_{o_k(i)}(\cdot)\right\|_\infty$. Also, $\|\Psi_{k,i}\|_{W_p^{1,2}(Q_{\sigma(k),t})} \leq Const \left\| f_{o_k(i)}(t,x,\widetilde{u})\right\|_p$.

If the $f_{o_k(i)}$ are polynomially bounded then we can conclude that $\|\Psi_{k,i}\|_{W_p^{1,2}(Q_{\sigma(k),t})}$ has a bound for every p. Applying the Sobolev embedding theorem gives us that $\|\Psi_{k,i}\|_{\infty,Q_{\sigma(k),t}}$ is bounded and the result follows.

D. Proof of Theorem 2.6

We will begin by proving the result for $n \leq 3$. Earlier we showed that R_+^m is invariant. Since $\overline{B_{2r,\infty}(z)} \cap R_+^m$ is invariant we know that (2.3) has a unique global solution. Let $d_{min} = min\{d_k\}$ and $a = 1/d_{min}$. Define $v(t,x) = u(at,x) - z$ where u solves (2.3). Then v satisfies the system of equations given component-wise by

$$\begin{cases} v_{k_t} = \widehat{d}_k \Delta v_k + a\widetilde{f}_k(v) & t > 0, \quad x \in \Omega_{\sigma(k)} \\ \frac{\partial v_k}{\partial \eta} = 0 & t > 0, \quad x \in \partial\Omega_{\sigma(k)} \\ v_k = u_{0_k} & t = 0, \quad x \in \overline{\Omega}_{\sigma(k)} \end{cases} \tag{4.31}$$

where $\widehat{d}_k = ad_k$ and $\widetilde{f}_k(v) = \widehat{f}_k(v+z)$.

We will show that v can be bounded in the sup-norm independent of r, and this will apply to the solution of (4.1).

We multiply the k^{th} component of (4.31) by v_k and integrate over the space-time cylinder $Q_{(\tau,T)}$ to obtain

$$\int_\tau^T \int_{\Omega_{\sigma(k)}} v_k v_{k_t} dxdt = \widehat{d}_k \int_\tau^T \int_{\Omega_{\sigma(k)}} v_k \Delta v_k dxdt + a \int_\tau^T \int_{\Omega_{\sigma(k)}} v_k \widetilde{f}_k(v) dxdt.$$

Integration by parts gives

$$\begin{aligned} &\tfrac{1}{2} \|v_k(T,\cdot)\|_{2,\Omega_{\sigma(k)}}^2 - \tfrac{1}{2} \|v_k(\tau,\cdot)\|_{2,\Omega_{\sigma(k)}}^2 \\ &\leq -\widehat{d}_k \int_\tau^T \int_{\Omega_{\sigma(k)}} |\nabla v_k|^2 \, dxdt + a \int_\tau^T \int_{\Omega_{\sigma(k)}} v_k \widetilde{f}_k(v) dxdt \end{aligned}$$

Since $\widehat{f}_k$ is Lipschitz with constant L_r on subdomain $\Omega_{\sigma(k)}$ we have

$$\begin{aligned} &\widehat{d}_k \int_\tau^T \int_{\Omega_{\sigma(k)}} |\nabla v_k|^2 \, dxdt \\ &\leq \tfrac{1}{2} \|v_k(\tau,\cdot)\|_{2,\Omega_{\sigma(k)}}^2 - \tfrac{1}{2} \|v_k(T,\cdot)\|_{2,\Omega_{\sigma(k)}}^2 + aL_r \int_\tau^T \int_{\Omega_{\sigma(k)}} |v_k| \sum_{k=1}^m |v_k| \, dxdt \end{aligned}$$

Applying the Mean Value Theorem we find for some $t \in (\tau, T)$

$$\int\limits_{\Omega_{\sigma(k)}} |\nabla v_k(t,x)|^2\, dx \leq (\frac{1}{2(T-\tau)} + aL_r)L_M \|v_k\|_{\infty,R_+\times\Omega_{\sigma(k)}}. \tag{4.32}$$

We can select an increasing sequence of time values $\{T_{1,i}\}_{i=1}^{\infty}$ satisfying

$$0 < T_{1,1} < 1$$

$$\tfrac{1}{2} < \Delta T_{1,i} < 1 \qquad \forall i$$

so that

$$\|v_k(T_{1,i},\cdot)\|^{(1)}_{2,\Omega_{\sigma(k)}} \leq (2+aL_r)^{1/2} L_M^{1/2} \|v_k\|^{1/2}_{\infty,R_+\times\Omega_{\sigma(k)}}$$

Taking $v_k(T_{1,i},\cdot)$ as initial data, we obtain a solution $v_k \in W_2^{1,2}(Q_{(T_{1,i},T_{1,i+1})})$.

From Lemma 3.15 we get a constant C_1 independent of $\widehat{d}_k$ so that for some $t \in (T_{1,i}, T_{1,i+1})$

$$\|\Delta v_k(t,\cdot)\|_{2,\Omega_{\sigma(k)}} \leq C_1(\left\|a\widetilde{f}_k\right\|_{2,(T_{1,j},T_{1,j+1})\times\Omega_{\sigma(k)}} + \|v_k(T_{1,i},\cdot)\|^{(1)}_{2,\Omega_{\sigma(k)}}).$$

Combining this with Lemma 3.12 we obtain

$$\|v_k(t,\cdot)\|^{(2)}_{2,\Omega_{\sigma(k)}} \leq (C_1+1)(1+aL_r+(2+aL_r)^{1/2})L_M^{1/2} \|v_k\|^{1/2}_{\infty,R_+\times\Omega_{\sigma(k)}}$$

for some $t \in (T_{1,i}, T_{1,i+1})$. In particular, there exists a sequence $\{T^*_{1,j}\}_{j=1}^{\infty}$ with $T^*_{1,2j-1} \in [T_{1,i}, T_{1,i}+1/2]$ and $T^*_{1,2j} \in [T_{1,i+1}-1/2, T_{1,i+1}]$ so that

$$\left\|v_k(T^*_{1,j},\cdot)\right\|^{(2)}_{2,\Omega_{\sigma(k)}} \leq (C_1+1)(aL_r+1+(2+aL_r)^{1/2})L_M^{1/2} \|v_k\|^{1/2}_{\infty,R_+\times\Omega_{\sigma(k)}}$$

From Theorem 3.2, we know $W_2^2(\Omega)$ imbeds continuously into $W_2^{2-2/q}(\Omega)$ for $q = \frac{2(n+2)}{n}$. Thus there exists a constant $\widehat{C}$ so that

$$\left\|v_k(T^*_{1,j},\cdot)\right\|^{(2-2/q)}_{q,\Omega_{\sigma(k)}} \leq \widehat{C} \left\|v_k(T^*_{1,j},\cdot)\right\|^{(2)}_{2,\Omega_{\sigma(k)}}$$

As a result, we can select a sequence of time values $\{T_{2,i}\}_{i=1}^{\infty}$ satisfying

$$T_{2,1} \leq 2$$

$$1/2 < \Delta T_{2,i} < 1 \qquad \forall i \epsilon N$$

and a constant C_2 so that

$$\|v_k(T_{2,i},\cdot)\|_{q,\Omega_{\sigma(k)}}^{(2-2/q)} \leq C_2(aL_r + 1 + (2 + aL_r)^{1/2}) \|v_k\|_{\infty,R_+\times\Omega_{\sigma(k)}}$$

Using $v_k(T_{2,i},\cdot)$ as initial data, we have from Lemma 3.16 a constant C_3 independent of d_k so that

$$\|Pv_k\|_{q,(T_{2,i},T_{2,i+1})\Omega_{\sigma(k)}}^{(1,2)} \leq C_3(a \left\|\widetilde{f}_k\right\|_{q,(T_{2,i},T_{2,i+1})\times\Omega_{\sigma(k)}} + \|v_k(T_{2,i},\cdot)\|_{q,\Omega_{\sigma(k)}}^{(2-2/q)}) \tag{4.33}$$

Hence, there exists a constant C_4 so that

$$\begin{aligned} &\|Pv_k\|_{q,(T_{2,i},T_{2,i+1})\times\Omega_{\sigma(k)}}^{(1,2)} \\ &\leq C_4(aL_r \|v\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{(q-1)/q} L_M^{1/q} + (aL_r + 1 + (2 + aL_r)^{1/2})) \|v\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{1/2} L_M^{1/2}) \end{aligned}$$

and consequently

$$\|Pv_k\|_{q,(T_{2,i},T_{2,i+1})\times\Omega_{\sigma(k)}}^{(1,2)} \leq C_4(2aL_r + 1 + (2 + aL_r)^{1/2}) \|v_k\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{p} L_M^{1/2}$$

where $p = \frac{q-1}{q}$ or $\frac{1}{2}$ is chosen to maximize $\|v\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{p}$. For $n \leq 3$, we have $q = \frac{2(n+2)}{n} > \frac{n+2}{2}$ and hence $W_q^{1,2}(Q_{(T_{2,i},T_{2,i+1})})$ imbeds continuously into $C(\overline{Q}_{(T_{2,i},T_{2,i+1})})$. Thus, there exists a constant C_5 so that

$$\|Pv_k\|_{\infty,(T_{2,i},T_{2,i+1})\times\Omega_{\sigma(k)}} \leq C_5(2aL_r + 1 + (2 + aL_r)^{1/2}) \|v_k\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{p} L_M^{1/2}.$$

Since this holds for every k, we have

$$\|v_k\|_{\infty,(T_{2,i},T_{2,i+1})\times\Omega_{\sigma(k)}} \leq C_5(2aL_r + 1 + (2+aL_r)^{1/2})\,\|v_k\|^p_{\infty,R_+\times\Omega_{\sigma(k)}}\,L_M^{1/2} + \frac{L_M}{\left|\Omega_{\sigma(k)}\right|}.$$

Because $v_k = 0$ for $x \in \overline{\Omega}^c_{\sigma(k)}$ we have

$$\|v_k\|_{\infty,(T_{2,i},T_{2,i+1})\times\Omega} \leq C_5(2aL_r + 1 + (2+aL_r)^{1/2})\,\|v_k\|^p_{\infty,R_+\times\Omega}\,L_M^{1/2} + \frac{L_M}{\left|\Omega_{\sigma(k)}\right|}.$$

Summing over the components we find that

$$\|v\|_{\infty,(T_{2,i},T_{2,i+1})\times\Omega} \leq mC_5(2aL_r + 1 + (2+aL_r)^{1/2})\,\|v\|^p_{\infty,R_+\times\Omega}\,L_M^{1/2} + \sum_{k=1}^{m}\frac{L_M}{\left|\Omega_{\sigma(k)}\right|}.$$

Applying Lemma 3.17 to this inequality we arrive at

$$\|v\|_{\infty,(T_{2,i},T_{2,i+1})\times\Omega} \leq (L_M^{1/2} mC_5(2aL_r + 1 + (2+aL_r)^{1/2}))^{\frac{1}{1-p}} + \frac{1}{1-p}\sum_{k=1}^{m}\frac{L_M}{\left|\Omega_{\sigma(k)}\right|}.$$

We now have a bound for the supremum of v on the interval $[T_{2,1},\infty)$. In order to complete the proof, we must find a bound for the supremum of v on the interval $[0,T_{2,1})$. From Theorem 3.6 that $d_k\Delta$ generates a strongly continuous semigroup of contractions $\{T_k(t)\}$ on $C(\overline{\Omega}_{\sigma(k)})$. By variation of parameters we have

$$v_k(t,\cdot) = T_k(t)v_{0_k} + \int_0^t T_k(t-s)a\widetilde{f}_k(v(s,\cdot))ds$$

and thus

$$\|v_k(t,\cdot)\|_{\infty,\Omega_{\sigma(k)}} = \|v_{0_k}\|_{\infty,\Omega_{\sigma(k)}} + aL_r\int_0^t \|v(s,\cdot)\|_{\infty,\Omega_{\sigma(k)}}\,ds$$

Applying Gronwall's inequality we find that

$$\begin{aligned}\|v_k(t,\cdot)\|_{\infty,\Omega_{\sigma(k)}} &\leq \|v_{0_k}\|_{\infty,\Omega_{\sigma(k)}}\,e^{aL_r t}\\ &\leq \|v_{0_k}\|_{\infty,\Omega_{\sigma(k)}}\,e^{2aL_r}\end{aligned}$$

Again, because $v_k = 0$ for $x \in \overline{\Omega}^c_{\sigma(k)}$ we have

$$\|v_k(t,\cdot)\|_{\infty,\Omega} \le \|v_{0_k}\|_{\infty,\Omega_{\sigma(k)}} e^{2aL_r}$$

Summing over the components we find

$$\|v\|_{\infty,R_+\times\Omega} \le m\,\|v_0\|_{\infty,\Omega}\, e^{2aL_r} \tag{4.34}$$

From inequalities (4.33) and (4.34) we find

$$\|v\|_{\infty,R_+\times\Omega} \le \tag{4.35}$$
$$max\{(L_M^{1/2} mC_5(2aL_r+1+(2+aL_r)^{1/2}))^{\frac{1}{1-p}} + \frac{1}{1-p}\sum_{k=1}^{m}\frac{L_M}{\left|\Omega_{\sigma(k)}\right|}, m\,\|v_0\|_{\infty,\Omega}\, e^{2aL_r}\}$$

Select r so that

$$r \ge max\{(3mL_MC_5)^{\frac{1}{1-p}} + \frac{1}{1-p}\sum_{k=1}^{m}\frac{L_M}{\left|\Omega_{\sigma(k)}\right|}, 2m\,\|v_0\|_{\infty,\Omega}\}$$

From (4.5) we can choose a small enough to force

$$\|v\|_{\infty,R_+\times\Omega} \le max\{(3mC_5)^{\frac{1}{1-p}} + \frac{1}{1-p}\sum_{k=1}^{m}\frac{L_M}{\left|\Omega_{\sigma(k)}\right|}, 2m\,\|v_0\|_{\infty,\Omega}\}.$$

This gives us that for sufficiently large diffusion

$$\|v(t,\cdot)\|_{\infty,\Omega} \le K_M \qquad \forall t \ge 0$$

where $K_M = max\{(3mC_5)^{\frac{1}{1-p}} + \frac{1}{1-p}\sum_{k=1}^{m}\frac{L_M}{\left|\Omega_{\sigma(k)}\right|}, 2m\,\|v_0\|_{\infty,\Omega}\}$.
If $L_M \to 0$ as $\|u_0 - \widehat{z}\|_{\infty,\Omega} \to 0$ then we can see that for a fixed r

$max\{(L_M^{1/2} mC_5(2aL_r+1+(2+aL_r)^{1/2}))^{\frac{1}{1-p}} + \frac{1}{1-p}\sum_{k=1}^{m}\frac{L_M}{\left|\Omega_{\sigma(k)}\right|}, m\,\|v_0\|_{\infty,\Omega}\, e^{2aL_r}\}$ can be made arbitrarily small.

This gives us that if the initial data is sufficiently close to z then no additional assumptions on the size of the diffusion coefficients are necessary to guarantee the solution u of (4.1) exists globally. This concludes the proof for $n \leq 3$.

The primary estimate that relied on $n \leq 3$ was (4.33). We will now establish this estimate for $n > 3$ and the result will then follow for an arbitrary n.

In order to extend the result to an arbitrary dimension n, we claim the following: For every $j \in N$, if $q_j = \left(\frac{n+2}{n}\right)^j$ there exists

i. a sequence $\{T_{j+1,i}\}_{i=1}^{\infty}$ such that

$$0 < T_{j+1,1} < j+1$$

$$\tfrac{1}{2} < \Delta T_{j+1,i} < 1$$

ii. a constant $C(j)$, with $0 < C(j) < 1$ and a function $K_{j+1} \in C(R_+, R_+)$ independent of $\widehat{d}_{\sigma(k),k}$ and truncation such that

$$\|v_k(T_{j+1,i},\cdot)\|_{q_j}^{(2-2/q_j)} \leq K_{j+1}(aL_r)\,\|v\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{C(j)} \tag{4.36}$$

We have that this holds for $j = 1$. We will now proceed by induction. Suppose that this holds for $j = l \geq 1$. We will now show that this is true for $j+1$.

Since (4.36) holds for $j = l$, there exists a sequence $\{T_{l+1,i}\}_{i=1}^{\infty}$ so that

$$\|v_k(T_{l+1,i},\cdot)\|_{q_l}^{(2-2/q_l)} \leq K_{l+1}(aL_r)\,\|v\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{C(l)}$$

Taking $v_k(T_{l+1,i},\cdot)$ as initial data, we obtain a solution $v_k \in W_{q_l}^{1,2}(Q_{(T_{l+1,i},T_{l+1,i+1})})$. By virtue of Lemma 3.14, there exists a constant C_6 independent of $\widehat{d}_k$ so that for some $t \in (T_{l+1,i}, T_{l+1,i+1})$

$$\|\Delta v_k(t,\cdot)\|_{q_l,\Omega_{\sigma(k)}} \leq C_6(a\left\|\widetilde{f}_k\right\|_{q_l,(T_{2,i},T_{2,i+1})\times\Omega_{\sigma(k)}} + \|v_k(T_{l+1,i},\cdot)\|_{q_l,\Omega_{\sigma(k)}}^{(2-2/q_l)}).$$

which gives us

$$\|\Delta v_k(t,\cdot)\|_{q_l,\Omega_{\sigma(k)}} \le C_6(aL_r\,\|v\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{\frac{q_l-1}{q_l}}\,L_M^{1/q_l} + K_{l+1}(aL_r)\,\|v_k(T_{l+1,i},\cdot)\|_{q_l,\Omega_{\sigma(k)}}^{C(l)}).$$

Using this with Lemma 3.12 gives us that for some $t \in (T_{l+1,i}, T_{l+1,i+1})$

$$\|v_k(t,\cdot)\|_{q_l,\Omega_{\sigma(k)}}^{(2)} \le (C_6+1)((aL_r+1)\,\|v\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{\frac{q_l-1}{q_l}}\,L_M^{1/q_l}+K_{l+1}(aL_r)\,\|v_k(T_{l+1,i},\cdot)\|_{q_l,\Omega_{\sigma(k)}}^{C(l)}).$$

Again, we can find a sequence $\{T^*_{l+1,i}\}_{i=1}^{\infty}$ with $T^*_{l+1,2j-1} \in [T_{l+1,i}, T_{l+1,i}+1/2]$ and $T^*_{l+1,2j} \in [T_{l+1,i+1}-1/2, T_{l+1,i+1}]$ so that

$$\left\|v_k(T^*_{1+1,i},\cdot)\right\|_{q_l,\Omega_{\sigma(k)}}^{(2)} \le (C_6+1)((aL_r+1)\,\|v_k\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{(q_l-1)/q_l}\,L_M^{1/q_l}+K_{l+1}(aL_r)\,\|v_k\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{K(l)}).$$

Applying Theorem 3.1, we find that $W^2_{q_l}(\Omega_{\sigma(k)})$ imbeds into $W^{2-2/q_l}_{q_l+1}(\Omega_{\sigma(k)})$. Thus there exists $C_7 \in R$ so that

$$\left\|v_k(T^*_{1+1,i},\cdot)\right\|_{q_{l+1},\Omega_{\sigma(k)}}^{(2-2/q_{l+1})} \le C_7\,\|v_k(T_{l+1,i},\cdot)\|_{q_l,\Omega_{\sigma(k)}}^{(2)}$$

It now follows that

$$\left\|v_k(T^*_{1+1,i},\cdot)\right\|_{q_{l+1},\Omega_{\sigma(k)}}^{(2-2/q_{l+1})} \le$$
$$C_7(C_6+1)((aL_r+1)\,\|v_k\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{(q_l-1)/q_l}\,L_M^{1/q_l} + K_{l+1}(aL_r)\,\|v_k\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{K(l)})$$

and hence

$$\left\|v_k(T^*_{1+1,i},\cdot)\right\|_{q,\Omega_{\sigma(k)}}^{(2-2/q)} \le K_{l+2}(aL_r)\,\|v_k\|_{\infty,R^+\times\Omega_{\sigma(k)}}^{K(l+1)}$$

where $K_{l+2}(x) = C_7(C_6+1)((x+1)L_M^{1/q_l} + K_{l+1}(x))$ and $C(l+1) = C(l)$ or $(q_l-1)/q_l$, whichever maximizes $\|v_k\|_{\infty,R_+\times\Omega_{\sigma(k)}}^{C(l+1)}$.

So, we can pick a sequence $\{T_{l+2,i}\}_{i=1}^{\infty}$ with $T_{l+2,1} < l+2$ and $\frac{1}{2} < \Delta T_{l+1,i} < 1$ with

$$\|v_k(T_{1+2,i},\cdot)\|_{q_{l+1},\Omega_{\sigma(k)}}^{(2-2/q_{l+1})} \le K_{l+2}(aL_r)\,\|v_k\|_{\infty,R^+\times\Omega_{\sigma(k)}}^{K(l+1)}$$

and our claim follows by induction.

Now, by choosing j sufficiently large so that $q_j > \frac{n+2}{2}$ and applying Lemma 3.16, we find that there is a constant C_9 independent of $\widehat{d}_k$ so that

$$\|Pv_k\|^{(1,2)}_{q_j,(T_{2,i},T_{2,i+1})\Omega_{\sigma(k)}} \leq C_9(a \left\|\widetilde{f}_k\right\|_{q_j,(T_{2,i},T_{2,i+1})\times\Omega_{\sigma(k)}} + \|v_k(T_{2,i},\cdot)\|^{(2-2/q_j)}_{q_j,\Omega_{\sigma(k)}}).$$

E. Proof of Theorem 2.7

Suppose that f satisfies conditions (1.3) and (1.10) and that u solves (2.3). Note that these conditions together imply that $f(x,0) = 0$ so that (2.2) holds with $z = 0$ and $u_k \geq 0$ on its interval of existence. Suppose that $0 \leq t < T_{max}$. Integrating over the k^{th} equation of (2.3) over $Q_{(0,t)}$ gives

$$\int\limits_{\Omega}(u_k(t,x) - u_k(0,x))dx = \int\limits_0^t \int\limits_{\Omega} \widehat{f}_k(x,u)dxdt$$

Since f satisfies (1.10) and $\Phi_r \geq 0$, we have

$$\sum_{k=1}^{m} c_k \widehat{f}_k(x,u) = \Phi_r \sum_{k=1}^{m} c_k f_k(x,u) \leq 0$$

Summing the components we find

$$\Phi_r \sum_{k=1}^{m} c_k f_k(x,u) = \sum_{k=1}^{m} c_k \int\limits_{\Omega}(u_k(t,x) - u_k(0,x))dx \leq 0$$

This gives us

$$\sum_{k=1}^{m} c_k \int\limits_{\Omega} u_k(t,x)dx \leq \sum_{k=1}^{m} c_k \int\limits_{\Omega} u_k(0,x)dx$$

It now follows that

$$\int\limits_{\Omega} \sum_{k=1}^{m} u_k(t,x)dx \leq \frac{max\{c_k\}}{min\{c_k\}} \int\limits_{\Omega} \sum_{k=1}^{m} u_k(0,x)dx$$

and we have our L^1 estimate of (2.3) independent of d_k and r.

The result now follows from Theorem 2.6.

F. Proof of Theorem 2.8

Suppose that f satisfies condition (2.2) and there exists and invariant region I and a convex seperable Lyapunov function associated with (2.1). Note that these conditions together imply that the equilibrium point, z, for f is the zero of the convex seperable Lyapunov function and $u_k \geq 0$ on its interval of existence.

Multplying the k^{th} component of the truncated system by $h_k^{'}(u_k(x,t))$, we obtain

$$h_k^{'}(u_k)\frac{\partial u_k}{\partial t} = h_k^{'}(u_k)d_k\Delta u_k + h_k^{'}(u_k)f_k(u_k) \tag{4.37}$$

Note:

$$\Delta h_k(u_k) = h_k^{''}(u_k)\left|\nabla u_k\right|^2 + h_k^{'}(u_k)\Delta u_k \tag{4.38}$$

So,

$$h_k^{'}(u_k)\frac{\partial u_k}{\partial t} = d_k h_k^{'}(u_k)\Delta u_k - d_k h_k^{''}(u_k)\left|\nabla u_k\right|^2 + h_k^{'}(u_k)f_k(u_k) \tag{4.39}$$

Suppose that z minimizes $\|H\|_{1,I}$. Without loss of generality assume $\|H(z)\|_{1,I} = 0$. We have

$$-d_k h_k^{''}(u_k)\left|\nabla u_k\right|^2 + h_k^{'}(u_k)f_k(u_k) \geq 0 \tag{4.40}$$

It follows that

$$h_k^{'}(u_k)f_k(u_k) \geq 0 \qquad \forall u \text{ such that } u_k = z_k \tag{4.41}$$

We can write a related system

$$\frac{\partial}{\partial t}(Q) = DQ + r \tag{4.42}$$

where

$$r_k = h_k^{'}(u_k) f_k(u_k). \tag{4.43}$$

Because r is quasipositive and balanced it follows that Q does not blow-up in finite time. Applying the maximum principle we see that $H(u) \leq Q$.

It now follows that u does not blow-up in finite time.

G. Proof of Theorem 2.9

This result is essentially the same as found in [4]. We will include the convergence of $u_k(t, \cdot)$ to $\overline{u}_k(t)$ in $L^2(\Omega_{\sigma(k)})$.

Define

$$\beta_k(t) = \frac{1}{2} \left\| \nabla u_k(t, \cdot) \right\|_{2,\Omega_{\sigma(k)}}^2$$

Then

$$\beta_k'(t) = \int\limits_{\Omega_{\sigma(k)}} \nabla u_k \cdot \nabla u_{k_t} dx = \int\limits_{\Omega_{\sigma(k)}} \nabla u_k \cdot \nabla (D \Delta u_k + f) dx$$

$$= \int\limits_{\Omega_{\sigma(k)}} \Delta u_k \cdot D \Delta u_k dx + \int\limits_{\Omega_{\sigma(k)}} \nabla u_k \cdot df_u \nabla u_k dx$$

Setting $\widehat{M}$ to the maximum value of $|df|$ over $\overline{B_{K_M}(0)}$ we find

$$\beta_k'(t) \leq -d_{min} \int\limits_{\Omega} \left| \Delta u_k \right|^2 dx + \widehat{M} \int\limits_{\Omega_{\sigma(k)}} \left| \nabla u_k \right|^2 dx$$

Applying Lemma 3.18 we find

$$\beta_k'(t) \leq (-\lambda_1 d_{min} + \widehat{M}) \int\limits_{\Omega_{\sigma(k)}} \left| \nabla u_k \right|^2 dx = (-\lambda_1 d_{min} + \widehat{M}) 2 \beta_k(t)$$

Setting

$$\sigma = \lambda_1 d_{min} - \widehat{M}$$

and applying Gronwall's inequality we obtain the following inequality

$$\beta_k(t) \leq \beta_k(0)e^{-2\sigma t}$$

and we have

$$\|\nabla u_k\|_{2,\Omega^2_{\sigma(k)}} \leq \|\nabla u_{k0}\|^2_{2,\Omega_{\sigma(k)}} e^{-2\sigma t}.$$

From Lemma 3.19 we find

$$\|u_k - \overline{u_k}\|^2_{2,\Omega_{\sigma(k)}} \leq \frac{1}{\lambda_1} \|\nabla u_{k0}\|^2_{2,\Omega_{\sigma(k)}} e^{-2\sigma t}$$

So for d_{min} sufficiently large, we have that the k-th component of the solution u of (2.1) converges exponentially to its spatial average in $L^2(\Omega_{\sigma(k)})$. Also, we have

$$\overline{u}_k'(t) = \tfrac{1}{\left|\Omega_{\sigma(k)}\right|} \int\limits_{\Omega_{\sigma(k)}} u_{k_t}(t,x)dx$$
$$= \tfrac{1}{\left|\Omega_{\sigma(k)}\right|} \int\limits_{\Omega_{\sigma(k)}} D\Delta u_k dx + \tfrac{1}{\left|\Omega_{\sigma(k)}\right|} \int\limits_{\Omega_{\sigma(k)}} f(u)dx$$

Integrating by parts implies that $\overline{u}_k'(t)$ satisfies

$$\overline{u}_k'(t) = \frac{1}{\left|\Omega_{\sigma(k)}\right|} \int\limits_{\Omega_{\sigma(k)}} f_k(x,u)dx$$

For $t > 0$, we have

$$\left| \frac{1}{\left|\Omega_{\sigma(k)}\right|} \int\limits_{\Omega_{\sigma(k)}} (f_k(x,u) - f(x,\overline{u}_k))dx \right| \leq \frac{\widehat{M}}{\left|\Omega_{\sigma(k)}\right|} \int\limits_{\Omega_{\sigma(k)}} |u_k - \overline{u}_k|\, dx$$

$$\leq \frac{\widehat{M}}{\left|\Omega_{\sigma(k)}\right|^{1/2}} \left(\int\limits_{\Omega_{\sigma(k)}} |u_k - \overline{u}_k|^2\, dx \right)^{1/2}$$

$$\leq \frac{\widehat{M}}{\left|\Omega_{\sigma(k)}\right|^{1/2}} \frac{1}{\lambda_1^{1/2}} \left\|\nabla u_{k_0}\right\|_{2,\Omega_{\sigma(k)}} e^{-\sigma t}$$

This gives us that

$$\overline{u}_{k_t} = f_k(\overline{u}(t)) + \epsilon_k(t). \tag{4.44}$$

It follows from [4] that the asymptotic behavior of (4.44) is determined by f.

CHAPTER V

APPLICATIONS AND CONCLUSION

A. Applications

In Chapter I we introduced a reaction diffusion system that is used to model the behavior of a simple population. The population is assumed to be spatially distributed, and the dispersion of the population is assumed to be modeled by Fickian diffusion. In this model there are three populations confined to separate habitats Ω_1, Ω_2 and Ω_3 such that

$$\Omega_1 \cap \Omega_2 \neq \emptyset,\ \Omega_2 \cap \Omega_3 \neq \emptyset,\ \Omega_1 \cap \Omega_3 = \emptyset \tag{5.1}$$

We model the habitats Ω_i as bounded domains in R^3 with smooth boundaries denoted by $\partial\Omega_i$ such that Ω_i lies locally on one side of itself. The population is divided into three groups, denoted by the host for a disease in Ω_1, the vector population in Ω_2, and the host for the disease in Ω_3. The populations in Ω_1, Ω_2 and Ω_3 subdivide into susceptibles and infectives. A susceptible host in Ω_1 interacts with an infective vector in Ω_2 to become an infective host in Ω_1. The interaction of an infective host in Ω_1 with a susceptible vector in Ω_2 creates an infective vector in Ω_2. Similarly, the interaction of an infective vector in Ω_2 with a susceptible host in Ω_3 results in an infective host in Ω_3, and the interaction of an infective host from Ω_3 with a susceptible vector from Ω_2 results in an infective vector from Ω_3. The reasoning behind the use of the terms "host" and "vector" stems from the assumption that the disease does not result in any mortality for the vector group. In the model below, we also assume that the host population in Ω_1 is resistant to the disease, and as a result, in some sense it also acts as a vector population. The host population in Ω_3 is not resistant to the disease. One simple model of this interaction is given in the system below.

$$
\begin{array}{lll}
\left(\begin{array}{ll} \phi_t = d_1\Delta\phi - k_1(x)\phi\beta + \lambda_1\psi & \text{for } x \in \Omega_1, t > 0 \\ \psi_t = d_2\Delta\psi + k_1(x)\phi\beta - \lambda_1\psi & \end{array}\right) & \text{host 1} & \\
\left(\begin{array}{ll} \alpha_t = d_3\Delta\alpha - k_2(x)\alpha\psi - k_3(x)\alpha v + \lambda_2\beta & \text{for } x \in \Omega_2, t > 0 \\ \beta_t = d_4\Delta\beta + k_2(x)\alpha\psi + k_3(x)\alpha v - \lambda_2\beta & \end{array}\right) & \text{vector} & (5.2) \\
\left(\begin{array}{ll} v_t = d_5\Delta v - k_4(x)v\beta & \text{for } x \in \Omega_3, t > 0 \\ w_t = d_6\Delta w + k_4(x)v\beta - \lambda_3 w & \end{array}\right) & \text{host 2} &
\end{array}
$$

where k_1, k_2, k_3 and k_4 are nonnegative functions, and λ_1, λ_2 and λ_3 are positive constants. Furthermore, the supports of k_1 and k_2 are contained in the intersection of Ω_1 and Ω_2, and the supports of k_3 and k_4 are contained in the intersection of Ω_2 and Ω_3. Finally, the values d_i and λ_j are positive constants for $i = 1, 2, ..., 6$ and $j = 1, 2, 3$. We augment the system above with homogeneous Neumann boundary conditions on each domain Ω_1, Ω_2, and Ω_3.

$$
\begin{array}{ll}
\partial\phi/\partial\eta = \partial\psi/\partial\eta = 0 & \text{for } x \in \partial\Omega_1, t > 0 \\
\partial\alpha/\partial\eta = \partial\beta/\partial\eta = 0 & \text{for } x \in \partial\Omega_2, t > 0 \\
\partial v/\partial\eta = \partial w/\partial\eta = 0 & \text{for } x \in \partial\Omega_3, t > 0
\end{array} \tag{5.3}
$$

and specify continuous nonnegative initial data.

$$
\begin{array}{lll}
\phi(x,0) = \phi_0(x), & \psi(x,0) = \psi_0(x) & \text{for } x \in \overline{\Omega}_1 \\
\alpha(x,0) = \alpha_0(x), & \beta(x,0) = \beta_0(x) & \text{for } x \in \overline{\Omega}_2 \\
v(x,0) = v_0(x), & w(x,0) = w_0(x) & \text{for } x \in \overline{\Omega}_3
\end{array} \tag{5.4}
$$

The vector field associated with the system above is given by $f = (f_i)$ where

$$
\begin{pmatrix} f_1\left(x,\phi,\psi,\alpha,\beta,v,w\right) \\ f_2\left(x,\phi,\psi,\alpha,\beta,v,w\right) \end{pmatrix} = \begin{pmatrix} -k_1(x)\phi\beta + \lambda_1\psi \\ k_1(x)\phi\beta - \lambda_1\psi \end{pmatrix} \tag{5.5}
$$

$$\begin{pmatrix} f_3(x,\phi,\psi,\alpha,\beta,v,w) \\ f_4(x,\phi,\psi,\alpha,\beta,v,w) \end{pmatrix} = \begin{pmatrix} -k_2(x)\alpha\psi - k_3(x)\alpha v + \lambda_2\beta \\ k_2(x)\alpha\psi + k_3(x)\alpha v - \lambda_2\beta \end{pmatrix} \tag{5.6}$$

$$\begin{pmatrix} f_5(x,\phi,\psi,\alpha,\beta,v,w) \\ f_6(x,\phi,\psi,\alpha,\beta,v,w) \end{pmatrix} = \begin{pmatrix} -k_4(x)v\beta \\ k_4(x)v\beta - \lambda_3 w \end{pmatrix} \tag{5.7}$$

It is a simple matter to verify that f is quasipositivity since for $(\phi,\psi,\alpha,\beta,v,w) \in R_+^6$ we have

$$f_1(x,0,\psi,\alpha,\beta,v,w) = \lambda_1\psi \geq 0 \tag{5.8}$$

$$f_2(x,\phi,0,\alpha,\beta,v,w) = k_1(x)\phi\beta \geq 0 \tag{5.9}$$

$$f_3(x,\phi,\psi,0,\beta,v,w) = \lambda_2\beta \geq 0 \tag{5.10}$$

$$f_4(x,\phi,\psi,\alpha,0,v,w) = k_2(x)\alpha\psi + k_3(x)\alpha v \geq 0 \tag{5.11}$$

$$f_5(x,\phi,\psi,\alpha,\beta,0,w) = 0 \tag{5.12}$$

$$f_6(x,\phi,\psi,\alpha,\beta,v,0) = k_4(x)v\beta \geq 0 \tag{5.13}$$

As a result, from Theorem 2.3, the system above has a unique, componentwise-nonnegative solution on its maximal interval of existence. Furthermore, the vector field f is clearly polynomially bounded since

$$|f_i(x,\phi,\psi,\alpha,\beta,v,w)| \leq K\left(\phi^2+\psi^2+\alpha^2+\beta^2+v^2+w^2+1\right) \tag{5.14}$$

for an appropriate choice of $K > 0$. Finally, the vector field f satisfies the linear intermediate sums condition since

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} f\left(x, \phi, \psi, \alpha, \beta, v, w\right) \leq \begin{pmatrix} \lambda_1 \psi \\ 0 \\ \lambda_2 \beta \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.15}$$

for all $x \in \Omega_1 \cup \Omega_2 \cup \Omega_3$ and $(\phi, \psi, \alpha, \beta, v, w) \in R^6_+$. Applying Theorem 2.5, we can conclude that the unique componentwise-nonnegative solution to this system exists globally.

In addition, we can apply Theorem 2.9 to find that if $u = (\phi, \psi, \alpha, \beta, v, w)$ then

$$\|u_i(t, .) - \overline{u}_i(t, .)\|_{\infty, \Omega_{\sigma(i)}} \to 0 \tag{5.16}$$

That is, there are no spatially dependent elements in the omega limit set for the system.

The analysis applied to the system above can also be used to analyze more complex population models. As a first extension, we consider populations on the habitats above which interactive through a criss-cross mechanism. In this setting we complicate the populations in each Ω_i to include two distinct populations, each containing susceptibles and infectives. To this end, we assume the two host populations in Ω_1 are given by $P_{1,1} = (\alpha_1, \beta_1)$ and $P_{1,2} = (\alpha_2, \beta_2)$ where α_i denotes a susceptible portion of the population $P_{1,i}$ and β_i denotes an infective portion of population $P_{1,i}$. Similarly, we assume the two vector populations in Ω_2 are given by $P_{2,1} = (\phi_1, \psi_1)$ and $P_{2,2} = (\phi_2, \psi_2)$ where ϕ_i denotes a susceptible portion of the population $P_{2,i}$ and ψ_i denotes an infective portion of population $P_{2,i}$. Finally, the two host populations in

Ω_3 are given by $P_{3,1} = (v_1, w_1)$ and $P_{3,2} = (v_2, w_2)$ where v_i denotes a susceptible portion of the population $P_{3,i}$ and w_i denotes an infective portion of population $P_{3,i}$. The criss-cross nature of the system arises from the assumption that α_i interacts with ψ_j for $i \neq j$, β_i interacts with ϕ_j for $i \neq j$, ϕ_i interacts with ω_j for $i \neq j$, and ψ_i interacts with w_j for $i \neq j$. Of course, in each case, we assume that interactions between susceptibles and infectives produce more infectives in the habitat of the susceptible. A model can be given for this type of interaction as an extension of the model above in the form

$$\begin{cases}
\left.\begin{aligned}
\phi_{1t} &= d_1\Delta\phi_1 - k_1(x)\phi_1\beta_2 + \lambda_1\psi_1\\
\phi_{2t} &= \widetilde{d}_1\Delta\phi_2 - \widetilde{k}_1(x)\phi_2\beta_1 + \widetilde{\lambda}_1\psi_2\\
\psi_{1t} &= d_2\Delta\psi_1 + k_1(x)\phi_1\beta_2 - \lambda_1\psi_1\\
\psi_{2t} &= \widetilde{d}_2\Delta\psi_2 + \widetilde{k}_1(x)\phi_2\beta_1 - \widetilde{\lambda}_1\psi_2
\end{aligned}\quad \text{for } x \in \Omega_1, t > 0\right) & \text{host 1}\\
\left.\begin{aligned}
\alpha_{1t} &= d_3\Delta\alpha_1 - k_2(x)\alpha_1\psi_2 - k_3(x)\alpha_1 v_2 + \lambda_2\beta_1\\
\alpha_{2t} &= \widetilde{d}_3\Delta\alpha_2 - \widetilde{k}_2(x)\alpha_2\psi_1 - \widetilde{k}_3(x)\alpha_2 v_1 + \widetilde{\lambda}_2\beta_2\\
\beta_{1t} &= d_4\Delta\beta_1 + k_2(x)\alpha_1\psi_2 + k_3(x)\alpha_1 v_2 - \lambda_2\beta_1\\
\beta_{2t} &= \widetilde{d}_4\Delta\beta_2 + \widetilde{k}_2(x)\alpha_2\psi_1 + \widetilde{k}_3(x)\alpha_2 v_1 - \widetilde{\lambda}_2\beta_2
\end{aligned}\quad \text{for } x \in \Omega_2, t > 0\right) & \text{vector}\\
\left.\begin{aligned}
v_{1t} &= d_5\Delta v_1 - k_4(x)v_1\beta_2\\
v_{2t} &= \widetilde{d}_5\Delta v_2 - \widetilde{k}_4(x)v_2\beta_1\\
w_{1t} &= d_6\Delta w_1 + k_4(x)v_1\beta_2 - \lambda_3 w_1\\
w_{2t} &= \widetilde{d}_6\Delta w_2 + \widetilde{k}_4(x)v_2\beta_1 - \widetilde{\lambda}_3 w_2
\end{aligned}\quad \text{for } x \in \Omega_3, t > 0\right) & \text{host 2}
\end{cases} \tag{5.17}$$

where k_1,$\widetilde{\mathrm{k}}_1$, k_2,$\widetilde{\mathrm{k}}_2$, k_3,$\widetilde{\mathrm{k}}_3$, k_4 and $\widetilde{\mathrm{k}}_4$, are nonnegative functions, and λ_1, $\widetilde{\lambda}_1$,λ_2, $\widetilde{\lambda}_2$, λ_3 and $\widetilde{\lambda}_3$ are positive constants. Furthermore, the supports of $k_1, \widetilde{k}_1, k_1$,and $\widetilde{\mathrm{k}}_2$ are contained in the intersection of Ω_1 and Ω_2, and the supports of $k_3, \widetilde{k}_3, k_4$,and $\widetilde{\mathrm{k}}_4$ are contained in the intersection of Ω_2 and Ω_3. Finally, the values $d_i, \widetilde{d}_i$ and λ_j are positive constants for $i = 1, 2, ..., 6$ and $j = 1, 2, 3$. We augment the system above with homogeneous Neumann boundary conditions on each domain Ω_1, Ω_2, and Ω_3.

$$\begin{aligned}
&\partial\phi_1/\partial\eta = \partial\phi_2/\partial\eta = \partial\psi_1/\partial\eta = \partial\psi_2/\partial\eta = 0 \quad \text{for } x \in \partial\Omega_1, t > 0 \\
&\partial\alpha_1/\partial\eta = \partial\alpha_2/\partial\eta = \partial\beta_1/\partial\eta = \partial\beta_2/\partial\eta = 0 \quad \text{for } x \in \partial\Omega_2, t > 0 \\
&\partial v_1/\partial\eta = \partial v_2/\partial\eta = \partial w_1/\partial\eta = \partial w_2/\partial\eta = 0 \quad \text{for } x \in \partial\Omega_3, t > 0
\end{aligned} \tag{5.18}$$

and specify continuous nonnegative initial data.

$$\begin{aligned}
\phi_1(x,0) &= \phi_{10}(x), \quad \phi_2(x,0) = \phi_{20}(x) \quad \text{for } x \in \overline{\Omega}_1 \\
\psi_1(x,0) &= \psi_{10}(x), \quad \psi_2(x,0) = \psi_{20}(x) \quad \text{for } x \in \overline{\Omega}_1 \\
\alpha_1(x,0) &= \alpha_{10}(x), \quad \alpha_2(x,0) = \alpha_{20}(x), \quad \text{for } x \in \overline{\Omega}_2 \\
\beta_1(x,0) &= \beta_{10}(x), \quad \beta_2(x,0) = \beta_{20}(x) \quad \text{for } x \in \overline{\Omega}_2 \\
v_1(x,0) &= v_{10}(x), \quad v_2(x,0) = v_{20}(x) \quad \text{for } x \in \overline{\Omega}_3 \\
w_1(x,0) &= w_{10}(x), \quad w_2(x,0) = w_{20}(x) \quad \text{for } x \in \overline{\Omega}_3
\end{aligned} \tag{5.19}$$

The analysis below verifies that this system can be analyzed in the same manner as the previous one.

$$f_1(x, 0, \phi_2, \psi_1, \psi_2, \alpha_1, \alpha_2, \beta_1, \beta_2, v_1, v_2, w_1, w_2) = \lambda_1\psi_1 \geq 0 \tag{5.20}$$

$$f_2(x, \phi_1, 0, \psi_1, \psi_2, \alpha_1, \alpha_2, \beta_1, \beta_2, v_1, v_2, w_1, w_2) = \widetilde{\lambda}_1\psi_2 \geq 0 \tag{5.21}$$

$$f_3(x, \phi_1, \phi_2, 0, \psi_2, \alpha_1, \alpha_2, \beta_1, \beta_2, v_1, v_2, w_1, w_2) = k_1(x)\phi_1\beta_2 \geq 0 \tag{5.22}$$

$$f_4(x, \phi_1, \phi_2, \psi_1, 0, \alpha_1, \alpha_2, \beta_1, \beta_2, v_1, v_2, w_1, w_2) = \widetilde{k}_1(x)\phi_2\beta_1 \geq 0 \tag{5.23}$$

$$f_5(x, \phi_1, \phi_2, \psi_1, \psi_2, 0, \alpha_2, \beta_1, \beta_2, v_1, v_2, w_1, w_2) = \lambda_2\beta_1 \geq 0 \tag{5.24}$$

$$f_6(x, \phi_1, \phi_2, \psi_1, \psi_2, \alpha_1, 0, \beta_1, \beta_2, v_1, v_2, w_1, w_2) = \widetilde{\lambda}_2\beta_2 \geq 0 \tag{5.25}$$

$$f_7(x, \phi_1, \phi_2, \psi_1, \psi_2, \alpha_1, \alpha_2, 0, \beta_2, v_1, v_2, w_1, w_2) = k_2(x)\alpha_1\psi_2 + k_3(x)\alpha_1 v_2 \geq 0 \tag{5.26}$$

$$f_8(x, \phi_1, \phi_2, \psi_1, \psi_2, \alpha_1, \alpha_2, \beta_1, 0, v_1, v_2, w_1, w_2) = \widetilde{k}_2(x)\alpha_2\psi_1 + \widetilde{k}_3(x)\alpha_2 v_1 \tag{5.27}$$

$$f_9(x, \phi_1, \phi_2, \psi_1, \psi_2, \alpha_1, \alpha_2, \beta_1, \beta_2, 0, v_2, w_1, w_2) \geq 0 \tag{5.28}$$

$$f_{10}(x, \phi_1, \phi_2, \psi_1, \psi_2, \alpha_1, \alpha_2, \beta_1, \beta_2, v_1, 0, w_1, w_2) \geq 0 \tag{5.29}$$

$$f_{11}(x, \phi_1, \phi_2, \psi_1, \psi_2, \alpha_1, \alpha_2, \beta_1, \beta_2, v_1, v_2, 0, w_2) = k_4(x)v_1\beta_2 \geq 0 \tag{5.30}$$

$$f_{12}(x, \phi_1, \phi_2, \psi_1, \psi_2, \alpha_1, \alpha_2, \beta_1, \beta_2, v_1, v_2, w_1, 0) = \widetilde{k}_4(x)v_2\beta_1 \geq 0 \tag{5.31}$$

Finally, the vector field f satisfies the linear intermediate sums condition. Consequently, more general criss-cross scenarios can also be analyzed in a similar fashion.

B. Conclusion

The primary results of this dissertation are three-fold. The work began with a well posedness result (Theorem 2.3) for the system (2.1). Then we obtained an extension

(Theorem 2.5) of the global existence result in Morgan [20]. Finally, we extended the work of Cupps [5] pertaining to systems of reaction-diffusion equations with large diffusion coefficients.

We intend to use these results in the future as a starting point to analyze more complex population models. In addition, we intend to explore the possibility of applying our results in other areas. For example, it should be possible to apply our results to biological systems from cell biology. In this setting, it is not uncommon for certain chemical species (due to molecular size) to pass freely through certain membranes, and be restricted by others. As a result, the membrane walls of organelles will serve as natural boundaries of domains of interaction of chemical species. Of course, this leads to the question of whether the analysis in this dissertation can be extended to systems which have moving boundaries, and growing domains (and boundaries). This setting will also serve as the basis for future work.

Finally, we remark that it seems possible to obtain a better result than the global existence result given in Theorem 2.5 via the assumption of quasipositivity and linear intermediate sums. If in addition, we assume that the system is balanced, then we can obtain a uniform $L^1(\Omega_j)$ estimate for each component of our unknown.

We can see that both systems analyzed in this section are balanced since

$$\sum_i f_i(t,x,u) \leq 0 \tag{5.32}$$

for each system.

Consequently, it does not seem unreasonable that the solutions to these systems (as well as general systems satisfying these hypotheses) should be uniformly bounded in the $L^\infty(\Omega_j)$ norm as well. In fact, our recent explorations indicate that these results can be obtained as an extension of work in Morgan [21].

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VITA

John Maurice-Car Ryan was born on August 1, 1969 in Tallahassee, Florida. He recieved a Bachelor of Arts in mathematics from the University of South Florida in 1991 and a Master of Science in mathematics from the University of Florida in 1993. In 1993, he started graduate school at Texas A&M University, where he received a Master of Science in finance in 1996. His current mailing address is 10203 Millridge North Dr., Houston, Texas 77070.