COSMOLOGY AND GRAVITY IN THE BRANE WORLD

A Dissertation

by

JAMES BLACKMAN DENT

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2005

Major Subject: Physics

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Approved by:

Chair of Committee,	Christopher Pope
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ABSTRACT

Cosmology and Gravity in the Brane World. (August 2005)

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Chair of Advisory Committee: Dr. Christopher Pope

The cosmology in the Hubble expansion era of the Horava-Witten M-theory compactified on a Calabi-Yau threefold is studied in the reduction to five-dimensions where the effects of the Calabi-Yau manifold are summarized by the volume modulus, and all perturbative potentials are included. Matter on the branes are treated as first order perturbations of the static vacuum solution, and all equations in the bulk and all boundary conditions on both end branes are imposed. It is found that for a static volume modulus and a static fifth dimension, y, one can recover the four dimensional Robertson-Friedmann-Walker cosmology for relativistic matter on the branes, but not for non-relativistic matter. For relativistic matter, the Hubble parameter H becomes independent of y to first order in matter density, and if a consistent solution for non-relativistic matter exists it would require H to be y dependent. These results hold also when an arbitrary number of 5-branes are included in the bulk. The five dimensional Horava-Witten model is compared with the Randall Sundrum phenomenology with a scalar field in the bulk where a bulk and brane potential are used so that the vacuum solutions can be rigorously obtained.(In the Appendix, the difficulty of obtaining approximate vacuum solutions for other potentials is discussed.) In this case non-relativistic matter is accommodated by allowing the distance between the branes to vary. It is suggested that non-perturbative potentials for the vacuum solution of

Horava-Witten theory are needed to remove the inconsistency that non-relativistic matter creates.

Also considered is the problem of gravitational forces between point particles on the branes in a Randall-Sundrum (R-S) two brane model with S^1/Z_2 symmetry. Matter is assumed to produce a perturbation to the R-S vacuum metric and all the 5D Einstein equations are solved to linearized order (for arbitrary matter on both branes). We show that while the gauge condition $h_{i5} = 0, i = 0, 1, 2, 3$ can always be achieved without brane bending, the condition $h_{55} = 0$ leads to large brane bending. The static potential arising from the zero modes and the corrections due to the Kaluza-Klein (KK) modes are calculated. Gravitational forces on the Planck ($y_1 = 0$) brane recover Newtonian physics with small KK corrections (in accord with other work). However, forces on the TeV (y_2) brane due to particles on that brane are strongly distorted by large R-S exponentials, making the model in disagreement with experiment if the TeV brane is the physical brane.

To Allison, Jill, Kathy, Lark, and Riley

ACKNOWLEDGMENTS

I would like to thank the members of my committee, Prof. Arnowitt, Prof. Fulling, Prof. Nanopoulos, and Prof. Pope, for their time, questions, and scheduling flexibility. I would like to thank Prof. Pope and Prof. Nanopoulos for all that I have learned in taking their courses, studying their notes, and the many conversations I have had with them through which I have benefitted immensely. I would like to thank Dr. Bhaskar Dutta for his incredible amount of help and guidance over the course of my research. I would like to especially thank Prof. Arnowitt for his dedication, patience, support, advice, insights, and for sharing his vast wealth of knowledge on a wide variety of subjects in and out of physics. Professor Arnowitt has been a wonderful advisor and I am honored to have worked with him over the course of my time at Texas A&M.

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CHAPTER I

INTRODUCTION

The Standard Model (SM) of particle physics is one of the great successes in the history of science. Its framework can be used to explain electromagnetic and nuclear interactions (strong and weak) among elementary particles and has been experimentally verified to astounding accuracy[1]. However, it is incomplete. The most glaring problem is that gravity is not incorporated in the SM. Other problems include: the gauge hierarchy problem (why is the weak scale so far below the GUT scale), coupling constant unification does not occur, many parameters (masses, couplings, number of generations) are put in by hand, neutrino masses can not be accommodated, and cosmological problems such as the origin and nature of dark energy and dark matter as well as inflation are not explained.

Supersymmetry (SUSY) (see [2] for a review of global and local supersymmetry) is an extension of the SM that can seemingly ameliorate some of the problems listed above. SUSY is a symmetry that relates bosons to fermions through a fermionic SUSY generator (which can be global or local). Matter multiplets (known as supermultiplets) are categorized as irreducible representations under the SUSY algebra and contain equal numbers of bosons and fermions. Particles within the same representation (termed superpartners) carry the same mass and charge under any gauge groups. This leads to an obvious incompatibility with experiment in that, for example, we do not observe a negatively charged scalar at .511MeV (which would be the electron's superpartner, the selectron[1]). Therefore SUSY is not a good symmetry of the cur-

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The journal model is Nuclear Physics B.

[1]The superpartners of SM fermions are named by adding an 's-' to the beginning

rent universe and must be broken at some higher energy. The difficulty of finding a natural mechanism for global SUSY breaking eventually led to the abandonment of global SUSY as a fundamental symmetry.

When SUSY is treated as a local symmetry it incorporates the spacetime symmetries of general relativity and is called supergravity (SUGRA). Supergravity provides a natural mechanism that solves the gauge hierarchy problem by cancelling the divergences coming from loop corrections to the scalar Higgs mass, supergravity gives a natural mechanism to break SUSY, and coupling constants unify in SUSY GUTs (here SUSY GUTs refers to locally supersymmetric grand unified theories). In addition SUSY GUTS can also include massive neutrinos, possess a natural candidate for dark matter in the form of the stable lightest supersymmetric particle (LSP), and inflation finds a more natural home since there exist fundamental scalars in supergravity models. However supergravity actually introduces more unconstrained parameters (although in some models such as mSUGRA [3] this number is small). Although gravity is included in supergravity it is still not compatible with quantum field theory in that it is non-renormalizable. There are other unresolved questions such as the origin and nature of both inflation and dark energy (the cosmological constant problem), the number of generations, why would the universe use certain SUSY GUTs as opposed to others, and why is spacetime 3+1 dimensional. Over the past twenty years string theory has emerged as the leading framework within which these questions may be considered.

String theory (for a review see [4]) dispenses with the notion of particles as mathematical points and treats particles as extended objects which allows gravity to be included in a finite manner. Supersymmetry is an ingredient of string theory

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of the SM particle name, e.g. stop, sbottom, stau, etc..., while the superpartners of SM bosons are named by appending an '-ino' to the name, e.g. Higgsino, gluino, etc...

(superstrings) and therefore it is hoped that all of its apparent successes will be found in any realistic superstring model. In order for string theory to be internally consistent one finds automatically that string theory gives the number of spacetime dimensions as a prediction (the superstring is formulated in ten dimensions whereas the bosonic string exists in 26). String theory is only consistent with certain gauge groups which can then provide the origin for SUSY GUT groups and string theory is the only theory that offers the possibility of calculating Yukawa coupling constants. The number of generations can also be calculated from topological considerations in superstring theory. However there are many problems that need addressing if string theory is to be a correct theory of nature. Among these are the fact that in string theory the problem of SUSY breaking re-emerges, also string theory predicts the existence of extra dimensions and there exist many additional fields known as moduli whose existence makes building a reasonable phenomenology difficult. Seemingly another problem is that superstring theory has not one but five vacuum states: Type I, IIA, IIB, heterotic $SO(32)\times SO(32)$ and $E_8\times E_8$. The fact that superstrings exist in ten dimensions also brings up another puzzle in that the highest dimension that one finds a consistent theory of supergravity is eleven. For example, we know that the Type IIA and Type IIB superstring theories can be realized as the low energy limit of the corresponging Type IIA and Type IIB supergravity theories in ten dimensions but there seems to be no place for the 11D supergravity in string theory. Also the problem of many parameters of SUSY reemerges more violently in M-theory in the form of many possible vacua, i.e. the $\mathcal{O}(10^5)$ CY manifolds (not to mention the landscape problem[5]). Some of these problems were addressed in the mid 1990s with the use of dualities, branes, and M-theory.

It was shown by the use of duality relations that one can relate each of the five superstring vacua to one another as well as 11D supergravity (for a review see [6]).

For example a particularly interesting theory was proposed by Horava and Witten (HW) [7, 8, 9]who showed that the strong coupling limit of the $E_8 \times E_8$ heterotic string is related to an 11D theory formulated on the orbifold $R^{10} \times S^1/Z_2$. It was therefore conjectured that there exists an underlying 11D quantum theory called M-theory that contains the five supertring formulations as well as 11D SUGRA as different points in its parameter space. One would like to explore the parameter space of the 11D M-theory and see if there are any vacua that give rise to 4D models which can correctly reproduce the SM of particle physics, explain the known cosmology (the Hubble and inflationary eras), reproduce 4D gravity, as well as give predictions concerning topics such as SUSY phenomenology, the cosmological constant problem, etc... In the remaining chapters we will discuss some of these problems in the framework of HW theory as well as a 5D phenomenological model based on HW theory known as the Randall-Sundrum (RS) model.

Next we will give a description of the Horava-Witten theory and its reduction to five dimensions. This will be followed by a study of Hubble era cosmology within the HW theory. Then we will introduce the 5D RS model and examine both cosmology and gravity within its framework.

CHAPTER II

HORAVA WITTEN M-THEORY

In this chapter we give a brief review of the construction of Horava-Witten M-theory and its compactification to five dimensions. Then we will explore the cosmology using the five dimensional action as a starting point. This chapter follows the work of [7, 8, 9, 10].

A. 11D Supergravity on an Orbifold

The low energy limit of M-theory is 11D supergravity. The field content of 11D supergravity is a graviton, g_{MN}, a gravitino, ψ^{α}_{M}, and a 3-form potential, C_{IJK}. It has $N = 1$ supersymmetry generated by a 32 component spinor. The 11D supergravity action was first formulated in [11] and for our purposes we will only need the bosonic piece which is

$$\begin{aligned} S_{SG} &= -\frac{1}{2/\kappa^2}\int_{M^{11}}\sqrt{-g}[R+\frac{1}{24}G_{IJKL}G^{IJKL} \\ &+ \frac{\sqrt{2}}{1728}\epsilon^{I_1\ldots I_{11}}C_{I_1I_2I_3}G_{I_4\ldots I_7}G_{I_8\ldots I_11}] \end{aligned} \tag{2.1}$$

where the 3-form C_{IJK} has field strength $G_{IJKL} = 24\partial_{[I}C_{JKL]}$.

It has been shown [12] that the strong coupling behavior of Type IIA string theory is related to 11D supergravity on $M^{10}\times S^1$ where the radius of the S^1 is related to the string coupling by

$$\rho = g_s^{2/3} \tag{2.2}$$

Here we will look at 11D supergravity formulated on the orbifold $M^{10}\times S^1/Z_2$ and show that it is related to the heterotic $E_8\times E_8$ string theory. The Z_2 symmetry acts on x^{11} as $x^{11} \to -x^{11}$. This gives two orbifold fixed points at $x^{11} = 0, \pi\rho$. Thus the universe appears as an eleven dimensional bulk space bounded by an interval of length $\pi\rho$ at whose endpoints are ten-dimensional hyperplanes. The action of 11D supergravity is invariant under the Z_2 symmetry with an additional sign change of C_{IJK}. The Z_2 symmetry also acts to kill half of the supersymmetries and one is left with a sixteen dimensional supersymmetry generator that is chiral from a ten-dimensional perspective. Therefore the low energy limit of the 11D supergravity on $M^{10}\times S^1/Z_2$ will be given by an $N = 1$ 10d supergravity. In [7] it was shown that this corresponds to the low energy structure of the $E_8\times E_8$ heterotic string theory due to arguments arising from space-time gravitational and gauge anomalies, the strong coupling behavior of the theory, as well as world-sheet anomaly considerations.

B. The Consequences of Anomaly Cancellation and Supersymmetry Invariance

By examining the arguments of space-time gravitational and gauge anomalies (for reviews of anomalies see [4, 13, 14]) as well as conditions for unbroken local supersymmetry given in [7, 8, 9] we will illuminate some remarkable properties of H-W M-theory. 11D supergravity formulated on a smooth manifold is anomaly free. This is because gravitational anomalies only occur in $4k + 2$ dimensions. Therefore when formulated on $M^{10}\times S^1/Z_2$, anomalies will be (evenly) supported on the 10D hyperplanes located at the orbifold fixed points. Theses anomalies are the usual 10D supergravity anomalies and can be dealt with in the standard way, i.e. factorizable anomalies cancel by the Green-Schwarz mechanism (the Green-Schwarz term now arises in a very natural way from the Chern-Simons $C \wedge G \wedge G$ term in the action) and unfactorizable anomalies must be cancelled by the introduction of 496 additional vector multiplets. However these multiplets must be (evenly) split since each hyperplane carries half of the usual 10D anomaly. Thus one is forced to introduced

248 vector multiplets at each fixed point by placing one E_8 gauge group on each hyperplane.

Now that there is an E_8 gauge group at each fixed point we have supermultiplets that live on each 10D hyperplane whose Yang-Mills action is given by

$$L_{YM} = -\frac{1}{\lambda^2}\int_{M^{10}} d^{10}x\sqrt{g}tr(\frac{1}{4}F_{AB}F^{AB} + \frac{1}{2}\bar{\chi}\Gamma^A D_A\chi) \tag{2.3}$$

where indices $A, B, ...$ are over $x^0, x^1, ...x^9$, λ is the gauge coupling constant, F_{AB} is the field strength, and χ is the gluino. One would like for this action plus the supergravity action to be invariant under local supersymmetry. In order to accomplish this one uses the standard Noether procedure and adds additional interaction terms which have the result of modifying the Bianchi identity of G_{IJKL} to be

$$dG_{11ABCD} = -3\sqrt{2}\frac{\kappa^2}{\lambda^2}\delta(x^{11})F^a_{[AB}F^a_{CD]} \tag{2.4}$$

This can be seen to arise due to a (step function) discontinuity across the boundary at the orbifold fixed points arising from G being odd under Z_2. This in turn means that C_{IJK} is no longer gauge invariant but transforms as

$$\delta C_{11AB} = -\frac{\kappa^2}{6\sqrt{2}\lambda^2}\delta(x^{11})tr(\epsilon F_{AB}) \tag{2.5}$$

It is clear from this equation that the $C \wedge G \wedge G$ term in the action is no longer gauge invariant which means that the classical theory is not gauge invariant. In order to cancel this anomalous variation one must look at the variation of the 10D fermionic effective action and use the anomaly arising from the quantum theory to cancel the gauge anomaly of the classical theory. What is found is that for a cancellation to occur there must exist a relation between the gauge coupling constant and the gravitational coupling constant i.e.

$$\lambda^2 = 2\pi(4\pi\kappa^2)^{2/3} \tag{2.6}$$

The theory is now anomaly free under gauge transformations, but in order to have a theory devoid of gravitational anomalies the Bianchi identity for G must be modified to read

$$dG_{11ABCD} = -3\sqrt{2}\frac{\kappa^2}{\lambda^2}\delta(x^{11})(F^a_{[AB}F^a_{CD]} - \frac{1}{2}tr(R_{[AB}R_{CD]})) \tag{2.7}$$

whose form can be seen to arise from the usual anomaly combination $trF^2-(1/2)trR^2$.

With these relations one can begin to see some remarkable consequences of H-W M-theory. As explained in [9] if one now compactifies the theory on a Calabi-Yau threefold there exists a prediction for the lower bound on the four dimensional Newton's constant that can be accommodated by experiment. This is in contrast to the case for the weakly coupled heterotic string theory where the value of Newton's constant is predicted to be much larger than the experimental value. One also finds a relation for the GUT coupling constant, α_{GUT}. In terms of the Calabi-Yau volume, V, the radius of the eleventh dimension, ρ, and the 11D gravitational coupling constant, κ, one has

$$G_N = \frac{\kappa^2}{16\pi^2 V\rho} \tag{2.8}$$

$$\alpha_{GUT} = \frac{(4\pi\kappa^2)^{2/3}}{2V} \tag{2.9}$$

where Eq.(2.6) has been utilized. It is natural to assume that the volume of the Calabi-Yau is of order M_{GUT}^{-6}. Then using the experimentally known values of G_N and α_{GUT} one finds that ρ is $\mathcal{O}(10)$ times the GUT scale and that the 11D gravitational constant is on the order of the GUT scale. This is remarkable in that it explains the discrepancy between the 4D Planck scale and the GUT scale (that plagued superstring theory in the 1980s). The 4D gravitational constant is now a derived quantity and it is the 11D Planck scale that is fundamental and it is the order of the GUT scale. Also the value of ρ allows one to consider a five-dimensional picture. Thus if one were to view spacetime along an ever increasing energy range one would first see the usual 3+1 spacetime. At an energy about $(1/\rho)$ an order of magnitude below the GUT scale (which is the compactification scale), one would begin to see a fifth dimension and above the GUT scale one could then access all eleven dimensions of spacetime. It is this view of a five-dimensional universe that we will be concerned with in the next section where we will be dealing with compactifications to five dimensions, and which allows one to consider phenomena at energies $\lesssim M_G$.

C. Compactification to Five Dimensions

We will now follow [15] as we examine the compactification of Horava-Witten theory to five dimensions. To lowest order, space is now of the form $X \times M_4 \times S^1/Z_2$ where X is a Calabi-Yau threefold. In [15] only the (1,1) sector of the Calabi-Yau was examined and here we will assume the number of (1,1) forms (denoted by $h^{1,1}$) is one, the Kähler form. The general form of the five dimensional theory will be that of $N = 1$ supergravity in the bulk and (after the reduction) $N = 1$ supersymmetry on the four dimensional planes at the orbifold fixed points. The field content arises due to a usual reduction on a Calabi-Yau of the field strength G and the potential C. One also must provide a solution to the modified Bianchi identity as well as the equation of motion for G. This is done in [15] using the standard embedding which leaves

$$(dG)_{11ABCD} = -\frac{1}{4\sqrt{2}\pi}(\frac{\kappa}{4\pi})^{2/3}[\delta(x^{11}) - \delta(x^{11} - \pi\rho)](trR \wedge R)_{ABCD} \qquad (2.10)$$

while the field equation is

$$D_I G^{IJKL} = 0 \tag{2.11}$$

For the purposes of obtaining a five-dimensional effective action one only needs to find the zero mode parts of the solution to these equations. This can be solved by introducing a basis of (2,2) forms and four-cycles C_i

$$\frac{1}{v}\int_X \omega_i \wedge \nu^j = \delta_i^j \tag{2.12}$$

$$\frac{1}{v^{2/3}}\int_{C_i} \nu^j = \delta_i^j \tag{2.13}$$

where v is a coordinate volume and ω_i $(i = 1, ..., h^{1,1})$ is a harmonic (1,1) form defined by the expansion of the Kähler form ω_{AB}

$$\omega_{AB} = a^i \omega_{iAB} \tag{2.14}$$

Here the a^i are moduli of the Calabi-Yau space. Now the zero mode part $trR \wedge R|_0$ is expanded in terms of ν^j

$$trR \wedge R\Big|_0 = -8\sqrt{2}\pi(\frac{4\pi}{\kappa})^{2/3}\alpha_i \nu^i \tag{2.15}$$

where α_i are given in terms of the first Pontrjagin class of the Calabi-Yau as

$$\alpha_i = \frac{\pi}{\sqrt{2}}(\frac{\kappa}{4\pi})^{2/3}\frac{1}{v^{2,3}}\beta_i \tag{2.16}$$

$$\beta_i = -\frac{1}{8\pi^2}\int_{C_i} tr(R \wedge R) \tag{2.17}$$

In the case being considered, i=1. Therefore we will call α_i, where i runs over the number of harmonic (1,1) forms, simply α and reserve in the future the notation α_i to indicate the value of α at the orbifold fixed points y_i where here $i = 1$ indicates the fixed point $y = 0 \equiv y_1$ and $i = 2$ indicates the fixed point at $y = \pi\rho \equiv y_2$. In this case

$$\alpha_i = (-1)^i \alpha \quad i = 1, 2 \tag{2.18}$$

With these relations, the zero mode pieces of the Bianchi identity and the equation of motion are solved by

$$\begin{aligned} G_{ABCD}\Big|_0 &= \alpha_i \nu^i_{ABCD}\epsilon(x^{11}) = \frac{1}{4V}\alpha^i \epsilon_{ABCD}{}^{EF}\omega_{iEF}\epsilon(x^{11}) &(2.19)\\ G_{ABC11}\Big|_0 &= 0 &(2.20)\end{aligned}$$

where $\epsilon(x^{11})$ is a step function and V is the volume modulus of the C-Y threefold which is defined as

$$V = \frac{1}{v}\int_X \sqrt{g_{CY}} \tag{2.21}$$

where v is a coordinate volume chosen to keep V dimensionless and g_{CY} is the determinant of the C-Y metric.

Discarding the shape moduli and keeping only the volume modulus of the C-Y threefold, the bosonic part of the reduced five-dimensional Lagrangian takes the following form

$$\begin{aligned} S &= -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{g}[R + \frac{1}{2}V^{-2}\partial_\alpha V\partial^\alpha V + \frac{3}{2}\alpha^2 V^{-2}] \\ &+ \frac{1}{\kappa_5^2}\sum_i \int_{M_4^{(i)}}\sqrt{-g}V^{-1}3(-1)^{i+1}\alpha \\ &- \frac{1}{16\pi\alpha_{GUT}}\sum_i\int_{M_4^{(i)}}\sqrt{-g}V tr F^{i2}_{\mu\nu} \\ &- \sum_i\int_{M_4^{(i)}}\sqrt{-g}\Big[(D_\mu C)^n(D_\mu\bar{C})^n + V^{-1}\frac{\partial W}{\partial C^n}\frac{\partial \bar{W}}{\partial \bar{C}^n} + D^{(\mu)}D^{(\mu)}\Big] \end{aligned} \tag{2.22}$$

where κ_5 is the five-dimensional Newton constant defined as

$$\kappa_5^2 = \frac{\kappa^2}{v} \tag{2.23}$$

R is the five-dimensional Ricci scalar, α_G is the GUT scale coupling constant, $F^{(i)}_{\mu\nu}$ are the gauge field strengths on the boundary orbifolds, C^n are complex scalars of chiral matter, W is the superpotential, and the last term represents the D term of the gauge theory on the branes. The parameter α, which is $\mathcal{O}(10^{15}\mathrm{GeV})$, fixes the bulk and brane cosmological constants.

CHAPTER III

COSMOLOGY IN HORAVA-WITTEN M-THEORY*

Now that we have a five-dimensional action we would like to study the cosmology of such a system. Specifically we would like to determine if the usual RWF cosmology can be recovered. We will use the vacuum solution given in [15]. This vacuum solution automatically fixes the bulk and brane cosmological constants (without any fine tuning) so that when matter is added to the branes the net cosmological constant is correctly zero. We will then add matter perturbatively and examine the resulting cosmology. We will also discuss the inclusion of bulk five-branes in the system. We do not include any mechanism for moduli stabilization.

Previous work on HW cosmology has been given in [16, 17, 18]. However, the first two papers do not impose all the boundary conditions, and hence do not see the difficulties found here. The last paper is concerned with the inflationary era rather that the Hubble expansion era being discussed here. A very general analysis for an arbitrary model was given in [19], but the authors do not seem to have noticed the difficulty discussed here. Within the M-theory framework, there have been several papers suggesting that stabilization of moduli can be achieved by turning on fluxes [20, 21, 22]. The first two give rise to large negative cosmological constants, while the last to a large positive cosmological constant. How to reduce these constants to their physical values in a natural way remains (as well as how to analyse the presence of matter).

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*Reprinted with permission from "Five Dimensional Cosmology in Horava-Witten M-theory" by R. Arnowitt, J. Dent, B. Dutta, 2004. *Physical Review D*, 70,126001:2-7.

A. 5D Equations

The starting point is the action given by Eq.(2.22) and the ansatz

$$ds^2 = a(t,y)^2 dx^k dx^k - n(t,y)^2 dt^2 + b(t,y)^2 dy^2, \tag{3.1}$$

where y$\equiv x^{11}$ is the coordinate of the fifth-dimension which extends from y=0 to y=$\pi\rho$, and t is time. The five-dimensional Einstein field equations are

$$\begin{aligned}
G^t{}_t &= \frac{3}{b^2}\left[\frac{a''}{a} + \frac{a'}{a}\left(\frac{a'}{a} - \frac{b'}{b}\right)\right] - \frac{3}{n^2}\frac{\dot a}{a}\left(\frac{\dot a}{a} + \frac{\dot b}{b}\right) \\
&= -\frac{1}{4}n^{-2}\dot\phi^2 - \frac{1}{4}b^{-2}\phi'^2 - \frac{3}{4}\alpha^2 e^{-2\phi} \qquad (3.2)\\
&\quad -\frac{1}{M_5^3}\sum_{i=1}^{2}\delta(y-y_i)b^{-1}\left(\rho_{ir}e^{\phi_i} + \rho_{inr}e^{-\phi_i} + 3M_5^3\alpha_i e^{-\phi_i}\right)
\end{aligned}$$

$$\begin{aligned}
G^k{}_k &= \frac{1}{b^2}\left[2\frac{a''}{a} + \frac{n''}{n} + \frac{a'}{a}\left(\frac{a'}{a} + 2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n} + 2\frac{a'}{a}\right)\right] \\
&\quad -\frac{1}{n^2}\left[2\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\dot a}{a}\left(\frac{\dot a}{a} - 2\frac{\dot n}{n}\right) + \frac{\dot b}{b}\left(2\frac{\dot a}{a} - \frac{\dot n}{n}\right)\right] \\
&= \frac{1}{4}n^{-2}\dot\phi^2 - \frac{1}{4}b^{-2}\phi'^2 - \frac{3}{4}\alpha^2 e^{-2\phi} \qquad (3.3)\\
&\quad +\frac{1}{M_5^3}\sum_{i=1}^{2}\delta(y-y_i)b^{-1}\left(p_{ir}e^{\phi_i} - 3M_5^3\alpha_i e^{-\phi_i}\right)
\end{aligned}$$

$$\begin{aligned}
G^y{}_y &= \frac{3}{b^2}\frac{a'}{a}\left(\frac{a'}{a} + \frac{n'}{n}\right) - \frac{3}{n^2}\left[\frac{\ddot a}{a} + \frac{\dot a}{a}\left(\frac{\dot a}{a} - \frac{\dot n}{n}\right)\right] \\
&= \frac{1}{4}n^{-2}\dot\phi^2 + \frac{1}{4}b^{-2}\phi'^2 - \frac{3}{4}\alpha^2 e^{-2\phi} \qquad (3.4)
\end{aligned}$$

$$G_{ty} = 3\left(\frac{n'}{n}\frac{\dot a}{a} + \frac{a'}{a}\frac{\dot b}{b} - \frac{\dot a'}{a}\right) = \frac{1}{2}\dot\phi\,\phi' \ , \tag{3.5}$$

where i = 1, 2 corresponds to the fixed points at y=0, $\pi\rho$, prime and dot denote derivatives with respect to y and t respectively, and V = e^{ϕ} where ϕ is the breathing modulus of the Calabi-Yau. The non-relativistic matter density on the i'th orbifold is ρ_{inr}, the relativistic matter is ρ_{ir}, and p_{ir} is the pressure. Their couplings to ϕ are determined from Eq.(2.22); one can see that ρ_r and p_r arise from the $trF^{i2}_{\mu\nu}$ term and are coupled to V while ρ_{nr} come from the term $\frac{\partial W}{\partial C^n}\frac{\partial \bar{W}}{\partial \bar{C}^n}$ which couples to 1/V (These V factors were incorrectly omitted in previous work[16, 17]). Also for cosmological considerations we make the usual assumption that matter is of the form of an ideal fluid

The δ-functions in Eqs.(3.2) and (3.3) imply boundary conditions at the orbifolds y=0, $\pi\rho$ given by

$$(-1)^i \frac{1}{b}\frac{a'}{a}\Big|_{y=y_i} = \frac{\rho_i}{6M_5^3} \ ; \ \rho_i = \rho_{ir} e^{\phi_i} + \rho_{inr} e^{-\phi_i} + 3M_5^3 \alpha_i e^{-\phi_i} \tag{3.6}$$

$$(-1)^i \frac{1}{b}\frac{n'}{n}\Big|_{y=y_i} = -\frac{2\rho_i + 3p_i}{6M_5^3} \ ; \ p_i = p_{ir} e^{\phi_i} - 3M_5^3 \alpha_i e^{-\phi_i} \tag{3.7}$$

$$\phi_i = \phi(y_i) \ ; \ \alpha_i = (-1)^i \alpha \tag{3.8}$$

where M_5 is the five-dimensional Planck mass given by $1/\kappa_5^2$ and ρ_i and p_i are the total matter density and pressure on the two orbifolds. Thus the bulk cosmological constants α, and the brane cosmological constant α_i are naturally correlated without any fine tuning.

In addition to the Einstein field equations, one can derive field equations and boundary conditions for the breathing modulus from the action:

$$-n^{-2}\left[\ddot{\phi} + \left(-\frac{\dot{n}}{n} + 3\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)\dot{\phi}\right] + b^{-2}\left[\phi'' + \left(\frac{n'}{n} + 3\frac{a'}{a} - \frac{b'}{b}\right)\phi'\right] + 3\,\alpha^2 e^{-2\phi} = 0 \tag{3.9}$$

$$\left(\phi' - (3b\alpha - \frac{b}{M^3}\rho_{inr})e^{-\phi}\right)\Big|_{y=y_i} = 0. \tag{3.10}$$

This differs from the result of Ref.[16] in that non-relativistic matter has been included in the boundary condition for ϕ as it should since ϕ is coupled to the gauge fields on the branes at y=0 and $\pi\rho$ by the factor $V^{-1}=e^{-\phi}$ in the superpotential term in Eq.(2.22). (Note that ϕ also couples to gauge fields in Eq.(2.22) with the factor $V=e^{\phi}$, but the coefficient $F_{\mu\nu}F^{\mu\nu}$ vanishes for the radiation fields).

B. Solution of the 5D Equations

We now proceed to solve the field equations using a perturbative expansion in powers of matter on the branes. This expansion is allowed since in the Hubble era the matter density is very small compared to the cosmological constants in the bulk and on the brane which are of GUT size. We start with the vacuum solutions and then include matter on the branes as a higher order correction. The vacuum solution which fully solves the bulk and boundary equations, preserves Poincaire invariance, and breaks 4 of the 8 supersymmetries (appropriate for getting N=1 supergravity when one descends to four dimensions) was given in [15] as

$$a(y) = f^{\frac{1}{2}}\,;\; n(y) = f^{\frac{1}{2}}\,;\; b(y) = b_o f^2\,;\; V(y) = b_o f^3. \tag{3.11}$$

Here b_o is a constant that is arbitrary due to the flat directions of the potential and f is given by

$$f(y) = c + \alpha|y|. \tag{3.12}$$

where c is a constant. To first order in ρ, the vacuum solutions are perturbed to take the following form:

$$a(y,t) = f^{1/2}(1+\delta a(y,t)) \tag{3.13}$$

$$n(y) = f^{1/2}(1+\delta n(y)) \tag{3.14}$$

$$b(y) = b_0 f^2(1+\delta b(y,t)) \tag{3.15}$$

$$V(y) = b_0 f^3(1+\delta V(y,t)) \tag{3.16}$$

It is convenient to introduce the notation

$$\Delta a' \equiv \delta a' + \frac{\alpha}{2f}\delta V - \frac{\alpha}{2f}\delta b \tag{3.17}$$

$$\Delta n' \equiv \delta n' + \frac{\alpha}{2f}\delta V - \frac{\alpha}{2f}\delta b \tag{3.18}$$

$$\Delta V' \equiv \delta V' + \frac{3\alpha}{f}\delta V - \frac{3\alpha}{f}\delta b \tag{3.19}$$

along with the definition of the Hubble constant

$$H \equiv \frac{\dot{a}}{a}. \tag{3.20}$$

The significance of the combinations of Eqs.(3.17-3.19) is that they are invariant under a first order coordinate change in the y coordinate: $\bar{y}$ = y + δ(y).

To first order in ρ, the Einstein equations G_{tt}, G_{kk}, G_{yy}, and the field equation for the breathing modulus become the following:

$$\Delta a'' = b_o^2 f^3 \left(H^2 + H\frac{\dot{b}}{b} - \frac{1}{12}\dot{\phi}^2 \right) \equiv b_o^2 f^3 A_1 \tag{3.21}$$

$$\Delta n'' + 2\Delta a'' = b_o^2 f^3 \left(3H^2 + 2\dot{H} + 2H\frac{\dot{b}}{b} + \frac{\ddot{b}}{b} + \frac{\dot{\phi}^2}{4} \right) \equiv b_o^2 f^3 A_2 \tag{3.22}$$

$$3\Delta a' + \Delta n' - \Delta V' = \frac{b_o^2 f^4}{\alpha} \left(4H^2 + 2\dot{H} - \frac{\dot{\phi}^2}{6} \right) \equiv \frac{b_o^2 f^4}{\alpha} A_3 \tag{3.23}$$

$$\Delta V'' + \frac{3\alpha}{f}\left(\Delta n' + 3\Delta a' - \Delta V'\right) = b_o^2 f^3 \left(\ddot{\phi} + 3H\dot{\phi} + \frac{\dot{b}}{b}\dot{\phi} \right) \equiv b_o^2 f^3 A_4 \tag{3.24}$$

As we will show in Section D, the G_{ty} equation is of higher order and will be discussed later concerning the possibility of including y-dependence in the Hubble constant. Inserting the metric ansatz into the boundary equations (3.6),(3.7), and (3,10) yields

$$\Delta a'_i = (-1)^i \left(\frac{b_o^2 f_i^5}{6M_5^3}\rho_{ir} + \frac{1}{6M_5^3 f_i}\rho_{inr} \right) \tag{3.25}$$

$$\Delta n'_i = (-1)^{i+1} \left(\frac{b_o^2 f_i^5}{6M_5^3}\left(2\rho_{ir} + 3p_{ir}\right) + \frac{2}{6M_5^3 f_i}\rho_{inr} \right) \tag{3.26}$$

$$\Delta V_i' = (-1)^{i+1} \frac{1}{M_5^3 f_i} \rho_{inr} \tag{3.27}$$

where f_i are the values of f(y) on the branes:

$$f_1 \equiv c \;\; ; \;\; f_2 \equiv c + \alpha_2 \pi \rho \tag{3.28}$$

It will also be helpful to use the combination of Eq.(3.25) and Eq.(3.26) that isolates ρ_{inr}:

$$(3\Delta a' + \Delta n')_{y_i} = (-1)^i \frac{1}{6 M_5^3 f_i} \rho_{inr} \tag{3.29}$$

where we have used p_{ir}=$\rho_{ir}/3$. The significance of the above results is that *both* the field equations *and* the boundary conditions can be expressed in terms of the y-invariant combinations of Eqs.(3.17-3.19).

We are now ready to examine the solution of the bulk equations, impose the boundary conditions on them, and check their consistency. First of all we notice that the equations G_{tt} and G_{kk} are easily solved by integration with respect to y. However, it will be more convenient to change integration variables from y to f and to use the combination $G_{tt} + G_{kk}$ instead of G_{kk}. One finds

$$\Delta a' = \frac{b_o^2}{\alpha} \left(\int_{f_1}^{f} df' f'^3 A_1 + c_1 \right) \tag{3.30}$$

$$3\Delta a' + \Delta n' = \frac{b_o^2}{\alpha} \left(\int_{f_1}^{f} df' f'^3 \left(A_1 + A_2 \right) + c_1 + c_2 \right) \tag{3.31}$$

and imposing the boundary conditions Eqs.(3.25) and (3.26) one obtains

$$c_1 = -\frac{\lambda}{6} \left(b_o^2 f_1^5 \rho_{1r} + \frac{1}{f_1} \rho_{1nr} \right) \;\; ; \;\; \lambda \equiv \frac{\alpha}{b_o^2 M_5^3} \tag{3.32}$$

$$\int_{f_1}^{f_2} df' f'^3 A_1 = \frac{\lambda}{6} \left(b_o^2 \left(\rho_{1r} f_1^5 + \rho_{2r} f_2^5 \right) + \left(\frac{\rho_{1nr}}{f_1} + \frac{\rho_{2nr}}{f_2} \right) \right) \tag{3.33}$$

$$c_1 + c_2 = -\frac{\lambda}{6 f_1} \rho_{1nr} \tag{3.34}$$

$$\int_{f_1}^{f_2} df' f'^3 (A_1 + A_2) = \frac{\lambda}{6} \left(\frac{1}{f_1} \rho_{1nr} + \frac{1}{f_2} \rho_{2nr} \right). \tag{3.35}$$

Eqs.(3.33) and (3.35), arising from the boundary conditions at the distant brane y=$\pi\rho$, thus produce constraints on the time derivatives (which enter in the A_i defined in Eqs.(3.21-3.24)) of the metric and ϕ. We next integrate the field equation for the breathing modulus, Eq.(3.24), using Eq.(3.23) to eliminate $3\Delta a' + \Delta n'$ -$\Delta V'$. Combined with the boundary condition Eq.(3.27) one obtains

$$\Delta V' = \frac{b_o^2}{\alpha} \left(\int_{f_1}^{f} df' f'^3 (A_4 - 3A_3) + c_3 \right) \ ; \ c_3 = \frac{\lambda}{f_1} \rho_{1nr} \tag{3.36}$$

$$\int_{f_1}^{f_2} df' f'^3 (A_4 - 3A_3) = -\lambda \left(\frac{1}{f_1} \rho_{1nr} + \frac{1}{f_2} \rho_{2nr} \right). \tag{3.37}$$

The remaining equation to be satisfied is Eq.(3.23). Inserting Eqs.(3.31) and (3.36) back into Eq.(3.23) gives the constraint

$$\frac{b_o^2}{\alpha} \left(\int_{f_1}^{f} df' f'^3 (A_1 + A_2 + 3A_3 - A_4) + c_1 + c_2 - c_3 \right) = \frac{b_o^2 f^4}{\alpha} A_3 \tag{3.38}$$

This constraint is a strong one as it must hold for all y.

We now examine the consistency of this system. We consider here the static case where $\dot{\phi} = 0 = \dot{b}$. Here A_4=0 and $A_3 = A_1 + A_2$. Thus multiplying Eq.(3.35) by 3 and adding to Eq.(3.37) gives

$$0 = -\frac{\lambda}{2} \left(\frac{1}{f_1} \rho_{1nr} + \frac{1}{f_2} \rho_{2nr} \right) \tag{3.39}$$

Thus a consistent solution without fine tuning requires (when $\dot{\phi} = 0 = \dot{b}$)

$$\rho_{nr} = 0 \tag{3.40}$$

i.e. only relativistic matter is consistent with Horava-Witten cosmological equations. However, in addition to Eq.(3.40) one must also make sure that the constraint

Eq.(3.38) is satisfied. In Section D we will show that the G_{ty} equation implies H' is $\mathcal{O}(\rho^{3/2})$ for the static case, and hence to $\mathcal{O}(\rho)$ that we are calculating one can consider H^2 and $\dot{H}$ to be independent of y. Hence we note that for the static case Eqs.(3.33) and (3.35) correctly reduce to the RFW cosmology equations for relativistic matter with G_N defined in terms of λ and f_i:

$$H^2 = \frac{8\pi}{3} G_N \left(\rho'_{1r} + \rho'_{2r}\right) \;;\; G_N = \frac{\lambda}{4\pi}\left(\frac{1}{f_2^4 - f_1^4}\right) \tag{3.41}$$

$$H^2 + 2\dot{H} = 0 \tag{3.42}$$

(where the rescaled $\rho'_{ir} = b_o f_i^5 \rho_{ir}$ are the mass densities as seen locally on the orbifold 3-branes). Eqs.(3.41) and (3.42) just incorporate the 4D relativistic matter equation of continuity: $\dot{\rho}$ = -3$H(\rho$ + p) = -4 $H\rho$. Since $A_1 + A_2 = A_3$, Eq.(3.38) is then identically satisfied as a consequence of Eqs.(3.40) and (3.42).

In summary we note that it is the boundary conditions on the distant brane at y $= \pi\rho$, Eqs.(3.35) and (3.37), that produces the constraint Eq.(3.40) on non-relativistic matter (which is why earlier analyses have not seen this). However, a satisfactory FRW cosmology does result for relativistic matter, with the brane cosmological constant naturally vanishing with no fine tuning required.

C. Inclusion of 5-Branes

We have shown that in the static case ($\dot{\phi}$, $\dot{b}$ = 0) the system of bulk equations with their boundary conditions imposed is inconsistent when non-relativistic matter is included in the system. Therefore we would like to examine other situations that might lead to a consistent solution when all types of matter are present. The only additional generalization available in the Horava-Witten theory is to include a set of 5-branes in the bulk transverse to the orbifold direction [9, 23]. We follow here the analysis of [24] of a single 5-brane residing at an arbitrary position y=Y in the bulk (which can easily be generalized to an arbitrary number of 5-branes). The fields that live on the 5-brane include an N=1 chiral multiplet and N=1 gauge multiplets but no superpotential. One must generalize the function f(y) and the definition of α to be (for 0≤y≤$\pi\rho$)

$$f(y) = c + h(y)\ ;\ h(y) = -\alpha_1 y + (-\alpha_5 y + \alpha_5 Y)\,\theta\,(y-Y) \tag{3.43}$$

$$h'(y) \equiv \alpha(y) = -\alpha_1 - \alpha_5\theta\,(y-Y) \tag{3.44}$$

$$\alpha(y=Y) \equiv \alpha_3 = \frac{\alpha_5}{2} \tag{3.45}$$

with the cohomology condition

$$\alpha_1 + \alpha_2 + \alpha_5 = 0 \tag{3.46}$$

but otherwise the vacuum solution has the same form as Eq.(3.11). Note that f(y) is continuous whereas α(y) = h'(y) is not

$$\alpha(y) = \begin{cases} -\alpha_1 & 0 \le y < Y \\ -\alpha_1 - \alpha_5 = \alpha_2 & Y < y \le \pi\rho \end{cases} \tag{3.47}$$

We use the same ansatz as before for the metric: Eqs. (3.12),(3.13),(3.14),(3.15), and (3.16). However, the Einstein equations are altered due to the fact that α is no longer a constant; thus α' no longer vanishes. We also must make some assumptions about the matter content of the five-brane in the bulk. We know that in order to give rise to the Big Bang at the end of inflation that the inflaton couples to matter on the physical orbifold at y=0 and we have assumed that it also couples to matter on the orbifold at y=$\pi\rho$. There is no a priori reason to believe that it also couples to any matter fields on the 5-brane in the bulk. However, we will make the assumption that it does couple to the five-brane matter with the same strength V as for the two orbifolds (this does not effect the general conclusions of this section).

Now we would like to solve the Einstein equations in the presence of this 5-brane. The calculation is very similar to that done in the previous section with the modifications that i runs from 1 to 3 and α' terms must now be included. For example in the G_{tt} equation we still have the definition

$$\Delta a' \equiv \delta a' + \frac{\alpha}{2f}\delta V - \frac{\alpha}{2f}\delta b. \tag{3.48}$$

However $\Delta a''$ is

$$\Delta a'' = \delta a'' + \frac{\alpha^2}{2f^2}\left(\delta b - \delta V\right) - \frac{\alpha}{2f}\left(\delta b' - \delta V'\right) - \frac{\alpha'}{2f'}\left(\delta b - \delta V\right) \tag{3.49}$$

and after making use of the ansatz, the G_{tt} equation becomes

$$\Delta a'' = -\frac{1}{M_5^3}\left(\rho_{3r} b_o^2 f_3^5 + \frac{1}{f_3}\rho_{3nr}\right)\delta\left(y - Y\right) + b_o^2 f^3 A_1 \tag{3.50}$$

where the δ-function term arises from a y-derivative of α, and A_1 is given in Eq.(3.21). The key point in this relation is the unexpected result that the α_5 terms cancel, not only at the vacuum order (which was already verified in showing Eq.(3.11) with f(y) of Eq.(3.43) is correctly the vacuum solution), but also at higher orders. This also applies to the other field equations.

Since there is a discontinuity in α when crossing the five-brane, we solve the G_{tt} equation in two domains: $0 \leq y < Y$ and $Y < y \leq \pi\rho$ and then match them across y=Y. We define the gauge covariant combinations in these regions as

$$\Delta a_1' \equiv \delta a' + \frac{\alpha_1}{2f}\left(\delta b - \delta V\right)\ ;\ 0 \leq y < Y \tag{3.51}$$

$$\Delta a_2' \equiv \delta a' - \frac{\alpha_2}{2f}\left(\delta b - \delta V\right)\ ;\ Y < y \leq \pi\rho. \tag{3.52}$$

These quantities obey the boundary conditions

$$\Delta a_1'(y=0) = -\frac{1}{M_5^3}\left(\rho_{1r} b_o^2 f_1^5 + \frac{1}{f_1}\rho_{1nr}\right) \tag{3.53}$$

$$\Delta a_2'(y=\pi\rho) = \frac{1}{M_5^3}\left(\rho_{2r} b_o^2 f_2^5 + \frac{1}{f_2}\rho_{2nr}\right). \tag{3.54}$$

We now solve the G_{tt} equation in these two domains to obtain

$$\Delta a_1' = \int_0^y dy' b_o^2 f^3 A_1 - \frac{1}{M_5^3}\left(\rho_{1r} b_o^2 f_1^5 + \frac{1}{f_1}\rho_{1nr}\right) \tag{3.55}$$

$$\Delta a_2' = -\int_y^{\pi\rho} dy' b_o^2 f^3 A_1 + \frac{1}{M_5^3}\left(\rho_{2r} b_o^2 f_2^5 + \frac{1}{f_2}\rho_{2nr}\right). \tag{3.56}$$

The discontinuity across the five-brane required by Eq.(3.50) gives

$$\int_{Y-\epsilon}^{Y+\epsilon} dy' \Delta a'' = (\Delta a_2' - \Delta a_1')_{y=Y} = -\frac{1}{M_5^3}\left(\rho_{3r} b_o^2 f_3^5 + \frac{1}{f_3}\rho_{3nr}\right) \tag{3.57}$$

Thus, subtracting (3.55) from (3.56) leads to

$$\int_0^{\pi\rho} dy' b_o^2 f^3 A_1 = \sum_{i=1}^{3} \frac{1}{M_5^3}\left(\rho_{ir} b_o^2 f_3^5 + \frac{1}{f_3}\rho_{inr}\right) \tag{3.58}$$

If we make the assumption as before that $\dot{\phi} = 0 = \dot{b}$ this equation will give the RFW relation for the Hubble parameter but now with matter from three separate branes included. However,the situation with no matter included on the 5-brane in the bulk does not reduce to the case with only two branes since α and therefore the function f(y) are modified from the case where only two branes were present. Therefore, the Hubble law (more specifically the Newton constant, G_N) is affected by the presence of the additional five brane even if it is empty.

The other field equations can be solved in a manner similar to that described above for the G_{tt} equation. Namely, we first modify the equations with the inclusion of terms involving the y-derivative of α, look at the equations separately in the regions, 0$\leq$ y $<$ Y and Y $<$ y $\leq \pi\rho$, and then match them across the five brane. The result for the ϕ equation in the bulk is

$$\Delta V'' + \frac{3\alpha}{f}\left(\Delta n' + 3\Delta a' - \Delta V'\right) = b_o^2 f^3 A_4 + \frac{1}{M_5^3 f_3}\rho_{3nr}\delta\left(y - Y\right) \tag{3.59}$$

which leads to

$$\int_0^{\pi\rho} dy' b_o^2 f^3 \left(A_4 - 3A_3\right) = -\sum_{i=1}^{3} \frac{1}{M_5^3 f_i}\rho_{inr}. \tag{3.60}$$

The G_{kk} equation gives

$$\int_0^{\pi\rho} dy' b_o^2 f^3 \left(A_1 + A_2\right) = \sum_{i=1}^{3} \frac{1}{6M_5^3 f_i}\rho_{inr}. \tag{3.61}$$

The G_{yy} equation remains unchanged from Eq.(3.23) since there are no matter sources present but where now the gauge invariant combinations Δa, Δn, and ΔV include the new definitions of α(y) and f(y). We can now proceed to check the consistency using the same steps that led to Eq.(3.39). The new relation is

$$\left(\frac{1}{f_1}\rho_{1nr} + \frac{1}{f_2}\rho_{2nr} + \frac{1}{f_3}\rho_{3nr}\right) = 0 \tag{3.62}$$

Once again we see that introducing non-relativistic matter into the system results in an inconsistency if there is no fine tuning of the matter on the different branes.

D. y-Dependence and Non-Relativistic Matter

Assuming the static case for ϕ and b we have shown that the system does not admit consistent solutions in the presence of non-relativistic matter. We would now like to relax this assumption and then examine the system. We consider here the case of no 5-branes present. To see what constraints are put on our assumptions, let us first look at two separate ways of evaluating the y-derivative of H and compare this with the G_{ty} equation. The definition of H is

$$H \equiv \frac{\dot{a}}{a}. \tag{3.63}$$

Therefore

$$H' = \frac{\dot{a}'}{a} - \frac{a'}{a}\frac{\dot{a}}{a} \tag{3.64}$$

and using the explicit form of $a(y,t)$ in Eq.(3.13) we find

$$\frac{\dot{a}'}{a} = \left(\frac{\alpha}{2f} + \delta a'\right) H + H'. \tag{3.65}$$

Alternately

$$\frac{a'}{a} = \frac{\alpha}{2f} + \delta a' \tag{3.66}$$

which gives

$$\frac{\dot{a}'}{a} = \left(\frac{\alpha}{2f} + \delta a'\right) H + \delta\dot{a}'. \tag{3.67}$$

Comparing Eq.(3.65) with Eq.(3.67) and using the fact that in the static case $\Delta\dot{a}' = \delta\dot{a}'$ we find

$$\Delta\dot{a}' = H'. \tag{3.68}$$

In the static case the G_{ty} equation is given by

$$\Delta\dot{a}' = (\Delta n' - \Delta a')\, H. \tag{3.69}$$

The right hand side is seen to be of order $\rho^{3/2}$ which shows from Eq.(3.68) that H' is also of order $\rho^{3/2}$. Thus the static case *requires* H to be independent of y to first order in ρ.

Let us next examine the situation when we let ϕ and b depend on time. The Einstein equations G_{tt}, $G_{kk} + G_{tt}$, G_{yy}, and G_{yy} plus the ϕ equation of motion

(Eqs.(3.33),(3.35),(3.37),(3.38)) give

$$\int_{f_1}^{f_2} df' f'^3 A_1 = \frac{\lambda}{6}\sum_{i=1}^{2}\left(b_o^2\rho_{ir} f_i^5 + \frac{\rho_{inr}}{f_i}\right) \tag{3.70}$$

$$\int_{f_1}^{f_2} df' f'^3 (A_1 + A_2) = \frac{\lambda}{6}\sum_{i=1}^{2}\frac{\rho_{inr}}{f_i} \tag{3.71}$$

$$\int_{f_1}^{f_2} df' f'^3 (3A_3 - A_4) = \lambda\sum_{i=1}^{2}\frac{\rho_{inr}}{f_i} \tag{3.72}$$

$$\int_{f_1}^{f} df' f'^3 (A_1 + A_2 + 3A_3 - A_4) = f^4 A_3(y) + \frac{7\lambda}{6}\frac{\rho_{1nr}}{f_1} \tag{3.73}$$

In particular evaluating Eq.(3.73) at y = y_1 gives

$$A_3(y_1) = -\frac{7\lambda\rho_{1nr}}{6f_1^5}. \tag{3.74}$$

However, considering the combination Eq.(3.71) + Eq.(3.72) - Eq.(3.73), where Eq.(3.73) is evaluated at y = y_2, yields

$$A_3(y_2) = \frac{7\lambda\rho_{2nr}}{6f_2^5} \tag{3.75}$$

and therefore we see that A_3 has non-trivial y-dependence. Since A_3 only depends on H and $\dot{\phi}$ this implies that $\dot{\phi}$ depends on y. With the static condition relaxed we can now see that the Einstein equations contain integrals over functions whose y-dependence is not known. Therefore relations such as Eq.(3.39) are no longer valid. Eq.(3.39) now becomes

$$\int_{f_1}^{f_2} df' \left(9H\frac{\dot{b}}{b} + 3\frac{\ddot{b}}{b} + \dot{\phi}^2 + \ddot{\phi} + \frac{\dot{b}}{b}\dot{\phi} + 3H\dot{\phi}\right) = -\frac{\lambda}{2}\left(\frac{1}{f_1}\rho_{1nr} + \frac{1}{f_2}\rho_{2nr}\right) \tag{3.76}$$

and we see that non-relativistic matter is now related to an integral over an unknown function of y. Without knowing the exact form of the integrand it is difficult to determine if there is any constraint on the non-relativistic matter with the static constraint relaxed.

CHAPTER IV

RANDALL SUNDRUM MODEL

The phenomenological Randall-Sundrum model (RS)[25, 26] is built upon the same geometrical framework as the reduced five dimensional Horava-Witten theory. It is a five-dimensional model with two 3-branes separated by an interval. (There are also Randall-Sundrum models that consist of only one brane but here we will consider the two brane model for cosmology and when discussing gravitational forces we will discuss some differences that arise between the one and two brane models). First we will discuss the general set-up for the model including the vacuum solutions and their consequences. Then we will describe a stabilization mechanism due to Goldberger and Wise (GW)[27] followed by an examination of cosmology in the model. We will then elaborate the differences between the RS model and the HW theory. Finally we will explore gravitational forces in the model.

A. The Randall Sundrum Vacuum Solution

In the Randall-Sundrum model (RS) the universe is five dimensional with the fifth dimension being an interval bounded by two 3-branes. This model is inspired by 5D compactifications of Horava-Witten theory. In RS the geometry is warped, i.e. the 4D metric has a dependence on the fifth dimension, y:

$$ds^2 = e^{-2A(y)} g_{ij} dx^i dx^j + dy^2 \tag{4.1}$$

$e^{-2A(y)}$ is called the warp factor and will have interesting consequences for determination of mass scales in the RS model. The warped geometry in this model is also attractive since warped geometries arise frequently in string theory, for example in type IIB orientifold compactifications with fluxes present[28, 29]. The fifth dimension has the symmetry of S^1/Z_2 just as in HW which is an interval with the endpoints identified and symmetric under $y \to -y$. There are two orbifold fixed points at $y_1 = 0$ and $y_2 = \pi\rho$. A 3-brane is located at each orbifold fixed point. The brane at y_1 is called the Planck brane and the brane at y_2 is the TeV brane where Standard Model particles and fields reside.

At the vacuum level one can consider RS to be a gravitational theory in the bulk with additional terms on the branes. (These terms will include the matter that resides on the brane as well as a potential term). The action is given by

$$
\begin{aligned}
S &= S_{bulk} + S_{brane} && (4.2)\\
S_{bulk} &= \int d^5x\sqrt{-^5g}\left(-\frac{R}{2\kappa^2} - \Lambda\right) && (4.3)\\
S_{brane} &= \sum_{\alpha}\int d^4x\sqrt{-^4g}(\mathcal{L}_{m_\alpha} - \mathcal{V}(y_\alpha))\delta(y - y_\alpha) \;\; \alpha = 1, 2 && (4.4)
\end{aligned}
$$

where $\mathcal{L}_{m_\alpha}$ is the matter Lagrangian on the brane located at y_α and $\mathcal{V}(y_\alpha)$ is the brane potential. Neglecting matter the metric Eq.(4.1) (with g_{ij} = the Lorentz metric η_{ij}) yields the Einstein equations for the vacuum

$$4A'^2 - A'' = -\frac{2\Lambda}{3M_5^3} - \sum_{\alpha}\frac{\delta(y - y_\alpha)}{3M_5^3}\mathcal{V}(y_\alpha) \tag{4.5}$$

$$4A'^2 - 4A'' = -\frac{2\Lambda}{3M_5^3} - \frac{4}{3}\sum_{\alpha}\frac{\delta(y - y_\alpha)}{3M_5^3}\mathcal{V}(y_\alpha) \tag{4.6}$$

where prime denotes a derivative with respect to y. The discontinuities that arise in the second derivatives of the metric across the branes lead to the boundary conditions

$$(-1)^{\alpha+1}A' = \frac{\mathcal{V}(y_\alpha)}{6M_5^3} \tag{4.7}$$

The solution to the bulk equations subject to these boundary conditions is

$$A = |y|\sqrt{-\frac{\Lambda}{6M_5^3}} \equiv \beta|y| \tag{4.8}$$

where

$$\beta \equiv \sqrt{-\frac{\Lambda}{6M_5^3}} \tag{4.9}$$

One also must choose the potentials $\mathcal{V}(y_\alpha)$ to be

$$\mathcal{V}(y_0) = -\mathcal{V}(y_1) = 6M_5^3\beta \tag{4.10}$$

One sees that the bulk space is a slice of AdS$_5$ (since $\Lambda < 0$) with radius of curvature given by $1/\beta$. Eq.(4.10) also has the effect of setting the 4D cosmological constant on the branes to zero (which is required since we have set $g_{ij} = \eta_{ij}$). This is one difference between the HW theory and the RS model. In the RS model the brane potentials are chosen as in Eq.(4.10) which is a fine-tuning that sets the brane cosmological constants to zero whereas in the HW theory the potentials are fixed by the internal consistency requirements of the theory and naturally produced a zero cosmological constant.

B. Mass Scales in the Model

The 5D metric now becomes

$$ds^2 = e^{-2\beta|y|}g_{ij}dx^i dx^j + b^2 dy^2 \tag{4.11}$$

which shows that there can be a large scale factor difference between the Planck and TeV brane locations due to the exponentially decreasing dependence on y of the warp factor. The relation $g_{ij}^{TeV} = e^{-2\beta y_2} g_{ij}^{Pl}$ has an affect on the physical masses as follows.

In the RS model all fundamental masses are taken to be of Planck size. Now imagine a scalar field with mass parameter m_o that resides on the TeV brane with the action

$$S_{TeV} = \int d^4x \sqrt{-g^{TeV}} (g^{ij\,TeV} D_i\phi D_j\phi - \frac{1}{2} m_o^2 \phi^2) \tag{4.12}$$

and then using the relation between the metric at the Planck brane and the TeV brane one finds

$$S_{TeV} = \int d^4x \sqrt{-g^{Pl}} e^{-4\beta y_2} (g^{ij\,Pl} e^{2\beta y_2} D_i\phi D_j\phi - \frac{1}{2} m_o^2 \phi^2) \tag{4.13}$$

Then after the renormalization $\tilde{\phi} = e^{-\beta y_2}\phi$ to achieve the canonical form for the kinetic energy term one has

$$S = \int d^4x \sqrt{-g^{Pl}} (g^{ij\,Pl} D_i\tilde{\phi} D_j\tilde{\phi} - \frac{1}{2} e^{-2\beta y_2} m_o^2 \tilde{\phi}^2) \tag{4.14}$$

The mass term can be seen to be renormalized which determines that a mass parameter m_o on the TeV brane gets a physical mass

$$m = m_o e^{-\beta y_2} \tag{4.15}$$

which can produce TeV masses from Planck masses for $\beta y_2 \simeq 50$. This is a method of creating large mass hierarchies from natural (not extremely large) quantities.

Now that we have studied the basic underlying structure of the RS model, in the next chapter we will turn to a discussion of Hubble era cosmology in this system.

CHAPTER V

COSMOLOGY IN THE RS MODEL*

In this chapter we will determine whether the conventional RWF 4D cosmology can be recovered in the RS model. This problem has been studied by many authors stimulated first by the work of Binetruy, Deffayet, and Langlois[30]. It has been argued that the standard four-dimensional RFW cosmology is obtained only if the fifth dimension is stabilized [31] (though counter arguments have been given in [32]). In order to achieve stabilization naturally, it has been suggested that one phenomenologically add a scalar field in the bulk with appropriate bulk and brane potentials [27, 33]. Vacuum solutions appropriate for these potentials are then obtained and solutions are discussed with brane matter viewed as perturbations on the vacuum in [34, 35]. The work of [27] and [35] however did not include the effect of back-reaction of the scalar field on the metric, which we will see is important.

In constructing solutions we will follow the same procedure as in the HW case, we will introduce matter perturbatively on the branes and look for solutions to the bulk Einstein equations along with the bulk scalar field equation subject to brane boundary conditions. In general it is very difficult to find vacuum solutions for the Einstein-scalar field equations with arbitrary brane and bulk potentials. Further, we have seen from our study of HW theory how important it is to have a rigorous vacuum solution since it is the careful imposition of boundary conditions on both branes that produces tension in the system. In Ref.[34], matter on the branes was added and treated perturbatively with respect to a particular choice of scalar field

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potentials. However, an explicit vacuum solution for their choice of potentials was not obtained, and the difficulty in obtaining a solution is discussed in Appendix A which tends to invalidate their analysis. However, a rigorous special class of solutions for the vacuum metric obeying all boundary conditions on the branes was constructed by deWolfe et. al [33] where the brane and bulk potentials are related to a single function of the scalar field. Here we will discuss the case of [33] with matter treated as a perturbative addition on the branes , since this case treats the vacuum solution rigorously.

We will also comment on the differences between the HW theory and the RS phenomenology which will become apparent in the following calculation.

A. 5D Equations and Solutions

We begin with the action which consists of gravity and a scalar field in the bulk along with potentials and matter on the branes:

$$S = \int d^5x\sqrt{-^5g}(-\frac{1}{4}R + \frac{1}{2}(\partial\phi)^2 - V(\phi)) + \sum_{\alpha=1,2}\int d^5x\sqrt{-^4g}(\mathcal{L}_{m_\alpha} + \lambda_\alpha(\phi))\delta(y-y_\alpha)$$

Where $V(\phi)$ is the bulk potential and $\lambda_i(\phi)$, i=1,2 are the potentials on the two branes at $y = y_1$ and $y = y_2$ respectively and $\mathcal{L}_{m_\alpha}$ are the matter Lagrangians on the branes. The metric is given by[1]

$$ds^2 = e^{2N(t,y)}dt^2 - e^{2A(t,y)}\sum_i dx_i^2 - b(t,y)^2dy^2 \tag{5.1}$$

with the perturbative expansions

$$N(t,y) \;\; = \;\; A_o(y) + \delta N(t,y) \tag{5.2}$$

[1]In this section we use the notation i=1,2,3 for the 4D spatial coordinates.

$$\begin{aligned} A(t,y) &= A_o(y) + \delta A(t,y) && (5.3)\\ b(t,y) &= 1 + \delta b(t,y) && (5.4)\\ \phi(t,y) &= \phi_o(y) + \delta\phi(t,y). && (5.5) \end{aligned}$$

A_o and ϕ_o are the vacuum fields and $\delta N, \delta A, \delta b$, and $\delta\phi$ are the perturbations due to matter. Since $V(\phi)$ and $\lambda_\alpha(\phi)$ are chosen to stabilize the system we will be working in the static case where $\dot{\phi} = 0 = \dot{b}$.

The vacuum equations in the bulk are

$$\phi_o'' + 4A_o'\phi_o' = V'(\phi_o) \tag{5.6}$$

$$A_o'' = -\frac{2}{3}\phi_o'^2 \tag{5.7}$$

$$A_o'^2 = -\frac{1}{3}V(\phi_o) + \frac{1}{6}\phi_o'^2 \tag{5.8}$$

where primes denote $\frac{\partial}{\partial y}$ except on $V(\phi)$ where it represents $\frac{\partial}{\partial \phi}$. The boundary conditions are given by

$$A_o'\Big|_{y=y_i} = (-1)^i \frac{b(y)}{3}\lambda_i(\phi_o)\Big|_{y=y_i} \tag{5.9}$$

$$\phi_o'\Big|_{y=y_i} = (-1)^{i+1}\frac{b(y)}{2}\frac{\partial\lambda_i(\phi_o)}{\partial\phi_o}\Big|_{y=y_i} \tag{5.10}$$

where i = 1,2 corresponds to the brane locations at y = y_1, y_2.

It was shown in [33] that if $V(\phi_o)$ has the form

$$V(\phi_o) = \frac{1}{8}\left(\frac{\partial W(\phi_o)}{\partial\phi_o}\right)^2 - \frac{1}{3}W(\phi_o)^2 \tag{5.11}$$

for some $W(\phi_o)$, then a solution of the bulk vacuum equations will also satisfy

$$\phi_o' = \frac{1}{2}\frac{\partial W(\phi_o)}{\partial\phi_o} \quad , \quad A_o' = -\frac{1}{3}W(\phi_o) \tag{5.12}$$

as long as the boundary conditions

$$W(\phi_o)\Big|_{y=y_i} = (-1)^{i+1}\lambda_i(\phi_o)\Big|_{y=y_i} \quad , \quad \frac{\partial W(\phi_o)}{\partial \phi_o}\Big|_{y=y_i} = (-1)^{i+1}\frac{\partial \lambda_i(\phi_o)}{\partial \phi_o}\Big|_{y=y_i} \tag{5.13}$$

are also satisfied. Once $W(\phi_o)$, λ_1, and λ_2 are chosen one is then left with a set of first order differential equations that can easily be solved.

In the example constructed in [33] $W(\phi_o)$, λ_1, and λ_2 were chosen to be

$$W(\phi_o) = \frac{3}{L} - b\phi_o^2 \tag{5.14}$$

$$\lambda_1(\phi_o) = W(\phi_o(y_1)) + W'(\phi_o(y_1))(\phi_o - \phi_o(y_1)) + \gamma_1(\phi_o - \phi_o(y_1))^2 \tag{5.15}$$

$$\lambda_2(\phi_o) = -W(\phi_o(y_2)) - W'(\phi_o(y_2))(\phi_o - \phi_o(y_2)) + \gamma_2(\phi_o - \phi_o(y_2))^2 \tag{5.16}$$

which gives

$$\phi_o(y) = \phi_1 e^{-\beta y} \tag{5.17}$$

$$A_o(y) = a_o - \frac{y}{L} - \frac{1}{6}\phi_1^2 e^{-2\beta y} \tag{5.18}$$

where a_o, L, γ_1, γ_2, ϕ_1, and ϕ_2 are all arbitrary parameters. The relationship between $\lambda_i(\phi_o)$ and $W(\phi_o)$ fine tunes the net cosmological constant on the branes to zero. While there is no a priori motivation for the choice (other than the fact it fine tunes the cosmological constant to zero), it leads to simple analytic forms, Eqs.(5.17),(5.18), which can be treated easily.

Next we define the quantities

$$\Delta A' = \delta A' + \frac{2}{3}\phi_o'\delta\phi - A_o'\delta b \tag{5.19}$$

$$\Delta N' = \delta N' + \frac{2}{3}\phi_o'\delta\phi - A_o'\delta b \tag{5.20}$$

$$\Delta V' \;=\; \phi_o' \delta\phi' - \phi_o'' \delta\phi - \phi_o'^2 \delta b. \tag{5.21}$$

These quantities are invariant under a first order coordinate change in the y coordinate: $\bar{y}$ = y + δ(y).

To first order in the matter the Einstein equations G_{tt}, G_{kk}, G_{yy}, and the field equation for ϕ in the bulk become

$$\Delta A'' + 4A_o'\Delta A' = e^{-2A_o(y)} H^2 \equiv A_1 \tag{5.22}$$

$$2\Delta A'' + 8A_o'\Delta A' + \Delta N'' + 4A_o'\Delta N' = e^{-2A_o(y)} \left(3H^2 + 2\dot{H}\right) \equiv A_2 \tag{5.23}$$

$$A_o'\left(3\Delta A' + \Delta N'\right) - \frac{2}{3}\Delta V' = e^{-2A_o(y)} \left(2H^2 + \dot{H}\right) \equiv A_3 \tag{5.24}$$

$$\Delta V'' + 4A_o'\Delta V' + \phi_o'^2\left(3\Delta A' + \Delta N'\right) = 0. \tag{5.25}$$

The boundary conditions are

$$\Delta A'\Big|_{y=y_i} = (-1)^i \frac{\rho_i}{3} \tag{5.26}$$

$$\left(3\Delta A' + \Delta N'\right)\Big|_{y=y_i} = (-1)^i \frac{\rho_{inr}}{3} \tag{5.27}$$

$$\Delta V'\Big|_{y=y_i} = \delta\phi_i \left((-1)^{i+1} \gamma_i \phi_o' - \phi_o''\right)\Big|_{y=y_i} \tag{5.28}$$

where $\rho_i = \rho_{ir} + \rho_{inr}$ and γ_i are free parameters in the model of Eqs.(5.15),(5.16) i.e.

$$\gamma_i = \frac{1}{2}\frac{\partial^2 \lambda_i}{\partial \phi^2}. \tag{5.29}$$

Immediately we see a difference between HW theory and the RS phenomenology manifesting itself in the boundary conditions for $\Delta V'$. In HW only the invariant quantity $\Delta V'$ appears in the boundary conditions whereas an additional $\delta\phi$ term appears in the RS model. One can easily check that in HW there is no free parameter analogous to γ_i, and the M-theory choice of potential makes the term in parenthesis on the right-hand side of Eq.(5.28) zero. This distinction will allow one to avoid the constraint on matter found in the HW theory.

We will now solve the system in the presence of arbitrary matter. The G_{tt} equation can be integrated to obtain

$$\Delta A' = e^{-4A_o(y)} \int_{y_1}^{y} dy' e^{2A_o(y')} H^2 + e^{-4A_o(y)} c_1 \tag{5.30}$$

and using the boundary conditions we get

$$c_1 = -\frac{\rho_1}{3} e^{4A_o(y_1)} \tag{5.31}$$

$$\int_{y_1}^{y_2} dy' e^{2A_o(y')} H^2 = \frac{\rho_1}{3} e^{4A_o(y_1)} + \frac{\rho_2}{3} e^{4A_o(y_2)} \tag{5.32}$$

If H is independent of y one can see that the usual Friedman relation is recovered. Eq.(5.30) now becomes

$$\Delta A' = e^{-4A_o(y)} \left(\int_{y_1}^{y} dy' e^{2A_o(y')} H^2 - \frac{\rho_1}{3} e^{4A_o(y_1)} \right) \tag{5.33}$$

In a similar manner we can solve the combination $G_{tt} + G_{ii}$ to obtain

$$\int_{y_1}^{y_2} dy' e^{2A_o(y')} (4H^2 + 2\dot{H}) = \frac{\rho_{1nr}}{3} e^{4A_o(y_1)} + \frac{\rho_{2nr}}{3} e^{4A_o(y_2)} \tag{5.34}$$

$$3\Delta A' + \Delta N' = e^{-4A_o(y)} \left(\int_{y_1}^{y} dy' e^{2A_o(y')} (4H^2 + 2\dot{H}) - \frac{\rho_{1nr}}{3} e^{4A_o(y_1)} \right). \tag{5.35}$$

The remaining equation is G_{yy} which can be solved for ΔV or equivalently for $\delta\phi(y)$ in terms of $\delta b(y)$

$$\begin{aligned} \delta\phi(y) = \phi_o' \int_{y_1}^{y} dy' \delta b(y') + \phi_o' \int_{y_1}^{y} dy' \frac{A_o'}{\phi_o'^2} (3\Delta A' + \Delta N') \\ -\phi_o' \int_{y_1}^{y} dy' \frac{e^{-2A_o(y')}}{\phi_o'^2} \left(2H^2 + \dot{H} \right) + \delta\phi(y_1). \end{aligned} \tag{5.36}$$

The boundary conditions on $\phi(y)$ are[2]

$$\delta\phi(y_i) = \frac{\frac{(-1)^i}{2} A_o'(y_i)\rho_{inr} - e^{-2A_o(y_i)}\left(2H^2 + \dot{H}\right)}{(-1)^{i+1}\gamma_i\phi_o'(y_i) - \phi_o''(y_i)} \tag{5.37}$$

One can then substitute the known expressions for A_o', ϕ_o' (for the specific model of [33] these are given by Eqs.(5.17),(5.18)), and $2H^2 + \dot{H}$ from Eq.(5.34) into Eq.(5.37). This then determines $\delta\phi(y_i)$ in terms of ρ_{inr}. However, there remains one additional relation involving $\delta\phi(y_i)$ for we may let y = y_2 in Eq.(114). Eliminating $\delta\phi(y_1)$ and $\delta\phi(y_2)$ using Eq.(5.37), then Eq.(5.36) at y = y_2 becomes a constraint on $\delta b(y)$:

$$\begin{aligned} \int_{y_1}^{y_2} dy'\delta b(y') &= \frac{\delta\phi(y_2) - \delta\phi(y_1)}{\phi_o'} + \int_{y_1}^{y_2} dy' \frac{e^{-2A_o(y')}}{\phi_o'^2}\left(2H^2 + \dot{H}\right) \\ &- \int_{y_1}^{y_2} dy' \frac{A_o'}{\phi_o'^2}\left(3\Delta A' + \Delta N'\right). \end{aligned} \tag{5.38}$$

Note that the right hand side of Eq.(5.38) depends only on the vacuum metric and ρ_{inr} which can be seen upon substitution of Eqs.(5.34),(5.35),(5.37)

$$\begin{aligned} & \int_{y_1}^{y_2} dy'\delta b(y') = \\ & \frac{\frac{A_o'(y_2)}{2}\rho_{2nr} - e^{-2A_o(y_2)}F^{-1}\sum_i \frac{\rho_{inr}}{6}e^{4A_o(y_i)}}{-\gamma_2\phi_o'^2(y_2) - \phi_o''(y_2)\phi_o'(y_2)} \\ + \; & \frac{\frac{A_o'(y_1)}{2}\rho_{1nr} + e^{-2A_o(y_1)}F^{-1}\sum_i \frac{\rho_{inr}}{6}e^{4A_o(y_i)}}{\gamma_1\phi_o'(y_1)\phi_o'(y_2) - \phi_o''(y_1)\phi_o'(y_2)} \\ + \; & \int_{y_1}^{y_2} dy' \frac{e^{-2A_o(y')}}{\phi_o'^2(y')} F^{-1}\sum_i \frac{\rho_{inr}}{6}e^{4A_o(y_i)} \\ + \; & \int_{y_1}^{y_2} dy' \frac{A_o'(y')}{\phi_o'^2(y')} e^{-4A_o(y')}\frac{\rho_{1nr}}{3}e^{4A_o(y_1)} \\ - \; & \int_{y_1}^{y_2} dy' \frac{A_o'(y')}{\phi_o'^2(y')} e^{-4A_o(y')} \int_{y_1}^{y'} dy'' e^{2A_o(y'')} F^{-1}\sum_i \frac{\rho_{inr}}{3}e^{4A_o(y_i)} \end{aligned} \tag{5.39}$$

[2]We note the limit $\gamma_i \to \infty$ is the stiff potential limit used in the analysis of [34]. It is unnecessary to make this assumption and our results hold for any γ_i.

where

$$F = \int_{y_1}^{y_2} dy' e^{2A_o(y')}. \tag{5.40}$$

Therefore a consistent solution for arbitrary non-relativistic matter is obtained. (It should be noted that although these equations hold in general, explicit values for $\delta\phi(y_i)$ can only be calculated if A'_o and ϕ'_o can be determined, which will be dependent on the choice of the bulk potential.)

We see now the meaning of the result that the $\delta\phi$ boundary condition depends on both the coordinate invariant combination $\Delta V'$ *and* on $\delta\phi_i$ (rather than just on $\Delta V'$ as in HW). Instead of putting a constraint on ρ_{inr}, this determines the integral of $\delta b(y)$ in Eq.(5.38) which is just the change of distance between the branes due to the presence of non-relativistic matter. This is possible with the phenomenological potentials of the RS model (e.g. the example of Eqs.(5.14)-(5.16)), but not in HW where the theory determines the potentials to automatically cancel the cosmological constant. Since ρ_{inr} decreases as $1/a(t)^3$, the brane separation is actually time dependent at higher order in RS so the distance between the branes cannot be fixed, and the RS model is also non-static. Here, however, the time variation of the invariant distance between the branes sets in at $\mathcal{O}(\rho_{inr}^{3/2})$ since $\dot{\rho} \sim H\rho = \mathcal{O}(\rho^{3/2})$ while in HW the time variation sets in at $\mathcal{O}(\rho)$.

The above illustrates the difference between the phenomenology of RS and the theory of HW. In the RS model one is free to add arbitrary bulk and brane potentials (characterised here by γ_i) for the scalar field while in HW the couplings of the volume modulus V are determined by the theory and one is not free to include ad hoc potentials. In our analysis of HW theory we have used all potentials that arise perturbatively. There are non-perturbative potentials in HW that have not been included that might relax the tension that non-relativistic matter produces. If this is the case, one would also have to modify the vacuum solution to take into account the additional interactions.

CHAPTER VI

GRAVITY IN THE RANDALL SUNDRUM MODEL*

In this chapter we will be concerned with the subject of whether the RS model gives rise to the correct 4D Newtonian gravitational potential (in leading order). To examine this one considers point particles on the branes and calculates the gravitational forces between them. Our calculation is performed to first order in the metric perturbation (due to the particles) in the static limit. There is a large literature on this subject as well and on calculations of corrections arising from the presence of an additional dimension[36, 37, 38, 39, 40, 41, 42, 43]. Previous analyses have only examined forces between particles on the $y_1 = 0$ brane[1] and also neglect the effects of the Goldberger-Wise scalar stabilization field ϕ.

The vacuum metric of the RS model has the usual form

$$ds^2 = e^{-2A(y)}\eta_{ij}dx^i dx^j + dy^2 \tag{6.1}$$

where A(y) is an increasing function of y. (We again use the notation i,j = 0,1,2,3; μ, ν = 0,1,2,3,5 and $x^5 \equiv y$.) Thus to account for the gauge hierarchy, the physical brane must be at y_2 (where $e^{-A(y_2)} \backsimeq 10^{-16}$) while in the HW theory one can assume that the physical brane is at $y_1 = 0$. Thus for RS one needs to calculate the gravitational forces between the two particles at y_2 (though it is also interesting to see what forces the theory predicts between one particle at $y_1 = 0$ and one at $y_2 = \pi\rho$). For this case

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[1] Ref.[38] considers a single brane model with the brane displaced from the origin. However this is different in that it does not have S^1/Z_2 boundary conditions imposed at $y = y_1$ and $y = y_2$.

the function A(y) reduces to

$$A(y) = \beta|y| \;\; ; \;\; y_1 - \epsilon \leq y \leq y_2 - \epsilon \;\; ; \;\; \epsilon > 0 \tag{6.2}$$

We examine here within this framework the general case of gravitational forces between particles on both branes as well as the size of the leading corrections due to the extra dimensions. We find that the force on a particle on $y_1 = 0$ due to other particles on y_1 and y_2 has a leading Newtonian form (in accord with previous work) though the Newton constant G_N is different for the two cases. However, the Newtonian force on a particle at y_2 due to another particle at y_2 contains terms that grow exponentially with y_2 which leads to an unsatisfactory theory.

In carrying out these calculations, it is important to take careful account of "brane bending" effects. Thus we assume that matter is added on the branes as a perturbation to the vacuum metric

$$ds^2 = e^{-2\beta y}(\eta_{ij} + h_{ij})dx^i dx^j + h_{i5} dy dx^i + (1 + h_{55})dy^2 \tag{6.3}$$

and then solve the Einstein equations to linear order in $h_{\mu\nu}$. The diffeomorphisms of a 5D theory with S^1/Z_2 symmetry are those of $R^4 \mathrm{x} S^1$ which commute with Z_2. This means that for the transformation

$$x^\mu \to x'^\mu + \xi^\mu \equiv x^\mu \tag{6.4}$$

one has that ξ^5 vanishes at the orbifold points, y_1 and y_2:

$$\xi^5(x^i, y_1) = 0 = \xi^5(x^i, y_2) \tag{6.5}$$

If one were to make a coordinate transformation with a non-vanishing ξ^5, then the branes become bent and this would create a complication when one imposes the Z_2 boundary constraint on the branes, leading to the so-called brane bending effects.

In previous analyses, the 5D Einstein equations were solved in Gaussian coordinates described by

$$h_{5\mu} = 0 \;\; ; \;\; \partial^j h_{ij} = 0 = \eta^{ij} h_{ij} \tag{6.6}$$

In general, these cannot be achieved without brane bending occurring (and thus can give wrong answers). We give here an alternate analysis which avoids these complications by making only coordinate transformations that satisfy Eq.(6.5).

In Sec.A we give the metric decomposition along with our gauge choices which will allow us to solve the Einstein equations in the presence of matter on the branes without introducing brane bending effects. In Secs.B and C we explicitly solve the bulk Einstein equations to first order in the metric perturbation in the static limit, and subject these solutions to the brane boundary conditions. In Sec.C we also find the poles of the transverse traceless piece of h_{ij} and show how the Kaluza-Klein modes produce corrections to the leading static potential. In Sec.D we give the form of the Newtonian potential and show that Newton's constant differs depending on whether the gravitational force is due to particles on coincident or separate branes. The leading corrections to the Newtonian terms are discussed more fully in Appendix B.

A. Coordinate Conditions

Following 4D analyses, it is convenient to decompose the metric into its transverse and longitudinal parts according to the ADM prescription[2]:

$$h_{ij} = h_{ij}^{TT} + h_{ij}^{T} + h_{i,j} + h_{j,i} \tag{6.7}$$

where h_{ij}^{TT} is transverse and traceless[3] ($\partial^i h_{ij}^{TT} \equiv 0 \equiv h_i^i$) and h_{ij}^T is transverse but (in general) possesses a trace:

$$\partial^i h_{ij}^T \equiv 0 \ ; \ (h^T)_i^i \equiv f^T \neq 0 \tag{6.8}$$

We also decompose h_i into transverse and longitudinal parts

$$h_i = h_i^T + \frac{1}{2} h_{,i}^L \ ; \ \partial^i h_i^T \equiv 0 \tag{6.9}$$

and can write

$$h_{ij}^T = \frac{1}{3}\pi_{ij} f^T \ ; \ \pi_{ij} \equiv \eta_{ij} - O_{ij} \tag{6.10}$$

where

$$O_{ij} \equiv \frac{\partial_i \partial_j}{\Box^2} \tag{6.11}$$

One can express each of the subparts in Eq.(6.7) in terms of h_{ij}. Thus taking the divergence of Eq.(6.7) gives

$$\partial^i h_{ij} = \partial_j \Box^2 h^L + \Box^2 h_j^T \tag{6.12}$$

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[2]This decomposition was first introduced in [44]. (Ref.[45] is a more accessible recent reprint summarizing the ADM formalism.) The generalization to 4-space is trivial except for the ambiguity in defining $1/\Box^2$ in Minkowski space. However, we will always be considering the static Newtonian limit here where $\Box^2 \to \nabla^2$, though the size (and correct definition of) the higher order dynamical effect are also of interest.

[3]Four dimensional indices are raised and lowered with the Lorentz metric η_{ij}. We use the notation $h_{i,j} \equiv \partial_j h_i$

and

$$\partial^i \partial^j h_{ij} = (\Box^2)^2 h^L \tag{6.13}$$

Thus

$$h^L_{,ij} = O_{ij} O_{kl} h^{kl} \tag{6.14}$$

and

$$h^T_{i,j} = O^k_j h_{ik} - O_{ij} O_{kl} h^{kl} \tag{6.15}$$

Taking the trace of Eq.(6.7) and using Eq.(6.14) determines f^T to be

$$f^T = \pi^{ij} h_{ij} \tag{6.16}$$

and since

$$h^{TT}_{ij} = h_{ij} - h^T_{ij} - h_{i,j} - h_{j,i} \tag{6.17}$$

one has

$$h^{TT}_{ij} = \pi_{ik} \pi_{jl} h^{kl} - \frac{1}{3} \pi_{ij} \pi_{kl} h^{kl} \tag{6.18}$$

Our general metric with matter on the branes has the form

$$ds^2 = e^{-2A(y)} (\eta_{ij} + h_{ij}) dx^i dx^j + h_{5i} dy dx^i + (1 + h_{55}) dy^2 \tag{6.19}$$

In order to discuss clearly the issues of brane bending, we assume here that there exists a frame with no brane bending (as e.g. is required in H-W theory) i.e. in this frame the 4D branes are orthogonal to the fifth dimension. The vacuum metric of Eq.(6.1) is indeed a solution of the 5D Einstein equations obeying the S^1/Z_2 boundary conditions in this frame (as can be seen below Eqs.(6.37, 6.38)). Since the perturbation due to matter, $h_{\mu\nu}$, in Eq.(6.19) is general, we can assume there is a choice of $h_{\mu\nu}$ that holds in a frame with no bending. We now ask what coordinate conditions can be imposed to simplify the metric but remain in a frame with unbent branes. We chose the coordinate condition

$$h_{5i} = 0 \ \ ; \ \ i = 0, 1, 2, 3 \tag{6.20}$$

To verify that this can be achieved without any brane bending, we consider the infinitesimal gauge transformation

$$h'_{5i}(x) = h_{5i}(x) + \xi_{5,i} + \xi_{i,5} - 2\Gamma^{\alpha}_{5i}\xi_{\alpha} \tag{6.21}$$

where $\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols for the vacuum metric ($h_{\mu\nu} = 0$). Hence

$$h'_{5i}(x) = h_{5i}(x) + \xi_{5,i} + e^{-2A}(e^{2A}\xi_i)_{,5} \tag{6.22}$$

Thus if in the initial gauge h_{5i} were not zero, one can always set $h'_{5i} = 0$ without brane bending e.g. by choosing $\xi_5 = 0$ and solving for ξ_i. Eqs.(6.20) are not a complete set of coordinate conditions (being only four in number), and one may ask what is the remaining gauge freedom that maintains $\delta h_{5i} = 0$. From Eq.(6.22), this implies

$$0 = \xi_{5,i} + e^{-2A}(e^{2A}\xi_i)_{,5} \tag{6.23}$$

Writing $\xi_i = \xi_i^T + \xi^L_{,i}$ where $\partial^i\xi_i^T \equiv 0$, one finds

$$\omega_i^T = F_i^T(x^i) \tag{6.24}$$

$$\omega_5 = -(\omega^L)_{,5} \tag{6.25}$$

where F_i^T is independent of y, and we have introduced the notation

$$\omega_\mu \equiv e^{2A(y)}\xi_\mu(x^i, y) \tag{6.26}$$

It is understood in Eq.(6.25) that from Eq.(6.5) $\omega_5(x^i,y_1) = 0 = \omega_5(x^i,y_2)$ to avoid brane bending, which consequently also constrains $(\omega^L)_{,5}$.

The remaining gauge freedom, Eqs.(6.24, 6.25), allows a residual gauge freedom in h_{ij} and h_{55}:

$$\delta h_{ij} = \omega_{i,j} + \omega_{j,i} - 2A'e^{-2A}\omega_5\eta_{ij} \tag{6.27}$$

$$\delta h_{55} = 2(e^{-2A}\omega_5)_{,5} \tag{6.28}$$

where $A'(y) \equiv \mathrm{d}A/\mathrm{dy}$. Decomposing h_{ij} into its transverse and longitudinal parts one finds using Eqs.(6.24, 6.25) that

$$\delta h_{ij}^{TT} = 0 \;\; ; \;\; \delta h_i^T = \omega_i^T(x^i) \tag{6.29}$$

$$\delta f^T = 6A'e^{-2A}(\omega^L)_{,5} = -6A'e^{-2A}\omega_5 \tag{6.30}$$

$$\delta(\Box^2 h^L) = 2\Box^2\omega^L + 2A'e^{-2A}(\omega^L)_{,5} \tag{6.31}$$

Since $\omega_5(x^i,y_\alpha) = 0$ we see that f^T is gauge invariant on the branes

$$\delta f^T(x^i, y_\alpha) = 0 \;\; ; \;\; \alpha = 1, 2 \tag{6.32}$$

Further, since $\omega_5(x^i$,y) is otherwise arbitrary, one may expand it around y = y_α

$$\omega_5(x^i, y) = (y - y_\alpha)\omega_5'(x^i, y_\alpha) + \frac{1}{2}(y - y_\alpha)^2\omega_5''(x^i, y_\alpha) + ... \tag{6.33}$$

so that

$$\delta(\partial_5 f^T(x^i, y_\alpha)) = -6A'e^{-2A}\omega_5'(x^i, y_\alpha) + ... \tag{6.34}$$

Hence one may choose $\omega_5'(x^i, y_\alpha)$ to set

$$\partial_5 f^T(x^i, y_\alpha) = 0 \tag{6.35}$$

which we will see below is a convenient further gauge choice.

B. Einstein Equations

The action for our system is

$$\begin{aligned} S &= \int d^5x\sqrt{-{}^5g}(-\frac{1}{2}M_5^3R + 6M_5^3\beta^2) \\ &+ \sum_{\alpha=1,2}\int d^5x\sqrt{-{}^4g}(\mathcal{L}_{m_\alpha} + (-1)^{\alpha+1}6M_5^3\beta)\delta(y-y_\alpha) \end{aligned} \tag{6.36}$$

where M_5 is the 5D Planck mass, $\mathcal{L}_{m_\alpha}$ are the Lagrangians for point particles on the branes $y_1 = 0$ and $y_2 = \pi\rho$, and we have fine tuned the bulk and brane cosmological constants so that the net cosmological constant is zero. The vacuum equations of motion for the metric of Eq.(6.1) read

$$\frac{1}{2}A'' = (-1)^{\alpha+1}\beta\delta(y-y_\alpha) \tag{6.37}$$

$$A'^2 = \beta^2 \tag{6.38}$$

so that $A = \beta y$ for $y_1 < y < y_2$ with S^1/Z_2 boundary conditions at the orbifold fixed points. The linearized first order equations read

$$R^{(1)}_{\mu\nu} = -\frac{1}{M_5^3}\sum_\alpha(T_{\mu\nu}(y_\alpha) - \frac{1}{3}g_{\mu\nu}T(y_\alpha))\delta(y-y_\alpha) \tag{6.39}$$

where $T \equiv g^{\mu\nu}T_{\mu\nu}$ and $T_{\mu\nu}(y_\alpha)$ are the 4D stress tensors for $\mathcal{L}_{m_\alpha}$. Hence $T_{5i} = 0 = T_{55}$. We will consider here only the static gravitational forces and so only T_{00} need to be taken non-zero[4].

The 15 Einstein equations are then

$$R^{(1)}_{5i} = 0 \ ; \ i = 0, 1, 2, 3 \tag{6.40}$$

[4]In obtaining Eq.(6.39), $R^{(1)}_{\mu\nu}$ is the first order part of $R_{\mu\nu}$ omitting terms proportional to the cosmological constant, since these terms are precisely canceled by the cosmological constant sources on the right hand side as a consequence of the zero'th order equations Eqs.(6.37, 6.38).

$$R_{55}^{(1)} = -\frac{1}{3M_5^3}\sum_\alpha e^{2A}T_{00}(y_\alpha)\delta(y-y_\alpha) \tag{6.41}$$

and

$$R_{ij}^{(1)} = -\frac{1}{M_5^3}\sum_\alpha (T_{ij}(y_\alpha) - \frac{1}{3}\eta_{ij}e^{-2A}T)\delta(y-y_\alpha) \tag{6.42}$$

The 10 equations Eq.(6.42) can be decomposed as

$$\eta^{ij}R_{ij}^{(1)} = -\frac{1}{3M_5^3}\sum_\alpha T_{00}(y_\alpha)\delta(y-y_\alpha) \tag{6.43}$$

$$\partial^i R_{ij}^{(1)} = -\frac{1}{M_5^3}\sum_\alpha (\partial^i T_{ij}(y_\alpha) + \frac{1}{3}\partial_j T_{00}(y_\alpha))\delta(y-y_\alpha) \tag{6.44}$$

and

$$R_{ij}^{TT(1)} = \frac{1}{M_5^3}\sum_\alpha T_{ij}^{TT}(y_\alpha)\delta(y-y_\alpha) \tag{6.45}$$

Eqs.(6.43) and (6.44) together pick out the $''T''$ components and $''L''$ component of R_{ij} while Eq.(6.45) picks out the TT components. In this section we solve Eqs.(6.40, 6.41, 6.43, 6.44). Eq.(6.45) is discussed in Sec.C below.

Eq.(6.40) reads

$$R_{5i}^{(1)} \equiv \frac{1}{2}\eta^{kl}\partial_5(\partial_i h_{kl} - \partial_l h_{ik}) + \frac{3}{2}A'\partial_i h_{55} = 0 \tag{6.46}$$

where $A' \equiv \partial_y A$. In terms of the decomposition of Eqs.(6.7) and (6.9), Eq.(6.46) reduces to

$$-\partial_i\partial_5 f^T - 3A'\partial_i h_{55} + \Box^2\partial_5 h_i^T = 0 \tag{6.47}$$

where $\Box^2 \equiv \nabla^2 - \partial_0^2$. In the static approximation, $\Box^2 \to \nabla^2$, the T part of Eq.(6.47) reads

$$\partial_5 h_i^T = 0 \tag{6.48}$$

which says that h_i^T is independent of y. We can thus use the gauge freedom of

Eq.(6.29) to set h_i^T to zero

$$h_i^T = 0 \tag{6.49}$$

The remaining longitudinal part of Eq.(6.47) yields

$$h_{55} = -\frac{1}{3A'}\partial_5 f^T \tag{6.50}$$

Note from Eqs.(6.28) and (6.30), the combination $3A'h_{55} + \partial_5 f^T$ is gauge invariant. However in the gauge of Eq.(6.35), one has that h_{55} vanishes on the branes, i.e.

$$h_{55}(x^i, y_\alpha) = 0 \tag{6.51}$$

though in general it is non-zero off the branes.

Eqs.(6.41) and (6.43) allow us to determine f^T and h^L. One has for $R_{55}^{(1)}$

$$R_{55}^{(1)} = (\frac{1}{2}\partial_5^2 - A'\partial_5)\eta^{ij}h_{ij} + \frac{1}{2}e^{2A}\Box^2 h_{55} + 2A'\partial_5 h_{55} \tag{6.52}$$

and so Eq.(6.41) becomes, using Eq.(6.50)

$$(\frac{1}{2}\partial_5^2 - A'\partial_5)(\Box^2 h^L - \frac{1}{3}f^T) - \frac{e^{2A}}{6A'}\Box^2\partial_5 f^T = -\frac{1}{3M_5^3}\sum_\alpha e^{2A}T_{00}\delta(y - y_\alpha) \tag{6.53}$$

In arriving at Eq.(6.53) we have made use of the fact that

$A''\partial_5 f^T \sim \delta(y - y_\alpha)\partial_5 f^T = 0$ in the gauge of Eq.(6.35).

The full $R_{ij}^{(1)}$ is

$$\begin{aligned} e^{2A}R_{ij}^{(1)} &= (\frac{1}{2}\partial_5^2 - 2A'\partial_5)h_{ij} - \frac{1}{2}A'\eta_{ij}\partial_5(\eta^{mk}h_{mk}) \\ &+ \eta_{ij}h_{55}(A'' - 4A'^2) + \frac{1}{2}A'\eta_{ij}\partial_5 h_{55} \\ &+ \frac{e^{2A}}{2}\partial_i\partial_j h_{55} + \frac{1}{2}e^{2A}\partial_i\partial_j(\eta^{mk}h_{mk}) \\ &+ \frac{1}{2}e^{2A}\eta^{mk}(\partial_k\partial_m h_{ij} - \partial_k\partial_i h_{jm} - \partial_j\partial_m h_{ik}) \end{aligned} \tag{6.54}$$

Hence Eq.(6.43) becomes

$$
\begin{aligned}
(\frac{1}{2}\partial_5^2 - 4A'\partial_5)(\Box^2 h^L - \frac{1}{3}f^T) - \frac{1}{6}\frac{e^{2A}}{A'}\Box^2\partial_5 f^T + e^{2A}\Box^2 f^T \\
= -\frac{1}{3M_5^3}\sum_\alpha e^{2A}T_{00}(y_\alpha)\delta(y-y_\alpha)
\end{aligned}
\tag{6.55}
$$

Subtracting Eq.(6.55) from Eq.(6.53) determines h^L in terms of f^T

$$
\Box^2 h^L = \frac{1}{3}f^T + \int_0^y dy'\frac{e^{2A}}{3A'}\Box^2 f^T + \phi(x) \tag{6.56}
$$

where the function of integration $\phi(x)$ is independent of y. However, from Eq.(6.31), one may set $\phi(x)$ to zero using a gauge transformation with $\omega^L(x^i)$. Further, Eqs.(6.53) and (6.55) imply the same boundary conditions

$$
\partial_5(\Box^2 h^L - \frac{1}{3}f^T)\Big|_{y=y_\alpha} = \frac{(-1)^\alpha}{3M_5^3}e^{2A(y_\alpha)}T_{00}(y_\alpha) \tag{6.57}
$$

Inserting Eq.(6.56) into Eq.(6.57) then gives

$$
\Box^2 f^T(x^i, y_\alpha) = \frac{(-1)^\alpha\beta}{M_5^3}T_{00}(y_\alpha) \tag{6.58}
$$

where in our static approximation $\Box^2 \to \nabla^2$. While f^T is not gauge invariant in the bulk, we saw that it was gauge invariant on the branes, which is why its value on each brane is determined by the physical quantities $T_{00}(y_\alpha)$.

If one now inserts Eq.(6.56) back into Eqs.(6.53) and (6.55), one sees that these equations are identically satisfied and so Eqs.(6.53) and (6.56) have no further content. Thus rather than determining h^L and f^T separately, Eqs.(6.53) and (6.55) determine only the gauge invariant combination Eq.(6.56).

To check the solution of Eq.(6.44), we first note that $\partial^i T_{ij}$ is second order and may be neglected. Using Eq.(6.54) and the coordinate conditions Eqs.(6.49) and

(6.51), and the fact that

$$A''h_{55} \sim \delta(y - y_\alpha)h_{55} = 0 \tag{6.59}$$

one sees that Eq.(6.44) reduces to

$$\begin{aligned}(\frac{1}{2}\partial_5^2 - \frac{5A'}{2}\partial_5)(\Box^2 h^L - \frac{1}{3}f^T) \quad &- \quad \frac{1}{6}\frac{e^{2A}}{A'}\Box^2\partial_5 f^T + \frac{1}{2}e^{2A}\Box^2 f^T = \\ &- \quad \frac{1}{3M_5^3}T_{00}(y_\alpha)\delta(y - y_\alpha)\end{aligned} \tag{6.60}$$

The boundary conditions implied by the right hand side of Eq.(6.60) are thus identical to Eq.(6.57), and inserting Eq.(6.56) one sees that Eq.(6.60) is identically satisfied.

We will see in the following section that the remaining Einstein equations, Eq.(6.45) uniquely determines h_{ij}^{TT} so that we have found solutions to all the Einstein equations. The undetermined function, $f^T(x^i, y)$ off the branes, is the remaining gauge freedom. However, note one cannot set $h_{55} = 0$ everywhere (as is conventionally done in other analyses) as Eq.(6.50) would then imply f^T is constant in y, which would be inconsistent with the boundary conditions Eq.(6.58) (which are in fact gauge invariant). We will see that Eq.(6.58) contributes a significant term to the static gravitational potential.

C. Solution for h_{ij}^{TT}

The remaining Einstein field equations, Eq.(6.45), can be obtained by taking the TT part of Eq.(6.54). We find

$$(\frac{1}{2}\partial_5^2 - 2A'\partial_5 + \frac{1}{2}e^{2A}\Box^2)h_{ij}^{TT} = -\frac{e^{2A}}{M_5^3}\sum_\alpha T_{ij}^{TT}(y_\alpha)\delta(y - y_\alpha) \tag{6.61}$$

and hence h_{ij}^{TT} obeys the boundary conditions

$$\partial_5 h_{ij}^{TT}\Big|_{y=y_\alpha} = (-1)^\alpha \frac{e^{2A}}{M_5^3}T_{ij}^{TT}(y_\alpha) \;\; ; \;\; \alpha = 1, 2 \tag{6.62}$$

The static potential is obtained from $h_{00}(x, y_\alpha)$ where

$$h_{00}(x^i, y_\alpha) = h_{00}^{TT}(x^i, y_\alpha) - \frac{1}{3} f^T(x^i, y_\alpha) \tag{6.63}$$

The corresponding source is then

$$T_{00}^{TT} = \pi_0^\rho \pi_0^\sigma T_{\rho\sigma} - \frac{1}{3}\pi_{00}\pi_\rho^\sigma T_\rho^\sigma \tag{6.64}$$

which in the static limit reduces to

$$T_{00}^{TT} = \frac{2}{3} T_{00} \tag{6.65}$$

To solve Eq.(6.61) we Fourier analyse h_{ij}^{TT}

$$h_{ij}^{TT}(x^i, y_\alpha) = \int d^4 p e^{ipx} h_{ij}^{TT}(p^i, y) \tag{6.66}$$

In the bulk then $h_{ij}^{TT}(p, y)$ obeys

$$(\frac{1}{2}\partial_5^2 - 2A'\partial_5 + \frac{1}{2} e^{2A} m^2) h_{ij}^{TT}(p, y) = 0 \tag{6.67}$$

where $m^2 \equiv -p^2 = p_o^2 - \vec{p}^2$. The solutions of Eq.(6.67) are Bessel and Neumann functions

$$h_{ij}^{TT}(p^i, y) = e^{2\beta y}[A_{ij}(p) J_2(\xi) + B_{ij}(p) N_2(\xi)] \tag{6.68}$$

where

$$\xi(y) = \frac{m}{\beta} e^{\beta y} \tag{6.69}$$

and m/β is short hand for $(m^2/\beta^2)^{1/2}$. The boundary conditions Eq.(6.62) determine A_{ij} and B_{ij}. One finds on the branes

$$h_{00}^{TT}(p; y_1) = -\frac{2}{3\beta M_5^3}[\frac{N_{11}(\xi_1, \xi_2)}{D} T_{00}(y_1) + \frac{N_{12}(\xi_1, \xi_2)}{D} T_{00}(y_2)] \tag{6.70}$$

where

$$D \equiv \frac{N_1(\xi_1)}{J_1(\xi_1)} - \frac{N_1(\xi_2)}{J_1(\xi_2)} \tag{6.71}$$

$$\xi_1 = \frac{m}{\beta} \;\; ; \;\; \xi_2 = \frac{m}{\beta} e^{\beta y_2} \tag{6.72}$$

$$m^2 \equiv -p^2 = (p^0)^2 - \vec{p}^2 \tag{6.73}$$

$$N_{11} \equiv \frac{J_2(\xi_1)}{\xi_1 J_1(\xi_1)} \left[\frac{N_2(\xi_1)}{J_2(\xi_1)} - \frac{N_1(\xi_2)}{J_1(\xi_2)} \right] \tag{6.74}$$

$$N_{12} \equiv \frac{J_2(\xi_1)}{\xi_2 J_1(\xi_2)} \left[\frac{N_2(\xi_1)}{J_2(\xi_1)} - \frac{N_1(\xi_1)}{J_1(\xi_1)} \right] \tag{6.75}$$

and

$$h_{00}^{TT}(p; y_2) = -\frac{2e^{2\beta y_2}}{3\beta M_5^3} [\frac{N_{21}(\xi_1, \xi_2)}{D} T_{00}(y_1) + \frac{N_{22}(\xi_1, \xi_2)}{D} T_{00}(y_2)] \tag{6.76}$$

where

$$N_{21}(\xi_1, \xi_2) = N_{12}(\xi_2, \xi_1) \;\; ; \;\; N_{22}(\xi_1, \xi_2) = N_{11}(\xi_2, \xi_1) \tag{6.77}$$

It is useful to consider $h_{00}^{TT}(p, y_\alpha)$ as a function of a complex variable $z = m^2$. In looking at the analytic behavior in z, the logarithmic branch cuts in the Neumann functions cancel in the differences such as $N_1(\xi_1)/J_1(\xi_1) - N_1(\xi_2)/J_1(\xi_2)$. Poles can arise from a number of sources. Thus in N_{11}, a pole might occur at the zeros of $J_1(\xi_1)$, but this is actually canceled by the zeros in $J_1(\xi_1)$ appearing in D. Similarly the zeros of $J_1(\xi_2)$ in N_{11} are canceled. Thus the only poles that occur are the pole at $m^2 = 0$ (arising e.g. from $N_2/J_2 \sim 1/m^4$ in N_{11}) and when D vanishes, i.e., at

$$D(m_n^2) = 0 = \frac{N_1(\xi_n)}{J_1(\xi_n)} - \frac{N_1(\xi_n e^{\beta y_2})}{J_1(\xi_n e^{\beta y_2})} \tag{6.78}$$

where $\xi_n \equiv (m_n^2/\beta^2)^{1/2}$. To find the residue at m_n^2, we expand $D(m_n^2)$ around the pole position

$$D(m^2) = \frac{\partial D(m^2)}{\partial m^2}\Big|_{m_n^2} (m^2 - m_n^2) + \ldots \tag{6.79}$$

and differentiating the Bessel and Neumann functions in D one has

$$\begin{aligned}\frac{\partial D(m^2)}{\partial m^2} &= \frac{1}{2\beta m}\frac{1}{J_1^2(\xi_1)}[J_1(\xi_1)N_1'(\xi_1) - N_1(\xi_1)J_1'(\xi_1)] \\ &- \frac{e^{\beta y_2}}{2\beta m}\frac{1}{J_1^2(\xi_2)}[J_1(\xi_2)N_1'(\xi_2) - N_1(\xi_2)J_1'(\xi_2)]\end{aligned} \tag{6.80}$$

Using the two Bessel function Wronskian identities

$$J_1N_1' - N_1J_1' = \frac{2}{\pi\xi} = N_1J_2 - J_1N_2 \tag{6.81}$$

Eq.(6.79) reduces to

$$D(m^2) = (\frac{1}{\pi m^2}[\frac{1}{J_1^2(\xi_1)} - \frac{1}{J_1^2(\xi_2)}])\Big|_{m_n}(m^2 - m_n^2) + \ldots \tag{6.82}$$

To obtain the residue at the pole, we need also the numerator N_{11} evaluated at $m^2 = m_n^2$

$$N_{11} = \frac{1}{\xi_1}\frac{1}{J_1^2(\xi_1)}[N_2(\xi_1)J_1(\xi_1) - N_1(\xi_1)J_2(\xi_1)]\Big|_{m_n} \tag{6.83}$$

and using Eq.(6.81) this reduces to

$$N_{11}(m_n^2) = -\frac{2\beta^2}{\pi m_n^2}\frac{1}{J_1^2(m_n^2/\beta^2)} \tag{6.84}$$

Hence the residue at m_n^2 is simply

$$R_n(m_n^2) = -\frac{2}{3\beta M_5^3}\frac{N_{11}}{D}\Big|_{m_n} = \frac{4\beta}{3M_5^3}\frac{1}{1 - \frac{J_1^2(m_n/\beta)}{J_1^2((m_n/\beta)e^{\beta y_2})}} \tag{6.85}$$

The residue at $m^2 = 0$ can be obtained by examining the limit when both ξ_1 and ξ_2 approach zero in $N_{11}(\xi_1, \xi_2)/D$. One finds

$$R_0 = -\frac{4\beta}{3M_5^3}\frac{1}{1 - e^{-2\beta y_2}} \tag{6.86}$$

where for the Randall-Sundrum model one may neglect the $e^{-2\beta y_2} \approx 10^{-32}$ in the denominator. A similar analysis to the above holds for the other three terms in

Eqs.(6.70) and (6.76).

One may now cast the results for $h_{00}^{TT}(z, y_\alpha)$ in a more convenient form. Using the asymptotic forms of Bessel and Neumann functions, one can see that $h_{00}^{TT}(z, y_\alpha)$ falls like 1/z for $|z|$ on a large circle. Hence integrating

$$g(z) \equiv \frac{h_{00}^{TT}(z, y_\alpha)}{z - m^2} \tag{6.87}$$

over a large circle in the complex plane we can express $h_{00}^{TT}(m^2, y_\alpha)$ as a sum of poles with the residues calculated above. One has then

$$\begin{aligned} h_{00}^{TT}(y_1) &= -\frac{4\beta}{3m^2 M_5^3}[T_{00}(y_1) + e^{-2\beta y_2} T_{00}(y_2)] \\ &+ \frac{4\beta}{M_5^3}\sum_{m_n} \frac{1}{m^2 - m_n^2}\left(\frac{J_2^2(\xi_2)}{J_2^2(\xi_2) - J_1^2(\xi_1)}\right)[T_{00}(y_1) + \frac{e^{-\beta y_2} J_1(\xi_1)}{J_1(\xi_2)} T_{00}(y_2)]\Big|_{m=m_n} \end{aligned} \tag{6.88}$$

and

$$\begin{aligned} h_{00}^{TT}(y_2) &= -\frac{4\beta}{3m^2 M_5^3}[T_{00}(y_1) + e^{-2\beta y_2} T_{00}(y_2)] \\ &+ \frac{4\beta}{M_5^3}\sum_{m_n} \frac{1}{m^2 - m_n^2}\left(\frac{J_1^2(\xi_1)}{J_1^2(\xi_2) - J_1^2(\xi_1)}\right)[T_{00}(y_2) + \frac{e^{\beta y_2} J_1(\xi_2)}{J_1(\xi_1)} T_{00}(y_1)]\Big|_{m=m_n} \end{aligned} \tag{6.89}$$

The first term of Eqs.(6.88) and (6.89) contributes to the Newtonian potential (since $m^2 = -p^2 \to \vec{p}^2$ in the static limit) while the other term gives 5D corrections to the Newtonian theory.

While Eq.(6.78) is a transcendental equation, one can obtain the positions of the poles analytically in certain limits. Thus if $\xi_n = m_n/\beta \ll 1$ but $\xi_n e^{\beta y_2} \gg 1$ (i.e. $\xi_n \gg 10^{-16}$) then inserting in the Bessel function asymptotic forms in the ratios of Eq.(6.78) gives

$$tan(\xi_n e^{\beta y_2} - \frac{3\pi}{4}) \cong -\frac{4}{\pi}(\frac{1}{\xi_n})^2 \tag{6.90}$$

which can be solved by iteration to give

$$\frac{m_n^2}{\beta^2} \cong [(n+\frac{5}{4})\pi + \epsilon_n]^2 e^{-2\beta y_2} \ \ ; \ \ \xi_n \ll 1 \ , \ \ \xi_n e^{\beta y_2} \gg 1 \tag{6.91}$$

where

$$\epsilon_n \cong \frac{\pi}{4}[(n+\frac{5}{4})\pi e^{-\beta y_2}]^2 \tag{6.92}$$

(We have included the first order correction ϵ_n as in some expressions the leading term can cancel out). The residues at the poles can then be calculated in this limit. Thus for the $T_{00}(y_1)$ term one finds for

$$R_n = -\frac{2}{3\beta M_5^3}\frac{N_{11}(\xi_1, \xi_2)}{D}\Big|_{m=m_n} \tag{6.93}$$

the result

$$R_n = -\frac{2\pi}{3M_5^3} m_n e^{-\beta y_2} \tag{6.94}$$

and the contribution to the scalar potential is

$$h_{00}^{TT} = -\frac{2\pi}{3M_5^3}\sum_{n=1}\frac{m_n e^{-\beta y_2}}{m^2 - m_n^2} T_{00}(y_1) \tag{6.95}$$

Since the poles are very dense

$$\Delta m_n \equiv m_{n+1} - m_n = \beta\pi e^{-\beta y_2} \tag{6.96}$$

one can approximate Eq.(6.95) by converting the sum to an integral ($m^2 = -p^2 = -\vec{p}^2$ in the static limit)

$$h_{00}^{TT} \cong \frac{2\pi}{3M_5^3}\int_0^\beta dm_n \frac{m_n}{\vec{p}^2 + m_n^2} T_{00}(y_1) \tag{6.97}$$

where we have cut off the integral at β since $\xi_1 = m_n/\beta \lesssim 1$. Returning to coordinate space this yields

$$h_{00}^{TT}(\vec{r}) = \frac{2}{3\beta M_5^3}\frac{m_0}{4\pi r^3}\int_0^{\beta r} d\alpha e^{-\alpha} \tag{6.98}$$

where m_0 is the mass of $T_{00}(y_1)$. For $r \gg 1/\beta$, i.e. for distances large compared to the warping parameter $1/\beta$, this is a $1/r^3$ correction to the leading Newtonian potential.

Eventually, for sufficiently large n, ξ_n becomes large (i.e. $n \gtrsim 10^{16}$), and the poles from Eq.(6.78) occur at

$$m_n = \frac{n\pi\beta}{e^{\beta y_2} - 1} \cong n\pi\beta e^{-\beta y_2} \;\; ; \;\; \xi_n \gg 1 \tag{6.99}$$

Then one finds for this contribution

$$h_{00}^{TT} = -\frac{4\beta}{3M_5^3} \sum_n \frac{e^{-\beta y_2}}{m^2 - m_n^2} T_{00}(y_1) \tag{6.100}$$

or in the continuum approximation

$$h_{00}^{TT} \cong \frac{4\beta}{3M_5^3} \int_\beta^\infty \frac{dm_n}{\vec{p}^2 + m_n^2} T_{00}(y_1) \tag{6.101}$$

In coordinate space one finds

$$h_{00}^{TT}(r) = \frac{m_0}{3M_5^3} \int_\beta^\infty \frac{dm_n e^{-m_n r}}{r} T_{00}(y_1) \tag{6.102}$$

and in the limit $r \ll 1/\beta$ one finds a $1/r^2$ correction to the Newtonian potential

$$h_{00}^{TT}(r) = \frac{m_0}{3M_5^3} \frac{1}{r^2} \tag{6.103}$$

Eqs.(6.98) and (6.103) agree with results obtained in [43](although there, a fine tuning of matter is needed on the second brane in order to get a consisitent solution of the Einstein equations).

One may carry out a similar analysis of the other three terms in Eqs.(6.70) and (6.76) (N_{12}/D, N_{21}/D, and N_{22}/D) and these results will be discussed further in the Appendix. In Eq.(6.97) it is conventional to extend the integral down to $m_n = 0$, and think of the continuum of poles as reaching down to $m^2 = 0$ without a gap. Actually, as can be seen from Eq.(91), the first discrete pole occurs at

$$m_n \cong (\frac{9}{4}\pi\beta)e^{-\beta y_2} \tag{6.104}$$

The size of the gap depends on the model. Thus for Randall-Sundrum one has

$$m_1 \approx (\frac{9}{4}\pi\beta)e^{-\beta y_2} \approx (10^{19}GeV)(10^{-16}) = 1TeV \tag{6.105}$$

since

$$\beta \approx M_{Pl} \tag{6.106}$$

On the Planck brane $y_1 = 0$, a TeV of energy is negligible (since masses are of order M_{Pl}). On the TeV brane y_2 however, it is sometimes argued that one should not consider phenomena $\gtrsim$ 1TeV. In this case one would neglect the Kaluza-Klein modes.

D. Newtonian Potential

The static Newtonian potential is the 1/r terms of $h_{00}(x^i, y_\alpha)$ of Eq.(6.63). These arise from the poles in momentum space at $m^2 = 0$. As discussed in Sec.C, these poles occur in h_{00}^{TT} from the fact that the numerator functions N_{ij} go as $N_{ij} \sim 1/m^4$ as $m^2 \to 0$ due to the $N_2(\xi_\alpha), \alpha = 1, 2$ terms, while the denominator function goes as $D \sim 1/m^2$ leading to a net $1/m^2$ term for small m^2. As seen from Eq.(6.58), $f^T(x^i, y_\alpha)$ is totally a $1/m^2$ term in momentum space. One can thus pick out the $m^2 = 0$ pole contributions on the two branes

$$h_{00}^N(y_1) = -\frac{4\beta}{3M_5^3}\frac{1}{m^2}[T_{00}(y_1) + e^{-2\beta y_2}T_{00}(y_2)] + \frac{\beta}{3M_5^3}\frac{1}{m^2}T_{00}(y_1) \tag{6.107}$$

and

$$h_{00}^N(y_2) = -\frac{4\beta}{3M_5^3}\frac{1}{m^2}[T_{00}(y_1) + e^{-2\beta y_2}T_{00}(y_2)] - \frac{\beta}{3M_5^3}\frac{1}{m^2}T_{00}(y_2) \tag{6.108}$$

In Eqs.(6.107) and (6.108) the first bracket is from h_{00}^{TT} and the second is from f^T. The stress tensor T_{ij} arising from the matter Lagrangian $\mathcal{L}_m$ is

$$T^{ij} = \frac{1}{\sqrt{-g}}\frac{\delta \mathcal{L}_m}{\delta g_{ij}} \tag{6.109}$$

where for a point particle on brane y_α

$$\mathcal{L}_{m_\alpha} = m_0 \int d\tau u^i u^j g_{ij}(x^i, y_\alpha)\delta^4(x^i - x^i(\tau)) \tag{6.110}$$

and $u^i = dx^i/d\tau$ with $d\tau^2 = -g_{ij}dx^i dx^j$. For our metric, $g_{ij} = e^{-2A}\hat{g_{ij}}$ where in the linearized approximation $\hat{g_{ij}} = \eta_{ij} + h_{ij}$. Thus defining

$$d\hat{\tau}^2 = \hat{g_{ij}} dx^i dx^j \ \ ; \ \ \hat{u}^i = \frac{dx^i}{d\hat{\tau}} \tag{6.111}$$

$\mathcal{L}_m$ reduces to

$$\mathcal{L}_{m_\alpha} = \bar{m}_\alpha(y_\alpha) \int d\hat{\tau} \hat{u}^i \hat{u}^j \hat{g_{ij}} \delta^4(x^i - x^i(\hat{\tau})) \tag{6.112}$$

where

$$\bar{m}(y) = e^{-A(y)} m_0 \tag{6.113}$$

showing the usual result that if m_0 is of Planck size, the effective mass seen on the TeV brane y_2 will be of TeV size. The Lagrangian of Eq.(6.112) will then correctly give rise to the (linearized) geodesic equation governed by $\hat{g_{ij}} = \eta_{ij} + h_{ij}$.

Returning to Eq.(6.109), the stress tensor is

$$T^{ij} = \frac{1}{\sqrt{-g}} m_0 \int d\tau u^i u^j \delta^4(x^i - x^i(\tau)) \tag{6.114}$$

and in the static approximation,

$$u^0 \cong e^A \ \ ; \ \ u^i \cong 0 \tag{6.115}$$

one has

$$T^{00} = e^{5A} m_0 \delta^3(r - r(t)) \tag{6.116}$$

so that

$$T_{00}(y_\alpha) = e^{2A(y_\alpha)} \bar{m}(y_\alpha) \delta^3(r - r(t)) \tag{6.117}$$

The interaction potential between the two particles may be defined by

$$V = -\int d^3 r \mathcal{L}_{m\ int} \tag{6.118}$$

where the total $\mathcal{L}_m$ is

$$\mathcal{L}_m = \sum_\alpha m_{0_\alpha} \int d\tau u^i u^j e^{-2A(y_\alpha)} (\eta_{ij} + h_{ij}(y_\alpha)) \delta^4(x - x_\alpha(\tau)) \tag{6.119}$$

Hence in the static limit

$$-\int d^3 r \mathcal{L}_m = -\sum_\alpha m_{0_\alpha} \int d\tau (\frac{dx^0_\alpha}{d\tau})^2 e^{-2A(y_\alpha)} (\eta_{00} + h_{00}(y_\alpha)) \delta^4(x - x_\alpha(\tau)) \tag{6.120}$$

Since $u^0_\alpha d\tau = dx^0_\alpha$ and

$$\frac{dx^0_\alpha}{d\tau} \cong \frac{1}{\sqrt{-g_{00}}} = \frac{e^A}{(-\eta_{00} - h_{00})^{1/2}} \tag{6.121}$$

one has

$$-\int d^3 r \mathcal{L}_m = \sum_\alpha \bar{m}_\alpha (-\eta_{00} - h_{00})^{1/2} \tag{6.122}$$

Expanding to first order gives for the interaction potential energy

$$V = -\frac{1}{2} \sum_\alpha \bar{m}_\alpha h_{00}(x^0_\alpha, y_\alpha) \tag{6.123}$$

Inserting Eq.(6.107) and (6.108) and returning to coordinate space ($m^2 = -\vec{p}^2$) one gets for the Planck brane the contribution

$$V(y_1) = -\frac{\beta}{8\pi M_5^3} \frac{1}{r} [\bar{m}_1 \bar{m}'_1 + \frac{4}{3} \bar{m}_1 \bar{m}_2] \tag{6.124}$$

where $\bar{m}_1$ is the mass of a second particle on the Planck brane, $\bar{m}_2$ a mass of a particle on the TeV brane ($\bar{m}_2 = e^{-\beta y_2} m_{20}$), and r is the 3D distance between the particles. Note that the fact that $\bar{m}_2$ is separated by additional distance in the fifth dimension ($y_2 - y_1 = \pi\rho$) does not enter in r.

We see from Eq.(6.124) that if the two particles are on the Planck brane, Eq.(6.126) correctly reproduces the Newtonian force law with

$$G_N \equiv \frac{\beta}{8\pi M_5^3} \tag{6.125}$$

(the conventional value for the Newton constant in Randall-Sundrum theory). The f^T contribution correctly changes the 4/3 factor in the first term of Eq.(6.107) to 1. However, if one particle is on the TeV brane, the Newton constant is modified by an extra factor of 4/3, since the f^T factor does not contribute. (The fact that matter on the TeV brane changes gravitational effects seen on the Planck brane has previously been noted in [37] in a different connection.)

For the potential energy seen on the TeV brane we use Eq.(6.108) in Eq.(6.123). One finds now from Eq.(6.117) that

$$\begin{aligned} V(y_2) = &-\frac{4\beta}{3M_5^3}\frac{1}{8\pi r}[\bar{m}_2\bar{m}_1 + \bar{m}_2\bar{m}_2'] \\ &-\frac{\beta}{3M_5^3}\frac{1}{8\pi r}\bar{m}_2\bar{m}_2' e^{2\beta y_2} \end{aligned} \tag{6.126}$$

where $\bar{m}_2'$ is a second particle on the y_2 brane. The interaction energy between $\bar{m}_2$ and $\bar{m}_1$ particles is as before as is the Newtonian potential between two particles on the TeV brane, m_2 and m_2', arising from h_{00}^{TT} (aside from the peculiar 4/3 factor). However, the f^T term gives an additional contribution to $V(y_2)$ scaled by $e^{2A(y_2)}$ (the factor from Eq.(6.117)) which would produce an anomolously large additional contribution. (Recall $e^{\beta y_2} \approx 10^{16}$ in the Randall-Sundrum model to account for the gauge hierarchy problem!) Thus the theory does not appear to give sensible results on the TeV brane.

E. Discussion on Previous Works

We briefly compare our analysis with some of the previous calculations for the static gravitational potential. In Ref.[37] it is assumed that in Gaussian coordinates

$$h_{\mu 5} = 0 \ ; \ \ \mu = 0, 1, 2, 3, 5 \tag{6.127}$$

there is no brane bending, and brane bending occurs only when one adds the coordinate conditions

$$\partial^i h_{ij} = 0 = h^i_i \ ; \ \ i, j = 0, 1, 2, 3 \tag{6.128}$$

One can easily check, however, that the extra condition $h_{55} = 0$ of Eq.(6.128) cannot be achieved without introducing brane bending. Thus to achieve $h_{55} = 0$, we see from Eqs.(6.28) and (6.50) one requires

$$\xi_5(x^i, y) = \frac{1}{6A'} f^T(x^i, y) + \phi_5(x) \tag{6.129}$$

where the function of integration $\phi_5(x)$ is independent of y and $f^T(x^i, y)$ is the value of f^T in the frame of Eq.(6.20). On the branes, therefore $f^T(y_\alpha)$ is given by Eq.(6.58), and one cannot choose $\phi_5(x)$ to make ξ_5 vanish on both branes. Thus if we choose $\phi_5(x) =$ -$f^T(x^i, y_1)/6A'$ (so that $\xi_5(y_1) = 0$) then $\xi_5(y_2)$ is proportional to $e^{2\beta y_2}\bar{m}(y_2)$ [by Eqs.(6.58) and (6.117)] and so there is a huge amount of brane bending on the TeV brane. (Alternately, the choice $\phi_5(x) = 0$ gives by Eq.(6.30) that $f^T(x^i, y) = 0$ but with brane bending on both branes.) This would presumably greatly modify the geodesic motion of particles on the TeV brane. Ref.[43] carries out the analysis in the frame of Eq.(6.128) assuming there is no brane bending in that frame. They define the gravitational potential by the diagram of two point mass stress tensors connected by a free field gravitational propagator. The f^T components vanish for free fields and so they miss the effects of f^T. Further, they find it necessary to fine tune the matter on the y_2 brane to get a consistent solution. In contrast, the analysis given here is valid for arbitrary matter on the y_1 and y_2 branes. Finally we note that none of the previous discussions have analysed gravitational forces involving two particles on the y_2 brane which is where difficulties arose.

CHAPTER VII

CONCLUSIONS

We have studied here the Hubble era cosmology in Horava-Witten M-theory. After compactification of the 11D space on a Calabi-Yau threefold, the system reduces to a 5D theory, the fifth dimension, y, bounded by two 3-branes with gravity and a scalar field (representing the volume modulus) in the bulk, and gauge and chiral matter on the 3-branes. The field equations were solved in the bulk and the boundary conditions on both 3-branes were imposed. We have shown that for the static solution, the standard RWF cosmology arises for relativistic matter on the branes, but the field equations cannot allow non-relativistic matter. This result arises from the constraint of satisfying the boundary conditions on *both* 3-branes. The same result maintains if one adds 5-branes in the bulk (the most general form of Horava-Witten M-theory). We have included all of the potentials that arise perturbatively in HW theory. However, there are non-perturbative potentials in the theory that have not been included and may allow the introduction of non-perturbative matter. Once these potentials are included, the vacuum structure of the theory will be altered and one will need to find a new set of vacuum solutions and then perturb for matter around these. We also have not addressed the issue of moduli stabilization that has been studied recently in the context of flux compactifications in [20, 22], and it is of interest to see if this modifies the above results.

We have also demonstrated the difference between the HW theory and the RS phenomenology. For the special class of potentials studied in [33] the RS model can accommodate arbitrary matter on the branes and thereby reproduce the RWF cosmology. This occurs due to the existence of free parameters in the brane potentials which can relax the constraint on non-relativistic matter found for HW theory by allowing one to solve for the change in brane separation due to the presence of matter rather than putting a constraint on the matter itself. In HW theory one is not allowed to introduce such free parameters as the form of the brane potentials are fully determined by the consistency conditions of the theory.

In Appendix A we have analysed the RS model discussed in [34] which includes a scalar potential that does not fall into the class of potentials studied in [33]. The vacuum solutions in this case are given by a series expansion and we have shown that one cannot generate the desired solution to the hierarchy problem if one truncates the series after the first term as was done in [34] (or if one includes the second term). The difficulty is that the vacuum solution must obey boundary conditions on both branes which precludes the hierarchy from developing. It is an interesting question of whether this difficulty is due to the truncation or whether formation of a hierarchy is sensitive to the choice of brane and bulk potentials.

We have also examined the gravitational forces between point particles in the static limit in the two brane Randall-Sundrum model. In contrast to previous analyses, we have chosen gauge conditions (coordinate frames) to solve the field equations that maintain the S^1/Z_2 boundary conditions, and hence produce no brane bending effects, and we also examine forces between particles on both branes, not just the $y_1 = 0$ brane. A convenient technique for solving the field equations is to introduce for the 4D generalization of the ADM decomposition [44, 45] for the metric perturbation

$$h_{ij} = h_{ij}^{TT} + h_{ij}^{T} + h_{i,j} + h_{j,i} \tag{7.1}$$

where $\partial^i h_{ij}^{TT} = 0 = \partial^i h_{ij}^{T}$, $\eta^{ij} h_{ij}^{TT} = 0$, and $\eta^{ij} h_{ij}^{T} \equiv f^T \neq 0$. The h_{ij}^{TT} contain the Kaluza-Klein modes while both the h_{ij}^{TT} and h_{ij}^{T} contribute to the static Newtonian potential (with pole at $\vec{p}^2 = 0$ in momentum space). One finds that a particle on the $y_1 = 0$ brane sees a Newtonian force from another particle on either the y_1 brane or the $y_2 = \pi\rho$ brane but with different Newtonian constants: $G_N = \beta/8\pi M_5^3$ and $G_N = \beta/6\pi M_5^3$ respectively. The difference arises from the fact that the f^T component of h_{00} enters with opposite sign for y_1 and y_2 particles, as seen in Eqs.(6.58) and (6.63). (The fact that matter at y_2 effects matter at y_1 differently from other matter at y_1 was also noted in [37] in another connection.) Note that the f^T contribution is precisely what is needed to give the conventional value $G_N = \beta/8\pi M_5^3$ on the y_1 brane.

A curious feature of the potential Eq.(6.124) is that the force depends only on the 3D distance, and is independent of any y separation. It would be interesting to see if this produces any causal questions i.e. if one jiggled the mass on y_2, how long does it take for the effect to become noticeable at the y_1 particle, a question involving dynamical rather than static solutions.

A more serious problem is the force seen by two particles on the TeV brane y_2. One sees from Eq.(6.126) that there is an attractive term arising from the f^T contribution which is $\mathcal{O}(e^{2\beta y_2}) \approx (10^{16})^2$ larger than the normal gravity and this occurs *after* one has correctly rescaled the y_2 masses to TeV size (as one normally does in the RS model). Thus one does not recover normal Newtonian gravitation in the static limit on the TeV brane. All analyses both in the literature and here up to now have neglected the Goldberger-Wise scalar field. Including it in might in some way cancel out the anomolous $e^{2\beta y_2}$ factor in Eq.(6.126). The analysis including the scalar field is much more complicated than the calculation given here and it is uncertain whether it will solve the problem.

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APPENDIX A

RS COSMOLOGY WITHOUT AN EXACT SOLUTION

In chapter 5 we discussed cosmology in the Randall-Sundrum model and showed that for a general class of bulk and brane potentials one can obtain vacuum solutions and then find consistent solutions to the Einstein and scalar field equations for arbitrary matter introduced perturbatively on the branes. In this appendix we will analyse the situation for a scalar field obeying the potentials chosen in [34],i.e. in the bulk

$$V(\Phi) = \frac{1}{2} m\Phi^2, \tag{A.1}$$

and on the boundaries $V_i(\Phi) = m_i(\Phi - v_i)^2$. Here m_i are the analogs of γ_i that arose in chapter 5. These potentials cannot be put into the form specified by Eq.(5.12) and therefore one will have to solve the second order differential equations to find the vacuum solutions. We will find that it is not easy to see how this choice of potentials leads to the desired hierarchy.

Throughout this section we follow the notation of [34]. The metric is

$$ds^2 = e^{-2N(t,y)}dt^2 - e^{-2A(t,y)} \sum_i dx_i^2 - b(t,y)^2 dy^2 \tag{A.2}$$

and the perturbative expansions are given by

$$\begin{aligned} N(t,y) &= A_o(y) + \delta N(t,y) && \text{(A.3)} \\ A(t,y) &= A_o(y) + \delta A(t,y) && \text{(A.4)} \\ b(t,y) &= b_o + \delta b(t,y) && \text{(A.5)} \\ \Phi(t,y) &= \Phi_o(y) + \delta\Phi(t,y). && \text{(A.6)} \end{aligned}$$

The Einstein equations and the scalar field equation at vacuum order are

$$A_o'^2 = \frac{\kappa^2}{12}\left(\Phi_o'^2 - m^2 b_o^2 \Phi_o^2\right) + k^2 b_o^2 \tag{A.7}$$

$$A_o'' = \kappa^2 \frac{1}{3} \Phi_o'^2 \tag{A.8}$$

$$\Phi_o'' = 4A_o'\Phi_o' + m^2 b_o^2 \Phi_o \tag{A.9}$$

where κ^2 is given by $\kappa^2 = 1/M^3$ where M is the 5D Planck scale and κ^2 and k^2 are related to the bulk cosmological constant Λ by Λ = -6k^2/κ^2. Only two of these equations are independent since the third equation can be generated by taking the y-derivative of the first and then inserting the second. Therefore a solution of any two of these equations will necessarily satisfy the third. We look for solutions of the form

$$A_o = a_o + \beta y + \sum_{n=1}^{\infty} a_n e^{-2n\alpha y} \tag{A.10}$$

$$\Phi_o = \sum_{n=1}^{\infty} c_n e^{-(2n-1)\alpha y} \tag{A.11}$$

where $\alpha = \epsilon k b_o$.

Ref.[34] assumed that truncating the series at n=1 represents a good approximation. However, it is easy to see that when this truncation is inserted into Eqs.(A.7-A.9) one generates the higher terms of Eqs.(A.10,A.11). We would like to examine the question of whether the higher terms can be ignored thereby giving the results found in [34]. While the equations are non-linear, it is still possible to obtain recursion relations. Inserting Eqs.(A.10) and (A.11) into the vacuum equation (A.8) generates relations between a_n and c_n, the first few of which are

$$4a_1 = \frac{\kappa^2}{3} c_1^2 \tag{A.12}$$

$$16a_2 = \frac{\kappa^2}{3} 6c_1c_2 \tag{A.13}$$

$$36a_3 = \frac{\kappa^2}{3}\left(10c_1c_3 + 9c_2^2\right). \tag{A.14}$$

One can also see that $\beta = kb_o$ from Eq.(A.7). Inserting Eqs.(A.10) and (A.11) into Eq.(A.9) we find

$$\sum_{n=1}^{\infty} c_n \left(\alpha^2 (2n-1)^2 + 4\beta\alpha(2n-1) - m^2 b_o^2\right) e^{-(2n-1)y} \tag{A.15}$$

$$= 8\alpha^2 \sum_{n,m=1}^{\infty} n(2m-1)a_n c_m e^{-(2n+2m-1)y}. \tag{A.16}$$

For n=1 this gives an equation for the coefficients of $e^{-\alpha y}$,

$$\alpha^2 + 4\alpha\beta - m^2 b_o^2 = 0 \tag{A.17}$$

or

$$\epsilon = -2 + \sqrt{4 + \frac{m^2}{k^2}} \tag{A.18}$$

where we have taken the positive root so that $\epsilon > 0$.(This is identical to the result found in [34].) For n = 2, after using the result of Eq.(A.12), we find the following relation between c_2 and c_1

$$c_2 = \frac{2\alpha^2\kappa^2}{3(9\alpha^2 + 12\beta\alpha - m^2 b_o^2)} c_1^3. \tag{A.19}$$

Using Eq.(A.17) reduces this to

$$c_2 = \frac{1}{12}\frac{\alpha\kappa^2}{\alpha + \beta} c_1^3 \tag{A.20}$$

which is an example of the general result

$$c_n \sim \kappa^{2n-2} c_1^{2n-1} \tag{A.21}$$

and subsequently

$$a_n \sim \kappa^{2n} c_1^{2n}. \tag{A.22}$$

Thus all the coefficients are determined by the constant of integration c_1.

Solutions to the bulk equations must also satisfy the boundary conditions

$$A'_o\Big|_{y=y_i} = (-1)^{i+1}\frac{\kappa^2}{6} b_o V_i(\Phi)\Big|_{y=y_i} \tag{A.23}$$

$$\Phi'_o\Big|_{y=y_i} = (-1)^{i+1}\frac{b_o}{2} V'_i(\Phi)\Big|_{y=y_i} \tag{A.24}$$

where i=1,2 refers to the boundary at y=0, and y=1 respectively and V_i is specified below Eq.(A.1). We will first try to satisfy these boundary conditions by keeping only the first term in the series for Φ_o and A_o and then use the relation Eq.(A.19) to determine if this is a reasonable approximation. Using

$$\Phi'_o = -\alpha c_1 e^{-\alpha y} \tag{A.25}$$

$$A'_o = \beta - \frac{\kappa^2}{6}\alpha c_1^2 e^{-2\alpha y} \tag{A.26}$$

in the Φ_o boundary conditions we find equations for c_1 and $e^{-\alpha}$ in terms of v_1 and v_2

$$c_1 = \frac{m_1 v_1}{m_1 + k\epsilon} \tag{A.27}$$

$$e^{-\alpha} = \frac{m_2 v_2}{m_2 - k\epsilon}\frac{m_1 + k\epsilon}{m_1 v_1}. \tag{A.28}$$

Note that if $m_{1,2} \gg k\epsilon$ (the "stiff potential" limit) then

$$e^{-\epsilon k b_o} \simeq \frac{v_2}{v_1} \tag{A.29}$$

which was obtained in [34] and used there to create a large hierarchy without the need for fine-tuning of parameters. Thus writing Eq.(A.29) as

$$e^{-k b_o} \simeq \left(\frac{v_2}{v_1}\right)^{1/\epsilon} \tag{A.30}$$

one see that for e.g. $\epsilon = 1/30$ one needs only assume $v_2/v_1 = 0.3$ to obtain

$$e^{-kb_o} \cong 2 \times 10^{-16} \tag{A.31}$$

However, v_1 and v_2 are not totally free parameters and using the boundary conditions for A_o we find relations for v_1 and v_2

$$v_1^2 = \frac{6}{\kappa^2}\frac{m_1 + \epsilon k}{\epsilon m_1} \tag{A.32}$$
$$v_2^2 = \frac{6}{\kappa^2}\frac{m_2 - \epsilon k}{\epsilon m_2} \tag{A.33}$$

When these relations are inserted into Eq.(A.28) we find that

$$e^{-\alpha} = \left(\frac{m_2}{m_2 - k\epsilon}\frac{m_1 + k\epsilon}{m_1}\right)^{1/2}. \tag{A.34}$$

This implies that $e^{-\alpha} \geq 1$ and therefore does not give a solution to the hierarchy problem. It should be noted that this result holds for any m_i.

One can also see that with $\epsilon \ll 1$ the n=2 term in the series is not small compared to the n=1 term. From Eq.(A.19) the ratio c_2/c_1 becomes

$$\frac{c_2}{c_1} \simeq \frac{\epsilon}{12}(\kappa c_1)^2 \tag{A.35}$$

when $\epsilon \ll 1$. Using Eq.(A.27) for c_1^2 and Eq.(A.32) for v_1^2 we find

$$\frac{c_2}{c_1} \simeq \frac{m_1}{2(m_1 + k\epsilon)} \tag{A.36}$$

which is of $\mathcal{O}(1)$, and in the stiff potential limit $c_2/c_1 \simeq 1/2$. Therefore truncating the series to the first term is not a valid approximation, as the small parameter ϵ in Eq.(A.35) cancels out in Eq.(A.36). It is the imposition of the A_o' boundary condition that makes $v_{1,2} \sim 1/\epsilon^{1/2}$.

We next examine the effect of retaining only the first two terms in the series expansions for Φ_o and A_o:

$$A_o = a_o + kb_o + a_1 e^{-2\alpha y} + a_2 e^{-4\alpha y} \tag{A.37}$$

$$\Phi_o = c_1 e^{-\alpha y} + c_2 e^{-3\alpha y}. \tag{A.38}$$

As was previously noted, all coefficients can be found in terms of c_1. The Φ_o boundary condition at y=0 becomes

$$\tilde{c_1}^3 + \frac{1+\delta_1}{1+3\delta_1}\tilde{c_1} - \frac{1}{1+3\delta_1}\tilde{v_1} = 0 \tag{A.39}$$

where we have introduced the notation

$$\tilde{c}_1 \equiv \left(\frac{\epsilon\kappa^2}{12(1+\epsilon)}\right)^{1/2} c_1 \tag{A.40}$$

$$\tilde{v}_1 \equiv \left(\frac{\epsilon\kappa^2}{12(1+\epsilon)}\right)^{1/2} v_1 \tag{A.41}$$

$$\delta_i \equiv \frac{k\epsilon}{m_i}. \tag{A.42}$$

Inserting Eqs.(A.37) and (A.38) into the y=0 boundary condition for A_o' gives

$$\frac{\delta_1}{2(1+\epsilon)} - \delta_1\tilde{c}_1\left(\tilde{c}_1 + 3\tilde{c}_1^3\right) = \left(\tilde{c}_1 + \tilde{c}_1^3 - \tilde{v}_1\right)^2. \tag{A.43}$$

Upon substituting for $\tilde{c}_1^3$ from Eq.(A.39), Eq.(A.43) becomes

$$2(1+\delta_1)\tilde{c}_1^2 - 3(1-\delta_1)\tilde{c}_1\tilde{v}_1 + \frac{(1+3\delta_1)^2}{2(1+\epsilon)} - 9\delta_1\tilde{v}_1^2 = 0. \tag{A.44}$$

This is easily solved for $\tilde{c}_1$ in terms of $\tilde{v}_1$:

$$\tilde{c}_1 = \frac{3\tilde{v}_1}{4(1+\delta_1)}\left(1 - \delta_1 \pm \left[(3\delta_1+1)^2 - \frac{4(1+\delta_1)(1+3\delta_1)^2}{9\tilde{v}_1^2(1+\epsilon)}\right]^{1/2}\right). \tag{A.45}$$

In the stiff potential limit, $\delta_1 \rightarrow 0$, this reduces to

$$\tilde{c}_1 = \frac{3\tilde{v}_1}{4}\left(1 \pm \left[1 - \frac{4}{9\tilde{v}_1^2(1+\epsilon)}\right]^{1/2}\right). \tag{A.46}$$

Taking the positive root and putting this into Eq.(A.39) leads to an equation for $\tilde{v}_1$ that can be solved to give

$$\tilde{v}_1 \cong .667. \tag{A.47}$$

No real solution is found if one takes the negative root in Eq.(A.46).

From the y=1 boundary conditions we obtain the equations

$$e^{-3\alpha}\tilde{c}_1^3 + \frac{1-\delta_2}{1-3\delta_2}e^{-\alpha}\tilde{c}_1 - \frac{1}{1-3\delta_2}\tilde{v}_2 = 0 \tag{A.48}$$

$$e^{-\alpha}\tilde{c}_1 = \frac{3\tilde{v}_2}{4(1-\delta_2)}\left(1+\delta_2 \pm \left[(1-3\delta_2)^2 - \frac{4(1-\delta_2)(1-3\delta_2)^2}{9\tilde{v}_2^2(1+\epsilon)}\right]^{1/2}\right). \tag{A.49}$$

which are identical to Eqs.(A.39) and (A.45) found at y=0 with $\tilde{c}_1$ replaced by $e^{-\alpha}\tilde{c}_1$ and δ_1 replaced by $(-\delta_2)$. In the stiff potential limit, $\delta_2 \to 0$, this gives $\tilde{v}_1 = \tilde{v}_2$ and $e^{-\alpha} = 1$ which again would not give the desired solution to the hierarchy problem just as in the n=1 case.

We can also determine the situation for δ_i small but non-zero. Table 1 gives some sample values. Thus a hierarchy is not obtained if we truncate at n=2. The above results suggest that keeping a finite number of terms in Eqs.(A.10) and (A.11) will not lead to a valid approximation, and it may be that truncating at n=1 does not approximate the rigorous solutions of Eqs.(A.7)-(A.9).

Table I. Example of determination of the hierarchy parameter $e^{-\alpha}$ for various choices of ϵ, δ_1, and δ_2. A valid hierarchy is obtained when $e^{-\beta} \approx 10^{-16}$, which requires $\tilde{v_1}/\tilde{v_2} \approx 1/3$ for $\epsilon = .03$ and $\tilde{v_1}/\tilde{v_2} \approx 2/3$ for $\epsilon = .01$

ϵ	0.03	0.03	0.01	0.01
δ_1	0.01	0.01	0.0001	0.0001
δ_2	-0.01	0.01	-0.0001	0.0001
$\tilde{v}_1$	$\pm.6613$	$\pm.6613$	$\pm.6578$	$\pm.6642$
$\tilde{v}_2$	$\pm.6613$	$\pm$ .6542	$\pm.6578$	$\pm.6640$
$\tilde{c}_1$	$\pm.5152$	$\pm.5152$	$\pm.5184$	$\pm.5219$
$e^{-\alpha}$	1	0.9876	1	1.0001
$e^{-\beta}$	1	0.6597	1	1.0101

APPENDIX B

KALUZA-KLEIN CORRECTIONS

In Sec.C of Chapter 6 we calculated the Kaluza-Klein (KK)corrections to h_{00}^{TT} on the $y_1 = 0$ brane in the case where both particles reside on the $y_1 = 0$ brane. In this Appendix we will calculate the other KK corrections for the two cases (i) $\xi_1 \ll 1$, $\xi_2 \gg 1$ and (ii) $\xi_1 \gg 1$, $\xi_2 \gg 1$. These can be most easily found in terms of what we have already shown for $h_{00}^{TT}(y_1)$ due to the presence of $T_{00}(y_1)$. These results were

$$h_{00}^{TT\ (i)}(1,1) = \sum_n \frac{R_n^i(1,1)}{m^2 - m_n^2} T_{00}(1)\ \ ;\ \ R_n^i(1,1) = -\frac{2\pi e^{-\beta y_2}}{3M_5^3} m_n \tag{B.1}$$

and

$$h_{00}^{TT\ (ii)}(1,1) = \sum_n \frac{R_n^{ii}(1,1)}{m^2 - m_n^2} T_{00}(1)\ \ ;\ \ R_n^{ii}(1,1) = -\frac{4\beta e^{-\beta y_2}}{3M_5^3} m_n \tag{B.2}$$

where we have denoted the contribution to h_{00}^{TT} on the i'th brane due to matter on the j'th brane by $h_{00}^{TT}(i,j)$ and (i), (ii) represent the two limits on ξ_1 and ξ_2 stated above.

We can convert these into coordinate space using

$$m^2 = -p^2 \cong -\vec{p}^2 \tag{B.3}$$

where the last approximation is true in the static limit. We then take the continuum limit where

$$\Delta m_n \to dm_n = \pi\beta e^{-\beta y_2} \tag{B.4}$$

Thus for Eq.(B.1)

$$h_{00}^{TT\ (i)}(1,1) = \frac{2}{3M_5^3\beta} \int_0^\beta dm_n d^3r \frac{e^{ip\cdot r} m_n}{\vec{p}^2 + m_n^2} T_{00}(1) \tag{B.5}$$

where we have taken the upper limit of the integral to be β since $\xi_1 \lesssim 1$. After performing the coordinate space integral we get

$$h_{00}^{TT\ (i)}(1,1) = \frac{2}{3M_5^3\beta}\int_0^\beta dm_n \frac{m_n e^{-m_n r}}{4\pi r} T_{00}(1) \tag{B.6}$$

Upon integration we find

$$h_{00}^{TT\ (i)}(1,1) = \frac{1}{6\pi M_5^3 \beta r^3}[1 - e^{-\beta r}(\beta r + 1)] T_{00}(1) \tag{B.7}$$

Thus in the limit $r \gg 1/\beta$ we have a $1/r^3$ correction

$$h_{00}^{TT\ (i)}(1,1) \cong \frac{1}{6\pi M_5^3 \beta r^3} T_{00}(1) \ \ ; \ \ \xi_1 \ll 1\, ;\, \xi_2 \gg 1 \tag{B.8}$$

Similarly from Eq.(B.2) we get

$$h_{00}^{TT\ (ii)}(1,1) = \frac{1}{3\pi^2 M_5^3 r^2} e^{-\beta r} T_{00}(1) \tag{B.9}$$

which in the limit $r \ll 1/\beta$ becomes a $1/r^2$ correction

$$h_{00}^{TT\ (ii)}(1,1) \cong \frac{1}{3\pi^2 M_5^3 r^2} T_{00}(1) \tag{B.10}$$

We consider now the other corrections arising from Eqs.(6.88) and(6.89). We have for the correction due to $T_{00}(2)$ on the y_1 brane

$$h_{00}^{TT}(1,2) = \frac{\xi_1 J_1(\xi_1) T_{00}(2)}{\xi_2 J_1(\xi_2) T_{00}(1)} h_{00}^{TT}(1,1) \tag{B.11}$$

For case (i) we have

$$\frac{\xi_1 J_1(\xi_1)}{\xi_2 J_1(\xi_2)} \cong e^{-\beta y_2} \frac{\xi_1}{2\sqrt{\frac{2}{\pi \xi_2}} cos(\xi_2 - \frac{3\pi}{4})} \tag{B.12}$$

From Eq.(6.90) $tan(\xi_2 - \frac{3\pi}{4}) = -4/(\pi\xi_1^2)$ and so

$$cos(\xi_2 - \frac{3\pi}{4}) = \frac{1}{\sqrt{1+\frac{16}{\pi^2\xi_1^4}}} \cong \frac{\pi\xi_1^2}{4} \tag{B.13}$$

Thus

$$\frac{\xi_1 J_1(\xi_1)}{\xi_2 J_1(\xi_2)} = (\frac{2\beta}{\pi m_n})^{1/2} e^{-\frac{\beta y_2}{2}} \tag{B.14}$$

Substituting this expression into Eq.(B.11) and after performing the coordinate space integration we find

$$h_{00}^{TT\ (i)}(1,2) = \frac{e^{-\frac{\beta y_2}{2}}}{6\pi^{3/2} M_5^3 r} \sqrt{\frac{2}{\beta}} \int_0^\beta dm_n m_n^{1/2} e^{-m_n r} T_{00}(2) \tag{B.15}$$

We can calculate the integral in the limit $r \gg 1/\beta$ which gives a $1/r^{5/2}$ correction

$$h_{00}^{TT\ (i)}(1,2) = \frac{\sqrt{2} e^{-\frac{\beta y_2}{2}}}{12\pi M_5^3} \frac{1}{\beta^{1/2} r^{5/2}} T_{00}(2) \tag{B.16}$$

For case (ii) we need

$$\frac{\xi_1 J_1(\xi_1)}{\xi_2 J_1(\xi_2)} \cong \sqrt{\frac{\xi_1}{\xi_2}} \frac{cos(\xi_1 - \frac{3\pi}{4})}{cos(\xi_2 - \frac{3\pi}{4})} = e^{-\frac{\beta y_2}{2}} (-1)^n \tag{B.17}$$

After substituting this into Eq.(B.11) and performing the coordinate space integral we find

$$h_{00}^{TT\ (ii)}(1,2) = \frac{\beta}{3\pi M_5^3} \frac{e^{-\frac{3\beta y_2}{2}}}{r} \sum_n (-1)^n e^{-m_n r} T_{00}(2) \tag{B.18}$$

Here since $\xi_1 \gtrsim 1$ the sum is over $m_n \gtrsim \beta$ and since $m_n = n\pi\beta e^{-\beta y_2}$ we require

$$n \gtrsim \frac{e^{\beta y_2}}{\pi} \equiv N \gg 1 \tag{B.19}$$

Thus Eq.(B.18) becomes

$$h_{00}^{TT\ (ii)}(1,2) \cong \frac{\beta}{3\pi M_5^3} \frac{e^{-\frac{3\beta y_2}{2}}}{r} \sum_{n=N}^{\infty} (-1)^n e^{-n\pi\beta r e^{-\beta y_2}} T_{00}(2) \tag{B.20}$$

Let

$$n = N + m \;\; ; \;\; m = 0, 1, 2, ... \tag{B.21}$$

Since

$$N\pi\beta r e^{-betay_2} = \beta r \tag{B.22}$$

one has

$$h_{00}^{TT\;(ii)}(1,2) \cong (-1)^N \frac{\beta}{3\pi M_5^3} \frac{e^{-\frac{3\beta y_2}{2}} e^{-\beta r}}{r} \sum_{m=0}^{\infty} (-1)^m e^{-m\pi\beta r e^{-\beta y_2}} T_{00}(2) \tag{B.23}$$

The sum is found to give

$$\sum_{m=0}^{\infty} (-1)^m e^{-m\pi\beta r e^{-\beta y_2}} T_{00}(2) = \frac{T_{00}(2)}{1 - e^{-\pi\beta r e^{-\beta y_2}}} \tag{B.24}$$

Thus in the limit $r \ll 1/\beta$

$$h_{00}^{TT\;(ii)}(1,2) = (-1)^N \frac{e^{-\frac{\beta y_2}{2}}}{3\pi^2 M_5^3 r^2} T_{00}(2) \tag{B.25}$$

For the KK corrections on the y_2 brane we have

$$h_{00}(2,1) = e^{2\beta y_2} h_{00}(1,2) \frac{T_{00}(y_1)}{T_{00}(y_2)} \tag{B.26}$$

Thus from Eqs.(B.16) and (B.25) we find

$$h_{00}^{TT\;(i)}(2,1) = \frac{\sqrt{2} e^{\frac{3\beta y_2}{2}}}{12\pi M_5^3} \frac{1}{\beta^{1/2} r^{5/2}} T_{00}(1) \;\; ; \beta r \gg 1 \tag{B.27}$$

$$h_{00}^{TT\;(ii)}(2,1) = (-1)^N \frac{e^{\frac{3\beta y_2}{2}}}{3\pi^2 M_5^3 r^2} T_{00}(1) \;\; ; \beta r \ll 1 \tag{B.28}$$

Similarly the corrections for both particles on the y_2 brane give

$$h_{00}^{TT}(2,2) = h_{00}^{TT}(1,1) \frac{J_1^2(\xi_1) T_{00}(2)}{J_1^2(\xi_2) T_{00}(1)} \tag{B.29}$$

For case (i) we need

$$\frac{J_1^2(\xi_1)}{J_1^2(\xi_2)} \cong \frac{m_n e^{\beta y_2}}{2\beta} \tag{B.30}$$

Hence

$$h_{00}^{TT\ (i)}(2,2) = \frac{\pi}{3\beta M_5^3} \sum_n \frac{m_n^2}{\vec{p}^2 + m_n^2} T_{00}(2) \tag{B.31}$$

and after going to the continuum limit and performing the integrations over coordinate space and the mass spectrum we find in the limit $\beta r \gg 1$

$$h_{00}^{TT\ (i)}(2,2) = \frac{e^{\beta y_2}}{6\pi\beta^2 M_5^3 r^4} T_{00}(2) \tag{B.32}$$

For case (ii) we have

$$\frac{J_1^2(\xi_1)}{J_1^2(\xi_2)} \cong \frac{\xi_2 cos^2(\xi_1 - \frac{3\pi}{4})}{\xi_1 cos^2(\xi_2 - \frac{3\pi}{4})} = e^{\beta y_2} \tag{B.33}$$

which in the limit $\beta r \ll 1$ gives

$$h_{00}^{TT\ (ii)}(2,2) = \frac{e^{\beta y_2}}{3\pi^2 M_5^3 r^2} T_{00}(2) \tag{B.34}$$

VITA

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