ASYMPTOTIC EXPANSIONS OF THE REGULAR SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS AND APPLICATIONS TO THE ANALYSIS OF THE HELICITY

A Dissertation

by

LUAN THACH HOANG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2005

Major Subject: Mathematics

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ABSTRACT

Asymptotic Expansions of the Regular Solutions to the 3D Navier–Stokes Equations and Applications to the Analysis of the Helicity. (May 2005)

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A new construction of regular solutions to the three dimensional Navier–Stokes equations is introduced and applied to the study of their asymptotic expansions. This construction and other Phragmen-Linderlöf type estimates are used to establish sufficient conditions for the convergence of those expansions. The construction also defines a system of inhomogeneous differential equations, called the extended Navier–Stokes equations, which turns out to have global solutions in suitably constructed normed spaces. Moreover, in these spaces, the normal form of the Navier–Stokes equations associated with the terms of the asymptotic expansions is a well-behaved infinite system of differential equations. An application of those asymptotic expansions of regular solutions is the analysis of the helicity for large times. The dichotomy of the helicity's asymptotic behavior is then established. Furthermore, the relations between the helicity and the energy in several cases are described.

To my Parents, Hoàng Thạch Thiết and Cao Thị Thu Cúc

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I believe that my beloved ones including my grandfather, Hoang Huu Dai, my aunt, Hoang Thi An, and my uncle, Nguyen Van Vinh, could not wait that long but can still be cheerful for my achievement somewhere above the sky. May my grandmother, Nguyen Thi Cam, who is not in good health for a while, feel pleased with my completion of this dissertation and regain her strength soon.

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CHAPTER I

INTRODUCTION

The Navier–Stokes equations describe the dynamics of the incompressible, viscous fluid flows. It is a long standing challenge to use the mathematical theory of the Navier–Stokes equations in order to explain phenomena in fluid mechanics. However that mathematical theory is still incomplete. For example, the basic question about the existence of regular solutions for large times in the three dimensional case is not yet answered. Even in cases when the regular solutions exist for all time, their dynamics is not well understood. One of the main difficulties in studying the three dimensional Navier–Stokes equations is the analysis of the role of the nonlinear terms in the equations. It is therefore appropriate to consider the simplest case when that role is minimal. One such case occurs when the solutions are periodic in the space variables and the body forces are potential.

As shown in [8, 9], under these circumstances, the regular solutions possess an asymptotic expansion and an associated normalization map. In particular, the normal form of the equations is constructed explicitly based on that normalization map of the initial data. The following three related questions are still open:

(i) When does the asymptotic expansion actually converge?

(ii) In what natural normed spaces does our normal form of the Navier–Stokes equations constitute a well-behaved infinite-dimensional system of ordinary differential equations?

(iii) What is the range of the normalization map?

One half of this dissertation is devoted to the study of those questions to which

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we give some partial answers. For instance, we show that if the asymptotic expansion of a regular solution $u(t)$ is absolutely convergent in the Sobolev space H^1 at time $t = 0$ then it converges in H^1 to $u(t)$ for all times t large enough (see Theorem V.1 and Proposition V.2; see also Corollaries V.1 and V.2 relevant to Question (*iii*)); also we give examples of normed spaces in which the answer to Question (*ii*) is positive (see Section C in Chapter V). Our method is a combination of a new construction of the regular solutions to the Navier–Stokes equations, which is well adapted to the study of their asymptotic expansions, and new Phragmen-Linderlöf type estimates which are made possible in carefully calculated domains of analyticity of the solutions.

With some new *a priori* estimates we show that the solutions obtained in this way are global solutions in suitable normed spaces to an extended system of the Navier–Stokes equations. These extended Navier–Stokes equations are then used to prove under certain conditions that the asymptotic expansions are convergent when time is large enough. These conditions are much simpler than those considered in Proposition 4.1 of [9]. Although, the convergence of the asymptotic expansions in the classical sense is still unknown for arbitrary values of the normalization map, we can show that there is convergence in very large normed spaces. Moreover, as a consequence of our estimates, the inverse of the normalization map is Lipschitz continuous in these normed spaces.

Another half of the dissertation is our study of the behavior of the helicity for large times using the above asymptotic expansions of regular solutions to the Navier–Stokes equations. The helicity and vorticity show the main differences between two-dimensional (2D) and three-dimensional (3D) fluid flows. In the 2D case, the velocity and vorticity are perpendicular, and hence the helicity is identically zero. This is not in general the case in 3D flows, where the helicity is an inviscid constant of motion [23, 20]. This may make the helicity invariant comparable to the enstrophy invariant in the context of 2D ideal fluids. For 2D fluids, an inverse energy cascade to the large length scales was conjectured [14] and supported by numerical simulations. In contrary, the helicity invariant in 3D fluids is suggested to allow the joint cascade of both energy and helicity to the small length scales ([4, 15, 3], recent paper [5] and the references therein). Moffatt [20] gave an interpretation of the helicity invariants in terms of topological invariants and dynamics of vortex tubes. Many studies have been devoted to understanding the role of the helicity in dissipative turbulent flows (see, for examples, [21, 22, 27] and the references therein). In particular helicity plays essential role in the investigations of helical structures of turbulent flows in [27]. In order to understand the helicity on a rigorous mathematical basis it is natural to start with the deterministic behavior of the solutions to the 3D Navier-Stokes equations and to consider the statistical behavior afterwards. We show below that, in the case of 3D periodic Navier–Stokes equations with potential forces, the helicity has only two possibilities: it is either identically zero (i.e., the flow is helicity-free) or eventually nonzero. The two cases turn out to be quite substantive so that neither can be neglected in the study of fluid flows. The helicity-free flows are “unstable”, in the sense that small perturbations can always produce non helicity-free flows. For these latter flows, the helicity eventually has one sign (positive or negative) and

$$\lim_{t\to\infty} \frac{\log |\int_\Omega (\nabla \times u(x,t)) \cdot u(x,t) dx|}{\nu t} = -2h_0\lambda_1,$$

where h_0 is a positive integer, $\Omega = (0,L)^3, L > 0$ is the domain of periodicity in the space variable, ν is the viscosity of the fluid, $u(x,t)$ is the velocity field, and $\lambda_1 = 4\pi^2/L^2$ is the first eigenvalue of the Laplacian operator $(-\Delta) = \nabla \times (\nabla \times \cdot)$ on L-periodic divergence-free fields with zero space averages; i.e., with zero integral over

Ω. It is worth recalling from [7] that when $u(x,t)$ is not identically zero we have

$$\lim_{t\to\infty} \frac{\log \int_\Omega |u(x,t)|^2 dx}{\nu t} = -2n_0\lambda_1,$$

where $n_0\lambda_1$ is an eigenvalue of the above operator $(-\Delta)$.

This dissertation is organized as follows.

In Chapter II we recall the functional setting of the Navier–Stokes equations and some known results about their regular solutions, particularly the asymptotic expansions, the normalization map and the resulting normal form. We prove the auxiliary inequalities for the nonlinear term of the Navier–Stokes equations, show some supplementary properties of the polynomial coefficients in the asymptotic expansion. Moreover, we give a general definition of asymptotic expansions by polynomials and exponential functions in normed spaces. We present some of its immediate consequences which will be used to find the asymptotic expansion of the helicity.

In Chapter III, we derive large domains of analyticity of the regular solutions to the Navier–Stokes equations. This is important for the establishment of the dichotomy of the helicity's asymptotic behavior in Chapter VI and is part of the study of the extended Navier–Stokes equations. We then obtain some Phragmen-Linderlöf type estimates for bounded analytic functions in such domains.

In Chapter IV, we introduce a new construction of regular solutions. Using this construction, we rediscover the classical results concerning the regular solutions of the 3D Navier–Stokes equations. The new system of equations is called the extended Navier–Stokes equations and is proved to have a global solution in certain normed spaces.

In Chapter V we study that new system of equations, establish some sufficient conditions for the convergence of the asymptotic expansion. We also give a partial description of the range of the normalization map. Furthermore, we construct some normed spaces in which the normal form of the Navier–Stokes equations associated with the asymptotic expansions is a well-behaved system of infinitely many differential equations. In those spaces, the normalization map is a homeomorphism in a neighborhood of the origin.

In Chapter VI, we derive the asymptotic expansion for the helicity and establish the dichotomy of its asymptotic behavior: the helicity is identically zero or eventually nonzero. We show that the set $\mathcal{R}_0$ of the regular initial data such that the solutions are regular and have zero helicity for all time $t \geq 0$ though is "small" compared to the set of regular initial data, but contains infinitely many invariant closed linear manifolds of infinite dimension. We also present examples of solutions for which the helicity and energy display similar and dissimilar asymptotic behavior. The helicity is, in fact, asymptotically decaying at least as fast as the energy. Moreover, we give examples showing that the exponential decay of the helicity may have an exponent which is much larger than that of the energy.

In the last chapter, Chapter VII, we summarize our main results and suggest a few open problems.

CHAPTER II

PRELIMINARIES

A. Mathematical settings

We consider the Navier–Stokes equations in the three dimensional space $\mathbb{R}^3$ with a potential body force

$$\begin{cases} \dfrac{du}{dt} + (u\cdot\nabla)u - \nu\Delta u = -\nabla p + \nabla P, \\ \operatorname{div}\, u = 0, \\ u(x,0) = u^0(x), \end{cases} \tag{2.1}$$

where $\nu > 0$ is the kinematic viscosity, $u = (u_1, u_2, u_3)$ is the unknown velocity field, p is the unknown pressure, ∇P is the body force specified by a given function P and the initial velocity u^0 is also known. We focus our study on periodic solutions such that

$$u(x + Le_j) = u(x) \quad \text{for all} \quad x \in \mathbb{R}^3, \quad j = 1, 2, 3, \tag{2.2}$$

where $\{\, e_j : j = 1, 2, 3 \,\}$ is the canonical basis in $\mathbb{R}^3$ and $L > 0$. We call functions satisfying (2.2) L-periodic functions. By changing the reference system, we may assume that the flow also satisfies the zero average condition

$$\int_\Omega u(x)dx = 0, \tag{2.3}$$

where $\Omega = (0, L)^3$. Throughout this dissertation we take $L = 2\pi$ and $\nu = 1$. The general case is recovered by a change of scale such as

$$\tilde{u}(\tilde{t}, \tilde{x}) = \frac{1}{\lambda_1^{1/2}\nu} u\Big(\frac{\tilde{t}}{\lambda_1 \nu}, \frac{\tilde{x}}{\lambda_1^{1/2}}\Big),$$

where $\lambda_1 = (2\pi/L)^2$.

Let $\mathcal{V}$ be the set of all L-periodic trigonometric polynomials on Ω with values in $\mathbb{R}^3$ which are divergence-free as well as satisfy the condition (2.3). We define

$$\begin{cases} H = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3, \\ V = \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \end{cases}$$

where $H^l(\Omega)$ with $l = 0, 1, 2, \ldots$ denotes the Sobolev space of functions $\varphi \in L^2(\Omega)$ such that for every multi-index α with $|\alpha| \leq l$ the distributional derivative $D^\alpha \varphi \in L^2(\Omega)$.

For $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$, define $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$ and $|a| = \sqrt{a \cdot a}$. Let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the scalar product and norm in $L^2(\Omega)^3$ given by

$$\langle u, v \rangle = \int_\Omega u(x) \cdot v(x) dx, \quad |u| = \langle u, u \rangle^{1/2}, \quad u, v \in L^2(\Omega)^3.$$

Note that we use $|\cdot|$ for the length of vectors in $\mathbb{R}^3$ as well as the L^2-norm of vector fields in $L^2(\Omega)^3$. In each case the context clarifies the precise meaning of this notation.

Let P_L denote the orthogonal projection in $L^2(\Omega)^3$ onto H. On V we consider the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ and the norm $\|\cdot\|$ defined by

$$\langle\langle u, v \rangle\rangle = \sum_{j,k=1}^{3} \int_\Omega \frac{\partial u_j(x)}{\partial x_k} \frac{\partial v_j(x)}{\partial x_k} dx \quad \text{and} \quad \|u\| = \langle\langle u, u \rangle\rangle^{1/2},$$

for $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in V.

Define the Stokes operator A with domain $\mathcal{D}_A = V \cap H^2(\Omega)^3$ by

$$Au = -\Delta u \quad \text{for all} \quad u \in \mathcal{D}_A.$$

The inner product of $u, v \in \mathcal{D}_A$ and the norm of $w \in \mathcal{D}_A$ are defined by $\langle Au, Av \rangle$ and $|Aw|$, respectively. Note for $w \in \mathcal{D}_A$ that (2.3) implies the norm $|Aw|$ is equivalent to the usual Sobolev norm of $H^2(\Omega)^3$. We also define the curl operator by

$$Tw = \nabla \times w \quad \text{for all } w \in V, \tag{2.4}$$

and the bilinear mapping associated with the nonlinear term in the Navier–Stokes equations by

$$B(u,v) = P_L(u \cdot \nabla v) \quad \text{for all } u, v \in \mathcal{D}_A. \tag{2.5}$$

The following relations are well-known

$$T^2 u = Au \quad \text{for all } u \in \mathcal{D}_A, \tag{2.6}$$

$$|Tw| = \|w\| \quad \text{for } w \in V, \tag{2.7}$$

$$B(u,u) = -P_L(u \times Tu) \quad \text{for all } u \in \mathcal{D}_A, \tag{2.8}$$

$$\langle B(u,v), v \rangle = 0 \quad \text{for } u, v \in \mathcal{D}_A, \tag{2.9}$$

$$\langle B(Tv,v), u \rangle = \langle B(u,v), Tv \rangle \quad \text{for } u, v \in \mathcal{D}_A \text{ such that } Tv \in \mathcal{D}_A, \tag{2.10}$$

$$\langle B(u,u), Tv \rangle + \langle B(u,v), Tu \rangle + \langle B(v,u), Tu \rangle = 0 \tag{2.11}$$

for $u, v \in \mathcal{D}_A$ such that $Tu, Tv \in \mathcal{D}_A$.

A classical result (tracking back to Leray's pioneering works in the 1930's, e.g. [19, 17, 18]) in the theory of the Navier–Stokes equations is that for any initial data u^0 in H there exists a weak solution defined for all $t > 0$ which *eventually* becomes analytic in space and time variables and which converges exponentially to zero as $t \to \infty$ ([16, 26, 6, 11]). Since in this dissertation we are interested in the asymptotic behavior of a solution for time $t \to \infty$, we will consider only regular solutions. For these solutions the Navier–Stokes equations can be given the functional form

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > 0, \\ u(0) = u^0 \in V, \end{cases} \tag{2.12}$$

where the equation holds in $\mathcal{D}_A$ for all $t > 0$ and $u(t)$ is continuous from $[0, \infty)$ into V. The set of all initial data u^0 in (2.12) will be denoted by $\mathcal{R}$. Also in the sequel, whenever not otherwise specified, the topology of $\mathcal{R}$ is that of V. It is known that $u(t) \in H^l(\Omega)^3$ for all $t > 0$ and $l = 1, 2, 3...$ In particular $u(t) \in \mathcal{D}_A$ and $Tu(t) \in \mathcal{D}_A$ for all $t > 0$. Now from (2.9) and (2.10) we obtain the energy balance

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + |Tu(t)|^2 = 0, \quad \text{for all } t \geq 0 \tag{2.13}$$

and the helicity balance

$$\frac{1}{2}\frac{d}{dt}\langle u(t), Tu(t)\rangle + \langle Tu(t), T^2u(t)\rangle = 0, \quad \text{for all } t > 0. \tag{2.14}$$

Above $\frac{1}{2}|u|^2$ and $\langle u, Tu\rangle$ are the total (kinetic) energy/mass and total helicity/mass of u.

The Stokes operator A has a sequence of eigenvalues $\{\lambda_j, j = 1, 2, 3, ...\} = \sigma(A)$ of the form $\lambda_j = |k|^2$ for some $k \in \mathbb{Z}^3 \setminus \{0\}$. Note that $\lambda_1 = 1 = |e_1|^2$ and hence the additive semigroup generated by these eigenvalues coincides with the set $\mathbb{N} = \{1, 2, 3, ...\}$ of all natural numbers. For $n \in \mathbb{N}$ we denote by R_n the orthogonal projection of H onto the eigenspace of A associated to n, namely,

$$R_nH = \{u \in H, Au = nu\}. \tag{2.15}$$

If n is an eigenvalue of A, R_nH is generated by functions of the forms

$$(a_k^1 + ia_k^2)e^{i(k\cdot x)} + (a_k^1 - ia_k^2)e^{-i(k\cdot x)}, \quad k \in \mathbb{Z}^3, \quad |k|^2 = n,$$

where

$$a_k^1, a_k^2 \in \mathbb{R}^3, \quad a_k^1 \cdot k = a_k^2 \cdot k = 0.$$

Otherwise, $R_n = 0$, for example, $R_7 = 0, R_{15} = 0, R_{23} = 0$, etc... Define

$$P_n = R_1 + R_2 + \cdots + R_n \qquad \text{and} \qquad Q_n = I - P_n. \tag{2.16}$$

B. The asymptotic behavior of solutions

Let us recall some known results on the asymptotic expansions and the normal form of the regular solutions to the Navier–Stokes equations (see [7, 8, 9, 12] for more details). First, for any $u^0 \in \mathcal{R}$ there is an eigenvalue n_0 of A such that

$$\lim_{t\to\infty} \frac{\|u(t)\|^2}{|u(t)|^2} = n_0 \quad \text{and} \quad \lim_{t\to\infty} u(t)e^{n_0 t} = w_{n_0}(u^0) \in R_{n_0}H \setminus \{0\}. \tag{2.17}$$

Furthermore, $u(t)$ has the asymptotic expansion (see also Definition II.2)

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \cdots , \tag{2.18}$$

where $q_j(t)$ is a polynomial in t of degree at most $j-1$ with values trigonometric polynomials in H. This means that for any $N \in \mathbb{N}$ the correction term $v_N(t) = u(t) - \sum_{j=1}^N q_j(t)e^{-jt}$ satisfies

$$|v_N(t)| = O\big(e^{-(N+\varepsilon)t}\big) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad \varepsilon = \varepsilon_N > 0. \tag{2.19}$$

In fact, $v_N(t)$ belongs to $C^1\big([0,\infty), V\big) \cap C^\infty\big((0,\infty), C^\infty(\mathbb{R}^3)\big)$, and for each $m \in \mathbb{N}$

$$\|v_N(t)\|_{H^m(\Omega)} = O\big(e^{-(N+\varepsilon)t}\big) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad \varepsilon = \varepsilon_{N,m} > 0. \tag{2.20}$$

Let $W(u^0) = W_1(u^0) \oplus W_2(u^0) \oplus \cdots$, where $W_j(u^0) = R_j q_j(0)$ for $j \in \mathbb{N}$. Then W is an one-to-one analytic mapping from $\mathcal{R}$ to the Frechet space $S_A = R_1H \oplus R_2H \oplus \cdots$ equipped with the component-wise topology. Also, $W'(0) = \mathbf{I}$, meaning

$$W'(0)u^0 = R_1u^0 \oplus R_2u^0 \oplus R_3u^0 \oplus \cdots . \tag{2.21}$$

Therefore, if Π_N denotes the canonical projection of S_A onto $R_1H \oplus \cdots \oplus R_NH$ then

$$\{\, \Pi_N W(u^0) : \|u^0\| < \rho \,\} \text{ is a neighborhood of 0 in } \Pi_N S_A \tag{2.22}$$

for all $\rho > 0$ and $N \in \mathbb{N}$.

The case (2.17) holds if and only if $W_1(u^0) = W_2(u^0) = \cdots = W_{n_0-1}(u^0) = 0$ and $W_{n_0}(u^0) \neq 0$. In this case

$$q_1 = q_2 = \cdots = q_{n_0-1} = 0 \quad \text{and} \quad q_{n_0} = w_{n_0}(u^0) = W_{n_0}(u^0). \tag{2.23}$$

If $u^0 \in \mathcal{R}$ and $W(u^0) = (\xi_1, \xi_2, \ldots)$, then the polynomials q_j are the unique polynomial solutions to the following equations

$$q_j'(t) + (A - j)q_j(t) + \beta_j(t) = 0, \quad t \in \mathbb{R}, \tag{2.24}$$

with

$$R_j q_j(0) = \xi_j, \tag{2.25}$$

where the terms $\beta_j(t)$ are defined by

$$\beta_1(t) = 0 \quad \text{and} \quad \beta_j(t) = \sum_{k+l=j} B(q_k(t), q_l(t)) \quad \text{for} \quad j > 1. \tag{2.26}$$

Given $\xi = (\xi_n)_{n=1}^\infty \in S_A$, the polynomials $q_j(t) = q_j(t, \xi)$ are explicitly given by the recursive formula

$$q_j(t,\xi) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau - \sum_{n \geq 0} (-1)^n [(A-j)(I-R_j)]^{-n-1} \frac{d^n}{dt^n} (I - R_j)\beta_j, \tag{2.27}$$

for $j \in \mathbb{N}$. Here $[(A-j)(I-R_j)]^{-n-1}$ is defined by

$$[(A-j)(I-R_j)]^{-n-1} u(x) = \sum_{|k|^2 \neq j} \frac{a_k}{(|k|^2 - j)^{n+1}} e^{ik\cdot x} \tag{2.28}$$

for $u(x) = \sum_{|k|^2 \neq j} a_k e^{ik\cdot x} \in \mathcal{V}$.

Finally, the S_A-valued function $\xi(t) = (\xi_n(t))_{n=1}^\infty = \big(W_n(u(t))\big)_{n=1}^\infty = W(u(t))$

satisfies the following system of differential equations

$$\begin{cases} \dfrac{d\xi_1(t)}{dt} + A\xi_1(t) = 0 \\ \dfrac{d\xi_n(t)}{dt} + A\xi_n(t) + \displaystyle\sum_{k+j=n} R_n B\big(q_k(0,\xi(t)), q_j(0,\xi(t))\big) = 0, \quad n > 1. \end{cases} \tag{2.29}$$

This system is the normal form of the Navier–Stokes equations (2.12) associated with the asymptotic expansion (2.18). It is easily to check that the solution of (2.29) with initial data $\xi^0 = (\xi_n^0)_{n=1}^\infty \in S_A$ is precisely $\big(R_n q_n(t,\xi^0)e^{-nt}\big)_{n=1}^\infty$. Thus, formula (2.27) yields the normal form and its solutions.

C. Complexification of the Navier–Stokes equations

We first introduce the Navier–Stokes equations with complex times and its analytic solutions. Let X be a real Hilbert space with scalar product $(\cdot,\cdot)_X$. We define the complexification of X as the following

$$X_{\mathbb{C}} = \{u + iv,\ u, v \in X\} \tag{2.30}$$

with the addition and scalar product defined by

$$(u_1 + iu_2) + (v_1 + iv_2) = (u_1 + v_1) + i(u_2 + v_2), \quad \text{for } u_1, u_2, v_1, v_2 \in X \tag{2.31}$$

and

$$(z_1 + iz_2)(u_1 + iu_2) = z_1u_1 - z_2u_2 + i(z_2u_1 + z_1u_2), \quad \text{for } z_1, z_2 \in \mathbb{R}, u_1, u_2 \in X, \tag{2.32}$$

respectively. The complexified space $X_{\mathbb{C}}$ is a Hilbert space with respect to the following inner product

$$(u+iv, u'+iv')_{X_{\mathbb{C}}} = (u,u')_X + (v,v')_X + i[(v,u')_X - (u,v')_X], \quad u, v, u', v' \in X. \tag{2.33}$$

When $X = H$ or $X = V$, we obtain the complexified spaces $H_{\mathbb{C}}$, $V_{\mathbb{C}}$ and their corresponding inner products and norms. For the sake of simplicity, we use the same notations $|\cdot|$ and $\|\cdot\|$ to denote $|\cdot|_{H_{\mathbb{C}}}$ and $\|\cdot\|_{V_{\mathbb{C}}}$, respectively.

The Stokes operator A is extended to the operator $A_{\mathbb{C}}$ defined on $\mathcal{D}_{A_{\mathbb{C}}} = (\mathcal{D}_A)_{\mathbb{C}}$ by the formula

$$A_{\mathbb{C}}(u + iv) = Au + iAv, \quad u, v \in \mathcal{D}_A. \tag{2.34}$$

The curl operator T is extended to $T_{\mathbb{C}}$ defined in $V_{\mathbb{C}}$ by

$$T_{\mathbb{C}}(u + iv) = Tu + iTv, \quad u, v \in V. \tag{2.35}$$

Similarly, $B(\cdot,\cdot)$ can be extended to a bounded bilinear map from $V_{\mathbb{C}} \times \mathcal{D}_{A_{\mathbb{C}}}$ to $H_{\mathbb{C}}$ as follows

$$B_{\mathbb{C}}(u + iv, u' + iv') = B(u, u') - B(v, v') + i[B(u, v') + B(v, u')] \tag{2.36}$$

for $u, v \in V,\ u', v' \in \mathcal{D}_A$. Note that, unlike the real case, we have

$$\langle B_{\mathbb{C}}(u, v), v\rangle_{H_{\mathbb{C}}} \not\equiv 0, \qquad \text{for } u, v \in \mathcal{D}_{A_{\mathbb{C}}}. \tag{2.37}$$

The Navier–Stokes equations with complex times is defined as

$$\frac{du(\zeta)}{d\zeta} + B_{\mathbb{C}}(u(\zeta), u(\zeta)) + A_{\mathbb{C}}u(\zeta) = 0, \tag{2.38}$$

$$u(\zeta_0) = u^\star, \tag{2.39}$$

where $\zeta_0 \in \mathbb{C}$ and $u^\star \in V_{\mathbb{C}}$ are given, $d/d\zeta$ denotes the derivative of $H_{\mathbb{C}}$-valued functions. For simplicity we write A and $B(\cdot,\cdot)$ instead of $A_{\mathbb{C}}$ and $B_{\mathbb{C}}(\cdot,\cdot)$.

D. Fourier formulation

For $a=(a_1,a_2,a_3), b=(b_1,b_2,b_3)\in\mathbb{C}^3$, let $a\cdot b=a_1b_1+a_2b_2+a_3b_3,\ a^*=(\bar{a_1},\bar{a_2},\bar{a_3})$ and $|a|=\sqrt{a\cdot a^*}$. Recall that every $u(x)\in\mathcal{V}$ has a Fourier series of the form

$$u(x)=\sum_{k\in\mathbb{Z}^3}a_ke^{ik\cdot x}, \tag{2.40}$$

where $a_k=a_k^1+ia_k^2\in\mathbb{C}^3,\ a_k\cdot k=0,\ a_{-k}=a_k^*,\ a_0=0$ and $a_k\neq 0$ for only finitely many vectors k of $\mathbb{Z}^3$. Also if $v(x)=\sum_{k\in\mathbb{Z}^3}v_ke^{ik\cdot x}$ is a $\mathbb{R}^3$-valued trigonometric polynomial (that is, if $v_{-k}=v_k^*$, $v_0=0$ and $v_k\neq 0$ for only finitely many k) then the Leray projection of $v(x)$ is simply given by

$$P_L(\sum_{k\in\mathbb{Z}^3}v_ke^{ik\cdot x})=\sum_{k\in\mathbb{Z}^3}a_ke^{ik\cdot x},\ \text{where } a_0=0,\ a_k=v_k-\frac{v_k\cdot k}{|k|^2}k \text{ for } k\neq 0. \tag{2.41}$$

We apply the curl and Stokes operators to $u(x)$ term by term and obtain

$$Tu(x)=\sum_{k\in\mathbb{Z}^3}ik\times a_ke^{ik\cdot x}, \tag{2.42}$$

$$Au(x)=\sum_{k\in\mathbb{Z}^3}|k|^2a_ke^{ik\cdot x}. \tag{2.43}$$

Furthermore, if $v(x)=\sum_{k\in\mathbb{Z}^3}b_ke^{ik\cdot x}\in\mathcal{V}$, the Fourier series of the bilinear mapping $B(u,v)$ is

$$B(u,v)=P_L(\sum_{n\in\mathbb{Z}^3}Q_ne^{in\cdot x})=\sum_{n\in\mathbb{Z}^3}B_ne^{in\cdot x}, \tag{2.44}$$

where $\sum_{n\in\mathbb{Z}^3}Q_ne^{in\cdot x}$ is a $\mathbb{R}^3$-valued trigonometric polynomial since

$$Q_n=i\sum_{k+l=n}(a_k\cdot j)b_j \quad \text{for } n\in\mathbb{Z}^3. \tag{2.45}$$

Therefore

$$B_0=0 \quad \text{and} \quad B_n=Q_n-\frac{Q_n\cdot n}{|n|^2}n \quad \text{for } n\neq 0. \tag{2.46}$$

We also have

$$\langle Tu, v\rangle = iL^3 \sum_{k\in\mathbb{Z}^3} (k \times a_k) \cdot b_k^* = iL^3 \sum_{k\in\mathbb{Z}^3} k \cdot (a_k \times b_k^*). \tag{2.47}$$

In particular, when $u = v$ this becomes

$$\begin{aligned}\langle Tu, u\rangle &= iL^3 \sum_{k\in\mathbb{Z}^3} k \cdot [(a_k^1 + ia_k^2) \times (a_k^1 - ia_k^2)] \\ &= 2L^3 \sum_{k\in\mathbb{Z}^3} k \cdot (a_k^1 \times a_k^2).\end{aligned} \tag{2.48}$$

Remark II.1. The general term of the last sum in Formula (2.48) vanishes if $a_k^1 \times a_k^2 = 0$. This happens when $a_k^2 = 0$ or $a_k^1 = 0$ or a_k^1 is parallel with a_k^2. In particular, if $a_k \in z_k\mathbb{R}^3$ for all k with some $z_k \in \mathbb{C}$, then $\langle Tu, u\rangle = 0$. We often use the case $a_k \in \mathbb{R}^3$, for all $k \in \mathbb{Z}^3$, or $a_k \in (i\mathbb{R}^3)$, for all $k \in \mathbb{Z}^3$, in our examples in Sections C and D of Section VI.

Remark II.2. Let $u(x)$ and $v(x)$ satisfy that whenever $a_k \neq 0$ and $b_j \neq 0$, the vectors k and j are parallel. For such k and j, $a_k \cdot j = 0$, hence Q_n given in (2.45) is zero, so is B_n. Therefore, $B(u, v) = 0$. Also, if $Tu = \mu u$ for some $\mu \in \mathbb{R}$, then by (2.8), the nonlinear term $B(u, u) = 0$.

We give here a simple description of eigenspaces of the curl operator T. Suppose that $u(x)$ is an eigenfunction of T corresponding to some eigenvalue μ. Then $Tu = \mu u$ and $Au = T^2u = \mu^2 u$. Thus μ^2 is an eigenvalue of A and u is also an eigenfunction of A corresponding to μ^2. Therefore $u \in R_{\mu^2}H$ and has the Fourier expansion of the form

$$u(x) = \sum_{|k|=|\mu|} a_k e^{ik\cdot x}. \tag{2.49}$$

Moreover, from (2.42) it follows that

$$ik \times a_k = \mu a_k \quad \text{or} \quad ik \times (a_k^1 + ia_k^2) = \mu(a_k^1 + ia_k^2).$$

This is equivalent to

$$k \times a_k^1 = \mu a_k^2, \quad k \times a_k^2 = -\mu a_k^1$$

or

$$a_k^2 = \frac{k}{\mu} \times a_k^1, \quad a_k^1 = -\frac{k}{\mu} \times a_k^2. \tag{2.50}$$

Obviously, (2.50) holds if and only if either $a_k^1 = a_k^2 = 0$, or, $|a_k^1| = |a_k^2| \neq 0$ and $\{k/\mu, a_k^1/|a_k^1|, a_k^2/|a_k^2|\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. The orientation in $\mathbb{R}^3$ considered here is that of the standard basis $\{e_1, e_2, e_3\}$. We sum up this discussion with the following.

Lemma II.1. *The spectrum of the curl operator T is*

$$\sigma(T) = \{\pm\sqrt{n}, n \in \sigma(A)\} = \{\pm|k|, k \in \mathbb{Z}^3 \setminus \{0\}\}. \tag{2.51}$$

Moreover, the eigenfunctions of T corresponding to an eigenvalue μ are the functions of the form (2.49) *such that nonzero coefficients $a_k = a_k^1 + ia_k^2$ in* (2.49) *satisfy*

i) $|a_k^1| = |a_k^2| \neq 0$ and

ii) $\{\frac{k}{\mu}, \frac{a_k^1}{|a_k^1|}, \frac{a_k^2}{|a_k^2|}\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3$.

E. Auxiliary inequalities

We prove below some estimates for the bilinear form $B(\cdot,\cdot)$ (see (2.5) for its definition).

Lemma II.2. *There is an absolute constant $C_1 > 0$ such that*

$$|P_n B(u,v)| \leq C_1 n^{1/4} \|u\| \|v\|, \quad u, v \in V, \tag{2.52}$$

$$\|P_n B(u,v)\| \leq C_1 n^{3/4} \|u\| \|v\|, \quad u, v \in V, \tag{2.53}$$

$$|B(u,v)| \leq C_1 \|u\|^{1/2} |Au|^{1/2} \|v\|, \quad u \in D_A, v \in V, \tag{2.54}$$

$$\|B(u,v)\| \leq C_1 \|u\|^{1/2} |Au|^{1/2} |Av|, \quad u, v \in D_A. \tag{2.55}$$

Proof. These inequalities are the consequences of elementary inequalities and the Sobolev and Agmon inequalities (see, for example, [1, 2]). In the estimates below, the positive constant C is generic. For $u, v, w \in \mathcal{V}$, we have

$$\begin{aligned}
|\langle P_n B(u,v), w\rangle| &\le C\|u\|_{L^6}\|v\|\|P_n w\|_{L^3} \le C\|u\|\|v\|\|P_n w\|_{L^2}^{1/2}\|P_n w\|_{L^6}^{1/2} \\
&\le C\|u\|\|v\||P_n w|^{1/2}\|P_n w\|^{1/2} \le C\|u\|\|v\||P_n w|^{1/2}|P_n w|^{1/2}(n^{1/2})^{1/2} \\
&\le C n^{1/4}\|u\|\|v\||w|.
\end{aligned}$$

Hence we obtain (2.52). Since $\|P_n B(u,v)\| = |A^{1/2}P_n B(u,v)|$, inequality (2.53) immediately implies (2.53). Inequality (2.54) follows from

$$\begin{aligned}
\big|\big\langle B(u,v), w\big\rangle\big| &\le C\|u\|_{L^\infty}|A^{1/2}v||w| \\
&\le C\|u\|^{1/2}|Au|^{1/2}\|v\||w|.
\end{aligned}$$

To prove (2.55), consider $u(x) = \sum_{0\neq k\in\mathbb{Z}^3} \hat{u}_k e^{ik\cdot x}$, define the scalar function

$$u'(x) = \sum_{0\neq k\in\mathbb{Z}^3} \hat{u}'_k e^{ik\cdot x} \text{ where } \hat{u}'_k = |\hat{u}_k|.$$

Similarly define v' and w'. We estimate

$$\begin{aligned}
\big|\big\langle B(u,v), Aw\big\rangle\big| &= \Big|\int_\Omega B(u,v)\cdot(Aw)^\star\Big| \\
&= L^3\Big|\sum_{h+k=l}(\hat{u}_h\cdot ik)\hat{v}_k\cdot|l|^2\hat{w}_l^\star\Big| \\
&\le L^3\sum_{h+k=l}|\hat{u}_h||k||\hat{v}_k||l|^2|\hat{w}_l| \\
&\le L^3\sum_{h+k=l}\big\{|\hat{u}_h||k|^2|\hat{v}_k||l||\hat{w}_l| + |h||\hat{u}_h||k||\hat{v}_k||l||\hat{w}_l|\big\} \\
&= \int_\Omega\big\{u'(-\Delta v')[(-\Delta)^{1/2}w'] + [(-\Delta)^{1/2}u'][(-\Delta)^{1/2}v'][(-\Delta)^{1/2}w']\big\} \\
&\le \|u'\|_{L^\infty}\|\Delta v'\|_{L^2}|\nabla w'| + \|\nabla u'\|_{L^3}\|\nabla v'\|_{L^6}|\nabla w'|
\end{aligned}$$

Then by using the Sobolev and Agmon inequalities, we continue

$$
\begin{aligned}
\left|\left\langle B(u,v),Aw\right\rangle\right| &\le C|\nabla u'|^{1/2}\|\Delta u'\|_{L^2}^{1/2}\|\Delta v'\|_{L^2}|\nabla w'| + C|\nabla u'|^{1/2}\|\Delta u'\|_{L^2}^{1/2}\|\Delta v'\|_{L^2}|\nabla w'| \\
&\le C|\nabla u'|^{1/2}\|\Delta u'\|_{L^2}^{1/2}\|\Delta v'\|_{L^2}|\nabla w'| \\
&= C\|u\|^{1/2}|Au|^{1/2}|Av|\|w\|.
\end{aligned}
$$

This inequality implies (2.55). □

F. Supplementary properties of the polynomials in the asymptotic expansion

Definition II.1. For $r \in \mathbb{N} \cup \{0\}$, we denote by $\mathcal{F}_r$ the set of functions $f(x) = \sum_{k\in\mathbb{Z}^3} f_k e^{ik\cdot x}$ belonging to $\mathcal{V}$ such that $f_k = 0$ whenever $|k|^2 \not\equiv r(mod\ 2)$.

It is clear that

(a) For each $r = 0, 1, 2, ...$, $\mathcal{F}_r$ is a linear space and $R_n H \subset \mathcal{F}_n$ for all $n \in \mathbb{N}$.

(b) $\mathcal{F}_r = \mathcal{F}_s$ if $r \equiv s(mod\ 2)$.

(c) $T\mathcal{F}_r \subset \mathcal{F}_r, R_n\mathcal{F}_r \subset \mathcal{F}_r, (I - R_n)\mathcal{F}_r \subset \mathcal{F}_r$, and $[(A-n)(I-R_n)]^{-1}\mathcal{F}_r \subset \mathcal{F}_r$ (see Formula (2.28)).

(d) If $u(t), v(t) \in \mathcal{F}_r$ for all $t > 0$ then the integral $\displaystyle\int_0^t u(\tau)d\tau$ and the derivatives $\dfrac{d^n v(t)}{dt^n}$, for $n = 0, 1, 2, 3...,\ t > 0$ are also in $\mathcal{F}_r$, whenever these quantities are defined.

(e) If $u \in \mathcal{F}_r, v \in \mathcal{F}_s$, and $r \not\equiv s(mod\ 2)$ then $\langle u, v\rangle = 0$. The action of the nonlinear term $B(\cdot,\cdot)$ on the classes $\mathcal{F}_r$ is given by the following.

Lemma II.3. *For any $r, s \in \mathbb{N} \cup \{0\}$, $B(\mathcal{F}_r, \mathcal{F}_s) \subset \mathcal{F}_{r+s}$.*

Proof. Let $u(x) = \sum_{k\in\mathbb{Z}^3} u_k e^{ik\cdot x} \in \mathcal{F}_r$, $v(x) = \sum_{l\in\mathbb{Z}^3} v_k e^{il\cdot x} \in \mathcal{F}_s$ and

$$B(u,v)(x) = \sum_{m\in\mathbb{Z}^3} B_m e^{im\cdot x}.$$

Then an elementary computation shows that

$$B_0 = 0 \text{ and for } m \neq 0,\ B_m = Q_m - \frac{m \cdot Q_m}{|m|^2} m, \qquad \text{where} \qquad Q_m = i \sum_{k+l=m} (u_k \cdot l) v_l.$$

Therefore, if $B_m \neq 0$, there must exist $k, l \in \mathbb{Z}^3$ such that $u_k \neq 0$, $v_l \neq 0$ and $m = k+l$. According Definition II.1, these k, l satisfy $|k|^2 \equiv r(mod\ 2)$ and $|l|^2 \equiv s(mod\ 2)$. Therefore,

$$|m|^2 = |k+l|^2 = |k|^2 + |l|^2 + 2k \cdot l \equiv (|k|^2 + |l|^2)(mod\ 2) \equiv (r+s)(mod\ 2).$$

This means that $B_m \neq 0$ implies $|m|^2 \equiv (r+s)(mod\ 2)$, thus $B(u,v) \in \mathcal{F}_{r+s}$. □

We can now establish a relation between polynomials $q_n(t)$ in the asymptotic expansion (2.18) and the classes $\mathcal{F}_r$.

Lemma II.4. *Let $\xi \in S_A$. Then the polynomials $q_n(t) = q_n(t, \xi)$ and $\beta_n(t)$ defined by* (2.27) *and* (2.26) *belong to $\mathcal{F}_n$ for all $n \in \mathbb{N}$ and $t \geq 0$.*

Proof. We will prove this by induction. Since $q_1 = \xi_1 \in R_1 H$, we have $q_1 \in \mathcal{F}_1$ by (a). Let $n > 1$ and suppose that the statement holds for all q_j with $j < n$. Consider $q_n(t)$ given by (2.27)

$$q_n(t) = \xi_n - \int_0^t R_n \beta_n(\tau) d\tau + \sum_{h \geq 0} (-1)^{h+1} [(A-n)(I-R_n)]^{-h-1} (\frac{d}{dt})^h (I-R_n)\beta_n.$$

Note that $\xi_n, R_n\beta_n \in R_n H$ hence they belong to $\mathcal{F}_n$. Furthermore, if $r+s=n$, then $q_r \in \mathcal{F}_r$ and $q_s \in \mathcal{F}_s$, hence $B(q_r, q_s) \in \mathcal{F}_{r+s} = \mathcal{F}_n$, by Lemma II.3. Thus $\beta_n = \sum_{r+s=n} B(q_r, q_s)$ is in $\mathcal{F}_n$. Thanks to (c) and (d), $\int_0^t R_n\beta_n(\tau)d\tau$ and $[(A-n)(I-R_n)]^{-h-1}(\frac{d}{dt})^h(I-R_n)\beta_n$, for $h = 0, 1, 2, ...$, are in $\mathcal{F}_n$. In conclusion, $q_n(t)$ is a sum of $\mathcal{F}_n$-functions, therefore $q_n(t) \in \mathcal{F}_n$, due to (a). □

In addition, we have

Lemma II.5. *Let $\xi \in S_A$. Then $q_n(t,\xi) \in P_{n^2}H$, for all $n \in \mathbb{N}, t \geq 0$,*

Proof. Based on the formula of $q_n(t,\xi)$ in (2.27) and $B(u,v)$ in (2.44)–(2.46) and by induction we can prove the following: for each $n \geq 1$, suppose

$$q_n(t,\xi)(x) = \sum_{k\in\mathbb{Z}^3} a(k,t)e^{ik\cdot x},$$

then we have that $|k| \leq n$ whenever $a(k,t) \neq 0$. □

G. Asymptotic expansions in normed spaces

Definition II.2. Let $(X, \|.\|_X)$ be a normed space and $u(t)$ be a map from $(0,\infty)$ to X. We say that $u(t)$ has the following *asymptotic expansion*

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + ..., \tag{2.56}$$

where $q_j(t),\ j = 1,2,3,...$, are polynomials in t, if for each N,

$$v_N(t) = u(t) - [q_1(t)e^{-t} + q_2(t)e^{-2t} + ... + q_N(t)e^{-Nt}] \tag{2.57}$$

satisfies

$$\|v_N(t)\|_X = O(e^{-(N+\varepsilon)t}), \qquad \text{as } t\to\infty,\ \text{for some } \varepsilon = \varepsilon_N > 0. \tag{2.58}$$

Definition II.2 has the following immediate consequences. The first is the uniqueness of the expansions.

Lemma II.6. *Suppose that $u(t)$ has an asymptotic expansion* (2.56)*, then the polynomials $q_j(t)$, $j = 1,2,3...$, are unique.*

Proof. Let $u(t)$ have two asymptotic expansions

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + ... \tag{2.59}$$

and

$$u(t) \sim p_1(t)e^{-t} + p_2(t)e^{-2t} + p_3(t)e^{-3t} + ... \tag{2.60}$$

Suppose that there is $j \geq 1$ such that $q_j \neq p_j$. Let N be the first of such orders, that is, $q_N \neq q_N$ and $q_j = p_j$ for $j = 1, 2, ..., N-1$. We denote

$$U_N(t) = u(t) - [q_1(t)e^{-t} + q_2(t)e^{-2t} + ... + q_N(t)e^{-Nt}], \tag{2.61}$$

$$V_N(t) = u(t) - [p_1(t)e^{-t} + p_2(t)e^{-2t} + ... + p_N(t)e^{-Nt}]. \tag{2.62}$$

According to Definition II.2, these two remainders of $u(t)$ satisfy

$$U_N(t) = O(e^{-(N+\varepsilon)t}) \quad \text{and} \quad V_N(t) = O(e^{-(N+\varepsilon)t}), \tag{2.63}$$

as $t \to \infty$, for some $\varepsilon > 0$. Hence

$$q_N(t) - p_N(t) = e^{Nt}(V_N(t) - U_N(t)) = O(e^{-\varepsilon t}) \quad \text{as } t \to \infty. \tag{2.64}$$

Since $q_N(t) - p_N(t)$ is a nonzero polynomial, it can not decay exponentially as $t \to \infty$, hence contradicting (2.64). Therefore, $q_j = p_j$ for all $j \in \mathbb{N}$. $\square$

Lemma II.7. *Suppose that $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ are two normed spaces and S is a bounded linear map from X to Y. Assume that $u(t)$ has the asymptotic expansion*

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + ... \quad \text{in } X. \tag{2.65}$$

Then the asymptotic expansion of $Su(t)$ in Y is

$$Su(t) \sim Sq_1(t)e^{-t} + Sq_2(t)e^{-2t} + Sq_3(t)e^{-3t} + ..., \tag{2.66}$$

Proof. Given $N \in \mathbb{N}$, we have $u(t) = \sum_{j=1}^{N} q_j(t)e^{-jt} + v_N(t)$, with $\|v_N(t)\|_X =$

$O(e^{-(N+\varepsilon)t})$ for $t \to \infty$ and some $\varepsilon > 0$. By the linearity of S,

$$Su(t) = \sum_{j=1}^{N} Sq_j(t)e^{-jt} + Sv_N(t),$$

where the last term decays exponentially as

$$\|Sv_N(t)\|_Y \leq \|S\|\|v_N(t)\|_Y = O(e^{-(N+\varepsilon)t}), \quad \text{for } t \to \infty,$$

here $\|S\|$ denotes the operator norm of S. Therefore, according to Definition II.2 $Su(t)$ has asymptotic expansion (2.66). □

Lemma II.8. *Suppose that* $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y), (Z, \|\cdot\|_Z)$ *are normed spaces and* $M(\cdot, \cdot)$ *is a bounded bilinear map from* $X \times Y$ *to* Z*. Assume that* $u : (0, \infty) \to X$ *and* $v : (0, \infty) \to Y$ *have corresponding asymptotic expansions*

$$u(t) \sim \sum_{j=1}^{\infty} u_j(t)e^{-jt} \text{ in } X \text{ and } v(t) \sim \sum_{j=1}^{\infty} v_j(t)e^{-jt} \text{ in } Y. \tag{2.67}$$

Then $M(u, v)$ *has the following asymptotic expansion in* Z

$$M(u(t), v(t)) \sim \sum_{j=1}^{\infty} M_j(t)e^{-jt}, \tag{2.68}$$

where $M_1 = 0$ *and for* $j > 1$*,*

$$M_j(t) = \sum_{k+l=j} M(u_k(t), v_l(t)). \tag{2.69}$$

Proof. By the boundedness of $M(\cdot, \cdot)$, there is $C > 0$ such that

$$\|M(x, y)\|_Z \leq C\|x\|_X\|y\|_Y \quad \text{for all } x \in X, y \in Y. \tag{2.70}$$

According to Definition II.2, we have

$$u(t) = \sum_{j=1}^{N} u_j(t)e^{-jt} + U_N(t), \quad v(t) = \sum_{j=1}^{N} v_j(t)e^{-jt} + V_N(t), \tag{2.71}$$

where

$$\|U_N(t)\|_X = O(e^{-(N+2\varepsilon)t}) \quad \text{and} \quad \|V_N(t)\|_Y = O(e^{-(N+2\varepsilon)t}), \tag{2.72}$$

for $t \to \infty$ and some $\varepsilon \in (0, 1/4)$. Thanks to the linearity of $M(\cdot, \cdot)$ with respect to each component, we have

$$\begin{aligned} M(u(t), v(t)) &= \sum_{k,l=1}^{N} e^{-(k+l)t} M(u_k(t), v_l(t)) + \sum_{k=1}^{N} e^{-kt} M(u_k(t), V_N(t)) \\ &\quad + \sum_{l=1}^{N} e^{-lt} M(U_N(t), v_l(t)) + M(U_N(t), V_N(t)). \end{aligned} \tag{2.73}$$

Since $u_k(t)$ and $v_l(t)$ are polynomials, their corresponding norms can not exceed exponential functions with any positive exponents as $t \to \infty$. Therefore finite sums of their norms can be bounded as

$$\sum_{k=1}^{N} \|u_k(t)\|_X = O(e^{\varepsilon t}), \quad \sum_{l=1}^{N} \|v_l(t)\|_Y = O(e^{\varepsilon t}), \tag{2.74}$$

hence

$$\sum_{k,l=1}^{N} \|u_k(t)\|_X \|v_l(t)\|_Y = O(e^{2\varepsilon t}) = O(e^{t/2}), \tag{2.75}$$

for $t \to \infty$. Combining (2.70),(2.72),(2.74) and (2.75), we derive

$$\|M(U_N(t), V_N(t))\|_Z \leq C \|U_N(t)\|_X \|V_N(t)\|_Y = O(e^{-2(N+2\varepsilon)t}),$$

$$\sum_{k=1}^{N} e^{-kt} \|M(u_k(t), V_N(t))\|_Z \leq C \sum_{k=1}^{N} \|u_k(t)\|_X \|V_N(t)\|_Y = O(e^{-(N+\varepsilon)t}),$$

and

$$\sum_{l=1}^{N} e^{-lt} \|M(U_N(t), v_l(t))\|_Z \leq C \sum_{l=1}^{N} \|U_N(t)\|_X \|v_l(t)\|_Y = O(e^{-(N+\varepsilon)t}),$$

for $t \to \infty$. Thus the last three terms on the right hand side of (2.73) are of $O(e^{-(N+\varepsilon)t})$

as $t \to \infty$. The remaining sum can be rewritten as

$$\sum_{k,l=1}^{N} e^{-(k+l)t} M(u_k(t), v_l(t)) = \sum_{j=1}^{N} M_j(t) e^{-jt} + R_N(t),$$

where

$$R_N(t) = \sum_{1 \leq k,l \leq n, k+l \geq N+1} e^{-(k+l)t} M(u_k(t), v_l(t)).$$

We bound $\|R_N(t)\|_Z$ by

$$\begin{aligned}\|R_N(t)\|_Z &\leq Ce^{-(N+1)t} \sum_{k,l=1}^{N} \|u_k(t)\|_X \|v_l(k)\|_Y = e^{-(N+1)t} O(e^{t/2}) \\ &= O(e^{-(N+1/2)t}) = O(e^{-(N+\varepsilon)t}),\end{aligned}$$

for $t \to \infty$. In summary, we have

$$M(u(t), v(t)) = \sum_{j=1}^{N} M_j(t) e^{-jt} + O(e^{-(N+\varepsilon_N)t}) \text{ in Z as } t \to \infty,$$

for all $N = 1, 2, 3, ...$ This proves the asymptotic expansion (2.68) of $M(u(t), v(t))$. $\square$

CHAPTER III

ANALYTIC DOMAINS AND PHRAGMEN-LINDERLÖF TYPE ESTIMATES

A. Analytic domains of regular solutions

We recall here a basic result on the analyticity of the complex valued solutions to the Navier–Stokes equations using the approach introduced in [10]. For our convenience, we define

$$F(\zeta_0, r_0) = \{\zeta_0 + se^{i\theta}; |\theta| < \pi/2 \ , \ 0 < s < \frac{\cos^3\theta}{4c_0 r_0^4}, \}, \qquad \text{for } \zeta_0 \in \mathbb{C}, r_0 \geq 0, \tag{3.1}$$

where $c_0 = 4C_1^4$, with C_1 being the constant introduced in Lemma II.2.

Theorem III.1. *Given $u_0 \in V_{\mathbb{C}}$ and $\zeta_0 \in \mathbb{C}$, there exists a unique $V_{\mathbb{C}}$-valued function $u(\zeta)$ defined in $\bar{F}(\zeta_0, \|u_0\|)$ satisfying the following properties*
i) $u(\zeta)$ is continuous from $\bar{F}(\zeta_0, \|u_0\|)$ to $V_{\mathbb{C}}$ and satisfies (2.39)*,*
ii) $u(\zeta)$ is analytic from $F(\zeta_0, \|u_0\|)$ to $\mathcal{D}_{A_{\mathbb{C}}}$ and satisfies (2.38)*.*

For the completeness we provide a sketch of the proof for Theorem III.1.

Proof of Theorem III.1. For $\zeta_0 \in \mathbb{C}$ and $u_0 \in V_{\mathbb{C}}$, let $u(\zeta)$ be the solution to a Galerkin approximation of (2.38) and (2.39). Fix $\theta \in (-\pi/2, \pi/2)$, from (2.38) we have the following *a priori* estimate along the ray $\{\zeta = \zeta_0 + se^{i\theta} \ ; \ s > 0\}$

$$\frac{d}{ds}\|u(\zeta_0 + se^{i\theta})\|^2 + \cos\theta |A_{\mathbb{C}} u(\zeta_0 + se^{i\theta})|^2 \leq \frac{c_0}{\cos^3\theta}\|u(\zeta_0 + se^{i\theta})\|^6, \tag{3.2}$$

where c_0 defined above is a positive constant independent of u_0, ζ_0, s, θ and of the dimension of the Galerkin approximation. Consequently,

$$\frac{d}{ds}\|u(\zeta_0 + se^{i\theta})\|^2 \leq \frac{c_0}{\cos^3\theta}\|u(\zeta_0 + se^{i\theta})\|^6. \tag{3.3}$$

Therefore,

$$\|u(\zeta_0 + se^{i\theta})\|^2 \leq \frac{\|u_0\|^2}{(1 - \frac{2c_0 s}{\cos^3\theta}\|u_0\|^4)^{1/2}} \leq \sqrt{2}\|u_0\|^2, \tag{3.4}$$

for $0 \leq s \leq \dfrac{\cos^3\theta}{4c_0\|u_0\|^4}$. We refer to [10] for further details. □

Remark III.1. The solution $u(\zeta)$ can be analytically extended beyond the boundary of $F(\zeta_0, \|u_0\|)$. However, with our limited choice of $F(\zeta_0, \|u_0\|)$ one also has

$$\|u(\zeta)\| \leq 2^{1/4}\|u_0\| \quad \text{for all } \zeta \in \bar{F}(\zeta_0, \|u_0\|). \tag{3.5}$$

For our purpose, we need to study the solutions' analytic domains in more details. The following proposition will give explicit descriptions of some regions contained in those domains. These regions turn out to be large enough for us to continue our analysis of the solutions and their helicity.

Proposition III.1. *For $\zeta_0 = 0$ and $u_0 \in V$ such that $\|u_0\|^2 < \dfrac{1}{2\sqrt{2c_0}}$, there exists a unique solution $u(\zeta)$ to* (2.38) *and* (2.39) *in the domain*

$$\mathcal{D} \stackrel{\text{def}}{=\!=} \bigcup_{t\geq 0}\{\zeta = \tau + i\sigma \ ; \ \tau > t, |\sigma| \leq \Gamma_t(\tau)\}, \tag{3.6}$$

where

$$\Gamma_t(\tau) = c(\tau - t)e^{\gamma(\tau+t)} \quad \textit{for} \quad \gamma = \frac{1}{8} \textit{ and } c = \frac{1}{4\sqrt{2}(2c_0)^{1/4}\|u_0\|}. \tag{3.7}$$

Moreover,

$$\|u(\zeta)\| \leq \|u_0\| \quad \textit{for all } \zeta \in \mathcal{D}. \tag{3.8}$$

Proof. For $t_0 \geq 0, v_0 \in V$, let $v(\zeta) = v(\zeta; t_0, v_0)$ be the analytic solution to the following problem

$$\begin{cases} \dfrac{dv(\zeta)}{d\zeta} + B_{\mathbb{C}}(v(\zeta), v(\zeta)) + A_{\mathbb{C}} v(\zeta) = 0, \\ v(t_0) = v_0, \end{cases} \tag{3.9}$$

Note that the existence and uniqueness of $v(\zeta)$ come from Theorem III.1. Fix $\theta \in (-\pi/2, \pi/2)$, from (3.9) we have the following *a priori* estimate along the ray $\{\zeta = t_0 + se^{i\theta}\ ;\ s > 0\}$

$$\frac{d}{ds}\|v(t_0 + se^{i\theta})\|^2 + \cos\theta |A_{\mathbb{C}} u(t_0 + se^{i\theta})|^2 \le \frac{c_0}{\cos^3\theta}\|v(t_0 + se^{i\theta})\|^6 \tag{3.10}$$

or

$$\frac{d}{ds}\|v(t_0 + se^{i\theta})\|^2 + \big(\cos\theta - \frac{c_0}{\cos^3\theta}\|v(t_0 + se^{i\theta})\|^4\big)\|v(t_0 + se^{i\theta})\|^2 \le 0. \tag{3.11}$$

If $\|v_0\|^2 < c_1 = 1/\sqrt{2c_0}$ and θ satisfies

$$\frac{c_0\|v_0\|^4}{\cos^3\theta} \le \frac{\cos\theta}{2}, \qquad \text{or equivalently, } \|v_0\|^2 \le c_1 \cos^2\theta, \tag{3.12}$$

then we have

$$\frac{d}{ds}\|v(t_0 + se^{i\theta})\|^2 \le 0 \qquad \text{for all } s > 0. \tag{3.13}$$

Thus $\|v(t_0 + se^{i\theta})\|$ is decreasing in s and

$$\|v(t_0 + se^{i\theta})\|^2 \le \|v_0\|^2 \le c_1 \cos^2\theta \qquad \text{for all } s > 0. \tag{3.14}$$

Therefore *a priori* estimate of $\|v(\zeta)\|$ is

$$\|v(\zeta)\| \le \|v_0\|, \quad \zeta \in \mathcal{D}(t_0, v_0) \stackrel{\text{def}}{=\!=} \{\zeta = t_0 + se^{i\theta} \in \mathbb{C}\ ;\ s > 0,\ \cos\theta \ge \frac{\|v_0\|}{\sqrt{c_1}}\}. \tag{3.15}$$

Then by Theorem III.1 the analytic domain of $v(\zeta)$ contains $\mathcal{D}(t_0, v_0)$. We now fix $u_0 \in \mathcal{R}$ and let $u(t)$ be the regular solution to the real Navier–Stokes equations

(2.12). For each $t \geq 0$, we have shown that $u(\cdot)$ can be extended to function $u(\zeta;t) = v(\zeta;t,u(t))$ which is analytic in ζ variable where ζ belongs to

$$D_t \overset{\text{def}}{=\!=} \mathcal{D}(t,u(t)) = \{\zeta = t + se^{i\theta}\ ;\ s > 0,\ \cos\theta \geq \frac{\|u(t)\|}{\sqrt{c_1}}\}. \tag{3.16}$$

By the uniqueness of analytic solutions, $u(\zeta,t,u(t)) = u(\zeta,t',u(t'))$ for any $t,t' > 0$ and $\zeta \in D_t \cap D_{t'}$. Therefore, $u(t), t > 0$ has an analytic extension $u(\zeta), \zeta \in \bigcup_{t\geq 0} D_t$. Set $t_0 = 0$, we know that

$$\|u(t)\|^2 < \|u_0\|^2 e^{-t/2} = Me^{-\Lambda t} \quad \text{for all } t \geq t_0, \tag{3.17}$$

where $\Lambda = 1/2$ and $M = \|u_0\|^2 < c_1$ and

$$Me^{-\Lambda t_0} \leq c_1/2 \quad \text{hence} \quad \|u(t)\|^2 < c_1 \quad \text{for all } t \geq t_0. \tag{3.18}$$

Therefore the analytic domain of $u(\zeta)$ contains

$$D = \bigcup_{t\geq t_0} D_t = \{\zeta = t + se^{i\theta}\ ;\ s > 0,\ t \geq t_0,\ \cos\theta \geq \frac{\|u(t)\|}{\sqrt{c_1}}\}. \tag{3.19}$$

Moreover, according to (3.14) and (3.18), for each $\zeta = t + se^{i\theta} \in D_t$, we have

$$\|u(\zeta)\| \leq \|u(t)\| < \sqrt{c_1} \quad \text{hence} \quad \|u(\zeta)\| < \sqrt{c_1} \quad \text{for all } \zeta \in D. \tag{3.20}$$

Because of the decay of the solution along the real axis as in (3.17), for each $t \geq t_0$, the set D_t defined in (3.16) above contains a subset

$$D'_t = \{\zeta = t + se^{i\theta}\ ;\ s > 0,\ \cos\theta \geq c_2 e^{-\Lambda t/2}\} \subset D_t, \quad \text{where } c_2 = \sqrt{M/c_1}. \tag{3.21}$$

Thus, $D \supset D' = \bigcup_{t\geq t_0} D'_t$. Fix $t \in \mathbb{R}, t \geq t_0$, given $\tau \in (t,\infty)$, consider $\zeta = \tau + i\sigma \in D'_{t'}$, where $t' = (t+\tau)/2$. We have $\zeta = t' + s'e^{i\theta'}$, where $s' = \sqrt{\sigma^2 + (\tau - t')^2} > 0$ and

$\cos\theta' \geq c_2 e^{-\Lambda t'/2}$. The last condition can be written in terms of τ and σ as

$$\left|\frac{\sigma}{\tau - t'}\right| = |\tan\theta'| = \sqrt{\frac{1}{\cos^2\theta'} - 1} \leq \sqrt{\frac{e^{\Lambda t'}}{c_2^2} - 1},$$

that is

$$|\sigma| \leq \frac{\tau - t}{2}\sqrt{\frac{c_1 e^{\Lambda(\tau+t)/2}}{M} - 1}. \tag{3.22}$$

On the other hand, from (3.18) we infer that for all $\tau \geq t \geq t_0$, we have

$$\frac{c_1 e^{\Lambda(\tau+t)/2}}{M} \geq \frac{c_1 e^{\Lambda t}}{M} \geq \frac{c_1 e^{\Lambda t_0}}{M} \geq 2,$$

then

$$\frac{\tau - t}{2}\sqrt{\frac{c_1 e^{\Lambda(\tau+t)/2}}{M} - 1} \geq \frac{\tau - t}{2}\sqrt{\frac{c_1 e^{\Lambda(\tau+t)/2}}{2M}} = c_3(\tau - t)e^{\gamma(\tau+t)}, \tag{3.23}$$

where

$$\gamma = \Lambda/4 = 1/8 \quad \text{and} \quad c_3 = \frac{\sqrt{c_1}}{2\sqrt{2M}} = \frac{1}{4\sqrt{2}(2c_0)^{1/4}\|u_0\|}. \tag{3.24}$$

For each $t \geq t_0$, let us denote

$$\Gamma_t(\tau) \stackrel{\text{def}}{=\!=} c_3(\tau - t)e^{\gamma(\tau+t)}, \qquad \text{for } \tau > t. \tag{3.25}$$

Then from (3.22) and (3.23), we infer that the sets D_t and D_t' contain

$$\mathcal{D}_t \stackrel{\text{def}}{=\!=} \{\zeta = \tau + i\sigma\ ;\ \tau > t, |\sigma| \leq \Gamma_t(\tau)\}. \tag{3.26}$$

Therefore

$$\mathcal{D} \stackrel{\text{def}}{=\!=} \bigcup_{t\geq t_0} \mathcal{D}_t = \bigcup_{t\geq t_0} \{\zeta = \tau + i\sigma\ ;\ \tau > t,\ |\sigma| \leq \Gamma_t(\tau)\} \subset D' \subset D. \tag{3.27}$$

Moreover, by (3.20), $\|u(\zeta)\|$ is bounded by $\sqrt{c_1}$ in D, hence also in $\mathcal{D}$. □

A small change in the above proof will give us another description of the domain which is connected to the normalization map.

Proposition III.2. *For $\zeta_0 = 0$ and $u_0 \in \mathcal{R}$, let n_0 be defined in* (2.17) *and $u(\zeta)$ be the unique solution to* (2.38) *and* (2.39)*. Then the analytic domain of $u(\zeta)$ contains*

$$[\bigcup_{0 \le t \le t_0} F(t, r_0)] \cup \mathcal{D}, \tag{3.28}$$

for some $t_0 \ge 0$, $r_0 = max\{\|u(t)\| \ ; \ 0 \le t \le t_0\}$ and

$$\mathcal{D} = \bigcup_{t \ge t_0} \{\zeta = \tau + i\sigma \ ; \ \tau > t, |\sigma| \le \Gamma_t(\tau)\}, \tag{3.29}$$

where $\Gamma_t(\tau) = c(\tau - t)e^{\gamma(\tau+t)}$ for

$$\gamma = \frac{n_0}{2} \quad \text{and} \quad c = \frac{1}{4(2c_0)^{1/4}\|W_{n_0}(u_0)\|}. \tag{3.30}$$

Moreover,

$$\|u(\zeta)\| \le (2c_0)^{-1/4} \quad \text{for all } \zeta \in \mathcal{D}. \tag{3.31}$$

Proof. Instead of having $t_0 = 0$ together with (3.17) and (3.18), we use (2.17) and (2.23) to infer that with $\Lambda = 2n_0$ and $M = 2\|W_{n_0}(u_0)\|^2$, there is some $t_0 \ge 0$ such that

$$Me^{-\Lambda t_0} \le c_1/2 \quad \text{and} \quad \|u(t)\|^2 < Me^{-\Lambda t} \quad \text{for all } t \ge t_0. \tag{3.32}$$

With the new values of Λ and M, (3.24) now becomes

$$\gamma = \Lambda/4 = n_0/2 \quad \text{and} \quad c_3 = \frac{\sqrt{c_1}}{2\sqrt{2M}} = \frac{1}{4(2c_0)^{1/4}\|W_{n_0}(u_0)\|}. \tag{3.33}$$

Combining with Theorem III.1, we have that the analytic domain of $u(\zeta)$ contains the subset defined by (3.28). □

Remark III.2. In fact, by a real time translating we can consider only solutions for which the time t_0 in Proposition III.2 is equal to zero. Therefore, the set $\mathcal{D}$ in

Proposition III.1 is defined by

$$\mathcal{D} = \bigcup_{t \geq 0} \{\zeta = \tau + i\sigma \ ; \ \tau > t, |\sigma| \leq \Gamma_t(\tau)\}. \tag{3.34}$$

Indeed, let $\zeta = \tilde{\zeta} + t_0$, $\mathbf{Re}\tilde{\zeta} \geq 0$ and $t = \tilde{t} + t_0$, $\tilde{t} \geq 0$. Let $\tilde{u}(\tilde{\zeta}) = u(\zeta)$. Then $\tilde{u}(\tilde{\zeta})$ is analytic in the domain

$$\tilde{\mathcal{D}} = \bigcup_{\tilde{t} \geq 0} \{\tilde{\zeta} = \tilde{\tau} + i\sigma \ ; \ \tilde{\tau} > \tilde{t}, |\sigma| \leq c_3(\tilde{\tau} - \tilde{t}) e^{\gamma(\tilde{\tau} + \tilde{t} + 2t_0)}\}.$$

Hence we can define

$$\tilde{\Gamma}_{\tilde{t}}(\tilde{\tau}) = c_3 e^{2\gamma t_0} (\tilde{\tau} - \tilde{t}) e^{\gamma(\tilde{\tau} + \tilde{t})}.$$

We next derive estimates for analytic functions in domains like $\mathcal{D}$.

B. Some Phragmen-Linderlöf type estimates

First we recall the following direct consequence of problem 325, page 168 in [24].

Theorem III.2 ([24], p. 168). *Let $f(\zeta)$ be analytic on the right half plane $H_0 = \{\zeta \in \mathbb{C} : \mathbf{Re}\zeta > 0\}$, bounded by a constant M and*

$$\sup_{x > 0} e^{\alpha x} |f(x)| < \infty, \tag{3.35}$$

where α is a positive number. Then

$$|f(\zeta)| \leq M e^{-\alpha \mathbf{Re}\zeta}, \ \zeta \in H_0. \tag{3.36}$$

We will establish a version of this theorem for some special domains arising in the study of the analytic solutions of the Navier–Stokes equations. By virtue of Propositions III.1 and III.2, we consider the following domains

$$D(c, \alpha) = \{\tau + i\sigma : \tau > 0, |\sigma| < c\tau e^{\alpha\tau}\}, \tag{3.37}$$

where c and α are positive numbers. To take the advantage of Theorem III.2, we need to convert $D(c, \alpha)$ to the right half plane. The following analytic transformation is what we need

$$\varphi_\alpha(\zeta) = \zeta - \frac{1}{\alpha} \log(1 + \alpha\zeta). \tag{3.38}$$

Lemma III.1. *Let $c \geq \sqrt{2}, \alpha > 0$, then function $\varphi_\alpha(\zeta)$ conformally maps $D(c, \alpha)$ to a set containing the right half plane $H_0 = \{\zeta \in \mathbb{C} : \mathbf{Re}\zeta > 0\}$. Moreover, $\varphi_\alpha([0, \infty)) = [0, \infty)$.*

Proof. For $\zeta \neq 0$ we have that

$$\varphi'_\alpha(\zeta) = 1 - \frac{1}{1 + \alpha\zeta} \neq 0.$$

It follows that φ_α is one-to-one on D. Moreover, if $\zeta \in \partial D(c, \alpha)$ then

$$\mathbf{Re}\varphi_\alpha(\zeta) = \tau - \frac{1}{2\alpha} \log\left((1 + \alpha\tau)^2 + (\alpha\sigma)^2\right) \leq 0$$

if and only if

$$e^{2\alpha\tau} \leq 1 + 2\alpha\tau + \alpha^2\tau^2 + \alpha^2 c^2 \tau^2 e^{2\alpha\tau}. \tag{3.39}$$

Since $c \geq \sqrt{2}$, then (3.39) is implied by

$$e^{2\alpha\tau} < 1 + 2\alpha\tau + 2\alpha^2\tau^2 e^{2\alpha\tau}, \quad \tau > 0.$$

It follows that $\varphi_\alpha^{-1}(H_0) \subset D(c, \alpha)$. For real τ we have

$$\varphi_\alpha(\tau) = \tau - \alpha^{-1} \log(1 + \alpha\tau).$$

Therefore $\varphi_\alpha(0) = 0$ and $\varphi_\alpha\big([0, \infty)\big) = [0, \infty)$. □

The following is a version of Theorem III.2 for functions in the domains $D(c, \alpha)$.

Corollary III.1. *Suppose $u(\zeta)$ is analytic in $D(c, \alpha)$ where $c \geq \sqrt{2}, \alpha > 0$, and* *satisfies*

$$|u(\zeta)| \leq M, \quad \zeta \in D(c,\alpha) \tag{3.40}$$

and

$$\sup_{t>0} e^{nt}|u(t)| < \infty, \tag{3.41}$$

where n is a positive constant. Then

$$|u(\zeta)| \leq Me^{-n\mathbf{Re}\zeta}|1+\alpha\zeta|^{n/\alpha}, \quad \zeta \in \varphi_\alpha^{-1}(H_0). \tag{3.42}$$

Proof. Let $C = \sup_{t>0} e^{nt}|u(t)|$ and $v(\eta) = u(\varphi_\alpha^{-1}(\eta)), \eta \in H_0$, then

$$|v(\eta)| \leq M$$

and

$$|v(\varphi_\alpha(\tau))| = |u(\tau)| \leq Ce^{-n\tau} \leq Ce^{-n(\tau - \frac{1}{\alpha}\log(1+\alpha\tau))} = Ce^{-n\varphi_\alpha(\tau)}, \quad \tau > 0.$$

Then

$$|v(\tau)| \leq Ce^{-n\tau}, \quad \tau > 0.$$

By Theorem III.2, we have

$$|v(\eta)| \leq Me^{-n\mathbf{Re}\eta}, \quad \eta \in H_0.$$

Hence for $\zeta \in \varphi_\alpha^{-1}(H_0)$,

$$\begin{aligned} |u(\zeta)| = |v(\varphi_\alpha(\zeta))| &\leq Me^{-n\mathbf{Re}(\zeta - \frac{1}{\alpha}\log(1+\alpha\zeta))} \\ &= Me^{-n\mathbf{Re}\zeta}e^{\frac{n}{\alpha}\log|1+\alpha\zeta|} = Me^{-n\mathbf{Re}\zeta}|1+\alpha\zeta|^{n/\alpha}. \end{aligned}$$

□

Corollary III.2. *Suppose $u(\zeta)$ is analytic and bounded in $D(c,\alpha)$ where $c \geq \sqrt{2}, \alpha >$*

0, *satisfying*

$$\lim_{t\to\infty} e^{nt}|u(t)| = 0, \tag{3.43}$$

for all $n > 0$. *Then* $u(\zeta)$ *is identically zero in* $D(c, \alpha)$.

Proof. For each $n > 0$, we have $\sup_{t>0} e^{nt}|u(t)| < \infty$. Apply Corollary III.1 noting that $\varphi_\alpha([0,\infty)) = [0,\infty)$, we imply

$$|u(t)| \le Me^{-nt}|1+\alpha t|^{n/\alpha}, \quad t > 0.$$

For each $t > 0$, letting $n \to \infty$ gives $u(t) = 0$ for all $t > 0$. Since $D(c, \alpha)$ is connected and contains $(0, \infty)$, it follows that $u(\zeta)$ is identically in $D(c, \alpha)$. □

We next describe subdomains of $\varphi_\alpha^{-1}(H_0)$ consisting of disks with certain specified centers and radii.

Lemma III.2. *Given* $c \ge \sqrt{2}, \alpha > 0$, *let*

$$T^\star = \frac{1+\sqrt{2}}{\alpha} \quad \textit{and} \quad g(\tau) = \frac{\sqrt{3}e^{\alpha\tau}}{2\alpha}.$$

For each $\tau > T^\star$, *let* $t(\tau) = \tau + \alpha g(\tau)^2$ *and* $r_t(\tau) = g(\tau)\sqrt{1+\alpha^2 g(\tau)^2}$. *Then*

$$1 \le \frac{t(\tau)}{r_t(\tau)} \le \frac{\tau}{\alpha g(\tau)^2} + 1, \quad t(\tau) - r_t(\tau) \ge \frac{\tau}{1+\sqrt{2}},$$

and the disk $B(t(\tau), r_t(\tau)) = \{\zeta \in \mathbb{C} \ : \ |\zeta - t(\tau)| \le r_t(\tau)\}$ *is a subset of* $\varphi_\alpha^{-1}(H_0) \subset D(c, \alpha)$.

Proof. We know that $\varphi_\alpha(\zeta)$ is a conformal map from $D(c, \alpha)$ to a domain containing the right half plane H_0 and $\varphi_\alpha\big([0,\infty)\big) = [0,\infty)$. For $\tau > 0$, and $\zeta = \tau + i\sigma$, $\sigma \in \mathbb{R}$, the condition that $\varphi_\alpha(\zeta) \in H_0$ is

$$\tau - \frac{1}{2\alpha}\log\left[(1+\alpha\tau)^2 + (\alpha\sigma)^2\right] > 0,$$

or

$$|\sigma| < \frac{\sqrt{e^{2\alpha\tau} - (1+\alpha\tau)^2}}{\alpha}.$$

Take $\tau^\star > 0$ that $e^{2\tau\alpha} - (1+\alpha\tau)^2 \geq (3/4)e^{2\tau\alpha}$ for all $\tau \geq \tau^\star$. Equivalently, we require that

$$e^{\alpha\tau} \geq 2(1+\alpha\tau) \quad \text{for all} \quad \tau \geq \tau^\star. \tag{3.44}$$

We obtain

$$U \overset{\text{def}}{=\!=} \left\{\tau + i\sigma \in D(c,\alpha) : \tau > \tau^\star, |\sigma| < g(\tau)\right\} \subset \varphi_\alpha^{-1}(H_0). \tag{3.45}$$

Since $e^y \geq 1 + y + y^2/2$ for $y > 0$, then the fact that

$$\frac{y^2}{2} \geq 1 + y \iff (y-1)^2 \geq 3$$

implies that (3.44) holds whenever $1 + \alpha\tau^\star \geq 2$, or equivalently when $\tau^\star \geq 1/\alpha$. We may further assume that $g(\tau) \leq c\tau e^{\alpha\tau}$ for all $\tau > \tau^\star$. This is guaranteed when $\tau^\star \geq \sqrt{3}/(2\alpha c)$. Note that $\sqrt{3}/(2\alpha c) < 1/\alpha$, hence we choose $\tau^\star = 1/\alpha$.

Now, for each $t > \tau^\star$ we find the maximum radius r_t such that the disk $B(t, r_t)$ centered at t with radius r_t is in U. Given $\tau > \tau^\star$, the line orthogonal to the graph of $g(\tau)$ at $(\tau, g(\tau))$ is

$$Y = -\frac{1}{g'(\tau)}(X - \tau) + g(\tau).$$

Note that $g'(\tau) = \alpha g(\tau)$. Therefore, this line intersects the real axis at

$$t(\tau) = \tau + g'(\tau)g(\tau) = \tau + \alpha g(\tau)^2$$

and

$$r_t(\tau) = \sqrt{(g(\tau)g'(\tau))^2 + g(\tau)^2} = g(\tau)\sqrt{1 + \alpha^2 g(\tau)^2}.$$

Taking limits we obtain $t(\tau)/r_t(\tau) \to 1$ as $\tau \to \infty$. More precisely, for $\tau > \tau^\star$,

$$1 \leq \sqrt{\frac{1}{\alpha^2 g(\tau)^2} + 1} \leq \frac{t(\tau)}{r_t(\tau)} \leq \frac{\tau}{\alpha g(\tau)^2} + 1 = \frac{4\alpha\tau}{3e^{2\alpha\tau}} + 1. \tag{3.46}$$

Also,

$$\begin{aligned} t(\tau) - r_t(\tau) &= \frac{t^2(\tau) - r_t^2(\tau)}{t(\tau) + r_t(\tau)} = \frac{\tau^2 + g'(\tau)^2 g(\tau)^2 + 2g'(\tau)g(\tau) - g'(\tau)^2 g(\tau)^2 - g(\tau)^2}{t(\tau) + r_t(\tau)} \\ &= \frac{\tau^2 + (2\alpha\tau - 1)g(\tau)^2}{\tau + \alpha g(\tau)^2 + g(\tau)\sqrt{1 + \alpha^2 g(\tau)^2}}. \end{aligned}$$

It follows that

$$\lim_{\tau \to \infty} \frac{t(\tau) - r_t(\tau)}{\tau} = \frac{2\alpha}{2\alpha} = 1.$$

If $\alpha\tau > \log(4/3)$, then $1 \leq \alpha^2 g(\tau)^2$ and moreover

$$t(\tau) - r_t(\tau) \geq \frac{\tau^2 + \alpha\tau g^2(\tau)}{\tau + (1 + \sqrt{2})\alpha g^2(\tau)} \geq \frac{\tau}{1 + \sqrt{2}}. \tag{3.47}$$

With $T^\star = (1+\sqrt{2})\tau^\star$, when $\tau > T^\star$, we have $\tau > 1/\alpha$ and $\tau/(1+\sqrt{2}) > \tau^\star$, therefore $B(t(\tau), r_t(\tau)) \subset U \subset \varphi_\alpha^{-1}(H_0)$. □

We now give an explicit estimate of a polynomial based on its growth rate in the domain $D(c, \alpha)$.

Lemma III.3. *Let $a > r > 0$ and $B(a, r)$ be the disk $\{\zeta \in \mathbb{C} : |\zeta - a| < r\}$. Suppose $q(\zeta)$ is a polynomial of degree less than or equal to p and*

$$|q(\zeta)| \leq M|1 + \alpha\zeta|^N, \quad \zeta \in \bar{B}(a, r), \tag{3.48}$$

where M and N are positive numbers. Then

$$|q(\zeta)| \leq M(p + 1)(1 + \alpha a + \alpha r)^N (\frac{|\zeta| + a}{r})^p, \quad \zeta \in \mathbb{C}. \tag{3.49}$$

Proof. Write $q(\zeta) = d_0 + d_1(\zeta - a) + \cdots + d_p(\zeta - a)^p$. Estimate the coefficients using the Cauchy formula

$$\begin{aligned}|d_j| &= \Big|\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-a)^{j+1}}d\zeta\Big|\\ &\le \frac{1}{2\pi}\int_0^{2\pi}\frac{|q(a+re^{i\theta})|}{r^{j+1}}rd\theta\\ &\le \frac{M}{2\pi r^j}\int_0^{2\pi}|1+\alpha(a+re^{i\theta})|^N d\theta\\ &\le \frac{M(1+\alpha(a+r))^N}{r^j}.\end{aligned}$$

This implies

$$|q(\zeta)| \le M(1+\alpha a+\alpha r)^N\sum_{j=0}^{p}\frac{|\zeta-a|^j}{r^j}, \quad \zeta\in\mathbb{C}.$$

Since $a/r > 1$ we obtain

$$\begin{aligned}|q(\zeta)| &\le M(1+\alpha a+\alpha r)^N\sum_{j=0}^{p}\Big(\frac{|\zeta|+a}{r}\Big)^p\\ &\le M(p+1)(1+\alpha a+\alpha r)^N\Big(\frac{|\zeta|+a}{r}\Big)^p.\end{aligned}$$

□

Combining Corollary III.1 and Lemma III.3, we obtain

Proposition III.3. *Given $c \ge \sqrt{2}, \alpha > 0$. Let $a > r > 0$ such that the disk $B(a,r)$ is a subset of $\varphi_\alpha^{-1}(H_0)$. Suppose $q(\zeta)$ is a polynomial of degree less than or equal to p and*

$$|e^{-N\zeta}q(\zeta)| \le M, \quad \zeta\in D(c,\alpha). \tag{3.50}$$

Then

$$|q(\zeta)| \le M|1+\alpha\zeta|^{N/\alpha}, \quad \zeta\in\varphi_\alpha^{-1}(H_0) \tag{3.51}$$

and

$$|q(\zeta)| \le M(p+1)(1+\alpha a+\alpha r)^{N/\alpha}\Big(\frac{|\zeta|+a}{r}\Big)^p, \quad \zeta\in\mathbb{C}. \tag{3.52}$$

In particular, when $p = N$,

$$|q(\zeta)| \leq M(N+1)C(\alpha,a,r)^N(|\zeta|+a)^N, \quad \zeta \in \mathbb{C}, \tag{3.53}$$

where

$$C(\alpha,a,r) = r^{-1}(1+\alpha a+\alpha r)^{1/\alpha}. \tag{3.54}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Apply Corollary III.1 to $u(\zeta) = e^{-N\zeta}q(\zeta)$ using the fact that $|u(\zeta)| \leq M$ and $|u(t)| = O(e^{-(N-\varepsilon)t})$ as $t \to \infty$. Hence,

$$|u(\zeta)| \leq Me^{-N\mathbf{Re}\zeta}|1+\alpha\zeta|^{(N-\varepsilon)/\alpha}, \quad \zeta \in \varphi_\alpha^{-1}(H_0).$$

Taking limits as $\varepsilon \to 0$ gives

$$|q(\zeta)| \leq M|1+\alpha\zeta|^{N/\alpha}, \quad \zeta \in \varphi_\alpha^{-1}(H_0).$$

Apply Lemma III.3 to finish the proof. □

Definition III.1. Let $\alpha_* = 1/2$. By virtue of Lemma III.2, we now fix $a_* > 0$ and $r_* \in (0, a_*)$ such that the disk $B(a_*, r_*)$ is contained in $\varphi_{\alpha_*}^{-1}(H_0) \subset D(\sqrt{2}, \alpha_*)$. We define the constants

$$C_2 = C(\alpha_*, a_*, r_*) = r_*^{-1}(1+\alpha_* a_* + \alpha_* r_*)^{1/\alpha_*} \quad \text{and} \quad C_3 = C_2/2. \tag{3.55}$$

CHAPTER IV

EXTENDED NAVIER–STOKES EQUATIONS

Motivated by the asymptotic expansion (2.18) of the regular solutions of the Navier–Stokes equations, we introduce a specific construction of the solutions. Using this construction, we rediscover the classical existence results for those solutions. Note that we will obtain a slightly different estimate for the time interval of the local solution's existence (see Remark IV.2). More importantly, this construction is useful for our further study of the asymptotic expansions of the solutions as well as the normal form of the Navier–Stokes equations in the next chapters. We also introduce a weighted normed space in which the constructed solutions $(u_n(t))_{n\in\mathbb{N}}$ exist for all times $t > 0$.

A. A new construction of regular solutions

Suppose we split the initial data u^0 in V as

$$u^0 = \sum_{n=1}^{\infty} u_n^0.$$

We try to find the solution $u(t)$ of the form

$$u(t) = \sum_{n=1}^{\infty} u_n(t),$$

where for each n

$$\begin{cases} \dfrac{du_n(t)}{dt} + Au_n(t) + B_n(t) = 0, \quad t > 0, \\ u_n(0) = u_n^0, \end{cases} \tag{4.1}$$

and where

$$B_1(t) = 0 \quad \text{and} \quad B_n(t) = \sum_{j+k=n} B\big(u_j(t), u_k(t)\big) \quad \text{for} \quad n > 1. \tag{4.2}$$

We extend equation (4.1) to the one with complexified times. Namely, given $\zeta_0 \in \mathbb{C}$ and $u^\star = \sum_{n=1}^\infty u_n^\star$ in V, consider the sum $u(\zeta) = \sum_{n=1}^\infty u_n(\zeta)$ with $\zeta \in \mathbb{C}$ such that

$$\begin{cases} \dfrac{du_n(\zeta)}{d\zeta} + Au_n(\zeta) + B_n(\zeta) = 0, \quad \zeta \in \mathbb{C}, \\ u_n(\zeta_0) = u_n^\star, \end{cases} \tag{4.3}$$

where

$$B_1(\zeta) = 0 \quad \text{and} \quad B_n(\zeta) = \sum_{j+k=n} B\big(u_j(\zeta), u_k(\zeta)\big) \quad \text{for} \quad n > 1.$$

Remark IV.1. Given $\rho > 0$, denote $v_n(\zeta) = \rho^n u_n(\zeta)$ for $n \in \mathbb{N}$ and $\zeta \in \mathbb{C}$. Then

$$\begin{cases} \dfrac{dv_n(\zeta)}{d\zeta} + Av_n(\zeta) + \tilde{B}_n(\zeta) = 0, \quad \zeta \in \mathbb{C}, \\ v_n(\zeta_0) = v_n^\star = \rho^n u_n^\star, \end{cases} \tag{4.4}$$

where

$$\tilde{B}_1(\zeta) = 0 \quad \text{and} \quad \tilde{B}_n(\zeta) = \sum_{j+k=n} B\big(v_j(\zeta), v_k(\zeta)\big) \quad \text{for} \quad n > 1. \tag{4.5}$$

Let $M \geq 1$, we solve the Galerkin approximation problem of (4.3)

$$\begin{cases} \dfrac{du_n^{(M)}(\zeta)}{d\zeta} + Au_n^{(M)}(\zeta) + \displaystyle\sum_{k+j=n} P_M B(u_k^{(M)}(\zeta), u_j^{(M)}(\zeta)) = 0, \quad \zeta \in \mathbb{C}, \\ u_n^{(M)}(\zeta_0) = P_M u_n^\star, \end{cases} \tag{4.6}$$

where $u_n^{(M)} \in P_M H$ for $n = 1, 2, 3, \ldots$ For each M, this is a system of differential equations in a Euclidean space. Therefore, it has a unique analytic solution locally in the complex plane, i.e., in an open neighborhood of ζ_0 (see [13]). As far as we can prove that the solution $u_n^{(M)}$ is *a priori* bounded in a domain, that solution actually exists in the whole domain also. Moreover, when we have a uniform bound for $|Au_n^{(M)}|$ independent of M in some domain, by extracting a convergent subsequence we can find the analytic solution $u_n(\zeta)$ to (4.3) (see, for example, [10] for details of this standard procedure). Therefore we prove below some *a priori* bounds for solutions to the Galerkin approximation problem.

B. A priori bounds

In this section we first obtain some *a priori* bounds for the Galerkin solutions $\|u_n^{(M)}(\zeta)\|$, $M > 0$, in terms of the norms $\|u_j^\star\|, 1 \le j \le n$. For our convenience in the next statements, we define the function

$$\xi(s,\theta) = \frac{C_1}{\cos\theta}(1 - e^{-s\cos\theta})^{1/4}, \quad s \ge 0, |\theta| < \pi/2, \tag{4.7}$$

where C_1 is the positive constant introduced in Lemma (II.2),

Proposition IV.1. *Let $(u_n^\star)_{n\ge 1}$ be a sequence in V_C. Given $\zeta_0 \in \mathbb{C}$ and $\theta \in (-\pi/2, \pi/2)$. Let $(u_n)_{n=1}^\infty$ be the solutions of the Galerkin approximation problem* (4.6) *for some $M > 0$. Then*

$$\|u_n(\zeta_0 + se^{i\theta})\| \le e^{-s\cos\theta}\gamma_n(s), \quad s > 0, n \in \mathbb{N}, \tag{4.8}$$

and when $u_n(\zeta_0 + se^{i\theta})$ is not identically zero for $s > 0$,

$$\int_0^s \frac{|Au_n(\zeta_0 + \rho e^{i\theta})|^2}{\|u_n(\zeta_0 + \rho e^{i\theta})\|} d\rho \le \frac{\gamma_n(s)}{\cos\theta}, \quad s > 0, n \in \mathbb{N}, \tag{4.9}$$

where

$$\begin{cases} \gamma_1(s) = \|u_1^\star\|, \\ \gamma_n(s) = \|u_n^\star\| + \xi(s) \sum_{k+j=n} \gamma_k(s)\gamma_j(s), \quad n > 1, s > 0. \end{cases} \tag{4.10}$$

Proof. Write $v_n(s) = u_n(\zeta_0 + se^{i\theta})$ and $\xi(s) = \xi(s,\theta)$ for $s > 0$. For *a priori* bounds, we assume that $v_n(s)$ is analytic on $[0,\infty)$. Note that since $\xi(s)$ is an increasing function of s, so is $\gamma_n(s)$. For $N = 1, 2, 3, \ldots$, the equation for dv_N/ds is given by

$$\frac{dv_N(s)}{ds} + e^{i\theta}Av_N(s) = -e^{i\theta}P_M B_N(s),$$

where $B_1(s) \equiv 0$ and $B_N(s) = \sum_{j+k=N} B(v_j(s), v_k(s))$, $N > 1$. Since

$$\frac{d}{ds}\|v_N\|^2 = \Big\langle \frac{dv_N}{ds}, Av_N \Big\rangle + \Big\langle Av_N, \frac{dv_N}{ds} \Big\rangle = 2\mathbf{Re}\big\{\langle dv_N/ds, Av_N\rangle\big\},$$

it follows that

$$\frac{1}{2}\frac{d}{ds}\|v_N\|^2 + \cos\theta|Av_N|^2 \le \Big|\big\langle B_N, Av_N\big\rangle\Big|. \tag{4.11}$$

Given $\varepsilon > 0$, let

$$v_{n,\varepsilon}(s) = (\|v_n(s)\|^2 + \varepsilon)^{1/2},$$

$$\mathcal{S}_{n,\varepsilon}(s) = \cos\theta \int_0^s \frac{\|v_n(\rho)\|^2}{v_{n,\varepsilon}^2(\rho)} d\rho.$$

and

$$\mathcal{G}_n(s) = \begin{cases} \frac{|Av_n(s)|^2}{\|v_n(s)\|}, & \text{when } v_n(s) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

When $N = 1$, we have $B_1 = 0$, hence from (4.11)

$$\|v_1(s)\|^2 \le e^{-2s\cos\theta}\|P_M u_1^\star\|^2 \le e^{-2s\cos\theta}\|u_1^\star\|^2.$$

Moreover,

$$\frac{d}{ds}v_{1,\varepsilon}(s) + \cos\theta\frac{|Av_1(s)|^2}{v_{1,\varepsilon}(s)} \le 0,$$

thus

$$\cos\theta \int_0^s \frac{|Av_N(\rho)|^2}{v_{1,\varepsilon}(\rho)} d\rho \le v_{1,\varepsilon}(0).$$

In the case $v_1(s)$ is not identically zero for all $s > 0$, its analyticity implies it has at most countably many zeros, hence

$$\frac{|Av_1(s)|^2}{v_{1,\varepsilon}} \nearrow \frac{|Av_1(s)|^2}{\|v_1(s)\|} \text{ when } \varepsilon \searrow 0 \text{ for almost every } s \in (0,\infty).$$

Thus letting $\varepsilon \to 0$ gives

$$\cos\theta \int_0^s \frac{|Av_1(\rho)|^2}{\|v_1(\rho)\|} d\rho \le \|P_M u_1^\star\| \le \|u_1^\star\|, \text{ or } \int_0^s \mathcal{G}_1(\rho) d\rho \le \frac{\gamma_1}{\cos\theta}.$$

For induction, let $N > 1$ and suppose

$$\|v_n(s)\| \le e^{-s\cos\theta}\gamma_n(s) \qquad \text{and} \qquad \int_0^s \mathcal{G}_n(\rho) d\rho \le \frac{\gamma_n(s)}{\cos\theta} \tag{4.12}$$

hold for $n < N$. By (4.13) and inequality (2.55) in Lemma II.2,

$$\frac{d}{ds}\|v_N\|^2 + \cos\theta |Av_N|^2 \le C_1 \sum_{j+k=N} \|v_j\|^{1/2}|Av_j|^{1/2}|Av_k|\|v_N\| \tag{4.13}$$

We derive

$$\begin{aligned} \frac{d}{ds}v_{N,\varepsilon} + \cos\theta \frac{|Av_N|^2}{v_{N,\varepsilon}^2} v_{N,\varepsilon} &\le C_1 \sum_{j+k=N} \|v_j\|^{1/2}|Av_j|^{1/2}|Av_k| \\ &= C_1 \sum_{j+k=N} \mathcal{G}_j^{1/4}\|v_j\|^{3/4}\mathcal{G}_k^{1/2}\|v_k\|^{1/2} \\ &= C_1 \sum_{j+k=N} \Big\{\mathcal{G}_j^{1/4}\|v_k\|^{1/4}\Big\}\Big\{\mathcal{G}_k^{1/2}\|v_j\|^{1/2}\Big\}\Big\{\|v_k\|^{1/4}\|v_j\|^{1/4}\Big\} \\ &\le C_1 \Big\{\sum_{j+k=N} \mathcal{G}_j\|v_k\|\Big\}^{3/4}\Big\{\sum_{j+k=N} \|v_k\|\|v_j\|\Big\}^{1/4}. \end{aligned}$$

Since $k, j < N$, then we may apply the induction hypothesis (4.12) to the sum to obtain

$$\begin{aligned} &\frac{d}{ds}v_{N,\varepsilon} + \cos\theta \frac{|Av_n|^2}{v_{n,\varepsilon}^2} v_{N,\varepsilon} \\ &\qquad \le C_1 e^{-(5/4)s\cos\theta}\Big\{\sum_{j+k=N} \mathcal{G}_j\gamma_k\Big\}^{3/4}\Big\{\sum_{j+k=N} \gamma_j\gamma_k\Big\}^{1/4}. \end{aligned} \tag{4.14}$$

Applying the Poincaré inequality yields

$$v_{N,\varepsilon}(s) \leq e^{-\mathcal{S}_{N,\varepsilon}(s)}\Big\{ v_{N,\varepsilon}(0) + C_1\Big\{\sum_{j+k=N}\gamma_j(s)\gamma_k(s)\Big\}^{1/4}\int_0^s e^{\mathcal{S}_{N,\varepsilon}(\rho)-(5/4)\rho\cos\theta}\Big\{\sum_{j+k=N}\mathcal{G}_j(\rho)\gamma_k(s)\Big\}^{3/4}d\rho\Big\}.$$

Now applying Hölder's inequality leads to

$$\begin{aligned}\int_0^s & e^{\mathcal{S}_{N,\varepsilon}(\rho)-(5/4)\rho\cos\theta}\Big\{\sum_{j+k=N}\mathcal{G}_j(\rho)\gamma_k(s)\Big\}^{3/4}d\rho \\ &\leq \Big\{\int_0^s e^{4\mathcal{S}_{N,\varepsilon}(\rho)-5\rho\cos\theta}d\rho\Big\}^{1/4}\Big\{\sum_{j+k=N}\gamma_k(s)\int_0^s\mathcal{G}_j(\rho)d\rho\Big\}^{3/4} \\ &\leq \Big\{\int_0^s e^{4\mathcal{S}_{N,\varepsilon}(\rho)-5\rho\cos\theta}d\rho\Big\}^{1/4}\Big\{\frac{1}{\cos\theta}\sum_{j+k=N}\gamma_k(s)\gamma_j(s)\Big\}^{3/4}.\end{aligned}$$

It follows that

$$v_{N,\varepsilon}(s) \leq e^{-\mathcal{S}_{N,\varepsilon}(s)}\Big\{v_{N,\varepsilon}(0) + C_1\Big\{\int_0^s e^{4\mathcal{S}_{N,\varepsilon}(\rho)-5\rho\cos\theta}d\rho\Big\}^{1/4}(\cos\theta)^{-3/4}\sum_{j+k=N}\gamma_k(s)\gamma_j(s)\Big\}.$$

Since $v_n(s)$ is assumed to be analytic in $[0,\infty)$, it has at most countably many zeros. Thus

$$\lim_{\varepsilon\to 0}\mathcal{S}_{N,\varepsilon}(s) = s\cos\theta, \quad s>0,$$

and hence by letting $\varepsilon\to 0$, we obtain

$$\begin{aligned}\|v_N(s)\| &\leq e^{-s\cos\theta}\Big\{\|v_N(0)\| + C_1\Big\{\int_0^s e^{-\rho\cos\theta}d\rho\Big\}^{1/4}(\cos\theta)^{-3/4}\sum_{j+k=N}\gamma_k(s)\gamma_j(s)\Big\} \\ &\leq e^{-s\cos\theta}\Big\{\|u_N^\star\| + (1-e^{-s\cos\theta})^{1/4}\frac{1}{\cos\theta}\sum_{j+k=N}\gamma_k(s)\gamma_j(s)\Big\} = e^{-s\cos\theta}\gamma_N(s),\end{aligned}$$

thereby showing the first inequality of the induction.

To obtain the second inequality in (4.12) for $n=N$, integrate (4.14) directly.

Thus,

$$\cos\theta \int_0^s \frac{|Av_N(\rho)|^2}{v_{N,\varepsilon}(\rho)} d\rho \le v_{N,\varepsilon}(0) + C_1 \Big\{ \sum_{j+k=N} \gamma_j(s)\gamma_k(s) \Big\}^{1/4} \int_0^s e^{-(5/4)\rho\cos\theta} \Big\{ \sum_{j+k=N} \mathcal{G}_j(\rho)\gamma_k(s) \Big\}^{3/4} d\rho.$$

Applying Hölder's inequality yields

$$\begin{aligned}
&\int_0^s e^{-(5/4)\rho\cos\theta} \Big\{ \sum_{j+k=N} \frac{|Av_j(\rho)|^2}{\|v_j(\rho)\|}\gamma_k(s) \Big\}^{3/4} d\rho \\
&\quad \le \Big\{ \int_0^s e^{-5\rho\cos\theta} d\rho \Big\}^{1/4} \Big\{ \sum_{k+j=N} \gamma_k(s) \int_0^s \frac{|Av_j(\rho)|^2}{\|v_j(\rho)\|} d\rho \Big\}^{3/4} \\
&\quad \le \Big\{ \int_0^s e^{-\rho\cos\theta} d\rho \Big\}^{1/4} \Big\{ \frac{1}{\cos\theta} \sum_{j+k=N} \gamma_j(s)\gamma_k(s) \Big\}^{3/4} \\
&\quad \le \Big\{ \frac{1 - e^{-s\cos\theta}}{\cos\theta} \Big\}^{1/4} \Big\{ \frac{1}{\cos\theta} \sum_{j+k=N} \gamma_j(s)\gamma_k(s) \Big\}^{3/4}.
\end{aligned}$$

It follows that

$$\cos\theta \int_0^s \frac{|Av_N(\rho)|^2}{v_{N,\varepsilon}(\rho)} d\rho \le v_{N,\varepsilon}(0) + \frac{C_1(1 - e^{-s\cos\theta})^{1/4}}{\cos\theta} \sum_{j+k=N} \gamma_j(s)\gamma_k(s).$$

Again, letting $\varepsilon \to 0$, we complete the inductive proof of (4.9). □

From Proposition IV.1 we can obtain the classical results on the regular solutions of the Navier–Stokes equations. Namely, the existence of local solutions for arbitrary initial data in V and the existence of global solutions for small initial data in V.

C. Local solutions

Since the bounds of $\|u_n(\cdot)\|$ are given by (4.8), the convergence of $\sum_{n=1}^{\infty} \|u_n(\cdot)\|$ is implied by the convergence of $\sum_{n=1}^{\infty} \gamma_n(s)$, where $\gamma_n(s)$ are nonnegative numbers defined recursively by (4.10). The convergence of such series is considered in the following lemma.

Lemma IV.1. *Let $\xi \geq 0$ and $a_n \geq 0$, $1 \leq n \leq N$. Define γ_n by*

$$\begin{cases} \gamma_1 = a_1, \\ \gamma_n = a_n + \xi \sum_{k+j=n} \gamma_k \gamma_j, \quad 1 < n \leq N. \end{cases}$$

If $4\xi \sum_{k=1}^{N} a_k \leq 1$, then

$$\sum_{k=1}^{N} \gamma_k \leq 2 \sum_{k=1}^{N} a_k \quad \textit{for} \quad 1 \leq n \leq N. \tag{4.15}$$

Proof. Let $S_n = \sum_{k=1}^{n} \gamma_k$ and $X = \sum_{k=1}^{N} a_k$. Then we have

$$S_n = \sum_{k=1}^{n} a_k + \xi \sum_{k+j \leq n} \gamma_k \gamma_j \leq X + \xi S_{n-1}^2, \quad 1 \leq n \leq N.$$

Denote by $S_\star$ the smaller solution of $\xi S^2 - S + X = 0$, *i.e.*,

$$S_\star = \frac{1 - \sqrt{1 - 4\xi X}}{2\xi} = \frac{2X}{1 + \sqrt{1 - 4\xi X}} \in [X, 2X].$$

Note that $S_1 = a_1 = \gamma_1 \leq X \leq S_\star$. Suppose $S_{n-1} \leq S_\star$ then

$$S_n \leq X + \xi S_{n-1}^2 \leq X + \xi S_\star^2 = S_\star$$

proves (4.15). □

We now prove the existence of local solutions in complexified time and also describe their domains of analyticity.

Theorem IV.1. *Let $S^\star = \sum_{n=1}^{\infty} \|u_n^\star\| < \infty$ and $(u_n(\zeta))_{n=1}^{\infty}$ be the solutions of* (4.3). *Let*

$$\begin{cases} \gamma_1 = \|u_1^\star\|, \\ \gamma_n = \|u_n^\star\| + \xi \sum_{k+j=n} \gamma_k \gamma_j, \quad n > 1, \end{cases} \tag{4.16}$$

where $\xi > 0$ such that $4S^\star \xi \leq 1$. Then $u(\zeta) = \sum_{n=1}^{\infty} u_n(\zeta)$ is the unique solution of the complexified Navier–Stokes equations (2.38) *with $u^\star \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} u_n^\star$ and ζ belonging* *to the following domain*

$$E(\zeta_0,\xi)=\Big\{\,\zeta_0+se^{i\theta}:\xi(s,\theta)<\xi,\;\; s>0\;\; and\; |\theta|<\pi/2\,\Big\}. \tag{4.17}$$

Note that $\xi(s,\theta)$ *is defined in* (4.7). *More precisely,*

$$\|u_n(\zeta)\|\le\gamma_n e^{-\mathbf{Re}(\zeta-\zeta_0)},\quad \zeta\in E(\zeta_0,\xi), \tag{4.18}$$

$$|Au_n(\zeta)|\le 2\delta\gamma_n(s+r)(\pi r^2)^{-1}[\min\{\cos(\theta+\delta),\cos(\theta-\delta)\}]^{-1}, \tag{4.19}$$

where $\zeta=\zeta_0+se^{i\theta}\in E(\zeta_0,\xi)$ *and* $r>0$ *such that the closed ball* $\bar{B}(\zeta,r)$ *is contained in*

$$\Big\{\,\zeta_0+\rho e^{i\omega}:\omega\in(\theta-\delta,\theta+\delta)\;\; and\; 0\le\rho\le s+r\,\Big\}\subset E(\zeta_0,\xi)$$

with $\delta=\arcsin(r/s)>0$, *and*

$$\sum_{n=1}^{\infty}\gamma_n\le 2S^\star. \tag{4.20}$$

Proof. We first consider $(u_n^{(M)})_{n\ge1}$ the Galerkin solutions to (4.6), for some $M>1$. Comparing (4.10) and (4.16), we note that $\gamma_n(s)\le\gamma_n$ provided $\xi(s,\theta)\le\xi$. Since $4\xi S^\star\le1$, apply Proposition IV.1 and Lemma IV.1, we obtain

$$\|u_n^{(M)}(\zeta)\|\le\gamma_n e^{-\mathbf{Re}(\zeta-\zeta_0)},\quad \zeta\in E(\zeta_0,\xi), \tag{4.21}$$

and (4.20). We estimate $|Au_n^{(M)}(\zeta)|$ next. Given $n\ge1$, for each $w\in H$, define the analytic function $f(\zeta)=\langle Au_n^{(M)}(\zeta),w\rangle$ for $\zeta\in E(\zeta_0,\xi)$. Given $\zeta\in E(\zeta_0,\xi)$, let r and δ be positive numbers described in the statement of the theorem. Using Cauchy's formula we have

$$f(\zeta)=\frac{1}{\pi r^2}\int_{B(\zeta_0,r)}f(\eta)d\eta,$$

Applying the Cauchy–Schwartz inequality followed by (4.8) and (4.9) obtains

$$\begin{aligned}
|f(\zeta)| &\leq \frac{1}{\pi r^2}\int_{\theta-\delta}^{\theta+\delta}\int_0^{s+r} |w||Au_n^{(M)}(\zeta_0+\rho e^{i\omega})|\rho d\rho d\omega \\
&\leq \frac{|w|}{\pi r^2}\int_{\theta-\delta}^{\theta+\delta}\left\{\int_0^{s+r}\frac{|Au_n^{(M)}(\zeta_0+\rho e^{i\omega})|^2}{\|u_n^{(M)}(\zeta_0+\rho e^{i\omega})\|}d\rho\int_0^{s+r}\|u_n^{(M)}(\zeta_0+\rho e^{i\omega})\|\rho^2 d\rho\right\}^{1/2} d\omega \\
&\leq \frac{\gamma_n|w|}{\pi r^2}\int_{\theta-\delta}^{\theta+\delta}\left\{\frac{1}{\cos\omega}\int_0^{s+r} e^{-\rho\cos\omega}\rho^2 d\rho\right\}^{1/2} d\omega \\
&\leq \frac{\gamma_n|w|}{\pi r^2}\int_{\theta-\delta}^{\theta+\delta}\frac{s+r}{\cos\omega}d\omega \\
&\leq \frac{2\delta(s+r)\gamma_n|w|}{\pi r^2\min\{\cos(\theta+\delta),\cos(\theta-\delta)\}},
\end{aligned}$$

hence

$$|Au_n^{(M)}(\zeta)| \leq \frac{2\delta(s+r)\gamma_n}{\pi r^2\min\{\cos(\theta+\delta),\cos(\theta-\delta)\}}. \tag{4.22}$$

By the upper bounds shown in (4.21), (4.22) and (4.20) which are independent of M, we can extract subsequences, denoted by $(u_n^{(M_p)})$, where $n, p \in \mathbb{N}, M_p \nearrow \infty$ when $p \nearrow \infty$, such that $u_n^{(M_p)}(\zeta)$ (for $p \to \infty$) is convergent in $\mathcal{D}_A$, for $\zeta \in E(\zeta_0, \xi)$. Its limit is, in fact, the solution $u_n(\zeta)$ of (4.3). Consequently, after passing to the limit when $M = M_p \to \infty$ in (4.21) and (4.22), we have (4.18) and (4.19), respectively. Now, by (4.18) and (4.20) the sum $\sum_{n=1}^{\infty} u_n(\zeta)$ is convergent in V uniformly for $\zeta \in E(\zeta_0, \xi)$. Hence

$$\lim_{\zeta\to\zeta_0}\sum_{n=1}^{\infty} u_n(\zeta) = \sum_{n=1}^{\infty}\lim_{\zeta\to\zeta_0} u_n(\zeta) = \sum_{n=1}^{\infty} u_n^{\star} = u^{\star},$$

where the limit in ζ is taken for $\zeta \in E(\zeta_0, \xi)$.

Given $N \in \mathbb{N}$, define $U_N(\zeta) = \sum_{n=1}^{N} u_n(\zeta)$ where $\zeta \in E(\zeta_0, \xi)$. We have

$$\frac{dU_N(\zeta)}{d\zeta} = -AU_N(\zeta) - \sum_{k+j\leq N} B(u_k(\zeta), u_j(\zeta)). \tag{4.23}$$

We have already shown that $U_N(\zeta)$ converges to $u(\zeta)$ in V uniformly for ζ in $E(\zeta_0, \xi)$ and $AU_N(\zeta)$ converges to $Au(\zeta)$ in H uniformly for ζ in any compact subset of $E(\zeta_0, \xi)$.

Now, using inequality (2.55) in Lemma II.2, we obtain $B(U_N(\zeta), U_N(\zeta))$ converges in V to $B(u(\zeta), u(\zeta))$ uniformly for ζ in compact subsets of $E(\zeta_0, \xi)$, and also estimate

$$\begin{aligned}\Big\|B(u(\zeta), u(\zeta)) - \sum_{k+j\le N} B(u_k(\zeta), u_j(\zeta))\Big\| &\le \sum_{k+j>N} \|B(u_k(\zeta), u_j(\zeta))\| \\ &\le \sum_{k+j>N} C_1\|u_k(\zeta)\|^{1/2}|Au_k(\zeta)|^{1/2}|Au_j(\zeta)| \\ &\le C_1(\zeta) \sum_{k+j>N} \gamma_k\gamma_j,\end{aligned}$$

where the positive constants $C_1(\zeta)$ are bounded on compact subsets of $E(\zeta_0, \xi)$. Let $c_m = \sum_{k+j=m} \gamma_k\gamma_j$, for $m \ge 2$. Then $\sum_{m=2}^{\infty} c_m$ is the Cauchy product of $\sum_{k=1}^{\infty} \gamma_k$ multiplied with itself which is convergent by (4.20). Therefore,

$$\lim_{N\to\infty} \sum_{k+j>N} \gamma_k\gamma_j = \lim_{N\to\infty} \sum_{m=N+1}^{\infty} c_m = 0.$$

Hence $\sum_{k+j\le N} B(u_k(\zeta), u_j(\zeta))$ converges to $B(u(\zeta), u(\zeta))$ in V uniformly for ζ in any compact subset of $E(\zeta_0, \xi)$. It follows that (4.23) can be passed to the limit in H as $N \to \infty$ and hence $u(\zeta)$ is the solution of (2.38). □

Remark IV.2. When $u^0 = \sum_{n=1}^{\infty} u_n^0 \in V \setminus \{0\}$ such that $\sum_{n=1}^{\infty} \|u_n^0\| < \infty$, take $\zeta_0 = 0$, $\theta = 0$, $u_n^\star = u_n^0$ and $\xi > 0$ satisfying $4\xi \sum_{n=1}^{\infty} \|u_n^0\| = 1$, Theorem IV.1 implies that $u(t) = \sum_{n=1}^{\infty} u_n(t)$ is the regular solution to the Navier–Stokes equations (2.12) on the set

$$\Big\{t > 0 : C_1(1 - e^{-t})^{1/4} < \big(4\sum_{n=1}^{\infty} \|u_n^0\|\big)^{-1}\Big\}. \tag{4.24}$$

In the case $4C_1 \sum_{n=1}^{\infty} \|u_n^0\| > 1$, the regular solution exists on $[0, T)$ where

$$T = -\log\Big[1 - \big\{4C_1 \sum_{n=1}^{\infty} \|u_n^0\|\big\}^{-4}\Big]$$

Note that setting $u_n^0 = 0$ for $n > 1$ and $u_1^0 = u^0$ gives an estimate

$$T = -\log\left[1 - \left\{4C_1\|u^0\|\right\}^{-4}\right] \tag{4.25}$$

of the time interval of existence for regular solutions to the Navier–Stokes equations with large initial data.

D. Global solutions with small initial data

If $4C_1\sum_{n=1}^{\infty}\|u_n^0\| \leq 1$, the regular solution $u(t) = \sum_{n=1}^{\infty}u_n(t)$ exists for all times $t > 0$, by (4.24). However, under this condition the domain of analyticity is not large enough for our purpose of estimating the terms in the asymptotic expansion of a regular solution. Therefore we will derive a larger domain of analyticity for $u_n(\zeta)$ and hence for $u(\zeta)$ when $\sum_{n=1}^{\infty}\|u_n^0\|$ satisfies a slightly more stringent condition.

Definition IV.1. We list here some absolute constants which are used in the remainder of this article. Recall that the constant $C_1 > 0$ is introduced in Lemma II.2. Define

$$\varepsilon_0 = (24C_1)^{-1}, \quad \varepsilon_1 = (8C_1)^{-1}, \quad C_2 = 8C_1, \quad \alpha_* = \frac{1}{2}, \quad c_0 = \sqrt{2}. \tag{4.26}$$

Theorem IV.2. *Given* $\epsilon \in (0, \varepsilon_1)$ *and* $S^0 = \sum_{n=1}^{\infty}\|u_n^0\| \leq \epsilon$. *Let* $(u_n(\zeta))_{n=1}^{\infty}$ *be the solutions to* (4.3) *with* $\zeta_0 = 0$, $u_n^\star = u_n^0$. *Then* $u(\zeta) = \sum_{n=1}^{\infty}u_n(\zeta)$ *is the unique solution of the complexified Navier–Stokes equations* (2.38) *with initial condition* $u(0) = u^0 \stackrel{\text{def}}{=\!=} \sum_{n=1}^{\infty}u_n^0 \in V$ *and* ζ *belonging to the domain*

$$E(\epsilon) = \left\{ t + se^{i\theta} : \frac{\epsilon e^{-t}}{\varepsilon_1\cos\theta} < 1,\ s > 0,\ t > 0 \text{ and } |\theta| < \pi/2 \right\}. \tag{4.27}$$

Moreover, define

$$\begin{cases} \gamma_1 = \|u_1^0\|, \\ \gamma_n = \|u_n^0\| + \frac{1}{4\epsilon}\sum_{k+j=n}\gamma_k\gamma_j, \ n>1 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\gamma}_1 = \gamma_1, \\ \tilde{\gamma}_n = \gamma_n + \frac{1}{8\epsilon}\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j, \ n>1. \end{cases}$$

Then, for each $n \in \mathbb{N}$,

$$\|u_n(t)\| \le \gamma_n e^{-t}, \quad t>0, \tag{4.28}$$

$$\|u_n(\zeta)\| \le \tilde{\gamma}_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in E(\epsilon), \tag{4.29}$$

and

$$\sum_{n=1}^{\infty}\tilde{\gamma}_n \le 2\sum_{n=1}^{\infty}\gamma_n \le 4S^0. \tag{4.30}$$

Consequently,

$$\sum_{n=1}^{\infty}\|u_n(t)\| \le 2S^0 e^{-t}, \quad t>0, \tag{4.31}$$

and

$$\sum_{n=1}^{\infty}\|u_n(\zeta)\| \le 4S^0 e^{-\mathbf{Re}\zeta}, \quad \zeta \in E(\epsilon). \tag{4.32}$$

Proof. First, apply Theorem IV.1 for $\zeta_0 = 0$, $\theta = 0$ and $\zeta = t_0 > 0$ noting that

$$4S^0\frac{1}{4\epsilon} \le 1 \quad \text{and} \quad \xi(t_0,0) = C_1(1-e^{-t_0})^{1/4} \le C_1 < \frac{1}{4\epsilon}.$$

We obtain

$$\|u_n(t_0)\| \le \gamma_n e^{-t_0} \tag{4.33}$$

and

$$S_n(t_0) \stackrel{\text{def}}{=\!=} \sum_{j=1}^{n}\|u_j(t_0)\| \le 2S^0 e^{-t_0}. \tag{4.34}$$

Hence

$$S_\infty(t_0) \stackrel{\text{def}}{=\!=} \sum_{n=1}^{\infty}\|u_n(t_0)\| \le 2S^0 e^{-t_0}. \tag{4.35}$$

For $t_0 > 0$, set

$$\begin{cases} \tilde{\gamma}_1(t_0) = \|u_1(\tau_0)\| \\ \tilde{\gamma}_n(t_0) = \|u_n(\tau_0)\| + \dfrac{e^{t_0}}{8\epsilon} \displaystyle\sum_{j+k=n} \tilde{\gamma}_j(\tau_0)\tilde{\gamma}_k(\tau_0). \end{cases}$$

Next, we apply Theorem IV.1 to $\zeta_0 = t_0$, noting that

$$4S_\infty(t_0)\frac{e^{t_0}}{8\epsilon} \le 4(2S^0 e^{-t_0})\frac{e^{t_0}}{8\epsilon} \le 1$$

and for $\zeta = t_0 + se^{i\theta} \in E(\epsilon)$ that

$$\xi(s,\theta) \le \frac{C_1}{\cos\theta} = \frac{\epsilon e^{-t_0}}{\varepsilon_1 \cos\theta} \cdot \frac{e^{t_0}}{8\epsilon} < \frac{e^{t_0}}{8\epsilon},$$

hence $\zeta \in E(t_0, e^{t_0}(8\epsilon)^{-1})$. It follows that

$$\|u(t_0 + se^{i\theta})\| \le \tilde{\gamma}_n(t_0)e^{-s\cos\theta}, \quad t_0 + se^{i\theta} \in E(\epsilon). \tag{4.36}$$

We can show by induction and the use of (4.33) that $\tilde{\gamma}_n(t_0) \le \tilde{\gamma}_n e^{-t_0}$. Therefore

$$\|u(\zeta)\| = \|u(t_0 + se^{i\theta})\| \le \tilde{\gamma}_n e^{-t_0 - s\cos\theta} = \tilde{\gamma}_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in E(\epsilon).$$

Moreover, Lemma IV.1 implies (4.30), thus yields (4.31) and (4.32). □

When S^0 is uniformly small, the analytic domains of regular solutions contain a common subregion which is significant to our subsequent study.

Proposition IV.2. *The domain $E(\varepsilon_0)$ defined by (4.27) contains the following subregion*

$$D = \{\tau + i\sigma : \tau > 0, |\sigma| < c_0 \tau e^{\alpha_* \tau}\}, \tag{4.37}$$

where the constants ε_0, c_* and α_* are defined in (4.26).

Proof. Let $\zeta = \tau + i\sigma = \tau/2 + se^{i\theta} \in E(\epsilon)$, $\epsilon \in (0, \varepsilon_1)$. Then, $\cos\theta > \varepsilon_1^{-1}\epsilon e^{-\tau/2}$ and

$$|\sigma| = \frac{\tau}{2}\tan\theta < \frac{\tau}{2}\sqrt{\frac{\varepsilon_1^2 e^{\tau}}{\epsilon^2} - 1}.$$

This holds if

$$|\sigma| < \frac{\tau e^{\tau/2}}{2}\sqrt{\frac{\varepsilon_1^2}{\epsilon^2} - 1}.$$

We consider those ϵ that satisfy

$$\frac{1}{2}\sqrt{\frac{\varepsilon_1^2}{\epsilon^2} - 1} \geq \sqrt{2}.$$

Hence $\epsilon \leq \varepsilon_1/3 = (24C_1)^{-1}$. This is our definition of ε_0. □

When only finitely many u_j^0 are given, we consider the finite sum $\sum_{j=1}^{n} \|u_j(\zeta)\|$ for some $n > 1$ rather than the infinite sum. The following proposition is an obvious consequence of Theorems IV.2 and Proposition IV.2 when we take $u_j^0 = 0$ for all $j > n$.

Proposition IV.3. *Given $n \geq 1$ and suppose that $S_n = \sum_{k=1}^{n} \|u_k^0\| < \varepsilon_0$. Then*

$$S_n(t) = \sum_{k=1}^{n} \|u_k^0(t)\| \leq 2S_n e^{-t}, \quad t \geq 0, \tag{4.38}$$

and

$$S_n(\zeta) = \sum_{k=1}^{n} \|u_k^0(\zeta)\| \leq 4S_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in E(\varepsilon_0), \tag{4.39}$$

where the analytic domain $E(\varepsilon_0)$ defined in (4.27) *contains the subdomain D defined in* (4.37)*. More specifically, let*

$$\begin{cases} \gamma_1 = \|u_1^0\|, \\ \gamma_m = \|u_m^0\| + \frac{1}{4\varepsilon_0}\sum_{k+j=m} \gamma_k\gamma_j, \ m > 1 \end{cases} \quad \textit{and} \quad \begin{cases} \tilde{\gamma}_1 = \gamma_1, \\ \tilde{\gamma}_m = \gamma_m + \frac{1}{8\varepsilon_0}\sum_{k+j=m} \tilde{\gamma}_k\tilde{\gamma}_j, \ m > 1. \end{cases}$$

Then

$$\|u_j(t)\| \leq \gamma_j e^{-t}, \quad t > 0, 1 \leq j \leq n, \tag{4.40}$$

$$\|u_j(\zeta)\| \leq \tilde{\gamma}_j e^{-\mathbf{Re}\zeta}, \quad \zeta \in D, 1 \leq j \leq n, \tag{4.41}$$

and

$$\sum_{k=1}^{n} \tilde{\gamma}_j \leq 2 \sum_{k=1}^{n} \gamma_j \leq 4S_n. \tag{4.42}$$

E. Extended Navier–Stokes equations

We will consider now the system of equations (4.1) in the space V^∞ of all sequences $\bar{u} = (u_n)_{n=1}^\infty$ where $u_n \in V$ for all $n = 1, 2, 3, \ldots$ We will refer to this system of inhomogeneous differential equations (i.e., inhomogeneous Stokes equations) as the *extended Navier–Stokes equations*. Note that in Section A we have implicitly proved that for each initial data $\bar{u}^0 \overset{\text{def}}{=\!=} (u_n^0)_{n\geq 1} \in V^\infty$ there exists a unique solution $\bar{u}(t) = (u_n(t))_{n\geq 1}$ for $t \geq 0$ of the extended Navier–Stokes equations. Moreover, we have also shown that if $\sum_{n=1}^\infty \|u_n^0\|$ is small enough then $\sum_{n=1}^\infty \|u_n(t)\|$ is absolutely convergent and its sum is a regular solution of the Navier–Stokes equations (2.12) for all $t \geq 0$. We do not know if solutions of the extended Navier–Stokes equations leave invariant the whole space

$$V_{*1} = \Big\{ u = (u_n)_{n=1}^\infty \in V^\infty : \|u\|_{*1} \overset{\text{def}}{=\!=} \sum_{n=1}^\infty \|u_n\| < \infty \Big\}.$$

However we will show that there exist norms of the form

$$\|(u_n)_{n=1}^\infty\|_* = \sum_{n=1}^\infty \rho_n \|u_n\|, \tag{4.43}$$

with positive $\rho_n \to 0$ such that those solutions leave invariant the spaces

$$V_* = \Big\{ u = (u_n)_{n=1}^\infty \in V^\infty : \|u\|_* < \infty \Big\}. \tag{4.44}$$

It will turn out that those spaces play a similar role for the normal form of the Navier–Stokes equations. We now specify the weights $(\rho_n)_{n=1}^\infty$ in (4.43).

Definition IV.2. Let $(\kappa_n)_{n\geq 2}$ be a sequence of positive numbers satisfying

$$\limsup_{n\to\infty} \kappa_n^{1/n} = 0. \tag{4.45}$$

Define a sequence of weights $(\rho_n)_{n\geq 1}$ by

$$\begin{cases} \rho_1 > 0, \\ \rho_n = \kappa_n C_1^{-1} \min\{\rho_k \rho_j \ : \ k, j \geq 1 \ \text{ and } \ k + j = n\}, \quad n > 1. \end{cases}$$

Let $\|\cdot\|_*$ and V_* be the norm and space defined by (4.43) and (4.44), respectively. Then $(V_*, \|\cdot\|_*)$ is a Banach space.

The following lemma will be used to estimate $\|\bar{u}(t)\|_*$ later.

Lemma IV.2. *Let $(\kappa_n)_{n\geq 2}$ be as in Definition IV.2. Let $a_n \geq 0$. Define*

$$\begin{cases} d_1 = a_1, \\ d_n = a_n + \kappa_n \sum_{k+j=n} d_k d_j, \quad n \geq 2. \end{cases}$$

If $\sum_{n=1}^{\infty} a_n$ is finite, then so is $\sum_{n=1}^{\infty} d_n$.

Proof. Let $K = \max\left\{1/2, \sum_{n=2}^{\infty}(n-1)\kappa_n\right\} < \infty$ and $M = \max\left\{1/2, \sum_{n=1}^{\infty} a_n\right\}$. Define $S_n = \sum_{m=1}^{n} d_m$. We will first prove by induction that

$$S_n \leq (2M)^n (2K)^{n-1} \quad \text{for} \quad n \in \mathbb{N}. \tag{4.46}$$

Since $S_1 = a_1 \leq M$, then (4.46) holds for $n = 1$. Let $N > 1$ and suppose (4.46) holds for all $n < N$. Then

$$S_N = \sum_{n=1}^{N} a_n + \sum_{n=2}^{N} \kappa_n \sum_{k+j=n} d_k d_j \leq M + \sum_{n=2}^{N} \Big(\kappa_n \sum_{k+j=n} S_k S_j\Big).$$

Using the induction hypothesis, we have

$$\begin{aligned} S_N &\le M + \sum_{n=2}^{N} \kappa_n \sum_{k+j=n} (2M)^k (2K)^{k-1} (2M)^j (2K)^{j-1} \\ &\le \frac{(2M)^N}{2} + \sum_{n=2}^{N} \kappa_n (2M)^n (2K)^{n-2} (n-1) \\ &\le \frac{(2M)^N (2K)^{N-1}}{2} + (2M)^N \frac{(2K)^{N-1}}{2K} \sum_{n=2}^{N} (n-1)\kappa_n \\ &\le (2M)^N (2K)^{N-1}. \end{aligned}$$

Hence (4.46) holds for all $n \ge 1$. This, by virtue of (4.45), implies that

$$\begin{aligned} S_n &\le \sum_{m=1}^{n} a_m + \sum_{m=2}^{n} (m-1)\kappa_m (2M)^m (2K)^{m-1} \\ &\le M \Big\{ 1 + 2 \sum_{m=2}^{\infty} (m-1)\kappa_m (4MK)^{m-1} \Big\} < \infty \end{aligned}$$

for each $n \ge 2$. □

It turns out that the extended Navier–Stokes equations has the global solution in the normed space $(V_*, \|\cdot\|_*)$.

Theorem IV.3. *Let* $\bar{u}^0 = (u_n^0)_{n\ge 1} \in V_*$ *and* $u_n(t),\ n \in \mathbb{N}$, *be the solutions of* (4.1) *for* $n \in \mathbb{N}$. *Then* $\bar{u}(t) = (u_n(t))_{n \ge 1}, t \ge 0$ *is the solution to the extended Navier–Stokes equations in the space* V_*. *Moreover,*

$$\|\bar{u}(t)\|_* \le e^{-t} \Big\{ \|\bar{u}^0\|_* + \sum_{n=2}^{\infty} \kappa_n (n-1) M_0^n \Big\} \quad \textit{for all} \quad t > 0, \tag{4.47}$$

where

$$M_0 = 4 \max\big(1/2, \|\bar{u}^0\|_*\big) \max\Big(1/2, \sum_{n=2}^{\infty} (n-1)\kappa_n \Big).$$

Proof. From the *a priori* estimates given in Proposition IV.1 with $\zeta_0 = 0$, $\theta = 0$ and

$s = t > 0$, we have $\|u_n(t)\| \le e^{-t}\gamma_n$ for $n \in \mathbb{N}$ where

$$\begin{cases} \gamma_1 = \|u_1^0\|, \\ \gamma_n = \|u_n^0\| + C_1 \sum_{k+j=n} \gamma_k \gamma_j, \quad n > 1. \end{cases}$$

Therefore, $\rho_1\gamma_1 = \rho_1\|u_1^0\|$ and

$$\rho_n\gamma_n \le \rho_n\|u_n^0\| + \kappa_n \sum_{k+j=n} (\rho_k\gamma_k)(\rho_j\gamma_j), \quad n > 1.$$

Apply Lemma IV.2 with $a_n = \rho_n\|u_n^0\|$ and $d_n = \rho_n\gamma_n$. We have

$$\|\bar{u}(t)\|_* \le e^{-t} \sum_{n=1}^{\infty} \rho_n\gamma_n \le e^{-t}\Big\{\|\bar{u}^0\|_* + \sum_{n=2}^{\infty} \kappa_n (n-1) M_0^n\Big\}, \quad t > 0.$$

□

CHAPTER V

ASYMPTOTIC EXPANSIONS AND THE NORMAL FORM

In this chapter, we consider the functions u_n related to the asymptotic expansions of the regular solutions to the Navier–Stokes equations. Let $\xi = (\xi_1, \xi_2, \dots) \in S_A$ and the polynomials $q_n(t) = q_n(t, \xi)$ be constructed as in Subsection B. We extend the polynomials $q_n(t), t \in \mathbb{R}$, to polynomials $q_n(\zeta), \zeta \in \mathbb{C}$, i.e., $q_n(t)$ and $q_n(\zeta)$ have the same $\mathcal{V}$-valued coefficients, and define $u_n(\zeta) = e^{-n\zeta} q_n(\zeta)$ for $\zeta \in \mathbb{C}$. Then $u_n(\zeta)$ satisfies (4.3) with initial condition

$$u_n(0) = u_n^0 \overset{\text{def}}{=\!=} q_n(0), \quad n \in \mathbb{N}. \tag{5.1}$$

We apply Corollary III.1 to estimate $\|q_n(\zeta)\|$ for $\zeta \in \mathbb{C}$.

A. Convergence of the asymptotic expansion

We apply Corollary III.1 to estimate $\|q_n(\zeta)\|$ for $\zeta \in \mathbb{C}$.

Lemma V.1. *Assume* $\sum_{n=1}^{N} \|u_n^0\| < \varepsilon_0$ *for some* $N > 0$ *where* ε_0 *is defined as in* (4.26). *Let* D *be the domain defined in Theorem IV.2. Let* $S_n = \sum_{k=1}^{n} \|u_k^0\|$ *and* $\tilde{\gamma}_n$ *be defined as in Proposition IV.3 for* $n = 1, 2, \dots, N$. *Then for each* $n = 1, 2, \dots, N$ *we have*

$$\|q_n(\zeta)\| \le \tilde{\gamma}_n |1 + \alpha_* \zeta|^{(n-1)/\alpha_*} \quad \textit{for} \quad \zeta \in \varphi_{\alpha_*}^{-1}(H_0), \tag{5.2}$$

and

$$\|q_n(\zeta)\| \le n \tilde{\gamma}_n C_2^{n-1} (|\zeta| + a_*)^{n-1} \quad \textit{for} \quad \zeta \in \mathbb{C}, \tag{5.3}$$

where φ_{α_*} *is defined by* (3.38), H_0 *is the right half plane and constants* a_*, C_2 *are defined in Definition III.1.*

Proof. For $n \geq 2, w \in V$, define $q(\zeta) = \langle\langle q_n(\zeta), w \rangle\rangle$, for $\zeta \in \mathbb{C}$. Note that the degree of $q(\zeta)$ is less than or equal to $(n-1)$. According to Proposition IV.3, we have

$$|e^{-(n-1)\zeta} q(\zeta)| \leq \|e^{\zeta} u_n(\zeta)\| \|w\| \leq \tilde{\gamma}_n \|w\|.$$

Apply Proposition III.3 to $q(\zeta)$ with $p = n - 1$ we have

$$|q_n(\zeta)| \leq \tilde{\gamma}_n |1 + \alpha\zeta|^{(n-1)/\alpha} \|w\|, \quad \zeta \in \varphi^{-1}(H_0)$$

and

$$|q(\zeta)| \leq n\tilde{\gamma}_n C_2^{n-1} (|\zeta| + a_*)^{n-1} \|w\|, \quad \zeta \in \mathbb{C},$$

thus obtaining (5.2) and (5.3). □

When the initial data in (5.1) are small, the above estimates are used to establish the convergence of the series $\sum_{n=1}^{\infty} u_n(t)$.

Proposition V.1. *Suppose* $\sum_{n=1}^{\infty} \|u_n^0\| < \varepsilon_0$. *Then* $u(t) = \sum_{n=1}^{\infty} u_n(t)$ *is the regular solution to the Navier–Stokes equations* (2.12) *with initial condition* $u(0) = u^0 \stackrel{\text{def}}{=\!=} \sum_{n=1}^{\infty} u_n^0 \in V$. *Moreover,* $\sum_{n=1}^{\infty} u_n(t)$ *is the asymptotic expansion of* $u(t)$. *More specifically, there exists* $\epsilon > 0$ *such that*

$$\Big\| u(t) - \sum_{n=1}^{N} u_n(t) \Big\| = O\big(e^{-(N+\epsilon)t}\big) \quad \text{as} \quad t \to \infty \tag{5.4}$$

for every $N > 0$.

Proof. The fact that $u(t) = \sum_{n=1}^{\infty} u_n(t)$ is the regular solution to the Navier–Stokes equations (2.12) with $u(0) = \sum_{n=1}^{\infty} u_n^0$ is stated in Theorem IV.2. From Lemma V.1 and (4.42), we have for each $t > 0$ that

$$\|u_n(t)\| \leq 4\varepsilon_0 n C_2^{n-1} (t + a_*)^{n-1} e^{-nt} = C_0 e^{-t} n \sigma(t)^{n-1},$$

where $C_0 = 4\varepsilon_0$ and $\sigma(t) = C_2(t + a_*)e^{-t}$. Let $t_0 > 0$ such that $\sigma = \sigma(t) \leq 1/2$ for all

$t \geq t_0$. Then, for $t > t_0$ we have

$$\sum_{n=N+1}^{\infty} n\sigma^{n-1} = \frac{d}{d\sigma} \sum_{n=N+1}^{\infty} \sigma^n = \frac{d}{d\sigma} \frac{\sigma^{N+1}}{1-\sigma} = \frac{(N+1)\sigma^N - N\sigma^{N+1}}{(1-\sigma)^2} \leq 4(N+1)\sigma^N.$$

Therefore,

$$\begin{aligned}\Big\| \sum_{n=N+1}^{\infty} u_n(t) \Big\| &\leq C_0 e^{-t} 4(N+1)(C_2(t+a_*)e^{-t})^N \\ &= 4C_0(N+1)C_2^N(t+a_*)^N e^{-(N+1)t} \\ &= o\big(e^{-(N+1/2)t}\big) \quad \text{as} \quad t \to \infty.\end{aligned}$$

This proves that $\sum_{n=1}^{\infty} u_n(t)$ is the asymptotic expansion of $u(t)$. □

When $\xi = W(u^0)$ we have $u_n^0 = W_n(0, u^0)$ and $q_n(t) = W_n(t, u^0)$. It follows, therefore, that $\sum_{n=1}^{\infty} u_n(t) = \sum_{n=1}^{\infty} W_n(t, u^0)e^{-nt}$ is the asymptotic expansion of the regular solution $u(t, u^0)$ of the Navier–Stokes equations (2.12).

Theorem V.1. *Suppose $u^0 \in \mathcal{R}$ with $\sum_{n=1}^{\infty} \|W_n(0, u^0)\| < \varepsilon_0$. Then the asymptotic expansion $\sum_{n=1}^{\infty} W_n(t, u^0)e^{-nt}$ converges in V for all $t > 0$ to the regular solution $u(t, u^0)$ of the Navier–Stokes equations* (2.12).

Proof. According to Proposition V.1, both $u(t) = \sum_{n=1}^{\infty} W_n(t, u^0)e^{-nt}$ and $u(t, u^0)$ are regular solutions to the Navier–Stokes equations with the same asymptotic expansion. Since the normalization map W is one-to-one (see Section B, Chapter II), the two solutions $u(t)$ and $u(t, u^0)$ coincide. □

When the initial data (5.1) are not quite so small, we prove the convergence of the asymptotic expansion only for large times.

Proposition V.2. *If $\sum_{n=1}^{\infty} \|u_n^0\| = S^0 \geq \varepsilon_0$, then there is $T > 0$ such that $u(t) = \sum_{n=1}^{\infty} u_n(t)$ is absolutely convergent in V uniformly for $t \in [T, \infty)$, $u(t)$ is a regular solution of the Navier–Stokes equations* (2.12) *for $t \geq T$, and $\sum_{n=1}^{\infty} u_n(t)$ is the* *asymptotic expansion of* $u(t)$. *In addition, if there is* $u^0 \in \mathcal{R}$ *such that* $u_n^0 = W_n(0, u^0)$ *for all* $n \in \mathbb{N}$, *then* $u(t)$ *is equal to the regular solution* $u(t, u^0)$ *of* (2.12) *for all* $t \in [T, \infty)$.

Proof. Let $\epsilon > 0$ be small enough such that $\rho = \varepsilon_0/S^0 - \epsilon \in (0, 1)$. Define $\tilde{u}_n(\zeta) = \rho^n u_n(\zeta)$ for $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} \|\tilde{u}_n(0)\| = \sum_{n=1}^{\infty} \rho^n \|u_n^0\| \leq \rho \sum_{n=1}^{\infty} \|u_n^0\| < \varepsilon_0.$$

As indicated in Remark IV.1 we can apply Theorem IV.2 to $\tilde{u}_n(\zeta)$. Hence,

$$\sum_{n=1}^{\infty} \|\tilde{u}_n(\zeta)\| = \sum_{n-1}^{\infty} \rho^n \|u_n(\zeta)\| \leq 4\varepsilon_0 e^{-\mathbf{Re}\zeta} \quad \text{for} \quad \zeta \in D.$$

Letting $\epsilon \to 0$ we obtain

$$\|u_n(\zeta)\| \leq 4\varepsilon_0 \Big(\frac{S^0}{\varepsilon_0}\Big)^n e^{-\mathbf{Re}\zeta} \quad \text{for} \quad \zeta \in D.$$

Similar to Lemma V.1, we derive for $t > 0$ that

$$\|u_n(t)\| \leq 4\varepsilon_0 \Big(\frac{S^0}{\varepsilon_0}\Big)^n n C_2^{n-1} (t + a_*)^{n-1} e^{-nt} = n\tilde{C}_0 \tilde{C}_2^{n-1} (t + a_*)^{n-1} e^{-nt}, \tag{5.5}$$

where $\tilde{C}_0 = 4S^0$ and $\tilde{C}_2 = S^0 C_2 \varepsilon_0^{-1}$.

Let $t_0 > 0$ such that $\tilde{C}_2(t + a_*)e^{-t}$ is small, say, less that 1/2. Then $u(t) = \sum_{n=1}^{\infty} u_n(t)$ is absolutely convergent in V uniformly for $t \in [t_0, \infty)$. As shown in Proposition V.1, $\sum_{n=1}^{\infty} u_n(t)$ is the asymptotic expansion of $u(t)$ in V. Moreover, by (5.5), we can choose $T > t_0$ and further large enough that

$$\sum_{n=1}^{\infty} \|u_n(T)\| < \varepsilon_0. \tag{5.6}$$

Note that $v_n(\tau) \stackrel{\text{def}}{=\!=} u_n(\tau + T)$ with $n \in \mathbb{N}$ is the solution of (4.1) for $\tau > 0$ with initial condition $v_n(0) = u_n(T)$. Theorem V.1 and (5.6) imply $v(\tau) \stackrel{\text{def}}{=\!=} \sum_{n=1}^{\infty} v_n(\tau) = u(\tau + T)$

is a regular solution of the Navier–Stokes equations for $\tau > 0$.

To finish the proof, suppose there is $u^0 \in \mathcal{R}$ such that $u_n^0 = W_n(0, u^0)$ for all $n \in \mathbb{N}$. We next show that the regular solution $u(t, u^0)$ is equal to the asymptotic expansion $u(t)$ for t large enough. Define $v^0 = u(T, u^0)$. Then $\tilde{v}(\tau) \stackrel{\text{def}}{=\!=} u(\tau + T, u^0)$ is the regular solution with initial condition v^0. Moreover, since the normalizing map is one-to-one we have that the asymptotic expansion of $\tilde{v}(\tau)$ is the same as the asymptotic expansion of $u(\tau + T, u^0)$. Thus,

$$\tilde{v}(\tau) \sim \sum_{n=1}^{\infty} v_n(\tau).$$

Therefore, Theorem V.1 implies that $\tilde{v}(\tau) = v(\tau) = \sum_{n=1}^{\infty} v_n(\tau) = u(\tau+T)$ for $\tau \geq 0$. It follows that $u(\tau + T, u^0) = \tilde{v}(\tau) = u(\tau + T)$ for $\tau \geq 0$. $\square$

Our next goal is to weaken the condition on the initial data in Proposition V.2. To achieve that we need to deal with bounds of each term $\|u_n(\zeta)\|$ rather than of the sum $\sum_{n=1}^{N} \|u_n(\zeta)\|$ for $N > 1$. Let us first define the following reference sequences.

Definition V.1. We define

$$\begin{cases} b_1 = 1/4, \\ b_n = \sum_{k+j=n} b_k b_j, \ \ n > 1 \end{cases} \quad \text{and} \quad \begin{cases} \mu_1 = 1, \\ \mu_n = \sum_{k+j=n} \mu_k \mu_j, \ \ n > 1. \end{cases}$$

One can verify that

$$\frac{1}{4^n} \leq b_n = \frac{\mu_n}{4^n} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \leq \frac{1}{2}$$

by Lemma IV.1.

Before working with the norms $\|u_n(\zeta)\|$, we first establish some properties of series of non-negative real numbers.

Lemma V.2. *Let $\chi > 0, r > 0$ and $a_n \geq 0$ for $n \in \mathbb{N}$. Assume that*

$$a_n \leq \frac{1}{2}(2\chi)^{n-1} r^n b_n \quad \textit{for} \quad n \in \mathbb{N}. \tag{5.7}$$

Define

$$\begin{cases} \gamma_1 = a_1, \\ \gamma_n = a_n + \chi \sum_{k+j=n} \gamma_k \gamma_j, \ n > 1 \end{cases} \quad \textit{and} \quad \begin{cases} \tilde{\gamma}_1 = \gamma_1, \\ \tilde{\gamma}_n = \gamma_n + \frac{\chi}{2} \sum_{k+j=n} \tilde{\gamma}_k \tilde{\gamma}_j, \ n > 1. \end{cases}$$

Then

$$\gamma_n \leq (2\chi)^{n-1} r^n b_n \tag{5.8}$$

and

$$\tilde{\gamma}_n \leq 2(2\chi)^{n-1} r^n b_n. \tag{5.9}$$

Proof. We prove (5.8) by induction. Clearly (5.8) holds for $n = 1$, by (5.7). Given $n > 1$, assume (5.8) holds for all $k < n$. Then

$$\begin{aligned} \gamma_n &\leq a_n + \chi \sum_{k+j=n} (2\chi)^{k-1} r^k b_k (2\chi)^{j-1} r^j b_j \\ &\leq \frac{1}{2}(2\chi)^{n-1} r^n b_n + \frac{(2\chi)^{n-1} r^n \chi}{2\chi} \sum_{k+j=n} b_k b_j \\ &= (2\chi)^{n-1} r^n b_n. \end{aligned}$$

Hence (5.8) holds for all $n \geq 1$.

To prove (5.9), apply the first part to $\chi' = \chi/2$, $r' = 2r$, $a'_n = \gamma_n$ and $\gamma'_n = \tilde{\gamma}_n$. Since

$$a'_n = \gamma_n \leq (2\chi)^{n-1} r^n b_n = \frac{1}{2}(2\chi')^{n-1}(r')^n b_n,$$

we obtain

$$\tilde{\gamma}_n \leq (2\chi')^{n-1}(r')^n b_n = 2(2\chi)^{n-1} r^n b_n.$$

□

We now relax the convergence condition on the initial sum in Proposition V.2.

Lemma V.3. *Let* $\sup_{n\geq 1}\|u_n^0\| = M_0 < \infty$. *Then*

$$\|u_n(\zeta)\| \leq 2\varepsilon_0\big\{4\max(1,\varepsilon_0^{-1}M_0)\big\}^n e^{-\mathbf{Re}\zeta} \quad \textit{for} \quad \zeta\in D, \tag{5.10}$$

where D *is the domain defined by* (4.37).

Proof. Given $N\geq 1$, let

$$M_N \stackrel{\text{def}}{=\!=} \max\left\{\Big(\frac{\|u_k^0\|}{\varepsilon_0}\Big)^{1/k} : k=1,2,\ldots,N\right\}.$$

If $M_N=0$, then $u_n(\zeta)=0$ for $\zeta\in D$ and $n=1,2,\ldots N$, hence (5.10) trivially holds for $n=N$. In the remaining case when $M_N\neq 0$, let $\chi=(4\varepsilon_0)^{-1}$ and choose $r>0$ small enough such that

$$\sum_{n=1}^{\infty}\frac{1}{2}(2\chi)^{n-1}r^n b_n < \varepsilon_0. \tag{5.11}$$

Let $\rho = r\chi/(2M_N)$, $v_n(\zeta)=\rho^n u_n(\zeta), \zeta\in D$, and $v_n^0=\rho^n u_n^0$, for $n=1,2,\ldots N$. Define

$$\begin{cases}\gamma_1 = \|v_1^0\|,\\ \gamma_n = \|v_n^0\| + \chi\sum_{k+j=n}\gamma_k\gamma_j,\ \ n>1\end{cases} \quad \text{and} \quad \begin{cases}\tilde\gamma_1=\gamma_1,\\ \tilde\gamma_n = \gamma_n + \frac{\chi}{2}\sum_{k+j=n}\tilde\gamma_k\tilde\gamma_j,\ \ n>1.\end{cases}$$

Estimate

$$\|v_n^0\| = \rho^n\|u_n^0\| \leq \varepsilon_0\Big(\frac{r\chi}{2}\Big)^n = \frac{1}{2}(2\chi)^{n-1}r^n\frac{1}{4^n} \leq \frac{1}{2}(2\chi)^{n-1}r^n b_n.$$

Setting $a_n=\|v_n^0\|$ for $n=1,2,\ldots,N$ and $a_n=0$ for $n>N$ yields a sequence of non-negative numbers that satisfy the conditions of Lemma V.2. Thus,

$$\tilde\gamma_n \leq 2(2\chi)^{n-1}r^n b_n \leq (2\chi)^{n-1}r^n \quad \text{for} \quad n=1,2,\ldots,N.$$

Also from (5.11) we have $\sum_{n=1}^{N} \|v_n^0\| < \varepsilon_0$. By virtue of Remark IV.1 and Proposition IV.3, $\|v_N(\zeta)\| \le \tilde{\gamma}_N e^{-\mathbf{Re}\zeta}$ for $\zeta \in D$. We obtain

$$\begin{aligned}
\|u_N(\zeta)\| &\le \rho^{-N}(2\chi)^{N-1} r^N e^{-\mathbf{Re}\zeta} \\
&\le (2\chi)^{N-1} r^N \Big(\frac{2}{r\chi}\Big)^N \Big\{ \max_{1\le k\le N} \Big(\frac{\|u_k^0\|}{\varepsilon_0}\Big)^{1/k}\Big\}^N e^{-\mathbf{Re}\zeta} \\
&\le \frac{4^N}{2\chi} \Big\{ \max_{1\le k\le N} \Big(\frac{M_0}{\varepsilon_0}\Big)^{1/k}\Big\}^N e^{-\mathbf{Re}\zeta} \\
&\le 2\varepsilon_0 \big\{4\max(1, \varepsilon_0^{-1} M_0)\big\}^N e^{-\mathbf{Re}\zeta} \quad \text{for} \quad \zeta \in D.
\end{aligned}$$

□

A direct consequence of Lemma V.3 allows $\|u_n^0\|$ to grow as a power function with exponent n.

Lemma V.4. *Suppose* $\limsup_{n\to\infty} \|u_n^0\|^{1/n} < \infty$. *Then*

$$\|u_n(\zeta)\| \le 2\varepsilon_0 M^n e^{-\mathbf{Re}\zeta}, \quad \zeta \in D, \tag{5.12}$$

for some $M > 0$, *where* D *is the domain defined by* (4.37).

Proof. Take $R > \limsup_{n\to\infty} \|u_n^0\|^{1/n}$. Then there is $M_0 > 0$ such that $\|u_n^0\| \le M_0 R^n$ for all $n \in \mathbb{N}$. According to Remark IV.1 we can apply Theorem V.3 to $v_n^0 = u_n^0/R^n$ and $v_n(\zeta) = u_n(\zeta)/R^n$. We obtain

$$\|v_n(\zeta)\| \le 2\varepsilon_0 \big\{4\max(1, \varepsilon_0^{-1} M_0)\big\}^n e^{-\mathbf{Re}\zeta} \quad \text{for} \quad \zeta \in D.$$

Taking $M = 4R\max(1, \varepsilon_0^{-1} M_0)$ finishes the proof. □

Combining Lemma V.4 and the proof of Proposition V.2 now gives the following results easily.

Proposition V.3. *Suppose* $\limsup_{n\to\infty} \|u_n^0\|^{1/n} < \infty$. *Then there is* $T > 0$ *such that* $u(t) = \sum_{n=1}^{\infty} u_n(t)$ *is absolutely convergent in* V *uniformly for* $t \in [T, \infty)$, $u(t)$ *is* *a regular solution of the Navier–Stokes equations for* $t \geq T$ *and* $\sum_{n=1}^{\infty} u_n(t)$ *is the asymptotic expansion of* $u(t)$*.*

Theorem V.2. *Suppose* $u^0 \in \mathcal{R}$ *and*

$$\limsup_{n\to\infty} \|W_n(0,u^0)\|^{1/n} < \infty.$$

Then there is $T > 0$ *such that*

$$u(t) = \sum_{n=1}^{\infty} W_n(t,u^0)e^{-nt}$$

is absolutely convergent in V *uniformly for* $t \in [T,\infty)$*,* $\sum_{n=1}^{\infty} W_n(t,u^0)e^{-nt}$ *is the asymptotic expansion of* $u(t)$*, and* $u(t)$ *is equal to the regular solution* $u(t,u^0)$ *of the Navier–Stokes equations* (2.12) *for all* $t \in [T,\infty)$*.*

Proposition V.1 can be applied to the study of the normalization map. Indeed, we have

Corollary V.1. *If* $\xi \in S_A$ *and* $\sum_{n=1}^{\infty} \|q_n(0,\xi)\| < \varepsilon_0$*, then* ξ *is in the range of the normalization map, i.e.,* $\xi = W(u^0)$ *where* $u^0 = \sum_{n=1}^{\infty} q_n(0,\xi) \in \mathcal{R}$*.*

Proof. From Proposition V.1 we know that $u(t) = \sum_{n=1}^{\infty} q_n(t,\xi)e^{-nt}$ is the regular solution of the Navier–Stokes equations with initial condition u^0 defined above. In addition, the series $\sum_{n=1}^{\infty} q_n(t,\xi)e^{-nt}$ is the asymptotic expansion of the solution $u(t)$. Therefore, by the definition of the normalization map in Subsection B and by (2.25) we have

$$W(u^0) = (R_n q_n(0,\xi))_{n\in\mathbb{N}} = \xi.$$

□

Similarly, Proposition V.3 has the following consequence concerning the solutions of the normal form.

Corollary V.2. *If $\xi \in S_A$ and $\limsup_{n\to\infty} \|q_n(0,\xi)\|^{1/n} < \infty$, then the solution $\xi(t) = (R_n q_n(t,\xi))_{n=1}^{\infty}$ of the normal form* (2.29) *with initial condition ξ is in the range of the normalization map for $t \in [T,\infty)$ for some $T > 0$. More precisely, there is $u^\star \in \mathcal{R}$ and $T > 0$ such that $\xi(t) = W(u(t-T,u^\star))$ for $t \in [T,\infty)$, where $u(\cdot,u^\star)$ denotes the regular solution of* (2.12) *with initial condition $u^\star$.*

Proof. From Proposition V.3, $u(t) = \sum_{n=1}^{\infty} q_n(t,\xi)e^{-nt}$ is a regular solution of the Navier–Stokes equations. Let $u^\star = u(T)$. Hence $u^\star \in \mathcal{R}$. For each $t \geq T$ fixed, let $v^\star = u(t) = u(t-T,u^\star)$. Then the regular solution $u(\tau,v^\star)$ with initial condition $v^\star$ has the asymptotic expansion $\sum_{n=1}^{\infty} q_n(\tau+t,\xi)e^{-n(\tau+t)}$ as $\tau \to \infty$. Hence

$$W(u(t)) = W(v^\star) = (R_n q_n(t,\xi)e^{-nt})_{n=1}^{\infty} = \xi(t).$$

□

B. Application to the normalization map

In this section, we continue to study the solution to the extended Navier–Stokes equations with initial conditions obtained from the normalization map as indicated in (5.1). We will study directly the relation between $\xi = (\xi_1,\xi_2,\ldots)$ and $\bar{u}(t) = (u_n(t))_{n\geq 1}$. Similar to Section E in Chapter IV, we consider the following type of sums

$$S_n(\zeta) = \sum_{j=1}^{n} \rho_j \|u_j(\zeta)\|, \qquad S_n = S_n(0), \qquad n \in \mathbb{N},$$

where $\rho_n > 0$ and

$$\rho_n \leq \min\{\rho_k\rho_j : k+j=n\}.$$

Note that all the results in Section A now apply to $\rho_n u_n(\zeta)$ and the above $S_n(\zeta)$. In particular, we have the following version of Proposition IV.3.

Proposition V.4. *Given $N \in \mathbb{N}$ suppose that $S_N = \sum_{j=1}^{N} \rho_j \|u_j^0\| < \varepsilon_0$, where $\rho_1 > 0$ and*

$$\rho_n \le \min\{\rho_k \rho_j, k + j = n\}, \quad 2 \le n \le N.$$

Define

$$\begin{cases} \gamma_1 = \rho_1 \|u_1^0\|, \\ \gamma_n = \rho_n \|u_n^0\| + \frac{1}{4\varepsilon_0} \sum_{k+j=n} \gamma_k \gamma_j, \quad 1 < n \le N \end{cases}$$

and

$$\begin{cases} \tilde{\gamma}_1 = \rho_1 \gamma_1, \\ \tilde{\gamma}_n = \rho_n \gamma_n + \frac{1}{8\varepsilon_0} \sum_{k+j=n} \tilde{\gamma}_k \tilde{\gamma}_j, \quad 1 < n \le N. \end{cases}$$

Then

$$\rho_n \|u_n(t)\| \le \gamma_n e^{-t}, \quad t > 0, 1 \le n \le N \tag{5.13}$$

$$\rho_n \|u_n(\zeta)\| \le \tilde{\gamma}_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in D, 1 \le n \le N, \tag{5.14}$$

where the domain D is defined in (4.37), and

$$\sum_{k=1}^{n} \tilde{\gamma}_j \le 2 \sum_{k=1}^{n} \gamma_j \le 4 S_n, \quad 1 \le n \le N. \tag{5.15}$$

In particular,

$$S_n(t) \le 2 S_n e^{-t}, \quad t \ge 0, 1 \le n \le N. \tag{5.16}$$

and

$$S_n(\zeta) \le 4 S_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in D, 1 \le n \le N. \tag{5.17}$$

We now estimate $\|q_n(0)\|$ provided $\|\xi_n\|$ and $\|q_j(0)\|$ for $j < n$ are given.

Lemma V.5. *Let $0 < \sigma_n \le \min\{\rho_k \rho_j : k + j = n \text{ and } k, j \in \mathbb{N}\}$ and suppose $S_{n-1} < \varepsilon_0$, then*

$$\|q_n(0)\| \le \|\xi_n\| + \frac{L_{3,n}}{\sigma_n} \sum_{k+j=n} \tilde{\gamma}_k \tilde{\gamma}_j, \tag{5.18}$$

where $\tilde{\gamma}_k$ *for* $k = 1, 2, \ldots, n-1$ *are defined in Proposition V.4 and* $L_{3,n} = L_{1,n} + L_{2,n}$, *where*

$$L_{1,n} = C_4 C_3^{n-2} (n-2)^{3/4} n^2 (n-2)!, \tag{5.19}$$

$$L_{2,n} = C_4 C_3^{n-2} n^{3/2} n^2 (n-2)!, \tag{5.20}$$

$$C_4 = C_4(a_*) = 8^{-1} C_1 e^{2a_*}, \tag{5.21}$$

and the positive numbers C_3 *and* a_* *are defined in Definition III.1.*

Proof. Let $\check{q}_n(t) = P_{n-1} q_n(t)$ where P_n is the projection given in (2.16). Note that Lemma II.4 implies that $\check{q}_n(t) = P_{n-2} q_n(t)$. Upon projecting (2.24) we have

$$\frac{d}{dt}\check{q}_n(t) + (A-n)\check{q}_n(t) + P_{n-2} B_n(t) = 0.$$

Then

$$e^{t'(A-n)P_{n-2}}\check{q}_n(t') - e^{t(A-n)P_{n-2}}\check{q}_n(t) = -\int_t^{t'} e^{\tau(A-n)P_{n-2}} P_{n-2} B_n(\tau) d\tau.$$

Letting $t' \to \infty$, we obtain

$$\check{q}_n(t) = \int_t^{\infty} e^{(\tau - t)(A-n)P_{n-2}} P_{n-2} B_n(\tau). \tag{5.22}$$

Now set $t = 0$ and use inequality (2.53) in Lemma II.2 to estimate

$$\begin{aligned} \|\check{q}_n(0)\| &\leq \int_0^{\infty} e^{-2\tau} \|P_{n-2} B_n(\tau)\| d\tau \\ &\leq C_1 (n-2)^{3/4} \int_0^{\infty} e^{-2\tau} \sum_{k+j=n} \|q_k(\tau)\| \|q_j(\tau)\| d\tau. \end{aligned}$$

Multiply the above inequality by σ_n and apply the estimates in Lemma V.1 for $\rho_n q_n$

instead of q_n alone to obtain

$$\begin{aligned}\sigma_n\|\check{q}_n(0)\| &\le C_1(n-2)^{3/4}\int_0^\infty e^{-2\tau}\sum_{k+j=n}\rho_k\|q_k(\tau)\|\rho_j\|q_j(\tau)\|d\tau\\ &\le C_1(n-2)^{3/4}C_2^{n-2}\Big(\sum_{k+j=n}k\tilde{\gamma}_k j\tilde{\gamma}_j\Big)\int_0^\infty e^{-2\tau}(\tau+a_*)^{n-2}d\tau\\ &\le C_1(n-2)^{3/4}\frac{n^2}{4}C_2^{n-2}\Big(\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j\Big)\int_0^\infty e^{-2\tau}(\tau+a_*)^{n-2}d\tau.\end{aligned}$$

Take $y=2(\tau+a_*)$ then

$$\int_0^\infty e^{-2\tau}(\tau+a_*)^{n-2}d\tau \le 2^{-1}e^{2a_*}\int_0^\infty e^{-y}(y/2)^{n-2}dy=\frac{e^{2a_*}}{2^{n-1}}\Gamma(n-1).$$

We obtain

$$\sigma_n\|P_{n-1}q_n(0)\| \le L_{1,n}\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j. \tag{5.23}$$

Let $\hat{q}(\zeta)=Q_nq_n(\zeta)$. By Lemmas II.4 and II.5 we write $\hat{q}(\zeta)=P_{n^2}Q_{n+1}q_n(\zeta)$. Similar to the estimate of $\check{q}_n(0)$, we have

$$\hat{q}_n(0)=-\int_{-\infty}^0 e^{\tau(A-n)(P_{n^2}-P_{n+1})}(P_{n^2}-P_{n+1})B_n(\tau)d\tau.$$

Thus,

$$\begin{aligned}\sigma_n\|\hat{q}_n(0)\| &\le C_1n^{3/2}\int_{-\infty}^0 e^{2\tau}\sum_{k+j=n}\rho_k\|q_k(\tau)\|\rho_j\|q_j(\tau)\|d\tau\\ &\le C_1n^{3/2}\frac{n^2}{4}C_2^{n-2}\Big(\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j\Big)\int_{-\infty}^0 e^{2\tau}(|\tau|+a_*)^{n-2}d\tau.\end{aligned}$$

We obtain

$$\sigma_n\|Q_nq_n(0)\| \le L_{2,n}\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j. \tag{5.24}$$

Finally, combine (5.23) and (5.24) along with (2.25) to obtain (5.18). □

We now bootstrap the above estimates to obtain bounds for the asymptotic expansions solely in terms of $\xi=(\xi_1,\xi_2,\ldots)$ using the weighted norm specified below.

Definition V.2. Let ε_0 be in Definition IV.1 and $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of numbers satisfying

$$\alpha_1 \geq 0, \quad \alpha_n > 0 \text{ for } n > 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n \leq 1/2. \tag{5.25}$$

Construct the sequence $(\rho_n)_{n\in\mathbb{N}}$ as follows: Let $\rho_1 = 1$ and for $n > 1$ define

$$\sigma_n = \min\{\, \rho_k \rho_j : k + j = n \text{ and } k, j \in \mathbb{N} \,\}. \tag{5.26}$$

Then let

$$0 < \rho_n = \frac{\sigma_n \alpha_n}{16\varepsilon_0 \max\{1, L_{3,n}, C_1 n^{3/2}\}}, \quad n > 1. \tag{5.27}$$

Proposition V.5. *If* $\sum_{j=1}^{\infty} \rho_j \|\xi_j\| = \delta < \varepsilon_0/2$ *then*

$$S(0) \stackrel{\text{def}}{=\!=} \sum_{j=1}^{\infty} \rho_j \|q_j(0)\| \leq 2\delta, \tag{5.28}$$

$$S(t) \stackrel{\text{def}}{=\!=} \sum_{j=1}^{\infty} \rho_j \|u_j(t)\| \leq 4\delta e^{-t} \quad \text{for} \quad t > 0 \tag{5.29}$$

and

$$S(\zeta) \stackrel{\text{def}}{=\!=} \sum_{j=1}^{\infty} \rho_j \|u_j(\zeta)\| \leq 8\delta e^{-\mathbf{Re}\zeta} < 4\varepsilon_0 e^{-\mathbf{Re}\zeta} \quad \text{for} \quad \zeta \in D, \tag{5.30}$$

where D is defined by (4.37).

Proof. We will prove by induction that

$$\sum_{j=1}^{n} \rho_j \|q_j(0)\| \leq 2\delta \quad \text{and} \quad \rho_n \|q_n(0)\| \leq \rho_n \|\xi_n\| + 2\delta\alpha_n \quad \text{for} \quad n \in \mathbb{N}. \tag{5.31}$$

Since $q_1(t) = \xi_1$ and $\rho_1 = 1$, then (5.31) holds for $n = 1$. Suppose (5.31) holds for all $n < N$. Since $S_{N-1} = \sum_{j=1}^{N-1} \rho_j \|q_j(0)\| \leq 2\delta < \varepsilon_0$, we have $\sum_{n=1}^{N-1} \tilde{\gamma}_n \leq 4S_{N-1} \leq 8\delta$

by Proposition V.4. By Lemma V.5 followed by (5.27) it follows that

$$\begin{aligned} \rho_n \|q_n(0)\| &\leq \rho_n \|\xi_n\| + \frac{\rho_n L_{3,n}}{\sigma_n} \Big(\sum_{n=1}^{n-1} \tilde{\gamma}_n \Big)^2 \\ &\leq \rho_n \|\xi_n\| + \frac{\alpha_n}{16\varepsilon_0} (8\delta)^2 \\ &< \rho_n \|\xi_n\| + 2\delta\alpha_n \quad \text{for} \quad 1 \leq n \leq N. \end{aligned} \tag{5.32}$$

In particular,

$$\rho_N \|q_N(0)\| \leq \rho_N \|\xi_N\| + 2\delta\alpha_N.$$

Summing up (5.32) from $n = 1$ to N yields

$$\sum_{n=1}^{N} \rho_n \|q_n(0)\| \leq \sum_{n=1}^{N} \rho_n \|\xi_n\| + 2\delta \sum_{n=1}^{N} \alpha_n \leq \delta + \delta = 2\delta.$$

Therefore, by the induction principle, (5.31) is true for all $n \in \mathbb{N}$. Consequently (5.28) holds. Now (5.29) and (5.30) follow from (5.16) and (5.17). □

C. Application to the normal form

Let $V_{\mathbb{C}}$ and $(R_n H)_{\mathbb{C}}$ for $n \in \mathbb{N}$ be the complexifications of V and $R_n H$. Define

$$V_{\mathbb{C}}^{\infty} = \big\{\, u = (u_n)_{n\in\mathbb{N}} : u_n \in V_{\mathbb{C}} \,\big\},$$

$$S_A = R_1 H \oplus R_2 H \oplus \cdots \quad \text{and} \quad (S_A)_{\mathbb{C}} = (R_1 H)_{\mathbb{C}} \oplus (R_2 H)_{\mathbb{C}} \oplus \cdots .$$

For $\bar{u} = (u_n)_{n\in\mathbb{N}} \in V_{\mathbb{C}}^{\infty}$ define

$$\|\bar{u}\|_\star = \sum_{n=1}^{\infty} \rho_n \|u_n\|,$$

where the sequence $(\rho_n)_{n\in\mathbb{N}}$ is given in Definition V.2. Let

$$V_{\mathbb{C}}^{\star} = \{\, u \in V_{\mathbb{C}} : \|u\|_\star < \infty \,\} \quad \text{and} \quad R^\star = \{\, u \in S_A : \|u\|_\star < \infty \,\}.$$

Define

$$F(\xi,\zeta)=\left(e^{-n\zeta}q_n(\zeta,\xi)\right)_{n\in\mathbb{N}}\in V_{\mathbb{C}}^{\infty},\quad \zeta\in D,\ \xi\in S_A,$$

and

$$G(\xi,\zeta)=\left(e^{-n\zeta}R_nq_n(\zeta,\xi)\right)_{n\in\mathbb{N}}\in (S_A)_{\mathbb{C}},\quad \zeta\in D,\ \xi\in S_A,$$

where the polynomials $q_n(\zeta,\xi)$ in ζ are generated by ξ as defined in (2.27). Using these notations we can restate Proposition V.5 as

Theorem V.3. *If $\xi\in R^\star$ and $\|\xi\|_\star<\varepsilon_0/2$, then $F(\xi,\zeta)\in V^\star$ for all $\zeta\in D$. Moreover*

$$\|F(\xi,0)\|_\star\le 2\|\xi\|_\star,$$

$$\|F(\xi,t)\|_\star\le 4\|\xi\|_\star e^{-t},\quad t>0,$$

and

$$\|F(\xi,\zeta)\|_\star\le 8\|\xi\|_\star e^{-\mathbf{Re}\zeta},\quad \zeta\in D.$$

In particular,

$$\|G(\xi,t)\|_\star\le 4\|\xi\|_\star e^{-t},\quad t\ge 0.$$

First we show a Lipschitz-like property of $F(\xi,\zeta)$ connected with $F(\xi,0)$.

Lemma V.6. *Suppose $\xi,\chi\in R^\star$ and $\|\xi\|_\star,\|\chi\|_\star<\varepsilon_0/2$. Then*

$$\|F(\xi,t)-F(\chi,t)\|_\star\le e^{1/8}e^{-t}\|F(\xi,0)-F(\chi,0)\|_\star,\quad t\ge 0,\tag{5.33}$$

and

$$\|F(\xi,\zeta)-F(\chi,\zeta)\|_\star\le e^{7/8}e^{-\mathbf{Re}\zeta}\|F(\xi,0)-F(\chi,0)\|_\star,\quad \zeta\in D,\tag{5.34}$$

where D is the domain defined by (4.37). *Moreover,*

$$\|F(\xi,t)-F(\chi,t)\|_\star\le e^{-7t/8}\|F(\xi,0)-F(\chi,0)\|_\star,\quad t\ge 0,\tag{5.35}$$

and

$$\|F(\xi,\zeta)-F(\chi,\zeta)\|_\star \le e^{-\mathbf{Re}\zeta/4}\|F(\xi,0)-F(\chi,0)\|_\star, \quad \zeta\in D. \tag{5.36}$$

Proof. Let $q_n(\zeta)=q_n(\zeta,\xi)$ and $p_n(\zeta)=q_n(\zeta,\chi)$ be the polynomials generated by ξ and χ, respectively. Let $u_n(\zeta)=e^{-n\zeta}q_n(\zeta)$, $v_n(\zeta)=e^{-n\zeta}p_n(\zeta)$ and $w_n(\zeta)=u_n(\zeta)-v_n(\zeta)$. Note that $u_n(\zeta), v_n(\zeta)$ and $w_n(\zeta)$ belong to $P_{n^2}H$, by Lemma II.5. Define

$$X_n(\zeta)=\sum_{j=1}^n \rho_j\|u_j(\zeta)\|, \qquad X_n=X_n(0),$$

$$Y_n(\zeta)=\sum_{j=1}^n \rho_j\|v_j(\zeta)\|, \qquad Y_n=Y_n(0),$$

and

$$Z_n(\zeta)=\sum_{j=1}^n \rho_j\|w_j(\zeta)\|, \qquad Z_n=Z_n(0).$$

According to Theorem V.3 we have for each $n\in\mathbb{N}$ that

$$X_n(t), Y_n(t)\le 2\varepsilon_0 e^{-t}, \quad t\ge 0 \tag{5.37}$$

and

$$X_n(\zeta), Y_n(\zeta)\le 4\varepsilon_0 e^{-\mathbf{Re}\zeta}, \quad \zeta\in D. \tag{5.38}$$

Subtracting the evolution equations (4.3) for u_n and v_n we find that w_n satisfies

$$\frac{d}{d\zeta}w_n(\zeta)+Aw_n(\zeta)+\sum_{k+j=n}\Big\{B(w_k(\zeta),u_j(\zeta))+B(v_j(\zeta),w_k(\zeta))\Big\}=0, \quad \zeta\in D. \tag{5.39}$$

Take the inner product of (5.39) with Aw_n and apply Lemma II.2 to obtain

$$\frac{d}{dt}\|w_n(t)\|+\|w_n(t)\|\le C_1 n^{3/2}\sum_{k+j=n}\|w_k(t)\|\big(\|u_j(t)\|+\|v_j(t)\|\big), \quad t>0.$$

Hence

$$\rho_n \frac{d}{dt}\|w_n(t)\| + \rho_n\|w_n(t)\| \le \frac{\rho_n C_1 n^{3/2}}{\sigma_n} \sum_{k+j=n} \rho_k \|w_k(t)\| \big(\rho_j\|u_j(t)\| + \rho_j\|v_j(t)\|\big)$$
$$\le \frac{\alpha_n}{16\varepsilon_0} \sum_{k+j=n} \rho_k \|w_k(t)\| \big(\rho_j\|u_j(t)\| + \rho_j\|v_j(t)\|\big)$$
$$\le \frac{1}{32\varepsilon_0} \sum_{k+j=n} \rho_k \|w_k(t)\| \big(\rho_j\|u_j(t)\| + \rho_j\|v_j(t)\|\big).$$

Sum up from $n = 1$ to N and use (5.37). We have

$$\frac{d}{dt}Z_N(t) + Z_N(t) \le \frac{1}{32\varepsilon_0} Z_{N-1}(t)(X_{N-1}(t) + Y_{N-1}(t))$$
$$\le \frac{4\varepsilon_0 e^{-t}}{32\varepsilon_0} Z_{N-1}(t) \le \frac{e^{-t}}{8} Z_N(t),$$

hence

$$\frac{d}{dt}Z_N(t) \le (\frac{1}{8e^t} - 1)Z_N(t). \tag{5.40}$$

Simple estimating the right hand side by $(-7Z_N(t)/8)$ gives

$$Z_N(t) \le Z_N e^{-7t/8}, \tag{5.41}$$

for all $N \in \mathbb{N}$ and $t \ge 0$. Letting $N \to \infty$ obtains (5.35). We also have from (5.40)

$$Z_N(t) \le Z_N e^{-t+\frac{1}{8}(1-e^{-t})} \le e^{1/8} Z_N e^{-t}, \quad N \in \mathbb{N}. \tag{5.42}$$

Letting $N \to \infty$ we obtain (5.33).

We next deal with $Z_n(\zeta)$ with $\zeta \in D$. First, from (5.39) and Lemma II.2 we have

$$\frac{d}{ds}\|w_n(\tau_0 + se^{i\theta})\| + \cos\theta\|w_n(\tau_0 + se^{i\theta})\|$$
$$\le C_1 n^{3/2} \sum_{k+j=n} \|w_k\| \big(\|u_j\| + \|v_j\|\big)\Big|_{\zeta=\tau_0+se^{i\theta}}.$$

Therefore

$$\rho_n \frac{d}{ds}\|w_n(\tau_0+se^{i\theta})\| + \rho_n\cos\theta\|w_n(\tau_0+se^{i\theta})\|$$
$$\leq \frac{C_1 n^{3/2}\rho_n}{\sigma_n}\sum_{k+j=n}\rho_k\|w_k\|\big(\rho_j\|u_j\|+\rho_j\|v_j\|\big)\Big|_{\zeta=\tau_0+se^{i\theta}}$$
$$\leq \frac{1}{32\varepsilon_0}\sum_{k+j=n}\rho_k\|w_k\|\big(\rho_j\|u_j\|+\rho_j\|v_j\|\big)\Big|_{\zeta=\tau_0+se^{i\theta}}.$$

Summing up

$$\frac{d}{ds}Z_N(\tau_0+se^{i\theta}) + (\cos\theta)Z_N(\tau_0+se^{i\theta})$$
$$\leq \frac{1}{32\varepsilon_0}Z_{N-1}(\tau_0+se^{i\theta})\big(X_{N-1}(\tau_0+se^{i\theta})+Y_{N-1}(\tau_0+se^{i\theta})\big)$$
$$\leq \frac{8\varepsilon_0 e^{-\tau_0-s\cos\theta}}{32\varepsilon_0}Z_{N-1}(\tau_0+se^{i\theta}),$$

hence

$$\frac{d}{ds}Z_N(\tau_0+se^{i\theta}) \leq (4^{-1}e^{-\tau_0}e^{-s\cos\theta}-\cos\theta)Z_N(\tau_0+se^{i\theta}). \tag{5.43}$$

Using Gronwall's inequality and then (5.42), we obtain

$$Z_N(\tau_0+se^{i\theta}) \leq Z_N(\tau_0)\exp\left\{-s\cos\theta+\frac{e^{-\tau_0}}{4\cos\theta}(1-e^{-s\cos\theta})\right\}$$
$$\leq Z_N e^{-\tau_0-s\cos\theta}e^{1/8}\exp\left\{\frac{e^{-\tau_0}}{4\cos\theta}\right\}.$$

Proposition IV.2 implies $\tau_0+se^{i\theta}\in D\subset E(\varepsilon_0)$. Therefore

$$\cos\theta > \varepsilon_1^{-1}\varepsilon_0 e^{-\tau_0} = \frac{1}{3}e^{-\tau_0}. \tag{5.44}$$

It follows that

$$Z_N(\tau_0+se^{i\theta}) \leq Z_N e^{-\tau_0-s\cos\theta}e^{1/8}e^{3/4} = e^{7/8}Z_N e^{-\tau_0-s\cos\theta}. \tag{5.45}$$

Letting $N \to \infty$ obtains (5.34). Now, using (5.44) in (5.43) gives

$$\frac{d}{ds} Z_N(\tau_0 + se^{i\theta}) \le -\frac{\cos\theta}{4} Z_N(\tau_0 + se^{i\theta}),$$

thus

$$Z_N(\tau_0 + se^{i\theta}) \le Z_N(\tau_0) e^{-\frac{1}{4}s\cos\theta} \le Z_N e^{-\frac{1}{4}(\tau_0 + s\cos\theta)},$$

where we used (5.41) to estimate $Z_N(\tau_0)$. Then (5.36) follows taking $N \to \infty$. □

We now connect $\|q_n(0) - p_n(0)\|$ with $\|\xi_j - \chi_j\|$, for $j = 1, 2, \ldots, n$ under the same conditions as in Lemma V.6.

Lemma V.7. *Suppose $\xi, \chi \in R^\star$ and $\|\xi\|_\star, \|\chi\|_\star < \varepsilon_0/2$. Let $r_n(\zeta) = q_n(\zeta) - p_n(\zeta)$, where $q_n(\zeta) = q_n(\zeta, \xi)$ and $p_n(\zeta) = q_n(\zeta, \chi)$. Then*

$$\rho_n \|r_n(\zeta)\| \le e^{7/8} n Z_n C_2^{n-1} (|\zeta| + a_*)^{n-1}, \quad \zeta \in \mathbb{C} \tag{5.46}$$

and

$$\|r_n(0)\| \le \|\xi_n - \chi_n\| + \frac{8e^{7/8}\varepsilon_0 L_{3,n}}{\sigma_n} Z_n, \tag{5.47}$$

where $Z_n = \sum_{j=1}^n \rho_j \|r_n(0)\|$, the constants C_2 and a_* are defined in Definition III.1.

Proof. Using the same notations as in Lemma V.6, we have

$$\rho_n \|e^{-(n-1)\zeta} r_n(\zeta)\| \le e^{7/8} Z_n, \quad \zeta \in D,$$

by (5.45) and the fact that $w_n(\zeta) = e^{-n\zeta} r_n(\zeta)$, for $n \ge 1$. Applying Proposition III.3, as in the proof of Lemma V.1, we obtain (5.46). From (2.24) we derive an evolution equation for r_n analogous to (5.39) expressed as

$$\frac{d}{dt} r_n(t) + (A - n) r_n(t) + \sum_{k+j=n} B(r_k(t), p_j(t)) + B(q_j(t), r_k(t)) = 0, \quad t > 0. \tag{5.48}$$

Estimate as in the proof of Lemma V.5 writing $r_n = \check{r}_n + R_n r_n + \hat{r}_n$ where $\check{r}_n = P_{n-2} r_n$

and $\hat{r}_n = P_{n^2}Q_{n+1}r_n$. Thus,

$$\check{r}_n(0) = \int_0^{-\infty} e^{\tau(A-n)P_{n-2}} P_{n-2} \sum_{k+j=n} \Big\{ B(r_k(\tau), p_j(\tau)) + B(q_j(\tau), r_k(\tau)) \Big\} d\tau$$

and applying Lemma II.2 gives

$$\sigma_n \|\check{r}_n(0)\| \leq C_1(n-2)^{3/4} \int_0^{\infty} e^{-2\tau} \sum_{k+j=n} \Big\{ \rho_k \|r_k(\tau)\| (\rho_j \|p_j(\tau)\| + \rho_j \|q_j(\tau)\|) \Big\} d\tau.$$

Using (5.46) and following the calculations in the proof of Lemma V.5, we have

$$\sigma_n \|\check{r}_n(0)\| \leq L_{1,n} e^{7/8} Z_n \sum_{j=1}^{n-1} (\tilde{\gamma}_{j,\xi} + \tilde{\gamma}_{j,\chi}) \leq 8 e^{7/8} \varepsilon_0 L_{1,n} Z_n,$$

where $\tilde{\gamma}_{j,\xi}$ and $\tilde{\gamma}_{j,\chi}$ for $j \in \mathbb{N}$ are defined in Proposition V.4 with $u_j^0 = q_j(0)$ and $u_j^0 = p_j(0)$, respectively. Similarly,

$$\sigma_n \|\hat{r}_n(0)\| \leq 8 e^{7/8} \varepsilon_0 L_{2,n} Z_n.$$

Therefore,

$$\|r_n(0)\| \leq \|\xi_n - \chi_n\| + \frac{8 e^{7/8} \varepsilon_0 L_{3,n}}{\sigma_n} Z_n.$$

□

We finish by proving that $F(\xi, \zeta)$ and $G(\xi, \zeta)$ are Lipschitz continuous in ξ, uniformly for $\zeta \in D$.

Theorem V.4. *Let $\xi, \chi \in R^\star$ and $\|\xi\|_\star < \varepsilon_0/2, \|\chi\|_\star < \varepsilon_0/2$, then*

$$\|F(\xi, 0) - F(\chi, 0)\|_\star \leq 4 \|\xi - \chi\|_\star, \tag{5.49}$$

$$\|G(\xi, t) - G(\chi, t)\|_\star \leq \|F(\xi, t) - F(\chi, t)\|_\star \leq 4 e^{1/8} e^{-t} \|\xi - \chi\|_\star, \quad t \geq 0, \tag{5.50}$$

$$\|G(\xi, \zeta) - G(\chi, \zeta)\|_\star \leq \|F(\xi, \zeta) - F(\chi, \zeta)\|_\star \leq 4 e^{7/8} e^{-\mathbf{Re}\zeta} \|\xi - \chi\|_\star, \quad \zeta \in D. \tag{5.51}$$

Proof. We adopt notations used in the previous two lemmas. Since $r_n(0) = w_n(0)$, then summing up (5.47) for $N > 1$ gives

$$\begin{aligned} Z_N = \sum_{n=1}^{N} \rho_n \|r_n(0)\| &\le \sum_{n=1}^{N} \rho_n \|\xi_n - \chi_n\| + \sum_{n=1}^{N} \frac{8e^{7/8}\varepsilon_0 L_{3,n}\rho_n}{\sigma_n} Z_n \\ &\le \sum_{n=1}^{N} \rho_n \|\xi_n - \chi_n\| + \frac{e^{7/8}}{2} Z_N \sum_{n=1}^{N} \alpha_n \\ &\le \sum_{n=1}^{N} \rho_n \|\xi_n - \chi_n\| + \frac{3}{4} Z_N. \end{aligned}$$

Therefore $Z_N \le 4\sum_{n=1}^{N} \rho_n \|\xi_n - \chi_n\|$. Letting $N \to \infty$, we obtain (5.49). We then apply Lemma V.6 to obtain (5.50) and (5.51). □

CHAPTER VI

ASYMPTOTIC ANALYSIS OF THE HELICITY

In this chapter, we study the behavior of the helicity $H(t) = \langle Tu(t), u(t)\rangle$ and its related quantity $J(t) = \langle T^2u(t), Tu(t)\rangle$ for time $t \to \infty$. For $u^0 \in \mathcal{R}$, these two satisfy the equation (see (2.14))

$$\frac{1}{2}\frac{dH(t)}{dt} + J(t) = 0. \tag{6.1}$$

This resembles the energy relation (2.13) connecting $|Tu(t)|^2$ and $|u(t)|^2$. We will find the analogues of the relations (2.17) for $J(t)/H(t)$ and $H(t)$. The asymptotic behavior of $H(t)$ and $J(t)$ as $t \to \infty$ will also be described.

A. The asymptotic expansion of the helicity

We will derive the *asymptotic expansion* of the helicity $H(t)$. This can be easily seen as a formal expansion of $\langle Tu(t), u(t)\rangle$ based on the expansion (2.18) of $u(t)$:

$$\langle Tu(t), u(t)\rangle \sim \sum_{j\geq 1} e^{-jt} \sum_{k+l=j} \langle Tq_k, q_l\rangle.$$

In fact, this formal computation is proved to be valid in the rigorous sense of Definition II.2. According to Definition II.2, we can say from (2.19) and (2.20) that the regular solution $u(t)$ to (2.12) has a unique asymptotic expansion

$$u(t) \sim \sum_{j\geq 1} q_j(t)e^{-jt} \quad \text{ in } H^m(\Omega) \text{ for } m = 0, 1, 2, ... \tag{6.2}$$

By virtue of Lemma II.7, the asymptotic expansions of Tu and T^2u are calculated by applying T and, respectively, T^2 to (6.2) term by term.

Lemma VI.1. *Let $u(t)$ be a regular solution of* (2.12) *with the asymptotic expansion*

(6.2). *Then* $Tu(t)$ *and* $T^2u(t)$ *have the asymptotic expansions*

$$Tu(t) \sim \sum_{j=1}^{\infty} Tq_j(t)e^{-jt} \tag{6.3}$$

and, respectively,

$$T^2u(t) \sim \sum_{j=1}^{\infty} T^2q_j(t)e^{-jt}, \tag{6.4}$$

in all $H^m(\Omega)$ *for* $m = 0, 1, 2, ...$

Lemma II.8 allows us to easily find the asymptotic expansions of $H(t)$ and $J(t)$.

Corollary VI.1. *Let* $u(t)$ *be a regular solution of* (2.12) *with the asymptotic expansion* (6.2). *Then the helicity* $H(t)$ *and* $J(t)$ *have the following asymptotic expansions*

$$H(t) = \langle Tu(t), u(t)\rangle \sim \sum \phi_n(t)e^{-nt}, \qquad \textit{where } \phi_n(t) = \sum_{j+l=n} \langle Tq_j(t), q_l(t)\rangle, \tag{6.5}$$

and, respectively,

$$J(t) = \langle T^2u(t), Tu(t)\rangle \sim \sum \psi_n(t)e^{-nt}, \qquad \textit{where } \psi_n(t) = \sum_{j+l=n} \langle T^2q_j(t), Tq_l(t)\rangle. \tag{6.6}$$

Proof. Apply Lemma II.8 with $X = Y = L^2(\Omega)$, and $Z = \mathbb{R}$ and the use of (6.2) as well as Lemma VI.1 for individual expansions of $u(t)$, $Tu(t)$ and $T^2u(t)$. □

So far we have obtained the asymptotic expansions (6.5) and (6.6) of $H(t)$ and $J(t)$, respectively. The functions $H(t)$ and $J(t)$ are connected by Equation (6.1). The corresponding connection between ϕ_n and ψ_n in those asymptotic expansions is given in the following.

Lemma VI.2. *For all* $n \in \mathbb{N}$ *and* $t > 0$,

$$\phi_n'(t) - n\phi_n(t) + 2\psi_n(t) = 0. \tag{6.7}$$

Proof. Taking the derivative of $\phi_n(t)$, we obtain

$$\begin{aligned}\phi_n'(t) &= \sum_{m+k=n} \langle Tq_m, q_k' \rangle + \sum_{m+k=n} \langle Tq_m', q_k \rangle = \sum_{m+k=n} \langle Tq_m, q_k' \rangle + \sum_{m+k=n} \langle q_m', Tq_k \rangle \\ &= 2 \sum_{m+k=n} \langle Tq_m, q_k' \rangle = -2 \sum_{m+k=n} \langle Tq_m, T^2 q_k - kq_k + \beta_k \rangle \quad \text{(by (2.24))} \\ &= -2 \sum_{m+k=n} \langle Tq_m, T^2 q_k \rangle + 2 \sum_{m+k=n} \langle Tq_m, kq_k \rangle - 2 \sum_{m+k=n} \langle Tq_m, \beta_k \rangle.\end{aligned}$$

The first sum is $-2\psi_n$. The second sum is

$$\sum_{m+k=n} \langle Tq_m, kq_k \rangle + \langle Tq_k, mq_m \rangle = \sum_{m+k=n} \langle Tq_m, (m+k) q_k \rangle = n \sum_{m+k=n} \langle Tq_m, q_k \rangle = n\phi_n.$$

The third sum can be written as

$$\begin{aligned}-2 \sum_{m+k=n} \langle Tq_m, \beta_k \rangle &= -2 \sum_{m+j+l=n} \langle Tq_m, B(q_j, q_l) \rangle \\ &= - \sum_{m+j+l=n} \left(\langle Tq_m, B(q_j, q_l) \rangle + \langle Tq_m, B(q_l, q_j) \rangle \right) \\ &= - \sum_{m+j+l=n} \langle q_m, T(B(q_j, q_l) + B(q_l, q_j)) \rangle \\ &= \sum_{m+j+l=n} \langle q_m, B(Tq_j, q_l) + B(Tq_l, q_j) - B(q_j, Tq_l) - B(q_l, Tq_j) \rangle \\ &= 2 \sum_{m+j+l=n} \langle q_m, B(Tq_j, q_l) \rangle - 2 \sum_{m+j+l=n} \langle q_m, B(q_j, Tq_l) \rangle.\end{aligned}$$

We have from (2.9)

$$2 \sum_{m+j+l=n} \langle q_m, B(Tq_j, q_l) \rangle = \sum_{m+j+l=n} \langle q_m, B(Tq_j, q_l) \rangle + \sum_{m+j+l=n} \langle q_l, B(Tq_j, q_m) \rangle = 0.$$

On the other hand,

$$-2\sum_{m+j+l=n}\langle q_m, B(q_j, Tq_l)\rangle = 2\sum_{m+j+l=n}\langle Tq_j, B(q_l, q_m)\rangle = 2\sum_{m+j+l=n}\langle Tq_m, B(q_j, q_l)\rangle$$
$$= 2\sum_{m+k=n}\langle Tq_m, \beta_k\rangle$$

Therefore, $-2\sum_{m+k=n}\langle Tq_m, \beta_k\rangle = 2\sum_{m+k=n}\langle Tq_m, \beta_k\rangle$, and hence it is zero. Thus, $\phi_n' = n\phi_n - 2\psi_n$ and this proves (6.7). $\square$

The behavior of $\phi_n(t)$ and $\psi_n(t)$ for large times t is constrained by Lemma VI.2 and the fact that those functions are polynomials. Indeed we also have the following.

Lemma VI.3. *For each $n \in \mathbb{N}$, either*

i) ϕ_n and ψ_n are identically zero, or

ii) ϕ_n and ψ_n are eventually nonzero.

In the second case,

$$\lim_{t\to\infty}\frac{\psi_n(t)}{\phi_n(t)} = \frac{n}{2}. \tag{6.8}$$

Proof. In the case ϕ_n is identically zero, so is ψ_n by the virtue of (6.7), hence (i) holds. If ϕ_n is not identically zero, since it is a polynomial, we have that deg $\phi_n' <$ deg ϕ_n. Hence $\phi_n' - n\phi_n$ is not identically zero and due to (6.7), neither is ψ_n. Since both ϕ_n and ψ_n are polynomials, it follows that for all large enough t, say, $t > t_0$, $\psi_n(t) \neq 0$ and $\phi_n(t) \neq 0$; this proves (ii). Moreover, for $t > t_0$, we have

$$\frac{\phi_n'}{\phi_n} - n + 2\frac{\psi_n}{\phi_n} = 0, \tag{6.9}$$

where the first term goes to zero as $t \to \infty$. Clearly, (6.9) now implies (6.8). $\square$

B. The dichotomy of the helicity's asymptotic behavior

Based on the asymptotic expansions (6.5) of the helicity $H(t)$ we first consider the case when the asymptotic expansion (6.5) is not identically zero, that is when at least one of the polynomials $\phi_n(t)$ $(n = 1, 2, 3, ..)$ is not identically zero. In this case, let N be the smallest $n \in \mathbb{N}$ such that the polynomial $\phi_n(t)$ is not identically zero. By Lemma VI.3, this number N is also the smallest $n \in \mathbb{N}$ such that the polynomial $\psi_n(t)$ in the asymptotic expansion (6.6) of $J(t)$ is not identically zero. Hence, we have

$$\phi_n(t) \equiv \psi_n(t) \equiv 0 \quad \text{for } 1 \leq n < N, \quad \phi_N(t) \not\equiv 0 \quad \text{and} \quad \psi_N(t) \not\equiv 0. \tag{6.10}$$

The asymptotic expansions (6.5) and (6.6) of $H(t)$ and, respectively, $J(t)$ now become

$$H(t) = \phi_N(t)e^{-Nt} + O(e^{-(N+\varepsilon)t}), \quad J(t) = \psi_N(t)e^{-Nt} + O(e^{-(N+\varepsilon)t}), \tag{6.11}$$

for some $\varepsilon > 0$. Therefore

$$\lim_{t\to\infty} \frac{J(t)}{H(t)} = \lim_{t\to\infty} \frac{\psi_N(t)}{\phi_N(t)} = \frac{N}{2}, \tag{6.12}$$

by (6.8). Thus, if we denote $d = deg\ \phi_N(t)$, then $d = deg\ \psi_N(t)$ and from (6.11), we have

$$\lim_{t\to\infty} \frac{H(t)}{t^d e^{-Nt}} \quad \text{and} \quad \lim_{t\to\infty} \frac{J(t)}{t^d e^{-Nt}} \quad \text{exist and are nonzero.} \tag{6.13}$$

A priori, it may happen that the limit number $N/2$ is not an integer and hence is not an eigenvalue of A. However, a more careful analysis will restrict the value $N/2$ to integers only. Using the results from Section F of Chapter II, we have the following supplementary property of the expansions (6.5) and (6.6).

Lemma VI.4. *For odd numbers n, the polynomials ϕ_n and ψ_n are identically zero.*

Proof. Suppose that n is odd, consider $\phi_n = \sum_{j+l=n} \langle Tq_j, q_l \rangle$. For each term of the sum, Lemma II.4 implies that $q_l \in \mathcal{F}_l$, $q_j \in \mathcal{F}_j$ and hence $Tq_j \in \mathcal{F}_j$ by the virtue of Property (c) after Definition II.1. Since $j+l=n$ is odd or equivalently $j \not\equiv l (mod\, 2)$, Property (e) after Definition II.1 yields $\langle Tq_j, q_l \rangle = 0$. It follows that $\phi_n = 0$. A similar argument applies to the polynomials ψ_n. □

The preceding lemma has the following direct consequence.

Corollary VI.2. *The integer N in* (6.10) *is even.*

We can sum up the discussions above as follows.

Proposition VI.1. *Let $u(t)$ be a regular solution of* (2.12) *such that its helicity $H(t)$ has a nonzero asymptotic expansion. Then there exist integers $d \geq 0$ and $h_0 > 0$ such that*

$$\lim_{t\to\infty} \frac{J(t)}{H(t)} = h_0 \tag{6.14}$$

and

$$\lim_{t\to\infty} \frac{H(t)e^{2h_0 t}}{t^d} \quad \text{exists and is nonzero.} \tag{6.15}$$

Proof. Set $N = 2h_0$ in (6.10), then (6.14) coincides with (6.12), while (6.15) is already contained in (6.13). Finally, the fact that h_0 is an integer was established in Corollary VI.2. □

When the helicity has identically zero asymptotic expansion, it turns out to be the zero function thanks to its large analytic domain and the very fast decay along the positive real axis.

Proposition VI.2. *Let $u(t), t \in [0,\infty)$, be a regular solution of* (2.12). *Then the asymptotic expansion* (6.5) *of its helicity $H(t) = \langle Tu(t), u(t) \rangle$ is identically zero if and only if $H(t)$ is identically zero.*

Proof. We prove the necessary condition. Assume that $u(t) \neq 0$ for all $t > 0$. Suppose all polynomials $\phi_n(t)$ in the asymptotic expansion (6.5) of the helicity are identically zero. This implies that

$$H(t) = o(e^{-nt}) \text{ as } t \to \infty, \text{ for all } n \in \mathbb{N}. \tag{6.16}$$

Let $H(\zeta), \zeta \in \mathcal{D}$, be the analytic extension of the real helicity $H(t), t > 0$. More precisely, if

$$u(\zeta) = u_1(\zeta) + iu_2(\zeta), \quad u_1(\zeta), u_2(\zeta) \in V, \quad \zeta \in \mathcal{D},$$

then

$$H(\zeta) = \langle Tu_1(\zeta), u_1(\zeta)\rangle - \langle Tu_2(\zeta), u_2(\zeta)\rangle + i[\langle Tu_1(\zeta), u_2(\zeta)\rangle + \langle Tu_2(\zeta), u_1(\zeta)\rangle]. \tag{6.17}$$

Let $t_0 > 0$ such that $\|u(t_0)\| \leq \big(8(2c_0)^{1/4}\big)^{-1}$. By Proposition III.1, $H(\zeta)$ is analytic in an open set

$$\{t_0 + \zeta \ , \ \zeta \in D(c, 1/8)\},$$

where

$$c = \big(4\sqrt{2}(2c_0)^{1/4}\|u(t_0)\|\big)^{-1} \geq \sqrt{2}.$$

It follows that $H(t_0 + \zeta)$ is bounded and analytic in $D(\sqrt{2}, 1/8)$, and satisfies

$$\lim_{t\to\infty} e^{nt}|H(t + t_0)| = 0,$$

for all $n > 0$. By Corollary III.2, we have $H(t) = 0$ for all $t > t_0$. Hence $H(t) = 0$ for all $t > 0$ by its analyticity on an open set containing $(0, \infty)$ (see Proposition III.2). The continuity of $H(t)$ at $t = 0$ implies $H(0) = 0$. □

We combine Propositions VI.1 and VI.2 to the following theorem.

Theorem VI.1. *The helicity $H(t)$ of a regular solution to* (2.12) *is either* *i) eventually nonzero and decaying when t goes to infinity as $t^d e^{-2h_0 t}$, where $d \geq 0$ and $h_0 > 0$ are integers depending on the solution and*

$$\lim_{t\to\infty} \frac{J(t)}{H(t)} = h_0; \tag{6.18}$$

or

ii) identically zero.

Definition VI.1. By virtue of Theorem VI.1, we can write $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$, where $\mathcal{R}_0 \cap \mathcal{R}_1 = \emptyset$, $\mathcal{R}_0$ is the set of all data in $\mathcal{R}$ such that the helicity is identically zero for all times, and $\mathcal{R}_1$ is the set of the data in $\mathcal{R}$ such that the helicity is eventually nonzero. Note that $\mathcal{R}_0$ and $\mathcal{R}_1$ are invariant in the following sense. A subset $\mathcal{S}$ of $\mathcal{R}$ is called *invariant* if for any $u^0 \in \mathcal{S}$, the regular solution $u(t)$ of (2.12) stays in $\mathcal{S}$ for all time $t \geq 0$.

C. On the flows with zero helicity

In this section, we will prove that the set $\mathcal{R}_0$ of all initial data in $\mathcal{R}$ for which the solutions have identically zero helicity is a closed, rich but nowhere dense subset of $\mathcal{R}$ endowed with the topology of V.

Theorem VI.2. *$\mathcal{R}_1$ is open in V and dense in $\mathcal{R}$, while $\mathcal{R}_0$ is closed and consists of an infinite union of invariant closed linear manifolds of infinite dimensions.*

Proof. First, let us recall that

$$\mathcal{R} \text{ is open in } V. \tag{6.19}$$

The reason is that for any $u^0 \in \mathcal{R}$, we know that $\|u(t)\| = O(e^{-n_0 t})$ as $t \to \infty$ for some $n_0 > 0$. Therefore $\int_0^\infty \|u(t)\|^4 dt < \infty$, and hence, there is a neighborhood of u^0 in V which is also a subset of $\mathcal{R}$ (see, for example, [25]).

We now show that $\mathcal{R}_0$ is closed in $\mathcal{R}$. Suppose that $u^0 \in \mathcal{R}$ and there is a sequence $(u_n^0)_{n\in\mathbb{N}}$ in $\mathcal{R}_0$ such that $\lim_{n\to\infty} \|u_n^0 - u^0\| = 0$. Let $u(t)$ and $u_n(t)$ be the regular solutions to (2.12) with initial data u^0 and, respectively, u_n^0. Again, in our case of potential forces,

$$\int_0^\infty \|u(t)\|^4 dt < \infty. \tag{6.20}$$

For large n, the norm difference $\|u_n^0 - u^0\|$ is so small that we have (cf. [25])

$$\|u_n(t) - u(t)\| \le \|u_n^0 - u^0\| e^{-M_1 t/2} e^{M(u^0) M_2},$$

for some constants $M(u^0), M_1, M_2 > 0$ independent of n. Thus for each $t > 0$,

$$\lim_{n\to\infty} \|u_n(t) - u(t)\| = 0.$$

Since $u_n^0 \in \mathcal{R}_0$, $\langle Tu_n(t), u_n(t)\rangle = 0$ for all $t \ge 0$. For each $t > 0$,

$$H(t) = \langle Tu(t), u(t)\rangle = \lim_{n\to\infty} \langle Tu_n(t), u_n(t)\rangle = 0.$$

This implies that $u^0 \in \mathcal{R}_0$, and hence $\mathcal{R}_0$ is closed in $\mathcal{R}$. Since $\mathcal{R}_0$ is closed in $\mathcal{R}$, its complement $\mathcal{R}_1$ is open in $\mathcal{R}$ and hence (due to (6.19)) in V.

We next prove that $\mathcal{R}_1$ is dense in $\mathcal{R}$. Given $u^0 \in \mathcal{R}_0$, the corresponding solution $u(t)$ of (2.12) satisfies $\langle Tu(t), u(t)\rangle = 0$ for all $t \ge 0$, in particular, $\langle Tu^0, u^0\rangle = 0$. If $u^0 = 0$, we can take $u_\varepsilon^0 = \varepsilon v_0$ where $\varepsilon > 0$ and $Tv_0 = \mu v_0 \ne 0$ with some $\mu \in \sigma(T)$ (see Lemma II.1). The solution of (2.12) with initial condition u_ε^0 is $u_\varepsilon(t) = \varepsilon e^{-\mu^2 t} v_0$. The helicity is

$$H_\varepsilon(t) = \langle Tu_\varepsilon(t), u_\varepsilon(t)\rangle = \varepsilon\mu |v_0|^2 e^{-2\mu^2 t} \ne 0.$$

Hence $u_\varepsilon^0 \in \mathcal{R}_1$ and $\|u_\varepsilon^0 - u^0\| = \|u_\varepsilon^0\| = \varepsilon \|v_0\|$ which goes to zero as ε goes to zero.

In the case when $u^0 \neq 0$, then $u^0(x)$ has the Fourier series

$$u^0(x) \sim \sum_{k\in\mathbb{Z}^3} a_k e^{ik\cdot x} \in V, \qquad \text{where} \qquad a_k = \frac{1}{L^3}\int_Q u^0(x)e^{-ik\cdot x}dx$$

and there is $k^0 \in \mathbb{Z}^3 \setminus \{0\}$ such that $a_{k^0} \neq 0$. Given $\varepsilon > 0$, consider the perturbed initial condition

$$u^0_\varepsilon(x) = u^0(x) + (be^{ik^0\cdot x} + b^* e^{-ik^0\cdot x}),$$

where $b = b(\varepsilon) = b^1 + ib^2$ with $b^1, b^2 \in \mathbb{R}^3$ specified later satisfying $k^0 \cdot b = 0$. By denoting $a^1_k = (a_k + a^*_k)/2$ and $a^2_k = (a_k - a^*_k)/2i$ for all $k \in \mathbb{Z}^3$, from Formula (2.48) for the helicity, we infer that

$$\begin{aligned}
H_\varepsilon(0) = \langle Tu^0_\varepsilon, u^0_\varepsilon\rangle &= 2L^3 \sum_{k\neq k^0,-k^0} k\cdot(a^1_k \times a^2_k) + 4L^3 k^0\cdot(a^1_{k^0} + b^1)\times(a^2_{k^0} + b^2) \\
&= \langle Tu^0, u^0\rangle + 4L^3 k^0 \cdot (a^1_{k^0} \times b^2 + b^1 \times a^2_{k^0} + b^1 \times b^2) \\
&= 4L^3 k^0 \cdot (a^1_{k^0} \times b^2 + b^1 \times a^2_{k^0} + b^1 \times b^2).
\end{aligned}$$

If $a^1_{k^0} \neq 0$, we choose $b^1 = 0$ and $b^2 = \varepsilon k^0 \times a^1_{k^0}$. Then $|b| = \varepsilon|k^0||a^1_{k^0}| \leq \varepsilon|k^0||a_{k^0}|$,

$$a^1_{k^0} \times b^2 + b^1 \times a^2_{k^0} + b^1 \times b^2 = a^1_{k^0} \times b^2 = \varepsilon k^0 |a^1_{k^0}|^2 - a^1_{k^0}(k^0 \cdot a^1_{k^0}) = \varepsilon k^0 |a^1_{k^0}|^2$$

and

$$H_\varepsilon(0) = 4L^3\varepsilon|k^0|^2|a^1_{k^0}|^2 \neq 0.$$

If $a^1_{k^0} = 0$, we must have $a^2_{k^0} \neq 0$, and consequently we can choose $b^2 = 0$ and $b^1 = \varepsilon k^0 \times a^2_{k^0}$. Then $|b| = \varepsilon|k^0||a^2_{k^0}| \leq \varepsilon|k^0||a_{k^0}|$,

$$a^1_{k^0} \times b^2 + b^1 \times a^2_{k^0} + b^1 \times b^2 = b^1 \times a^2_{k^0} = -\varepsilon k^0 |a^2_{k^0}|^2$$

and

$$H_\varepsilon(0) = -4L^3\varepsilon|k^0|^2|a^2_{k^0}|^2 \neq 0.$$

Therefore, in both cases (i.e., regardless if $a^1_{k^0}$ is zero or not), we have

$$H_\varepsilon(0) \neq 0 \quad \text{and} \quad |b| \leq \varepsilon |k^0||a_{k^0}| \quad \text{for all } \varepsilon > 0. \tag{6.21}$$

Moreover, for the difference between the two initial conditions we have

$$\|u^0_\varepsilon - u^0\|^2 \leq 2L^3 |k^0|^2 |b|^2 \leq 2\varepsilon^2 |k^0|^4 |a_{k^0}|^2 \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.22}$$

From (6.19) and (6.22), we have $u^0_\varepsilon \in \mathcal{R}$ for ε is small enough, hence the corresponding perturbed solution $u_\varepsilon(t)$ exists for all time $t \geq 0$ and the first property in (6.21) shows that $u^0_\varepsilon \in \mathcal{R}_1$. We have proved that in any neighborhood (in V) of $u^0 \in \mathcal{R}_0$, there exists some $u^0_\varepsilon \in \mathcal{R}_1$, therefore $\mathcal{R}_1$ is a dense subset of $\mathcal{R}$.

To complete our proof we must show that $\mathcal{R}_0$ contains many infinite dimensional invariant linear manifolds. We will present three different types of such manifolds. Our first example is the class of 2D solutions to the 3D Navier–Stokes equations. We define

$$\mathcal{M}^{2D}_{1,2} = \{u \in V; u(x) \sim \sum_{k \in \mathbb{Z}\times\mathbb{Z}\times\{0\}} a_k e^{ik\cdot x}, a_k \in \mathbb{C}\times\mathbb{C}\times\{0\}\}, \tag{6.23}$$

where $\mathbb{Z}\times\mathbb{Z}\times\{0\} = \{(k_1,k_2,0)/k_1,k_2 \in \mathbb{Z}\}$ and $\mathbb{C}\times\mathbb{C}\times\{0\} = \{(z_1,z_2,0)/z_1,z_2 \in \mathbb{C}\}$. From the theory of 2D Navier–Stokes equations, for $u^0 \in \mathcal{M}^{2D}_{1,2}$ the corresponding regular solution $u(t)$ exists for all time $t \geq 0$ and belongs to $\mathcal{M}^{2D}_{1,2}$, hence $\mathcal{M}^{2D}_{1,2}$ is invariant. Note that in this case $Tu(x,t) \in \mathbb{R}e_3$ which is orthogonal to $u(x,t) \in (\mathbb{R}e_1 + \mathbb{R}e_2)$ as vectors in $\mathbb{R}^3$. Therefore,

$$H(t) = \int_\Omega Tu(x,t)\cdot u(x,t)dx = 0, \quad t \geq 0,$$

and hence $\mathcal{M}^{2D}_{1,2} \subset \mathcal{R}_0$. We can also equally define the other two similar manifolds, namely,

$$\mathcal{M}_{1,3}^{2D} = \{u \in V; u(x) \sim \sum_{k \in \mathbb{Z} \times \{0\} \times \mathbb{Z}} a_k e^{ik \cdot x},\ a_k \in \mathbb{C} \times \{0\} \times \mathbb{C}\} \tag{6.24}$$

and

$$\mathcal{M}_{2,3}^{2D} = \{u \in V; u(x) \sim \sum_{k \in \{0\} \times \mathbb{Z} \times \mathbb{Z}} a_k e^{ik \cdot x},\ a_k \in \{0\} \times \mathbb{C} \times \mathbb{C}\}. \tag{6.25}$$

Then $\mathcal{M}_{1,3}^{2D}$ and $\mathcal{M}_{2,3}^{2D}$ are invariant submanifolds of $\mathcal{R}_0$.

The second example is an extension of the family of manifolds given in Remark 7 of [7]. Let $\vec{a}$ be a vector in $\mathbb{R}^3$ such that its orthogonal plane has nontrivial intersection with $\mathbb{Z}^3$, this means

$$\vec{a}^{\perp} := \{k \in \mathbb{Z}^3, k \cdot \vec{a} = 0\} \neq \{0\}.$$

Define the linear manifold $\mathcal{M}_{\vec{a}^{\perp}}$ in V as

$$\mathcal{M}_{\vec{a}^{\perp}} = \{u \in V : u(x) \sim \sum_{k \in \vec{a}^{\perp}} a_k e^{ik \cdot x},\ a_k \text{ is (complex) colinear to } \vec{a},\ \text{for all } k \in \vec{a}^{\perp}\}. \tag{6.26}$$

For $u^0 \in \mathcal{M}_{\vec{a}^{\perp}}$, $u(t) = e^{-tA}u^0$ belongs to $\mathcal{M}_{\vec{a}^{\perp}}$ for all $t \in [0, \infty)$ and solves the linearized Navier–Stokes equations

$$\begin{cases} \dfrac{du}{dt} + Au = 0, \quad t > 0, \\ u(0) = u^0 \in V. \end{cases} \tag{6.27}$$

as well as the Navier–Stokes equations (2.12) (the nonlinear term $B(u, u)$ vanishes according to Remark II.2). Also, in view of Remark II.1, $\langle Tu, u\rangle = 0$ for all $u \in \mathcal{M}_{\vec{a}^{\perp}}$, hence $H(t) = \langle Tu(t), u(t)\rangle = 0$ for all $t \geq 0$. Therefore, $u^0 \in \mathcal{R}_0$ and $\mathcal{M}_{\vec{a}^{\perp}}$ is an invariant linear manifold in $\mathcal{R}_0$. □

Before moving on to the next manifold, let us state a simple lemma on closed linear subspaces of H and invariant sets (see Definition VI.1).

Lemma VI.5. *Let X be a closed linear subspace of H such that $P_n u \in X$ for all $u \in X,\ n \in \mathbb{N}$, where $P_n = R_1 + R_2 + \cdots + R_n$ and $Au, B(u,u) \in X$ for all $u \in X \cap \mathcal{V}$. Then $X \cap \mathcal{R}$ is invariant.*

Proof. Let u^0 be the initial condition in $X \cap \mathcal{R}$ and $u(t)$ be the corresponding regular solution. For each $n \in \mathbb{N}$, we consider the Galerkin approximation of the Navier–Stokes equations on the finite dimensional subspace $P_n X \subset \mathcal{V}$ with initial condition $P_n u^0 \in P_n X$. Since $Av, P_n B(v,v) \in P_n X$ for $v \in P_n X$, this approximation problem has a unique solution $u_n(t) \in P_n X,\ t \in [0,T]$, for some $T > 0$. From the theory of the Navier–Stokes equations, there is a subsequence $\{u_{n'}\}$ such that

$$\lim_{n' \to \infty} u_{n'}(t) = u(t) \text{ in H for all most every } t \in (0,T). \tag{6.28}$$

Since X is closed in H, we have $u(t) \in X$ for all most every $t \in (0,T)$. By the strong continuity (in H) of the regular solution, $u(t) \in X$ for all $t \in [0,T)$. Because $u(t)$ is a global solution, we can take $T = \infty$. □

Proof of Proposition VI.2 (continued). The last manifold we present in this proof is the one formed by anti-symmetric (odd) functions $u \in V$. Namely, define

$$\begin{aligned} H_{odd} &= \{u \in H; u(-x) = -u(x),\ \text{a.e. in } \mathbb{R}^3\} \\ &= \{u \in H\ ;\ u \sim \sum_{k \in \mathbb{Z}^3} a_k e^{ik \cdot x},\ a_k \in i\mathbb{R}^3 \text{ for all } k \in \mathbb{Z}^3\}. \end{aligned} \tag{6.29}$$

For each $u \in H_{odd} \cap \mathcal{V}$, using (2.43)-(2.46) one can verify that $Au, B(u,u) \in H_{odd}$. It is also clear from (6.29) that H_{odd} is a closed linear subspace of H and $P_n u \in H_{odd}$ for all $n \in \mathbb{N}, u \in H_{odd}$. By the virtue of Lemma VI.5, we have

$$H_{odd} \cap \mathcal{R} \text{ is invariant.} \tag{6.30}$$

Moreover, if $u \in H_{odd} \cap V$, then it follows from (6.29) and Remark II.1 that $\langle Tu, u \rangle =$

0. Therefore, due to (6.30),

$$H_{odd} \cap \mathcal{R} \subset \mathcal{R}_0. \tag{6.31}$$

Also, it is well-known that for $\delta > 0$ small enough and $\|u^0\| < \delta$, the solution $u(t)$ exists for all times $t \geq 0$ and $d\|u(t)\|^2/dt \leq 0$ for $t \geq 0$. Therefore, $\|u(t)\|^2 \leq \|u^0\|^2 < \delta^2$, for $t \geq 0$ and hence $\mathcal{M}^\delta = \{u \in V : \|u\| < \delta\}$ is invariant. Together with Properties (6.30) and (6.31) we infer that $\mathcal{M}^\delta_{odd} = H_{odd} \cap \mathcal{M}^\delta$ is a subset of $\mathcal{R}_0$ and invariant. □

Remark VI.1. The manifolds $\mathcal{M}^{2D}_{1,2}, \mathcal{M}^{2D}_{1,3}, \mathcal{M}^{2D}_{2,3}$ and $\mathcal{M}_{\vec{a}^\perp}$ above are closed linear subspaces of V, while $\mathcal{M}^\delta_{odd}$ is only a portion of the closed linear subspace $V \cap H_{odd}$. Clearly, these manifolds are of infinite dimension.

D. Rates of exponential decay of the helicity

In this section we will discuss in more details the possible values that N, h_0 and d (see Proposition VI.1) can take. This will give a clearer picture of the difference between the asymptotic behavior of the helicity and that of the energy of the flow. Recall that the helicity $H(t) = \langle Tu(t), u(t)\rangle$ has the asymptotic expansion (6.5). We consider the case when one of the polynomials $\phi_n(t)$ in this expansion is not identically zero. The previously obtained asymptotic properties of $H(t) = \langle Tu(t), u(t)\rangle$ and $J(t) = \langle T^2u(t), Tu(t)\rangle$ in this case are summarized in Proposition VI.1. We start with comparing the limits

$$h_0 = \lim_{t\to\infty} \frac{J(t)}{H(t)} \text{ in (6.14)}, \quad \lim_{t\to\infty} \frac{\|u(t)\|^2}{|u(t)|^2} = n_0 \in \sigma(A) \text{ in (2.17)}$$

and

$$\alpha = \lim_{t\to\infty} \frac{H(t)}{|u(t)|^2}.$$

The last limit exists due to the asymptotic expansions of the helicity and energy and its value is related to the normalization map $W(u^0)$ (see Chapter II).

Proposition VI.3. *Let $u^0 \in \mathcal{R} \setminus \{0\}$ and $u(t)$ be the corresponding regular solution and n_0 be defined as in* (2.17)*. Then*

$$\alpha = \lim_{t\to\infty} \frac{H(t)}{|u(t)|^2} = \frac{\langle TW_{n_0}(u^0), W_{n_0}(u^0)\rangle}{|W_{n_0}(u^0)|^2}. \tag{6.32}$$

Proof. From the general asymptotic expansion (6.2) of $u(t)$ and specific properties (2.17), (2.23) we know that $u(t)$ has the following asymptotic expansion in $H^m(\Omega)$ for $m = 0, 1, 2, ...$

$$u(t) \sim W_{n_0}(u^0)e^{-n_0 t} + q_{n_0+1}(t)e^{-(n_0+1)t} + ..., \quad \text{where } W_{n_0}(u^0) \in R_{n_0}H \setminus \{0\}. \tag{6.33}$$

This implies that

$$u(t) = W_{n_0}(u^0)e^{-n_0 t} + O(e^{-(n_0+\varepsilon)t}) \text{ in H, for some } \varepsilon > 0, \tag{6.34}$$

and

$$|u(t)|^2 = |W_{n_0}(u^0)|^2 e^{-2n_0 t} + O(e^{-(2n_0+\varepsilon)t}). \tag{6.35}$$

By the asymptotic expansion (6.5) of the helicity $H(t) = \langle Tu(t), u(t)\rangle$ and possibly some adjustment in the values of $\varepsilon > 0$ appearing in (6.33) and (6.34), we have

$$\langle Tu(t), u(t)\rangle = \langle TW_{n_0}(u^0), W_{n_0}(u^0)\rangle e^{-2n_0 t} + O(e^{-(2n_0+\varepsilon)t}). \tag{6.36}$$

Therefore,

$$\lim_{t\to\infty} \frac{\langle Tu(t), u(t)\rangle}{|u(t)|^2} = \frac{\langle TW_{n_0}(u^0), W_{n_0}(u^0)\rangle}{|W_{n_0}(u^0)|^2}.$$

This completes our proof. □

We notice that

$$\frac{|H(t)|}{|u(t)|^2} \leq \frac{|Tu(t)||u(t)|}{|u(t)|^2} = \frac{\|u(t)\|}{|u(t)|} \to \sqrt{n_0} \quad \text{as } t \to \infty.$$

Hence $|\alpha| \leq \sqrt{n_0} \in \sigma(T)$. In the sequel, we will present some samples of solutions with different relations between the limits h_0, n_0 and α.

The first example will show that n_0 and h_0 can be any eigenvalue of A while α can attain the eigenvalues $\pm\sqrt{n_0}$ of T.

Example VI.1. Let $n \in \sigma(A)$. Using Lemma II.1, we can find $u^0_\pm \in R_n H \setminus \{0\}$ such that $Tu^0_\pm = \pm\sqrt{n} u^0_\pm$. By (2.8) one has $B(u^0_\pm, u^0_\pm) = 0$ and the solution to (2.12) is

$$u_\pm(t) = e^{-nt} u^0_\pm, \quad t \geq 0.$$

Therefore

$$|u_\pm(t)|^2 = e^{-2nt}|u^0_\pm|^2, \quad \|u_\pm(t)\|^2 = ne^{-2nt}|u^0_\pm|^2,$$

$$H^\pm(t) = \langle Tu_\pm(t), u_\pm(t)\rangle = \pm\sqrt{n}e^{-2nt}\langle u^0_\pm, u^0_\pm\rangle = \pm\sqrt{n}|u_\pm(t)|^2$$

and

$$J^\pm(t) = \langle T^2 u_\pm(t), Tu_\pm(t)\rangle = \pm n\sqrt{n}e^{-2nt}\langle u^0_\pm, u^0_\pm\rangle = nH^\pm(t).$$

Hence the corresponding limits $n^\pm_0, h^\pm_0$ and $\alpha^\pm$ above are given by ${n_0}^\pm = {h_0}^\pm = n \in \sigma(A)$ and $\alpha^\pm = \pm\sqrt{n} = \pm\sqrt{{n_0}^\pm} \in \sigma(T)$. □

Moreover, the following theorem describes all the possible values of α.

Theorem VI.3. *Let $u^0 \in \mathcal{R} \setminus \{0\}$ and let $u(t)$ be the regular solution of* (2.12) *and let n_0 be defined by* (2.17) *then*

$$\lim_{t\to\infty} \frac{\langle Tu(t), u(t)\rangle}{|u(t)|^2} = \alpha \quad \text{and} \quad \alpha \in [-\sqrt{n_0}, \sqrt{n_0}]. \tag{6.37}$$

Moreover, for any $n \in \sigma(A)$ these limits α cover $[-\sqrt{n}, \sqrt{n}]$.

Proof. Let $u^0, u(t), n_0$ be as in Theorem VI.3. By Proposition VI.3, we have

$$\alpha = \lim_{t\to\infty} \frac{H(t)}{|u(t)|^2} = \frac{\langle TW_{n_0}(u^0), W_{n_0}(u^0)\rangle}{|W_{n_0}(u^0)|^2}. \tag{6.38}$$

Note that,

$$\frac{|\langle TW_{n_0}(u^0), W_{n_0}(u^0)\rangle|}{|W_{n_0}(u^0)|^2} \leq \frac{|TW_{n_0}(u^0)||W_{n_0}(u^0)|}{|W_{n_0}(u^0)|^2} = \frac{\|W_{n_0}(u^0)\|}{|W_{n_0}(u^0)|} = \sqrt{n_0},$$

since $W_{n_0}(u^0) \in R_{n_0}H$. Hence $\alpha \in [-\sqrt{n_0}, \sqrt{n_0}]$.

We now fix $n \in \sigma(A)$ and $\alpha' \in [-\sqrt{n}, \sqrt{n}]$. Let $k \in \mathbb{Z}^3$ satisfy $|k|^2 = n$. Let a_k^1 and a_k^2 be two real vectors in the plane perpendicular to k such that $|a_k^1| = |a_k^2| \neq 0$. Moreover, since $\alpha'/\sqrt{n} \in [-1, 1]$, we can adjust the angle between a_k^1 and a_k^2 to have

$$k \cdot (a_k^2 \times a_k^2) = \frac{\alpha'}{\sqrt{n}}|k||a_k^1||a_k^2| = \alpha'|a_k^1|^2.$$

Set $a_k = a_k^1 + i a_k^2$ and $u^0 = a_k e^{ik\cdot x} + a_k^* e^{-ik\cdot x}$. Note that $u^0 \in R_n H$ and that $B(u^0, u^0) = 0$, by Remark II.2. Therefore the solution of the Navier–Stokes equations (2.12) in this case is

$$u(t) = e^{-nt}u^0, \quad t \geq 0.$$

By (2.48), the quotient between the helicity and the energy is

$$\frac{H(t)}{|u(t)|^2} = \frac{\langle Tu^0, u^0\rangle}{|u^0|^2} = \frac{4L^3 k\cdot(a_k^1 \times a_k^2)}{2L^3|a_k|^2} = \frac{4L^3\alpha'|a_k^1|^2}{4L^3|a_k^1|^2} = \alpha'.$$

Letting $t \to \infty$ gives $\alpha = \alpha'$. □

Remark VI.2. Let n_0 and N be defined in (2.17) and (6.10), respectively. The asymptotic expansion (6.36) of the helicity shows that $N \geq 2n_0$. From the proof of Proposition VI.1, the limit $h_0 = \lim_{t\to\infty} J(t)/H(t)$ is exactly $N/2$, hence $h_0 \geq n_0$. On the other hand, if $h_0 = n_0$ we have $N = 2n_0$ and therefore in the expansion (6.36), $\langle TW_{n_0}(u^0), W_{n_0}(u^0)\rangle \neq 0$. Then (6.32) implies that $\alpha \neq 0$.

The next example shows that there are initial data u^0 for which the limits h_0 are no more in the spectrum of A although we always have $n_0 \in \sigma(A)$.

Example VI.2. There is $u^0 \in \mathcal{R}$ such that the corresponding limit h_0 in (6.14) is 7 which does not belong to the spectrum of A.

Proof. Part I. We choose $\xi_4 \in R_4 H$ and $\xi_5 \in R_5 H$ such that $|\xi_4|$ and $|\xi_5|$ are very small satisfying

$$\langle T\xi_4, \xi_4\rangle = 0 = \langle T\xi_5, \xi_5\rangle \quad \text{and} \quad \langle T\xi_4, B(\xi_5, \xi_5)\rangle \neq 0. \tag{6.39}$$

A concrete choice of such ξ_4 and ξ_5 will be given in Part II of this proof. Let $\xi = (0, 0, 0, \xi_4, \xi_5, 0, 0, ..., 0) \in \Pi_{14} S_A$. Since ξ is small, (2.22) implies that $\xi = \Pi_{14} W(u^0)$ for some $u^0 \in \mathcal{R}$. Trivial calculations using (2.26) and (2.27) yield

$$q_1 = 0, \quad \beta_1 = \beta_2 = 0, \quad q_2 = 0, \quad \beta_3 = 0, \quad q_3 = 0, \quad \beta_4 = 0, \tag{6.40}$$

$$q_4 = \xi_4, \quad \beta_5 = 0, \quad q_5 = \xi_5, \quad \beta_6 = 0, \tag{6.41}$$

$$q_6 = 0, \quad \beta_7 = 0, \quad q_7 = 0 \tag{6.42}$$

By Lemma VI.4, for odd integers n, the polynomials ϕ_n in the asymptotic expansion (6.5) of $H(t)$ are identically zero. Concerning the polynomials ϕ_n for even n, from (6.40)-(6.42) we easily obtain that

$$\phi_2 = \phi_4 = \phi_6 = 0, \tag{6.43}$$

as well as

$$\phi_8 = \langle Tq_4, q_4\rangle + 2 \sum_{k<4, k+l=8} \langle Tq_k, q_l\rangle = \langle Tq_4, q_4\rangle = \langle T\xi_4, \xi_4\rangle = 0, \tag{6.44}$$

$$\phi_{10} = \langle Tq_5, q_5\rangle + 2\langle Tq_4, q_6\rangle + 2 \sum_{k<4, k+l=10} \langle Tq_k, q_l\rangle = \langle Tq_5, q_5\rangle = \langle T\xi_5, \xi_5\rangle = 0, \tag{6.45}$$

where we also used the condition (6.39). Since $\xi_4 \in R_4H$, the Fourier expansion (2.40) of $u = \xi_4$ is $\xi_4(x) = \sum_{|k|=2} a_k e^{ik\cdot x}$. Therefore, by (2.44), we have

$$\beta_8(x) = B(q_4, q_4) = B(\xi_4, \xi_4) = \sum_m B_m e^{im\cdot x},$$

where the summation is over all $m = k + k'$ for $k, k' \in \mathbb{Z}^3, |k| = |k'| = 2, k \neq -k'$. Moreover, for each k as of above, Remark II.2 implies that $Q_{2k} = B_{2k} = 0$. Therefore, $B_m \neq 0$ only if m belongs to the set

$$\begin{aligned} E &= \{k \pm k'; k, k' \in \mathbb{Z}^3, |k| = |k'| = 2, k \neq \pm k'\} \\ &= \{(\epsilon 2, \epsilon' 2, 0), (0, \epsilon 2, \epsilon' 2), (\epsilon 2, 0, \epsilon' 2); \epsilon, \epsilon' = 1, -1\}. \end{aligned}$$

Thus $\beta_8(x) = \sum_{m\in E} B_m e^{im\cdot x}$ and $|m|^2 = 8$ for each $m \in E$, hence $\beta_8 \in R_8H$ and $q_8(t) = -t\beta_8$, by (2.27). Because $q_4 \in R_4H$ and $q_8 \in R_8H$, $\phi_{12} = 2\langle Tq_4, q_8\rangle = 0$. From the relations (2.26), (2.27) and (2.28) one can readily infer that

$$\beta_9 = B(q_5, q_4) + B(q_4, q_5), \tag{6.46}$$

$$q_9(t) = -tR_9\beta_9 - [(A-9)(I-R_9)]^{-1}(I-R_9)\beta_9, \tag{6.47}$$

$$\beta_{10} = B(q_5, q_5) = B(\xi_5, \xi_5), \tag{6.48}$$

$$q_{10}(t) = -tR_{10}\beta_{10} - [(A-10)(I-R_{10})]^{-1}(I-R_{10})\beta_{10}, \tag{6.49}$$

and consequently

$$\phi_{14} = 2(Tq_5, q_9) + 2(Tq_4, q_{10}), \tag{6.50}$$

by (6.5). Now, using (6.47),(6.46), (2.28) and (2.11), we obtain

$$\begin{aligned} \langle Tq_5, q_9\rangle &= \langle T\xi_5, -[(A-9)(I-R_9)]^{-1}(I-R_9)\beta_9\rangle \\ &= \langle T\beta_9, -[(A-9)(I-R_9)]^{-1}(I-R_9)\xi_5\rangle = \frac{1}{4}\langle T\xi_5, \beta_9\rangle \\ &= \frac{1}{4}\langle T\xi_5, B(\xi_4, \xi_5) + B(\xi_5, \xi_4)\rangle = -\frac{1}{4}\langle T\xi_4, B(\xi_5, \xi_5)\rangle, \end{aligned}$$

and similarly,

$$\begin{aligned}\langle Tq_4, q_{10}\rangle &= \langle T\xi_4, -[(A-10)(I-R_{10})]^{-1}(I-R_{10})\beta_{10}\rangle \\ &= \frac{1}{6}\langle T\xi_4, \beta_{10}\rangle = \frac{1}{6}\langle T\xi_4, B(\xi_5,\xi_5)\rangle.\end{aligned}$$

Thus (6.50) finally gives us

$$\phi_{14} = 2(-\frac{1}{4}+\frac{1}{6})\langle T\xi_4, B(\xi_5,\xi_5)\rangle = -\frac{1}{6}\langle T\xi_4, B(\xi_5,\xi_5)\rangle,$$

which is not zero by our condition (6.39). We conclude that the number N identified in (6.10) is 14 and it follows from (6.12) that $h_0 = \lim_{t\to\infty}\dfrac{J(t)}{H(t)} = N/2 = 7$ which is not an eigenvalue of the Stokes operator A.

Part II. Fix $\varepsilon > 0$, let $k_1 = (0,2,0)$, $a_{k_1} = \varepsilon(1,0,0)$ and define

$$\xi_4 = a_{k_1}(e^{ik_1\cdot x} + e^{-ik_1\cdot x}) \in R_4H. \tag{6.51}$$

We choose $m_1 = (2,1,0)$, $m_2 = (-2,1,0)$, $b_{m_1} = \varepsilon(-1,2,0)$, $b_{m_2} = \varepsilon(0,0,1)$ and let

$$\xi_5 = b_{m_1}(e^{im_1\cdot x} + e^{-im_1\cdot x}) + b_{m_2}(e^{im_2\cdot x} + e^{-im_2\cdot x}) \in R_5H. \tag{6.52}$$

Since the Fourier coefficients $a_{k_1}, b_{m_1}, b_{m_2}$ belong to $\mathbb{R}^3$, we have

$$\langle T\xi_4, \xi_4\rangle = 0 = \langle T\xi_5, \xi_5\rangle, \tag{6.53}$$

due to Remark II.1. It is also obvious that $|\xi_4|$ and $|\xi_5|$ can be made as small as necessary in Part I by choosing $\varepsilon > 0$ suitably small. By writing $B(\xi_5,\xi_5) = \sum B_m e^{im\cdot x}$ and using (2.47) for $u = \xi_4$ and $v = B(\xi_5,\xi_5)$ we obtain

$$\langle T\xi_4, B(\xi_5,\xi_5)\rangle = 2L^3 Re[i(k_1 \times a_{k_1})\cdot B^*_{k_1}]. \tag{6.54}$$

Note that $m_1 + m_2 = k_1$ and by (2.45) and (2.46)

$$B_{k_1} = Q_{k_1} - \frac{Q_{k_1} \cdot k_1}{|k_1|^2} k_1,$$

where

$$Q_{k_1} = i[(b_{m_1} \cdot m_2) b_{m_2} + (b_{m_2} \cdot m_1) b_{m_1}].$$

Therefore,

$$\begin{aligned}\langle T\xi_4, B(\xi_5, \xi_5)\rangle &= 2L^3 Re[i(k_1 \times a_{k_1}) \cdot Q_{k_1}^*] \\ &= 2L^3 (k_1 \times a_{k_1}) \cdot [(b_{m_1} \cdot m_2) b_{m_2} + (b_{m_2} \cdot m_1) b_{m_1}].\end{aligned}$$

Finally, we have $k_1 \times a_{k_1} = -2\varepsilon(0, 0, 1)$, and consequently also

$$\langle T\xi_4, B(\xi_5, \xi_5)\rangle = -4L^3\varepsilon^3(-1, 2, 0) \cdot (-2, 1, 0) = -16L^3\varepsilon^3.$$

Hence $\langle T\xi_4, B(\xi_5, \xi_5)\rangle$ is not zero. With this and (6.53), we conclude that ξ_4 and ξ_5 specified in (6.51) and, respectively, (6.52) also satisfy Condition (6.39). □

As far as the asymptotic behaviors (as $t \to \infty$) of $J(t)$, $H(t)$ and $|u(t)|^2$ are concerned in the case $u^0 \in \mathcal{R}_1$, we know from (6.14) of Proposition VI.1 that those of $H(t)$ and $J(t)$ are essentially the same. Also in the case when the limit α in (6.37) is not zero, the same is true for the asymptotic behaviors of the helicity $H(t)$ and the energy $|u(t)|^2$. We will pay more attention to the case $\alpha = 0$, for $u^0 \in \mathcal{R}_1$. We recall from (2.17) and (6.15) that as $t \to \infty$, $|u(t)|^2$ behaves like $e^{-2n_0 t}$, where $n_0 \in \sigma(A)$ while $H(t)$ behaves like $t^d e^{-2h_0 t}$. Example VI.2 shows that we can find $u^0 \in \mathcal{R}_1$ such that $\lim_{t\to\infty} H(t)e^{2h_0 t}$ exists and is not zero, whereas $h_0 = 7 \notin \sigma(A)$. This means that the exponents in the exponential decay of $H(t)$ and $|u(t)|^2$ are quite different. The next example shows that the difference goes farther than that. More precisely, the degree d in (6.15) can be positive.

Example VI.3. There is $u^0 \in \mathcal{R}$ such that $\lim_{t\to\infty} H(t)t^{-1}e^{4t}$ exists and is not zero, i.e., $d = 1 > 0$.

Proof. Part I. We choose $\xi_1 \in R_1H$ and $\xi_2 \in R_2H$ satisfying the following conditions

$$\langle T\xi_1, \xi_1\rangle = 0 \quad \text{and} \quad \langle TB(\xi_1,\xi_1), B(\xi_1,\xi_1)\rangle = 0, \tag{6.55}$$

$$\langle T\xi_2, \xi_2\rangle = 0, \tag{6.56}$$

and

$$\langle T\xi_2, B(\xi_1,\xi_1)\rangle \neq 0. \tag{6.57}$$

We again will explicitly specify such ξ_1 and ξ_2 in the second part of this proof. By scaling ξ_1 and ξ_2, we can assume in addition to Conditions (6.55), (6.56) and (6.57) that $|\xi_1|$ and $|\xi_2|$ are small enough in order that $\xi = (\xi_1, \xi_2, 0, 0) \in \Pi_4 S_A$ should be in $\Pi_4 W(\mathcal{R})$ (see Property (2.22)). Then $\xi = \Pi_4 W(u^0)$ for some $u^0 \in \mathcal{R}$. Let $u(t)$ be the regular solution with the initial data u^0. The first three polynomials $q_n(t)$ in the asymptotic expansions (2.18) and (6.2) of $u(t)$ are calculated below by using (2.26)-(2.28).

$$q_1 = \xi_1, \quad \beta_2 = B(q_1, q_1) = B(\xi_1,\xi_1) \in R_2H, \tag{6.58}$$

$$q_2 = \xi_2 - t\beta_2 \in R_2H, \tag{6.59}$$

$$\begin{aligned}\beta_3 &= B(q_1,q_2) + B(q_2,q_1) = B(\xi_1, \xi_2 - t\beta_2) + B(\xi_2 - t\beta_2, \xi_1) \\ &= B(\xi_1,\xi_2) + B(\xi_2,\xi_1) - t[B(\xi_1,\beta_2) + B(\beta_2,\xi_1)] = \beta_3^0 + t\beta_3^1,\end{aligned} \tag{6.60}$$

where

$$\beta_3^0 = B(\xi_1,\xi_2) + B(\xi_2,\xi_1) \quad \text{and} \quad \beta_3^1 = -[B(\xi_1,\beta_2) + B(\beta_2,\xi_1)], \tag{6.61}$$

and

$$q_3 = -\int_0^t R_3\beta_3(\tau)d\tau - [(A-3)(I-R_3)]^{-1}(I-R_3)(\beta_3^0+t\beta_3^1) + [(A-3)(I-R_3)]^{-2}(I-R_3)\beta_3^1. \tag{6.62}$$

By Lemma VI.4, in the asymptotic expansion (6.5) of $H(t)$ we have

$$\phi_1 = \phi_3 = 0. \tag{6.63}$$

Condition (6.55) implies that

$$\phi_2 = \langle Tq_1, q_1\rangle = \langle T\xi_1, \xi_1\rangle = 0. \tag{6.64}$$

The next polynomial in this expansion is

$$\phi_4 = 2\langle Tq_1, q_3\rangle + \langle Tq_2, q_2\rangle. \tag{6.65}$$

Using (6.59) and Conditions (6.55) and (6.56), we have

$$\begin{aligned}\langle Tq_2, q_2\rangle &= \langle T(\xi_2 - t\beta_2), \xi_2 - t\beta_2\rangle = \langle T\xi_2, \xi_2\rangle - 2t\langle T\xi_2, \beta_2\rangle + t^2\langle T\beta_2, \beta_2\rangle \\ &= -2t\langle T\xi_2, \beta_2\rangle.\end{aligned}$$

On the other hand, thanks to (6.62) and (2.28), one obtains

$$\begin{aligned}\langle Tq_1, q_3\rangle &= -\langle Tq_1, [(A-3)(I-R_3)]^{-1}(I-R_3)(\beta_3^0 + t\beta_3^1)\rangle \\ &\quad + \langle Tq_1, [(A-3)(I-R_3)]^{-2}(I-R_3)\beta_3^1\rangle \\ &= 1/2\langle Tq_1, \beta_3^0 + t\beta_3^1\rangle + 1/4\langle Tq_1, \beta_3^1\rangle \\ &= \langle Tq_1, \beta_3^0\rangle/2 + \langle Tq_1, \beta_3^1\rangle/4 + t\langle Tq_1, \beta_3^1\rangle/2.\end{aligned}$$

Thus, (6.65) becomes

$$\phi_4 = \langle Tq_1, \beta_3^0\rangle + 1/2\langle Tq_1, \beta_3^1\rangle + t\Big[\langle Tq_1, \beta_3^1\rangle - 2\langle T\xi_2, \beta_2\rangle\Big].$$

By using (6.61) and the identity (2.11),

$$\langle Tq_1, \beta_3^1\rangle = -\langle T\xi_1, B(\xi_1,\beta_2)+B(\beta_2,\xi_1)\rangle = \langle T\beta_2, B(\xi_1,\xi_1)\rangle = \langle T\beta_2,\beta_2\rangle = 0,$$

by Condition (6.55). Therefore

$$\phi_4 = \langle Tq_1,\beta_3^0\rangle - 2t\langle T\xi_2,\beta_2\rangle,$$

where $\langle T\xi_2,\beta_2\rangle \neq 0$ by Condition (6.57). Hence, according to (6.10) and (6.13), we have $N=4$ and $\lim_{t\to\infty} H(t)t^{-d}e^{Nt}$ exists and is nonzero where $d = deg\ \phi_4 = 1$.

Note. In fact, we have $\phi_4 = -(1+2t)\langle T\xi_2,\beta_2\rangle$, for, again by (6.61) and (2.11),

$$\langle Tq_1,\beta_3^0\rangle = \langle T\xi_1, B(\xi_1,\xi_2)+B(\xi_2,\xi_1)\rangle = -\langle T\xi_2, B(\xi_1,\xi_1)\rangle = -\langle T\xi_2,\beta_2\rangle.$$

Part II. (A selection of ξ_1 and ξ_2 satisfying Conditions (6.55), (6.56) *and* (6.57)*).* Let ξ_1 be defined by

$$\xi_1 = ae^{ik\cdot x} + a^*e^{-ik\cdot x} + be^{ik'\cdot x} + b^*e^{-ik'\cdot x} \in R_1H, \tag{6.66}$$

where

$$k = e_1 = (1,0,0), \quad a = e_2 = (0,1,0),$$

$$k' = e_2 = (0,1,0), \quad b = e_1 + e_3 = (1,0,1),$$

and ξ_2 be defined by

$$\xi_2 = ce^{il\cdot x} + c^*e^{-il\cdot x} \in R_2H, \tag{6.67}$$

where $l = (1,1,0)$ and $c = (1,-1,1)$. Since $a,b,c \in \mathbb{R}^3$, we have

$$\langle T\xi_1,\xi_1\rangle = \langle T\xi_2,\xi_2\rangle = 0, \tag{6.68}$$

by Remark II.1. In order to compute β_2, we first use (2.45) to find

$$(\xi_1 \cdot \nabla)\xi_1 = Q_m e^{im\cdot x} + Q_m^* e^{-im\cdot x} + Q_{m'} e^{im'\cdot x} + Q_{m'}^* e^{-im'\cdot x}, \tag{6.69}$$

where

$$m = k + k' = (1,1,0) = l, \quad m' = k - k' = (1,-1,0),$$

$$Q_m = i(a\cdot k')b + i(b\cdot k)a = i(1,1,1) \in i\mathbb{R}^3, \tag{6.70}$$

$$Q_{m'} = -i(a\cdot k')b + i(b\cdot k)a \in i\mathbb{R}^3. \tag{6.71}$$

Then applying the Leray projection to (6.69), we obtain

$$\beta_2 = B(\xi_1,\xi_1) = P_L[(\xi_1\cdot\nabla)\xi_1] = B_m e^{im\cdot x} + B_m^* e^{-im\cdot x} + B_{m'} e^{im'\cdot x} + B_{m'}^* e^{-im'\cdot x},$$

where B_m and $B_{m'}$ are calculated by using (2.46). As seen in (6.70),(6.71) both Q_m and $Q_{m'}$ are in $i\mathbb{R}^3$, hence so are B_m and $B_{m'}$. From Remark II.1 we infer that $\langle T\beta_2, \beta_2\rangle = 0$. Together with (6.68), we have proven that ξ_1 and ξ_2 satisfy Conditions (6.55) and (6.56). We derive from (2.47) that

$$\begin{aligned}\langle T\xi_2, \beta_2\rangle &= 2L^3 Re[i(l\times c)\cdot B_m^*] = 2L^3 Re[i(l\times c)\cdot Q_m^*] \\ &= 2L^3(1,-1,-2)\cdot(1,1,1) = -4L^3 \neq 0.\end{aligned}$$

This verifies the last condition (6.57). □

From the construction of Examples VI.2 and VI.3 it easily follows that the corresponding limits $n_0 = \lim_{t\to\infty}\|u(t)\|^2/|u(t)|^2$, $h_0 = \lim_{t\to\infty} J(t)/H(t)$ are given by $n_0 = 4$, $h_0 = 7$ and $n_0 = 2$, $h_0 = 4$, respectively. We will see that the quotient $H(t)/|u(t)|^2$ between the helicity and the energy behaves even more differently than the two quotients above as $t \to \infty$. It is clear that in Theorem VI.3, if $\alpha \neq 0$ then $u(t)$ is in the case (i) of Theorem VI.1 with $d = 0$ and $h_0 = n_0 \in \sigma(A)$. In fact,

$h_0 = n_0$ if and only if $\alpha \neq 0$ (see Remark VI.2). The other situation when $\alpha = 0$ will be analyzed further in the next theorem.

Theorem VI.4. *For any $n \in \sigma(A)$, $M > 0$, there exists an initial data $u^0 \in \mathcal{R}$ such that the corresponding regular solution $u(t)$ of* (2.12) *satisfies* (2.17)*, is in the case* (*i*) *of Theorem VI.1, $n_0 = n, h_0 \geq n_0 + M$ and*

$$\frac{\langle Tu(t), u(t)\rangle}{\langle u(t), u(t)\rangle} = O(e^{-2Mt}) \quad \text{as } t \to \infty. \tag{6.72}$$

Also, there are solutions with the helicity satisfying the condition

$$\lim_{t\to\infty} H(t)t^{-d}e^{2h_0 t} \text{ exists and is not zero,} \tag{6.73}$$

where $d > 0$ or h_0 is not an eigenvalue of A.

Proof. The last statement follows directly from Examples VI.2 and VI.3. We now prove the remaining part of the theorem. Let $n \in \sigma(A)$ and $k_n \in \mathbb{Z}^3$ such that $|k_n|^2 = n$ and let $a_{k_n} \in \mathbb{R}^3$ be a nonzero vector such that $a_{k_n} \cdot k_n = 0$. Set

$$\xi_n(x) = a_{k_n}(e^{ik_n\cdot x} + e^{-ik_n\cdot x}) \neq 0.$$

For $s \in \mathbb{N}, s \geq 2$, take $m = s^2 n$ and $k_m = sk_n$. Since $m = |k_m|^2 \in \sigma(A)$ and by Lemma II.1, we can find $a_{k_m} \in \mathbb{C}^3$ such that

$$\xi_m(x) = a_{k_m}e^{ik_m\cdot x} + a^*_{k_m}e^{-ik_m\cdot x} \in R_m H \setminus \{0\} \text{ and } T\xi_m = \sqrt{m}\xi_m \tag{6.74}$$

From the Remarks II.2 and II.1 we have

$$B(\xi_n, \xi_n) = 0, \quad \langle T\xi_n, \xi_n\rangle = 0 \quad \text{and} \quad B(\xi_n, \xi_m) = B(\xi_m, \xi_n) = 0. \tag{6.75}$$

Take $\xi_j = 0$ for all $j \neq n, m$ and let $\xi = \oplus_{j=1}^{2m}\xi_j \in \Pi_{2m}S_A$. By rescaling to have small

$|\xi_n|$ and $|\xi_m|$ we can assume that $\xi = \Pi_{2m} W(u^0)$ for some $u^0 \in \mathcal{R}$. Therefore

$$n_0 = n \quad \text{and} \quad W_{n_0}(u^0) = \xi_n. \tag{6.76}$$

Thanks to Proposition VI.3 and (6.75), we infer that

$$\alpha = \frac{\langle T\xi_n, \xi_n\rangle}{|\xi_n|^2} = 0. \tag{6.77}$$

The asymptotic expansion (2.18) or (6.2) of the corresponding regular solution $u(t)$ in this case is

$$u(t) \sim \xi_n + \sum_{j>n} q_j(t) e^{-jt}. \tag{6.78}$$

This means that $q_1 = q_2 = ... = q_{n-1} = 0$ and $q_n = \xi_n$. Since $s \geq 2, m = s^2 n > 2n$, we have that $\beta_j = 0$ for all $j \in [1, 2n)$, hence $q_j = 0$ for $j \in (n+1, 2n)$. By (6.75), we have $\beta_{2n} = B(\xi_n, \xi_n) = 0$ hence $q_{2n} = 0$. Using (2.26) and (2.27) to calculate β_j and q_j for $j > 2n$ we obtain $q_j = 0$ for all $j = 1, 2, ..., m-1$ except for $j = n$ and $\beta_j = 0$ for all $j = 1, 2, ..., m$. In particular, $\beta_m = 0$ hence $q_m = \xi_m$. Once again, using the formulas (2.26) and (2.27), one gets $q_{m+1} = q_{m+2} = ... = q_{m+n-1} = 0$, and also by (6.75), $\beta_{m+n} = B(\xi_m, \xi_n) + B(\xi_n, \xi_m) = 0$, thus, $q_{m+n} = 0$. Similar arguments lead us to the overall identification of the first $(2m-1)$ polynomials q_j's in the asymptotic expansion of $u(t)$ as the following

$$q_n = \xi_n \in R_n H, \quad q_m = \xi_m \in R_m H \quad \text{and} \quad q_j = 0 \text{ for } j \in [1, 2m) \setminus \{n, m\}.$$

Consequently, the polynomials in the asymptotic expansion (6.5) of $H(t)$ are $\phi_j = 0$ for all $j < 2m, j \neq 2n, m+n$. By (6.75) we have $\phi_{2n} = \langle T\xi_n, \xi_n\rangle = 0$. Since $q_n \in R_n H$ and $q_m \in R_m H$, $\phi_{n+m} = 2\langle Tq_n, q_m\rangle = 0$. The next polynomial in the expansion of $H(t)$ is

$$\phi_{2m} = \langle Tq_m, q_m\rangle = \langle T\xi_m, \xi_m\rangle = \sqrt{m}|\xi_m|^2 \neq 0,$$

by the virtue of (6.74). This shows that the number N in (6.11) is given by $N = 2m$. Therefore

$$\frac{H(t)}{|u(t)|^2} = \frac{\phi_{2m} e^{-2mt} + O(e^{-(2m+\varepsilon)t})}{|\xi_n|^2 e^{-2nt} + O(e^{-(2n+\varepsilon)t})} = O(e^{-2(m-n)t}), \quad \text{as } t \to \infty, \tag{6.79}$$

where ε is some adequate positive number. For each $M > 0$, take $s > 0$ such that

$$h_0 = N/2 = m = s^2 n \geq n + M = n_0 + M.$$

Then (6.79) above implies

$$\frac{H(t)}{|u(t)|^2} = O(e^{-2Mt}) \quad \text{as } t \to \infty,$$

and the proof is complete. □

CHAPTER VII

CONCLUSIONS

We have introduced a new construction of regular solutions to the Navier–Stokes equations and a new system of differential equations, the extended Navier–Stokes equations. Studying those two, we used Phragmen-Linderlöf type estimates in large domains of analyticity to find simple conditions under which the asymptotic expansion of a regular solution converges exactly to that solution. We also constructed suitable normed spaces in which the extended Navier–Stokes equations has global solutions and the normal form of the Navier–Stokes equations associated to the terms of the asymptotic expansions is a well-behaved infinite system of differential equations.

However, we need sharper estimates for the terms of those asymptotic expansions so that the convergence in H or V may follow at least when the normalization map has small values. In addition, the relation between the global solutions to the extended Navier–Stokes equations and the classical Leray weak solutions need to be clarified.

We then used the asymptotic expansion of regular solutions to the Navier–Stokes equations to establish the dichotomy of the helicity's asymptotic behavior. Namely, we can split the set of regular initial data $\mathcal{R}$ into $\mathcal{R}_0$ and $\mathcal{R}_1$, where $\mathcal{R}_0$ contains all regular initial data such that the helicity is identically zero for all times $t \geq 0$ and $\mathcal{R}_1$ is the set of all regular initial data such that the helicity is eventually nonzero when time t is large. Moreover, $\mathcal{R}_1$ is open and dense in $\mathcal{R}$, with respect to the topology of V, and $\mathcal{R}_0$ contains infinite union of invariant closed linear manifold of infinite dimension. We presented a few examples of those manifolds and more of them need to be discovered.

We also proved that when the initial data u^0 is in $\mathcal{R}_1$, the solution $u(t)$ and the helicity $H(t)$ satisfy

$$\lim_{t\to\infty} \frac{H(t)}{t^d e^{-2h_0 t}} \text{ and } \alpha = \lim_{t\to\infty} \frac{H(t)}{|u(t)|^2} \text{ exist,}$$

where $d \geq 0$ and $h_0 > 0$ are some integers. In fact, we know

$$h_0 = h_0(u^0) = \lim_{t\to\infty} J(t)/H(t),$$

where $J(t) =< T^2u(t), Tu(t) >$, T is the curl operator. We have identified all the possible values of $n_0 = \lim_{t\to\infty} \|u(t)\|^2/|u(t)|^2$ and α. The set of values of h_0 is known to be strictly larger than the spectrum of the Stokes operator A. However we do not know whether h_0 can be any arbitrarily natural number.

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