GENERALIZATION OF ROTATIONAL MECHANICS AND APPLICATION TO AEROSPACE SYSTEMS

A Dissertation

by

ANDREW JAMES SINCLAIR

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2005

Major Subject: Aerospace Engineering

GENERALIZATION OF ROTATIONAL MECHANICS

AND APPLICATION TO AEROSPACE SYSTEMS

A Dissertation

by

ANDREW JAMES SINCLAIR

Submitted to Texas A&M University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Approved as to style and content by:

John L. Junkins (Co-Chair of Committee)	John E. Hurtado (Co-Chair of Committee)
Srinivas R. Vadali (Member)	John H. Painter (Member)
Helen L. Reed (Head of Department)	

May 2005

Major Subject: Aerospace Engineering

ABSTRACT

Generalization of Rotational Mechanics and Application to Aerospace Systems.

(May 2005)

Andrew James Sinclair, B.S., University of Florida;

M.S., University of Florida

Co–Chairs of Advisory Committee: Dr. John L. Junkins
Dr. John E. Hurtado

This dissertation addresses the generalization of rigid-body attitude kinematics, dynamics, and control to higher dimensions. A new result is developed that demonstrates the kinematic relationship between the angular velocity in N-dimensions and the derivative of the principal-rotation parameters. A new minimum-parameter description of N-dimensional orientation is directly related to the principal-rotation parameters.

The mapping of arbitrary dynamical systems into N-dimensional rotations and the merits of new quasi velocities associated with the rotational motion are studied. A Lagrangian viewpoint is used to investigate the rotational dynamics of N-dimensional rigid bodies through Poincaré's equations. The N-dimensional, orthogonal angular-velocity components are considered as quasi velocities, creating the Hamel coefficients. Introducing a new numerical relative tensor provides a new expression for these coefficients. This allows the development of a new vector form of the generalized Euler rotational equations.

An N-dimensional rigid body is defined as a system whose configuration can be completely described by an $N \times N$ proper orthogonal matrix. This matrix can be related to an $N \times N$ skew-symmetric orientation matrix. These Cayley orientation variables and the angular-velocity matrix in N-dimensions provide a new connection between general mechanical-system motion and abstract higher-dimensional rigid-body rotation. The resulting representation is named the Cayley form.

Several applications of this form are presented, including relating the combined attitude and orbital motion of a spacecraft to a four-dimensional rotational motion. A second example involves the attitude motion of a satellite containing three momentum wheels, which is also related to the rotation of a four-dimensional body.

The control of systems using the Cayley form is also covered. The wealth of work on three-dimensional attitude control and the ability to apply the Cayley form motivates the idea of generalizing some of the three-dimensional results to N-dimensions. Some investigations for extending Lyapunov and optimal control results to N-dimensional rotations are presented, and the application of these results to dynamical systems is discussed.

Finally, the nonlinearity of the Cayley form is investigated through computing the nonlinearity index for an elastic spherical pendulum. It is shown that whereas the Cayley form is mildly nonlinear, it is much less nonlinear than traditional spherical coordinates.

ACKNOWLEDGMENTS

Special thanks to my advisors Dr. John L. Junkins and Dr. John E. Hurtado. Their deep insight into dynamics and control and their collegial philosophy has benefited me greatly here at Texas A&M and will continue to do so throughout my career. I thank my committee members Dr. Rao Vadali and Dr. John Painter for their guidance. I am also grateful to Dr. Daniele Mortari for many educational, constructive, and entertaining conversations on the subject of generalized rotational mechanics. Thanks to Lisa Willingham and Karen Knabe for all of their assistance throughout my degree program.

I wish to thank F. Landis Markley, Robert Bauer, Jackie Schandua, and Michael Swanzy for their helpful comments and suggestions in the course of preparing this work. Thanks also to Christian Bruccoleri and Puneet Singla for use of their Matlab code to select initial conditions for computing nonlinearity indices. I gratefully acknowledge the support of the National Defense Science and Engineering Graduate Fellowship.

I feel deeply fortunate to have wonderful friends who have made my time in College Station a special part of my life: Roshawn Bowers, Eddie Caicedo, Todd Griffith, Bjoern Kiefer, Luciano Machado, Josh O'Neil, Gary Seidel, Lesley Weitz, and Matt Wilkins. Finally, I offer my gratitude to my family for their advice, encouragement, and belief in me.

TABLE OF CONTENTS

CHAPTER Page

CHAPTER Page

LIST OF TABLES

LIST OF FIGURES

FIGURE Page

CHAPTER I

INTRODUCTION

This dissertation deals with the generalization of rotational mechanics to describe N-dimensional rotations. The field of rotational mechanics has been developed to describe the orientation, kinematics, dynamics, and control of rigid bodies and has been key in the development of aerospace vehicles. The first elements of this theory were laid down over 250 years ago, and the field has been the subject of continued attention over the past fifty years with the development of spacecraft technology. Although this field has been developed to describe physical, three-dimensional bodies, many of the concepts that have been developed can be extended to mathematically describe higher-dimensional bodies. The first half of this dissertation reviews the kinematics and dynamics of N-dimensional rotations, as well as presenting several new ideas. The second half of the dissertation presents and investigates the implications of a new idea to use N-dimensional rotational concepts to describe the motion of real, physical systems. An example of this approach is applied to spacecraft orbital and attitude dynamics, and a new approach for feedback control design is presented.

Although not as old as the study of three-dimensional rotations, the field of N-dimensional rotations has developed for more than 150 years. Much of Chapter II of this dissertation deals with reviewing generalizations of three-dimensional attitude descriptions to N-dimensional orientation. Fundamental to establishing a geometric interpretation of N-dimensional rotations is the extension of Euler's theorem to higher dimensions. Although there is only one principal plane for any rotation in three-dimensions, an additional principal plane is added with every increase in dimen-

---

The journal model is *IEEE Transactions on Automatic Control.*

sion by two dimensions. Several of the attitude representations for three-dimensional rotations can be directly extended to N-dimensional rotations, and like many of the concepts in this dissertation, three-dimensional representations are actually just a special case of the general form. In particular, attitude representations related to the Cayley transform and higher-order Cayley transforms have direct N-dimensional generalizations. These generalizations carry over relationships to the principal rotations and singularity conditions, for which the parameters are undefined, similar to the three-dimensional special cases. Chapters II and III also present several new ideas for describing N-dimensional orientations and their evolution.

In Chapters IV and V the focus shifts from kinematics to dynamics. Again, the dynamics of N-dimensional rotations have been studied for over 125 years. Equations of motion have been developed for these rotations by extending the concept of angular-momentum conservation under the assumption of a symmetric, unforced system. Here though, two new derivations of these equations are presented. The new derivations link N-dimensional rotational mechanics to Lagrangian dynamics for the first time, which is made possible by the introduction of a new numerical relative tensor. Some important features of the new derivations are that they provide a vector form of the equations of motion and, by removing the assumptions of symmetry and unforced motion, they allow for applied forces and coordinate dependence.

Considering applied forces and coordinate dependence is necessary to allow the application of these rotational equations to a broader class of problems, which is the subject of the second half of the dissertation. The new idea of describing the motion of real, physical systems using the N-dimensional kinematic and dynamic equations is presented. This idea associates the rotation of an N-dimensional rigid body with the motion of any given system, or in other words, views the motion as an N-dimensional rotation. This concept is called the Cayley form. Specifically, the generalized coordinates of the system are treated as orientation variables of an N-dimensional rigid body, and the Cayley form defines a set of quasi velocities for the system that are equal to the angular velocity of the N-dimensional body. In Chapter VI two examples are presented treating the dynamics of physical systems using the Cayley form, including treating the combined orbital and attitude dynamics of a spacecraft as pure rotation of a four-dimensional body.

Chapter VII presents some results for designing feedback controllers using the Cayley form. The idea behind this approach is to generalize spacecraft attitude controllers to N-dimensional rotations and then use the Cayley form to apply these to general systems. The motivation for this is the possibility to leverage the wealth of work that has been produced over the past fifty years for spacecraft attitude control in application to broader classes of systems. An example is shown, though, that spacecraft attitude control can be based on the special properties of three-dimensional rotations that do not hold for general N-dimensional kinematics. The rotational dynamics, however, do not appear as sensitive to generalization to N-dimensions, and an example is presented taking advantage of this in designing a stabilizing controller for a three-link manipulator system.

Finally, some issues dealing with the complexity or nonlinearity of the Cayley form are addressed in Chapter VIII. Several measures of nonlinearity are computed for an elastic spherical pendulum. It is shown that whereas applying the Cayley form to an originally linear system produces a mildly nonlinear system, the Cayley form can also be much less nonlinear than traditional alternative representations.

Index notation is used extensively throughout the dissertation. The elements of a matrix or tensor, $\boldsymbol{A}$, are expressed as A_{ij} and the elements of a vector, $\boldsymbol{a}$, as a_i. The Einstein summation convention is that if any index is repeated twice within a term, then the term represents the summation for every possible value of the index.

An index must not be repeated more than twice in a term. Indices that appear only once in each term of an equation are free indices, and the equation is valid for each possible value of the index. The Kronecker delta, δ_{ij}, is equal to unity if $i = j$ and is equal to zero otherwise.

CHAPTER II

KINEMATICS OF N-DIMENSIONAL PRINCIPAL ROTATIONS

A. Introduction

An important description of three-dimensional rotations is provided by Euler's theorem that describes any general orientation in terms of a single principal rotation. The principal rotation concept also extends to N-dimensional rotations [1], however, for higher-dimensional spaces a general orientation requires $N/2$ principal rotations for even N and $(N-1)/2$ principal rotations for odd N. In general the number of required rotations can be expressed as $L = \lfloor N/2 \rfloor$. These rotations take place on completely orthogonal planes, called the principal planes. For even dimensions these planes completely occupy the space. For odd dimensions, however, one axis is left out of the rotational motion and is referred to as a principal axis.

In addition to Euler's theorem, key representations of an orientation in N-dimensional space are the proper-orthogonal rotation matrix, the extended Rodrigues parameters (ERPs), and the Euler Matrix. For any given orientation, the principal rotations (planes and angles) can be computed from a variety of methods. These methods are different decompositions of the representations mentioned above. Some of the methods are a spectral decomposition described by Mortari [2], a principal rotation matrix decomposition also developed by Mortari [2], a canonical form described by Bauer [3], and a minimum-parameter canonical form developed by Sinclair and Hurtado [4]. Of course each of these representations is related, and the above references largely deal with the interconnections between the different decompositions.

As will be described, the canonical form decomposes the various representation matrices into a block-diagonal form with a 2×2 block associated with each principal plane and a 1×1 block associated with the principal axis if it exists. In illustrating these blocks, the notation $[\boldsymbol{A}]\,(a:b)$ will be used to refer to the block on the diagonal of the matrix $\boldsymbol{A}$ from the ath row and column to the bth row and column. Additionally, in the following sections the spectral decomposition and canonical form of the various rotation variables will be illustrated for odd N. The corresponding forms for even N can be constructed by simply deleting the Nth row and column which will be associated with the principal axis.

B. Review of N-Dimensional Rotations

Much of the description of N-dimensional orientations in this section was given by Mortari [2] and Bauer [3]. The current work attempts to follow their notation and conventions as closely as possible with one exception. In discussing a rotated frame both of the above authors define representations of the rotation as the mapping from the rotated frame back to a reference frame. Here the convention will be to give the mapping from the reference frame to the rotated frame. Therefore, many of the definitions given below correspond to the transpose of the proper rotation matrices given by Mortari and Bauer.

1. Rotation Matrix

The transformation of an N-dimensional vector by a proper orthogonal matrix, $\boldsymbol{C}$, describes a rotation in N-dimensional space. The following equation describes the transformation from a column matrix parameterizing a vector in a reference coordinate system, the $\boldsymbol{n}$ frame, to a column matrix parameterizing the vector in a rotated coordinate system, the $\boldsymbol{b}$ frame.

$$[\boldsymbol{r}]_b = [\boldsymbol{C}]\,[\boldsymbol{r}]_n \tag{2.1}$$

Alternatively, Eq. (2.1) can be viewed as simply an orthogonal projection of the components of an arbitrary vector. This matrix $\boldsymbol{C}$ is called a rotation matrix and is the most fundamental representation of N-dimensional rotations. Two complementary representations of the rotation matrix that will be considered are the spectral decomposition and the canonical form.

The spectral decomposition of $\boldsymbol{C}$ is discussed by Mortari [2] and is shown below.

$$\boldsymbol{C} = \boldsymbol{W}_C \boldsymbol{\Lambda}_C \boldsymbol{W}_C^{\dagger} \tag{2.2}$$

Here, the columns of $\boldsymbol{W}_C$ are the unit eigenvectors, $\boldsymbol{\Lambda}_C$ is the diagonal matrix of eigenvalues, and $(\quad)^{\dagger}$ indicates the conjugate transpose. The kth complex-conjugate pair of eigenvectors and eigenvalues are related to the kth principal rotation. The eigenvectors are related to the principal planes, and the eigenvalues are related to the principal angles. If N is odd then one eigenvalue will be equal to positive one, and the corresponding real eigenvector is the principal axis. These matrices are shown below for odd N. The representations are similar for even N, except the omission of the principal axis and real eigenvalue.

$$[\boldsymbol{W}_C] = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_{N-2} & \boldsymbol{w}_{N-1} & \boldsymbol{w}_N \end{bmatrix} \tag{2.3}$$

$$\boldsymbol{w}_1 = \frac{\sqrt{2}}{2}\left(\boldsymbol{p}_1 + i\boldsymbol{p}_2\right) \quad ; \quad \boldsymbol{w}_2 = \frac{\sqrt{2}}{2}\left(\boldsymbol{p}_1 - i\boldsymbol{p}_2\right) \tag{2.4}$$

$$\vdots$$

$$\boldsymbol{w}_{N-2} = \frac{\sqrt{2}}{2}\left(\boldsymbol{p}_{N-2} + i\boldsymbol{p}_{N-1}\right) \quad ; \quad \boldsymbol{w}_{N-1} = \frac{\sqrt{2}}{2}\left(\boldsymbol{p}_{N-2} - i\boldsymbol{p}_{N-1}\right) \tag{2.5}$$

$$\boldsymbol{w}_N = \boldsymbol{p}_N \tag{2.6}$$

Here, the vectors $\boldsymbol{p}_k$ are real unit vectors lying in the principal planes except $\boldsymbol{p}_N$, which lies along the principal axis. The matrix of eigenvalues, $\boldsymbol{\Lambda}_C$, is diagonal with values $\lambda_k^{(C)}$.

$$\lambda_1^{(C)} = \cos(\phi_1 + 2\pi n_1) + i\sin(\phi_1 + 2\pi n_1) \tag{2.7}$$

$$\lambda_2^{(C)} = \cos(\phi_1 + 2\pi n_1) - i\sin(\phi_1 + 2\pi n_1) \tag{2.8}$$

$$\vdots$$

$$\lambda_{N-2}^{(C)} = \cos(\phi_L + 2\pi n_L) + i\sin(\phi_L + 2\pi n_L) \tag{2.9}$$

$$\lambda_{N-1}^{(C)} = \cos(\phi_L + 2\pi n_L) - i\sin(\phi_L + 2\pi n_L) \tag{2.10}$$

$$\lambda_N^{(C)} = +1 \tag{2.11}$$

Here, each angle $-\pi \leq \phi_k \leq \pi$ is the value of rotation in the kth principal plane, and the values n_k can be any integer. It is important to note that the matrices $\boldsymbol{W}_C$ and $\boldsymbol{\Lambda}_C$ are not unique. This is because of the ambiguity in selecting the vectors $\boldsymbol{p}_k$ (they can lie anywhere in the principal plane) as well as the existence of multiple choices for ordering the eigenvectors and eigenvalues within $\boldsymbol{W}_C$ and $\boldsymbol{\Lambda}_C$ (i.e., labeling the principal planes one through L).

The canonical representation of $\boldsymbol{C}$ is related to the spectral decomposition [3].

$$\boldsymbol{C} = \boldsymbol{P}^T \boldsymbol{C}' \boldsymbol{P} \quad ; \quad \boldsymbol{C}' = \boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^T \tag{2.12}$$

Here, $\boldsymbol{P}$ is a proper orthogonal matrix, and $\boldsymbol{C}'$ is a block-diagonal proper orthogonal matrix. The rows of $\boldsymbol{P}$ are the coordinatization of the principal coordinate vectors in the $\boldsymbol{b}$ frame.

$$[\boldsymbol{P}]^T = \begin{bmatrix} [\boldsymbol{p}_1]_b & [\boldsymbol{p}_2]_b & \ldots & [\boldsymbol{p}_N]_b \end{bmatrix} \tag{2.13}$$

The matrix $\boldsymbol{C}'$ is related to the principal angles. The kth block on the diagonal of $\boldsymbol{C}'$ has the following form.

$$[\boldsymbol{C}'](2k-1:2k) = \begin{bmatrix} \cos(\phi_k + 2\pi n_k) & \sin(\phi_k + 2\pi n_k) \\ -\sin(\phi_k + 2\pi n_k) & \cos(\phi_k + 2\pi n_k) \end{bmatrix} \tag{2.14}$$

For odd N the (N, N) element of $\boldsymbol{C}'$ forms a 1×1 block and is equal to $+1$.

The matrix $\boldsymbol{P}$ is itself an N-dimensional rotation matrix that describes the transformation from the rotated frame to a third frame, the principal frame. This coordinate system has coordinate vectors $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N\}$ which are aligned with the principal planes of the rotation described by $\boldsymbol{C}$: $(\boldsymbol{p}_1, \boldsymbol{p}_2)$, $(\boldsymbol{p}_3, \boldsymbol{p}_4)$, etc. Note that consistent with his convention mentioned earlier, Bauer defines $\boldsymbol{P}$ as the transpose of the definition given here; thus it is the mapping from the principal to rotated frame.

2. Principal Rotation Matrices

Another representation of N-dimensional orientation that is closely related to the principal planes and angles involves the principal rotation matrices. These are L proper orthogonal matrices, each describing one of the principal rotations that compose a general orientation [2].

$$\boldsymbol{C} = \boldsymbol{R}_1 \boldsymbol{R}_2 \ldots \boldsymbol{R}_L = \boldsymbol{R}_1 + \boldsymbol{R}_2 + \ldots + \boldsymbol{R}_L - (L-1)\,\boldsymbol{I} \tag{2.15}$$

The remarkable fact that $\boldsymbol{C}$ can be expressed as either a product or sum of the principal rotation matrices is due to the complete orthogonality of the planes in which the rotations occur. The elegant decomposition of Eq. (2.15) was discovered by Mortari [2]. Note that Mortari uses the convention that a rotation is an orientation that can be described by only one nonzero principal rotation (hence, the terms rotation and orientation are equivalent only for $N = 2$ or 3) and simply refers to the $\boldsymbol{R}$ matrices as rotation matrices; this terminology is not entirely adopted here.

Mortari gives the relationship between the principal rotation matrices and the principal planes and angles [2]. These expressions are written in terms of the rows of $\boldsymbol{P}$ arranged in $N \times 2$ matrices and the 2×2 symplectic matrix.

$$[\boldsymbol{P}_k] = \begin{bmatrix} [\boldsymbol{p}_{2k-1}]_b & [\boldsymbol{p}_{2k}]_b \end{bmatrix} \quad ; \quad [\boldsymbol{J}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.16}$$

The principal rotation matrices are given as follows [2].

$$\boldsymbol{R}_k\left(\boldsymbol{P}_k, \phi_k\right) = \boldsymbol{I} + \left(\cos\phi_k - 1\right)\boldsymbol{P}_k\boldsymbol{P}_k^T + \boldsymbol{P}_k\boldsymbol{J}\boldsymbol{P}_k^T\sin\phi_k \tag{2.17}$$

Note that this expression is identical to the one given by Mortari in spite of the fact that the symplectic matrix defined in Eqs. (2.16) is the transpose of the matrix used by Mortari. The implication of this change in $\boldsymbol{J}$ is that, whereas the principal rotation matrix $\boldsymbol{R}_k\left(\boldsymbol{P}_k, \phi_k\right)$ defined by Mortari describes a rotation of $-\phi_k$, the current definitions describe a rotation of positive ϕ_k. Conversely the rotation matrix as defined by Mortari can be seen as the transformation matrix from a frame that is rotated by ϕ_k back to some reference frame, whereas the current definition is the rotation from the reference to the rotated frame.

3. Extended Rodrigues Parameters

The rotation matrix has N^2 elements and is subject to $N^2 - M$ orthogonality constraints, where the minimum number of parameters necessary to represent an N-dimensional rotation is $M = \frac{1}{2}N(N-1)$. A commonly used minimum parameter representation is the extended Rodrigues parameters (ERPs) [5, 6]. These parameters are defined by the Cayley transform, which relates proper orthogonal and skew-symmetric matrices [7]. Cayley discovered the forward relationship while investigating some properties of "left systems" [8].

$$\text{Forward:} \quad \boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} + \boldsymbol{Q})^{-1} = (\boldsymbol{I} + \boldsymbol{Q})^{-1}(\boldsymbol{I} - \boldsymbol{Q}) \tag{2.18}$$

$$\text{Inverse:} \quad \boldsymbol{Q} = (\boldsymbol{I} - \boldsymbol{C})(\boldsymbol{I} + \boldsymbol{C})^{-1} = (\boldsymbol{I} + \boldsymbol{C})^{-1}(\boldsymbol{I} - \boldsymbol{C}) \tag{2.19}$$

Here, $\boldsymbol{Q}$ is an $N \times N$ skew-symmetric matrix, and $\boldsymbol{I}$ is the identity matrix. The M distinct elements of the matrix $\boldsymbol{Q}$ comprise the ERPs. Although the forward transformation is valid for all $\boldsymbol{Q} = -\boldsymbol{Q}^T$, the inverse transformation is singular for the "180° rotations" where $\det(\boldsymbol{I} + \boldsymbol{C})$ vanishes.

The eigenvalues and eigenvectors of $\boldsymbol{Q}$ can be found by substituting the spectral decomposition of $\boldsymbol{C}$ into the Cayley transform, Eq. (2.19).

$$\begin{aligned}\boldsymbol{Q} &= (\boldsymbol{I} + \boldsymbol{C})^{-1}(\boldsymbol{I} - \boldsymbol{C}) = \left(\boldsymbol{I} + \boldsymbol{W}_C \boldsymbol{\Lambda}_C \boldsymbol{W}_C^{\dagger}\right)^{-1}\left(\boldsymbol{I} - \boldsymbol{W}_C \boldsymbol{\Lambda}_C \boldsymbol{W}_C^{\dagger}\right) \\ &= \left[\boldsymbol{W}_C(\boldsymbol{I} + \boldsymbol{\Lambda}_C)\boldsymbol{W}_C^{\dagger}\right]^{-1}\left[\boldsymbol{W}_C(\boldsymbol{I} - \boldsymbol{\Lambda}_C)\boldsymbol{W}_C^{\dagger}\right] = \boldsymbol{W}_C(\boldsymbol{I} + \boldsymbol{\Lambda}_C)^{-1}(\boldsymbol{I} - \boldsymbol{\Lambda}_C)\boldsymbol{W}_C^{\dagger}\end{aligned} \tag{2.20}$$

Therefore, the eigenvectors of $\boldsymbol{Q}$ can be set equal to the eigenvectors of $\boldsymbol{C}$: $\boldsymbol{W}_Q = \boldsymbol{W}_C$. The following is concluded for the eigenvalues of $\boldsymbol{Q}$.

$$\boldsymbol{\Lambda}_Q = (\boldsymbol{I} + \boldsymbol{\Lambda}_C)^{-1}(\boldsymbol{I} - \boldsymbol{\Lambda}_C) \tag{2.21}$$

Because each of the above matrices are diagonal, the individual eigenvalues are related as follows.

$$\lambda_k^{(Q)} = \frac{1 - \lambda_k^{(C)}}{1 + \lambda_k^{(C)}} \tag{2.22}$$

Comparing this result with Eqs. (2.7) to (2.11) relates the eigenvalues of $\boldsymbol{Q}$ to the principal angles.

$$\lambda_1^{(Q)} = -i \tan\left(\frac{\phi_1 + 2\pi n_1}{2}\right) \quad ; \quad \lambda_2^{(Q)} = i \tan\left(\frac{\phi_1 + 2\pi n_1}{2}\right) \tag{2.23}$$

$$\vdots$$

$$\lambda_{N-2}^{(Q)} = -i \tan\left(\frac{\phi_L + 2\pi n_L}{2}\right) \quad ; \quad \lambda_{N-1}^{(Q)} = i \tan\left(\frac{\phi_L + 2\pi n_L}{2}\right) \tag{2.24}$$

$$\lambda_N^{(Q)} = 0 \tag{2.25}$$

The canonical form of $\boldsymbol{Q}$ relates the ERPs to the canonical form of $\boldsymbol{C}$. The canonical representation of a skew-symmetric matrix decomposes the matrix into a proper orthogonal matrix and a block-diagonal skew-symmetric matrix [3, 9]. These matrices can be found by substituting the canonical form of $\boldsymbol{C}$ into the Cayley transform, Eq. (2.19).

$$\begin{aligned} \boldsymbol{Q} &= (\boldsymbol{I} + \boldsymbol{C})^{-1} (\boldsymbol{I} - \boldsymbol{C}) = \left(\boldsymbol{I} + \boldsymbol{P}^T \boldsymbol{C}' \boldsymbol{P}\right)^{-1} \left(\boldsymbol{I} - \boldsymbol{P}^T \boldsymbol{C}' \boldsymbol{P}\right) \\ &= \left[\boldsymbol{P}^T (\boldsymbol{I} + \boldsymbol{C}') \boldsymbol{P}\right]^{-1} \left[\boldsymbol{P}^T (\boldsymbol{I} - \boldsymbol{C}') \boldsymbol{P}\right] = \boldsymbol{P}^T (\boldsymbol{I} + \boldsymbol{C}')^{-1} (\boldsymbol{I} - \boldsymbol{C}') \boldsymbol{P} \end{aligned} \tag{2.26}$$

The Cayley transform of $\boldsymbol{C}'$ is a block-diagonal, skew-symmetric matrix and is defined as $\boldsymbol{Q}'$.

$$\boldsymbol{Q}' = (\boldsymbol{I} + \boldsymbol{C}')^{-1} (\boldsymbol{I} - \boldsymbol{C}') \quad ; \quad \boldsymbol{C}' = (\boldsymbol{I} + \boldsymbol{Q}')^{-1} (\boldsymbol{I} - \boldsymbol{Q}') \tag{2.27}$$

Therefore, the same proper orthogonal matrix $\boldsymbol{P}$ transforms $\boldsymbol{C}$ and $\boldsymbol{Q}$ to canonical form.

$$\boldsymbol{Q} = \boldsymbol{P}^T \boldsymbol{Q}' \boldsymbol{P} \quad ; \quad \boldsymbol{Q}' = \boldsymbol{P} \boldsymbol{Q} \boldsymbol{P}^T \tag{2.28}$$

The elements of this new skew-symmetric matrix $\boldsymbol{Q}'$ are referred to as the *canonical ERPs*. The similarity transformation enforces that $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ share the same eigenvalues and their eigenvectors are related through $\boldsymbol{P}$. By convention the following form is chosen for $\boldsymbol{Q}'$ for odd N.

$$[\boldsymbol{Q}'] = \begin{bmatrix} 0 & Q'_{12} & \cdots & 0 & 0 & 0 \\ -Q'_{12} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & Q'_{N-1,N} & 0 \\ 0 & 0 & \cdots & -Q'_{N-1,N} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{2.29}$$

In the canonical representation of $\boldsymbol{Q}$, the matrix $\boldsymbol{Q}'$ is related to the principal angles, and $\boldsymbol{P}$ is related to the principal planes. The kth block on the diagonal of $\boldsymbol{Q}'$ is related to the angle of the kth principal rotation as follows.

$$Q'_{2k-1,2k} = -\tan\left(\frac{\phi_k + 2\pi n_k}{2}\right) \tag{2.30}$$

The sign convention above is chosen to be consistent with the canonical form of $\boldsymbol{C}$ in Eq. (2.14).

The canonical form of $\boldsymbol{Q}$ is also directly related to the principal rotation matrices, $\boldsymbol{R}_k$. These rotation matrices are simply the canonical transformation of the Cayley transform of each block on the diagonal of $\boldsymbol{Q}'$. The individual blocks of $\boldsymbol{Q}'$ can be separated as follows.

$$\boldsymbol{Q}' = \boldsymbol{Q}'_1 + \boldsymbol{Q}'_2 + \ldots + \boldsymbol{Q}'_L \tag{2.31}$$

Here, each $\boldsymbol{Q}'_k$ contains only one of the blocks on the diagonal of $\boldsymbol{Q}'$. For example, in a five-dimensional rotation $\boldsymbol{Q}'$ will consist of $\boldsymbol{Q}'_1$ and $\boldsymbol{Q}'_2$.

$$[\boldsymbol{Q}'_1] = \begin{bmatrix} 0 & Q'_{12} & 0 & 0 & 0 \\ -Q'_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad [\boldsymbol{Q}'_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q'_{34} & 0 \\ 0 & 0 & -Q'_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.32}$$

Returning to Eq. (2.27), the two terms involving $\boldsymbol{Q}'$ can be expanded using the matrices $\boldsymbol{Q}'_k$.

$$(\boldsymbol{I} + \boldsymbol{Q}')^{-1} = (\boldsymbol{I} + \boldsymbol{Q}'_1)^{-1} \ldots (\boldsymbol{I} + \boldsymbol{Q}'_L)^{-1} \tag{2.33}$$

$$(\boldsymbol{I} - \boldsymbol{Q}') = (\boldsymbol{I} - \boldsymbol{Q}'_1) \ldots (\boldsymbol{I} - \boldsymbol{Q}'_L) \tag{2.34}$$

These expansions can be used to rewrite the canonical form of $\boldsymbol{C}$.

$$\boldsymbol{C} = \boldsymbol{P}^T (\boldsymbol{I} + \boldsymbol{Q}'_1)^{-1} \ldots (\boldsymbol{I} + \boldsymbol{Q}'_L)^{-1} (\boldsymbol{I} - \boldsymbol{Q}'_1) \ldots (\boldsymbol{I} - \boldsymbol{Q}'_L) \boldsymbol{P} \tag{2.35}$$

Because of the special form of the $\boldsymbol{Q}'_k$ matrices, this product can be rearranged as follows.

$$\boldsymbol{C} = \boldsymbol{P}^T (\boldsymbol{I} + \boldsymbol{Q}'_1)^{-1} (\boldsymbol{I} - \boldsymbol{Q}'_1) \ldots (\boldsymbol{I} + \boldsymbol{Q}'_L)^{-1} (\boldsymbol{I} - \boldsymbol{Q}'_L) \boldsymbol{P} \tag{2.36}$$

Comparing this result with Eq. (2.15), one can choose the principal rotation matrices as shown below.

$$\boldsymbol{R}_k = \boldsymbol{P}^T (\boldsymbol{I} + \boldsymbol{Q}'_k)^{-1} (\boldsymbol{I} - \boldsymbol{Q}'_k) \boldsymbol{P} \tag{2.37}$$

4. Euler Matrix

A final representation of N-dimensional rotations that will be useful for the current purposes is the Euler matrix. Whereas this matrix has been used tangentially in previous works [2, 10], it will be developed more fully here. The Euler matrix is a skew-symmetric matrix, $\boldsymbol{E}$, that can be related to the rotation matrix using properties of the matrix exponential and determinant [11].

$$\exp(\boldsymbol{E})\left(\exp(\boldsymbol{E})\right)^T = \exp(\boldsymbol{E})\exp\left(\boldsymbol{E}^T\right) = \exp\left(\boldsymbol{E}+\boldsymbol{E}^T\right) = \exp(\boldsymbol{0}) = \boldsymbol{I} \tag{2.38}$$

$$\det\left(\exp(\boldsymbol{E})\right) = \exp\left(\mathrm{Tr}(\boldsymbol{E})\right) = \exp(0) = +1 \tag{2.39}$$

Because $\exp(\boldsymbol{E})$ is proper orthogonal, the following relationship to the rotation matrix can be considered the definition of the Euler matrix.

$$\boldsymbol{C} = \exp(\boldsymbol{E}) \tag{2.40}$$

Based on this definition it is possible to relate the Euler matrix to the principal rotations and solve for $\boldsymbol{E}$. Although it is tempting to simply write $\boldsymbol{E} = \ln(\boldsymbol{C})$, it will be shown that this mapping is not unique because infinitely many solutions for $\boldsymbol{E}$ correspond to any particular orientation. Additionally, the matrix logarithm can suffer from limited range of convergence [11].

To relate the Euler matrix to the principal rotations, the spectral decomposition of $\boldsymbol{E}$ will be considered.

$$\boldsymbol{E} = \boldsymbol{W}_E\boldsymbol{\Lambda}_E\boldsymbol{W}_E^\dagger \tag{2.41}$$

Here, $\boldsymbol{W}_E$ is a matrix of the unit eigenvectors of $\boldsymbol{E}$. This matrix is not unique, however, because of the ambiguity in the complex-conjugate pairs of eigenvectors as well as the existence of multiple choices for ordering the eigenvectors within $\boldsymbol{W}_E$. The matrix exponential of Eq. (2.41) is shown below.

$$\exp(\boldsymbol{E}) = \boldsymbol{W}_E\exp(\boldsymbol{\Lambda}_E)\boldsymbol{W}_E^\dagger \tag{2.42}$$

Comparing this expression with the definition of the Euler matrix relates the eigenvalues of $\boldsymbol{C}$ and $\boldsymbol{E}$. By properly choosing the eigenvector ordering within $\boldsymbol{W}_C$ and $\boldsymbol{W}_E$, the following matrix equality can be set.

$$\boldsymbol{\Lambda}_C = \exp\left(\boldsymbol{\Lambda}_E\right) \tag{2.43}$$

Of course, because the eigenvalue matrices are diagonal the matrix exponential simplifies to the exponential of each eigenvalue element. Comparing this result with Eqs. (2.7) to (2.11) gives the eigenvalues of $\boldsymbol{E}$.

$$\lambda_1^{(E)} = i\left(\phi_1 + 2\pi n_1\right) \quad ; \quad \lambda_2^{(E)} = -i\left(\phi_1 + 2\pi n_1\right) \tag{2.44}$$

$$\vdots$$

$$\lambda_{N-2}^{(E)} = i\left(\phi_L + 2\pi n_L\right) \quad ; \quad \lambda_{N-1}^{(E)} = -i\left(\phi_L + 2\pi n_L\right) \tag{2.45}$$

$$\lambda_N^{(E)} = 0 \tag{2.46}$$

Next, the canonical form of the Euler matrix can be considered which relates $\boldsymbol{E}$ to a canonical transformation matrix and a block-diagonal skew-symmetric matrix $\boldsymbol{E}'$.

$$\boldsymbol{E} = \boldsymbol{P}^T \boldsymbol{E}' \boldsymbol{P} \tag{2.47}$$

Taking the exponential of Eq. (2.47) and comparing to Eq. (2.12) demonstrates that the same canonical transformation matrix $\boldsymbol{P}$ applied to $\boldsymbol{C}$ and $\boldsymbol{Q}$ can also be applied to $\boldsymbol{E}$. The matrix $\boldsymbol{C}'$ is the matrix exponential of $\boldsymbol{E}'$. The matrices $\boldsymbol{E}$ and $\boldsymbol{E}'$ share the same eigenvalues. Because $\boldsymbol{E}'$ is block-diagonal, however, its kth pair of eigenvalues are simply $\lambda_{2k-1}^{(E')} = iE'_{2k-1,2k}$ and $\lambda_{2k}^{(E')} = -iE'_{2k-1,2k}$. Comparing this with

Eqs. (2.44) to (2.46) gives the elements of $\boldsymbol{E}'$.

$$[\boldsymbol{E}'] = \begin{bmatrix} 0 & \phi_1 + 2\pi n_1 & \cdots & 0 & 0 & 0 \\ -\phi_1 - 2\pi n_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \phi_L + 2\pi n_L & 0 \\ 0 & 0 & \cdots & -\phi_L - 2\pi n_L & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{2.48}$$

Again, the sign convention above is chosen to be consistent with $\boldsymbol{C}'$ in Eq. (2.14). Because of the ambiguity in n_k, infinitely many values of $\boldsymbol{E}'$ and $\boldsymbol{E}$ exist which correspond to any particular $\boldsymbol{C}$.

C. Kinematics of Principal Rotations

In the previous section several methods to describe N-dimensional rotations in terms of the principal planes and angles were reviewed. In this section these results will be extended to relate the N-dimensional angular velocity and the derivatives of the principal planes and angles. This will result in kinematic differential equations for $\dot{\boldsymbol{P}}$ and $\dot{\phi}_k$.

Traditionally the kinematic evolution of N-dimensional rotations have not been related to the principal rotations. Instead, equations for the derivatives of $\boldsymbol{C}$ or $\boldsymbol{Q}$ are used directly. The first is provided by Poisson's equation.

$$\dot{\boldsymbol{C}} = -\boldsymbol{\Omega}\boldsymbol{C} \tag{2.49}$$

Here, $\boldsymbol{\Omega}$ is the N-dimensional skew-symmetric angular-velocity matrix. This equation and the Cayley transform can be used to derive the Cayley-transform kinematic relationships, which connect the derivative of $\boldsymbol{Q}$ to the angular-velocity matrix; these results were first developed by Junkins and Kim [12].

$$\boldsymbol{\Omega} = 2\left(\boldsymbol{I}+\boldsymbol{Q}\right)^{-1}\dot{\boldsymbol{Q}}\left(\boldsymbol{I}-\boldsymbol{Q}\right)^{-1} \quad ; \quad \dot{\boldsymbol{Q}} = \frac{1}{2}\left(\boldsymbol{I}+\boldsymbol{Q}\right)\boldsymbol{\Omega}\left(\boldsymbol{I}-\boldsymbol{Q}\right) \tag{2.50}$$

Whereas both Poisson's equation and the Cayley-transform kinematic relationships hold for any value of N, for three-dimensional rotations there also exists relationships between the angular velocity and the derivatives of the principal angle, ϕ, and principal axis, $\hat{\boldsymbol{a}}$. These can be developed by writing the rotation matrix $\boldsymbol{C}$ as a function of ϕ and $\hat{\boldsymbol{a}}$ and then taking the derivative. These expressions are then substituted into Eq. (2.49), which is solved for $\boldsymbol{\Omega}$ [13]. In fact, a similar procedure can be used to relate the N-dimensional angular velocity to the derivatives of $\boldsymbol{P}$ and ϕ_k.

The kinematic equations for the principal planes and angles of a four-dimensional rotation will be developed here. For four dimensions the canonical transformation matrix has the following form.

$$[\boldsymbol{P}]^T = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 \end{bmatrix} = \begin{bmatrix} [\boldsymbol{p}_1]_b & [\boldsymbol{p}_2]_b & [\boldsymbol{p}_3]_b & [\boldsymbol{p}_4]_b \end{bmatrix} \tag{2.51}$$

Again, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are orthogonal vectors lying in the first principal plane, whereas $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$ are orthogonal vectors lying in the second principal plane, completely orthogonal to the first. To develop the kinematic equations, the rotation matrix is written in terms of the two principal rotation matrices.

$$\begin{aligned} \boldsymbol{C} &= \boldsymbol{R}_1 + \boldsymbol{R}_2 - \boldsymbol{I} \\ &= \boldsymbol{I} + \left(\cos\phi_1 - 1\right)\boldsymbol{P}_1\boldsymbol{P}_1^T + \sin\phi_1\boldsymbol{P}_1\boldsymbol{J}\boldsymbol{P}_1^T + \left(\cos\phi_2 - 1\right)\boldsymbol{P}_2\boldsymbol{P}_2^T + \sin\phi_2\boldsymbol{P}_2\boldsymbol{J}\boldsymbol{P}_2^T \end{aligned} \tag{2.52}$$

$$\boldsymbol{C}^T = \boldsymbol{I} + \left(\cos\phi_1 - 1\right)\boldsymbol{P}_1\boldsymbol{P}_1^T - \sin\phi_1\boldsymbol{P}_1\boldsymbol{J}\boldsymbol{P}_1^T + \left(\cos\phi_2 - 1\right)\boldsymbol{P}_2\boldsymbol{P}_2^T - \sin\phi_2\boldsymbol{P}_2\boldsymbol{J}\boldsymbol{P}_2^T \tag{2.53}$$

$$
\begin{aligned}
\dot{\boldsymbol{C}} = &-\dot{\phi}_1 \sin\phi_1 \boldsymbol{P}_1\boldsymbol{P}_1^T + (\cos\phi_1 - 1)\left(\dot{\boldsymbol{P}}_1\boldsymbol{P}_1^T + \boldsymbol{P}_1\dot{\boldsymbol{P}}_1^T\right) + \dot{\phi}_1 \cos\phi_1 \boldsymbol{P}_1\boldsymbol{J}\boldsymbol{P}_1^T \\
&+ \sin\phi_1\left(\dot{\boldsymbol{P}}_1\boldsymbol{J}\boldsymbol{P}_1^T + \boldsymbol{P}_1\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\right) - \dot{\phi}_2 \sin\phi_2 \boldsymbol{P}_2\boldsymbol{P}_2^T + (\cos\phi_2 - 1)\left(\dot{\boldsymbol{P}}_2\boldsymbol{P}_2^T + \boldsymbol{P}_2\dot{\boldsymbol{P}}_2^T\right) \\
&+ \dot{\phi}_2 \cos\phi_2 \boldsymbol{P}_2\boldsymbol{J}\boldsymbol{P}_2^T + \sin\phi_2\left(\dot{\boldsymbol{P}}_2\boldsymbol{J}\boldsymbol{P}_2^T + \boldsymbol{P}_2\boldsymbol{J}\dot{\boldsymbol{P}}_2^T\right)
\end{aligned}
\tag{2.54}
$$

The product of Eqs. (2.53) and (2.54) is evaluated to find $\boldsymbol{\Omega} = -\dot{\boldsymbol{C}}\boldsymbol{C}^T$. This expansion is simplified using the following identities.

$$
\boldsymbol{p}_i^T\boldsymbol{p}_j = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{2.55}
$$

This implies the following.

$$
\boldsymbol{P}_1^T\boldsymbol{P}_1 = \boldsymbol{P}_2^T\boldsymbol{P}_2 = \boldsymbol{I} \quad ; \quad \boldsymbol{P}_1^T\boldsymbol{P}_2 = \boldsymbol{0} \tag{2.56}
$$

Additionally the square of the symplectic matrix is given by $\boldsymbol{J}\boldsymbol{J} = -\boldsymbol{I}$. Using these identities, the terms of the product are collected into terms containing the derivatives $\dot{\phi}_1$ and $\dot{\phi}_2$ and terms containing the derivatives $\dot{\boldsymbol{P}}_1$ and $\dot{\boldsymbol{P}}_2$.

$$\begin{aligned}\boldsymbol{\Omega} = & -\dot{\phi}_1 \boldsymbol{P}_1 \boldsymbol{J} \boldsymbol{P}_1^T - \dot{\phi}_2 \boldsymbol{P}_2 \boldsymbol{J} \boldsymbol{P}_2^T + \Big[(1-\cos\phi_1) \left(\boldsymbol{P}_1 \dot{\boldsymbol{P}}_1^T - \dot{\boldsymbol{P}}_1 \boldsymbol{P}_1^T \right) \\ & - \sin\phi_1 \left(\dot{\boldsymbol{P}}_1 \boldsymbol{J} \boldsymbol{P}_1^T + \boldsymbol{P}_1 \boldsymbol{J} \dot{\boldsymbol{P}}_1^T \right) - (1-\cos\phi_1)(1-\cos\phi_2) \boldsymbol{P}_1 \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_2 \boldsymbol{P}_2^T \\ & - (1-\cos\phi_1) \sin\phi_2 \boldsymbol{P}_1 \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_2 \boldsymbol{J} \boldsymbol{P}_2^T + \sin\phi_1 (1-\cos\phi_2) \boldsymbol{P}_1 \boldsymbol{J} \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_2 \boldsymbol{P}_2^T \\ & + \sin\phi_1 \sin\phi_2 \boldsymbol{P}_1 \boldsymbol{J} \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_2 \boldsymbol{J} \boldsymbol{P}_2^T - (1-\cos\phi_1)^2 \boldsymbol{P}_1 \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_1 \boldsymbol{P}_1^T \\ & + \sin\phi_1 (1-\cos\phi_1) \left(\boldsymbol{P}_1 \boldsymbol{J} \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_1 \boldsymbol{P}_1^T - \boldsymbol{P}_1 \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_1 \boldsymbol{J} \boldsymbol{P}_1^T \right) + \sin^2\phi_1 \boldsymbol{P}_1 \boldsymbol{J} \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_1 \boldsymbol{J} \boldsymbol{P}_1^T \Big] \\ & + \Big[(1-\cos\phi_2) \left(\boldsymbol{P}_2 \dot{\boldsymbol{P}}_2^T - \dot{\boldsymbol{P}}_2 \boldsymbol{P}_2^T \right) - \sin\phi_2 \left(\dot{\boldsymbol{P}}_2 \boldsymbol{J} \boldsymbol{P}_2^T + \boldsymbol{P}_2 \boldsymbol{J} \dot{\boldsymbol{P}}_2^T \right) \\ & - (1-\cos\phi_1)(1-\cos\phi_2) \boldsymbol{P}_2 \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_1 \boldsymbol{P}_1^T + \sin\phi_1 (1-\cos\phi_2) \boldsymbol{P}_2 \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_1 \boldsymbol{J} \boldsymbol{P}_1^T \\ & + (1-\cos\phi_1) \sin\phi_2 \boldsymbol{P}_2 \boldsymbol{J} \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_1 \boldsymbol{P}_1^T + \sin\phi_1 \sin\phi_2 \boldsymbol{P}_2 \boldsymbol{J} \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_1 \boldsymbol{J} \boldsymbol{P}_1^T \\ & - (1-\cos\phi_2)^2 \boldsymbol{P}_2 \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_2 \boldsymbol{P}_2^T + \sin\phi_2 (1-\cos\phi_2) \left(\boldsymbol{P}_2 \boldsymbol{J} \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_2 \boldsymbol{P}_2^T - \boldsymbol{P}_2 \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_2 \boldsymbol{J} \boldsymbol{P}_2^T \right) \\ & + \sin^2\phi_2 \boldsymbol{P}_2 \boldsymbol{J} \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_2 \boldsymbol{J} \boldsymbol{P}_2^T \Big] \end{aligned} \tag{2.57}$$

Further simplifications can be made by investigating the derivatives of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. These are found by using the orthogonality of $\boldsymbol{P}$.

$$\dot{\boldsymbol{P}} \boldsymbol{P}^T = \begin{bmatrix} \dot{\boldsymbol{P}}_1^T \\ \dot{\boldsymbol{P}}_2^T \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_1 & \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_2 \\ \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_1 & \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_2 \end{bmatrix} \tag{2.58}$$

Because $\dot{\boldsymbol{P}}$ must obey Poisson's equation the above product must be skew-symmetric (and in fact is related to the angular-velocity matrix of the principal frame relative to the rotated frame). This implies the following.

$$\dot{\boldsymbol{P}}_1^T \boldsymbol{P}_1 = -\boldsymbol{P}_1^T \dot{\boldsymbol{P}}_1 \quad ; \quad \dot{\boldsymbol{P}}_2^T \boldsymbol{P}_2 = -\boldsymbol{P}_2^T \dot{\boldsymbol{P}}_2 \quad ; \quad \dot{\boldsymbol{P}}_1^T \boldsymbol{P}_2 = -\left(\dot{\boldsymbol{P}}_2^T \boldsymbol{P}_1 \right)^T = -\boldsymbol{P}_1^T \dot{\boldsymbol{P}}_2 \tag{2.59}$$

Finally, the terms involving $\boldsymbol{J}$ can be collected using the following identities.

$$\dot{\boldsymbol{P}}_1^T \boldsymbol{P}_1 = \begin{bmatrix} \dot{\boldsymbol{p}}_1^T \\ \dot{\boldsymbol{p}}_2^T \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{p}}_1^T \boldsymbol{p}_1 & \dot{\boldsymbol{p}}_1^T \boldsymbol{p}_2 \\ \dot{\boldsymbol{p}}_2^T \boldsymbol{p}_1 & \dot{\boldsymbol{p}}_2^T \boldsymbol{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & \dot{\boldsymbol{p}}_1^T \boldsymbol{p}_2 \\ -\dot{\boldsymbol{p}}_1^T \boldsymbol{p}_2 & 0 \end{bmatrix} \tag{2.60}$$

$$\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 \\ -\dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 & 0 \end{bmatrix} = \begin{bmatrix} -\dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 & 0 \\ 0 & -\dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 \end{bmatrix} \tag{2.61}$$

$$\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_1\boldsymbol{J} = \begin{bmatrix} 0 & \dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 \\ -\dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 & 0 \\ 0 & -\dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 \end{bmatrix} = \boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_1 \tag{2.62}$$

$$\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_1\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 & 0 \\ 0 & -\dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 \\ \dot{\boldsymbol{p}}_1^T\boldsymbol{p}_2 & 0 \end{bmatrix} = -\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_1 \tag{2.63}$$

Of course, the identities analogous to those shown for $\boldsymbol{P}_1$ and $\dot{\boldsymbol{P}}_1$ in Eqs. (2.60) through (2.63) also hold for $\boldsymbol{P}_2$ and $\dot{\boldsymbol{P}}_2$. These lead to the following expression for $\dot{\boldsymbol{C}}\boldsymbol{C}^T$.

$$\begin{aligned}
\boldsymbol{\Omega} = &-\dot{\phi}_1\boldsymbol{P}_1\boldsymbol{J}\boldsymbol{P}_1^T - \dot{\phi}_2\boldsymbol{P}_2\boldsymbol{J}\boldsymbol{P}_2^T + \Big[(1-\cos\phi_1)\left(\boldsymbol{P}_1\dot{\boldsymbol{P}}_1^T - \dot{\boldsymbol{P}}_1\boldsymbol{P}_1^T\right) \\
&- \sin\phi_1\left(\dot{\boldsymbol{P}}_1\boldsymbol{J}\boldsymbol{P}_1^T + \boldsymbol{P}_1\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\right) - (1-\cos\phi_1)(1-\cos\phi_2)\,\boldsymbol{P}_1\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{P}_2^T \\
&- (1-\cos\phi_1)\sin\phi_2\boldsymbol{P}_1\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{J}\boldsymbol{P}_2^T + \sin\phi_1(1-\cos\phi_2)\,\boldsymbol{P}_1\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{P}_2^T \\
&+ \sin\phi_1\sin\phi_2\boldsymbol{P}_1\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{J}\boldsymbol{P}_2^T - 2(1-\cos\phi_1)\,\boldsymbol{P}_1\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_1\boldsymbol{P}_1^T\Big] \\
&+ \Big[(1-\cos\phi_2)\left(\boldsymbol{P}_2\dot{\boldsymbol{P}}_2^T - \dot{\boldsymbol{P}}_2\boldsymbol{P}_2^T\right) - \sin\phi_2\left(\dot{\boldsymbol{P}}_2\boldsymbol{J}\boldsymbol{P}_2^T + \boldsymbol{P}_2\boldsymbol{J}\dot{\boldsymbol{P}}_2^T\right) \\
&- (1-\cos\phi_1)(1-\cos\phi_2)\,\boldsymbol{P}_2\dot{\boldsymbol{P}}_2^T\boldsymbol{P}_1\boldsymbol{P}_1^T - \sin\phi_1(1-\cos\phi_2)\,\boldsymbol{P}_2\dot{\boldsymbol{P}}_2^T\boldsymbol{P}_1\boldsymbol{J}\boldsymbol{P}_1^T \\
&+ (1-\cos\phi_1)\sin\phi_2\boldsymbol{P}_2\boldsymbol{J}\dot{\boldsymbol{P}}_2^T\boldsymbol{P}_1\boldsymbol{P}_1^T + \sin\phi_1\sin\phi_2\boldsymbol{P}_2\boldsymbol{J}\dot{\boldsymbol{P}}_2^T\boldsymbol{P}_1\boldsymbol{J}\boldsymbol{P}_1^T \\
&-2(1-\cos\phi_2)\,\boldsymbol{P}_2\dot{\boldsymbol{P}}_2^T\boldsymbol{P}_2\boldsymbol{P}_2^T\Big]
\end{aligned} \tag{2.64}$$

This expression can be simplified by mapping it to the principal coordinate frame. This is done by applying the following similarity transformation.

$$\left[\boldsymbol{P}\boldsymbol{\Omega}\boldsymbol{P}^T\right] = \begin{bmatrix} \boldsymbol{P}_1^T \\ \boldsymbol{P}_2^T \end{bmatrix} [\boldsymbol{\Omega}] \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}_1^T\boldsymbol{\Omega}\boldsymbol{P}_1 & \boldsymbol{P}_1^T\boldsymbol{\Omega}\boldsymbol{P}_2 \\ \boldsymbol{P}_2^T\boldsymbol{\Omega}\boldsymbol{P}_1 & \boldsymbol{P}_2^T\boldsymbol{\Omega}\boldsymbol{P}_2 \end{bmatrix} \tag{2.65}$$

Whereas the similarity transformation $\boldsymbol{P}$ mapped the skew-symmetric matrices $\boldsymbol{Q}$ and $\boldsymbol{E}$ into a canonical form, here $\boldsymbol{\Omega}$ is a different matrix and will have a different canonical transformation. Equation (2.65) is skew-symmetric but not block-diagonal in general. The individual terms of Eq. (2.65) can be evaluated by applying the indicated transformations to Eq. (2.64).

$$\begin{aligned}\boldsymbol{P}_1^T\boldsymbol{\Omega}\boldsymbol{P}_1 = &-\dot{\phi}_1\boldsymbol{J} + (1-\cos\phi_1)\left(\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_1 - \boldsymbol{P}_1^T\dot{\boldsymbol{P}}_1\right)\\ &-\sin\phi_1\left(\boldsymbol{P}_1^T\dot{\boldsymbol{P}}_1\boldsymbol{J} + \boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_1\right) - 2(1-\cos\phi_1)\,\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_1\end{aligned} \tag{2.66}$$

These simplifications have taken advantage of the relationships between $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ in Eq. (2.56); however, additional simplifications can be made using the first of Eqs. (2.59) and Eq. (2.62).

$$\boldsymbol{P}_1^T\boldsymbol{\Omega}\boldsymbol{P}_1 = -\dot{\phi}_1\boldsymbol{J} \tag{2.67}$$

A similar procedure gives a corresponding result for the second block on the diagonal of Eq. (2.65).

$$\boldsymbol{P}_2^T\boldsymbol{\Omega}\boldsymbol{P}_2 = -\dot{\phi}_2\boldsymbol{J} \tag{2.68}$$

Equations (2.67) and (2.68) demonstrate that the components of the angular velocity in the $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ plane and $(\boldsymbol{p}_3, \boldsymbol{p}_4)$ plane are $-\dot{\phi}_1$ and $-\dot{\phi}_2$. The minus signs in these components are artifacts of the convention chosen to define the angular-velocity matrix as $\boldsymbol{\Omega} = -\dot{\boldsymbol{C}}\boldsymbol{C}^T$. These minus signs are entirely equivalent to the convention in defining the $(1, 2)$ component of the three-dimensional angular-velocity matrix as $-\omega_3$. Of course, this choice is made to make angular-velocity matrix multiplication equivalent to the angular-velocity vector cross product. These conventions are maintained for N-dimensions even though the angular-velocity vector and cross product lose their physical significance.

Clearly, the blocks on the diagonal of Eq. (2.65) are related to the derivatives of the principal angles. To relate the angular velocity and the derivatives of the principal planes, $\dot{\boldsymbol{P}}_1$ and $\dot{\boldsymbol{P}}_2$, the off-diagonal blocks of Eq. (2.65) must be computed.

$$\begin{aligned}\boldsymbol{P}_1^T\boldsymbol{\Omega}\boldsymbol{P}_2 = &\left(1-\cos\phi_1\right)\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2 - \sin\phi_1\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\\ &-\left(1-\cos\phi_1\right)\left(1-\cos\phi_2\right)\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2 - \left(1-\cos\phi_1\right)\sin\phi_2\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{J}\\ &+\sin\phi_1\left(1-\cos\phi_2\right)\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2 + \sin\phi_1\sin\phi_2\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{J}\\ &-\left(1-\cos\phi_2\right)\boldsymbol{P}_1^T\dot{\boldsymbol{P}}_2 - \sin\phi_2\boldsymbol{P}_1^T\dot{\boldsymbol{P}}_2\boldsymbol{J}\end{aligned}\tag{2.69}$$

Using the last of Eqs. (2.59), the derivatives of $\boldsymbol{P}_2$ can be recast as derivatives of $\boldsymbol{P}_1$.

$$\begin{aligned}\boldsymbol{P}_1^T\boldsymbol{\Omega}\boldsymbol{P}_2 = &\left(1-\cos\phi_1\right)\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2 - \sin\phi_1\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\\ &-\left(1-\cos\phi_1\right)\left(1-\cos\phi_2\right)\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2 - \left(1-\cos\phi_1\right)\sin\phi_2\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{J}\\ &+\sin\phi_1\left(1-\cos\phi_2\right)\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2 + \sin\phi_1\sin\phi_2\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{J}\\ &+\left(1-\cos\phi_2\right)\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2 + \sin\phi_2\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{J}\end{aligned}\tag{2.70}$$

Collecting terms gives the following final expression.

$$\begin{aligned}\boldsymbol{P}_1^T\boldsymbol{\Omega}\boldsymbol{P}_2 = &\left(1-\cos\phi_1\cos\phi_2\right)\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2 + \cos\phi_1\sin\phi_2\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{J}\\ &-\sin\phi_1\cos\phi_2\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2 + \sin\phi_1\sin\phi_2\boldsymbol{J}\dot{\boldsymbol{P}}_1^T\boldsymbol{P}_2\boldsymbol{J}\end{aligned}\tag{2.71}$$

Additionally, the last of Eqs. (2.59) can be used to rewrite this expression in terms of $\dot{\boldsymbol{P}}_1^T$, and the skew-symmetry of Eq. (2.65) can be used to find $\boldsymbol{P}_2^T\boldsymbol{\Omega}\boldsymbol{P}_1$ from Eq. (2.71).

$$\begin{aligned}\boldsymbol{P}_2^T\boldsymbol{\Omega}\boldsymbol{P}_1 = &\left(1-\cos\phi_1\cos\phi_2\right)\dot{\boldsymbol{P}}_2^T\boldsymbol{P}_1 + \sin\phi_1\cos\phi_2\dot{\boldsymbol{P}}_2^T\boldsymbol{P}_1\boldsymbol{J}\\ &-\cos\phi_1\sin\phi_2\boldsymbol{J}\dot{\boldsymbol{P}}_2^T\boldsymbol{P}_1 + \sin\phi_1\sin\phi_2\boldsymbol{J}\dot{\boldsymbol{P}}_2^T\boldsymbol{P}_1\boldsymbol{J}\end{aligned}\tag{2.72}$$

Equations (2.67), (2.68), (2.71), and (2.72) relate the angular velocity and the derivatives of the principal angles and planes for four-dimensional rotations. Knowing the current principal rotations and their derivatives, the angular velocity can be easily calculated using these equations and the similarity transformation. Knowing the angular velocity and the current principal planes, the derivatives of the principal angles can be easily calculated using Eqs. (2.67) and (2.68). Solving for the derivatives of the principal planes, however, using Eqs. (2.71) and (2.72) is more complicated.

The results relating the angular velocity and the principal-angle derivatives can be extended to general N-dimensions using the canonical form of $\boldsymbol{C}$. The derivative of the canonical form is taken as follows.

$$\dot{\boldsymbol{C}} = \dot{\boldsymbol{P}}^T \boldsymbol{C}' \boldsymbol{P} + \boldsymbol{P}^T \dot{\boldsymbol{C}}' \boldsymbol{P} + \boldsymbol{P}^T \boldsymbol{C}' \dot{\boldsymbol{P}} \tag{2.73}$$

As mentioned $\boldsymbol{P}$ itself is a rotation matrix describing the transformation from the rotated to principal frame. Its derivative is related to the angular velocity of the principal frame relative to the rotated frame. This skew-symmetric matrix is defined as $\boldsymbol{\Psi} = -\dot{\boldsymbol{P}}\boldsymbol{P}^T$, and $\boldsymbol{P}$ satisfies the Poisson equation $\dot{\boldsymbol{P}} = -\boldsymbol{\Psi}\boldsymbol{P}$. This is used to rewrite the derivative of $\boldsymbol{C}$.

$$\dot{\boldsymbol{C}} = \boldsymbol{P}^T \left(\dot{\boldsymbol{C}}' + \boldsymbol{\Psi}\boldsymbol{C}' - \boldsymbol{C}'\boldsymbol{\Psi} \right) \boldsymbol{P} \tag{2.74}$$

The angular velocity $\boldsymbol{\Omega}$ can now be written in terms of the canonical form $\boldsymbol{C}'$ and its derivative $\dot{\boldsymbol{C}}'$.

$$\begin{aligned} \boldsymbol{\Omega} = -\dot{\boldsymbol{C}}\boldsymbol{C}^T &= -\boldsymbol{P}^T \left(\dot{\boldsymbol{C}}' + \boldsymbol{\Psi}\boldsymbol{C}' - \boldsymbol{C}'\boldsymbol{\Psi} \right) \boldsymbol{P}\boldsymbol{P}^T \boldsymbol{C}'^T \boldsymbol{P} \\ &= -\boldsymbol{P}^T \left(\dot{\boldsymbol{C}}'\boldsymbol{C}'^T + \boldsymbol{\Psi} - \boldsymbol{C}'\boldsymbol{\Psi}\boldsymbol{C}'^T \right) \boldsymbol{P} \end{aligned} \tag{2.75}$$

The blocks on the diagonal of $\boldsymbol{C}'$ have the form shown in Eq. (2.14), and the derivative of $\boldsymbol{C}'$ will of course also be block diagonal with blocks of the following form.

$$\left[\dot{\boldsymbol{C}}'\right](2k-1:2k) = \dot{\phi}_k \begin{bmatrix} -\sin(\phi_k + 2\pi n_k) & \cos(\phi_k + 2\pi n_k) \\ -\cos(\phi_k + 2\pi n_k) & -\sin(\phi_k + 2\pi n_k) \end{bmatrix} \tag{2.76}$$

For odd N the (N, N) element of $\dot{\boldsymbol{C}}'$ forms a 1×1 block and is equal to zero. Because both $\boldsymbol{C}'$ and $\dot{\boldsymbol{C}}'$ are block diagonal, the product $\dot{\boldsymbol{C}}'\boldsymbol{C}'^T$ will also be block diagonal with blocks of the following form.

$$\begin{aligned} \left[\dot{\boldsymbol{C}}'\boldsymbol{C}'^T\right](2k-1:2k) &= \left[\dot{\boldsymbol{C}}'\right](2k-1:2k)\left[\boldsymbol{C}'^T\right](2k-1:2k) \\ &= \begin{bmatrix} 0 & \dot{\phi}_k \\ -\dot{\phi}_k & 0 \end{bmatrix} = \left[\dot{\phi}_k \boldsymbol{J}\right] \end{aligned} \tag{2.77}$$

Again, for odd N the (N, N) element of the product is zero. The remaining terms in parentheses on the right-hand side of Eq. (2.75) are skew symmetric but in general not block diagonal. To develop the relationship between the angular velocity and the principal angle derivatives, however, the blocks on the diagonal of these terms will be investigated. Because $\boldsymbol{C}'$ is block diagonal, the third term has the following form where the $2\pi n_k$ terms in $\boldsymbol{C}'$ have been dropped for convenience.

$$\begin{aligned} \left[\boldsymbol{C}'\boldsymbol{\Psi}\boldsymbol{C}'^T\right](2k-1:2k) &= \left[\boldsymbol{C}'\right](2k-1:2k)\left[\boldsymbol{\Psi}\right](2k-1:2k)\left[\boldsymbol{C}'^T\right](2k-1:2k) \\ &= \begin{bmatrix} \cos(\phi_k) & \sin(\phi_k) \\ -\sin(\phi_k) & \cos(\phi_k) \end{bmatrix} \begin{bmatrix} 0 & \Psi_{2k-1,2k} \\ -\Psi_{2k-1,2k} & 0 \end{bmatrix} \begin{bmatrix} \cos(\phi_k) & -\sin(\phi_k) \\ \sin(\phi_k) & \cos(\phi_k) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \Psi_{2k-1,2k} \\ -\Psi_{2k-1,2k} & 0 \end{bmatrix} \end{aligned} \tag{2.78}$$

The blocks on the diagonal of $\boldsymbol{C}'\boldsymbol{\Psi}\boldsymbol{C}'^T$ are identical to the blocks on the diagonal of $\boldsymbol{\Psi}$. Therefore, the only contribution to the blocks on the diagonal of Eq. (2.75) comes from Eq. (2.77).

$$\left[\boldsymbol{P\Omega P}^{T}\right](2k-1:2k) = -\left[\dot{\phi}_k \boldsymbol{J}\right] \tag{2.79}$$

This generalizes to any dimension with any number of planes the result found for the two principal planes of $N = 4$ in Eqs. (2.67) and (2.68).

D. Optimal Kinematic Maneuvers

It is well known that the minimum angular distance between two orientations in three dimensions is the principal angle associated with the rotation matrix relating them. A rigid-body rotational maneuver about the corresponding Euler axis through the principal angle is called an eigenaxis rotation, and this maneuver is nearly time optimal in most situations.

In higher-dimensional spaces, the concept of an eigenaxis rotation is understood as a collection of principal-plane rotations, each of which occurs on a principal plane through a principal angle. In this section it is demonstrated that the minimum angular distance between two orientations in higher-dimensional spaces is the L_2 vector norm of the vector arrangement of principal angles.

To begin, the distance between two orientations must be defined. The selection of a particular definition is analogous to the selection in three-dimensional mechanics of a definition for attitude error used in attitude estimation. Two such popular definitions of three-dimensional attitude error are the "multiplicative error" and the "additive error" [14]. Whereas the multiplicative error may have greater geometric significance, the additive error can be algebraically simpler to work with. For the current work, the following definition will be used for the distance between two N-dimensional orientations.

$$\|\boldsymbol{C}(t+\mathrm{d}t) - \boldsymbol{C}(t)\| = \|\mathrm{d}\boldsymbol{C}\| \tag{2.80}$$

Here, $\| \quad \|$ indicates the Frobenius matrix norm, $\|\boldsymbol{C}\| \equiv \sqrt{\mathrm{tr}\,(\boldsymbol{C}\boldsymbol{C}^T)}$. The integration of $\|\mathrm{d}\boldsymbol{C}\|$ can be used to obtain the minimum distance between the two orientations associated with a rotational path.

$$J_1 = \int_0^T \|\mathrm{d}\boldsymbol{C}\| = \int_0^T \left\|\dot{\boldsymbol{C}}\right\| \mathrm{d}t \tag{2.81}$$

The matrix $\dot{\boldsymbol{C}}$ obeys Poisson's equation.

$$\dot{\boldsymbol{C}} = -\boldsymbol{\Omega}\boldsymbol{C} \tag{2.82}$$

This kinematic equation can be used in Eq. (2.81) to obtain the following.

$$J_1 = \int_0^T \|-\boldsymbol{\Omega}\boldsymbol{C}\| \,\mathrm{d}t = \int_0^T \sqrt{\mathrm{tr}\,(\boldsymbol{\Omega}\boldsymbol{\Omega}^T)}\mathrm{d}t \tag{2.83}$$

The minimum distance between the two orientations is now given as the minimization of Eq. (2.83) subject to the kinematic equations, Eq. (2.82).

It is convenient to investigate a slightly modified problem.

$$J_2 = \frac{1}{2}\int_0^T \mathrm{tr}\left(\boldsymbol{\Omega}\boldsymbol{\Omega}^T\right) \mathrm{d}t \tag{2.84}$$

The minimization of Eq. (2.84) is now sought subject to the kinematic equations, Eq. (2.82). Because Eqs. (2.83) and (2.84) are related via a monotone transformation of the integrand, the minimization of one also minimizes the other.

Solving the optimal control problem will require manipulating the individual elements of the angular velocity, the rotation matrix, and the costates. Therefore it will be convenient to express the cost function and kinematic equations in index notation.

$$\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Omega}^T\boldsymbol{\Omega}\right) = \frac{1}{2}\left[\boldsymbol{\Omega}^T\boldsymbol{\Omega}\right]_{kk} = \frac{1}{2}\Omega_{jk}\Omega_{jk} \tag{2.85}$$

$$J_2 = \frac{1}{2}\int_0^T \Omega_{jk}\Omega_{jk}\mathrm{d}t \tag{2.86}$$

Here, the initial time is zero, and the final time is T. The problem is subject to the kinematic equations in Eq. (2.82) and repeated here in index notation.

$$\dot{C}_{ik} = -\Omega_{ij} C_{jk} \tag{2.87}$$

Without loss of generality, the boundary conditions are chosen such that the rotated frame is initially aligned with the reference frame, $\boldsymbol{C}(0) = \boldsymbol{I}$, and the final orientation is $\boldsymbol{C}(T) = \boldsymbol{F}$. The Hamiltonian for the problem is shown below.

$$H = \frac{1}{2}\Omega_{jk}\Omega_{jk} + \lambda_{ik}\left(-\Omega_{ij} C_{jk}\right) \tag{2.88}$$

From the Hamiltonian the first-order necessary conditions are found.

$$\dot{C}_{rs} = \frac{\partial H}{\partial \lambda_{rs}} = -\delta_{ir}\delta_{ks}\Omega_{ij}C_{jk} = -\Omega_{rj}C_{js} \tag{2.89}$$

$$\dot{\lambda}_{rs} = -\frac{\partial H}{\partial C_{rs}} = \lambda_{ik}\Omega_{ij}\delta_{jr}\delta_{ks} = \lambda_{is}\Omega_{ir} \tag{2.90}$$

$$\frac{\partial H}{\partial \Omega_{rs}} = \frac{1}{2}\delta_{jr}\delta_{ks}\Omega_{jk} + \frac{1}{2}\Omega_{jk}\delta_{jr}\delta_{ks} - \lambda_{ik}C_{jk}\delta_{ir}\delta_{js} = 0 \tag{2.91}$$

The third condition gives the following.

$$\Omega_{rs} = \lambda_{rk} C_{sk} \tag{2.92}$$

Next, the derivative of Eq. (2.92) is taken, and the first two conditions, Eqs. (2.89) and (2.90), are substituted to find the derivative of the optimal angular velocity.

$$\dot{\Omega}_{rs} = \dot{\lambda}_{rk}C_{sk} + \lambda_{rk}\dot{C}_{sk} = \lambda_{ik}\Omega_{ir}C_{sk} - \lambda_{rk}\Omega_{si}C_{ik} \tag{2.93}$$

Equation (2.92) itself can now be used, as well as the skew-symmetry of $\boldsymbol{\Omega}$.

$$\dot{\Omega}_{rs} = \Omega_{is}\Omega_{ir} - \Omega_{ri}\Omega_{si} = \Omega_{is}\Omega_{ir} - \Omega_{ir}\Omega_{is} = 0 \tag{2.94}$$

Therefore, the optimal angular velocity is a constant.

For constant angular velocity the kinematic equations have the following solution.

$$\boldsymbol{C}(t) = \exp(-\boldsymbol{\Omega} t)\, \boldsymbol{C}(0) \tag{2.95}$$

The value of the angular velocity can be related to the boundary conditions.

$$\boldsymbol{F} = \exp(-\boldsymbol{\Omega} T) \tag{2.96}$$

The implication of this is that for the optimal solution, the matrix $-\boldsymbol{\Omega} T$ is an Euler matrix of the final orientation. Of course, it was observed earlier that infinitely many Euler matrices exist for any particular orientation. The optimal angular velocity can be written using the canonical form of the Euler matrix.

$$[\boldsymbol{\Omega}] = \frac{-1}{T}[\boldsymbol{P}]^T \begin{bmatrix} 0 & \phi_1 + 2\pi n_1 & \cdots & 0 & 0 & 0 \\ -\phi_1 - 2\pi n_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \phi_L + 2\pi n_L & 0 \\ 0 & 0 & \cdots & -\phi_L - 2\pi n_L & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} [\boldsymbol{P}] \tag{2.97}$$

Therefore, infinitely many angular velocities satisfy the first-order optimality conditions, and it remains to be shown that selecting $n_k = 0$ gives the global minimum of the cost function.

Because the optimal angular velocity has been found to be a constant the cost function evaluated for the optimal solution can be rewritten as follows.

$$J_2 = \frac{T}{2}\mathrm{tr}\left(\boldsymbol{\Omega}^T \boldsymbol{\Omega}\right) \tag{2.98}$$

In the previous section it was observed that applying the canonical transformation $\boldsymbol{P}$ of the rotation matrix to the angular velocity does not in general produce a block-diagonal form. Because the optimal angular velocity is proportional to the Euler matrix of $\boldsymbol{F}$, however, Eq. (2.97) shows that in this case $\boldsymbol{P\Omega P}^T$ will be block diagonal. This will be defined as $\boldsymbol{\Omega}'$.

$$J_2 = \frac{T}{2}\mathrm{tr}\left(\boldsymbol{P}^T\boldsymbol{\Omega}'^T\boldsymbol{\Omega}'\boldsymbol{P}\right) = \frac{T}{2}\mathrm{tr}\left(\boldsymbol{\Omega}'^T\boldsymbol{\Omega}'\right) \tag{2.99}$$

The product $\boldsymbol{\Omega}'^T\boldsymbol{\Omega}'$, however, is a diagonal matrix, and its trace can be used to write the cost function as shown below.

$$J_2 = \frac{1}{2T}\left[(\phi_1 + 2\pi n_1)^2 + \ldots + (\phi_L + 2\pi n_L)^2\right] \tag{2.100}$$

Therefore, finding the optimal-angular velocity is reduced to minimizing each of the parenthetical terms above. This is clearly done by selecting each n_k equal to zero.

This gives the final result for the optimal angular velocity.

$$[\boldsymbol{\Omega}] = \frac{-1}{T}[\boldsymbol{P}]^T \begin{bmatrix} 0 & \phi_1 & \cdots & 0 & 0 & 0 \\ -\phi_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \phi_L & 0 \\ 0 & 0 & \cdots & -\phi_L & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} [\boldsymbol{P}] \tag{2.101}$$

Therefore the optimal maneuver is a rotation in each of the principal planes relating the initial and final orientations with a rotational rate of ϕ_k/T. As mentioned, this solution for the minimization of J_2 must also minimize J_1. The optimal cost for J_1 can be evaluated using the found angular velocity.

$$J_1 = T\sqrt{\mathrm{tr}\left(\boldsymbol{\Omega}\boldsymbol{\Omega}^T\right)} = \sqrt{\phi_1^2 + \ldots + \phi_L^2} \tag{2.102}$$

This demonstrates that the minimum angular distance between two N-dimensional orientations is indeed the L_2 vector norm of the vector arrangement of principal angles.

E. Conclusion

The ERPs and Euler matrix reviewed in this chapter are examples of an entire family of parameterizations that generalize three-dimensional attitude representations related to principal rotations. In these three-dimensional representations and their N-dimensional generalizations the parameters are undefined for certain orientations. For the Rodrigues parameters this occurs at principal rotations of $\pm\pi$. The modified Rodrigues parameters move this singularity back to $\pm 2\pi$. The asymptotic limit of this family, the Euler matrix, moves the singularity to $\pm\infty$. Of course, additional singularities occur for which the kinematic equations of these parameters are undefined.

The price of moving back the configuration singularity, however, is a loss of uniqueness. For the modified Rodrigues parameters two sets of parameters correspond to any particular orientation, and infinitely many Euler matrices describe the same orientation. Only the ERPs have a one-to-one mapping with the orientation matrix, which is given by the forward and inverse forms of the Cayley transform.

Whereas the description of N-dimensional orientations in terms of principal rotations has been studied previously, this chapter extended that description to the kinematic evolution of those orientations in time. The key result was found that the components of the angular velocity lying in the principal planes directly give the derivatives of the principal angles. Relating the angular velocity with the derivatives of the principal planes, however, was found to be more complicated for higher dimensions. The result for the principal angles, though, was used to find the minimum angular distance between two N-dimensional orientations. The optimal maneuver was shown to be the constant angular-rate principal-rotation reorientation.

CHAPTER III

MINIMUM-PARAMETER REPRESENTATIONS OF N-DIMENSIONAL PRINCIPAL ROTATIONS*

A. Introduction

In aerospace engineering the attitude of rigid bodies can be described using Euler's theorem or the Cayley transform. Euler's theorem describes any given orientation in terms of a principal rotation. The Cayley transform provides a minimum-parameter representation of an orientation. Of course, the relationship between these two descriptions is well known.

The attitude of three-dimensional bodies, however, is a subset of N-dimensional isometries. This chapter deals with proper linear isometries in $\Re^N$ as represented by the group of proper (or special) orthogonal matrices, $SO(N)$. Euler's theorem and the Cayley transform have both been generalized to describe $SO(N)$. The generalization of the Cayley transform was first performed by Cayley himself [8], and the generalization of Euler's theorem was developed by Schoute [1]. Recent treatments of these topics have been written by Bar-Itzhack [10], Bar-Itzhack and Markley [5], Mortari [2], and Bauer [3].

The relationship between the principal-rotation and Cayley-transform descriptions of three-dimensional orientations has been an important tool in the study of spacecraft attitude dynamics, control, and estimation. Recent studies have developed a representation of general mechanical-system dynamics as N-dimensional rota-

*Reprinted with permission from "Minimum-Parameter Representations of N-Dimensional Principal Rotations" by A. J. Sinclair and J. E. Hurtado, 2004. *Proceedings of the 6th International Conference on Dynamics and Control of Systems and Structures in Space: 2004* by Cranfield University Press, Cranfield, UK.

tional motions based on the Cayley transform [15–17]. This motivates a desire for an improved understanding of the relationship between principal-rotation and Cayley-transform descriptions of $SO(N)$. These concepts and N-dimensional isometries, in general, have been extensively studied from the perspective of N-dimensional Euclidean geometry. The focus of this chapter is to develop this relationship from an engineering perspective, rather than to add to the rigorous developments that have been achieved in N-dimensional Euclidian geometry.

In the second section of the chapter the descriptions of $SO(N)$ using Euler's theorem and the Cayley transform are reviewed and compared to the familiar concepts in spacecraft attitude. In the third section a new minimal parameterization of $SO(N)$ is proposed that is directly related to the principal rotations. Two numerical examples are also provided for representative values of N.

B. Review of N-Dimensional Orientations

The smallest dimensioned space that can allow rotational motion is two-dimensional. In this planar space, clearly any orientation can be achieved from any other by a single rotation. For higher-dimensional spaces, however, the situation becomes more complicated. Rotations in two-dimensional spaces, though, form the kernel by which higher-dimensional rotations are built up.

This is the basis of the principal-plane description of rotations in N-dimensional spaces, which was described by Mortari [2]. For even-dimensioned spaces, $N/2$ principal planes exist that are completely orthogonal to each other, and any given orientation can be described by rotations in such planes. For odd-dimensional spaces, the number of coordinate vectors has been increased by one over the next smallest even-dimensional space, and although this increases the dimensionality of the space, it is not enough to hold another principal plane. Therefore one vector is left out of the rotational motion. This is the principal axis of the odd-dimensioned space. Identifying this principal axis reduces rotations in odd-dimensions to the next smallest even-dimensioned space.

Rotations in higher dimensions have important differences from rotations in three-dimensions. For even-dimensioned spaces the dimensions are fully utilized in holding principal planes, and no principal axis exists. For spaces with odd dimension a principal axis does exist, but unlike the three-dimensional case, it will have more than one plane orthogonal to it. The even-dimensioned subspace orthogonal to the principal axis will hold several planes, and the rotation on each must be given to specify a particular orientation.

The mathematical representation of this principal-rotation description, which forms the N-dimensional Euler's theorem, comes from the eigenanalysis of the N-dimensional orientation matrix and was discussed by Mortari [2]. The transformation of an N-dimensional vector, $\boldsymbol{r}$, due to a rotation is given by $\boldsymbol{C}$, a proper orthogonal matrix [7].

$$\boldsymbol{r}' = \boldsymbol{C}\boldsymbol{r} \tag{3.1}$$

The eigenvalues of a proper orthogonal matrix lie on the unit circle in the complex plane and are conjugate pairs. If the matrix is odd dimensioned, then the "left-over" eigenvalue will equal $1 + i0$. The eigenvector associated with this eigenvalue is the principal axis of the rotation and is the only unit vector untransformed by the rotation. The eigenvectors associated with the kth conjugate pair of eigenvalues are themselves a conjugate pair, $\frac{\sqrt{2}}{2}(\boldsymbol{p}_{2k-1} \pm i\boldsymbol{p}_{2k})$. The normalized, real and imaginary parts of the pair, $\boldsymbol{p}_{2k-1}$ and $\boldsymbol{p}_{2k}$, are orthogonal unit vectors that form a principal plane. The kth conjugate pair of eigenvalues are related to the angle of rotation in this plane.

$$\lambda_k^{(C)} = \cos\phi_k \pm i \sin\phi_k \tag{3.2}$$

Another characterization of N-dimensional orientation matrices is provided by the Cayley transform.

$$\boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} + \boldsymbol{Q})^{-1} = (\boldsymbol{I} + \boldsymbol{Q})^{-1}(\boldsymbol{I} - \boldsymbol{Q}) \tag{3.3}$$

$$\boldsymbol{Q} = (\boldsymbol{I} - \boldsymbol{C})(\boldsymbol{I} + \boldsymbol{C})^{-1} = (\boldsymbol{I} + \boldsymbol{C})^{-1}(\boldsymbol{I} - \boldsymbol{C}) \tag{3.4}$$

The upper-triangular elements of the skew-symmetric matrix $\boldsymbol{Q}$ are a minimum-parameter representation of $\boldsymbol{C}$ and thus form an orientation representation. The number of independent elements in an $N \times N$ orthogonal matrix and the minimum number of parameters required to describe an N-dimensional orientation is $M = N(N-1)/2$. For $N = 3$ the elements of $\boldsymbol{Q}$ are the Rodrigues parameters. For higher dimensions the elements of $\boldsymbol{Q}$ are referred to as the extended Rodrigues parameters (ERPs) [5].

Euler's theorem and the Cayley-transform description can be somewhat linked by comparing the eigenvalues and eigenvectors of $\boldsymbol{C}$ and $\boldsymbol{Q}$. These matrices have the same eigenvectors, and their eigenvalues are related as shown below [2].

$$\lambda^{(C)} = \frac{1 - \lambda^{(Q)}}{1 + \lambda^{(Q)}} \quad ; \quad \lambda^{(Q)} = \frac{1 - \lambda^{(C)}}{1 + \lambda^{(C)}} \tag{3.5}$$

This implies the following relationship between the eigenvalues of $\boldsymbol{Q}$ and the rotation angles.

$$\lambda_k^{(Q)} = \mp i \tan\left(\frac{\phi_k}{2}\right) \tag{3.6}$$

Equation (3.5) shows that for odd N, the eigenvalue of $\boldsymbol{C}$ associated with the principal axis becomes a zero eigenvalue of $\boldsymbol{Q}$.

For $N = 3$ the relationship between the Rodrigues parameters and the principal rotation extends beyond the eigenvalues and eigenvectors of $\boldsymbol{Q}$. This connection, however, makes intrinsic use of the special properties of $N = 3$ such as plane-vector equivalency and $N = M$. In the remainder of this section a canonical representation of the ERPs is reviewed. This will be used to develop a minimum-parameter representation that is directly related to the principal rotations for all values of N.

The canonical representation of a skew-symmetric matrix decomposes the matrix into a proper orthogonal matrix, $\boldsymbol{P}$, and a block-diagonal skew-symmetric matrix, $\boldsymbol{Q}'$ [3, 9].

$$\boldsymbol{Q} = \boldsymbol{P}^T \boldsymbol{Q}' \boldsymbol{P} \quad ; \quad \boldsymbol{Q}' = \boldsymbol{P} \boldsymbol{Q} \boldsymbol{P}^T \tag{3.7}$$

The elements of this new skew-symmetric matrix $\boldsymbol{Q}'$ are referred to as the *canonical ERPs*. The similarity transformation enforces that $\boldsymbol{Q}$ and $\boldsymbol{Q}'$ share the same eigenvalues and their eigenvectors are related through $\boldsymbol{P}$. By convention the following form is chosen for $\boldsymbol{Q}'$ for even N.

$$[\boldsymbol{Q}'_{N \text{ even}}] = \begin{bmatrix} 0 & Q'_{12} & \cdots & 0 & 0 \\ -Q'_{12} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & Q'_{N-1,N} \\ 0 & 0 & \cdots & -Q'_{N-1,N} & 0 \end{bmatrix} \tag{3.8}$$

For odd N the form is similar with an appended row and column of zeros.

Equation (3.7) implies the following interpretation for a general set of ERPs, $\boldsymbol{Q}$, which represent the orientation of a coordinate system with coordinate vectors $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_N\}$ (i.e., a body frame) that is rotated relative to the reference coordinate system with coordinate vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \ldots, \boldsymbol{i}_N\}$ by the rotation matrix $\boldsymbol{C}$. For any set of ERPs there exists another coordinate system with coordinate vectors

$\{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N\}$ in which the principal rotation planes are aligned with the planes formed by these vectors: $(\boldsymbol{p}_1, \boldsymbol{p}_2)$, $(\boldsymbol{p}_3, \boldsymbol{p}_4)$, etc. Because of this alignment these vectors are called *principal coordinate vectors* and compose a *principal frame*. The rotation viewed in this frame results in a matrix $\boldsymbol{Q}'$ of block-diagonal form. The elements of this matrix are related to the principal rotations through Eq. (3.6). This explicitly decomposes the N-dimensional orientation problem into its constituent two-dimensional rotations. The mapping from the $\boldsymbol{b}$ frame to the $\boldsymbol{p}$ frame is given by $\boldsymbol{P}$.

For even N, Eq. (3.7) is rewritten as follows.

$$Q_{ij\text{ even}} = Q'_{12}\left(P_{1i}P_{2j} - P_{2i}P_{1j}\right) + \ldots + Q'_{N-1,N}\left(P_{N-1,i}P_{N,j} - P_{N,i}P_{N,j}\right) \tag{3.9}$$

A similar expression is obtained for the odd N case. The component P_{mi} represents the projection of the mth principal-coordinate vector, $\boldsymbol{p}_m$, into the ith body vector, $\boldsymbol{b}_i$. Therefore the product $P_{mi}P_{nj}$ can be considered as the projection of the $(\boldsymbol{p}_m, \boldsymbol{p}_n)$ principal plane into the $(\boldsymbol{b}_i, \boldsymbol{b}_j)$ body plane. Equation (3.9) shows that the (i, j) component of $\boldsymbol{Q}$ represents the projection of each of the principal rotations into the $(\boldsymbol{b}_i, \boldsymbol{b}_j)$ body plane. This constitutes a physical interpretation because, while the entire N-dimensional space cannot be physically visualized, each principal rotation is simply a two-dimensional rotation and is physically intuitive.

It is a well known property of the Rodrigues parameters that a singularity is encountered if the magnitude of the principal rotation is equal to π rad. Equations (3.6) and (3.9) show that the ERPs have a similar condition. The ERPs encounter a singularity as the magnitudes of any of the principal angles approach π rad because one or more elements of $\boldsymbol{Q}' \to \infty$.

C. Minimal Representations of Principal Rotations

The canonical ERP set, $\boldsymbol{Q}'$, has $N/2$ independent elements for the even case and $(N-1)/2$ independent elements for the odd case. Additionally, due to orthogonality constraints the $N \times N$ matrix $\boldsymbol{P}$ contains M independent elements. The general ERP set $\boldsymbol{Q}$, however, has only a total of M independent elements. This implies that infinitely many values of $\boldsymbol{P}$ will perform the mapping shown in Eq. (3.7).

A particular $\boldsymbol{P}$ can be considered by again applying the Cayley transform.

$$\boldsymbol{P} = (\boldsymbol{I} - \boldsymbol{S})(\boldsymbol{I} + \boldsymbol{S})^{-1} = (\boldsymbol{I} + \boldsymbol{S})^{-1}(\boldsymbol{I} - \boldsymbol{S}) \tag{3.10}$$

$$\boldsymbol{S} = (\boldsymbol{I} - \boldsymbol{P})(\boldsymbol{I} + \boldsymbol{P})^{-1} = (\boldsymbol{I} + \boldsymbol{P})^{-1}(\boldsymbol{I} - \boldsymbol{P}) \tag{3.11}$$

This Cayley transform defines a skew-symmetric matrix $\boldsymbol{S}$, which is an M-parameter representation of $\boldsymbol{P}$. This representation thus describes the orientation of the $\boldsymbol{p}$ frame relative to the $\boldsymbol{b}$ frame. In this form the elements of $\boldsymbol{S}$ that lie in the principal planes defined by $\boldsymbol{Q}'$ are clearly arbitrary. This suggests an *ansatz* for $\boldsymbol{S}$ that depends on N.

$$[\boldsymbol{S}_{N \text{ even}}] = \begin{bmatrix} 0 & 0 & \cdots & S_{1,N-1} & S_{1,N} \\ 0 & 0 & \cdots & S_{2,N-1} & S_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -S_{1,N-1} & -S_{2,N-1} & \cdots & 0 & 0 \\ -S_{1,N} & -S_{2,N} & \cdots & 0 & 0 \end{bmatrix} \tag{3.12}$$

$$[\boldsymbol{S}_{N\ \mathrm{odd}}] = \begin{bmatrix} 0 & 0 & \cdots & S_{1,N-2} & S_{1,N-1} & S_{1,N} \\ 0 & 0 & \cdots & S_{2,N-2} & S_{2,N-1} & S_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -S_{1,N-2} & -S_{2,N-2} & \cdots & 0 & 0 & S_{N-2,N} \\ -S_{1,N-1} & -S_{2,N-1} & \cdots & 0 & 0 & S_{N-1,N} \\ -S_{1,N} & -S_{2,N} & \cdots & -S_{N-2,N} & -S_{N-1,N} & 0 \end{bmatrix} \tag{3.13}$$

The off-diagonal elements that are non-zero in $\boldsymbol{Q}'$ are arbitrarily set to zero in $\boldsymbol{S}$, whereas the off-diagonal elements that are zero in $\boldsymbol{Q}'$ are non-zero in $\boldsymbol{S}$. The elements of these two matrices therefore combine to form a minimum-parameter (i.e., M-parameter) orientation representation in terms of the principal rotations. For any particular value of $\boldsymbol{Q}$, the associated $\boldsymbol{Q}'$ and $\boldsymbol{S}$ can be found by substituting the assumed forms for these matrices (Eqs. (3.8) and (3.12) for even N and similar for odd N) into Eq. (3.7). Using Eq. (3.10) to expand Eq. (3.7) gives the following.

$$\boldsymbol{Q}' = (\boldsymbol{I} - \boldsymbol{S})(\boldsymbol{I} + \boldsymbol{S})^{-1}\boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{S})^{-1}(\boldsymbol{I} + \boldsymbol{S}) \tag{3.14}$$

Evaluating these equations and setting the appropriate elements of $\boldsymbol{Q}'$ to zero provides $M - N/2$ (for even N) or $M - (N-1)/2$ (for odd N) equations for the non-zero elements of $\boldsymbol{S}$. This process is straightforward but for large values of N the expanded product in Eq. (3.14) can clearly involve a large number of terms. The equations can be easily developed, however, using a symbolic manipulator such as Maple. Once obtained, the equations are solved for the non-zero elements of $\boldsymbol{S}$ corresponding to any particular value of $\boldsymbol{Q}$. These elements can then be substituted into Eq. (3.14) to produce the non-zero elements of $\boldsymbol{Q}'$. It will be seen that for $N = 3$ the solution for $\boldsymbol{S}$ can be obtained analytically, and simulation results for higher dimensions indicate that a solution can be obtained numerically for general N.

The equations for the elements of $\boldsymbol{S}$ for $N = 3$ are developed as follows. The Rodrigues parameters have the form shown below.

$$[\boldsymbol{Q}] = \begin{bmatrix} 0 & Q_{12} & Q_{13} \\ -Q_{12} & 0 & Q_{23} \\ -Q_{13} & -Q_{23} & 0 \end{bmatrix} \tag{3.15}$$

For the sake of generality in notation, the components of $\boldsymbol{Q}$ are retained in their matrix notation instead of applying the vector notation specialized for $N = 3$. For this case $\boldsymbol{Q}'$ and $\boldsymbol{S}$ have the following forms.

$$[\boldsymbol{Q}'] = \begin{bmatrix} 0 & Q'_{12} & 0 \\ -Q'_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad [\boldsymbol{S}] = \begin{bmatrix} 0 & 0 & S_{13} \\ 0 & 0 & S_{23} \\ -S_{13} & -S_{23} & 0 \end{bmatrix} \tag{3.16}$$

From this form of $\boldsymbol{S}$ the Cayley transform is used to find $\boldsymbol{P}$.

$$[\boldsymbol{P}] = \frac{1}{1+S_{13}^2+S_{23}^2} \begin{bmatrix} 1 - S_{13}^2 + S_{23}^2 & -2S_{13}S_{23} & -2S_{13} \\ -2S_{13}S_{23} & 1 + S_{13}^2 - S_{23}^2 & -2S_{23} \\ 2S_{13} & 2S_{23} & 1 - S_{13}^2 - S_{23}^2 \end{bmatrix} \tag{3.17}$$

This, of course, is similar to the general expression for a rotation matrix in terms of a Rodrigues parameter set (with one identically-zero parameter). The elements of $\boldsymbol{Q}$ and $\boldsymbol{S}$ are related to the elements of $\boldsymbol{Q}'$, and Eqs. (3.15) and (3.17) are substituted into Eq. (3.7). This product can be expanded to produce the elements of $\boldsymbol{Q}'$. Setting Q'_{13} and Q'_{23} to zero yields the following two equations for the two unknown elements of $\boldsymbol{S}$.

$$2Q_{12}S_{23} + Q_{13}S_{13}^2 + 2Q_{23}S_{23}S_{13} - Q_{13}S_{23}^2 + Q_{13} = 0 \tag{3.18}$$

$$2Q_{12}S_{13} + Q_{23}S_{13}^2 - 2Q_{13}S_{23}S_{13} - Q_{23}S_{23}^2 - Q_{23} = 0 \tag{3.19}$$

For this case of $N = 3$ these equations can be solved analytically. First, for the situation $Q_{12} = Q_{13} = Q_{23} = 0$, infinitely many solutions exist: $S_{13} \in \Re$ and $S_{23} \in \Re$. For this case there is no rotation, and any plane can be considered the principal plane. Next, for the situation $Q_{12} \neq 0$ and $Q_{13} = Q_{23} = 0$ the equations admit a unique solution: $S_{13} = S_{23} = 0$. For this case the rotation is in the $(\boldsymbol{b}_1, \boldsymbol{b}_2)$ body plane, and the principal frame is aligned with the body frame.

Two additional special cases can be solved directly from Eqs. (3.18) and (3.19). For $Q_{13}=0$ and $Q_{23}\neq 0$ two real solutions exist: $S_{13}=(-Q_{12}\pm\sqrt{Q_{12}^2+Q_{23}^2})/Q_{23}$ and $S_{23} = 0$. For $Q_{13} \neq 0$ and $Q_{23} = 0$ there are also two real solutions: $S_{13} = 0$ and $S_{23} = \left(Q_{12} \pm \sqrt{Q_{12}^2 + Q_{13}^2}\right)/Q_{13}$.

The remaining general case of $Q_{13} \neq 0$ and $Q_{23} \neq 0$ can be solved by computing the Gröbner basis [18] of Eqs. (3.18) and (3.19). The result of this computation with stronger weight on S_{13} is a factorable, fourth-order polynomial in S_{23} and a second polynomial linear in S_{13}.

$$\begin{aligned}&\left[\left(Q_{13}^2 + Q_{23}^2\right) S_{23}^2 - 2Q_{12}Q_{13}S_{23} - Q_{13}^2\right]\\ &\quad\times\left[\left(Q_{13}^2 + Q_{23}^2\right) S_{23}^2 - 2Q_{12}Q_{13}S_{23} + Q_{12}^2 + Q_{23}^2\right] = 0\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\left(Q_{13}^3 Q_{23} + Q_{13}Q_{23}^3 + Q_{12}^2 Q_{13}Q_{23}\right) S_{13} + Q_{12}Q_{13}^3 + \left(Q_{13}^2 + Q_{23}^2\right)^2 S_{23}^3\\ &-3Q_{12}Q_{13}\left(Q_{13}^2 + Q_{23}^2\right) S_{23}^2 + \left(Q_{12}^2\left(2Q_{13}^2 + Q_{23}^2\right) - Q_{13}^4 + Q_{23}^4\right) S_{23} = 0\end{aligned} \tag{3.21}$$

The real solutions of Eq. (3.20) are shown below.

$$S_{23} = \frac{Q_{12}Q_{13} \pm Q_{13}\sqrt{Q_{12}^2 + Q_{13}^2 + Q_{23}^2}}{Q_{13}^2 + Q_{23}^2} \tag{3.22}$$

Associated with each real solution of S_{23} there is a unique solution for S_{13} from Eq. (3.21). Therefore Eqs. (3.18) and (3.19) have, in general, two real solutions with two exceptions for which there exists either a unique solution or infinitely many solutions.

Next, the relationship between these two solutions can be determined. For the general case of two solutions the convention is adopted that $\boldsymbol{S}_1$ is associated with the plus solution of Eq. (3.22), and $\boldsymbol{S}_2$ is associated with the minus solution. The Cayley transform can be applied to these two solutions to produce $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, respectively. These two proper orthogonal matrices define the transformation from the body frame to two different principal frames, $\boldsymbol{p}^{(1)}$ and $\boldsymbol{p}^{(2)}$. The relationship between the two principal frames can be examined by considering the mapping from the first to the second. This is found to be the following.

$$\boldsymbol{P}_2\boldsymbol{P}_1^T = \frac{1}{Q_{13}^2 + Q_{23}^2}\begin{bmatrix} Q_{13}^2 - Q_{23}^2 & 2Q_{13}Q_{23} & 0 \\ 2Q_{13}Q_{23} & -Q_{13}^2 + Q_{23}^2 & 0 \\ 0 & 0 & -Q_{13}^2 - Q_{23}^2 \end{bmatrix} \tag{3.23}$$

This transformation has the form of a rotation of π rad about an axis in the ($\boldsymbol{p}_1^{(1)}$,$\boldsymbol{p}_2^{(1)}$) plane. Therefore, the two principal frames differ by a *flipping* of the principal plane, and the ($\boldsymbol{p}_1^{(1)}$,$\boldsymbol{p}_2^{(1)}$) and ($\boldsymbol{p}_1^{(2)}$,$\boldsymbol{p}_2^{(2)}$) planes are coplanar. The flipping axis in the principal plane and the associated transformation are shown in Fig. 1. Equation (3.23) can be rewritten in terms of the angle this axis makes with the $\boldsymbol{p}_1^{(1)}$ vector.

$$\boldsymbol{P}_2\boldsymbol{P}_1^T = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) & 0 \\ \sin(2\theta) & -\cos(2\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.24}$$

The upper-left, two-by-two partition of this matrix performs the flipping operation on the principal plane. Partitions of this form are called *flipping partitions* and appear repeatedly in the higher-dimensional examples discussed in the next sections. Finally,

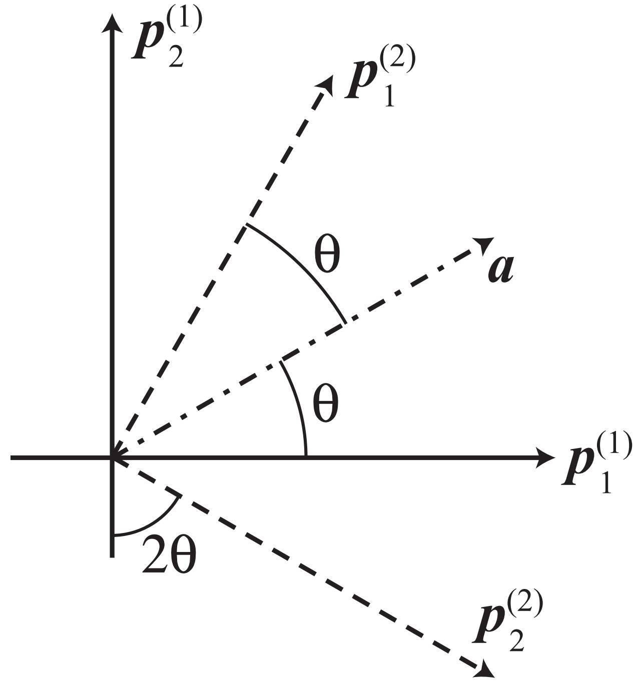

Fig. 1. Coordinatization of the principal plane by $\boldsymbol{p}^{(1)}$ and $\boldsymbol{p}^{(2)}$ frames, which are related by a flipping about the axis $\boldsymbol{a}$.

the angle θ can be found from the elements of $\boldsymbol{Q}$.

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2Q_{13}Q_{23}}{Q_{13}^2 - Q_{23}^2}\right) \tag{3.25}$$

So far, the analytical solution for the minimal-parameter representation was described for $N = 3$ as well as the relationship between the two solutions. Although many superior attitude representations exist for three dimensions, these results will serve as a basis for the study of higher-dimension representations in the following sections. For these higher dimensions, analytic solutions have not been found, and numerical analysis is discussed instead.

1. Numeric Analysis for $N = 4$

For the four-dimensional representation, equations for the non-zero elements of $\boldsymbol{S}$ are generated by setting the Q'_{13}, Q'_{14}, Q'_{23}, and Q'_{24} elements of Eq. (3.14) to zero. The resulting equations are fairly extended, however, and are not shown here. Additionally, analytic solutions to these equations have not been found. Monte Carlo simulation can be used, though, to gain some confidence in the existence of a solution.

A Monte Carlo simulation was performed to show that solutions for $\boldsymbol{S}$ exist for a distribution of $\boldsymbol{Q}$ values. Random values were generated for the independent elements of $\boldsymbol{Q}$, and then a numerical solution process was employed to find $\boldsymbol{S}$. Values for $\boldsymbol{Q}$ were selected with a uniform distribution between -1000 and 1000. The convergence of each trial was tested to show that each identically zero value of $\boldsymbol{Q}'$ evaluated from $\boldsymbol{Q}$ and the computed $\boldsymbol{S}$ was less than 10^{-8}. Ten thousand trials were run, and each of these cases successfully converged to an accurate solution. This gives some confidence, at least in an engineering sense, in the existence of a solution in general.

More detailed numerical studies were performed for the following example [2].

$$[\boldsymbol{C}] = \begin{bmatrix} 0.1003 & 0.2496 & -0.8894 & -0.3697 \\ 0.9593 & -0.0238 & -0.0153 & 0.2810 \\ -0.1172 & -0.8638 & -0.3828 & 0.3059 \\ -0.2366 & 0.4370 & -0.2495 & 0.8311 \end{bmatrix} \tag{3.26}$$

From this rotation matrix a set of ERPs is computed using the Cayley transform.

$$[\boldsymbol{Q}] = \begin{bmatrix} 0 & 1.0600 & 1.3893 & -0.1929 \\ -1.0600 & 0 & -1.5467 & -0.1091 \\ -1.3893 & 1.5467 & 0 & -0.6849 \\ 0.1929 & 0.1091 & 0.6849 & 0 \end{bmatrix} \tag{3.27}$$

Using this value of $\boldsymbol{Q}$, a second Monte Carlo simulation was performed to determine solutions for $\boldsymbol{S}$. This simulation selected random initial guesses for $\boldsymbol{S}$ using the same distribution used for $\boldsymbol{Q}$ in the first simulation. The resulting set of solutions for $\boldsymbol{S}$ was then analyzed to determine the number of different solutions. A total of eight different solutions were found. Six were found directly from the Monte Carlo simulation, and the geometric relationships between these six enabled the construction of two additional solutions.

These eight solutions can be labeled $\boldsymbol{S}_1$ through $\boldsymbol{S}_8$, and the Cayley transform defines the corresponding matrices $\boldsymbol{P}_1$ through $\boldsymbol{P}_8$. For concreteness, the solution $\boldsymbol{S}_1$ is shown below.

$$[\boldsymbol{S}_1] = \begin{bmatrix} 0 & 0 & 1.1900 & -0.1273 \\ 0 & 0 & 1.0787 & 0.0854 \\ -1.1900 & -1.0787 & 0 & 0 \\ 0.1273 & -0.0854 & 0 & 0 \end{bmatrix} \tag{3.28}$$

Similar to the multiple solutions for the $N = 3$ case, each value of $\boldsymbol{P}$ defines a principal frame: $\boldsymbol{p}^{(1)}$ to $\boldsymbol{p}^{(8)}$. The relationships between the various solutions are demonstrated by the products of the $\boldsymbol{P}$ matrices. The forms of these products are illustrated below.

The principal frames defined by $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are related by the following relative transformation.

$$\boldsymbol{P}_2\boldsymbol{P}_1^T = \begin{bmatrix} 0.1021 & 0.9948 & 0 & 0 \\ 0.9948 & -0.1021 & 0 & 0 \\ 0 & 0 & 0.9989 & -0.0464 \\ 0 & 0 & -0.0464 & -0.9989 \end{bmatrix} \tag{3.29}$$

Both of the blocks on the diagonal have the form of flipping partitions. Similar to the relationship between the two solutions for $N = 3$, this mapping can be viewed as the flipping of both principal planes.

The third solution is related to $\boldsymbol{P}_1$ as follows.

$$\boldsymbol{P}_3\boldsymbol{P}_1^T = \begin{bmatrix} 0 & 0 & -0.7266 & 0.6871 \\ 0 & 0 & -0.6871 & -0.7266 \\ 0.7266 & 0.6871 & 0 & 0 \\ -0.6871 & 0.7266 & 0 & 0 \end{bmatrix} \tag{3.30}$$

This transformation maps vectors in the $(\boldsymbol{p}_1^{(1)}, \boldsymbol{p}_2^{(1)})$ plane to vectors in the $(\boldsymbol{p}_3^{(3)}, \boldsymbol{p}_4^{(3)})$ plane, and vice versa. The mapping, therefore, geometrically represents a swapping of the two principal planes. The principal plane coordinatized by $\boldsymbol{p}_1^{(1)}$ and $\boldsymbol{p}_2^{(1)}$ in the first principal frame is coordinatized by $\boldsymbol{p}_3^{(3)}$ and $\boldsymbol{p}_4^{(3)}$ in the third principal frame.

The two geometric operations also define a fourth frame that is both flipped and swapped relative to the first frame.

$$\boldsymbol{P}_4\boldsymbol{P}_1^T = \begin{bmatrix} 0 & 0 & -0.7577 & -0.6526 \\ 0 & 0 & -0.6526 & 0.7577 \\ 0.7577 & 0.6526 & 0 & 0 \\ 0.6526 & -0.7577 & 0 & 0 \end{bmatrix} \tag{3.31}$$

In addition to these solutions, another set of four corresponding solutions were also found. For each solution $\boldsymbol{P}_1$ through $\boldsymbol{P}_4$ a *shadow solution* exists, $\boldsymbol{P}_5$ through $\boldsymbol{P}_8$, that has the relationship illustrated by $\boldsymbol{P}_1$ and $\boldsymbol{P}_5$.

$$\boldsymbol{P}_5\boldsymbol{P}_1^T = -\boldsymbol{I} \tag{3.32}$$

This transformation represents an inversion of each of the coordinate vectors. This is equivalent to a rotation of π rad in each of the principal planes. Equation (3.32) is equivalent to the following relationship between $\boldsymbol{S}_1$ and $\boldsymbol{S}_5$.

$$\boldsymbol{S}_5 = (\boldsymbol{I} - \boldsymbol{P}_5)(\boldsymbol{I} + \boldsymbol{P}_5)^{-1} = (\boldsymbol{I} + \boldsymbol{P}_1)(\boldsymbol{I} - \boldsymbol{P}_1)^{-1} = \boldsymbol{S}_1^{-1} \tag{3.33}$$

In summary, the eight solutions that were found for this example are related through the geometric properties of flipping, swapping, and shadow solutions. The transformations between the first four solutions are depicted in Fig. 2. The elements of $\boldsymbol{Q}'$ are computed from $\boldsymbol{Q}$ using Eq. (3.7). All solutions for $\boldsymbol{P}$ produce identical magnitudes for the $\boldsymbol{Q}'$ elements, however, the flipping and swapping operations change the sign and location of the magnitudes within $\boldsymbol{Q}'$. The $\boldsymbol{Q}'$ associated with $\boldsymbol{P}_1$ is shown below.

$$[\boldsymbol{Q}_1'] = \begin{bmatrix} 0 & -2.4396 & 0 & 0 \\ 2.4396 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1132 \\ 0 & 0 & -0.1132 & 0 \end{bmatrix} \tag{3.34}$$

From these canonical ERPs the principal-rotation angles are computed from Eq. (3.6).

$$\phi_1 = 2.3636 \text{ rad} \qquad ; \qquad \phi_2 = -0.2254 \text{ rad} \tag{3.35}$$

For $N = 3$ the two solutions were shown to be related through a flipping process, however, two additional geometric operations were found for $N = 4$. Clearly, the swapping process involves multiple principal planes and is not possible in three dimensions. The absence of shadow solutions for $N = 3$, however, is less obvious. The relationship between $\boldsymbol{P}_1$ for $N = 3$ and a candidate shadow solution $\boldsymbol{P}_3$ would represent a π-rad rotation in the principal plane.

$$\boldsymbol{P}_3\boldsymbol{P}_1^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \boldsymbol{H} \tag{3.36}$$

From Eq. (3.7) this candidate solution would clearly produce the correct $\boldsymbol{Q}'$.

$$\boldsymbol{P}_3\boldsymbol{Q}\boldsymbol{P}_3^T = \boldsymbol{H}\boldsymbol{P}_1\boldsymbol{Q}\boldsymbol{P}_1^T\boldsymbol{H} = \boldsymbol{H}\boldsymbol{Q}'\boldsymbol{H} = \boldsymbol{Q}' \tag{3.37}$$

The matrix $\boldsymbol{P}_1$, however, satisfies the form given in Eq. (3.17). Applying the mapping $\boldsymbol{H}$ to this form shows that $\boldsymbol{P}_3$ is a symmetric matrix. Therefore, each eigenvalue of $\boldsymbol{P}_3$ is either 1 or -1. Because each $\boldsymbol{P}$ is defined to be proper orthogonal, the eigenvalues of $\boldsymbol{P}_3$ are either $\{-1,-1,1\}$ or $\{1,1,1\}$. If the eigenvalues are $\{-1,-1,1\}$, then $\boldsymbol{P}_3$ represents a π-rad rotation away from the body frame and can not be described by any corresponding $\boldsymbol{S}_3$ (i.e., the shadow solution is at infinity). If the eigenvalues are $\{1,1,1\}$ then $\boldsymbol{P}_3$ is the identity matrix, and $\boldsymbol{P}_1 = \boldsymbol{H}$. This, however, implies that $\boldsymbol{S}_1$ does not exist and contradicts the results from the previous section. Therefore, while shadow solutions for $\boldsymbol{P}$ exist for $N = 3$ they represent π-rad rotations from the body frame and can not be described by a corresponding shadow $\boldsymbol{S}$. For $N = 4$ the application of $-\boldsymbol{I}$ produces no such symmetry in $\boldsymbol{P}$.

2. Numeric Analysis for $N = 5$

Equations for the five-dimensional representation of $\boldsymbol{S}$ are generated by setting the Q'_{13}, Q'_{14}, Q'_{15}, Q'_{23}, Q'_{24}, Q'_{25}, Q'_{35}, and Q'_{45} elements of Eq. (3.14) to zero. The resulting equations are easily generated symbolically, however, they are fairly extended and are not shown here. Again, analytic solutions to these equations have not been

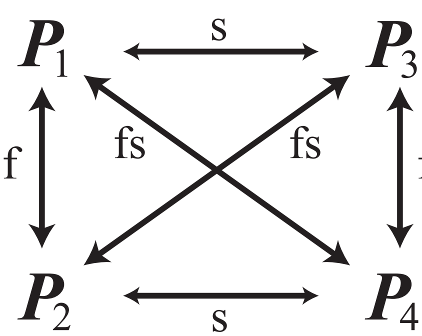

Fig. 2. Relationships between four of the eight solutions for $N = 4$: f - flip, s - swap, fs - flip and swap.

found. Monte Carlo simulation can be used, though, to gain some confidence in the existence of a solution.

Similar to the four-dimensional case, a Monte Carlo simulation was performed to show that solutions for $\boldsymbol{S}$ exist for a distribution of $\boldsymbol{Q}$ values. Random values were generated for the independent elements of $\boldsymbol{Q}$, and then a numerical solution process was employed to find $\boldsymbol{S}$. Values for $\boldsymbol{Q}$ were selected using the same distribution as the four-dimensional case, and the converged solutions were checked using the same test of numerical accuracy. Again ten thousand trials were run, and each of these cases successfully converged to an accurate solution.

Similar to the four-dimensional case a more detailed numerical study of the following example was performed.

$$[\boldsymbol{C}] = \begin{bmatrix} -0.5708 & -0.2224 & 0.4317 & -0.2972 & -0.5917 \\ 0.6799 & -0.6616 & 0.1856 & -0.1815 & -0.1806 \\ 0.4241 & 0.6000 & -0.0183 & 0.0554 & -0.6758 \\ -0.0505 & -0.0280 & -0.6987 & -0.7067 & -0.0955 \\ -0.1719 & -0.3899 & -0.5392 & 0.6134 & -0.3892 \end{bmatrix} \tag{3.38}$$

From this rotation matrix a set of ERPs is again computed using the Cayley transform.

$$[\boldsymbol{Q}] = \begin{bmatrix} 0 & 2.8562 & -0.3120 & -0.1737 & 1.4409 \\ -2.8562 & 0 & 0.7635 & 0.7395 & -1.5106 \\ 0.3120 & -0.7635 & 0 & -2.1241 & 0.8507 \\ 0.1737 & -0.7395 & 2.1241 & 0 & 2.4560 \\ -1.4409 & 1.5106 & -0.8507 & -2.4560 & 0 \end{bmatrix} \tag{3.39}$$

Another Monte Carlo simulation was performed using this value of $\boldsymbol{Q}$ to attempt to find multiple solutions for $\boldsymbol{S}$. Initial guesses for $\boldsymbol{S}$ were selected from the same distribution used previously, and the results from the numerical solution procedure were analyzed. A total of sixteen solutions were found for this example. Thirteen solutions were found directly from the Monte Carlo simulation, and three more were constructed from the geometric relations implied by the thirteen. For concreteness the solution $\boldsymbol{S}_1$ is shown below.

$$[\boldsymbol{S}_1] = \begin{bmatrix} 0 & 0 & -0.3111 & 0.1450 & -0.4123 \\ 0 & 0 & 0.0959 & 0.3898 & -0.2318 \\ 0.3111 & -0.0959 & 0 & 0 & -0.3816 \\ -0.1450 & -0.3898 & 0 & 0 & 0.0551 \\ 0.4123 & 0.2318 & 0.3816 & -0.0551 & 0 \end{bmatrix} \tag{3.40}$$

The rotation matrix, $\boldsymbol{P}_1$, associated with $\boldsymbol{S}_1$ is computed using the Cayley transform. Because the fifth axis of the principal frame was chosen to be aligned with the principal axis, the fifth column of every $\boldsymbol{P}^T$ (the fifth row of $\boldsymbol{P}$) is aligned with the principal axis in body coordinates.

In addition to the flipping, swapping, and shadow solutions observed for $N = 4$ a new geometric relationship was found for the $N = 5$ case. The new geometric operation is a flipping of one principal plane and a rotation in the other principal plane. Defining $\boldsymbol{P}_2$ to be a *flipped-rotated solution* relative to $\boldsymbol{P}_1$, the relative transformation matrix was found to be the following.

$$\boldsymbol{P}_2\boldsymbol{P}_1^T = \begin{bmatrix} -0.4203 & -0.9074 & 0 & 0 & 0 \\ -0.9074 & 0.4203 & 0 & 0 & 0 \\ 0 & 0 & 0.8879 & -0.4601 & 0 \\ 0 & 0 & 0.4601 & 0.8879 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.41}$$

Here, the (1-2,1-2) block has the form of a flipping partition. The (3-4,3-4) block, however, has the form of a rotation in a plane and is a *rotation partition*. Another solution, $\boldsymbol{P}_3$, is related to $\boldsymbol{P}_1$ by a rotation in the first principal plane and a flipping of the second plane. The relative transformation for this *rotated-flipped solution* is similar.

$$\boldsymbol{P}_3\boldsymbol{P}_1^T = \begin{bmatrix} 0.8600 & 0.5103 & 0 & 0 & 0 \\ -0.5103 & 0.8600 & 0 & 0 & 0 \\ 0 & 0 & -0.9952 & 0.0977 & 0 \\ 0 & 0 & 0.0977 & 0.9952 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.42}$$

Similar to the $N = 4$ case another solution exists that is related by a flipping of both principal planes.

$$\boldsymbol{P}_4\boldsymbol{P}_1^T = \begin{bmatrix} -0.6301 & 0.7765 & 0 & 0 & 0 \\ 0.7765 & 0.6301 & 0 & 0 & 0 \\ 0 & 0 & -0.9743 & -0.2253 & 0 \\ 0 & 0 & -0.2253 & 0.9743 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.43}$$

The relationships between the first four solutions are summarized in Fig. 3.

In addition to these four solutions another four solutions are related to $\boldsymbol{P}_1$ by a swapping of the principal planes. These relative transformations contain either a flipping or rotation partition in the (1-2,3-4) and (3-4,1-2) blocks and a positive or negative one in the $(5,5)$ element. Each of these eight solutions also has a shadow solution that is related by the following relative transformation.

$$\boldsymbol{P}_9\boldsymbol{P}_1^T = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.44}$$

Similar to the $N = 4$ case, this transformation performs a π-rad rotation in both principal planes.

The existence of flipped-rotated and rotated-flipped solutions for $N = 5$ but not $N = 4$ is related to the differences in sign of the $(5,5)$ elements of Eqs. (3.41) to (3.43). Because $\boldsymbol{P}$ is defined as a proper orthogonal matrix, the relative transformation between two solutions must also be proper. The determinant of the relative transformation will be equal to the product of the eigenvalues of each of the blocks on the diagonal. The product of the eigenvalues of a rotation partition is positive one. The product of the eigenvalues of a flipping partition, however, is negative one.

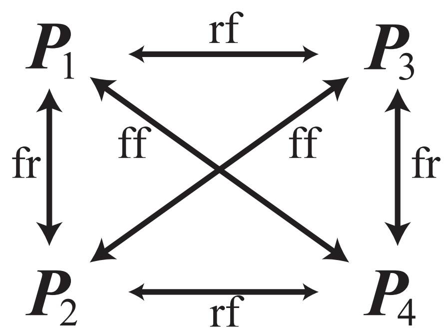

Fig. 3. Relationships between four of the sixteen solutions for $N = 5$: fr - flip-rotate, rf - rotate-flip, ff - flip-flip.

For $N = 4$ a double-flipping relative transformation has two blocks with negative one determinants and results in an overall proper transformation. For $N = 5$ the double flip can be accompanied by an identity transformation of the principal axis to also result in a proper transformation. The presence of the principal axis, however, introduces another possibility. A single flip and a rotate can be accompanied by an inversion of the principal axis to preserve properness. In even spaces flips must appear in pairs, but in odd spaces flips can either appear in pairs or appear singly along with inversion of the principal axis.

From $\boldsymbol{P}$ and $\boldsymbol{Q}$ the elements of $\boldsymbol{Q}'$ can once again be computed using Eq. (3.7). The matrix $\boldsymbol{Q}'_1$ is shown below.

$$[\boldsymbol{Q}'_1] = \begin{bmatrix} 0 & 4.0838 & 0 & 0 & 0 \\ -4.0838 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.8920 & 0 \\ 0 & 0 & 2.8920 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.45}$$

These canonical ERPs give the following principal rotation angles.

$$\phi_1 = -2.6613 \text{ rad} \qquad ; \qquad \phi_2 = 2.4758 \text{ rad} \tag{3.46}$$

D. Discussion

For $N = 3$ the Rodrigues parameters form a minimum-parameter attitude representation that is directly related to the principal-rotation description. The direct extension of these parameters to higher dimensions, the ERPs, maintain the minimum-parameter property, however, they lose the direct connection to the principal rotations. Historically, the connection between the ERPs and the principal-rotation description has been established using eigenanalysis or canonical forms. These techniques by themselves, however, destroy the minimum-parameter property. Through the evaluation of $\boldsymbol{Q}'$ and $\boldsymbol{S}$ this chapter has demonstrated one method of describing the principal rotations while maintaining a minimum-parameter representation. Perhaps most usefully, these matrices provide an interpretation for the ERP elements. Because these matrices form an orientation representation it should be possible to develop kinematic equations to directly relate their derivatives to the N-dimensional angular velocity. A result as elegant as the Cayley-transform kinematic relations for the ERP rates, however, is not anticipated, and the idea is not pursued further in this dissertation.

The evaluation of $\boldsymbol{Q}'$ and $\boldsymbol{S}$ that has been presented is similar to the eigenanalysis of $\boldsymbol{C}$ because both methods produce the principal angles, principal planes, and principal axis (if it exists) of any arbitrary orientation. The key difference between these two approaches comes in the representation of the principal planes. An important aspect of the canonical ERPs is that they provide a principal frame in which the various principal rotations become geometrically decoupled. This is a useful tool for describing N-dimensional orientations and relating them to the more familiar two and three-dimensional rotations. In particular this could be applied to the study of mechanical-system dynamics and control as represented by N-dimensional rotations.

CHAPTER IV

HAMEL COEFFICIENTS FOR THE ROTATIONAL MOTION OF AN N-DIMENSIONAL RIGID BODY*

A. Introduction

Although the field of mechanics has focused on continua and rigid bodies existing in three-dimensional space, there has also been interest in extending the principles that have been discovered to higher dimensions. In particular, methods have been developed to describe the orientation, rotation, and angular velocity of N-dimensional rigid bodies [2, 5–7, 10, 12, 19]. Work has also been done on developing the equations that govern the motion of rigid bodies in higher-dimensional spaces. In this regard, the primary focus has been on different representations of the equations and their integrability.

The various forms of the N-dimensional dynamic equations reported in the literature were constructed using geometric methods, namely, the Hamiltonian method of mechanics. This method takes a geometric view of mechanics, focusing on geometric structures called symplectic or Poisson structures [20]. These various forms were obtained from the accepted principle that, even in higher-dimensional spaces, the time derivative of the angular-momentum matrix equals the applied skew-symmetric matrix of torques; in the absence of torques, the time derivative of the angular-momentum matrix equals zero.

According to Fedorov and Kozlov, the idea of generalizing the rigid-body rotational equations was first put forth by Cayley [8,21]. The generalized Euler equations,

*Reprinted with permission from "Hamel coefficients for the rotational motion of an N-dimensional rigid body" by J. E. Hurtado and A. J. Sinclair, 2004. *Proceedings of the Royal Society of London Series A*, 460, 3613–3630. 2004 by The Royal Society.

which are also called the Euler–Frahm equations, represent the equations that govern the torque-free rotational motion of an N-dimensional rigid body. Fedorov and Kozlov mention that these equations were first developed by Frahm in the form given below [21, 22].

$$(I_i + I_j)\dot{\Omega}_{ij} = (I_i - I_j)\Omega_{ik}\Omega_{kj} \quad ; \qquad \text{no sum on } i, j; \quad i < j = 1, \ldots, N \tag{4.1}$$

Here, I_i is an element of the diagonal (principal) mass tensor in the moving frame. Frahm also considered the kinematics of the N-dimensional motion $\dot{\boldsymbol{C}} = -\boldsymbol{\Omega}\boldsymbol{C}$ and found a collection of first integrals for the combined kinematic and dynamic system [21]. Another early derivation of the generalized Euler equations shown in Eq. (4.1) was presented by Weyl [23].

More recently, Ratiu developed a form of the generalized Euler equations by investigating the free evolution of the angular-momentum matrix using geometric constructs [24]. Additionally, he developed the equations by extending a matrix equation of Dubrovin *et al.* [25]. The form of the equations, given below, is different in form than Eq. (4.1).

$$\dot{\boldsymbol{L}} = [\boldsymbol{L}, \boldsymbol{\Omega}] \tag{4.2}$$

Here, $\boldsymbol{L}$ is the skew-symmetric, angular-momentum matrix and $\boldsymbol{\Omega}$ is the skew-symmetric, angular-velocity matrix. Both of these matrices are measured in the moving-body frame and will be discussed later in the chapter. The operation on the right-hand side of Eq. (4.2) is the matrix Lie bracket defined by $[\boldsymbol{L}, \boldsymbol{\Omega}] \equiv \boldsymbol{L}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{L}$ [20]. The form exhibited in Eq. (4.2) is called a Lax pair representation. Ratiu [24] primarily discusses the complete integrability of the generalized Euler equations. He proves, in two separate ways, the involution of Manakov's integrals [26] and discusses another set of integrals discovered earlier by Mishchenko [27].

Fedorov and Kozlov present a discussion on 'viewing' the free motion of an N-dimensional rigid body [21]. A technique is presented in which one considers an M-dimensional rigid body ($M = N(N-1)/2$), called the kinematical body. The kinematical body lends itself to being analyzed using a generalized Poinsot model, but only in a limited way: the generalized Poinsot model can only provide information about the positions of the kinematical body but not its motion.

Bloch *et al.* give a remarkable symmetric representation of the N-dimensional rigid-body equations [28]. This representation arises from studying the free motion of the N-dimensional rigid body as an optimal-control problem [29, 30].

In this chapter, the equations of motion are developed for N-dimensional rigid bodies from Lagrange's equations using the angular-velocity components as quasi velocities. Here an N-dimensional rigid body will be defined as a system whose configuration can be completely defined by an $N \times N$ proper orthogonal matrix. A wide variety of dynamical systems can be modeled as N-dimensional rigid bodies using this definition [16], which relaxes some conditions used in earlier work. The chapter begins with a review of N-dimensional kinematics. Next, a new numerical relative tensor, χ^j_{ik}, is introduced and defined to allow the expression of N-dimensional kinematics in index notation. This symbol is then used to compute the Hamel coefficients for N-dimensional rotations. Finally, these coefficients allow the development of the rotational equations of motion for general rigid bodies.

B. Review of N-Dimensional Kinematics

The kinematics of rigid bodies in N-dimensional spaces has been developed through the work of many researchers. The transformation of an N-dimensional position vector, $\boldsymbol{p}$, due to a rotation is given by a proper orthogonal matrix, $\boldsymbol{C}$ [7].

$$\boldsymbol{r}' = \boldsymbol{C}\boldsymbol{r} \tag{4.3}$$

The orthogonality of $\boldsymbol{C}$ can be used to find its derivative.

$$\boldsymbol{C}\boldsymbol{C}^T = \boldsymbol{I} \tag{4.4}$$

$$\dot{\boldsymbol{C}}\boldsymbol{C}^T + \boldsymbol{C}\dot{\boldsymbol{C}}^T = \boldsymbol{0} \tag{4.5}$$

$$-\dot{\boldsymbol{C}}\boldsymbol{C}^T = \boldsymbol{C}\dot{\boldsymbol{C}}^T = (\dot{\boldsymbol{C}}\boldsymbol{C}^T)^T \tag{4.6}$$

This matrix will be denoted as $\boldsymbol{\Omega}$ and called the *angular-velocity matrix*.

$$-\dot{\boldsymbol{C}}\boldsymbol{C}^T = \boldsymbol{\Omega} \tag{4.7}$$

Consequently, the evolution of the orthogonal matrix $\boldsymbol{C}$ is governed by the following.

$$\dot{\boldsymbol{C}} = -\boldsymbol{\Omega}\boldsymbol{C} \tag{4.8}$$

For an N-dimensional space, the skew-symmetric angular-velocity matrix has M independent elements.

$$M = (N-1) + (N-2) + \ldots + 1 = \frac{1}{2}N\,(N-1) \tag{4.9}$$

The fact that $N = M$ for three-dimensional spaces leads to many remarkable simplifications and elegant solutions for the rotational dynamics in this familiar case. It is noteworthy that in an N-dimensional space, the N orthogonal coordinate vectors define M orthogonal planes. For example, in the case $N = 6$ the coordinate vectors **a**, **b**, **c**, **d**, **e** and **f** define fifteen orthogonal planes (see Table I). A physically sensible interpretation of the angular-velocity matrix, therefore, is to associate its (i, j) element with the rate of rotation in the plane defined by the ith and jth coordinate

Table I. EXAMPLE OF ORTHOGONAL PLANES FOR $N = 6$

ab	***ac***	***ad***	***ae***	***af***	5
bc	***bd***	***be***	***bf***		4
cd	***ce***	***cf***			3
de	***df***				2
ef					1
Total					15

vectors. This physical interpretation, however, will not be proven here and is not necessary for the current purposes.

Based on the angular-velocity matrix, an M-dimensional *angular-velocity vector*, $\boldsymbol{\omega}$, is defined. In general this vector is of different dimension than the body it corresponds to and does not exist in the same space. The elements, ω_i, of this vector are related to the matrix as shown below.

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & -\omega_M & \cdots & & & \cdots & \\ \omega_M & 0 & & & & \cdots & \\ \vdots & & \ddots & & & \cdots & \\ & & & 0 & -\omega_6 & \omega_5 & -\omega_4 \\ & & & \omega_6 & 0 & -\omega_3 & \omega_2 \\ \vdots & \vdots & \vdots & -\omega_5 & \omega_3 & 0 & -\omega_1 \\ & & & \omega_4 & -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{4.10}$$

Any $N \times N$ skew-symmetric matrix can be defined in terms of an M-dimensional vector using the above form, not just the angular-velocity matrix. This vector is referred to as the *generating vector* of the matrix. The definition given in Eq. (4.10) is somewhat arbitrary and is chosen to be consistent with the familiar three-dimensional form. In that special case, multiplication of the angular-velocity matrix and any arbitrary vector is equivalent to taking the cross product of the angular-velocity vector and the arbitrary vector. In general, however, this is not true. The general N-dimensional cross product can be represented as multiplication of a skew-symmetric matrix and a vector, but the matrix is actually composed of $N-2$ orthogonal, N-dimensional vectors [19]. Therefore, these *cross-product vectors* have $N(N-2)$ elements subject to P orthogonality constraints, leaving R independent elements.

$$P = (N-3) + (N-4) + \ldots + 1 = \frac{1}{2}N^2 - \frac{5}{2}N + 3 \tag{4.11}$$

$$R = N(N-2) - P = \frac{1}{2}N^2 + \frac{1}{2}N - 3 \tag{4.12}$$

The facts that for $N = 3$, one has $N = M = R$ and the single cross-product vector is the angular-velocity vector, are additional remarkable properties of rotational motion in three dimensions. For higher dimensions, R is larger than M and multiple cross-product vectors give the same skew-symmetric, angular-velocity matrix. This makes it difficult to use these vectors to parameterize the angular velocity of an N-dimensional body. If a method could be found, however, to select a particular set of cross-product vectors, then it seems possible to study N-dimensional rotations in terms of these vectors. This idea is not pursued further in this dissertation.

C. Definition of the Numerical Relative Tensor χ_{ik}^{j}

For the derivations in the subsequent sections it will be useful to have a compact form in index notation to describe the relations between the elements of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ given by Eq. (4.10). For $N = 3$ this is given by the Levi-Civita permutation symbol [31].

$$\Omega_{ik} = \epsilon_{ijk}\omega_j \tag{4.13}$$

For N-dimensions a new symbol, χ_{ik}^{j}, must be defined that will perform the same operation as ϵ_{ijk} in the three-dimensional case.

$$\Omega_{ik} = \chi_{ik}^{j}\omega_j \tag{4.14}$$

Like ϵ_{ijk}, the symbol χ_{ik}^{j} is a numerical relative tensor [32]. It is a generalization of the Levi-Civita symbol that for the case $N = 3$ simplifies to ϵ_{ijk}. The symbol χ_{ik}^{j} is not, however, the generalized Levi-Civita symbol discussed by Papastavridis and others [33, 34]. Also, whereas here χ_{ik}^{j} is used to relate the angular-velocity matrix and vector, it can be used with any skew-symmetric matrix and its generating vector. In this section some properties of this new numerical relative tensor will be examined.

First, it is important to notice that in Eq. (4.14) the index j is summed from 1 to M, whereas the indices i and k take on values from 1 to N. It is another remarkable and elegant property of the $N = 3$ case that the number of indices of χ_{ik}^{j} (always three) is equal to the range of each of the indices (three only for $N = 3$). From examination of Eq. (4.10) the values of χ_{ik}^{j} are clearly limited to $+1$, -1, and 0. Additionally it is clear that for a particular value of i and k, at most one value of j will give a nonzero value of χ_{ik}^{j}.

Another property of χ_{ik}^{j} can be deduced from the skew-symmetry of $\mathbf{\Omega}$.

$$\begin{aligned}
\Omega_{ik} &= -\Omega_{ki} \\
\chi_{ik}^{j}\omega_j &= -\chi_{ki}^{j}\omega_j \\
\left(\chi_{ik}^{j} + \chi_{ki}^{j}\right)\omega_j &= 0 \\
\chi_{ik}^{j} &= -\chi_{ki}^{j}
\end{aligned} \tag{4.15}$$

The final step above holds because the elements ω_j are independent. Equation (4.15) demonstrates that if $i = k$, then χ_{ik}^{j} equals zero for all values of j.

Table II. CORRESPONDING VALUES OF i, j, AND k

i	k	j
1	2	10
1	3	9
1	4	8
1	5	7
2	3	6
2	4	5
2	5	4
3	4	3
3	5	2
4	5	1

Based on these properties it is possible to deduce the values of χ_{ik}^{j} for any i, j, k, and N. Just as there are many possible ways to state the values of the Levi-Civita tensor, the values of χ_{ik}^{j} can be given by several equivalent definitions. One straightforward method would be to simply write out the i, k, and j groupings that produce +1 values for χ_{ik}^{j} for a particular value of N. Computation of χ_{ik}^{j} could then be performed using a series of logic tests. For the sake of generality and explicitness, however, a functional form of χ_{ik}^{j} is developed using a minimal amount of branching. In pursuit of this, the specific example of $N = 5$ is considered.

From Table II one observes that for a particular value of i, the largest value of j is given by the following.

$$\begin{aligned} j_{max} &= (N-i) + (N-i-1) + \ldots + 1 \\ &= (N-i) + \frac{1}{2}\left[(N-i)^2 - (N-i)\right] \\ &= \frac{1}{2}\left[(N-i)^2 + (N-i)\right] \end{aligned} \tag{4.16}$$

For these values of i and j, the value of k is $i+1$ and this is seen to be the minimum value of k. As k increases, j decreases from the maximum value given by Eq. (4.16). Therefore, the following relationship holds between i, j, and k for which $\chi^j_{ik} = (-1)^{i+k}$.

$$\begin{aligned} j &= \frac{1}{2}\left[(N-i)^2 + (N-i)\right] + (i+1) - k \\ &= \frac{1}{2}i^2 + \left(\frac{1}{2} - N\right) i - k + \frac{1}{2}N^2 + \frac{1}{2}N + 1 \end{aligned} \tag{4.17}$$

In developing the above relationship the upper-triangular elements of the skew-symmetric matrix were considered, assuming $k > i$. To use this expression for any element of the matrix, the pair k and i must be replaced with the pair x and y, where x is the larger value of the original pair and y is the smaller. In expressing x, y, and χ^j_{ik}, a useful function is $f(i,k)$ which returns $+1$ for $k > i$ and -1 for $k < i$.

$$f(i,k) = \frac{k-i}{\sqrt{(k-i)^2}} \tag{4.18}$$

Therefore the form of χ^j_{ik} is given by the following expression, which happens to be a function of the dimension N because of the form chosen in Eq. (4.10).

$$\chi^j_{ik} = \begin{cases} (-1)^{i+k} f(i,k) & \text{for} \quad j = z \quad \text{and} \quad i \neq k \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

where $x = \frac{1}{2}\left(i + k + \sqrt{(k-i)^2}\right)$; $y = \frac{1}{2}\left(i + k - \sqrt{(k-i)^2}\right)$;

$$z = \frac{1}{2}y^2 + \left(\frac{1}{2} - N\right) y - x + \frac{1}{2}N^2 + \frac{1}{2}N + 1$$

The inverse of χ^j_{ik} will also be considered. Equations (4.14) and (4.19) provide a mapping or sorting of the elements of a generating vector into the elements of the skew-symmetric matrix. The mapping from the matrix elements to the vector elements can also be considered. To accomplish this the numerical relative tensor ψ^j_{ik} is introduced which satisfies the following equation.

$$\omega_j = \psi^j_{ik}\Omega_{ik} \tag{4.20}$$

Equation (4.14) can be substituted into the above equation.

$$\begin{aligned} \delta_{jl}\omega_l &= \psi^j_{ik}\chi^l_{ik}\omega_l \\ \left(\delta_{jl} - \psi^j_{ik}\chi^l_{ik}\right)\omega_l &= 0 \\ \psi^j_{ik}\chi^l_{ik} &= \delta_{jl} \end{aligned} \tag{4.21}$$

Therefore ψ^j_{ik} can be considered the inverse of χ^l_{ik}. The properties of ψ^j_{ik} can now be investigated similar to χ^j_{ik}. First note, however, that Eq. (4.20) represents a mapping of N^2 matrix elements onto M vector elements. Therefore Eq. (4.20) does not completely define ψ^j_{ik} and several additional properties must be arbitrarily chosen to uniquely define ψ^j_{ik}. In particular, ψ^j_{ik} is selected such that it is nonzero if and only if the combination i, j and k produces a nonzero value of χ^j_{ik} and such that $\psi^j_{ik} = -\psi^j_{ki}$. This implies that in the summation of Eq. (4.20), elements Ω_{ik} not containing $\pm\omega_j$ will be multiplied by a coefficient $\psi^j_{ik} = 0$. Equation (4.10) therefore shows that only two nonzero terms of Eq. (4.20) will exist for any value of j. These terms will correspond with some pair of particular values i and k, and the summation in Eq. (4.20) can be written explicitly as the following.

$$\begin{aligned} \omega_j &= \psi^j_{ik}\Omega_{ik} + \psi^j_{ki}\Omega_{ki} \qquad \text{(no sum on } i,\ j, \text{ and } k) \\ &= \psi^j_{ik}\chi^j_{ik}\omega_j + \psi^j_{ki}\chi^j_{ki}\omega_j \\ &= \left(\psi^j_{ik}\chi^j_{ik} + \psi^j_{ki}\chi^j_{ki}\right)\omega_j \end{aligned} \tag{4.22}$$

Consequently, because the elements ω_j can be nonzero, the following equation must hold.

$$\psi^j_{ik}\chi^j_{ik} + \psi^j_{ki}\chi^j_{ki} = 1 \qquad \text{(no sum on } i,\ j, \text{ and } k) \tag{4.23}$$

Without the loss of generality, it may be assumed that the current values of i, j, and k produce $\chi^j_{ik} = 1$ and $\chi^j_{ki} = -1$. Additionally, it is true that ψ^j_{ik} must equal some nonzero value α, whereas ψ^j_{ki} must equal $-\alpha$.

$$(\alpha)(1) + (-\alpha)(-1) = 1 \quad \Longrightarrow \quad \alpha = \frac{1}{2} \tag{4.24}$$

Thus Eq. (4.24) and the above definitions show that for any possible values of i, j, and k the inverse symbol ψ^j_{ik} is equal to half of χ^j_{ik}. Therefore the ψ^j_{ik} notation is dropped as redundant, leaving the following useful relations.

$$\begin{aligned} \Omega_{ik} &= \chi^j_{ik}\omega_j \\ \omega_j &= \frac{1}{2}\chi^j_{ik}\Omega_{ik} \\ \frac{1}{2}\chi^j_{ik}\chi^l_{ik} &= \delta_{jl} \end{aligned} \tag{4.25}$$

In fact, Eq. (4.25) can be derived without explicitly introducing the inverse tensor, as shown in Appendix A.

Several additional useful properties of χ^j_{ik} can be established. The first is a generalization of the 'ϵ-δ identity': $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$. For N-dimensional systems the following 'χ-δ identity' will be proven.

$$\chi^i_{jk}\chi^i_{mn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \equiv \delta^{jk}_{mn} \tag{4.26}$$

The symbol on the far right-hand side of Eq. (4.26) is the generalized Kronecker delta [32]. Consider the inner-product of two arbitrary generating vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$. This product can be written in terms of the related skew-symmetric tensors.

$$u_i v_i = \left(\frac{1}{2}\chi^i_{jk}U_{jk}\right)\left(\frac{1}{2}\chi^i_{mn}V_{mn}\right) = \frac{1}{4}\chi^i_{jk}\chi^i_{mn}U_{jk}V_{mn} \tag{4.27}$$

This inner-product can also be written in the following manner.

$$u_i v_i = \left(\frac{1}{2}\chi^i_{jk}U_{jk}\right)v_i = \frac{1}{2}U_{jk}V_{jk} \tag{4.28}$$

Comparing equations (4.27) and (4.28) and using Kronecker delta substitution gives the following result.

$$\frac{1}{2}\chi^i_{jk}\chi^i_{mn}U_{jk}V_{mn} = U_{jk}V_{jk} = \delta_{jm}\delta_{kn}U_{jk}V_{mn} \tag{4.29}$$

Due to the skew-symmetry of $\boldsymbol{U}$ one can also write the following.

$$\frac{1}{2}\chi^i_{jk}\chi^i_{mn}U_{jk}V_{mn} = -U_{kj}V_{jk} = -\delta_{jn}\delta_{km}U_{kj}V_{nm} = -\delta_{jn}\delta_{km}U_{jk}V_{mn} \tag{4.30}$$

Next, equations (4.29) and (4.30) are summed.

$$\chi^i_{jk}\chi^i_{mn}U_{jk}V_{mn} = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\,U_{jk}V_{mn} \tag{4.31}$$

$$\left[\chi^i_{jk}\chi^i_{mn} - (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\right]\{U_{jk}V_{mn}\} = 0 \tag{4.32}$$

Equation (4.32) represents a series summing on the indices j, k, m, and n, and for convenience is rewritten in the following manner.

$$[c_{jkmn}]\,\{x_{jkmn}\} = 0 \tag{4.33}$$

Because the elements x_{jkmn} are not independent, the coefficients in the square brackets can not be set immediately to zero. The elements of x_{jkmn} have a certain structure due to the skew-symmetry of $\boldsymbol{U}$ and $\boldsymbol{V}$ described by the following two properties: (1) some elements x_{jkmn} are identically zero; (2) certain elements of x_{jkmn} are related to each other.

The first property implies that for particular values of j, k, m, and n the value of c_{jkmn} can not be determined from Eq. (4.33). These values, however, can be determined from direct evaluation of c_{jkmn}. An element x_{jkmn} will be identically equal to zero if $j = k$ or $m = n$. In either of these cases, Eq. (4.32) shows that one has $c_{jkmn} = 0$.

The second property implies that certain terms of the series in Eq. (4.33) will be related to each other. A four-term subseries of the summation in Eq. (4.33) is explicitly written below for particular values of j, k, m, and n.

$$\begin{aligned} &c_{jkmn}x_{jkmn} + c_{kjmn}x_{kjmn} + c_{jknm}x_{jknm} + c_{kjnm}x_{kjnm} \\ &\qquad = (c_{jkmn} - c_{kjmn} - c_{jknm} + c_{kjnm})\,x_{jkmn} \qquad \text{(no sum on } j,\ k,\ m, \text{ and } n) \end{aligned} \tag{4.34}$$

In writing the above equation the skew-symmetry of $\boldsymbol{U}$ and $\boldsymbol{V}$ were again used to switch indices of the elements x_{jkmn}. The parenthetical term above can be simplified using the definition of c_{jkmn}.

$$\begin{aligned}
&c_{jkmn} - c_{kjmn} - c_{jknm} + c_{kjnm} \qquad\qquad \text{(no sum on } j,\ k,\ m, \text{ and } n)\\
&\quad = \left[\chi^i_{jk}\chi^i_{mn} - (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\right] - \left[\chi^i_{kj}\chi^i_{mn} - (\delta_{km}\delta_{jn} - \delta_{kn}\delta_{jm})\right]\\
&\qquad - \left[\chi^i_{jk}\chi^i_{nm} - (\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn})\right] + \left[\chi^i_{kj}\chi^i_{nm} - (\delta_{kn}\delta_{jm} - \delta_{km}\delta_{jn})\right]\\
&\quad = 4\left[\chi^i_{jk}\chi^i_{mn} - (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\right]\\
&\quad = 4\, c_{jkmn}
\end{aligned} \tag{4.35}$$

Thus, the subseries of related terms can be written as $4\,(c_{jkmn})\,(x_{jkmn})$ (no sum on j, k, m, and n).

Using this result, the series in Eq. (4.33) can be expressed in the following condensed form.

$$[c_{jkmn}]\,\{y_{jkmn}\} = 0 \quad ; \quad j \le k; \quad m \le n \tag{4.36}$$

In this form, certain elements of y_{jkmn} are identically zero but are otherwise arbitrary. For the terms $y_{jkmn} = 0$ it has been shown that $c_{jkmn} = 0$; for the nonzero elements of y_{jkmn}, it must then be that $c_{jkmn} = 0$ for $j \le k$ and $m \le n$. By the skew-symmetry properties illustrated in Eq. (4.35) one can see that this implies $c_{jkmn} = 0$ for all j, k, m, and n, and the proof of Eq. (4.26) is complete.

Setting the index m equal to j in the χ-δ identity reveals another property.

$$\chi^i_{jk}\chi^i_{jn} = \delta_{jj}\delta_{kn} - \delta_{jn}\delta_{kj} = (N-1)\,\delta_{kn} \tag{4.37}$$

The properties of the χ^j_{ik} are summarized in Table III. As mentioned earlier, χ^j_{ik} is a generalization of the Levi-Civita permutation symbol. Due to the properties of $N = 3$ described in this and the previous sections, the Levi-Civita symbol combines properties relating to permutations, the cross-product operation, and skew-symmetric matrices. For higher dimensions, however, these properties become distinct and can

Table III. SUMMARY OF PROPERTIES RELATED TO χ_{ik}^{j}

generating vector to skew-symmetric matrix	$U_{ik} = \chi_{ik}^{j} u_j$
skew-symmetric matrix to generating vector	$u_j = \frac{1}{2}\chi_{ik}^{j} U_{ik}$
skew-symmetry in the lower indices	$\chi_{ik}^{j} = -\chi_{ki}^{j}$
upper-index identity	$\chi_{ik}^{j}\chi_{ik}^{l} = 2\,\delta_{jl}$
χ-δ identity	$\chi_{jk}^{i}\chi_{mn}^{i} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$
lower-index identity	$\chi_{jk}^{i}\chi_{jn}^{i} = (N-1)\,\delta_{kn}$

no longer be captured by a single symbol. In general, these concepts are represented by the generalized Levi-Civita permutation symbol, the N-dimensional cross product, and χ_{ik}^{j}; each of which simplify to the Levi-Civita symbol for $N = 3$. In the following sections the above kinematics for N-dimensional rotations will be used to develop N-dimensional rotational equations of motion using the elements of the angular-velocity vector as quasi velocities.

D. N-Dimensional Hamel Coefficients

Let $\boldsymbol{q}$ be a set of M generalized coordinates representing the orientation of an N-dimensional rigid body. Several possible sets of such parameters have been developed [2, 5, 6]. Let $\dot{\boldsymbol{q}}$ be the corresponding generalized velocities and let $\boldsymbol{\omega}$ be a set of quasi velocities which is the angular-velocity vector of the body. Assume that the generalized velocities and the quasi velocities can be related by the following linear mappings.

$$\boldsymbol{\omega} = \boldsymbol{B}(\boldsymbol{q})\,\dot{\boldsymbol{q}} \quad ; \quad \dot{\boldsymbol{q}} = \boldsymbol{A}(\boldsymbol{q})\,\boldsymbol{\omega} \tag{4.38}$$

Lagrange's equations of motion in terms of generalized coordinates and generalized velocities are shown below.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_0}{\partial \dot{q}_k}\right) - \frac{\partial T_0}{\partial q_k} = f_{0k} \tag{4.39}$$

Lagrange's equations of motion in terms of generalized coordinates and quasi velocities can also be established.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k}\right) + \gamma^r_{ka}\omega_a\frac{\partial T_1}{\partial \omega_r} - A_{rk}\frac{\partial T_1}{\partial q_r} = f_{1k} \tag{4.40}$$

Here, $T = T_0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $T = T_1(\boldsymbol{q}, \boldsymbol{\omega})$ are two different functional forms of the same kinetic-energy expression, and the generalized forces f_{1k} are defined as $f_{1k} \equiv A_{rk}f_{0r}$. Equation (4.40) is simply called the Lagrange equations for quasi coordinates [35]. Related to these equations are the Euler-Poincaré equations [20,36]. In one sense, the Lagrange equations for quasi coordinates are more general than the Euler-Poincaré equations because the Euler-Poincaré equations only apply to left-invariant Lagrangian systems. On the other hand, the Euler-Poincaré equations are more general because they govern the motion of Lagrangian systems corresponding to general Lie groups.

The three-index symbol γ^r_{ka} that appears in Eq. (4.40) represents the *Hamel coefficients*, which may be given by the following expression.

$$\gamma^r_{ka} \equiv \left(\frac{\partial B_{rj}}{\partial q_i} - \frac{\partial B_{ri}}{\partial q_j}\right) A_{jk}A_{ia} \tag{4.41}$$

These coefficients are also known as the *Hamel–Volterra transitivity coefficients* or the *Ricci–Boltzmann–Hamel three-index symbols* [33]. Note that the Hamel coefficient tensor $\boldsymbol{\gamma}$ is skew-symmetric in the lower indices:

$$\begin{aligned}
\gamma^r_{ak} &= \left(\frac{\partial B_{rj}}{\partial q_i} - \frac{\partial B_{ri}}{\partial q_j}\right) A_{ja} A_{ik} \\
&= \left(\frac{\partial B_{ri}}{\partial q_j} - \frac{\partial B_{rj}}{\partial q_i}\right) A_{ia} A_{jk} \\
&= -\left(\frac{\partial B_{rj}}{\partial q_i} - \frac{\partial B_{ri}}{\partial q_j}\right) A_{jk} A_{ia} \\
&= -\gamma^r_{ka}
\end{aligned} \tag{4.42}$$

One remarkable truth regarding the Hamel coefficients is that they are unique for a set of quasi coordinates and invariant to particular generalized coordinates [37].

Consider the rotational motion of an N-dimensional rigid body. Let the quasi velocities ω_k in Eq. (4.40) be elements of the angular-velocity vector. Recall Eq. (4.8), repeated here for convenience.

$$\dot{\boldsymbol{C}} = -\boldsymbol{\Omega}\boldsymbol{C} \tag{4.43}$$

The matrix $\boldsymbol{C}(\boldsymbol{q})$ is the orthogonal matrix that relates an N-dimensional inertial reference frame to an N-dimensional body-fixed reference frame, and $\boldsymbol{\Omega}$ is the skew-symmetric matrix representation of the angular velocity. This evolution equation is true regardless of the choice in orientation parameters that are used to relate the two reference frames. Defining $\boldsymbol{D} \equiv \boldsymbol{C}^T$, Eq. (4.43) may be rewritten.

$$\dot{\boldsymbol{D}} = \boldsymbol{D}\boldsymbol{\Omega} \tag{4.44}$$

Using index notation, Eq. (4.43) can be represented by the following.

$$-\dot{C}_{il} D_{lk} = \Omega_{ik} = \chi^j_{ik} \omega_j \tag{4.45}$$

Because the matrix $\boldsymbol{C}$ is a function of the generalized coordinates, Eq. (4.45) may be written in terms of partial derivatives.

$$\chi_{ik}^{j}\omega_j = -\frac{\partial C_{il}}{\partial q_p} D_{lk}\dot{q}_p \tag{4.46}$$

The partial derivative of this equation is taken with respect to the generalized coordinates.

$$\chi_{ik}^{j}\frac{\partial \omega_j}{\partial q_s} = -\frac{\partial^2 C_{il}}{\partial q_p \partial q_s} D_{lk}\dot{q}_p - \frac{\partial C_{il}}{\partial q_p}\frac{\partial D_{lk}}{\partial q_s}\dot{q}_p \tag{4.47}$$

And taking the partial derivative of this equation with respect to the generalized velocities gives the following.

$$\begin{aligned}\chi_{ik}^{j}\frac{\partial^2 \omega_j}{\partial q_s \partial \dot{q}_a} &= -\frac{\partial^2 C_{il}}{\partial q_p \partial q_s} D_{lk}\delta_{ap} - \frac{\partial C_{il}}{\partial q_p}\frac{\partial D_{lk}}{\partial q_s}\delta_{ap} \\ &= -\frac{\partial^2 C_{il}}{\partial q_a \partial q_s} D_{lk} - \frac{\partial C_{il}}{\partial q_a}\frac{\partial D_{lk}}{\partial q_s}\end{aligned} \tag{4.48}$$

Using index notation, the first of Eq. (4.38) becomes $\omega_j = B_{ja}\dot{q}_a$. Taking the partial derivatives of this expression gives the following.

$$\frac{\partial \omega_j}{\partial q_s} = \frac{\partial B_{ja}}{\partial q_s}\dot{q}_a \quad ; \quad \frac{\partial^2 \omega_j}{\partial q_s \partial \dot{q}_a} = \frac{\partial B_{ja}}{\partial q_s} \tag{4.49}$$

Consequently, Eq. (4.48) may be rewritten.

$$\chi_{ik}^{j}\frac{\partial B_{ja}}{\partial q_s} = -\frac{\partial^2 C_{il}}{\partial q_a \partial q_s} D_{lk} - \frac{\partial C_{il}}{\partial q_a}\frac{\partial D_{lk}}{\partial q_s} \tag{4.50}$$

Similarly, the free indices s and a can be exchanged.

$$\chi_{ik}^{j}\frac{\partial B_{js}}{\partial q_a} = -\frac{\partial^2 C_{il}}{\partial q_s \partial q_a} D_{lk} - \frac{\partial C_{il}}{\partial q_s}\frac{\partial D_{lk}}{\partial q_a} \tag{4.51}$$

Next, Eq. (4.51) is subtracted from Eq. (4.50) and the result is multiplied by $\dot{q}_a$.

$$\chi_{ik}^{j}\left(\frac{\partial B_{ja}}{\partial q_s} - \frac{\partial B_{js}}{\partial q_a}\right)\dot{q}_a = \frac{\partial C_{il}}{\partial q_s}\frac{\partial D_{lk}}{\partial q_a}\dot{q}_a - \frac{\partial C_{il}}{\partial q_a}\frac{\partial D_{lk}}{\partial q_s}\dot{q}_a \tag{4.52}$$

The partial derivatives in the first term on the right-hand side can be rewritten as follows.

$$\frac{\partial D_{lk}}{\partial q_a}\dot{q}_a = \dot{D}_{lk} = D_{lp}\chi^r_{pk}\omega_r \tag{4.53}$$

Similar steps are performed on the partial derivatives from the second term on the right-hand side.

$$\frac{\partial C_{il}}{\partial q_a}\dot{q}_a = \dot{C}_{il} = -\chi^r_{ip}\omega_r C_{pl} \tag{4.54}$$

These results can be substituted into Eq. (4.52).

$$\chi^j_{ik}\left(\frac{\partial B_{ja}}{\partial q_s} - \frac{\partial B_{js}}{\partial q_a}\right)\dot{q}_a = \frac{\partial C_{il}}{\partial q_s}D_{lp}\chi^r_{pk}\omega_r + \frac{\partial D_{lk}}{\partial q_s}\chi^r_{ip}\omega_r C_{pl} \tag{4.55}$$

Notice that substituting for $\omega_r = B_{rs}\dot{q}_s$ in Eqs. (4.53) and (4.54) gives the following.

$$\frac{\partial D_{lk}}{\partial q_s} = D_{lv}\chi^c_{vk}B_{cs} \quad ; \quad \frac{\partial C_{il}}{\partial q_s} = -\chi^v_{ic}C_{cl}B_{vs} \tag{4.56}$$

Using these expressions, Eq. (4.55) is rewritten.

$$\chi^j_{ik}\left(\frac{\partial B_{ja}}{\partial q_s} - \frac{\partial B_{js}}{\partial q_a}\right)\dot{q}_a = -\chi^v_{ic}C_{cl}B_{vs}D_{lp}\chi^r_{pk}\omega_r + D_{lv}\chi^c_{vk}B_{cs}\chi^r_{ip}\omega_r C_{pl} \tag{4.57}$$

The first term on the right-hand side may be manipulated in the following way.

$$\begin{aligned} -\chi^v_{ic}C_{cl}B_{vs}D_{lp}\chi^r_{pk}\omega_r &= -\chi^v_{ic}\delta_{cp}B_{vs}\chi^r_{pk}\omega_r \\ &= -\chi^v_{ic}B_{vs}\chi^r_{ck}\omega_r \end{aligned} \tag{4.58}$$

The second term on the right-hand side may be manipulated in a similar manner.

$$\begin{aligned} D_{lv}\chi^c_{vk}B_{cs}\chi^r_{ip}\omega_r C_{pl} &= \delta_{pv}\chi^c_{vk}B_{cs}\chi^r_{ip}\omega_r \\ &= \chi^c_{vk}B_{cs}\chi^r_{iv}\omega_r \\ &= \chi^v_{ck}B_{vs}\chi^r_{ic}\omega_r \end{aligned} \tag{4.59}$$

Using Eqs. (4.58) and (4.59) in Eq. (4.57) gives the following expression.

$$\chi^{j}_{ik}\left(\frac{\partial B_{ja}}{\partial q_s}-\frac{\partial B_{js}}{\partial q_a}\right)\dot{q}_a=\left(\chi^{v}_{ck}\chi^{r}_{ic}-\chi^{v}_{ic}\chi^{r}_{ck}\right)B_{vs}\omega_r \tag{4.60}$$

Now note the following.

$$\frac{\partial B_{ja}}{\partial q_s}=\frac{\partial B_{ja}}{\partial q_l}\delta_{ls} \quad ; \quad \frac{\partial B_{js}}{\partial q_a}=\frac{\partial B_{jl}}{\partial q_a}\delta_{ls} \tag{4.61}$$

These expressions may be used on the left-hand side of Eq. (4.60).

$$\begin{aligned}\chi^{j}_{ik}\left(\frac{\partial B_{ja}}{\partial q_l}-\frac{\partial B_{jl}}{\partial q_a}\right)\delta_{ls}\dot{q}_a&=\left(\chi^{v}_{ck}\chi^{r}_{ic}-\chi^{v}_{ic}\chi^{r}_{ck}\right)B_{vs}\omega_r\\ \chi^{j}_{ik}\left(\frac{\partial B_{ja}}{\partial q_l}-\frac{\partial B_{jl}}{\partial q_a}\right)A_{lv}B_{vs}\dot{q}_a&=\\ \chi^{j}_{ik}\left(\frac{\partial B_{ja}}{\partial q_l}-\frac{\partial B_{jl}}{\partial q_a}\right)A_{lv}B_{vs}A_{ar}\omega_r&=\end{aligned} \tag{4.62}$$

Using the definition of the Hamel coefficients from Eq. (4.41), and the skew-symmetry property $\gamma^{j}_{rv}=-\gamma^{j}_{vr}$, Eq. (4.62) is rewritten.

$$-\chi^{j}_{ik}B_{vs}\omega_r\gamma^{j}_{vr}=\left(\chi^{v}_{ck}\chi^{r}_{ic}-\chi^{v}_{ic}\chi^{r}_{ck}\right)B_{vs}\omega_r \tag{4.63}$$

The following can then be concluded.

$$\chi^{j}_{ik}\gamma^{j}_{vr}=\chi^{v}_{ic}\chi^{r}_{ck}-\chi^{v}_{ck}\chi^{r}_{ic} \tag{4.64}$$

Applying Eq. (4.25) allows the following simplifications.

$$\begin{aligned}\frac{1}{2}\chi^{m}_{ik}\chi^{j}_{ik}\gamma^{j}_{vr}&=\frac{1}{2}\chi^{m}_{ik}\left(\chi^{v}_{ic}\chi^{r}_{ck}-\chi^{v}_{ck}\chi^{r}_{ic}\right)\\ \delta_{mj}\gamma^{j}_{vr}&=\frac{1}{2}\chi^{m}_{ik}\left(\chi^{v}_{ic}\chi^{r}_{ck}-\chi^{v}_{ck}\chi^{r}_{ic}\right)\\ \gamma^{m}_{vr}&=\frac{1}{2}\chi^{m}_{ik}\left(\chi^{v}_{ic}\chi^{r}_{ck}-\chi^{v}_{ck}\chi^{r}_{ic}\right)\end{aligned} \tag{4.65}$$

Equation (4.65) gives the Hamel coefficients for N-dimensional rigid-body rotational motion.

The result in Eq. (4.65) can be specialized for the $N = 3$ case where $\chi^j_{ik} = \epsilon_{ijk}$.

$$\gamma^m_{vr} = \frac{1}{2}\epsilon_{imk}\left(\epsilon_{ivc}\epsilon_{crk} - \epsilon_{cvk}\epsilon_{irc}\right) \tag{4.66}$$

The ϵ-δ identity can be used to simplify the expression for this special case.

$$\begin{aligned}\gamma^m_{vr} &= \frac{1}{2}\epsilon_{imk}\left(\delta_{ir}\delta_{vk} - \delta_{ik}\delta_{rv} - \delta_{vi}\delta_{kr} + \delta_{vr}\delta_{ki}\right)\\ &= \frac{1}{2}\epsilon_{imk}\left(\delta_{ir}\delta_{vk} - \delta_{vi}\delta_{kr}\right)\\ &= \frac{1}{2}\left(\epsilon_{rmv} - \epsilon_{vmr}\right)\\ &= \epsilon_{rmv} = \epsilon_{vrm}\end{aligned} \tag{4.67}$$

This expression is identical to the result found from computing the Hamel coefficients strictly for three-dimensional rigid-body rotational motion [37].

E. Lagrange's Equations for N-Dimensional Angular Velocities

The result for the Hamel coefficients derived in the previous section can now be substituted into Eq. (4.40) to develop the equations of motion for N-dimensional, rotational dynamics. First the product in the second term of Eq. (4.40) is expanded.

$$\begin{aligned}\gamma^r_{ka}\omega_a &= \frac{1}{2}\chi^r_{ij}\left(\chi^k_{ic}\chi^a_{cj} - \chi^k_{cj}\chi^a_{ic}\right)\omega_a\\ &= \frac{1}{2}\chi^r_{ij}\left(\chi^k_{ic}\Omega_{cj} - \chi^k_{cj}\Omega_{ic}\right)\end{aligned} \tag{4.68}$$

This expression is then substituted into Eq. (4.40).

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k}\right) + \frac{1}{2}\chi^r_{ij}\left(\chi^k_{ic}\Omega_{cj} - \chi^k_{cj}\Omega_{ic}\right)\frac{\partial T_1}{\partial \omega_r} - A_{rk}\frac{\partial T_1}{\partial q_r} = f_{1k} \tag{4.69}$$

Equation (4.69) is the vector form of the rotational equations of motion of N-dimensional rigid bodies.

These M general equations of motion for N-dimensional space can be simplified to the familiar form for $N = 3$. Returning to Eq. (4.40), the three-dimensional Hamel coefficient found in Eq. (4.67) is substituted.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k}\right) + \epsilon_{kar}\omega_a \frac{\partial T_1}{\partial \omega_r} - A_{rk}\frac{\partial T_1}{\partial q_r} &= f_{1k} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k}\right) + \Omega_{kr} \frac{\partial T_1}{\partial \omega_r} - A_{rk}\frac{\partial T_1}{\partial q_r} &= f_{1k} \end{aligned} \tag{4.70}$$

As mentioned in the review of N-dimensional kinematics, for $N = 3$ the skew-symmetric matrix $\boldsymbol{\Omega}$ represents a cross product with the generating vector, $\boldsymbol{\omega}$. Additionally, for the case of rotational motion of a rigid body about a fixed point the kinetic energy can be represented by the following.

$$T_1 = \frac{1}{2} I_{ij}\omega_i\omega_j \tag{4.71}$$

This is independent of q_r, and therefore the equations of motion simplify.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k}\right) + \Omega_{kr} \frac{\partial T_1}{\partial \omega_r} = f_{1k} \tag{4.72}$$

In vector notation this leads to the familiar representation of Euler's rotational equations of motion about a fixed point.

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{f}_1 \tag{4.73}$$

The vector $\boldsymbol{f}_1$ is the vector of applied torques.

F. The Lax Pair Form Via the Lagrangian Method

From the vector form of the N-dimensional, rotational equations of motion, a matrix form of the N-dimensional rigid-body equations can be obtained in a straightforward way. Substituting the matrix forms of the generalized coordinates, $q_i = \frac{1}{2}\chi^i_{jk} Q_{jk}$, and the angular velocity, $\omega_i = \frac{1}{2}\chi^i_{jk}\Omega_{jk}$, gives a new expression for the kinetic energy: $T = \widetilde{T}_1\left(\boldsymbol{Q}, \boldsymbol{\Omega}\right)$. The equations of motion can then be simplified by using the chain rule.

$$\frac{\partial \widetilde{T}_1}{\partial \Omega_{ij}} = \frac{\partial T_1}{\partial \omega_r}\frac{\partial \omega_r}{\partial \Omega_{ij}} = \frac{1}{2}\chi^r_{ij}\frac{\partial T_1}{\partial \omega_r}$$

Using the above, Eq. (4.69) is rewritten as follows.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k}\right) + \left(\chi^k_{ic}\Omega_{cj} - \chi^k_{cj}\Omega_{ic}\right)\frac{\partial \widetilde{T}_1}{\partial \Omega_{ij}} - A_{rk}\frac{\partial T_1}{\partial q_r} = f_{1k} \tag{4.74}$$

This form is mapped into a matrix form by multiplication with χ^k_{jl}.

$$\chi^k_{jl}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k}\right) + \chi^k_{jl}\left(\chi^k_{ic}\Omega_{cd} - \chi^k_{cd}\Omega_{ic}\right)\frac{\partial \widetilde{T}_1}{\partial \Omega_{id}} = \chi^k_{jl}\left(f_{1k} + A_{rk}\frac{\partial T_1}{\partial q_r}\right) \tag{4.75}$$

For convenience the parenthetical term on the right-hand side is defined as g_k, and the vector $\boldsymbol{g}$ is used to generate the skew-symmetric matrix $\boldsymbol{G}$. The chain rule is again used to rewrite the derivatives in terms of the matrix components.

$$2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{jl}}\right) + \left(\chi^k_{jl}\chi^k_{ic}\Omega_{cd} - \chi^k_{jl}\chi^k_{cd}\Omega_{ic}\right)\frac{\partial \widetilde{T}_1}{\partial \Omega_{id}} = G_{jl} \tag{4.76}$$

To follow convention, the partial derivative of the kinetic energy with respect to $\boldsymbol{\omega}$ is defined as the angular-momentum vector, $\boldsymbol{l}$.

$$\frac{\partial T_1}{\partial \omega_k} = l_k \tag{4.77}$$

The partial derivative of the kinetic energy with respect to $\boldsymbol{\Omega}$ is expressed using the chain rule.

$$\frac{\partial T_1}{\partial \omega_k} = \frac{\partial \widetilde{T}_1}{\partial \Omega_{ij}} \frac{\partial \Omega_{ij}}{\partial \omega_k} = \chi^k_{ij} \frac{\partial \widetilde{T}_1}{\partial \Omega_{ij}} \tag{4.78}$$

The angular-momentum vector is used to generate the angular-momentum matrix, $\boldsymbol{L}$.

$$l_k = \frac{1}{2} \chi^k_{ij} L_{ij} \tag{4.79}$$

This gives the following result.

$$\frac{\partial \widetilde{T}_1}{\partial \Omega_{ij}} = \frac{1}{2} L_{ij} \tag{4.80}$$

The equations of motion can now be rewritten in terms of the angular-momentum matrix.

$$2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} L_{jl} \right) + \left(\chi^k_{jl} \chi^k_{ic} \Omega_{cd} - \chi^k_{jl} \chi^k_{cd} \Omega_{ic} \right) \frac{1}{2} L_{id} = G_{jl} \tag{4.81}$$

Next, the χ-δ identity is used.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} (L_{jl}) + \left(\left(\delta_{ji} \delta_{lc} - \delta_{jc} \delta_{li} \right) \Omega_{cd} - \left(\delta_{jc} \delta_{ld} - \delta_{jd} \delta_{lc} \right) \Omega_{ic} \right) \frac{1}{2} L_{id} = G_{jl} \\
\frac{\mathrm{d}}{\mathrm{d}t} (L_{jl}) + \frac{1}{2} \left(\Omega_{ld} L_{jd} - \Omega_{jd} L_{ld} - \Omega_{ij} L_{il} + \Omega_{il} L_{ij} \right) = G_{jl} \\
\frac{\mathrm{d}}{\mathrm{d}t} (L_{jl}) + \left(-L_{jd} \Omega_{dl} + \Omega_{jd} L_{dl} \right) = G_{jl}
\end{aligned} \tag{4.82}$$

In matrix notation, Eq. (4.82) is written as the following, where the square brackets represent the matrix Lie bracket [20].

$$\dot{\boldsymbol{L}} = (\boldsymbol{L}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{L}) + \boldsymbol{G} = [\boldsymbol{L}, \boldsymbol{\Omega}] + \boldsymbol{G} \tag{4.83}$$

Due to the inverse property of χ^i_{jk}, an alternative path from Eq. (4.74) to Eq. (4.83) can be taken by extracting the factor χ^i_{jk} instead of multiplying by χ^i_{jk}; however, this is not shown here. Note that because of the skew-symmetry of the angular-momentum matrix $\boldsymbol{L}$, there are M independent, first-order differential equations in this $N \times N$ matrix differential equation. Although extracting the M independent equations is straightforward, they will be functions of $\boldsymbol{L}$ and $\boldsymbol{\Omega}$, each with N^2 elements. To use these equations one must also recognize that each of these matrices has only M independent elements. A set of kinematic equations is also needed to complete the motion description. Two choices are the M-dimensional vector equation $\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{\omega}$, or M independent elements of the matrix equation $\dot{\boldsymbol{C}} = -\boldsymbol{\Omega}\boldsymbol{C}$.

Equation (4.83) is the same Lax pair representation that is obtained using the Hamiltonian method of mechanics [21,24,28]. This equation is the matrix form of the generalized Euler equations. Its derivation here differs from those that are typically reported. Typically, a path along the Hamiltonian method of mechanics is followed to arrive at the free form ($\boldsymbol{G} = \boldsymbol{0}$) of Eq. (4.83). This method takes a strong geometric view of mechanics while using the principle that the time derivative of the angular momentum of a free N-dimensional rigid body is zero. Abstract algebra is then used to express the time derivative in an arbitrary coordinate system as shown in Eq. (4.83). Note that $\boldsymbol{G} = \boldsymbol{0}$ means that the Hamiltonian approach considers systems that have kinetic-energy functions that are left-invariant (do not depend on the generalized coordinates) and are not subject to externally applied torques. Here, Eq. (4.83) is derived by following the Lagrangian method of mechanics. The Lagrangian method is more focused on variational principles than on geometric structures. Also note that the derivation of Eq. (4.83) has allowed a forcing term on the right-hand side as well as the possibility that the kinetic-energy function depends on the generalized coordinates.

G. Conclusions

To the student of three-dimensional rotational motion, many concepts are understood through an intuition gained from experience in the three-dimensional world. Study of N-dimensional rotations reveals which of these concepts apply to rotational motion in general and which are peculiar to three-dimensional space. In this chapter three remarkable properties of the $N = 3$ special case were discussed: (1) the number of angular-velocity components is equal to N; (2) the number of cross-product components is equal to N; and (3) the range of the indices of χ_{ik}^{j} is equal to the number of indices. The first two of these properties are related and derive from the fact that for $N = 3$ any vector has only one plane perpendicular to it. There is a vector-plane equivalency in three-dimensional space. The third property, however, is independent. It derives from the fact that, combined, a matrix and vector have three dimensions. This itself may be an artifact of the development of linear algebra to match our experience with $N = 3$.

A key to the development shown in this chapter for the equations of motion for N-dimensional rotations was the use of Lagrange's method. Unlike previous derivations, the current development provides a convenient vector form of the equations, allows the study of systems with forcing functions, and allows for the sensitivity of the kinetic energy to the generalized coordinates. In this chapter, a useful new numerical relative tensor was also developed that allows for the straightforward mapping between skew-symmetric and vectorial representations of motion variables and equations.

In methods based on directly generalizing Euler's equations it is difficult to include the concept of applied moments. Newton's and Euler's laws are examples of principles based on experience with $N = 3$: they are empirical relationships that have been found to model the motion of three-dimensional bodies. No one has any experience upon which to base an extension of these principles to higher dimensions. The concept of generalized forces in Lagrange's equations, however, provide a mathematical way to describe these concepts even if one can not physically understand them.

The current derivation shares a certain similarity to the previous derivations in regard to the fact that Lagrange's equation can be derived from Hamilton's principle. Therefore, by applying these equations to N-dimensional bodies it has been assumed that Hamilton's principle holds in the spaces that these bodies occupy. This is similar to the Hamiltonian-based derivations that have been produced in the past. The derivation of Lagrange's equations themselves is based on differential geometry of the coordinate space, whose dimension is independent of the dimension of the bodies it describes.

CHAPTER V

CAYLEY KINEMATICS AND THE CAYLEY FORM OF DYNAMIC EQUATIONS*

A. Introduction

There have been many studies to extend the principles behind rigid-body mechanics in three-dimensional space to principles that govern motion in higher-dimensional spaces. Some progress has been made in extending the kinematic concepts of orientation, rotation, and angular velocity [2, 5–7, 10, 12, 19], and the dynamic concepts of inertia, momenta, and impressed forces [20–24, 28].

The expression that defines the angular-velocity matrix in N-dimensional space is $\boldsymbol{\Omega} = -\dot{\boldsymbol{C}}\boldsymbol{C}^T$ [7]. The matrix $\boldsymbol{C}$ is an $N \times N$ proper orthogonal matrix which can represent a rotation about the origin, and the angular-velocity matrix, $\boldsymbol{\Omega}$, is an $N \times N$ skew-symmetric matrix with $M = N(N-1)/2$ independent elements. The rearranged form $\dot{\boldsymbol{C}} = -\boldsymbol{\Omega}\boldsymbol{C}$ is sometimes called *Poisson's equation*. Many properties of the N-dimensional angular-velocity matrix have been established [7], and perhaps one of the most significant is that only in three-dimensional space can the M independent elements of the angular-velocity matrix be considered as components of an angular-rate vector (i.e., only in three-dimensional space does $N = M$) [10].

Rigid rotations in N-dimensional space take place on an invariant plane and are related to the concept of the N-dimensional vector cross-product operation [2, 19]. A natural way to parameterize an N-dimensional rigid rotation is with the principal rotation angle and a set of two N-dimensional orthogonal vectors that define

*Reprinted with permission from "Cayley kinematics and the Cayley form of dynamic equations" by A. J. Sinclair and J. E. Hurtado, 2005. *Proceedings of the Royal Society of London Series A*, 461, 761–781. 2005 by The Royal Society.

the invariant (principal) plane [2, 3]. It is important to note that N-dimensional orientation is different than N-dimensional rotation: an arbitrary orientation in N-dimensional space is a product of a minimum set of rigid rotations and only for the $N = 3$ case can an arbitrary orientation be realized with one rigid rotation.

The Cayley transform and the Cayley-transform kinematic relationship are another set of results that have relevance in N-dimensional orientations and rotations. These important results have a direct bearing on this chapter and will be discussed more fully later.

The governing equations of motion for N-dimensional rigid bodies are often presented in a matrix form. The traditional approach to deriving the equations has been the Hamiltonian method of mechanics [20, 21, 24]. That method takes a geometric view and uses concepts in abstract algebra and coordinate-free differential forms to arrive at the matrix Lax pair representation of the N-dimensional generalized Euler equations. The focus in many of these studies has been on different representations of the equations and their integrability [24, 26–28]. Consequently, the systems that have been addressed are those that are not subject to externally applied torques and do not have kinetic-energy functions that depend on the generalized coordinates. In some earlier work the vector form of the N-dimensional generalized Euler equations was developed by using the Lagrangian method of mechanics [15]. This vector form is easily mapped into the matrix Lax pair form. The vector form of the equations was developed without any *a priori* selection of orientation parameters for the N-dimensional rigid body.

Here an N-dimensional rigid body will be defined as a system whose configuration can be completely defined by an $N \times N$ proper orthogonal matrix. It will be seen that a wide variety of mechanical systems can be modeled as N-dimensional rigid bodies using this definition, which relaxes some conditions used in earlier work. In this current chapter, the Lax pair form of the N-dimensional generalized Euler equations of motion are developed following an *a priori* decision to describe the system using the Cayley orientation and kinematic variables. The Cayley orientation and kinematic variables are then used to relate the motion of general mechanical systems to the motion of higher-dimensional rigid bodies.

B. Cayley Kinematics

Some of the most important and elegant concepts in N-dimensional kinematics relate to the Cayley transform. This famous relationship provides a unique mapping between proper orthogonal and skew-symmetric matrices [7].

$$\boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} + \boldsymbol{Q})^{-1} = (\boldsymbol{I} + \boldsymbol{Q})^{-1}(\boldsymbol{I} - \boldsymbol{Q}) \tag{5.1}$$

$$\boldsymbol{Q} = (\boldsymbol{I} - \boldsymbol{C})(\boldsymbol{I} + \boldsymbol{C})^{-1} = (\boldsymbol{I} + \boldsymbol{C})^{-1}(\boldsymbol{I} - \boldsymbol{C}) \tag{5.2}$$

Here, $\boldsymbol{C}$ is an $N \times N$ proper orthogonal matrix, whereas $\boldsymbol{Q}$ is an $N \times N$ skew-symmetric matrix. The elements of $\boldsymbol{Q}$ are an M-dimensional set of parameters that represent N-dimensional attitude. In fact $\boldsymbol{q}$, the generating vector of $\boldsymbol{Q}$, is the vector of extended Rodrigues parameters for N-dimensional spaces [5, 6].

In this section, the Cayley transform and the Cayley-transform kinematic relationship will be used to find a general expression for the extended Rodrigues parameter rates, $\dot{\boldsymbol{q}}$, in terms of the angular-velocity vector, $\boldsymbol{\omega}$. In the derivations that follow, it will be convenient to make the following designations.

$$\boldsymbol{I} + \boldsymbol{Q} \equiv \boldsymbol{A}^{+} \quad ; \quad \boldsymbol{I} - \boldsymbol{Q} \equiv \boldsymbol{A}^{-} \tag{5.3}$$

$$(\boldsymbol{I} + \boldsymbol{Q})^{-1} \equiv \boldsymbol{B}^{+} \quad ; \quad (\boldsymbol{I} - \boldsymbol{Q})^{-1} \equiv \boldsymbol{B}^{-} \tag{5.4}$$

The Cayley-transform kinematic relationships connect the derivatives of the M independent parameters of $\boldsymbol{Q}$ to the angular-velocity matrix [12].

$$\boldsymbol{\Omega} = 2\left(\boldsymbol{I}+\boldsymbol{Q}\right)^{-1}\dot{\boldsymbol{Q}}\left(\boldsymbol{I}-\boldsymbol{Q}\right)^{-1} = 2\boldsymbol{B}^{+}\dot{\boldsymbol{Q}}\boldsymbol{B}^{-} \tag{5.5}$$

$$\dot{\boldsymbol{Q}} = \frac{1}{2}\left(\boldsymbol{I}+\boldsymbol{Q}\right)\boldsymbol{\Omega}\left(\boldsymbol{I}-\boldsymbol{Q}\right) = \frac{1}{2}\boldsymbol{A}^{+}\boldsymbol{\Omega}\boldsymbol{A}^{-} \tag{5.6}$$

Equations (5.5) and (5.6) represent a linear mapping between the generalized velocities, $\dot{\boldsymbol{Q}}$, and a set of quasi velocities, $\boldsymbol{\Omega}$, for N-dimensional rotations. Whereas these equations are represented as linear transformations of $N \times N$, skew-symmetric matrices, they can be rewritten in the familiar form in terms of the linear transformation of an M-dimensional vector. In index notation Eq. (5.6) is written as follows.

$$\dot{Q}_{vp} = \frac{1}{2}A^{+}_{vk}\Omega_{kl}A^{-}_{lp} \tag{5.7}$$

The generalized-velocity and angular-velocity matrices are mapped into their vector forms using χ^{i}_{jk}.

$$\chi^{j}_{vp}\dot{q}_{j} = \frac{1}{2}A^{+}_{vk}\chi^{m}_{kl}\omega_{m}A^{-}_{lp} \tag{5.8}$$

Both sides of the above equation are multiplied by χ^{i}_{vp}.

$$\chi^{i}_{vp}\chi^{j}_{vp}\dot{q}_{j} = \frac{1}{2}\chi^{i}_{vp}A^{+}_{vk}\chi^{m}_{kl}\omega_{m}A^{-}_{lp} \tag{5.9}$$

$$2\delta_{ij}\dot{q}_{j} = \frac{1}{2}\chi^{i}_{vp}\chi^{m}_{kl}A^{+}_{vk}A^{-}_{lp}\omega_{m} \tag{5.10}$$

$$\dot{q}_{i} = \frac{1}{4}\chi^{i}_{vp}\chi^{m}_{kl}A^{+}_{vk}A^{-}_{lp}\omega_{m} \equiv A_{im}\omega_{m} \tag{5.11}$$

The two-index variable A_{im} represents the components of the matrix $\boldsymbol{A}$ that performs a linear mapping of the angular-velocity vector onto the generalized-velocity vector. Similarly, an expression can be found for the matrix $\boldsymbol{B}$ that performs the inverse mapping; however, this matrix is a function of $\boldsymbol{B}^{+}$ and $\boldsymbol{B}^{-}$ and thus less convenient to evaluate. The expression for A_{im} can be expanded using the definitions of $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$.

$$\begin{aligned} A_{im} &= \frac{1}{4}\chi^i_{vp}\chi^m_{kl}A^+_{vk}A^-_{lp} \\ &= \frac{1}{4}\chi^i_{vp}\chi^m_{kl}\left(\delta_{vk}+Q_{vk}\right)\left(\delta_{lp}-Q_{lp}\right) \\ &= \frac{1}{4}\chi^i_{vp}\chi^m_{kl}\left(\delta_{vk}\delta_{lp}-\delta_{vk}Q_{lp}+\delta_{lp}Q_{vk}-Q_{vk}Q_{lp}\right) \end{aligned} \tag{5.12}$$

This expression can be simplified by analyzing the first three terms.

$$\frac{1}{4}\chi^i_{vp}\chi^m_{kl}\delta_{vk}\delta_{lp} = \frac{1}{4}\chi^i_{vp}\chi^m_{vp} = \frac{1}{2}\delta_{im} \tag{5.13}$$

$$-\frac{1}{4}\chi^i_{vp}\chi^m_{kl}\delta_{vk}Q_{lp} = -\frac{1}{4}\chi^i_{vp}\chi^m_{vl}Q_{lp} \tag{5.14}$$

$$\frac{1}{4}\chi^i_{vp}\chi^m_{kl}\delta_{lp}Q_{vk} = \frac{1}{4}\chi^i_{vp}\chi^m_{kp}Q_{vk} = \frac{1}{4}\chi^i_{pv}\chi^m_{lv}Q_{pl} = -\frac{1}{4}\chi^i_{vp}\chi^m_{vl}Q_{lp} \tag{5.15}$$

These are substituted back into Eq. (5.12) to give the final result.

$$A_{im} = \frac{1}{2}\left(\delta_{im} - \chi^i_{vp}\chi^m_{vl}Q_{lp} - \frac{1}{2}\chi^i_{vp}\chi^m_{kl}Q_{vk}Q_{lp}\right) \tag{5.16}$$

For the special case $N = 3$, the equation for the elements of A can be simplified by substituting ϵ_{ijk} for χ^j_{ik}.

$$A_{im} = \frac{1}{2}\left(\delta_{im} - \epsilon_{vip}\epsilon_{vml}Q_{lp} - \frac{1}{2}\epsilon_{vip}\epsilon_{kml}Q_{vk}Q_{lp}\right) \tag{5.17}$$

The 'ϵ-δ identity' can be applied to the second term of this equation. Additionally, the fact that Q_{ii} equals zero, because $\boldsymbol{Q}$ is skew-symmetric, is used.

$$\epsilon_{vip}\epsilon_{vml}Q_{lp} = \left(\delta_{im}\delta_{pl} - \delta_{il}\delta_{pm}\right)Q_{lp} = -Q_{im} \tag{5.18}$$

The third term of Eq. (5.17) can also be rewritten using the generalized Kronecker delta [32].

$$\epsilon_{vip}\epsilon_{kml} = \delta^{kml}_{vip} = \begin{vmatrix} \delta^k_v & \delta^k_i & \delta^k_p \\ \delta^m_v & \delta^m_i & \delta^m_p \\ \delta^l_v & \delta^l_i & \delta^l_p \end{vmatrix} \tag{5.19}$$

$$= \delta_{vk}\left(\delta_{im}\delta_{pl} - \delta_{pm}\delta_{il}\right) + \delta_{ik}\left(\delta_{pm}\delta_{vl} - \delta_{vm}\delta_{pl}\right) + \delta_{pk}\left(\delta_{vm}\delta_{il} - \delta_{im}\delta_{vl}\right)$$

$$= \delta_{vk}\delta_{im}\delta_{pl} - \delta_{vk}\delta_{pm}\delta_{il} + \delta_{ik}\delta_{pm}\delta_{vl} - \delta_{ik}\delta_{vm}\delta_{pl} + \delta_{pk}\delta_{vm}\delta_{il} - \delta_{pk}\delta_{im}\delta_{vl}$$

The third term of Eq. (5.17) therefore becomes the following.

$$\begin{aligned} \epsilon_{vip}\epsilon_{kml}Q_{vk}Q_{lp} &= Q_{vi}Q_{vm} + Q_{mp}Q_{ip} - \delta_{im}Q_{vp}Q_{vp} \\ &= 2Q_{vi}Q_{vm} - \delta_{im}Q_{vp}Q_{vp} \end{aligned} \tag{5.20}$$

This expression is now rewritten in terms of the generating vector elements q_j and the ϵ-δ identity is used once again.

$$\begin{aligned} \epsilon_{vip}\epsilon_{kml}Q_{vk}Q_{lp} &= 2\epsilon_{vri}q_r\epsilon_{vsm}q_s - \delta_{im}\epsilon_{vrp}q_r\epsilon_{vsp}q_s \\ &= 2\left(\delta_{rs}\delta_{im} - \delta_{rm}\delta_{is}\right)q_rq_s - \delta_{im}\left(\delta_{rs}\delta_{pp} - \delta_{rp}\delta_{sp}\right)q_rq_s \\ &= 2\delta_{im}q_rq_r - 2q_iq_m - 3\delta_{im}q_rq_r + \delta_{im}q_pq_p \\ &= -2q_iq_m \end{aligned} \tag{5.21}$$

Equations (5.18) and (5.21) are now substituted into Eq. (5.17) to give the familiar form for the mapping from the angular velocity to the Rodrigues parameter rates [38].

$$A_{im} = \frac{1}{2}\left(\delta_{im} + Q_{im} + q_iq_m\right) \tag{5.22}$$

For reference, the complete form of $\boldsymbol{A}$ is given below for three- and four-dimensional bodies.

$$[\boldsymbol{A}]_{M=3} = \frac{1}{2}\begin{bmatrix} 1+q_1^2 & q_1q_2-q_3 & q_1q_3+q_2 \\ q_2q_1+q_3 & 1+q_2^2 & q_2q_3-q_1 \\ q_3q_1-q_2 & q_3q_2+q_1 & 1+q_3^2 \end{bmatrix} \tag{5.23}$$

$$[\boldsymbol{A}]_{M=6} = \frac{1}{2}\begin{bmatrix} 1+q_1^2 & q_1q_2-q_3 & q_1q_3+q_2 & q_1q_4-q_5 & q_1q_5+q_4 & q_2q_5-q_3q_4 \\ q_2q_1+q_3 & 1+q_2^2 & q_2q_3-q_1 & q_2q_4-q_6 & q_3q_4+q_1q_6 & q_2q_6+q_4 \\ q_3q_1-q_2 & q_3q_2+q_1 & 1+q_3^2 & q_2q_5-q_1q_6 & q_3q_5-q_6 & q_3q_6+q_5 \\ q_4q_1+q_5 & q_4q_2+q_6 & q_5q_2-q_1q_6 & 1+q_4^2 & q_4q_5-q_1 & q_4q_6-q_2 \\ q_5q_1-q_4 & q_3q_4+q_1q_6 & q_5q_3+q_6 & q_5q_4+q_1 & 1+q_5^2 & q_5q_6-q_3 \\ q_2q_5-q_3q_4 & q_6q_2-q_4 & q_6q_3-q_5 & q_6q_4+q_2 & q_6q_5+q_3 & 1+q_6^2 \end{bmatrix} \tag{5.24}$$

It is noteworthy that many researchers reference Cayley's work 'On the Motion of Rotation of a Solid Body' [39] as the origin of the Cayley transform, Eqs. (5.1) and (5.2). That paper contains scalar expressions for the case $N = 3$ which are equivalent to the Cayley transform (p. 225) and the Cayley-transform kinematic relationship (p. 227). There is no indication, however, that the results apply, nor lead, to the general N-dimensional case, nor is there any indication of the elegant, matrix forms that are central to the Cayley transform and kinematic relationship that are recognized today.

To the current authors' knowledge, these features first appear in Cayley's work 'Sur quelques propriétés des déterminants gauches' [8]. For reasons that are not mentioned in that paper, Cayley investigates some properties of what he calls 'left systems'. A left system is defined as a square collection of quantities, λ_{ij}, that satisfy the following relationships.

$$\lambda_{ij} = -\lambda_{ji},\ i \neq j \quad ; \quad \lambda_{ii} = 1,\ \text{no sum on } i \tag{5.25}$$

The indices i and j range from 1 to N. Note that the quantities λ_{ij} are essentially the elements of an $N \times N$ matrix, $\boldsymbol{\lambda}$, which may be expressed as $\boldsymbol{\lambda} = \boldsymbol{I} - \boldsymbol{Q}$, where $\boldsymbol{Q}$ is a skew-symmetric matrix. He goes on to introduce the 'inverse left system,' $\boldsymbol{\Lambda}$, which satisfies $\boldsymbol{\lambda}^T\boldsymbol{\Lambda} = K\boldsymbol{I}$, where K is the determinant of $\boldsymbol{\lambda}$. While investigating some transformation properties of left systems, Cayley discovers his famous transform (bottom of p. 120):

> 'On a donc le théorème suivant: Les coëfficients propres à la transformation de coordonnées rectangulaires, peuvent être exprimés rationnellement au moyen de quantités arbitraires λ_{rs}, soumises aux conditions $\lambda_{rs} = -\lambda_{sr}$, $[r \neq s]$; $\lambda_{rr} = 1$. Pour les développer, il faut d'abord former le déterminant K dé ce système, puis le système inverse $\Lambda_{rs}, \ldots$ et écrire $K\alpha_{rs} = 2\Lambda_{rs}\,[r \neq s]$; $K\alpha_{rr} = 2\Lambda_{rr} - K$; ce qui donne le système cherché.'

Cayley's coefficients α_{rs} are equivalent to the elements of $\boldsymbol{C}$, and the relationship in Cayley's theorem can be written $K\boldsymbol{C} = 2\boldsymbol{\Lambda} - K\boldsymbol{I}$. Using $\boldsymbol{\lambda}^T\boldsymbol{\Lambda} = K\boldsymbol{I}$, the relationship is seen to become $K\boldsymbol{C} = \boldsymbol{\lambda}\boldsymbol{\Lambda}$. When $\boldsymbol{\lambda}$ is written as $\boldsymbol{\lambda} = \boldsymbol{I} - \boldsymbol{Q}$, the inverse system is $\boldsymbol{\Lambda} = K(\boldsymbol{I} + \boldsymbol{Q})^{-1}$, which leads to $\boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} + \boldsymbol{Q})^{-1}$. From this result, it is straightforward to write the inverse form of the Cayley transform relating skew-symmetric $\boldsymbol{Q}$ to the proper orthogonal matrix $\boldsymbol{C}$. Note that although Cayley does not use matrix notation, he does use what is recognized today as a form of index notation to manipulate system quantities (i.e., matrix elements). A generalization of the Cayley transform in matrix notation appears in Cayley's works 'Sur la transformation d'une fonction quadratique en elle même par des substitutions linéaires' (p. 288) [40] and 'A Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function' (p. 44) [41]. These papers consider the transformation of the variables of quadratic functions and not just the rotation of an orthogonal coordinate system. In the following section, the Cayley kinematics are used (as generalized coordinates and motion variables) to directly derive the equations for rotational motion of an N-dimensional body.

C. Tensor Form of Lagrange's Equations

Previously, the tensor form of the N-dimensional rotational equations of motion have been derived by generalizing Euler's equations via a Hamiltonian approach [21, 24, 28] or by mapping the vector form of Lagrange's equations in terms of the Hamel coefficients [15]. In this section a new approach is presented using Lagrange's equations and Cayley kinematics to directly derive the tensor form of the equations of motion.

Lagrange's equations of motion in terms of the generalized coordinates and velocities are given by the following familiar form.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_0}{\partial \dot{q}_i}\right) - \frac{\partial T_0}{\partial q_i} = f_i \tag{5.26}$$

In this equation $T_0 = T_0\left(\boldsymbol{q}, \dot{\boldsymbol{q}}\right)$ is a function of the generalized-coordinate and velocity vectors and is equal to the kinetic energy of the system. The generalized forces, f_i, are associated with the generalized coordinates.

This equation can be rewritten in terms of the generalized-coordinate and velocity matrices. To do this the skew-symmetric matrix elements are substituted using the relationships given below.

$$q_i = \frac{1}{2}\chi^i_{jk} Q_{jk} \qquad ; \qquad \dot{q}_i = \frac{1}{2}\chi^i_{jk} \dot{Q}_{jk} \tag{5.27}$$

Substituting Eqs. (5.27) into the expression for T_0 for any given system will define a new kinetic-energy function $\widetilde{T}_0 = \widetilde{T}_0(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$. Whereas Eqs. (5.27) give the vector elements in terms of the matrix elements, the inverse mapping can also be considered.

$$Q_{jk} = \chi^l_{jk} q_l \qquad ; \qquad \dot{Q}_{jk} = \chi^l_{jk} \dot{q}_l \tag{5.28}$$

The partial derivatives of these equations can be used to find a matrix form of Lagrange's equations.

$$\frac{\partial Q_{jk}}{\partial q_i} = \chi^l_{jk} \frac{\partial q_l}{\partial q_i} = \chi^l_{jk} \delta_{li} = \chi^i_{jk} \tag{5.29}$$

$$\frac{\partial \dot{Q}_{jk}}{\partial \dot{q}_i} = \chi^l_{jk} \frac{\partial \dot{q}_l}{\partial \dot{q}_i} = \chi^l_{jk} \delta_{li} = \chi^i_{jk} \tag{5.30}$$

Although the elements of $\boldsymbol{Q}$, $\dot{\boldsymbol{Q}}$, and $\boldsymbol{\Omega}$ satisfy the skew-symmetry constraint, in the following derivations one has the choice of enforcing the constraint or not. Either option will produce the correct, final equation of motion as long as it is applied consistently (see Appendix B). For convenience, the constraint is not enforced in the following derivations, and in essence, the elements of these skew-symmetric matrices are treated as independent. This allows the partial derivatives of the kinetic-energy function with respect to the matrix elements to be written using the following chain rules without *a priori* considering the form of the constraint.

$$\frac{\partial T_0}{\partial q_i} = \frac{\partial \widetilde{T}_0}{\partial Q_{jk}} \frac{\partial Q_{jk}}{\partial q_i} = \chi^i_{jk} \frac{\partial \widetilde{T}_0}{\partial Q_{jk}} \tag{5.31}$$

$$\frac{\partial T_0}{\partial \dot{q}_i} = \frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{jk}} \frac{\partial \dot{Q}_{jk}}{\partial \dot{q}_i} = \chi^i_{jk} \frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{jk}} \tag{5.32}$$

Substituting these expressions into Lagrange's equations gives the following.

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\chi^i_{jk} \frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{jk}} \right) - \chi^i_{jk} \frac{\partial \widetilde{T}_0}{\partial Q_{jk}} = f_i \tag{5.33}$$

$$\chi^i_{jk} \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{jk}} \right) - \frac{\partial \widetilde{T}_0}{\partial Q_{jk}} \right\} = \frac{1}{2} \chi^i_{jk} F_{jk} \tag{5.34}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{ij}} \right) - \frac{\partial \widetilde{T}_0}{\partial Q_{ij}} = \frac{1}{2} F_{ij} \tag{5.35}$$

Here, $\boldsymbol{F}$ represents the skew-symmetric matrix of generalized forces. Equation (5.35) is the matrix form of Lagrange's equations in terms of the generalized coordinates and velocities. The factor of one-half appears on the right-hand side because the contribution to T_0 due to the generalized coordinate q_i (or the generalized velocity $\dot{q}_i$) is shared in $\widetilde{T}_0$ equally between the corresponding Q_{jk} and Q_{kj} (or $\dot{Q}_{jk}$ and $\dot{Q}_{kj}$) elements.

The matrix form of Lagrange's equations in terms of generalized coordinates and quasi velocities will now be developed. To do this another kinetic-energy function, $\widetilde{T}_1 = \widetilde{T}_1(\boldsymbol{Q}, \boldsymbol{\Omega})$, is defined by substituting the Cayley-transform kinematic relationship into $\widetilde{T}_0$. The partial derivatives of $\widetilde{T}_0$ can be expressed in terms of $\widetilde{T}_1$ by using the chain rule.

$$\frac{\partial \widetilde{T}_0}{\partial Q_{ij}} = \frac{\partial \widetilde{T}_1}{\partial Q_{ij}} + \frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}} \frac{\partial \Omega_{kl}}{\partial Q_{ij}} \tag{5.36}$$

$$\frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{ij}} = \frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}} \frac{\partial \Omega_{kl}}{\partial \dot{Q}_{ij}} \tag{5.37}$$

These expansions are substituted into Eq. (5.35).

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}} \frac{\partial \Omega_{kl}}{\partial \dot{Q}_{ij}} \right) - \left(\frac{\partial \widetilde{T}_1}{\partial Q_{ij}} + \frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}} \frac{\partial \Omega_{kl}}{\partial Q_{ij}} \right) = \frac{1}{2} F_{ij} \tag{5.38}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\frac{\partial \Omega_{kl}}{\partial \dot{Q}_{ij}}\right) - \frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\frac{\partial \Omega_{kl}}{\partial Q_{ij}} = \frac{1}{2}F_{ij} + \frac{\partial \widetilde{T}_1}{\partial Q_{ij}} \tag{5.39}$$

In order to expand the terms on the left-hand side of this equation, the derivatives of the elements of $\boldsymbol{\Omega}$ must be considered. Thus, Eq. (5.5) is rewritten in index notation.

$$\Omega_{kl} = 2B^+_{kv}\dot{Q}_{vp}B^-_{pl} \tag{5.40}$$

In order to develop the first term of Eq. (5.39) the derivative of Ω_{kl} with respect to $\dot{Q}_{ij}$ is considered.

$$\frac{\partial \Omega_{kl}}{\partial \dot{Q}_{ij}} = 2B^+_{kv}\frac{\partial \dot{Q}_{vp}}{\partial \dot{Q}_{ij}}B^-_{pl} = 2B^+_{kv}\delta_{iv}\delta_{jp}B^-_{pl} = 2B^+_{ki}B^-_{jl} \tag{5.41}$$

The first term of Eq. (5.39) is therefore given by the following.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\frac{\partial \Omega_{kl}}{\partial \dot{Q}_{ij}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(2\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}B^+_{ki}B^-_{jl}\right) \\ &= 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\right)B^+_{ki}B^-_{jl} + 2\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\left(\dot{B}^+_{ki}B^-_{jl} + B^+_{ki}\dot{B}^-_{jl}\right) \end{aligned} \tag{5.42}$$

Clearly the matrices $\dot{\boldsymbol{B}}^+$ and $\dot{\boldsymbol{B}}^-$ must be investigated. These are found using the definitions in Eqs. (5.3) and (5.4).

$$\boldsymbol{A}^+\boldsymbol{B}^+ = \boldsymbol{I} \tag{5.43}$$

Taking a derivative gives the following.

$$\dot{\boldsymbol{A}}^+\boldsymbol{B}^+ + \boldsymbol{A}^+\dot{\boldsymbol{B}}^+ = \boldsymbol{0} \tag{5.44}$$

$$\dot{\boldsymbol{B}}^+ = -\boldsymbol{B}^+\dot{\boldsymbol{A}}^+\boldsymbol{B}^+ \tag{5.45}$$

From the definition of $\boldsymbol{A}^+$, however, it is true that $\dot{\boldsymbol{A}}^+ = \dot{\boldsymbol{Q}}$.

$$\dot{\boldsymbol{B}}^+ = -\boldsymbol{B}^+\dot{\boldsymbol{Q}}\boldsymbol{B}^+ \tag{5.46}$$

By applying similar steps to the product $\boldsymbol{A}^{-}\boldsymbol{B}^{-}$, the derivative of $\boldsymbol{B}^{-}$ can be found.

$$\dot{\boldsymbol{B}}^{-} = \boldsymbol{B}^{-}\dot{\boldsymbol{Q}}\boldsymbol{B}^{-} \tag{5.47}$$

These expressions for $\dot{\boldsymbol{B}}^{+}$ and $\dot{\boldsymbol{B}}^{-}$ are substituted back into Eq. (5.42).

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\frac{\partial \Omega_{kl}}{\partial \dot{Q}_{ij}}\right) =& 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\right) B^{+}_{ki}B^{-}_{jl} \\ &+ 2\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\left(-B^{+}_{kr}\dot{Q}_{rs}B^{+}_{si}B^{-}_{jl} + B^{+}_{ki}B^{-}_{jr}\dot{Q}_{rs}B^{-}_{sl}\right) \end{aligned} \tag{5.48}$$

Now the second term of Eq. (5.39) is considered. First, the derivative of Ω_{kl} with respect to Q_{ij} is given by the following.

$$\frac{\partial \Omega_{kl}}{\partial Q_{ij}} = 2\frac{\partial B^{+}_{kv}}{\partial Q_{ij}}\dot{Q}_{vp}B^{-}_{pl} + 2B^{+}_{kv}\dot{Q}_{vp}\frac{\partial B^{-}_{pl}}{\partial Q_{ij}} \tag{5.49}$$

This result is used to rewrite the second term of Eq. (5.39).

$$\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\frac{\partial \Omega_{kl}}{\partial Q_{ij}} = 2\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\left(\frac{\partial B^{+}_{kv}}{\partial Q_{ij}}\dot{Q}_{vp}B^{-}_{pl} + B^{+}_{kv}\dot{Q}_{vp}\frac{\partial B^{-}_{pl}}{\partial Q_{ij}}\right) \tag{5.50}$$

The derivatives of $\boldsymbol{B}^{+}$ and $\boldsymbol{B}^{-}$ with respect to the elements of $\boldsymbol{Q}$ must be investigated.

$$A^{+}_{rk}B^{+}_{kv} = \delta_{rv} \tag{5.51}$$

$$\frac{\partial A^{+}_{rk}}{\partial Q_{ij}}B^{+}_{kv} + A^{+}_{rk}\frac{\partial B^{+}_{kv}}{\partial Q_{ij}} = 0 \tag{5.52}$$

$$\frac{\partial B^{+}_{sv}}{\partial Q_{ij}} = -B^{+}_{sr}\frac{\partial A^{+}_{rk}}{\partial Q_{ij}}B^{+}_{kv} \tag{5.53}$$

The derivatives of A^{+}_{rk} can be found using the definition of $\boldsymbol{A}^{+}$.

$$\frac{\partial A^{+}_{rk}}{\partial Q_{ij}} = \frac{\partial Q_{rk}}{\partial Q_{ij}} = \delta_{ir}\delta_{jk} \tag{5.54}$$

Based on this, the derivative of B^+_{sv} is given by the following.

$$\frac{\partial B^+_{sv}}{\partial Q_{ij}} = -B^+_{sr}\delta_{ir}\delta_{jk}B^+_{kv} = B^+_{si}B^+_{jv} \tag{5.55}$$

By applying similar steps to $\boldsymbol{B}^-$ the derivatives of its elements are found.

$$\frac{\partial B^-_{sv}}{\partial Q_{ij}} = B^-_{si}B^-_{jv} \tag{5.56}$$

Substituting Eqs. (5.55) and (5.56) into Eq. (5.50) gives the following for the second term of Eq. (5.39).

$$\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\frac{\partial \Omega_{kl}}{\partial Q_{ij}} = 2\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\left(-B^+_{ki}B^+_{jv}\dot{Q}_{vp}B^-_{pl} + B^+_{kv}\dot{Q}_{vp}B^-_{pi}B^-_{jl}\right) \tag{5.57}$$

Finally, Eqs. (5.48) and (5.57) can be substituted back into Eq. (5.39).

$$\begin{aligned} 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\right)B^+_{ki}B^-_{jl} + 2\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\left(-B^+_{kr}\dot{Q}_{rs}B^+_{si}B^-_{jl} + B^+_{ki}B^-_{jr}\dot{Q}_{rs}B^-_{sl}\right) \\ - 2\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\left(-B^+_{ki}B^+_{jv}\dot{Q}_{vp}B^-_{pl} + B^+_{kv}\dot{Q}_{vp}B^-_{pi}B^-_{jl}\right) = \frac{1}{2}F_{ij} + \frac{\partial \widetilde{T}_1}{\partial Q_{ij}} \end{aligned} \tag{5.58}$$

The second and third terms of the left-hand side can be grouped.

$$\begin{aligned} 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\right)B^+_{ki}B^-_{jl} + 2\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\dot{Q}_{rs}\left(-B^+_{kr}B^+_{si}B^-_{jl} + B^+_{ki}B^-_{jr}B^-_{sl}\right. \\ \left. + B^+_{ki}B^+_{jr}B^-_{sl} - B^+_{kr}B^-_{si}B^-_{jl}\right) = \frac{1}{2}F_{ij} + \frac{\partial \widetilde{T}_1}{\partial Q_{ij}} \end{aligned} \tag{5.59}$$

$$\begin{aligned} 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\right)B^+_{ki}B^-_{jl} + 2\frac{\partial \widetilde{T}_1}{\partial \Omega_{kl}}\dot{Q}_{sr}\left(-B^+_{ks}B^+_{ri}B^-_{jl} + B^+_{ki}B^-_{js}B^-_{rl}\right. \\ \left. + B^+_{ki}B^+_{js}B^-_{rl} - B^+_{ks}B^-_{ri}B^-_{jl}\right) = \frac{1}{2}F_{ij} + \frac{\partial \widetilde{T}_1}{\partial Q_{ij}} \end{aligned} \tag{5.60}$$

$$2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}\right)B^+_{ki}B^-_{jl}+\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}A^+_{sa}\Omega_{ab}A^-_{br}\left(B^+_{ki}B^-_{rl}\big(B^-_{js}+B^+_{js}\big)\right.$$
$$\left.-B^+_{ks}B^-_{jl}\big(B^+_{ri}+B^-_{ri}\big)\right)=\frac{1}{2}F_{ij}+\frac{\partial\widetilde{T}_1}{\partial Q_{ij}} \tag{5.61}$$

Both sides of the above equation are now multiplied by $A^+_{id}A^-_{cj}$ to allow for future simplifications.

$$2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}\right)B^+_{ki}B^-_{jl}A^+_{id}A^-_{cj}+\Omega_{ab}\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}A^+_{id}A^-_{cj}A^+_{sa}A^-_{br}\left(B^+_{ki}B^-_{rl}\big(B^-_{js}+B^+_{js}\big)\right.$$
$$\left.-B^+_{ks}B^-_{jl}\big(B^+_{ri}+B^-_{ri}\big)\right)=\left(\frac{1}{2}F_{ij}+\frac{\partial\widetilde{T}_1}{\partial Q_{ij}}\right)A^+_{id}A^-_{cj} \tag{5.62}$$

Both terms on the left-hand side of the above equation can be simplified. It is convenient to start with the first term.

$$B^+_{ki}B^-_{jl}A^+_{id}A^-_{cj}=\delta_{kd}\delta_{cl} \tag{5.63}$$

This result is used to rewrite the first term of Eq. (5.62).

$$2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}\right)B^+_{ki}B^-_{jl}A^+_{id}A^-_{cj}=2\delta_{kd}\delta_{cl}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}\right)$$
$$=2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{T}_1}{\partial\Omega_{dc}}\right) \tag{5.64}$$

The second term of Eq. (5.62) is now examined.

$$A^+_{id}A^-_{cj}A^+_{sa}A^-_{br}\left(B^+_{ki}B^-_{rl}\big(B^-_{js}+B^+_{js}\big)-B^+_{ks}B^-_{jl}\big(B^+_{ri}+B^-_{ri}\big)\right)$$
$$=A^+_{sa}A^-_{br}\left(\delta_{kd}A^-_{cj}B^-_{rl}\big(B^-_{js}+B^+_{js}\big)-\delta_{cl}A^+_{id}B^+_{ks}\big(B^+_{ri}+B^-_{ri}\big)\right)$$
$$=A^+_{sa}A^-_{br}\left(\delta_{kd}B^-_{rl}\left(\delta_{cs}+C_{cs}\right)-\delta_{cl}B^+_{ks}\left(\delta_{dr}+C_{dr}\right)\right) \tag{5.65}$$

The elements of $\boldsymbol{C}+\boldsymbol{I}$ can be found using the Cayley transform.

$$\begin{aligned}
\boldsymbol{C}+\boldsymbol{I} &= (\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}+\boldsymbol{Q})^{-1}+\boldsymbol{I} \\
&= (\boldsymbol{I}+\boldsymbol{Q})^{-1}-\boldsymbol{Q}(\boldsymbol{I}+\boldsymbol{Q})^{-1}+\boldsymbol{I} \\
&= (\boldsymbol{I}+\boldsymbol{Q})^{-1}-\boldsymbol{Q}(\boldsymbol{I}+\boldsymbol{Q})^{-1}+(\boldsymbol{I}+\boldsymbol{Q})(\boldsymbol{I}+\boldsymbol{Q})^{-1} \\
&= 2(\boldsymbol{I}+\boldsymbol{Q})^{-1}=2\boldsymbol{B}^{+}
\end{aligned} \tag{5.66}$$

This result can be used to further simplify Eq. (5.65).

$$\begin{aligned}
&A^{+}_{id}A^{-}_{cj}A^{+}_{sa}A^{-}_{br}\left(B^{+}_{ki}B^{-}_{rl}\left(B^{-}_{js}+B^{+}_{js}\right)-B^{+}_{ks}B^{-}_{jl}\left(B^{+}_{ri}+B^{-}_{ri}\right)\right) \\
&\qquad = A^{+}_{sa}A^{-}_{br}\left(2\delta_{kd}B^{-}_{rl}B^{+}_{cs}-2\delta_{cl}B^{+}_{ks}B^{+}_{dr}\right) \\
&\qquad = 2A^{+}_{sa}A^{-}_{br}\left(\delta_{kd}B^{-}_{rl}B^{+}_{cs}-\delta_{cl}B^{-}_{sk}B^{+}_{dr}\right)
\end{aligned} \tag{5.67}$$

This result is used to rewrite the second term of Eq. (5.62).

$$\begin{aligned}
&\Omega_{ab}\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}A^{+}_{id}A^{-}_{cj}A^{+}_{sa}A^{-}_{br}\left(B^{+}_{ki}B^{-}_{rl}\left(B^{-}_{js}+B^{+}_{js}\right)-B^{+}_{ks}B^{-}_{jl}\left(B^{+}_{ri}+B^{-}_{ri}\right)\right) \\
&\qquad = 2\Omega_{ab}\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}A^{+}_{sa}A^{-}_{br}\left(\delta_{kd}B^{-}_{rl}B^{+}_{cs}-\delta_{cl}B^{-}_{sk}B^{+}_{dr}\right)
\end{aligned} \tag{5.68}$$

The skew-symmetry of $\boldsymbol{\Omega}$ is used in the following manner.

$$\begin{aligned}
-\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}\delta_{cl}B^{-}_{sk}B^{+}_{dr} &= \frac{\partial\widetilde{T}_1}{\partial\Omega_{lk}}\delta_{cl}B^{-}_{sk}B^{+}_{dr} \\
&= \frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}\delta_{ck}B^{-}_{sl}B^{+}_{dr}
\end{aligned} \tag{5.69}$$

This expression is substituted into Eq. (5.68).

$$\Omega_{ab}\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}A^+_{id}A^-_{cj}A^+_{sa}A^-_{br}\left(B^+_{ki}B^-_{rl}\left(B^-_{js}+B^+_{js}\right)-B^+_{ks}B^-_{jl}\left(B^+_{ri}+B^-_{ri}\right)\right)$$
$$\begin{aligned}&=2\Omega_{ab}\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}A^+_{sa}A^-_{br}\left(\delta_{kd}B^-_{rl}B^+_{cs}+\delta_{ck}B^-_{sl}B^+_{dr}\right)\\&=2\Omega_{ab}\frac{\partial\widetilde{T}_1}{\partial\Omega_{kl}}\left(\delta_{kd}\delta_{ca}\delta_{bl}+\delta_{ck}\delta_{la}\delta_{bd}\right)\\&=2\Omega_{cl}\frac{\partial\widetilde{T}_1}{\partial\Omega_{dl}}+2\Omega_{ld}\frac{\partial\widetilde{T}_1}{\partial\Omega_{cl}}\\&=2\frac{\partial\widetilde{T}_1}{\partial\Omega_{dl}}\Omega_{cl}-2\frac{\partial\widetilde{T}_1}{\partial\Omega_{cl}}\Omega_{dl}\end{aligned}\tag{5.70}$$

Equations (5.64) and (5.70) can now be substituted back into Eq. (5.62).

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{T}_1}{\partial\Omega_{dc}}\right)+\frac{\partial\widetilde{T}_1}{\partial\Omega_{dl}}\Omega_{cl}-\frac{\partial\widetilde{T}_1}{\partial\Omega_{cl}}\Omega_{dl}=\frac{1}{2}\left(\frac{1}{2}F_{ij}+\frac{\partial\widetilde{T}_1}{\partial Q_{ij}}\right)A^+_{id}A^-_{cj}\tag{5.71}$$

The above equation is the matrix form of Lagrange's equations for generalized coordinates and the $N\times N$ angular-velocity matrix (or Cayley quasi velocity matrix). For convenience the right-hand side is defined as $\frac{1}{2}G_{dc}$, where $\boldsymbol{G}$ is related to the generating vector $\boldsymbol{g}$ whose elements can be shown to be $g_k=A_{rk}\left(f_r+\partial T_1/\partial q_r\right)$. The equivalent vector representation of the equations is the following.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial\omega_k}\right)+\frac{1}{2}\chi^r_{ij}\left(\chi^k_{ic}\Omega_{cj}-\chi^k_{cj}\Omega_{ic}\right)\left(\frac{\partial T_1}{\partial\omega_r}\right)=g_k\tag{5.72}$$

To express these equations in the Lax pair form, the derivative of T_1 with respect to the angular-velocity components is defined as the angular-momentum vector $\boldsymbol{l}$.

$$\frac{\partial T_1}{\partial\omega_k}=l_k\tag{5.73}$$

The partial derivative of the kinetic energy with respect to $\boldsymbol{\Omega}$ is expressed using the chain rule.

$$\frac{\partial T_1}{\partial\omega_k}=\frac{\partial\widetilde{T}_1}{\partial\Omega_{ij}}\frac{\partial\Omega_{ij}}{\partial\omega_k}=\chi^k_{ij}\frac{\partial\widetilde{T}_1}{\partial\Omega_{ij}}\tag{5.74}$$

The angular-momentum vector is used to generate the angular-momentum matrix $\boldsymbol{L}$.

$$l_k = \frac{1}{2}\chi_{ij}^{k} L_{ij} \tag{5.75}$$

This gives the following result.

$$\frac{\partial \widetilde{T}_1}{\partial \Omega_{ij}} = \frac{1}{2} L_{ij} \tag{5.76}$$

Equation (5.71) can now be rewritten in terms of the angular-momentum matrix.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}L_{dc}\right) + \frac{1}{2}L_{dl}\Omega_{cl} - \frac{1}{2}L_{cl}\Omega_{dl} = \frac{1}{2}G_{dc} \tag{5.77}$$

In matrix notation this result gives the following Lax pair form.

$$\dot{\boldsymbol{L}} = (\boldsymbol{L\Omega} - \boldsymbol{\Omega L}) + \boldsymbol{G} = [\boldsymbol{L}, \boldsymbol{\Omega}] + \boldsymbol{G} \tag{5.78}$$

D. Cayley Quasi Velocities and the Cayley Form

An M-degree-of-freedom (DOF) mechanical system can be intimately related to an N-dimensional rigid body through the Cayley kinematic equations. Recall the Cayley kinematic equations.

$$\text{Forward relationship: } \boldsymbol{\Omega} = 2(\boldsymbol{I}+\boldsymbol{Q})^{-1}\,\dot{\boldsymbol{Q}}\,(\boldsymbol{I}-\boldsymbol{Q})^{-1} \tag{5.79}$$

$$\text{Inverse relationship: } \dot{\boldsymbol{Q}} = \frac{1}{2}(\boldsymbol{I}+\boldsymbol{Q})\,\boldsymbol{\Omega}\,(\boldsymbol{I}-\boldsymbol{Q}) \tag{5.80}$$

And recall also the forward Cayley transform expression and Poisson's equation for $\dot{\boldsymbol{C}}$.

$$\text{Forward relationship: } \boldsymbol{C} = (\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}+\boldsymbol{Q})^{-1} = (\boldsymbol{I}+\boldsymbol{Q})^{-1}(\boldsymbol{I}-\boldsymbol{Q}) \tag{5.81}$$

$$\text{Poisson's equation: } \dot{\boldsymbol{C}} = -\boldsymbol{\Omega C} \tag{5.82}$$

These lead to the following remarkable result:

> If the M-dimensional ($M = N(N-1)/2$) generalized-coordinate vector, $\boldsymbol{q}$, of an M-DOF mechanical system is considered the generating vector of an $N \times N$ skew-symmetric matrix $\boldsymbol{Q}$, and if $\boldsymbol{Q}$ and its derivative are used to define the quasi velocities $\boldsymbol{\Omega}$ via the forward Cayley kinematic equation, then $\boldsymbol{Q}$ generates a proper $N \times N$ orthogonal matrix, $\boldsymbol{C}$, via the forward Cayley transform that evolves according to $\dot{\boldsymbol{C}} = -\boldsymbol{\Omega}\boldsymbol{C}$. Furthermore, the motion of the system is governed by the following matrix differential equations.
>
> $$\dot{\boldsymbol{Q}} = \frac{1}{2}(\boldsymbol{I}+\boldsymbol{Q})\,\boldsymbol{\Omega}\,(\boldsymbol{I}-\boldsymbol{Q}) \quad ; \quad \dot{\boldsymbol{L}} = [\boldsymbol{L},\boldsymbol{\Omega}] + \boldsymbol{G}$$
>
> Consequently, the motion of an M-DOF mechanical system can be viewed as the rotational motion of an N-dimensional rigid body. When $\boldsymbol{q}$ is used in this manner (i.e., a generating vector), its elements are called the *extended Rodrigues parameters*; the elements of $\boldsymbol{\Omega}$ are given the name *Cayley quasi velocities*; and this treatment is called the *Cayley Form*.

The equivalent body is defined as an N-dimensional rigid body because the generalized coordinates that define the extended Rodrigues parameters completely describe the configuration of the system. This is an extension of the kinematic definition of a three-dimensional rigid body. Previous researchers have also extended certain dynamic properties relating to three-dimensional rigid bodies in their definitions of an N-dimensional rigid body [21,24,28]. These definitions, however, preclude the study of systems whose Lagrangian functions depend on the generalized coordinates. The governing equations presented herein for an M-DOF mechanical system or, equivalently, an N-dimensional rigid body are summarized in Table IV. The nu-

Table IV. THE CAYLEY FORM OF DYNAMIC EQUATIONS

	M-dimensional vector forms
kinematics	$\dot{q}_i = A_{ij}\omega_j$
dynamics	$\mathrm{d}(\partial T_1/\partial\omega_k)/\mathrm{d}t + \frac{1}{2}\chi^r_{ij}\left(\chi^k_{ic}\Omega_{cj} - \chi^k_{cj}\Omega_{ic}\right)(\partial T_1/\partial\omega_r) = g_k$
	$N \times N$ matrix forms
kinematics	$\dot{\boldsymbol{Q}} = \frac{1}{2}(\boldsymbol{I}+\boldsymbol{Q})\,\boldsymbol{\Omega}\,(\boldsymbol{I}-\boldsymbol{Q})$
dynamics	$\dot{\boldsymbol{L}} = [\boldsymbol{L}, \boldsymbol{\Omega}] + \boldsymbol{G}$

merical relative tensor χ^k_{jl} can be used to map the equations from one form to the other.

One application of the Cayley form is the representation of the general motion of an N-dimensional body as the pure rotation of an $(N+1)$-dimensional body. The general motion of an N-dimensional body consists of M rotational and N translational degrees of freedom.

$$\begin{aligned} M + N &= \frac{1}{2}N\,(N-1) + N \\ &= \frac{1}{2}\,(N+1)\,N \\ &= \frac{1}{2}\,(N+1)\,((N+1)-1) \end{aligned} \tag{5.83}$$

This total number of degrees of freedom is therefore equal to the number of rotational degrees of freedom for an $(N+1)$-dimensional body, and the Cayley form can be used to relate these two motions. In the following section the general motion of a two-dimensional body and pure rotations of an equivalent three-dimensional body are presented in detail.

E. Planar Motion Example

An enormous variety of mechanical systems can be studied using the Cayley form. One of the easiest to visualize, however, is the general motion of a two-dimensional body. This planar motion consists of three degrees of freedom (see Fig. 4) and can therefore be related to the rotational motion of a three-dimensional body. The generalized coordinates for the problem are $[\boldsymbol{q}] = [\theta \quad x \quad y]^T$. To analyze this problem in the Cayley form, the matrix form of the kinematic equations is used to solve for the vector form. The generalized coordinates and Cayley quasi velocities are arranged into the skew-symmetric matrices $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ as follows.

$$[\boldsymbol{Q}] = \begin{bmatrix} 0 & -y & x \\ y & 0 & -\theta \\ -x & \theta & 0 \end{bmatrix} \qquad ; \qquad [\boldsymbol{\Omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{5.84}$$

From the Cayley kinematic relationship the linear mapping between the generalized-velocity and quasi-velocity vectors is found.

$$[\dot{\boldsymbol{q}}] = [\boldsymbol{A}]\,[\boldsymbol{\omega}] = \frac{1}{2}\begin{bmatrix} 1+\theta^2 & \theta x - y & \theta y + x \\ \theta x + y & 1 + x^2 & xy - \theta \\ \theta y - x & xy + \theta & 1 + y^2 \end{bmatrix}[\boldsymbol{\omega}] \tag{5.85}$$

The form of Eq. (5.85) is identical to the relationship between the three-dimensional Rodrigues parameter rates and the angular velocity.

The dynamic equations can now be developed using the vector form in Eq. (5.72). The kinetic energy of the system in terms of the generalized velocities is described by the following, where I is the rotational inertia of the body and m is the body's mass.

$$T_0\left(\dot{\boldsymbol{q}}\right) = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{J}\dot{\boldsymbol{q}} \tag{5.86}$$

The matrix $\boldsymbol{J}$ is now the system mass matrix given by the following.

$$[\boldsymbol{J}] = \begin{bmatrix} I & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{5.87}$$

Using Eq. (5.85) the kinetic energy can now be expressed as a function of the generalized coordinates and the Cayley quasi velocities.

$$T_1(\boldsymbol{q}, \boldsymbol{\omega}) = \frac{1}{2}\boldsymbol{\omega}^T \left(\boldsymbol{A}^T \boldsymbol{J} \boldsymbol{A}\right) \boldsymbol{\omega} \tag{5.88}$$

The Cayley form of Lagrange's equations can now be applied to T_1. First, T_1 is rewritten using index notation and the various derivatives are then computed.

$$T_1 = \frac{1}{2}\omega_i A_{li} J_{lm} A_{mj} \omega_j \tag{5.89}$$

$$\frac{\partial T_1}{\partial q_r} = \frac{1}{2}\omega_i \left(\frac{\partial A_{li}}{\partial q_r} J_{lm} A_{mj} + A_{li} J_{lm} \frac{\partial A_{mj}}{\partial q_r}\right) \omega_j \tag{5.90}$$

$$\frac{\partial T_1}{\partial \omega_r} = A_{lr} J_{lm} A_{mj} \omega_j \tag{5.91}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k}\right) = \dot{A}_{lr} J_{lm} A_{mj} \omega_j + A_{lr} J_{lm} \dot{A}_{mj} \omega_j + A_{lr} J_{lm} A_{mj} \dot{\omega}_j \tag{5.92}$$

The derivatives of $\boldsymbol{A}$ are computed as follows.

$$\dot{A}_{lr} = \frac{\partial A_{lr}}{\partial q_s}\dot{q}_s = \frac{\partial A_{lr}}{\partial q_s} A_{st} \omega_t \tag{5.93}$$

The generalized forces in terms of the generalized velocities are equal to the moment and force components applied to the body: $[\boldsymbol{f}] = [M \quad F_x \quad F_y]^T$. Using these generalized forces and Eqs. (5.90) to (5.93), the equations of motion can be assembled and solved for the components $\dot{\omega}_j$.

Figures 5–7 show simulation results from the integration of these equations of motion and the Cayley kinematic equations, using mass property values of $I = m = 1$.

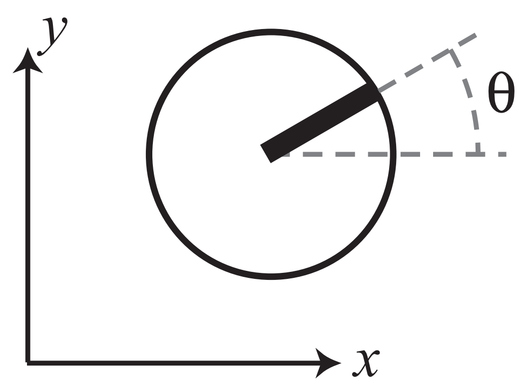

Fig. 4. Planar rigid body.

The initial conditions, forces, and moment were chosen such that the body translated with constant speed through a circular path of unit radius over a period of ten time units. During this period the body also performed two complete rotations. This motion is depicted in Fig. 8. Figures 5 and 6 show the solutions for the generalized coordinates and velocities obtained using a traditional Lagrangian formulation. Figure 7 shows the results obtained for the quasi velocities using the Cayley form. Of course, the solution for the generalized coordinates matched that obtained from the traditional approach. The implication of the Cayley form is that there is an equivalent rotational motion of a three-dimensional body that corresponds to the motion of the two-dimensional body. This equivalent motion is described by Rodrigues-parameter and angular-velocity trajectories equal to the solutions shown in Figs. 5 and 7, respectively.

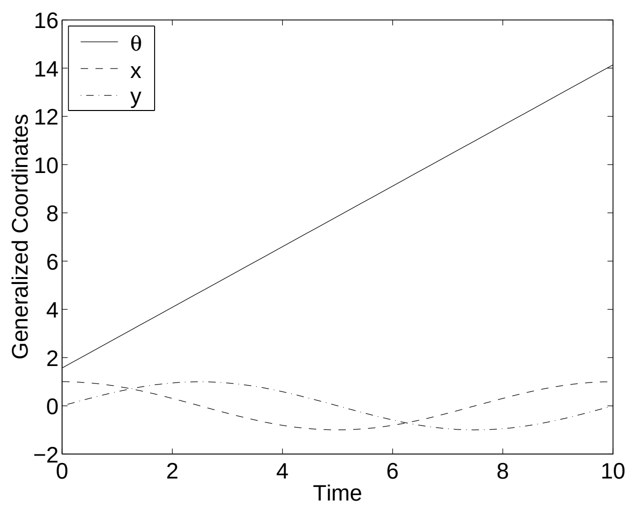

Fig. 5. Generalized coordinates.

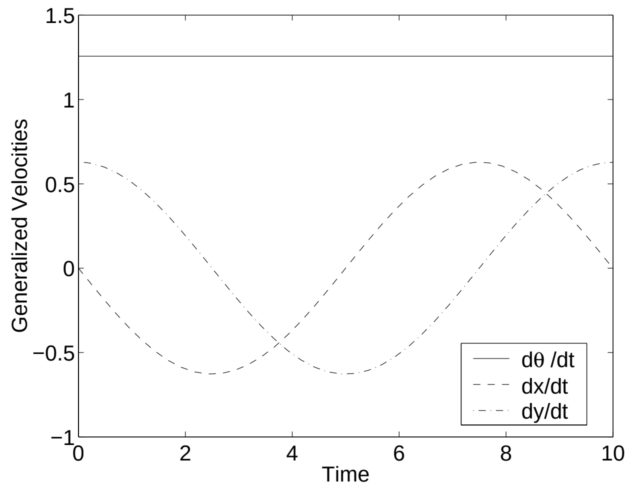

Fig. 6. Generalized velocities.

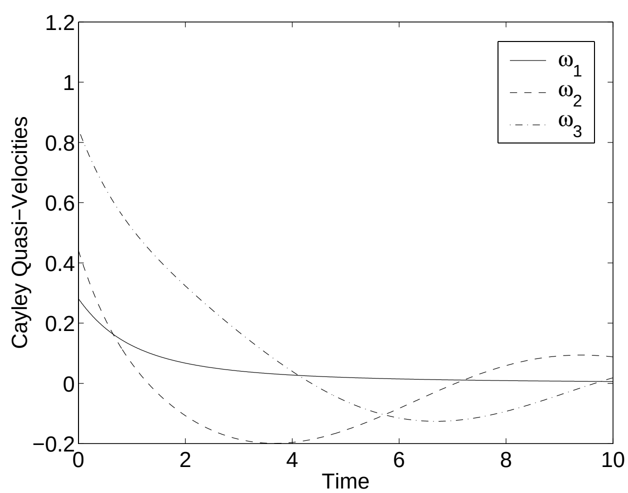

Fig. 7. Cayley quasi velocities.

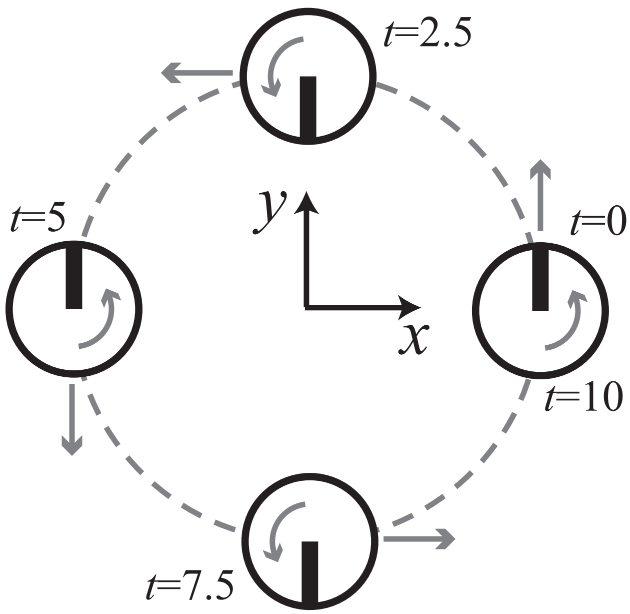

Fig. 8. Example planar motion.

F. Discussion

This chapter has focused on the Cayley transform and the Cayley-transform kinematic relationship, and their use in developing the matrix form of the generalized Euler equations of motion for N-dimensional rigid bodies. Related to these equations are the more general Euler-Poincaré equations, which govern the motion of left-invariant Lagrangian systems corresponding to general Lie groups [20, 36]. The Hamel coefficients for an N-dimensional rigid body [15] were not explicitly encountered in the development because of the *a priori* decision to describe the system using the Cayley orientation and kinematic variables.

The Cayley transform and the Cayley-transform kinematic relationship allowed the realization that the motion of a general M-DOF ($M = N(N-1)/2$) mechanical system is related to the rotational motion of an N-dimensional rigid body. The definition of an N-dimensional rigid body used here is kinematics based and hinges on three facts: (1) the extended Rodrigues parameters, which completely describe the configuration of the system, parameterize an $N \times N$ proper orthogonal matrix, (2) an $N \times N$ proper orthogonal matrix can be used to relate the orientation of one N-dimensional reference frame to another, and (3) the orientation of a rigid body can be defined as the orientation of a reference frame that is imbedded within it.

A useful relationship that was developed was the mapping of the skew-symmetric matrix $\boldsymbol{Q}$, which comprises the extended Rodrigues parameters for N-dimensional spaces, to the matrix $\boldsymbol{A}$ that appears in the equation $\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{\omega}$. This mapping allows the equations that govern the rotational motion of an N-dimensional rigid body to be written in an M-dimensional vector form.

The benefits and implications of the relationship between M-DOF mechanical systems and N-dimensional rigid bodies are still being investigated. Further research is needed to investigate the extent to which the elegant tools that have been developed for analyzing, controlling, and approximating the motion of three-dimensional rigid bodies can be modified and applied to N-dimensional rigid bodies. The use of the Cayley form will then allow these tools to be applied to a wide variety of physical systems. Further research is also needed to investigate the 'in-between' M situation, i.e., $M \neq N(N-1)/2$. One straightforward approach is to pad the vector of extended Rodrigues parameters with additional fictitious coordinates until $M = N(N-1)/2$. The additional fictitious coordinates then represent constrained degrees of freedom.

CHAPTER VI

APPLICATION OF THE CAYLEY FORM TO GENERAL SPACECRAFT MOTION

A. Introduction

Whereas the study of mechanics has been motivated by the desire to explain the three-dimensional, physical universe, the mathematical models that have resulted are in no way limited to three-dimensions. Higher-dimensional bodies can be kinematically defined, and by assuming that principles such as conservation of angular momentum and Hamilton's principle apply in higher-dimensional spaces, their dynamics can also be developed. In particular, one can consider the mechanics of an N-dimensional rigid body, which can be defined as a system whose configuration can be completely defined by an $N \times N$ proper orthogonal matrix.

The descriptions of N-dimensional bodies are not simply mathematical curiosities: they can be used to describe real systems. This is done by linking the motion of general systems to the rotation of a higher-dimensional, rigid body. Three-dimensional analogs to this approach have been used in the past. For example, Junkins and Turner developed an analogy between spacecraft orbital motion and rigid body rotations [42]. In that work a physical reference frame was defined using the spacecraft position and velocity vectors. The orbital motion could then be studied by describing the evolution of this frame. Because of the osculation constraint implied in the definition of this frame, however, its motion does not fully capture the orbital dynamics. The approach required explicit reintroduction of Newton's second law to describe the behavior of the radial distance. Additionally, whereas the kinematic analogy to a rigid body is clear, dynamically it was found that the gyroscopic equations contained variable inertia due to changes in the radial distance.

This chapter presents a new analogy, called the Cayley form [16], between the combined attitude and orbital motion of a spacecraft and rotational motion of a four-dimensional, rigid body. In addition to incorporating both the attitude and orbital motion, the new analogy more fully incorporates the dynamics in a general sense (i.e., osculation constraints are not imposed nor are explicit reintroductions required). Incorporating both attitude and orbital motion in a single dynamic representation could be seen as a disadvantage, because it does not take advantage of the decoupling that occurs between these two motions for the special case of unforced dynamics. Many other choices for representing translational and rotational motion, however, also exhibit coupling in the motion variables, an example being the body components of translational velocity [35]. The disadvantage is mitigated, however, by the fact that for most spacecraft systems the attitude and orbital motions are in fact coupled by forcing terms such as the rigid-body gravity potential or fixed-direction thrusters. A second example is presented to further illustrate the new analogy. It involves the attitude motion of a satellite containing three momentum wheels, which is also related to the rotational motion of a four-dimensional body. One potential advantage of the Cayley form is that incorporating all system degrees of freedom into a single dynamic representation can be beneficial in designing controllers. Although this is not fully treated in this chapter, some preliminary results have demonstrated superior performance in some cases for controllers designed using the Cayley form versus classical approaches [43].

In each of these examples, the relations are made by associating each point in the six-dimensional configuration space with a particular orientation in four-dimensional space. Similar to the Junkins and Turner analogy, however, the new analogy does not match all of the dynamics properties associated with rigid bodies. In the following sections of the chapter the concepts of N-dimensional kinematics and dynamics are reviewed, and their relationship to general systems is discussed. This is then used to analyze general spacecraft motion.

B. Cayley Kinematics

The Cayley transform is a remarkable set of relationships between proper orthogonal and skew-symmetric matrices [7]. Cayley discovered the forward relationship while investigating some properties of "left systems" [8].

$$\text{Forward:} \quad \boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} + \boldsymbol{Q})^{-1} = (\boldsymbol{I} + \boldsymbol{Q})^{-1}(\boldsymbol{I} - \boldsymbol{Q}) \tag{6.1}$$

$$\text{Inverse:} \quad \boldsymbol{Q} = (\boldsymbol{I} - \boldsymbol{C})(\boldsymbol{I} + \boldsymbol{C})^{-1} = (\boldsymbol{I} + \boldsymbol{C})^{-1}(\boldsymbol{I} - \boldsymbol{C}) \tag{6.2}$$

Here, $\boldsymbol{C}$ is an $N \times N$ proper orthogonal matrix, whereas $\boldsymbol{Q}$ is an $N \times N$ skew-symmetric matrix, and $\boldsymbol{I}$ is the identity matrix. The matrix $\boldsymbol{Q}$ comprises a set of M parameters that represent the orientation of an N-dimensional reference frame. In fact $\boldsymbol{q}$, the generating vector of $\boldsymbol{Q}$, is the vector of extended Rodrigues parameters (ERPs) for N-dimensional spaces [5, 6].

For N-dimensions the angular velocity is defined through the evolution of the rotation matrix.

$$\boldsymbol{\Omega} = \boldsymbol{C}^T \dot{\boldsymbol{C}} \tag{6.3}$$

In general, the angular velocity is a set of quasi velocities related to rotational motion (see Baruh, Section 7.6, pp. 378-380 [44]). Combining the angular velocity with the Cayley transform produces the Cayley-transform kinematic relationships. These relationships connect the derivatives of the M independent parameters of $\boldsymbol{Q}$ to the angular-velocity matrix [12].

$$\boldsymbol{\Omega} = 2\left(\boldsymbol{I}+\boldsymbol{Q}\right)^{-1}\dot{\boldsymbol{Q}}\left(\boldsymbol{I}-\boldsymbol{Q}\right)^{-1} \tag{6.4}$$

$$\dot{\boldsymbol{Q}} = \frac{1}{2}\left(\boldsymbol{I}+\boldsymbol{Q}\right)\boldsymbol{\Omega}\left(\boldsymbol{I}-\boldsymbol{Q}\right) \tag{6.5}$$

Equations (6.4) and (6.5) represent a linear mapping between the generalized velocities, $\dot{\boldsymbol{Q}}$, and a set of quasi velocities, $\boldsymbol{\Omega}$, for N-dimensional rotations. These equations involve operations on $N \times N$ matrices, but they can be rewritten in the more familiar M-dimensional vector form.

$$\dot{q}_i = A_{im}\omega_m \tag{6.6}$$

This was carried out by Sinclair and Hurtado [16] using the numerical relative symbol $\boldsymbol{\chi}$. The elements of $\boldsymbol{A}$ are given as a function of $\boldsymbol{Q}$ and the symbol $\boldsymbol{\chi}$ in the following.

$$A_{im} = \frac{1}{2}\left(\delta_{im} - \chi^{i}_{vp}\chi^{m}_{vl}Q_{lp} - \frac{1}{2}\chi^{i}_{vp}\chi^{m}_{kl}Q_{vk}Q_{lp}\right) \tag{6.7}$$

C. N-Dimensional Rigid Body Dynamics

Based on the kinematics of N-dimensional bodies the question of studying the dynamics of N-dimensional bodies naturally arises. Indeed this was first suggested by Cayley [8]. The equations of motion for rotations of N-dimensional bodies were first found by Frahm [22], who approached the problem by considering the motion of a collection of particles in N-dimensional space. More recently geometric methods have been used to describe the free evolution of the angular-momentum matrix, $\boldsymbol{L}$, of an N-dimensional body [20,21,24,29]. That work has resulted in an elegant form for the equations of motion called the Lax pair form.

The equations of motion for N-dimensional rigid bodies can also be achieved using Lagrange's equations in terms of quasi velocities. One method for doing this uses Hamel coefficients [15]. The Hamel coefficients for N-dimensional rotations are independent of the generalized coordinates and are given by the following.

$$\gamma_{vr}^{m} = \frac{1}{2}\chi_{ik}^{m}\left(\chi_{ic}^{v}\chi_{ck}^{r} - \chi_{ck}^{v}\chi_{ic}^{r}\right) \tag{6.8}$$

These values are used in Lagrange's equations to produce the vector form of the equations of motion as shown in Table IV, where T is the kinetic energy as a function of the generalized coordinates and angular velocity and g_k is given below.

$$g_k \equiv A_{rk}\left(f_r + \frac{\partial T}{\partial q_r}\right) \tag{6.9}$$

Here, f_r are the generalized forces and include potential and nonpotential forces. An alternative choice would be to write the equations of motion in terms of the Lagrangian and include only nonpotential terms in the generalized forces. Another method for deriving the equations of motion arises from choosing the ERPs as generalized coordinates [16]. The Cayley kinematics can then be used directly to derive the dynamic equations. Both methods produce the vector form of the equations of motion shown in Table IV. Additionally, the tensor $\boldsymbol{\chi}$ can be used to map the vector form into the classic Lax pair matrix form using $\boldsymbol{g}$ as the generating vector for the matrix $\boldsymbol{G}$.

Whereas the kinematics and dynamics equations given in Table IV were originally derived to describe the motion of N-dimensional rigid bodies, they can also be applied to any M-degree-of-freedom (DOF) mechanical system [16]. This leads to the interesting result that the motion of any M-DOF mechanical system is equivalent to the pure rotation of an N-dimensional rigid body. The simplest example of this is the mapping from 1-DOF translation to 1-DOF rotation. Although these spaces are topologically dissimilar, the equivalence is made possible by mapping the entire translational space onto the singularity-free portion of the rotational space using a parameterization with values including the entire real line. The M-dimensional vectors of generalized coordinates and generalized velocities, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, for any system can be used to generate skew-symmetric matrices $\boldsymbol{Q}$ and $\dot{\boldsymbol{Q}}$. The generalized coordinates can then be substituted into the Cayley transform to generate an orthogonal matrix, $\boldsymbol{C}$, that describes the orientation of an equivalent N-dimensional rigid body. The generalized coordinates of the system are equal to the ERPs of this rigid body. Additionally, these generalized coordinates and velocities can be used to define a set of Cayley quasi velocities via the Cayley-transform kinematic relationship. This set of quasi velocities is equal to the angular velocity of the equivalent N-dimensional rigid body. The Cayley quasi velocities are therefore governed by the N-dimensional rotational equations of motion described above. This view of Lagrangian dynamics is called the Cayley form [16].

D. General Spacecraft Motion

A large variety of mechanical systems can be studied using the Cayley form. This section presents an example of the combined orbital and rotational motion of a spacecraft acted on by a gravity-gradient torque. For this problem a spherically-symmetric Earth with gravitational parameter $\mu = 398600.4415\,\mathrm{km}^3/\mathrm{s}^2$ is assumed. Additionally, higher-order effects such as higher-order gravitational torques and the influence of attitude on the orbital motion are ignored. The satellite to be considered will have a circular orbit with a radius of 8000 km.

The general motion of a three-dimensional rigid body has six degrees of freedom and can therefore be related to the pure rotation of a four-dimensional rigid body. In fact, the general motion of any N-dimensional body can be viewed as pure rotation of an $(N + 1)$-dimensional body. The study of a three-dimensional rigid body using the Cayley form is slightly complicated by the fact that three-dimensional rotational motion is conventionally studied in terms of the three-dimensional angular velocity, already a set of quasi velocities. Therefore, this portion of the problem must first be converted back to a generalized-velocity expression. Throughout the example, primes are used to denote three-dimensional rotational variables, as opposed to the four-dimensional Cayley variables.

The generalized coordinates for the problem are $[\boldsymbol{q}] = [s'_1 \quad s'_2 \quad s'_3 \quad x_1 \quad x_2 \quad x_3]^T$, where s'_1, s'_2, and s'_3 are the three-dimensional modified Rodrigues parameters (MRPs) and x_1, x_2, and x_3 are the Cartesian position coordinates with respect to an Earth-centered inertial reference frame, $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$, in units of Earth radii (ER=6378 km). The choice of MRPs is somewhat arbitrary, and any three-parameter, attitude representation could be used equally well. The MRPs relate the orientation of a body-fixed frame $\left(\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\right)$ to the local vertical local horizontal (LVLH) frame. In this example the LVLH frame is defined with $\hat{\boldsymbol{a}}_1$ aligned with the radial direction, $\hat{\boldsymbol{a}}_3$ oriented out of the orbital plane, and $\hat{\boldsymbol{a}}_2$ completing the orthogonal triad. The LVLH frame was chosen as a reference to facilitate simulation over complete orbits while avoiding MRP singularities.

In terms of the velocity, $\boldsymbol{v}$ (ER/sec), and the angular velocity of the body with respect to the inertial reference, $\boldsymbol{\omega}'$ (rad/sec), the kinetic energy is given by the following, where $\boldsymbol{J}'$ is the three-dimensional inertia tensor and m is the body's mass.

$$T = \frac{1}{2}\boldsymbol{\omega}'^T \boldsymbol{J}' \boldsymbol{\omega}' + \frac{1}{2} m \boldsymbol{v}^T \boldsymbol{v} \tag{6.10}$$

Both $\boldsymbol{\omega}'$ and $\boldsymbol{J}'$ are coordinatized in the body frame. For the current example the following inertia properties were used.

$$[\boldsymbol{J}'] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 200 \end{bmatrix} \mathrm{kg\,m^2} \tag{6.11}$$

For this choice of the inertia tensor the gravity-gradient stable configuration occurs at alignment of the $\hat{\boldsymbol{b}}_1$ and $\hat{\boldsymbol{a}}_1$ vectors.

The angular velocity of the spacecraft with respect to the inertial reference is the sum of the angular velocity with respect to the LVLH frame, $\breve{\boldsymbol{\omega}}'$, and the angular velocity of the LVLH frame with respect to the inertial frame, $\bar{\boldsymbol{\omega}}'$. To simplify the following developments the known solution for the angular velocity of the LVLH frame in a circular orbit will be used directly.

$$\boldsymbol{\omega}' = \breve{\boldsymbol{\omega}}' + \bar{\boldsymbol{\omega}}' = \breve{\boldsymbol{\omega}}' + n\boldsymbol{C}_1\hat{\boldsymbol{a}}_3 \tag{6.12}$$

Here $\boldsymbol{C}_1$ is the rotation matrix from the LVLH to the body frame and is a function of the first three generalized coordinates, and n is the mean motion of the orbit. This expression is used to expand the kinetic energy.

$$T = \frac{1}{2}\breve{\boldsymbol{\omega}}'^T\boldsymbol{J}'\breve{\boldsymbol{\omega}}' + \breve{\boldsymbol{\omega}}'^T\boldsymbol{J}'\bar{\boldsymbol{\omega}}' + \frac{1}{2}\bar{\boldsymbol{\omega}}'^T\boldsymbol{J}'\bar{\boldsymbol{\omega}}' + \frac{1}{2}m\boldsymbol{v}^T\boldsymbol{v} \tag{6.13}$$

The three-dimensional angular velocity is related to the MRP rates by the familiar kinematic differential equation.

$$\dot{\boldsymbol{s}}' = \boldsymbol{A}'\breve{\boldsymbol{\omega}}' \qquad ; \qquad \breve{\boldsymbol{\omega}}' = \boldsymbol{B}'\dot{\boldsymbol{s}}' \tag{6.14}$$

$$[\boldsymbol{A}'] = \frac{1}{4}\begin{bmatrix} 1+{s'_1}^2-{s'_2}^2-{s'_3}^2 & 2\left(s'_1 s'_2 - s'_3\right) & 2\left(s'_1 s'_3 + s'_2\right) \\ 2\left(s'_1 s'_2 + s'_3\right) & 1-{s'_1}^2+{s'_2}^2-{s'_3}^2 & 2\left(s'_2 s'_3 - s'_1\right) \\ 2\left(s'_1 s'_3 - s'_2\right) & 2\left(s'_2 s'_3 + s'_1\right) & 1-{s'_1}^2-{s'_2}^2+{s'_3}^2 \end{bmatrix} \tag{6.15}$$

$$\boldsymbol{B}' = \boldsymbol{A}'^{-1} \tag{6.16}$$

Using these three-dimensional kinematics, the kinetic energy can be defined in terms of the generalized coordinates and velocities.

$$\begin{aligned} T_0\left(\boldsymbol{q}, \dot{\boldsymbol{q}}\right) &= \frac{1}{2}\dot{\boldsymbol{s}}'^T \boldsymbol{B}'^T \boldsymbol{J}' \boldsymbol{B}' \dot{\boldsymbol{s}}' + \dot{\boldsymbol{s}}'^T \boldsymbol{B}'^T \boldsymbol{J}' \bar{\boldsymbol{\omega}}' + \frac{1}{2}\bar{\boldsymbol{\omega}}'^T \boldsymbol{J}' \bar{\boldsymbol{\omega}}' + \frac{1}{2} m \boldsymbol{v}^T \boldsymbol{v} \\ &= \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{J} \dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T \boldsymbol{K} + \frac{1}{2}\bar{\boldsymbol{\omega}}'^T \boldsymbol{J}' \bar{\boldsymbol{\omega}}' \end{aligned} \tag{6.17}$$

$$[\boldsymbol{J}\left(\boldsymbol{q}\right)] = \begin{bmatrix} \boldsymbol{B}'^T \boldsymbol{J}' \boldsymbol{B}' & \boldsymbol{0} \\ \boldsymbol{0} & m\boldsymbol{I} \end{bmatrix} \quad ; \quad [\boldsymbol{K}\left(\boldsymbol{q}\right)] = \begin{bmatrix} \boldsymbol{B}'^T \boldsymbol{J}' \bar{\boldsymbol{\omega}}' \\ \boldsymbol{0} \end{bmatrix} \tag{6.18}$$

Here, the matrix $\boldsymbol{I}$ is the 3×3 identity matrix. The linear mapping from the Cayley quasi velocities to the generalized velocities, $\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{\omega}$, is found from the Cayley kinematic definition and used to write the kinetic energy in terms of the generalized coordinates and Cayley quasi velocities.

$$\begin{aligned} T_1\left(\boldsymbol{q}, \boldsymbol{\omega}\right) &= \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{A}^T \boldsymbol{J} \boldsymbol{A} \boldsymbol{\omega} + \boldsymbol{\omega}^T \boldsymbol{A}^T \boldsymbol{K} + \frac{1}{2}\bar{\boldsymbol{\omega}}'^T \boldsymbol{J}' \bar{\boldsymbol{\omega}}' \\ &= \frac{1}{2}\omega_i A_{li} J_{lm} A_{mj} \omega_j + \omega_i A_{li} K_l + \frac{1}{2}\bar{\omega}'_i J'_{ij} \bar{\omega}'_j \end{aligned} \tag{6.19}$$

The necessary partial derivatives of T_1 can now be taken. Note that $\boldsymbol{J}$ and $\boldsymbol{K}$ are functions of the first three generalized coordinates and are not constant.

$$\begin{aligned} \frac{\partial T_1}{\partial q_r} =& \frac{1}{2}\omega_i \left(\frac{\partial A_{li}}{\partial q_r} J_{lm} A_{mj} + A_{li} \frac{\partial J_{lm}}{\partial q_r} A_{mj} + A_{li} J_{lm} \frac{\partial A_{mj}}{\partial q_r} \right) \omega_j \\ &+ \omega_i \left(\frac{\partial A_{li}}{\partial q_r} K_l + A_{li} \frac{\partial K_l}{\partial q_r} \right) + \frac{\partial \bar{\omega}'_i}{\partial q_r} J'_{ij} \bar{\omega}'_j \end{aligned} \tag{6.20}$$

$$\frac{\partial T_1}{\partial \omega_r} = A_{lr} J_{lm} A_{mj} \omega_j + A_{lr} K_l \tag{6.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_1}{\partial \omega_k}\right) = \dot{A}_{lk}J_{lm}A_{mj}\omega_j + A_{lk}\dot{J}_{lm}A_{mj}\omega_j + A_{lk}J_{lm}\dot{A}_{mj}\omega_j + A_{lk}J_{lm}A_{mj}\dot{\omega}_j + \dot{A}_{lr}K_l + A_{lr}\dot{K}_l \tag{6.22}$$

The derivatives of $\boldsymbol{A}$ are computed using the chain rule.

$$\dot{A}_{lk} = \frac{\partial A_{lk}}{\partial q_s}\dot{q}_s = \frac{\partial A_{lk}}{\partial q_s}A_{st}\omega_t \tag{6.23}$$

The partial derivatives of $\boldsymbol{A}$ are directly evaluated from Eq. (6.7). The derivatives of $\boldsymbol{J}$ must now be considered.

$$\frac{\partial J_{lm}}{\partial q_r} = \begin{bmatrix} \frac{\partial \boldsymbol{B}'}{\partial q_r}^T \boldsymbol{J}'\boldsymbol{B}' + \boldsymbol{B}'^T\boldsymbol{J}'\frac{\partial \boldsymbol{B}'}{\partial q_r} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}_{lm} \tag{6.24}$$

$$\dot{J}_{lm} = \frac{\partial J_{lm}}{\partial q_i}\dot{q}_i \tag{6.25}$$

The derivatives of $\boldsymbol{B}$ are computed in the following manner.

$$\frac{\partial \boldsymbol{B}'}{\partial q_r} = -\boldsymbol{B}'\frac{\partial \boldsymbol{A}'}{\partial q_r}\boldsymbol{B}' \tag{6.26}$$

The derivatives of $\boldsymbol{K}$ are also computed.

$$\frac{\partial K_l}{\partial q_r} = \begin{bmatrix} \frac{\partial \boldsymbol{B}'^T}{\partial q_r}\boldsymbol{J}'\bar{\boldsymbol{\omega}}' + \boldsymbol{B}'^T\boldsymbol{J}'\frac{\partial \bar{\boldsymbol{\omega}}'}{\partial q_r} \\ \boldsymbol{0} \end{bmatrix}_l \tag{6.27}$$

$$\dot{K}_l = \frac{\partial K_l}{\partial q_i}\dot{q}_i \tag{6.28}$$

Finally, the partial derivatives of $\boldsymbol{A}'$ are directly evaluated from Eq. (6.15), and clearly only the derivatives with respect to the first three generalized coordinates are nonzero. Similarly, the partial derivatives of $\bar{\boldsymbol{\omega}}$ are evaluated by directly taking the derivatives of the rotation matrix $\boldsymbol{C}_1$ with respect to the MRP coordinates.

The generalized forces in terms of the generalized velocities are related to the moment vector $\boldsymbol{M}$ (coordinatized in the body frame) and the force vector $\boldsymbol{F}$ (coordinatized in the inertial frame) applied to the body. These are computed using the familiar gravitational models where the vector $\boldsymbol{r}$ has components consisting of the three position coordinates and a magnitude of r, and $\boldsymbol{C}_2$ is the rotation matrix from the inertial to LVLH frame. This matrix is assembled from the inertial components of $\boldsymbol{r}$ and $\boldsymbol{v}$. Additionally, a damping moment is included with the coefficient $k = 0.1\,\mathrm{kg\,m^2/s}$ to simulate a nutation damper onboard the spacecraft.

$$\boldsymbol{M} = 3\frac{\mu}{r^3}\boldsymbol{a} \times \boldsymbol{J}'\boldsymbol{a} - k\boldsymbol{\omega}' \qquad ; \qquad \boldsymbol{a} = \boldsymbol{C}_1\boldsymbol{C}_2\boldsymbol{r}/r \tag{6.29}$$

$$\boldsymbol{F} = -\frac{\mu m}{r^3}\boldsymbol{r} \tag{6.30}$$

The generalized forces with respect to the first and fourth generalized coordinates are shown below.

$$f_1 = \boldsymbol{F}^T\frac{\partial \boldsymbol{v}}{\partial \dot{q}_1'} + \boldsymbol{M}^T\frac{\partial \boldsymbol{\omega}'}{\partial \dot{q}_1'} = \boldsymbol{M}^T\boldsymbol{B}'\hat{\boldsymbol{b}}_1 \tag{6.31}$$

$$f_4 = \boldsymbol{F}^T\frac{\partial \boldsymbol{v}}{\partial \dot{x}_1} + \boldsymbol{M}^T\frac{\partial \boldsymbol{\omega}'}{\partial \dot{x}_1} = \boldsymbol{F}^T\hat{\boldsymbol{e}}_1 \tag{6.32}$$

The other generalized forces are similar to these two.

$$[\boldsymbol{f}] = \begin{bmatrix}\boldsymbol{M}^T\boldsymbol{B}' & \boldsymbol{F}^T\end{bmatrix}^T \tag{6.33}$$

Using Eqs. (6.20) to (6.33), the equations of motion as shown in Table IV can be assembled and solved for the components $\dot{\omega}_j$. These equations can then be integrated to solve directly for the motion of the satellite in terms of the generalized coordinates and the Cayley quasi velocities. Due to the coupling of all six degrees of freedom, however, these equations are somewhat more difficult to integrate than the traditional implementation. Solving for $\dot{\boldsymbol{\omega}}$ requires taking the inverse of a 6×6 matrix. The difficulty can be somewhat relieved through scaling. Integrating over one orbital period using the Matlab algorithm ODE45 required 769 integration steps for the traditional Euler and orbit equations and 989 integration steps for the Cayley form, with identical settings. Integration of the traditional implementation can also be used to verify the results from the Cayley form. The Cayley-transform kinematic relationship can be applied to the results of the traditional equations of motion to compute the quasi velocities from the generalized coordinates and velocities. The results of these numerical solutions are shown in Figs. 9 and 10.

Figure 9 shows the solutions for the generalized coordinates, three-dimensional angular velocity, and translational velocity. The simulation results represent the motion of the satellite through one complete orbit around the Earth. The orbit has an inclination of thirty degrees. The spacecraft is initially rotated 10 degrees relative to the LVLH frame. Due to the initial conditions and the gravity-gradient torque the spacecraft remains within 33 degrees of the LVLH reference. As can be seen by s_3' in Fig. 9, for the first sixty minutes the spacecraft attitude lags behind the LVLH frame due to the damping moment. As this happens the gravity-gradient moment increases and during the second sixty minutes the spacecraft begins to restore alignment with the LVLH frame. Figure 10 shows the results obtained for the Cayley quasi velocities. The implication of the Cayley form is that there is an equivalent rotational motion of a four-dimensional body that corresponds to the general motion of the satellite. For this equivalent motion the six components of the ERPs have trajectories equal to the solutions shown in Fig. 9 for the MRPs and position coordinates. The angular-velocity components are equal to the trajectories shown in Fig. 10.

These results also demonstrate some interesting features of the Cayley form. One feature is due to the nondimensionality of the ERPs. Because the Cayley form sets the generalized coordinates of the system numerically equal to the ERPs of the equivalent N-dimensional rotation, the mapping is sensitive to the choice of coordinates and

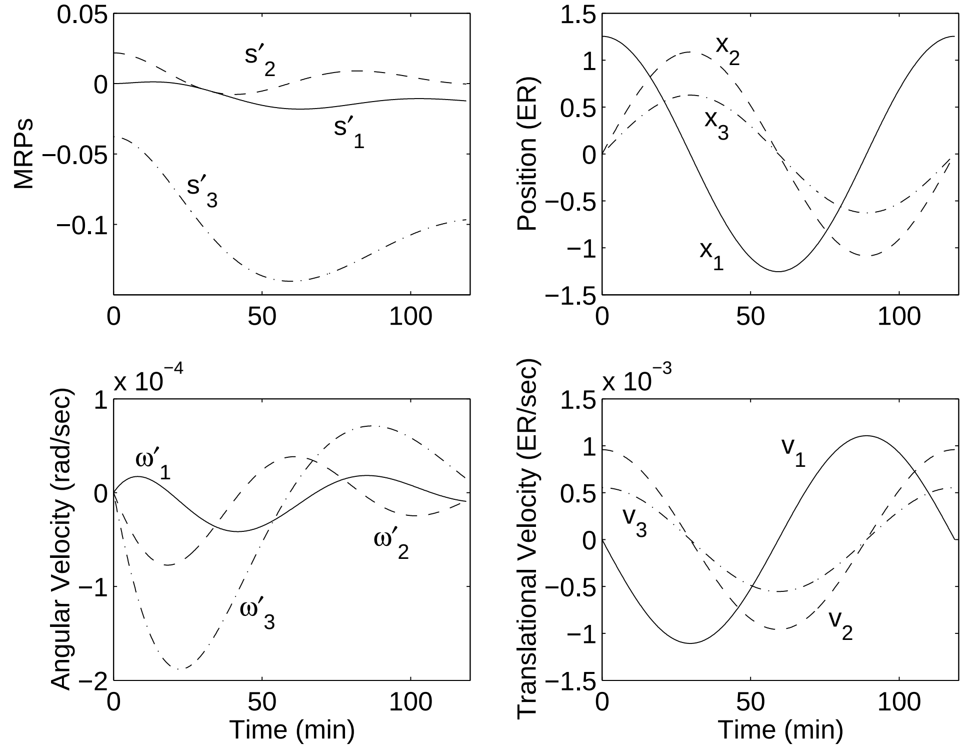

Fig. 9. Attitude and orbital motion variables.

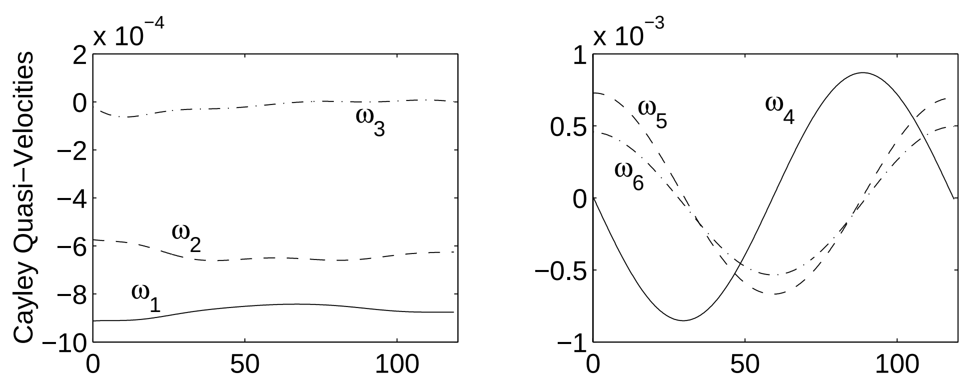

Fig. 10. Attitude and orbital motion Cayley quasi velocities.

units used to describe the original system. For example, the use of kilometer units to describe the above orbital dynamics would produce a different time history of the ERPs and thus a different four-dimensional rotation.

Another feature has to do with the presence of singularities in the ERP description of N-dimensional orientation. Exactly analogous to the familiar Rodrigues parameter singularities, certain $N \times N$ proper orthogonal matrices result in ERP elements that tend toward infinity [2]. Because in the Cayley form the ERPs are equal to the generalized coordinates of the original system, though, these rotation variables can only go to infinity if the original generalized coordinates do the same. The MRPs used in the above example suffer from singularities in the conventional equations of motion; however, applying the Cayley form does not add any new singularities. For cases in which the generalized coordinates of the original problem are guaranteed to avoid divergence to infinity, then the Cayley form will automatically be constrained to avoid the singular N-dimensional configurations.

E. Satellite with Three Momentum Wheels

The Cayley form can also be applied to multibody systems. Again, the dimension of the equivalent body in pure rotation is a function of the total number of degrees in freedom of the system. Here, the rotational motion of a satellite with three momentum wheels will be considered. Similar to the previous example, this system has six degrees of freedom and is equivalent to the rotational motion of a four-dimensional body.

The generalized coordinates for this problem consist of two sets: (1) the Rodrigues parameters describing the orientation of an axes set $\left(\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\right)$ attached to the satellite body relative to an inertial set $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ and (2) the angles describing the orientation of each wheel relative to the satellite.

$$[\boldsymbol{q}] = [q_1' \quad q_2' \quad q_3' \quad \theta_1 \quad \theta_2 \quad \theta_3]^T \equiv \left[\boldsymbol{q}'^T \quad \boldsymbol{\theta}^T\right]^T \tag{6.34}$$

The body axes attached to the satellite are assumed to be the principal axes of the satellite body. The three wheels are identical, symmetric, and aligned with the body axes. Additionally, for convenience, the wheels are assumed to be located at the system center of mass.

The kinetic energy of the system is given by the following.

$$T = \frac{1}{2}\boldsymbol{\omega}_b^T \boldsymbol{J}_b \boldsymbol{\omega}_b + \frac{1}{2}\boldsymbol{\omega}_{w1}^T \boldsymbol{J}_{w1} \boldsymbol{\omega}_{w1} + \frac{1}{2}\boldsymbol{\omega}_{w2}^T \boldsymbol{J}_{w2} \boldsymbol{\omega}_{w2} + \frac{1}{2}\boldsymbol{\omega}_{w3}^T \boldsymbol{J}_{w3} \boldsymbol{\omega}_{w3} \tag{6.35}$$

The subscript b refers to the satellite body and the subscripts $w1$, $w2$, and $w3$ refer to the three wheels. The angular velocity of wheel 1, for example, is $\boldsymbol{\omega}_{w1} = \boldsymbol{\omega}_b + \dot{\theta}_1 \hat{\boldsymbol{b}}_1$. For simplicity the inertia of the satellite body was chosen as $\boldsymbol{J}_b = 100\boldsymbol{I}\ \mathrm{kg\,m}^2$. The axial component of inertia for each wheel was chosen to be $J_a = 10\,\mathrm{kg\,m}^2$, and the transverse component was $J_t = 1\mathrm{kg\,m}^2$. For convenience, the sum $\boldsymbol{J}' = \boldsymbol{J}_b + \boldsymbol{J}_{w1} + \boldsymbol{J}_{w2} + \boldsymbol{J}_{w3}$ is defined. The kinetic energy in terms of the body angular velocity and the wheel angle rates is shown below.

$$T = \frac{1}{2}\boldsymbol{\omega}_b^T \boldsymbol{J}' \boldsymbol{\omega}_b + J_a \boldsymbol{\omega}_b^T \dot{\boldsymbol{\theta}} + \frac{1}{2} J_a \dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} \tag{6.36}$$

Using the three-dimensional kinematics this is converted to an expression in terms of the generalized coordinates and velocities.

$$T_0 = \frac{1}{2}\left[\dot{\boldsymbol{q}}'^T \quad \dot{\boldsymbol{\theta}}^T\right] \begin{bmatrix} \boldsymbol{B}'^T \boldsymbol{J}' \boldsymbol{B}' & J_a \boldsymbol{B}'^T \\ J_a \boldsymbol{B}' & J_a \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}' \\ \dot{\boldsymbol{\theta}} \end{bmatrix} \equiv \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{J} \dot{\boldsymbol{q}} \tag{6.37}$$

Again, this expression is rewritten in terms of the Cayley quasi velocities using the four-dimensional Cayley kinematics.

$$T_1(\boldsymbol{q},\boldsymbol{\omega}) = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{A}^T\boldsymbol{J}\boldsymbol{A}\boldsymbol{\omega} = \frac{1}{2}\omega_i A_{li} J_{lm} A_{mj}\omega_j \tag{6.38}$$

The derivatives of T_1 are computed in an identical manner as was done in the previous example. These expressions are somewhat simpler than in the previous example due to the natural form of the kinetic energy in this problem. Another difference is that the derivatives of $\boldsymbol{J}$ now have the following form.

$$\frac{\partial J_{lm}}{\partial q_r} = \begin{bmatrix} \frac{\partial \boldsymbol{B}'}{\partial q_r}^T \boldsymbol{J}'\boldsymbol{B}' + \boldsymbol{B}'^T\boldsymbol{J}'\frac{\partial \boldsymbol{B}'}{\partial q_r} & J_a\frac{\partial \boldsymbol{B}'^T}{\partial q_r} \\ J_a\frac{\partial \boldsymbol{B}'}{\partial q_r} & \boldsymbol{0} \end{bmatrix}_{lm} \tag{6.39}$$

The generalized forces are computed considering an external moment $\boldsymbol{M}$ (coordinatized in the body frame) and internal moments $\boldsymbol{u}_1 = u_1\hat{\boldsymbol{b}}_1$, $\boldsymbol{u}_2 = u_2\hat{\boldsymbol{b}}_2$, and $\boldsymbol{u}_3 = u_3\hat{\boldsymbol{b}}_3$. The internal moments are applied to each wheel, respectively. The external moment and internal moments $(-\boldsymbol{u} = -\boldsymbol{u}_1 - \boldsymbol{u}_2 - \boldsymbol{u}_3)$ are applied to the satellite body. The generalized forces associated with the first and fourth generalized velocities are shown below.

$$f_1 = (\boldsymbol{M}-\boldsymbol{u})^T\frac{\partial \boldsymbol{\omega}_b}{\partial \dot{q}_1'} + \boldsymbol{u}_1^T\frac{\partial \boldsymbol{\omega}_{w1}}{\partial \dot{q}_1'} + \boldsymbol{u}_2^T\frac{\partial \boldsymbol{\omega}_{w2}}{\partial \dot{q}_1'} + \boldsymbol{u}_3^T\frac{\partial \boldsymbol{\omega}_{w3}}{\partial \dot{q}_1'} = \boldsymbol{M}^T\frac{\partial \boldsymbol{\omega}_b}{\partial \dot{q}_1'} = \boldsymbol{M}^T\boldsymbol{B}'\hat{\boldsymbol{b}}_1 \tag{6.40}$$

$$f_4 = (\boldsymbol{M}-\boldsymbol{u})^T\frac{\partial \boldsymbol{\omega}_b}{\partial \dot{\theta}_1} + \boldsymbol{u}_1^T\frac{\partial \boldsymbol{\omega}_{w1}}{\partial \dot{\theta}_1} + \boldsymbol{u}_2^T\frac{\partial \boldsymbol{\omega}_{w2}}{\partial \dot{\theta}_1} + \boldsymbol{u}_3^T\frac{\partial \boldsymbol{\omega}_{w3}}{\partial \dot{\theta}_1} = u_1 \tag{6.41}$$

The other generalized forces are similar to these two: $[\boldsymbol{f}] = \begin{bmatrix}\boldsymbol{M}^T\boldsymbol{B}' & \boldsymbol{u}^T\end{bmatrix}^T$.

Using the derivatives of T_1 and the generalized forces, the equations of motion can be assembled and solved for the components $\dot{\omega}_j$. Figures 11 and 12 show simulation results from the integration of these equations of motion and the Cayley kinematic equations. The initial conditions were chosen such that the body was initially rotating about the $\hat{\boldsymbol{b}}_1$ axis. Beginning at time zero, a constant moment of 5 Nm is applied to the second momentum wheel. The external moment was set to zero in the simulation. Figure 11 shows the solutions for the generalized coordinates, three-dimensional angular velocity, and wheel-angle rates obtained using a traditional formulation. Figure 12 shows the results obtained for the quasi velocities using the Cayley form. Again, the implication of the Cayley form is that there is an equivalent rotational motion of a four-dimensional body that corresponds to the motion of the satellite and momentum-wheel system. For this equivalent motion the six components of the ERPs have trajectories equal to the solutions shown in Fig. 11 for the Rodrigues parameters and wheel angles. The angular velocity components are equal to the trajectories shown in Fig. 12.

F. Discussion

The most common representation of motion that is applied to general systems is the motion of a point in M-dimensional state space. The application of the Cayley form in this chapter, however, is one example of the broad set of other possible representations. Another, more famous, example is the use of the Frenet formulas to describe particle motion in terms of a rotating reference frame, and in fact the approach of Junkins and Turner [42] is an example of this. The classic Frenet frame is defined by the velocity and acceleration vectors, whereas Junkins and Turner used position and velocity. Their analogy and the Cayley form can both be used to represent three-dimensional translations as three-dimensional rotations. Therefore, combined orbital and attitude motion, the rotation of two rigid bodies, and the rotation of a four-dimensional rigid body represent three members of a family of systems that are in a

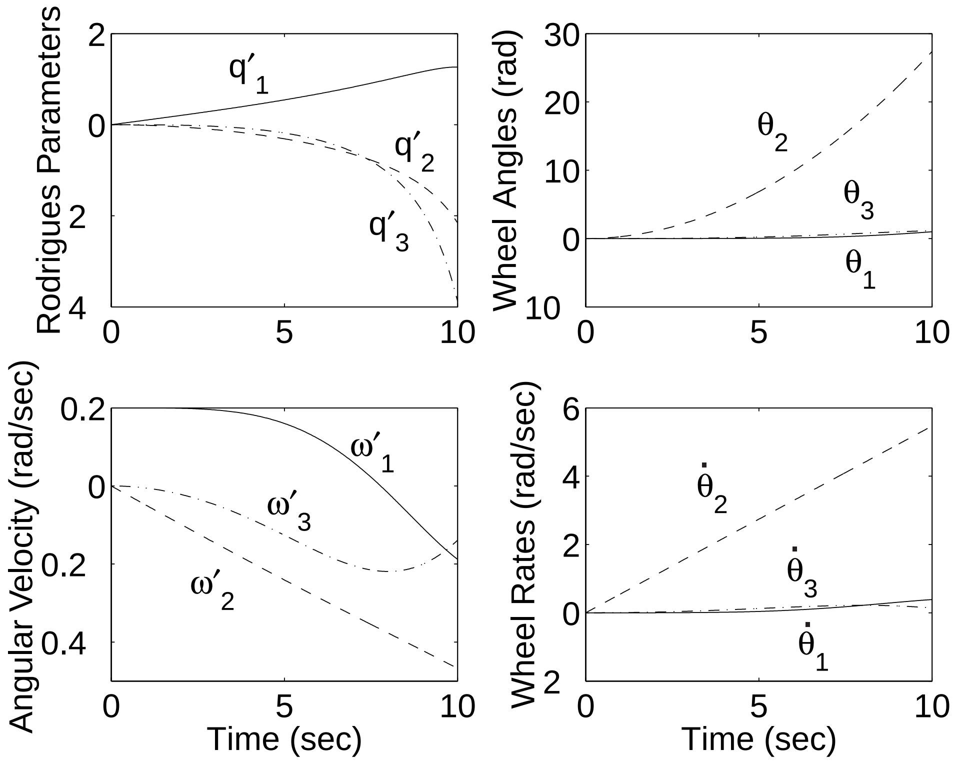

Fig. 11. Satellite system motion variables.

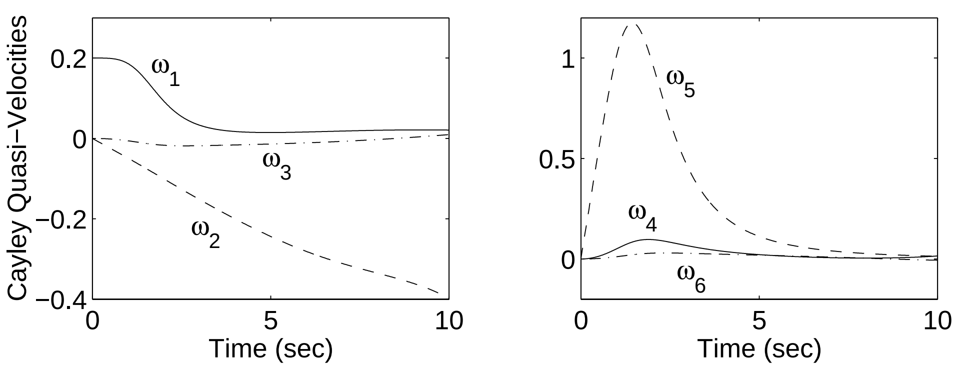

Fig. 12. Satellite system Cayley quasi velocities.

sense equivalent. Figure 13 presents a schematic of some of the possible mappings between these systems. Finally, notice that the use of a generalized coordinate vector represents each of these systems as the motion of a point in six-dimensional state space.

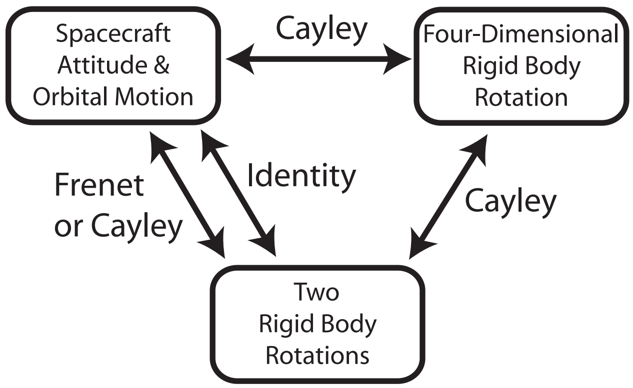

Fig. 13. Possible mappings between three equivalent systems.

Whereas the Cayley form and Frenet formulas have significant differences, the Frenet formulas could potentially also be generalized to higher dimensions. Unlike the Cayley form, which associates an orientation with every point in the configuration space, the Frenet frame is defined by the generalized velocities and accelerations. In higher dimensions, these vectors could form two coordinate vectors of an M-dimensional Frenet frame, leaving an $(M-2)$-dimensional orthogonal subspace for which a unique coordinatization would need to be chosen. This would then associate an M-dimensional reference frame with the M-DOF motion, again unlike the Cayley form which uses an N-dimensional reference frame.

As mentioned, the avoidance of any osculation constraint in the Cayley form allows a more complete incorporation of the dynamics. Similar to the variable inertia term in the Junkins and Turner analogy, however, in the Cayley form the kinetic energy was found to be a function of the generalized coordinates. While both analogies match the kinematic properties of rigid bodies, the dynamic properties of rigid bodies (i.e., constant inertia) do not precisely hold. The analogies can only go so far.

G. Conclusions

In this chapter the kinematics and dynamics of N-dimensional bodies have been discussed and their application to general spacecraft motion has been demonstrated. This Cayley form results in new equations of motion for the orbital and attitude dynamics. A perceived disadvantage of this form is the resulting coupling that takes place between the attitude and orbital motion. This coupling, however, reveals a possible application of the Cayley form. Many special techniques have been developed for attitude determination and control in three-dimensional space. If these techniques can be generalized to higher dimensions, then the Cayley form can be used to apply them to a broader class of mechanical systems. Although not treated in this chapter, some preliminary results involving feedback control using the Cayley form have been developed.

Possible applications include combined attitude and orbit determination and control. This could be of particular interest for systems that already exhibit a high degree of attitude and orbital coupling, such as low-thrust, fixed thrust-direction spacecraft and very large spacecraft subject to higher-order gravitational torques and forces. A more theoretical direction for future research is the study of the relation of ERPs to the principal planes and angles of N-dimensional orientation and the interpretation of ERP singularities.

CHAPTER VII

STABILIZATION AND CONTROL OF DYNAMICAL SYSTEMS IN THE CAYLEY FORM

A. Introduction

Beginning with Euler the rotational motion of three-dimensional bodies has been studied for over two hundred years. The development of spacecraft technology over the past fifty years has inspired a continuing focus on this problem that has produced many significant developments in attitude representations and control. An example of this, that will be a focus of this chapter, is the proof discovered by Tsiotras for global asymptotic stability using linear feedback of angular velocity and the Rodrigues parameters [45, 46].

Although not receiving as much attention, the study of N-dimensional rotational motion also has an impressive history [1, 8, 22, 23]. Developments have been made to generalize many of the kinematic and dynamical concepts from three-dimensional rotations to higher dimensions [2, 5, 6, 15, 21, 24, 28]. Another interesting aspect of this topic is that the resulting equations of motion can be used to describe the behavior of real physical systems. One example of this is called the Cayley form and represents the generalized coordinates and quasi velocities of a general system as the extended Rodrigues parameters (ERPs) and angular velocity of an N-dimensional rotating body [16, 17].

The wealth of work that has been done for three-dimensional attitude control and the ability to represent general systems as N-dimensional rigid bodies motivates the idea of generalizing some of these three-dimensional results to higher dimensions. This chapter presents some investigations for extending Lyapunov and optimal control results to N-dimensional rotations. Further investigations are also conducted into the application of these results to natural dynamical systems.

Section B of this chapter reviews the set of quasi velocities used in the Cayley form related to N-dimensional rotations. Section C reviews some elegant attitude control results discovered by Tsiotras, approaching them as a three-dimensional special case of the Cayley form. Next, the equations of motion for N-dimensional rotations are covered and the corresponding work/energy-rate expression is developed in Section D. Then, it is shown in Section E how properties of the N-dimensional kinematics disallow an exact generalization of Tsiotras's results, and several alternative controller designs are presented. An alternative set of quasi velocities is presented in Section F that does provide globally asymptotically stable linear feedback. In Sections G and H the controllers from the Cayley form are analyzed for their optimality and performance.

B. Definition of Cayley Quasi Velocities

The Cayley form describes N-dimensional rotations using the ERPs, $\boldsymbol{Q}$ or $\boldsymbol{q}$, which are related to the N-dimensional orientation matrix by the Cayley transform. The Cayley-transform kinematic relationship gives a mapping for N-dimensional rotations from the ERP rates to the angular velocity in skew-symmetric form [12].

$$\boldsymbol{\Omega} = 2\left(\boldsymbol{I}+\boldsymbol{Q}\right)^{-1}\dot{\boldsymbol{Q}}\left(\boldsymbol{I}-\boldsymbol{Q}\right)^{-1} \equiv 2\boldsymbol{B}^{+}\dot{\boldsymbol{Q}}\boldsymbol{B}^{-} \tag{7.1}$$

$$\dot{\boldsymbol{Q}} = \frac{1}{2}\left(\boldsymbol{I}+\boldsymbol{Q}\right)\boldsymbol{\Omega}\left(\boldsymbol{I}-\boldsymbol{Q}\right) \equiv \frac{1}{2}\boldsymbol{A}^{+}\boldsymbol{\Omega}\boldsymbol{A}^{-} \tag{7.2}$$

In the Cayley-form representation of mechanical systems these expressions also form the definition of the Cayley quasi velocities, $\boldsymbol{\Omega}$. For an M-degree-of-freedom (DOF) system, the Cayley form equates the generalized coordinates of the system with the

ERPs of an N-dimensional rotation, and the Cayley quasi velocities are equivalent to the N-dimensional angular velocity. From these relationships the linear mapping between the vector forms of the ERP rates and the angular velocity can be found, which also serves as the traditional vector-transformation definition of the quasi velocities.

$$\boldsymbol{\omega} = \boldsymbol{B}\dot{\boldsymbol{q}} \qquad ; \qquad \dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{\omega} \tag{7.3}$$

Relating Eq. (7.2) and the second of Eqs. (7.3) the elements of $\boldsymbol{A}$ are found.

$$\dot{Q}_{vp} = \frac{1}{2} A^{+}_{vk} \Omega_{kl} A^{-}_{lp} \tag{7.4}$$

$$\chi^{j}_{vp} \dot{q}_j = \frac{1}{2} A^{+}_{vk} \chi^{m}_{kl} \omega_m A^{-}_{lp} \tag{7.5}$$

$$2\delta_{ij} \dot{q}_j = \frac{1}{2} \chi^{i}_{vp} \chi^{m}_{kl} A^{+}_{vk} A^{-}_{lp} \omega_m \tag{7.6}$$

$$\dot{q}_i = \frac{1}{4} \chi^{i}_{vp} \chi^{m}_{kl} A^{+}_{vk} A^{-}_{lp} \omega_m \tag{7.7}$$

$$\begin{aligned} A_{im} \equiv \frac{1}{4} \chi^{i}_{vp} \chi^{m}_{kl} A^{+}_{vk} A^{-}_{lp} &= \frac{1}{4} \chi^{i}_{vp} \chi^{m}_{kl} \left(\delta_{vk} + Q_{vk}\right) \left(\delta_{lp} - Q_{lp}\right) \\ &= \frac{1}{4} \chi^{i}_{vp} \chi^{m}_{kl} \left(\delta_{vk}\delta_{lp} - \delta_{vk} Q_{lp} + \delta_{lp} Q_{vk} - Q_{vk} Q_{lp}\right) \end{aligned} \tag{7.8}$$

The first term of Eq. (7.8) simplifies as follows.

$$\frac{1}{4} \chi^{i}_{vp} \chi^{m}_{kl} \delta_{vk} \delta_{lp} = \frac{1}{4} \chi^{i}_{vp} \chi^{m}_{vp} = \frac{1}{2} \delta_{im} \tag{7.9}$$

The second term of Eq. (7.8) can also be simplified.

$$-\frac{1}{4} \chi^{i}_{vp} \chi^{m}_{kl} \delta_{vk} Q_{lp} = -\frac{1}{4} \chi^{i}_{vp} \chi^{m}_{vl} Q_{lp} \tag{7.10}$$

The third term is identical to the second.

$$\frac{1}{4} \chi^{i}_{vp} \chi^{m}_{kl} \delta_{lp} Q_{vk} = \frac{1}{4} \chi^{i}_{vp} \chi^{m}_{kp} Q_{vk} = \frac{1}{4} \chi^{i}_{pv} \chi^{m}_{lv} Q_{pl} = -\frac{1}{4} \chi^{i}_{vp} \chi^{m}_{vl} Q_{lp} \tag{7.11}$$

Substituting these terms back into Eq. (7.8) gives the following expression for the elements of $\boldsymbol{A}$.

$$A_{im} = \frac{1}{2}\left(\delta_{im} - \chi^{i}_{vp}\chi^{m}_{vl}Q_{lp} - \frac{1}{2}\chi^{i}_{vp}\chi^{m}_{kl}Q_{vk}Q_{lp}\right) \tag{7.12}$$

For the special case $N = 3$, the equation for the elements of A can be simplified by substituting ϵ_{ijk} for χ^{j}_{ik}.

$$A_{im} = \frac{1}{2}\left(\delta_{im} - \epsilon_{vip}\epsilon_{vml}Q_{lp} - \frac{1}{2}\epsilon_{vip}\epsilon_{kml}Q_{vk}Q_{lp}\right) \tag{7.13}$$

The "ϵ-δ identity" can be applied to the second term of this equation. The fact that Q_{ii} equals zero, because $\boldsymbol{Q}$ is skew-symmetric, is also used.

$$\epsilon_{vip}\epsilon_{vml}Q_{lp} = \left(\delta_{im}\delta_{pl} - \delta_{il}\delta_{pm}\right)Q_{lp} = -Q_{im} \tag{7.14}$$

The third term of Eq. (7.13) can also be rewritten using the generalized Kronecker delta [32].

$$\begin{aligned}\epsilon_{vip}\epsilon_{kml} = \delta^{kml}_{vip} = &\begin{vmatrix} \delta_{vk} & \delta_{ik} & \delta_{pk} \\ \delta_{vm} & \delta_{im} & \delta_{pm} \\ \delta_{vl} & \delta_{il} & \delta_{pl} \end{vmatrix} \\ &= \delta_{vk}\delta_{im}\delta_{pl} - \delta_{vk}\delta_{pm}\delta_{il} + \delta_{ik}\delta_{pm}\delta_{vl} - \delta_{ik}\delta_{vm}\delta_{pl} + \delta_{pk}\delta_{vm}\delta_{il} - \delta_{pk}\delta_{im}\delta_{vl}\end{aligned} \tag{7.15}$$

The third term of Eq. (7.13) therefore becomes the following.

$$\epsilon_{vip}\epsilon_{kml}Q_{vk}Q_{lp} = Q_{vi}Q_{vm} + Q_{mp}Q_{ip} - \delta_{im}Q_{vp}Q_{vp} = 2Q_{vi}Q_{vm} - \delta_{im}Q_{vp}Q_{vp} \tag{7.16}$$

This expression is now rewritten in terms of the vector elements q_j and the ϵ-δ identity is used once again.

$$\begin{aligned}
\epsilon_{vip}\epsilon_{kml}Q_{vk}Q_{lp} &= 2\epsilon_{vri}q_r\epsilon_{vsm}q_s - \delta_{im}\epsilon_{vrp}q_r\epsilon_{vsp}q_s \\
&= 2\left(\delta_{rs}\delta_{im} - \delta_{rm}\delta_{is}\right)q_r q_s - \delta_{im}\left(\delta_{rs}\delta_{pp} - \delta_{rp}\delta_{sp}\right)q_r q_s \\
&= 2\delta_{im}q_r q_r - 2q_i q_m - 3\delta_{im}q_r q_r + \delta_{im}q_p q_p \\
&= -2q_i q_m \qquad (7.17)
\end{aligned}$$

Equations (7.14) and (7.17) are now substituted into Eq. (7.13) to give the familiar form for the mapping from the angular velocity to the Rodrigues parameter rates [38].

$$A_{im} = \frac{1}{2}\left(\delta_{im} + Q_{im} + q_i q_m\right) \qquad (7.18)$$

C. Linear Rodrigues-Parameter Feedback

In the previous section a simple form was found for the elements, A_{im}, of the matrix that maps the angular velocity to the Rodrigues parameter rates for $N = 3$. In this section several special properties of this three-dimensional form will be demonstrated that lead to a proof of global asymptotic stability for linear feedback of the Rodrigues parameters and angular velocity. First, for this special case it will be shown that $\boldsymbol{q}$ is an eigenvector of $\boldsymbol{A}^T$, using the fact that the product $\boldsymbol{Qq}$ equals zero.

$$\left[\boldsymbol{A}^T q\right]_m = A_{im}q_i = \frac{1}{2}\left(q_m + Q_{im}q_i + q_i q_m q_i\right) = \frac{1}{2}\left(1 + q_i q_i\right)q_m \equiv \lambda q_m \qquad (7.19)$$

Therefore, the eigenvalue associated with $\boldsymbol{q}$ is $\frac{1}{2}\left(1 + q_i q_i\right)$.

A similar derivation can be used to show that $\boldsymbol{q}$ is also an eigenvector of $\boldsymbol{A}$. This fact, however, can also be understood from a physical interpretation. Consider the situation of $\boldsymbol{\omega}$ being aligned with $\boldsymbol{q}$, $\boldsymbol{\omega} = \alpha\boldsymbol{q}$. In this case the body is simply "spinning up" because the vector of Rodrigues parameters is always parallel with the principal axis of rotation. Therefore, the direction of $\boldsymbol{q}$ is constant, and only its magnitude is changing. This means that $\dot{\boldsymbol{q}}$ is also parallel to $\boldsymbol{q}$, $\dot{\boldsymbol{q}} = \alpha\lambda\boldsymbol{q}$. In this case, the second of Eqs. (7.3) becomes the following.

$$\alpha\lambda\boldsymbol{q} = \boldsymbol{A}\left(\alpha\boldsymbol{q}\right) \tag{7.20}$$

The proportionality factors α and λ are simply scalars, and thus $\boldsymbol{q}$ being an eigenvector of $\boldsymbol{A}$ is physically expected.

The eigenvalue of $\boldsymbol{A}^T$ associated with $\boldsymbol{q}$ has the following remarkable property.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(1 + q_i q_i\right)\right) = q_i \dot{q}_i = q_i A_{im}\omega_m = \frac{1}{2}\left(1 + q_i q_i\right) q_m \omega_m \tag{7.21}$$

Equations (7.19) and (7.21) allow the following elegant proof discovered by Tsiotras [45]. Consider the following Lyapunov function where the first term is the rotational kinetic energy of a three-dimensional rigid body and the second is a fictitious potential energy.

$$V = \frac{1}{2}\omega_i J_{ij}\omega_j + \ln\left(1 + q_i q_i\right) \tag{7.22}$$

Here, J_{ij} are the elements of the principal inertia matrix. As it will be seen, the kinetic-energy term of the Lyapunov functions leads to a *stabilization* term of the control law that brings the body to rest, and the potential-energy term leads to a *regulation* control term that drives the body to the reference orientation. Using Euler's equations the time derivative of the kinetic energy is given by the following.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\omega_i J_{ij}\omega_j\right) = \dot{\omega}_i J_{ij}\omega_j = \omega_j\left(\epsilon_{jil} J_{ik}\omega_l\omega_k + f_j^{(\omega)}\right) = \omega_j f_j^{(\omega)} \tag{7.23}$$

Here, $f_j^{(\omega)}$ are the applied moment components acting on the body and are assumed to be control torques. This leads to the following result for the Lyapunov function derivative.

$$\dot{V} = \omega_i f_i^{(\omega)} + \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(1 + q_i q_i\right)}{1 + q_i q_i} = \omega_i f_i^{(\omega)} + q_i \omega_i \tag{7.24}$$

If the control elements, $f_i^{(\omega)}$, are chosen to make Eq. (7.24) negative semi-definite, then the system will be globally asymptotically stable by LaSalle's theorem. This can be done in the following fashion.

$$\dot{V} = \omega_i f_i^{(\omega)} + q_i \omega_i \equiv -\omega_i \omega_i \tag{7.25}$$

$$f_i^{(\omega)} = -\omega_i - q_i \tag{7.26}$$

Therefore, Eq. (7.26) is a linear, globally asymptotically stable controller for the attitude stabilization and regulation of three-dimensional bodies. This controller is very attractive, and the following sections will attempt to determine if it can be extended to guarantee global asymptotic stability for general N-dimensional bodies or M-DOF systems.

D. Work/Energy-Rate Expression for N-Dimensional Dynamics

An important step in the developments of the previous section is the simplification of the derivative of the kinetic energy under Euler's equation. To generalize these results it needs to be determined if these terms follow similar simplifications under the general N-dimensional dynamics. In this section the work/energy-rate expression will be developed for the general N-dimensional dynamics, recalling that by using the Cayley form these dynamics can represent any M-DOF physical system. These equations of motion can be derived using Lagrange's equations for quasi velocities, which are also known as Poincaré's equations [35].

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \omega_k}\right) + \gamma_{ka}^{r} \omega_a \frac{\partial T}{\partial \omega_r} - A_{rk} \frac{\partial T}{\partial q_r} = f_k^{(\omega)} \tag{7.27}$$

Here, $T = T(\boldsymbol{q}, \boldsymbol{\omega})$ is the kinetic energy, and the components $f_k^{(\omega)}$ can either be considered as the moments applied to an N-dimensional rigid body or functions of the generalized forces, $\boldsymbol{f}$, applied to an M-DOF physical system: $f_k^{(\omega)} = A_{jk} f_j$. Therefore, these components $f_k^{(\omega)}$ are the generalized forces associated with the quasi velocities, and are here simply called the *quasi forces*. The three-index symbol γ_{ka}^r that appears in Eq. (7.27) represents the Hamel coefficients which are given by the following expression [15].

$$\gamma_{vr}^m = \frac{1}{2} \chi_{ik}^m \left(\chi_{ic}^v \chi_{ck}^r - \chi_{ck}^v \chi_{ic}^r \right) \tag{7.28}$$

The kinetic energy of a natural system is represented using the Cayley form as follows.

$$T = \frac{1}{2} \omega_i J_{ij}(\boldsymbol{q}) \omega_j \tag{7.29}$$

Here, J_{ij} are the elements of the symmetric mass matrix associated with the Cayley quasi velocities. For an N-dimensional rotating body, $\boldsymbol{J}$ can be selected without loss of generality to be diagonal and is independent of the generalized coordinates. For other systems mapped by the Cayley form, however, these properties do not need to be assumed. Additionally because the current interest is in feedback controllers, it is assumed that the system is not explicitly time dependent, and thus the focus is placed on natural systems.

To evaluate Lagrange's equations the following derivatives of T are taken.

$$\frac{\partial T}{\partial q_r} = \frac{1}{2} \omega_i \frac{\partial J_{ij}}{\partial q_r} \omega_j \tag{7.30}$$

$$\frac{\partial T}{\partial \omega_r} = J_{rj} \omega_j \tag{7.31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \omega_k} \right) = J_{kj} \dot{\omega}_j + \frac{\partial J_{kj}}{\partial q_s} \dot{q}_s \omega_j \tag{7.32}$$

These are substituted into the equations of motion.

$$J_{kj}\dot{\omega}_j + \frac{\partial J_{kj}}{\partial q_s}\dot{q}_s\omega_j + \gamma^r_{ka}\omega_a J_{rj}\omega_j - \frac{1}{2}A_{rk}\omega_i\frac{\partial J_{ij}}{\partial q_r}\omega_j = f^{(\omega)}_k \tag{7.33}$$

Similar to the previous section the time derivative of the kinetic energy needs to be found due to these dynamic equations.

$$\begin{aligned}
\dot{T} &= \frac{\partial T}{\partial \omega_r}\dot{\omega}_r + \frac{\partial T}{\partial q_r}\dot{q}_r = J_{rj}\omega_j\dot{\omega}_r + \frac{1}{2}\omega_i\frac{\partial J_{ij}}{\partial q_r}\omega_j\dot{q}_r \\
&= \left(-\frac{\partial J_{jk}}{\partial q_s}\dot{q}_s\omega_k - \gamma^r_{ja}\omega_a J_{rk}\omega_k + \frac{1}{2}A_{rj}\omega_i\frac{\partial J_{ik}}{\partial q_r}\omega_k + f^{(\omega)}_j\right)\omega_j + \frac{1}{2}\omega_i\frac{\partial J_{ij}}{\partial q_r}\omega_j\dot{q}_r \\
&= -\frac{\partial J_{jk}}{\partial q_s}\dot{q}_s\omega_k\omega_j - \gamma^r_{ja}\omega_a J_{rk}\omega_k\omega_j + \frac{1}{2}\omega_i\frac{\partial J_{ik}}{\partial q_r}\omega_k\dot{q}_r + f^{(\omega)}_j\omega_j + \frac{1}{2}\omega_i\frac{\partial J_{ij}}{\partial q_r}\omega_j\dot{q}_r \\
&= -\gamma^r_{ja}\omega_a J_{rk}\omega_k\omega_j + f^{(\omega)}_j\omega_j
\end{aligned} \tag{7.34}$$

This expression is further simplified by considering the definition of the Hamel coefficients in Eq. (7.28) and the properties of $\boldsymbol{\chi}$.

$$\begin{aligned}
\gamma^r_{ja}\omega_a J_{rk}\omega_k\omega_j &= \frac{1}{2}J_{rk}\chi^r_{de}\left(\chi^j_{dc}\chi^a_{ce} - \chi^j_{ce}\chi^a_{dc}\right)\omega_a\omega_k\omega_j \\
&= \frac{1}{2}J_{rk}\chi^r_{de}\left(\chi^j_{dc}\chi^a_{ce}\omega_a\omega_j - \chi^j_{ce}\chi^a_{dc}\omega_a\omega_j\right)\omega_k \\
&= -\frac{1}{2}J_{rk}\chi^r_{ed}\left(\chi^j_{dc}\chi^a_{ce}\omega_a\omega_j - \chi^j_{ce}\chi^a_{dc}\omega_a\omega_j\right)\omega_k \\
&= -\frac{1}{2}J_{rk}\chi^r_{de}\left(\chi^j_{ec}\chi^a_{cd}\omega_a\omega_j - \chi^j_{cd}\chi^a_{ec}\omega_a\omega_j\right)\omega_k \\
&= -\frac{1}{2}J_{rk}\chi^r_{de}\left(\chi^j_{ce}\chi^a_{dc}\omega_a\omega_j - \chi^j_{dc}\chi^a_{ce}\omega_a\omega_j\right)\omega_k \\
&= -\frac{1}{2}J_{rk}\chi^r_{de}\left(\chi^a_{ce}\chi^j_{dc}\omega_j\omega_a - \chi^a_{dc}\chi^j_{ce}\omega_j\omega_a\right)\omega_k = 0
\end{aligned} \tag{7.35}$$

Hence the Hamel coefficient term is nonworking. This leaves the identical work/energy-rate expression as the three-dimensional special case: $\dot{T} = \omega_j f^{(\omega)}_j$. For the three-dimensional case several feedback attitude controllers have been developed using a Lyapunov function consisting of the sum of the kinetic energy and various fictitious potential energies. The next section will demonstrate the behavior of some of these potential energies under the general N-dimensional Cayley kinematics.

E. Feedback Control for N-Dimensional Rotations

Recall from an earlier section that a key to developing the linear controller for $N = 3$ was the recognition that $\boldsymbol{q}$ is an eigenvector of $\boldsymbol{A}^T$. Also, for the three-dimensional special case it was noted that $\boldsymbol{q}$ is an eigenvector of both $\boldsymbol{A}^T$ and $\boldsymbol{A}$. To extend these developments to higher dimensions it will be determined if these properties are true in general. This can be approached in general using the definition of $\boldsymbol{A}$.

$$\left[\boldsymbol{A}^T\boldsymbol{q}\right]_m = A_{im}q_i = \frac{1}{2}\left(\delta_{im}q_i - \chi^i_{vp}\chi^m_{vl}Q_{lp}q_i - \frac{1}{2}\chi^i_{vp}\chi^m_{kl}Q_{vk}Q_{lp}q_i\right) \tag{7.36}$$

Using the properties of the Kronecker delta and $\boldsymbol{\chi}$ it can be seen that $A_{im}q_i = A_{mi}q_i$. This is obviously true for the first term above, and similar to the three-dimensional case, the second term is equal to zero as shown below. The third term is demonstrated through manipulation of the repeated indices as also shown below.

$$-\chi^i_{vp}\chi^m_{vl}Q_{lp}q_i = \chi^m_{vl}Q_{pl}Q_{vp} = -\chi^m_{lv}Q_{lp}Q_{pv} = -\chi^m_{vl}Q_{vp}Q_{pl} = 0 \tag{7.37}$$

$$-\frac{1}{2}\chi^i_{vp}\chi^m_{kl}Q_{vk}Q_{lp}q_i = -\frac{1}{2}\chi^i_{kl}\chi^m_{vp}Q_{kv}Q_{pl}q_i = -\frac{1}{2}\chi^i_{kl}\chi^m_{vp}Q_{vk}Q_{lp}q_i \tag{7.38}$$

Therefore, $\boldsymbol{q}$ is an eigenvector of $\boldsymbol{A}^T$ if and only if it is also an eigenvector of $\boldsymbol{A}$. For $\boldsymbol{q}$ to be an eigenvector of $\boldsymbol{A}$ the following relation must be true: $A_{mi}q_i = \lambda q_m$. From Eq. (7.36) it is seen that the term linear in q_i obviously satisfies this relation, and the quadratic term has been shown to equal zero. Clearly the cubic term is critical.

Similar to the physical description of $N = 3$, this eigenproblem can be studied by considering the special case for which the angular velocity is proportional to the ERPs: $\boldsymbol{\omega} = \alpha\boldsymbol{q}$. Assuming $\boldsymbol{q}$ is indeed an eigenvector of $\boldsymbol{A}$, then in this case the derivatives of these parameters will also be proportional to the angular velocity.

$$\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{\omega} = \alpha\boldsymbol{A}\boldsymbol{q} = \alpha\lambda\boldsymbol{q} = \lambda\boldsymbol{\omega} \tag{7.39}$$

Here, λ is the eigenvalue of $\boldsymbol{A}$ assumed to be associated with $\boldsymbol{q}$. This proportionality can also be expressed using the original matrix form of the kinematic equations.

$$\dot{\boldsymbol{Q}} = \frac{1}{2}\left(\boldsymbol{I}+\boldsymbol{Q}\right)\boldsymbol{\Omega}\left(\boldsymbol{I}-\boldsymbol{Q}\right) = \frac{\alpha}{2}\left(\boldsymbol{I}+\boldsymbol{Q}\right)\boldsymbol{Q}\left(\boldsymbol{I}-\boldsymbol{Q}\right) = \frac{\alpha}{2}\left(\boldsymbol{Q}-\boldsymbol{Q}\boldsymbol{Q}\boldsymbol{Q}\right) = \alpha\lambda\boldsymbol{Q} = \lambda\boldsymbol{\Omega} \tag{7.40}$$

Again, the cubic term is critical to the proposed proportionality, and the simplification to $\alpha\lambda\boldsymbol{Q}$ requires the assumed eigenvector property.

Any skew-symmetric matrix can be put into a canonical, block-diagonal form using a similarity transformation [9]. This is expressed as follows for the ERPs.

$$\boldsymbol{Q} = \boldsymbol{P}^T\boldsymbol{Q}'\boldsymbol{P} \tag{7.41}$$

Here, $\boldsymbol{P}$ is an $N \times N$ proper orthogonal matrix, and $\boldsymbol{Q}'$ is a block-diagonal, skew-symmetric matrix. Note, however, that this canonical decomposition is not unique. For one value of $\boldsymbol{Q}$ several values of $\boldsymbol{Q}'$ can be defined, which are related by interchanges and sign changes of the blocks, and infinitely many values of $\boldsymbol{P}$ can be chosen. Because of the proportionality between $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$, the canonical forms of these matrices are related. Both sides of Eq. (7.41) can be multiplied by the proportionality factor α.

$$\boldsymbol{\Omega} = \boldsymbol{P}^T\alpha\boldsymbol{Q}'\boldsymbol{P} \tag{7.42}$$

$$\boldsymbol{\Omega}' \equiv \alpha\boldsymbol{Q}' \tag{7.43}$$

Similar steps are performed for the ERP rates.

$$\dot{\boldsymbol{Q}} = \boldsymbol{P}^T\alpha\lambda\boldsymbol{Q}'\boldsymbol{P} \tag{7.44}$$

$$\left(\dot{\boldsymbol{Q}}\right)' \equiv \alpha\lambda\boldsymbol{Q}' \tag{7.45}$$

Therefore, under the current assumption and given special condition the canonical forms of all three matrices should be proportional. Note that Eq. (7.45) is a definition for the canonical form of the ERP rates. This is a different quantity than the derivative of the canonical form of the ERPs. In general $\left(\dot{\boldsymbol{Q}}\right)' \neq \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}'\right)$, however, the relationship between the two values and their equivalency in special cases will be discussed later.

The canonical forms for $\boldsymbol{Q}$, $\boldsymbol{\Omega}$, and $\dot{\boldsymbol{Q}}$ can be substituted into the Cayley-transform kinematic relationship to give the following.

$$\left(\dot{\boldsymbol{Q}}\right)' = \frac{1}{2}\left(\boldsymbol{I} + \boldsymbol{Q}'\right)\boldsymbol{\Omega}'\left(\boldsymbol{I} - \boldsymbol{Q}'\right) \tag{7.46}$$

The canonical forms obey the same kinematic relationship for this special case of proportionality. In this form, however, it can be seen that the proposed proportionality does not hold for higher dimensions. Consider for example the case $N = 5$. For this dimension the canonical values will have the following forms.

$$\left[(\dot{\boldsymbol{Q}})'\right] = \frac{1}{2}\begin{bmatrix} 1 & Q'_{12} & 0 & 0 & 0 \\ -Q'_{12} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & Q'_{34} & 0 \\ 0 & 0 & -Q'_{34} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & \Omega'_{12} & 0 & 0 & 0 \\ -\Omega'_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega'_{34} & 0 \\ 0 & 0 & -\Omega'_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -Q'_{12} & 0 & 0 & 0 \\ Q'_{12} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -Q'_{34} & 0 \\ 0 & 0 & Q'_{34} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & (1+Q'^2_{12})\Omega'_{12} & 0 & 0 & 0 \\ -(1+Q'^2_{12})\Omega'_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1+Q'^2_{34})\Omega'_{34} & 0 \\ 0 & 0 & -(1+Q'^2_{34})\Omega'_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.47}$$

Clearly $\left(\dot{\boldsymbol{Q}}\right)'$ is not proportional to $\boldsymbol{\Omega}'$ because the (1-2,1-2) and (3-4,3-4) blocks, which describe the two principal planes of the five-dimensional rotation, have different scaling factors. The case $N = 4$ is very similar to this example, and any higher dimension will follow a similar pattern only with even more principal planes. Only for the two or three-dimensional cases with one principal plane will the proportionality hold. Therefore, assuming that $\boldsymbol{A}\boldsymbol{q} = \lambda\boldsymbol{q}$ in general has led to a contradiction. Thus, for these higher dimensions $\boldsymbol{q}$ is not an eigenvector of $\boldsymbol{A}$ or $\boldsymbol{A}^T$.

Although the above demonstration is sufficient for the controller design considered in this section, it is interesting to take a small diversion to investigate the relationship between the two quantities $\left(\dot{\boldsymbol{Q}}\right)'$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}'\right)$. In general the canonical form of $\dot{\boldsymbol{Q}}$ will require a proper orthogonal matrix $\hat{\boldsymbol{P}}$ different from the canonical form for $\boldsymbol{Q}$.

$$\dot{\boldsymbol{Q}} = \hat{\boldsymbol{P}}^T\left(\dot{\boldsymbol{Q}}\right)'\hat{\boldsymbol{P}} \tag{7.48}$$

For comparison with this canonical form, the derivative of Eq. (7.41) is taken.

$$\dot{\boldsymbol{Q}} = \dot{\boldsymbol{P}}^T \boldsymbol{Q}' \boldsymbol{P} + \boldsymbol{P}^T \frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{Q}' \right) \boldsymbol{P} + \boldsymbol{P}^T \boldsymbol{Q}' \dot{\boldsymbol{P}} \tag{7.49}$$

The matrix $\boldsymbol{P}$ describes the transformation from the rotated coordinate frame to a principal coordinate frame [4], and its derivative can be described by a Poisson equation.

$$\dot{\boldsymbol{P}} = -\boldsymbol{\Psi} \boldsymbol{P} \tag{7.50}$$

Here, $\boldsymbol{\Psi}$ is the skew-symmetric, angular-velocity matrix of the principal frame relative to the rotated frame. Making this substitution into Eq. (7.49) gives the following.

$$\dot{\boldsymbol{Q}} = \boldsymbol{P}^T \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{Q}' \right) + \boldsymbol{\Psi} \boldsymbol{Q}' - \boldsymbol{Q}' \boldsymbol{\Psi} \right] \boldsymbol{P} \tag{7.51}$$

At this point Eq. (7.51) is simply a similarity decomposition of $\dot{\boldsymbol{Q}}$ and is not equivalent in general to the canonical form in Eq. (7.48). The special cases for which the two forms are equivalent can be studied, however, by analyzing the bracketed terms in Eq. (7.51). First, the term $\frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{Q}' \right)$ must be block diagonal and skew-symmetric because $\boldsymbol{Q}'$ is defined to be block diagonal and skew-symmetric for all times. The third term can be rewritten as follows.

$$\boldsymbol{Q}' \boldsymbol{\Psi} = \left(\boldsymbol{Q}' \right)^T \boldsymbol{\Psi}^T = \left(\boldsymbol{\Psi} \boldsymbol{Q}' \right)^T \tag{7.52}$$

Therefore the second and third terms combined are twice the skew-symmetric component of $\boldsymbol{\Psi} \boldsymbol{Q}'$. Expanding this product for the four-dimensional case illustrates the general form of this matrix.

$$[\boldsymbol{\Psi}\boldsymbol{Q}'] = \begin{bmatrix} 0 & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ -\Psi_{12} & 0 & \Psi_{23} & \Psi_{24} \\ -\Psi_{13} & -\Psi_{23} & 0 & \Psi_{34} \\ -\Psi_{14} & -\Psi_{24} & -\Psi_{34} & 0 \end{bmatrix} \begin{bmatrix} 0 & Q'_{12} & 0 & 0 \\ Q'_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q'_{34} \\ 0 & 0 & Q'_{34} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -\Psi_{12}Q'_{12} & 0 & -\Psi_{14}Q'_{34} & \Psi_{13}Q'_{34} \\ 0 & -\Psi_{12}Q'_{12} & -\Psi_{24}Q'_{34} & \Psi_{23}Q'_{34} \\ \Psi_{23}Q'_{12} & -\Psi_{13}Q'_{12} & -\Psi_{34}Q'_{34} & 0 \\ \Psi_{24}Q'_{12} & -\Psi_{14}Q'_{12} & 0 & -\Psi_{34}Q'_{34} \end{bmatrix} \tag{7.53}$$

This implies the following form for the sum of the second and third terms.

$$[\boldsymbol{\Psi}\boldsymbol{Q}' - \boldsymbol{Q}'\boldsymbol{\Psi}] =$$
$$\begin{bmatrix} 0 & 0 & -\Psi_{14}Q'_{34}-\Psi_{23}Q'_{12} & \Psi_{13}Q'_{34}-\Psi_{24}Q'_{12} \\ 0 & 0 & -\Psi_{24}Q'_{34}+\Psi_{13}Q'_{12} & \Psi_{23}Q'_{34}+\Psi_{14}Q'_{12} \\ \Psi_{14}Q'_{34}+\Psi_{23}Q'_{12} & \Psi_{24}Q'_{34}-\Psi_{24}Q'_{12} & 0 & 0 \\ -\Psi_{13}Q'_{34}+\Psi_{13}Q'_{12} & -\Psi_{23}Q'_{34}-\Psi_{14}Q'_{12} & 0 & 0 \end{bmatrix} \tag{7.54}$$

This result is skew-symmetric but is clearly not block diagonal, and will have a similar form for any dimension. For the special case, however, that $\boldsymbol{\Psi}\boldsymbol{Q}' - \boldsymbol{Q}'\boldsymbol{\Psi} = \boldsymbol{0}$, Eq. (7.51) can be simplified as shown below.

$$\dot{\boldsymbol{Q}} = \boldsymbol{P}^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}'\right) \boldsymbol{P} \tag{7.55}$$

Because $\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}'\right)$ is block diagonal, Eq. (7.55) now represents a canonical representation of $\dot{\boldsymbol{Q}}$. Therefore, $\boldsymbol{Q}'$ and $\boldsymbol{P}$ can be chosen as follows.

$$\boldsymbol{P} = \hat{\boldsymbol{P}} \quad ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}'\right) = \left(\dot{\boldsymbol{Q}}\right)' \tag{7.56}$$

Conversely, for motions prescribed such that $\boldsymbol{Q}$ and $\dot{\boldsymbol{Q}}$ are brought into canonical form by the same transformation matrix $\boldsymbol{P} = \hat{\boldsymbol{P}}$, the following must be true.

$$\boldsymbol{\Psi Q}' - \boldsymbol{Q}'\boldsymbol{\Psi} = \boldsymbol{0} \quad ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}'\right) = \left(\dot{\boldsymbol{Q}}\right)' \tag{7.57}$$

Therefore, for these special motions, which can be described by either specifying $\boldsymbol{\Psi}$ and $\boldsymbol{Q}'$ or $\boldsymbol{P}$ and $\hat{\boldsymbol{P}}$, differentiation and the canonical transformation of the ERPs are commutative.

Returning to the topic of controller design, because the eigenvalue property of $\boldsymbol{A}^T$ does not extend to N-dimensions, the Lyapunov function used by Tsiotras can not be utilized to develop a linear feedback controller for N-dimensional rigid bodies. This function, $V = \frac{1}{2}\omega_i J_{ij}\omega_j + \ln(1 + q_i q_i)$, and others can be used, however, to develop globally asymptotically stable nonlinear feedback controllers. Applying the work/energy-rate expression from the previous section gives the following for the derivative of the Lyapunov function.

$$\begin{aligned}
\dot{V} = \omega_i f_i^{(\omega)} + \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(1 + q_i q_i\right)}{1 + q_i q_i} &= \frac{1}{2}\Omega_{jk}F_{jk}^{(\omega)} + \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(1 + \frac{1}{2}Q_{jk}Q_{jk}\right)}{1 + q_i q_i} \\
= \frac{1}{2}\Omega_{jk}F_{jk}^{(\omega)} + \frac{Q_{jk}\dot{Q}_{jk}}{1 + q_i q_i} &= \frac{1}{2}\Omega_{jk}F_{jk}^{(\omega)} + \frac{Q_{jk}\left(\delta_{jr} + Q_{jr}\right)\Omega_{rs}\left(\delta_{sk} - Q_{sk}\right)}{2(1 + q_i q_i)} \\
= \frac{1}{2}\Omega_{jk}F_{jk}^{(\omega)} &+ \frac{\left(Q_{rk} + Q_{jk}Q_{jr}\right)\Omega_{rs}\left(\delta_{sk} - Q_{sk}\right)}{2(1 + q_i q_i)} \\
= \frac{1}{2}\Omega_{jk}F_{jk}^{(\omega)} &+ \frac{\Omega_{rs}\left(Q_{rs} - Q_{rk}Q_{sk} + Q_{js}Q_{jr} - Q_{jk}Q_{jr}Q_{sk}\right)}{2(1 + q_i q_i)} \\
= \frac{1}{2}\Omega_{jk}F_{jk}^{(\omega)} &+ \frac{\Omega_{rs}\left(Q_{rs} - Q_{rj}Q_{jk}Q_{ks}\right)}{2(1 + q_i q_i)}
\end{aligned} \tag{7.58}$$

The following nonlinear feedback controller will set the Lyapunov function derivative to be negative semi-definite.

$$F_{jk}^{(\omega)} = -\Omega_{jk} - \frac{\left(Q_{jk} - Q_{jm}Q_{mn}Q_{nk}\right)}{1 + q_i q_i} \tag{7.59}$$

Here, the first term of the control law is referred to as a stabilization term, and the second term is referred to as a regulation term. For three-dimensions this expression can be simplified by extracting a factor of $\boldsymbol{Q}$ from the numerator of the regulation term leading to cancelation of the denominator. This cancelation, however, uses the property that $\boldsymbol{q}$ is an eigenvector of $\boldsymbol{A}^T$, which does not hold in general as has just been shown. For general N-dimensions no further simplifications can be made.

Another more typical Lyapunov function that can be used is the sum of kinetic energy and a quadratic product of the coordinates.

$$V = \frac{1}{2}\omega_i J_{ij}\omega_j + q_i q_i = \frac{1}{2}\omega_i J_{ij}\omega_j + \frac{1}{2}Q_{jk}Q_{jk} \tag{7.60}$$

The derivative of this function is shown below.

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\Omega_{lm}F^{(\omega)}_{lm} + Q_{jk}\dot{Q}_{jk} = \frac{1}{2}\Omega_{lm}F^{(\omega)}_{lm} + \frac{1}{2}Q_{jk}\left(\delta_{jl} + Q_{jl}\right)\Omega_{lm}\left(\delta_{mk} - Q_{mk}\right) \\
&= \frac{1}{2}\Omega_{lm}\left(F^{(\omega)}_{lm} + \left(Q_{lk} + Q_{jk}Q_{jl}\right)\left(\delta_{mk} - Q_{mk}\right)\right) \\
&= \frac{1}{2}\Omega_{lm}\left(F^{(\omega)}_{lm} + Q_{lm} - Q_{lk}Q_{mk} + Q_{jm}Q_{jl} - Q_{jk}Q_{jl}Q_{mk}\right) \\
&= \frac{1}{2}\Omega_{lm}\left(F^{(\omega)}_{lm} + Q_{lm} - Q_{jk}Q_{jl}Q_{mk}\right)
\end{aligned} \tag{7.61}$$

For global asymptotic stability the control is chosen as follows.

$$F^{(\omega)}_{lm} = -\Omega_{lm} - Q_{lm} + Q_{lj}Q_{jk}Q_{km} \tag{7.62}$$

This can also be expressed in matrix notation.

$$\boldsymbol{F}^{(\omega)} = -\boldsymbol{\Omega} - \boldsymbol{Q} + \boldsymbol{QQQ} = -\boldsymbol{\Omega} - \boldsymbol{Q}\left(\boldsymbol{I} - \boldsymbol{QQ}\right) = -\boldsymbol{\Omega} - \boldsymbol{Q}\left(\boldsymbol{I} + \boldsymbol{Q}^T\boldsymbol{Q}\right) \tag{7.63}$$

Again, this controller contains both stabilization and regulation terms. In two following sections these terms will be further analyzed. The regulation terms for both of the controllers developed in this section will be shown to optimize certain cost functions for control of the kinematic equations. Additionally, numerical simulation will be used to compare the stabilization behavior of Cayley quasi-velocity feedback against generalized-velocity feedback and another choice of quasi-velocity feedback. First, however, a second set of quasi velocities is developed that allows globally asymptotically stable linear feedback.

F. Quasi Velocities for Linear Feedback

In Section C several special properties of the angular-velocity/Rodrigues-parameter kinematics were discussed that led to a proof for global asymptotic stability of linear feedback. Those kinematics are described by the transformation matrix $\boldsymbol{A}$, whose elements are repeated here for convenience.

$$A_{im} = \frac{1}{2}\left(\delta_{im} + Q_{im} + q_i q_m\right) \tag{7.64}$$

Sections D and E presented an attempt to extend that proof to M-DOF physical systems by recognizing that the three-dimensional motion variables are simply a special case of the Cayley form. That approach used a general form of Eq. (7.64) to describe N-dimensional rotations. It was found, however, that the complexity of the N-dimensional kinematics produced difficulty in extending the proof. This section presents an alternative approach to extending that proof by defining a new set of quasi velocities. Instead of extending the rotational kinematics of Eq. (7.64) to higher dimensions like the Cayley quasi velocities, the new quasi velocities are defined by simply modifying the functional form of Eq. (7.64).

To do this a new $M \times M$ transformation matrix $\bar{\boldsymbol{A}}$ will be defined. Two difficulties exist, however, in directly extending the functional form of Eq. (7.64) to any M-DOF system. Both are directly related to the second term, Q_{im}. First $\boldsymbol{Q}$ can only be formed for certain values of M, and second $\boldsymbol{Q}$ is $N \times N$ in general, not $M \times M$. This suggests the following mapping to define a new set of quasi velocities $\boldsymbol{u}$.

$$\dot{\boldsymbol{q}} = \bar{\boldsymbol{A}}\boldsymbol{u} \qquad ; \qquad \boldsymbol{u} = \bar{\boldsymbol{B}}\dot{\boldsymbol{q}} \tag{7.65}$$

$$\bar{A}_{im} = \frac{1}{2}\left(\delta_{im} + q_i q_m\right) \tag{7.66}$$

This set of quasi velocities can be applied to a general system with any number of generalized coordinates. For Eqs. (7.65) and (7.66) to constitute a valid quasi-velocity definition, however, the inverse of $\bar{\boldsymbol{A}}$ (i.e., $\bar{\boldsymbol{B}}$) must exist. In vector/matrix notation $2\bar{\boldsymbol{A}}$ is equal to $\boldsymbol{I} + \boldsymbol{q}\boldsymbol{q}^T$, and the eigenvalues of $2\bar{\boldsymbol{A}}$ are found by adding one to the eigenvalues of $\boldsymbol{q}\boldsymbol{q}^T$. To show that $\bar{\boldsymbol{A}}$ is nonsingular it simply needs to be demonstrated that negative one can not be an eigenvalue of $\boldsymbol{q}\boldsymbol{q}^T$. If it is assumed that $\boldsymbol{q}\boldsymbol{q}^T$ does have an eigenvalue of negative one and a corresponding eigenvector $\boldsymbol{e}$, then the following eigenproblem can be considered.

$$\boldsymbol{q}\boldsymbol{q}^T\boldsymbol{e} = \left(\boldsymbol{q}^T\boldsymbol{e}\right)\boldsymbol{q} = -\boldsymbol{e} \tag{7.67}$$

This implies that $\boldsymbol{e}$ must be proportional to $\boldsymbol{q}$.

$$\boldsymbol{e} = \alpha\boldsymbol{q} \tag{7.68}$$

$$\alpha\left(\boldsymbol{q}^T\boldsymbol{q}\right)\boldsymbol{q} = -\alpha\boldsymbol{q} \tag{7.69}$$

Thus assuming that $\boldsymbol{q}\boldsymbol{q}^T$ has an eigenvalue of negative one has led to the contradiction $\boldsymbol{q}^T\boldsymbol{q} = -1$. Therefore, $\bar{\boldsymbol{A}}$ is nonsingular, and $\boldsymbol{u}$ is a valid set of quasi velocities.

Before continuing with the new quasi velocities, it is worth revisiting the work/energy-rate expression for general quasi velocities. For natural systems the work/energy-rate expression with respect to the generalized velocities can be expressed as $\dot{T} = \dot{\boldsymbol{q}}^T\boldsymbol{f}$. Considering a set of quasi velocities as defined by Eq. (7.65), Poincaré's equations define the quasi forces.

$$\boldsymbol{f}^{(u)} = \bar{\boldsymbol{A}}^T \boldsymbol{f} \qquad ; \qquad \boldsymbol{f} = \bar{\boldsymbol{B}}^T \boldsymbol{f}^{(u)} \tag{7.70}$$

Equations (7.65) and (7.70) can be used to write the work/energy-rate expression for any set of quasi velocities, once and for all.

$$\dot{T} = \dot{\boldsymbol{q}}^T \boldsymbol{f} = \boldsymbol{u}^T \bar{\boldsymbol{A}}^T \bar{\boldsymbol{B}}^T \boldsymbol{f}^{(u)} = \boldsymbol{u}^T \boldsymbol{f}^{(u)} \tag{7.71}$$

Of course this expression is consistent with the work/energy-rate expression found for N-dimensional rotations in Section D.

Returning to the particular set of quasi velocities defined by Eq. (7.66), linear feedback of $\boldsymbol{q}$ and $\boldsymbol{u}$ can be proven to be globally asymptotically stable. Consider again the Lyapunov function defined by Tsiotras for the sum of the kinetic energy and a fictitious potential energy, where $\boldsymbol{J}(\boldsymbol{q})$ is the mass matrix associated with $\boldsymbol{u}$.

$$V = \frac{1}{2} u_i J_{ij} u_j + \ln (1 + q_i q_i) \tag{7.72}$$

The time derivative is given by the following.

$$\dot{V} = u_i f_i^{(u)} + \frac{2 q_i \dot{q}_i}{1 + q_j q_j} = u_i f_i^{(u)} + \frac{2 q_i \bar{A}_{im} u_m}{1 + q_j q_j} \tag{7.73}$$

Due to the choice of $\bar{\boldsymbol{A}}$, however, $\boldsymbol{q}$ is now an eigenvector of $\bar{\boldsymbol{A}}^T$ for any value of M.

$$q_i \bar{A}_{im} = \frac{1}{2} (\delta_{im} q_i + q_i q_m q_i) = \frac{1}{2} (q_m + q_i q_i q_m) = \frac{1}{2} (1 + q_i q_i) q_m \equiv \lambda q_m \tag{7.74}$$

Therefore, the time derivative of the Lyapunov function can be simplified.

$$\dot{V} = u_i f_i^{(u)} + q_i u_i \tag{7.75}$$

If the control elements, $f_i^{(u)}$, are chosen to make Eq. (7.24) negative definite, then the system will be globally asymptotically stable. This can be done in the following fashion.

$$\dot{V} = u_i f_i^{(u)} + q_i u_i \equiv -u_i u_i \tag{7.76}$$

$$f_i^{(u)} = -u_i - q_i \tag{7.77}$$

Equation (7.77) demonstrates that linear feedback of the generalized coordinates and the new quasi velocities, defined by Eqs. (7.65) and (7.66), will provide global asymptotic stability for any system.

G. Optimality Results for Regulation Terms

In the previous sections the control of a complete system, both dynamic and kinematic equations, using quasi velocities was discussed. In this section the control of only the kinematic equations, returning to the Cayley quasi velocities, will be considered. This approach treats the Cayley quasi velocities as control variables used to control the generalized coordinates. The cost functions optimized by feedback laws with the form of the regulation terms from the previous section will be developed.

First for notational convenience the vector $\bar{\boldsymbol{q}}$ is defined as the generating vector of the skew-symmetric, cubic product $\boldsymbol{QQQ}$.

$$\bar{q}_a = \frac{1}{2}\chi^a_{ml} Q_{mk} Q_{kj} Q_{jl} \tag{7.78}$$

The minimization of the following cost function subject to the kinematic equations will be considered.

$$J = \frac{1}{2}\int_0^\infty \left[k^2 \left(\boldsymbol{q} - \bar{\boldsymbol{q}}\right)^T \left(\boldsymbol{q} - \bar{\boldsymbol{q}}\right) + \boldsymbol{\omega}^T \boldsymbol{\omega}\right] \mathrm{d}t \tag{7.79}$$

$$\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{\omega} \tag{7.80}$$

Introducing the costates $\boldsymbol{\lambda}$, the Hamiltonian is written below.

$$H(\boldsymbol{q}, \boldsymbol{\omega}, \boldsymbol{\lambda}) = \frac{1}{2}k^2 (\boldsymbol{q} - \bar{\boldsymbol{q}})^T (\boldsymbol{q} - \bar{\boldsymbol{q}}) + \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{\omega} + \boldsymbol{\lambda}^T \boldsymbol{A}\boldsymbol{\omega} \tag{7.81}$$

Next, the following three conditions on the optimal trajectory are imposed.

$$H = 0 \qquad ; \qquad \frac{\partial H}{\partial \boldsymbol{\omega}} = \boldsymbol{\omega} + \boldsymbol{A}^T \frac{\partial V}{\partial \boldsymbol{q}} = 0 \qquad ; \qquad \boldsymbol{\lambda} = \frac{\partial V}{\partial \boldsymbol{q}} \tag{7.82}$$

Here, V is the unknown optimal-cost function. These conditions are substituted into Eq. (7.81) to develop the Hamilton-Jacobi-Bellman equation for this problem.

$$k^2 (\boldsymbol{q} - \bar{\boldsymbol{q}})^T (\boldsymbol{q} - \bar{\boldsymbol{q}}) - \left(\frac{\partial V}{\partial \boldsymbol{q}}\right)^T \boldsymbol{A}\boldsymbol{A}^T \frac{\partial V}{\partial \boldsymbol{q}} = 0 \tag{7.83}$$

Now, the candidate optimal-cost function $V = k\boldsymbol{q}^T\boldsymbol{q}$ is considered. The optimal control implied by this candidate solution is found using Eq. (7.82).

$$\boldsymbol{\omega} = -\boldsymbol{A}^T \frac{\partial V}{\partial \boldsymbol{q}} = -2k\boldsymbol{A}^T\boldsymbol{q} \tag{7.84}$$

This is evaluated using Eqs. (7.36) and (7.37).

$$\begin{aligned} A_{im}q_i &= \frac{1}{2}\left(\delta_{im}q_i - \frac{1}{2}\chi^i_{vp}\chi^m_{kl}Q_{vk}Q_{lp}q_i\right) = \frac{1}{2}\left(q_m - \frac{1}{2}\chi^m_{kl}Q_{vk}Q_{lp}Q_{vp}\right) \\ &= \frac{1}{2}\left(q_m - \frac{1}{2}\chi^m_{kl}Q_{kv}Q_{vp}Q_{pl}\right) = \frac{1}{2}(q_m - \bar{q}_m) \end{aligned} \tag{7.85}$$

Equations (7.84) and (7.85) can be substituted into Eq. (7.83) to show that the candidate optimal-cost function is indeed a solution of the Hamilton-Jacobi-Bellman equation. The final form of the optimal control is shown below.

$$\boldsymbol{\omega} = -k(\boldsymbol{q} - \bar{\boldsymbol{q}}) \tag{7.86}$$

Notice that this feedback controller is equal to the regulation term of the second controller from Section E, and the optimal-cost function is equal to the regulation term of the Lyapunov function corresponding to that controller.

A second optimal-control problem of the kinematic equations can be considered using the following cost function.

$$J = \frac{1}{2}\int_0^\infty \left[k^2 \frac{(\boldsymbol{q}-\bar{\boldsymbol{q}})^T(\boldsymbol{q}-\bar{\boldsymbol{q}})}{(1+\boldsymbol{q}^T\boldsymbol{q})^2} + \boldsymbol{\omega}^T\boldsymbol{\omega} \right] \mathrm{d}t \tag{7.87}$$

The corresponding Hamiltonian is shown below.

$$H(\boldsymbol{q},\boldsymbol{\omega},\boldsymbol{\lambda}) = \frac{1}{2}k^2 \frac{(\boldsymbol{q}-\bar{\boldsymbol{q}})^T(\boldsymbol{q}-\bar{\boldsymbol{q}})}{(1+\boldsymbol{q}^T\boldsymbol{q})^2} + \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{\omega} + \boldsymbol{\lambda}^T\boldsymbol{A}\boldsymbol{\omega} \tag{7.88}$$

For this problem the optimality conditions are identical to Eq. (7.82) and can be expanded to give the following Hamilton-Jacobi-Bellman equation.

$$k^2 \frac{(\boldsymbol{q}-\bar{\boldsymbol{q}})^T(\boldsymbol{q}-\bar{\boldsymbol{q}})}{(1+\boldsymbol{q}^T\boldsymbol{q})^2} - \left(\frac{\partial V}{\partial \boldsymbol{q}}\right)^T \boldsymbol{A}\boldsymbol{A}^T\frac{\partial V}{\partial \boldsymbol{q}} = 0 \tag{7.89}$$

For this problem the candidate optimal-cost function $V = k\ln\left(1+\boldsymbol{q}^T\boldsymbol{q}\right)$ is considered. To demonstrate that this function satisfies Eq. (7.89), first, the gradient of V is taken.

$$\frac{\partial V}{\partial \boldsymbol{q}} = \frac{2k\boldsymbol{q}}{1+\boldsymbol{q}^T\boldsymbol{q}} \tag{7.90}$$

$$\boldsymbol{A}^T\frac{\partial V}{\partial \boldsymbol{q}} = \frac{2k\boldsymbol{A}^T\boldsymbol{q}}{1+\boldsymbol{q}^T\boldsymbol{q}} = \frac{k(\boldsymbol{q}-\bar{\boldsymbol{q}})}{1+\boldsymbol{q}^T\boldsymbol{q}} \tag{7.91}$$

Clearly the candidate optimal-cost function is a solution of the Hamilton-Jacobi-Bellman equation. The optimal control implied by the second of Eqs. (7.82) is shown below.

$$\boldsymbol{\omega} = -\frac{k(\boldsymbol{q}-\bar{\boldsymbol{q}})}{1+\boldsymbol{q}^T\boldsymbol{q}} \tag{7.92}$$

Again, notice that this control is equal to the regulation term of the first controller from Section E, and the optimal-cost function is the regulation term of the associated Lyapunov function.

H. Stabilization Using Velocity Feedback

Whereas the previous section analyzed the regulation terms of the Cayley quasi velocity feedback controllers and Lyapunov functions, this section focuses on the stabilization terms. In this section the kinetic energy alone, shown in Eq. (7.29), is used as a Lyapunov function. In a previous section the derivative of the kinetic energy was found to be the work done by the quasi forces associated with the Cayley quasi velocities.

$$\dot{T} = \boldsymbol{\omega}^T \boldsymbol{f}^{(\omega)} \tag{7.93}$$

The generalized forces can be selected to make this derivative negative semi-definite.

$$\boldsymbol{f}^{(\omega)} = -\boldsymbol{D}\boldsymbol{\omega} \tag{7.94}$$

Here, $\boldsymbol{D}$ is a symmetric, positive-definite matrix. For simplicity, in previous sections $\boldsymbol{D}$ was chosen to be the identity matrix.

$$\boldsymbol{f}^{(\omega)} = -\boldsymbol{\omega} \tag{7.95}$$

$$\boldsymbol{f} = -\boldsymbol{B}^T\boldsymbol{\omega} = -\boldsymbol{B}^T\boldsymbol{B}\dot{\boldsymbol{q}} \tag{7.96}$$

Another choice that will be investigated in this section is $\boldsymbol{D} = P^{\omega}\left(\boldsymbol{A}\boldsymbol{A}\right)^T\left(\boldsymbol{A}\boldsymbol{A}\right)$, which is positive definite because $\boldsymbol{A}$ is full rank, and where P^{ω} is a control gain. This choice leads to the following controller.

$$\boldsymbol{f}^{(\omega)} = -P^{\omega}\left(\boldsymbol{A}\boldsymbol{A}\right)^T\left(\boldsymbol{A}\boldsymbol{A}\right)\boldsymbol{\omega} \tag{7.97}$$

$$\boldsymbol{f} = -P^{\omega}\boldsymbol{B}^T\left(\boldsymbol{A}\boldsymbol{A}\right)^T\left(\boldsymbol{A}\boldsymbol{A}\right)\boldsymbol{\omega} = -P^{\omega}\boldsymbol{A}^T\boldsymbol{A}\dot{\boldsymbol{q}} \tag{7.98}$$

The controller in Eq. (7.98) is referred to as the $\boldsymbol{\omega}$ control law. The performance of this controller will be compared to two other feedback controllers: one using the generalized velocities and one designed by Schaub and Junkins [47] using a different set of quasi velocities. The controller designed by Schaub and Junkins is summarized below.

Using a spectral decomposition with square-root factorization of the eigenvalue matrix, the system mass matrix can be written as the following.

$$\boldsymbol{M} = \boldsymbol{C}^T\boldsymbol{S}^T\boldsymbol{S}\boldsymbol{C} \tag{7.99}$$

Here, $\boldsymbol{C}$ is proper orthogonal, and $\boldsymbol{S}$ is diagonal. The eigenfactor quasi velocities are defined as $\boldsymbol{\eta} = \boldsymbol{S}\boldsymbol{C}\dot{\boldsymbol{q}}$ and can be used to rewrite the kinetic energy.

$$T = \frac{1}{2}\boldsymbol{\eta}^T\boldsymbol{\eta} \tag{7.100}$$

Through the development of the dynamic equations in terms of $\boldsymbol{\eta}$, Schaub and Junkins showed that the derivative of the kinetic energy is equal to the work done by the quasi forces associated with the eigenfactor quasi velocities.

$$\dot{T} = \boldsymbol{\eta}^T\boldsymbol{f}^{(\eta)} \tag{7.101}$$

From this a globally asymptotically stable control law is chosen to be the following.

$$\boldsymbol{f}^{(\eta)} = -P^{\eta}\boldsymbol{\eta} \tag{7.102}$$

This corresponds to the following controller for the generalized forces in terms of the generalized velocities, and is referred to as the $\boldsymbol{\eta}$ control law.

$$\boldsymbol{f} = -P^{\eta}\boldsymbol{M}\dot{\boldsymbol{q}} \tag{7.103}$$

In addition to the eigenfactor and Cayley quasi-velocity controllers, a conventional controller using the generalized velocities was implemented. This is developed using the kinetic energy written as $T = \frac{1}{2}\dot{\boldsymbol{q}}\boldsymbol{M}\dot{\boldsymbol{q}}$, and results in the following $\dot{\boldsymbol{q}}$ control law.

$$\boldsymbol{f} = -P^{\dot{q}}\dot{\boldsymbol{q}} \tag{7.104}$$

Again following Schaub and Junkins [47], the controllers in Eqs. (7.98), (7.103), and (7.104) were applied to a planar three-link manipulator system. The generalized coordinates of the system are $[\boldsymbol{q}] = [\ \theta_1 \quad \theta_2 \quad \theta_3\ ]^T$ and are the absolute angles of each joint. The generalized forces are related to the motor torques acting at each joint. The system mass matrix was given by Schaub and Junkins.

$$[\boldsymbol{M}] = \begin{bmatrix} (m_1+m_2+m_3)\,l_1^2 & (m_2+m_3)\,l_1 l_2 \cos(\theta_2-\theta_1) & m_3 l_1 l_3 \cos(\theta_3-\theta_1) \\ (m_2+m_3)\,l_1 l_2 \cos(\theta_2-\theta_1) & (m_2+m_3)\,l_2^2 & m_3 l_2 l_3 \cos(\theta_3-\theta_2) \\ m_3 l_1 l_3 \cos(\theta_3-\theta_1) & m_3 l_2 l_3 \cos(\theta_3-\theta_2) & m_3 l_3^2 \end{bmatrix} \tag{7.105}$$

Here, m_1, m_2, and m_3 are the masses of point masses located at the tip of each link, and l_1, l_2, and l_3 are the lengths of each link.

Simulations were performed using initial states of $[\boldsymbol{q}(0)] = [\ 93 \quad -110 \quad -73\ ]^T$ degrees. The initial velocities were chosen to be $[\dot{\boldsymbol{q}}(0)] = [\ -90 \quad 30 \quad 0\ ]^T$ degrees/second. The values m_1, m_2, m_3, l_1, l_2, and l_3 were set to one. The control gains were chosen for each controller such that the absolute control effort encountered was equal. The values $P^{\dot{q}} = 1.1$, $P^{\eta} = 0.59$, and $P^{\omega} = 0.38$ were chosen. The numerical integration was performed using a fourth and fifth order Runge-Kutta method and a simulation duration of 15 seconds.

The results from the simulation are shown in Figs. 14 and 15. The plot of the control effort, the magnitude of $\boldsymbol{f}$, on a logarithmic scale shows that each of the control laws goes through a peak in the first few seconds, but then begin to drop off

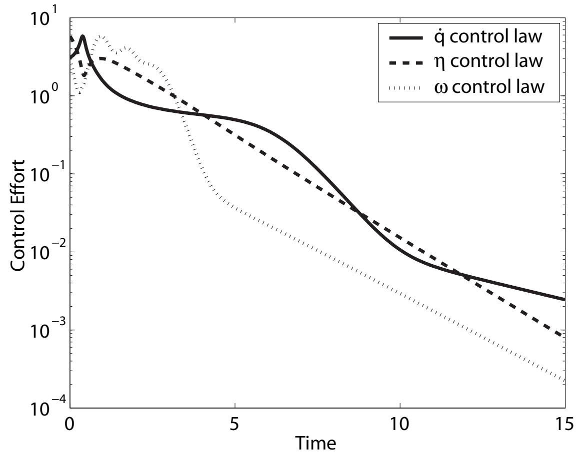

Fig. 14. Control vector magnitude time history.

quickly. The maximum control effort encountered for each control law was slightly greater than 5.8. The kinetic energy is a measure of the error motion of the system, and Fig. 15 shows that all three control laws quickly converge towards rest. The $\boldsymbol{\eta}$ control law quickly converges to a linear trend on the logarithmic scale indicating exponential convergence. The behavior of the $\boldsymbol{\omega}$ control law is not as smooth, however. Approximately two to three seconds into the simulation the control effort of the $\boldsymbol{\omega}$ control law has not dropped off as much as the other control laws, and the kinetic energy decays dramatically. After this the kinetic energy for the $\boldsymbol{\omega}$ control law is lower than both of the other controllers for the remainder of the simulation.

The lower kinetic energy with the $\boldsymbol{\omega}$ control law indicates that for this particular example stabilization using the Cayley quasi velocities outperforms the eigenfactor quasi velocities and the generalized velocities. These results, however, were highly sensitive to the maximum control effort which was selected. The $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$ control laws share a similarity in that both add a state-dependent influence matrix to the velocity feedback. The addition of this state information to the stabilization feedback is probably related to the superior performance of these two control laws compared

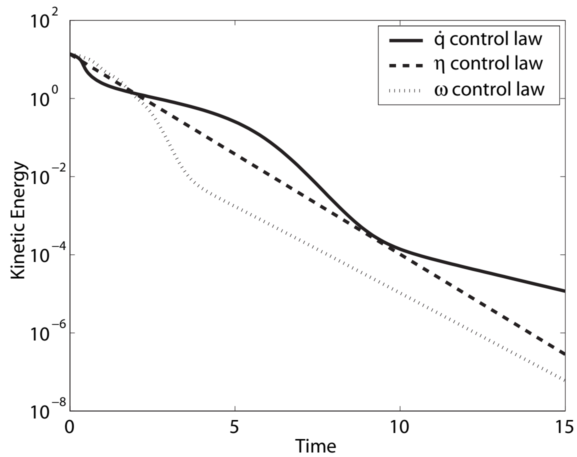

Fig. 15. Kinetic energy time history.

to the generalized-velocity feedback. For the $\boldsymbol{\eta}$ control law this influence matrix is the mass matrix, $\boldsymbol{M}$, and is dependent on knowledge of system parameters. The $\boldsymbol{\omega}$ control law, however, uses $\boldsymbol{A}^T\boldsymbol{A}$ which is only dependent on kinematic definitions and is not susceptible to system uncertainty.

I. Conclusion

This chapter has demonstrated a novel approach for the design of feedback controllers for natural mechanical systems. This approach is enabled by the Cayley form, which allows systems to be represented as N-dimensional rigid bodies. Several examples of feedback controllers were developed by extending spacecraft attitude controllers to N-dimensional rotations. It was found, however, that the complications of N-dimensional kinematics (i.e., multiple principal planes) clouded, or perhaps destroyed, some of the elegance of the three-dimensional results.

The controllers developed in this chapter were further analyzed for optimality and performance. First, the cost functions optimized by the regulation terms were developed. Second, numerical simulation was used to compare the stabilization performance of Cayley quasi-velocity feedback, eigenfactor quasi-velocity feedback, and generalized-velocity feedback. Results were presented for one example showing superior performance for Cayley quasi-velocity feedback. Of course, this does not demonstrate superiority in any global sense (nor is it believed to exist). Such global demonstrations of performance are a very difficult problem in nonlinear control.

The results that are presented, however, do provide an example of the significance of control coupling. A disadvantage of the Cayley form from a dynamics perspective is that it produces coupling in the equations of motion between coordinates that might otherwise be uncoupled. From a control perspective, however, having a controller that reflects the coupling that already exists in the dynamics can be advantageous. Both the Cayley and eigenfactor quasi-velocity feedbacks represent a coupling between each control and every velocity. The generalized-velocity feedback, however, matches the controls to the velocities in a one-to-one fashion, out of simplicity, that does not reflect the dynamics. Whereas this simplicity is attractive in itself, it does not guarantee superior performance.

CHAPTER VIII

NONLINEARITY INDEX OF THE CAYLEY FORM

A. Introduction

The previous two chapters presented applications of the Cayley form for developing representations of system dynamics and for designing feedback controllers. In both of these chapters it was significant that the Cayley form produces coupled system representations. In Chapter VI it was noted that the resulting equations of motion from the Cayley form can be more complicated than alternative methods that result in decoupled equations of motion, and in Chapter VII numerical-simulation results indicated that designing controllers based on coupled representations can in some cases result in superior performance. These issues motivate a desire to analyze the Cayley form to quantitatively measure these properties related to complexity and coupling.

The Cayley form, of course, is just one example of the generally infinite possibilities for representing dynamic systems. Due to the broad variety of system representations, the idea of comparing different representations of a physical system (or indeed, representations of different physical systems) is not a new one. Methods have been developed to analyze and compare system representations. One of these is the nonlinearity index developed by Junkins [48, 49]. In this chapter the nonlinearity index of the Cayley form and two alternative representations will be computed for a sample problem. First, however, the definition of the nonlinearity index is reviewed.

B. Nonlinearity Index

As mentioned, generally infinite possibilities exist for coordinate choices to represent any given physical system. Much of the history of analytical mechanics has spawned from the development of new coordinate choices. Several issues effect the choice of a particular coordinate system. A typical approach defines one set of *position-level coordinates* to describe the configuration of the system and a second set of *velocity-level coordinates* to describe the evolution of the system. One issue that effects the choice of position-level coordinates is the presence of singularities, e.g., configurations which can not be described by a particular set of coordinates or configurations for which the coordinates are undefined. A classic example of this is the variety of popular choices for representing the orientation of a rigid body. Choices for velocity-level coordinates (such as the Cayley form) generally provide canonical representations for the dynamics of broad classes of problems. Examples of this are the conjugate momenta and quasi velocities. Of course, alternatives to the split position and velocity-level coordinates also exist, such as the classic orbital elements that describe both the position and velocity of a spacecraft in a single set of variables.

Along with the issues of singularities and canonical representation, another issue related to coordinate choice is the linearity or nonlinearity of the resulting dynamical system. Of course, linear equations are desirable; however, it is difficult to make general comparisons of nonlinear dynamical systems. One approach to do this is the nonlinearity index developed by Junkins, which provides a measure for the nonlinearity of a dynamical system and a particular initial condition. Consider the following dynamical system.

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(t, \boldsymbol{x}\right) \qquad ; \qquad \boldsymbol{x}\left(t_0\right) = \boldsymbol{x}_0 \tag{8.1}$$

The first-order sensitivity of the trajectory to the initial conditions are described by the state-transition matrix.

$$\boldsymbol{\Phi}(t, t_0) = \frac{\partial \boldsymbol{x}(t)}{\partial \boldsymbol{x}(t_0)} \tag{8.2}$$

The state-transition matrix satisfies the following differential equation.

$$\dot{\boldsymbol{\Phi}}(t, t_0) = \boldsymbol{F}\boldsymbol{\Phi}(t, t_0) \quad ; \quad \boldsymbol{F} = \frac{\partial \boldsymbol{f}(t, \boldsymbol{x})}{\partial \boldsymbol{x}(t)} \quad ; \quad \boldsymbol{\Phi}(t_0, t_0) = \boldsymbol{I} \tag{8.3}$$

Of course, for a linear system the Jacobian matrix $\boldsymbol{F}$ is constant. Therefore, for linear systems $\boldsymbol{\Phi}(t, t_0)$ is independent of the initial condition $\boldsymbol{x}(t_0)$. In other words, the state-transition matrix $\bar{\boldsymbol{\Phi}}(t, t_0)$ evaluated along a nominal trajectory with initial condition $\bar{\boldsymbol{x}}(t_0)$ will be exactly equal to the state-transition matrix $\boldsymbol{\Phi}(t, t_0)$ evaluated along any neighboring trajectory with initial condition $\boldsymbol{x}(t_0)$. This suggests using the magnitude of the difference between state-transition matrices evaluated along neighboring trajectories as a measure of nonlinearity. In particular the following nonlinearity index was suggested by Junkins [48, 49].

$$\nu(t, t_0) \equiv \sup_{i=1\ldots n} \frac{\left\|\boldsymbol{\Phi}_i(t, t_0) - \bar{\boldsymbol{\Phi}}(t, t_0)\right\|}{\left\|\bar{\boldsymbol{\Phi}}(t, t_0)\right\|} \tag{8.4}$$

Here, $\boldsymbol{\Phi}_i(t, t_0)$ is the state-transition matrix evaluated along the trajectory corresponding to the ith initial condition from a family of n neighboring initial conditions.

The selection of the neighboring initial conditions is clearly an important issue in computing the nonlinearity index for any system and nominal initial condition. In the following study, this selection was performed using a method suggested by Junkins and Singla of populating an M-dimensional sphere surrounding the nominal initial condition in state space [49]. The initial conditions are found by distributing points approximately uniformly on the M-dimensional sphere using an optimization process. This is done by considering the points as identical attracting particles on

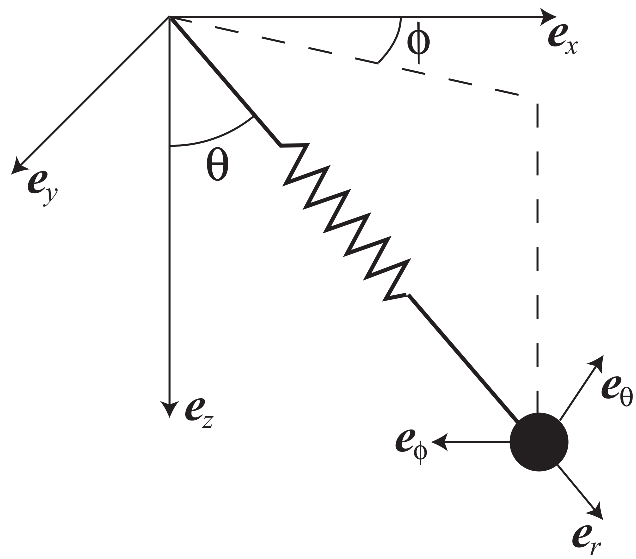

Fig. 16. Elastic spherical pendulum.

the sphere and computing the configuration that minimizes the associated potential function. Initially the points are distributed randomly, and then iteratively the points are moved along the local gradient of the potential function.

C. Elastic Spherical Pendulum

In this section the nonlinearity index will be computed for the elastic spherical pendulum shown in Fig. 16. The pendulum bob is considered as a particle with mass m. The linear spring has a spring constant k. There is also a gravitational acceleration of $g\hat{\boldsymbol{e}}_z$. Three representations of the physical system will be considered: Cartesian coordinates, Cartesian coordinates with the Cayley quasi velocities, and spherical coordinates.

In Cartesian coordinates the position of the particle relative to the origin is given by $\boldsymbol{r} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z$. The potential energy due to the elastic and gravitational potentials is given by $V = k(x^2+y^2+z^2)/2-mgz$. From these the following decoupled, linear equations of motion are found.

$$\begin{aligned} m\ddot{x} + kx &= 0 \\ m\ddot{y} + ky &= 0 \\ m\ddot{z} + kz - mg &= 0 \end{aligned} \tag{8.5}$$

In spherical coordinates the position vector is given by $\boldsymbol{r} = r\hat{\boldsymbol{e}}_r$, and the angular velocity of the body-fixed frame is given by $\boldsymbol{\omega}_{B/I} = \dot{\phi}\hat{\boldsymbol{e}}_z + \dot{\theta}\hat{\boldsymbol{e}}_\phi$. The potential energy is given by $V = kr^2/2 - mgr\cos(\theta)$. Unlike the Cartesian coordinates, the equations of motion for the spherical coordinates are coupled and nonlinear.

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 + r\dot{\phi}^2\sin^2(\theta) - \frac{k}{m}r + g\cos(\theta) \\ \ddot{\theta} &= \dot{\phi}^2\sin(\theta)\cos(\theta) - 2\frac{\dot{r}\dot{\theta}}{r} - \frac{g}{r}\sin(\theta) \\ \ddot{\phi} &= -2\frac{\dot{r}\dot{\phi}}{r} - 2\dot{\phi}\dot{\theta}\frac{\cos(\theta)}{\sin(\theta)} \end{aligned} \tag{8.6}$$

In deriving these equations, it is found that ϕ is a cyclic coordinate and the motion constant $h_z = r^2\dot{\phi}\sin^2(\theta)$ exists. This constant is the vertical component of the angular momentum about the origin.

The final representation of the elastic spherical pendulum that will be considered is Cartesian coordinates in the Cayley form. This uses the Cartesian coordinates for generalized coordinates, $[\boldsymbol{q}] = [x\ y\ z]^T$, and the associated Cayley quasi velocities, $[\boldsymbol{\omega}] = [\omega_1\ \omega_2\ \omega_3]^T$, for velocity-level coordinates. The Cayley quasi velocities for this

3-degree-of-freedom system are related to the generalized velocities as shown below.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = [\boldsymbol{A}] \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \quad ; \quad [\boldsymbol{A}] = \frac{1}{2} \begin{bmatrix} 1+x^2 & xy-z & xz+y \\ yx+z & 1+y^2 & yz-x \\ zx-y & zy+x & 1+z^2 \end{bmatrix} \tag{8.7}$$

This relationship can be used to write the kinetic energy as a function of the generalized coordinates and quasi velocities.

$$T = \frac{1}{2} m \dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}} = \frac{1}{2} \boldsymbol{\omega}^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{\omega} \tag{8.8}$$

The equations of motion are then developed by applying Lagrange's equations for quasi velocities, where the Hamel coefficients for this three-dimensional problem are the Levi-Civita permutation symbol.

$$\begin{aligned} \dot{\omega}_1 &= -\left[\omega_1 \left(\omega_1 x + \omega_2 y + \omega_3 z\right) + \frac{2\frac{k}{m}x + 2gy}{1+x^2+y^2+z^2}\right] \\ \dot{\omega}_2 &= -\left[\omega_2 \left(\omega_1 x + \omega_2 y + \omega_3 z\right) + \frac{2\frac{k}{m}y - 2gx}{1+x^2+y^2+z^2}\right] \\ \dot{\omega}_3 &= -\left[\omega_3 \left(\omega_1 x + \omega_2 y + \omega_3 z\right) + \frac{2\frac{k}{m}z - 2g}{1+x^2+y^2+z^2}\right] \end{aligned} \tag{8.9}$$

The three representations are summarized in Table V.

Table V. ELASTIC SPHERICAL PENDULUM REPRESENTATIONS

Representation	Position-Level Coordinates	Velocity-Level Coordinates	Kinematics	Dynamics
Cartesian coordinates	x, y, z	$\dot{x}, \dot{y}, \dot{z}$	linear	linear
Spherical coordinates	r, θ, ϕ	$\dot{r}, \dot{\theta}, \dot{\phi}$	linear	nonlinear
Cayley form	x, y, z	$\omega_1, \omega_2, \omega_3$	nonlinear	nonlinear

In order to integrate the state-transition matrix and compute the nonlinearity index, the Jacobian of each dynamical system was found. This is done by taking the partial derivatives of Eq. (8.5) and the associated kinematics (e.g., $\dot{x} = \dot{x}$), Eq. (8.6) and the associated kinematics, and Eqs. (8.7) and (8.9) with respect to the corresponding state variables. Although these matrices are not shown here, it is important to note that the linear system corresponding to the Cartesian coordinate representation produces a constant Jacobian. For the other two representations the values of the Jacobian matrices vary with the state variables.

The nonlinearity index which was described above gives one measure for the nonlinearity of each of these representations. In computing this index, normalization is performed with respect to the nominal trajectory as represented by each set of coordinates. Therefore, the nonlinearity index represents a measurement of nonlinearity within the context of each individual coordinate system. This is perhaps most useful in judging the nonlinearity in an absolute sense, such as determining the integrability of the equations of motion.

Another concept for measuring nonlinearity, however, is to check how well some property of interest related to the motion is captured by linear portions of the equations of motion. This is done by integrating the linearized departure motion from the nominal trajectory.

$$\dot{\boldsymbol{x}}_{dep} = \boldsymbol{F}\left(\bar{\boldsymbol{x}}\right)\boldsymbol{x}_{dep} \quad ; \quad \boldsymbol{x}_{dep}\left(t_0\right) = \boldsymbol{x}\left(t_0\right) - \bar{\boldsymbol{x}}\left(t_0\right) \tag{8.10}$$

The linear prediction of a neighboring trajectory is given by $\boldsymbol{x}_{dep}(t) + \bar{\boldsymbol{x}}(t)$ and can be analyzed to determine how well the linearized state equations capture the motion. For this example the total energy E and the vertical angular momentum h_z, both constants, will be computed.

D. Numerical Results

The nonlinearity index of each representation was computed for the trajectory associated with the following initial condition in Cartesian coordinates.

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.1 \\ 1.0 \end{bmatrix} \quad ; \quad \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.15 \\ 0.1 \end{bmatrix} \tag{8.11}$$

In order to investigate the behavior of each system in the neighborhood of this trajectory, a set of 500 initial conditions were selected on a six-dimensional sphere in the Cartesian-coordinate state space with radius 0.01 surrounding the nominal initial point. The points were distributed approximately uniformly. In order to perform the computations for the spherical-coordinate and Cayley-form representations, these points were transformed to the corresponding variables using the appropriate nonlinear coordinate transformations. The parameter values $m = k = g = 1$ were used.

As mentioned, the Jacobian matrix for the Cartesian coordinate representation is a constant. Therefore the state-transition matrix for these coordinates has the solution $\boldsymbol{\Phi}(t, t_0) = \exp(\boldsymbol{F}(t - t_0))$ and is independent of the initial condition. Therefore the nonlinearity index for this representation is identically zero, as expected for a linear system. Also, the linearized departure equations for the Cartesian coordinates are the true equations of motion, and therefore they exactly predict the correct energy and vertical angular momentum. The solution for the nominal trajectory in Cartesian coordinates over an interval of ten time units is shown in Fig. 17.

For the nonlinear systems associated with the spherical-coordinate and Cayley-form representations, the nonlinearity index was computed by integrating the trajectories and state-transition matrices over ten time units for each initial condition and then evaluating Eq. (8.4). The nominal trajectory in spherical-coordinate and

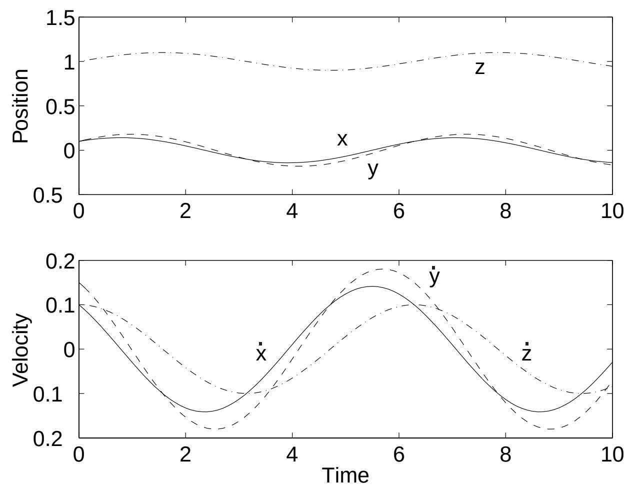

Fig. 17. Nominal trajectory in Cartesian coordinates.

Cayley-form representations are shown in Figs. 18 and 19. The nonlinearity indices found for each representation are shown in Figs. 20 and 21. The average value of the nonlinearity index over the time interval and the maximum value are shown for both spherical coordinates and the Cayley form in Table VI. These results show much lower nonlinearity indices for the Cayley form than the spherical coordinates.

In addition to the nonlinearity index, the errors in linear prediction of E and h_z were also computed. The linearized departure equations were integrated using the same nominal trajectory, set of initial conditions, and parameter values. The maximum error in E and h_z over the set of initial conditions was computed for each point in time. For the spherical coordinates the errors in these constants are shown in Fig. 22, and for the Cayley form the errors are shown in Fig. 23. The average values of the errors over the time interval and the maximum values are shown for both spherical coordinates and the Cayley form in Table VI. These results show that the linearized departure equations for the Cayley form perform much better than the spherical coordinates in predicting the correct values for the constants E and h_z.

Table VI. NUMERICAL RESULTS FOR NONLINEARITY

	Spherical coordinates	Cayley form
ν, average	0.4747	0.0114
ν, maximum	6.7956	0.0179
E, average	5.3768×10^{-4}	1.1347×10^{-5}
E, maximum	8.1235×10^{-3}	1.6418×10^{-5}
h_z, average	2.3004×10^{-4}	7.2015×10^{-6}
h_z, maximum	1.2375×10^{-3}	1.2244×10^{-5}

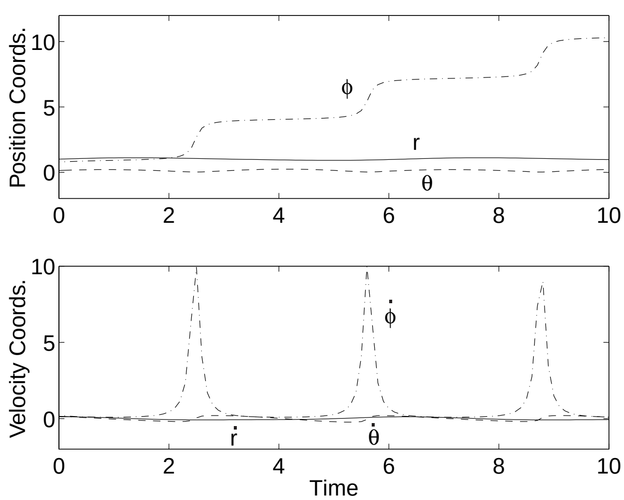

Fig. 18. Nominal trajectory in spherical coordinates.

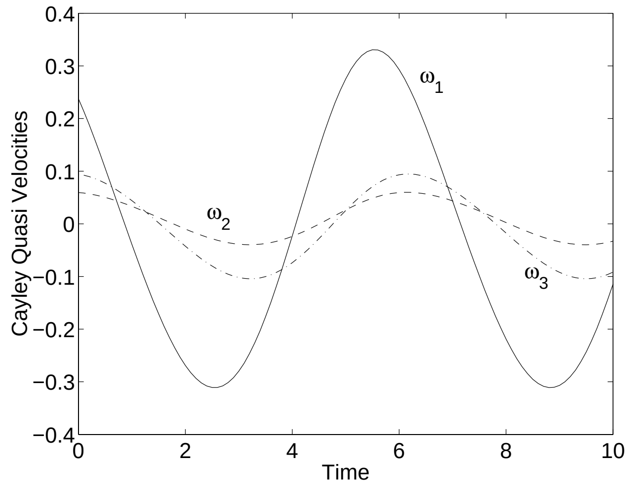

Fig. 19. Nominal trajectory for Cayley quasi velocities.

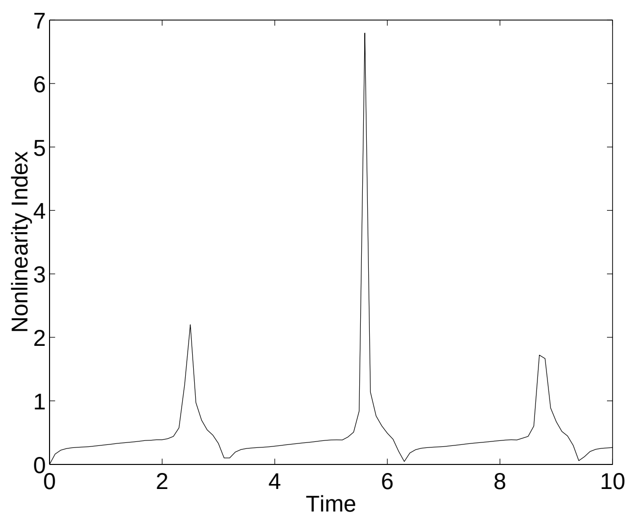

Fig. 20. Nonlinearity index for spherical coordinates.

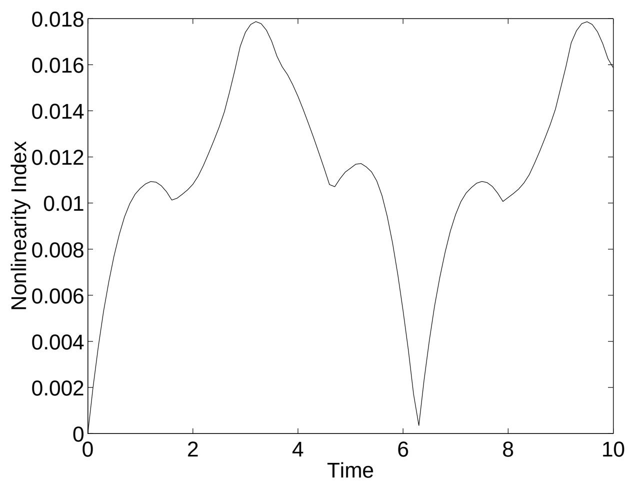

Fig. 21. Nonlinearity index for Cayley form.

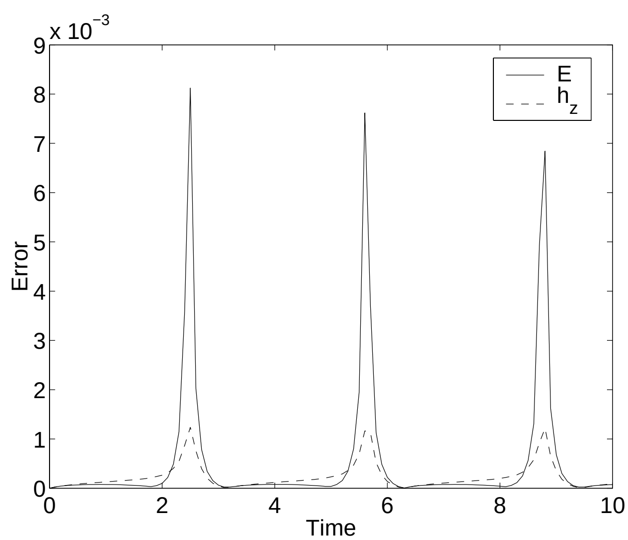

Fig. 22. Linearization error in motion constants for spherical coordinates.

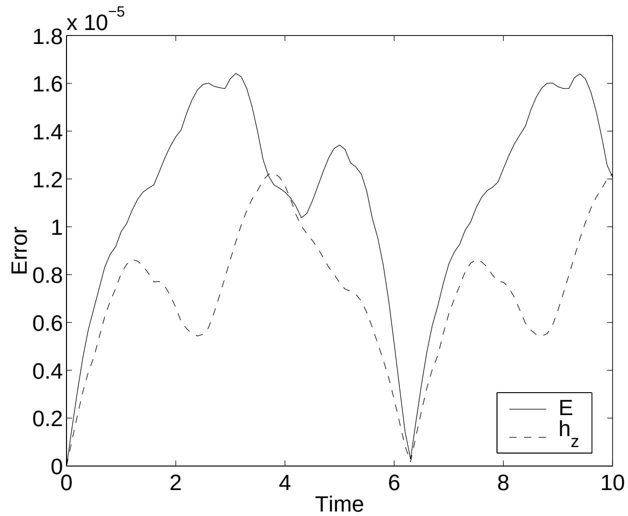

Fig. 23. Linearization error in motion constants for Cayley form.

E. Discussion

The results above show good agreement between the two types of nonlinearity measurement. Both measurements show lower nonlinearity for the Cayley form than the spherical coordinates and also agree in several features in the time history. For the spherical-coordinate representation the nonlinearity index and motion-constant errors experience sharp peaks at the points along the trajectory where the coordinate ϕ goes through large changes in value. At these points the trajectory approaches the singularity in the spherical coordinates. In particular, the coordinate ϕ is undefined when $\theta = 0$. The constant of motion described above shows that $\dot{\phi}$ can diverge as θ approaches zero. (Of course, another singularity exists such that both θ and ϕ are undefined for $r = 0$.) The nonlinearity index indicates that the spherical-coordinate representation is, in general, moderately nonlinear and highly nonlinear in the neighborhood of the singular configuration.

Alternatively, the Cayley-form representation is singularity free; neither the Cartesian position-level coordinates nor the Cayley kinematics suffer from singularities. Compared with the spherical-coordinate representation, the nonlinearity index for the Cayley form shows only mild nonlinearity. Related to the issue of singularities is the fact that the Cayley form introduces polynomial nonlinearities into the kinematics and dynamics, whereas the spherical coordinates produce trigonometric nonlinearities in the dynamic equations. These results indicate that the Cayley form can be used without too great a penalty in nonlinearity versus alternative representations.

CHAPTER IX

SUMMARY

Results were developed in this dissertation for generalizing broad areas of rotational mechanics, including kinematics, dynamics, and control, to N-dimensional rotations. One main impact of this work is that studying N-dimensional mechanics serves as a pedagogical tool for better understanding the three-dimensional special case. Investigating the general N-dimensional case illuminates the properties of rotational motion in general and those properties that only hold for rotations in three-dimensions. Some examples of this related to understanding of Euler's theorem and the definition of angular velocity.

In three-dimensions strong connections exist between Euler's theorem and the family of attitude representations related to the Cayley transform and higher-order Cayley transforms. The results of Chapters II and III focused on strengthening the connection between the N-dimensional extensions of these concepts. Similar to their three-dimensional relative, the ERPs contain a singularity for principal rotations of ± 180 degrees. Although other members of this family have larger singularity free regions, the price of this is a loss of uniqueness in the parameter values associated with any given orientation. One reason why the Cayley form focused on the use of ERPs is that they give an invertible relationship between the configuration of a general system and the orientation of the associated N-dimensional rigid body. The ERPs are useful parameters because they can take on values from $-\infty$ to $+\infty$ and can therefore be related to a wide variety of generalized coordinates.

The results contained in the dissertation show how the field of N-dimensional rotations can be applied to the description of real, physical systems. The development of the Cayley form provides a new method for viewing the motion of a broad class of problems. The Cayley form, in turn, can be applied in developing feedback controllers. The results shown in the dissertation for feedback control using Cayley quasi velocities and results from the literature on other quasi velocity forms, such as the eigenfactor quasi velocities, suggest an important issue for the selection of variables used in controller design. In addition to the issues involved in coordinate choice such as simplicity or nonlinearity of equations of motion, it also appears that the selection of variables can have an impact on the performance of resulting controller designs. This issue seems analogous to the selection of a Lyapunov function in Lyapunov control. Once designed, a controller can be expressed in any desired coordinates and is not dependent on the variables used to aid in its design. Additionally, no general method is apparent for choosing desirable quasi velocities for a particular system.

REFERENCES

[1] P. H. Schoute, "Le déplacement le plus général dans l'espace à n dimensions," *Annales de L'Ecole Polytechnique de Delft*, vol. 7, pp. 139–158, 1891.

[2] D. Mortari, "On the rigid rotation concept in n-dimensional spaces," *The Journal of the Astronautical Sciences*, vol. 49, no. 3, pp. 401–420, 2001.

[3] R. Bauer, "Euler's theorem on rigid body displacements generalized to n dimensions," *The Journal of the Astronautical Sciences*, vol. 50, no. 3, pp. 305–309, 2002.

[4] A. Sinclair and J. Hurtado, "Minimum-parameter representations of N-dimensional principal rotations," *The Journal of the Astronautical Sciences*, 2004, (in review).

[5] I. Y. Bar-Itzhack and F. L. Markley, "Minimal parameter solution of the orthogonal matrix differential equation," *IEEE Transactions on Automatic Control*, vol. 35, no. 3, pp. 314–317, 1990.

[6] H. Schaub, P. Tsiotras, and J. L. Junkins, "Principal rotation representations of proper n×n orthogonal matrices," *International Journal of Engineering Science*, vol. 33, no. 15, pp. 2277–2295, 1995.

[7] O. Bottema and B. Roth, *Theoretical Kinematics.* Amsterdam: North-Holland Publishing Company, 1979, chap. 1 & 2.

[8] A. Cayley, "Sur quelques propriétés des déterminant gauches," *Journal für die reine und angewandte Mathematik*, vol. 32, pp. 119–123, 1846, also in *The Collected Mathematical Papers of Arthur Cayley*, Cambridge, U.K.: Cambridge University Press, 1889-1898, vol. 1, pp. 332-336.

[9] R. A. Horn and C. R. Johnson, *Matrix Analysis.* Cambridge, U.K.: Cambridge University Press, 1987, pp.105–107.

[10] I. Y. Bar-Itzhack, “Extension of Euler’s theorem to n-dimensional spaces,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 25, no. 6, pp. 903–909, 1989.

[11] M. L. Curtis, *Matrix Groups.* New York: Springer-Verlag, 1979, chap. 4.

[12] J. L. Junkins and Y. Kim, *Introduction to Dynamics and Control of Flexible Structures.* Washington, DC: American Institute of Aeronautics and Astronautics, 1993, sect. 2.4.2.

[13] P. C. Hughes, *Spacecraft Attitude Dynamics.* New York: John Wiley & Sons, 1986, pp. 24–25.

[14] E. J. Lefferts, F. L. Markley, and M. D. Shuster, “Kalman filtering for spacecraft attitude estimation,” *Journal of Guidance, Control, and Dynamics*, vol. 5, no. 5, pp. 417–429, 1982.

[15] J. Hurtado and A. Sinclair, “Hamel coefficients for the rotational motion of an N-dimensional rigid body,” *Proceedings of the Royal Society of London Series A*, vol. 460, no. 2052, pp. 3613–3630, 2004.

[16] A. Sinclair and J. Hurtado, “Cayley kinematics and the Cayley form of dynamic equations,” *Proceedings of the Royal Society of London Series A*, vol. 461, no. 2055, pp. 761–781, 2005.

[17] A. Sinclair, J. Hurtado, and J. Junkins, “Application of the Cayley form to general spacecraft motion,” *Journal of Guidance, Control, and Dynamics*, 2004, (in review).

[18] W. W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, ser. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 1994, vol. 3.

[19] D. Mortari, "n-Dimensional cross product and its application to matrix eigenanalysis," *Journal of Guidance, Control, and Dynamics*, vol. 20, no. 3, pp. 509–515, 1997.

[20] A. Bloch, J. Baillieul, P. Crouch, and J. Marsden, *Nonholonomic Mechanics and Control*. New York: Springer Verlag, 2003.

[21] Y. N. Fedorov and V. V. Kozlov, "Various aspects of n-dimensional rigid body dynamics," in *American Mathematical Society Translations*, ser. 2. Providence, RI: American Mathematical Society, 1995, vol. 168, pp. 141–171.

[22] W. Frahm, "Ueber gewisse differentialgleichungen," *Mathematische Annalen*, vol. 8, pp. 35–44, 1875.

[23] H. Weyl, *Space-Time-Matter*. New York: Dover Publications, Inc., 1950.

[24] T. Ratiu, "The motion of the free n-dimensional rigid body," *Indiana University Mathematics Journal*, vol. 29, no. 4, pp. 609–629, 1980.

[25] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, *Non-linear Equations of Korteweg-deVries Type, Finite Zone Linear Operators, and Abelian Varieties*, ser. Russian Math. Surveys. London: London Mathematical Society, 1976, vol. 31.

[26] S. V. Manakov, "Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body," *Functional Analysis and Its Applications*, vol. 10, pp. 328–329, 1976.

[27] A. S. Mishchenko, "Integral geodesics of a flow on lie groups," *Functional Analysis and Its Applications*, vol. 4, pp. 232–235, 1970.

[28] A. Bloch, P. Crouch, J. Marsden, and T. Ratiu, "The symmetric representation of the rigid body equations and their discretization," *Nonlinearity*, vol. 15, no. 4, pp. 1309–1341, 2002.

[29] A. Bloch and P. Crouch, "Optimal control and geodesic flows," *Systems and Control Letters*, vol. 28, pp. 65–72, 1996.

[30] A. E. Bryson and Y.-C. Ho, *Applied Optimal Control: Optimization, Estimation, and Control*. Washington, DC: Hemisphere Publishing Corp., 1975.

[31] D. Frederick and T. S. Chang, *Continuum Mechanics*. Cambridge, MA: Scientific Publishers, Inc., 1965, chap. 1.

[32] D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles*. New York: Dover Publications, Inc., 1989, chap. 4, sect. 2.

[33] J. G. Papastavridis, *Tensor Calculus and Analytical Dynamics*. Boca Raton, FL: CRC Press, 1999, chap. 3, sect. 5, pp.126-128.

[34] R. M. Bowen and C.-C. Wang, *Introduction to Vectors and Tensors*. New York: Plenum Press, 1976, vol. 1: Linear and Multilinear Algebra, p. 109.

[35] L. Meirovitch, *Methods of Analytical Dynamics*. New York: McGraw-Hill, 1970, section 4.13, pp. 160-162.

[36] J. Marsden and T. Ratiu, *Introduction to mechanics and symmetry*, 2nd ed. New York: Springer Verlag, 1999, p. 437.

[37] J. Hurtado, "Hamel coefficients for the rotational motion of a rigid body," *The Journal of the Astronautical Sciences*, vol. 52, no. 1 & 2, pp. 129–147, 2004.

[38] H. Schaub, "Novel coordinates for nonlinear multibody motion with applications to spacecraft dynamics and control," Ph.D. dissertation, Texas A&M University, College Station, Texas, 1998.

[39] A. Cayley, "On the motion of rotation of a solid body," *Cambridge Mathematical Journal*, vol. 3, pp. 224–232, 1843, also in *Collected Mathematical Papers of Arthur Cayley*, Cambridge, U.K.: Cambridge University Press, 1889-1898, vol. 1, pp. 28-35.

[40] ——, "Sur la transformation d'une fonction quadratique en elle même par des substitutions linéaires," *Journal für die reine und angewandte Mathematik*, vol. 50, pp. 288–299, 1855, also in *The collected mathematical papers of Arthur Cayley*, Cambridge, U.K.: Cambridge University Press, 1889–1898, vol. 2, pp. 192–201.

[41] ——, "A memoir on the automorphic linear transformation of a bipartite quadric function," *Philosophical Transactions of the Royal Society of London*, vol. 148, pp. 39–46, 1858, also in *Collected Mathematical Papers of Arthur Cayley*, Cambridge, U.K.: Cambridge University Press, 1889-1898, vol. 2, pp. 497-505.

[42] J. L. Junkins and J. D. Turner, "On the analogy between orbital dynamics and rigid body dynamics," *The Journal of the Astronautical Sciences*, vol. 27, no. 4, pp. 345–358, 1979.

[43] A. Sinclair, J. Hurtado, and J. L. Junkins, "Investigations on the use of the Cayley form for feedback controller design," in *AIAA Guidance, Navigation, and Control Conference*, Providence, RI, August 2004, paper AIAA 2004-4761.

[44] H. Baruh, *Analytical Dynamics*. New York: McGraw-Hill, 1999.

[45] P. Tsiotras, "New control laws for the attitude stabilization of rigid bodies," in 13^{th} *IFAC Symposium on Automatic Control in Aerospace*, Palo Alto, California, September 1994, pp. 316–321.

[46] ——, "Stabilization and optimality results for the attitude control problem," *Journal of Guidance, Control, and Dynamics*, vol. 19, no. 4, pp. 772–779, 1996.

[47] H. Schaub and J. L. Junkins, "Feedback control law using the eigenfactor quasi-coordinate velocity vector," *Journal of the Chinese Society of Mechanical Engineers*, vol. 19, no. 1, pp. 51–59, 1997.

[48] J. L. Junkins, "Adventures on the interface of dynamics and control," *Journal of Guidance, Control, and Dynamics*, vol. 20, no. 6, pp. 1058–1071, 1997.

[49] J. L. Junkins and P. Singla, "How nonlinear is it? a tutorial on the nonlinearity of orbit and attitude dynamics," *The Journal of Astronautical Sciences*, vol. 52, no. 1 & 2, pp. 7–60, 2004.

APPENDIX A

ALTERNATIVE DERIVATION OF INVERSE PROPERTY

The inverse property of χ_{ik}^{j} can alternatively be demonstrated by considering the product $\chi_{ik}^{l}\chi_{ik}^{j}$. In this product any two particular values of l and j can be chosen while i and k are summed from 1 to N. However, for a particular value of l only one pair of values for i and k will produce nonzero values of χ_{ik}^{l}. Because the same is true for j, each term in the summation will contain at least one zero unless $l = j$. Therefore, the following relationship expresses the value of the product.

$$\chi_{ik}^{l}\chi_{ik}^{j} = \left[(1)(1) + (-1)(-1)\right]\delta_{lj} = 2\delta_{lj} \tag{A.1}$$

Using this relation the inverse mapping from the matrix elements to the vector elements can be found directly, starting with Eq. (4.14)

$$\begin{aligned}
\Omega_{ik} &= \chi_{ik}^{j}\omega_{j} \\
\chi_{ik}^{l}\Omega_{ik} &= \chi_{ik}^{l}\chi_{ik}^{j}\omega_{j} \\
\chi_{ik}^{l}\Omega_{ik} &= 2\delta_{lj}\omega_{j} \\
\frac{1}{2}\chi_{ik}^{l}\Omega_{ik} &= \omega_{l}
\end{aligned} \tag{A.2}$$

As noted in the text, $\frac{1}{2}\chi_{ik}^{l}$ is only one of infinitely many numerical relative tensors that satisfies Eq. (A.2).

APPENDIX B

PARTIAL DERIVATIVES OF $\boldsymbol{Q}$, $\dot{\boldsymbol{Q}}$, AND $\boldsymbol{\Omega}$

In the derivation of the N-dimensional rotational equations of motion contained in the text above, the elements of the skew-symmetric matrices $\boldsymbol{Q}$, $\dot{\boldsymbol{Q}}$, and $\boldsymbol{\Omega}$ were treated as independent. This allowed partial derivatives with respect to these elements to be written in the following manner.

$$\frac{\partial T_0}{\partial q_i} = \frac{\partial \widetilde{T}_0}{\partial Q_{jk}} \frac{\partial Q_{jk}}{\partial q_i} = \chi^i_{jk} \frac{\partial \widetilde{T}_0}{\partial Q_{jk}} \tag{B.1}$$

$$\frac{\partial \dot{Q}_{vp}}{\partial \dot{Q}_{ij}} = \delta_{iv}\delta_{jp} \tag{B.2}$$

Another option in performing the derivation is to treat the elements of the skew-symmetric matrices as dependent by considering the form of the skew-symmetric constraint. Of course, the constraint specifies that for each non-zero element of a skew-symmetric matrix there will be an equal and opposite element. This means that the partial derivative of $\widetilde{T}_0$ with respect to Q_{jk} will produce two equal terms: one corresponding to the contribution of Q_{jk} and the other corresponding to the contribution of Q_{kj}. To maintain the desired result a factor of one-half is needed. The partial derivatives thus take on the new form given below.

$$\frac{\partial T_0}{\partial q_i} = \frac{1}{2} \frac{\partial \widetilde{T}_0}{\partial Q_{jk}} \frac{\partial Q_{jk}}{\partial q_i} = \frac{1}{2} \chi^i_{jk} \frac{\partial \widetilde{T}_0}{\partial Q_{jk}} \tag{B.3}$$

$$\frac{\partial \dot{Q}_{vp}}{\partial \dot{Q}_{ij}} = \delta_{iv}\delta_{jp} - \delta_{ip}\delta_{jv} \tag{B.4}$$

The above equation produces $+1$ if $i = v$ and $j = p$ or -1 if $i = p$ and $j = v$. Consider the three-dimensional example $T_0 = \frac{1}{2}\dot{q}_3^2$ and $\partial T_0/\partial \dot{q}_3 = \dot{q}_3$. This will produce the following form for $\widetilde{T}_0$.

$$\widetilde{T}_0 = \frac{1}{4}\dot{Q}_{12}^2 + \frac{1}{4}\dot{Q}_{21}^2 \tag{B.5}$$

If the elements are treated as independent, clearly the partial derivative with respect to $\dot{q}_3$ will be the following.

$$\begin{aligned}\frac{\partial T_0}{\partial \dot{q}_3} &= \frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{12}}\frac{\partial \dot{Q}_{12}}{\partial \dot{q}_3} + \frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{21}}\frac{\partial \dot{Q}_{21}}{\partial \dot{q}_3} \\ &= -\frac{1}{2}\dot{Q}_{12} + \frac{1}{2}\dot{Q}_{21} \\ &= \dot{q}_3 \end{aligned} \tag{B.6}$$

If the elements are treated as dependent then the same result will be achieved as shown below.

$$\begin{aligned}\partial T_0/\partial \dot{q}_3 &= \frac{1}{2}\frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{12}}\frac{\partial \dot{Q}_{12}}{\partial \dot{q}_3} + \frac{1}{2}\frac{\partial \widetilde{T}_0}{\partial \dot{Q}_{21}}\frac{\partial \dot{Q}_{21}}{\partial \dot{q}_3} \\ &= -\frac{1}{2}\left(\frac{1}{2}\dot{Q}_{12} + \frac{1}{2}\dot{Q}_{21}\frac{\partial \dot{Q}_{21}}{\partial \dot{Q}_{12}}\right) + \frac{1}{2}\left(\frac{1}{2}\dot{Q}_{12}\frac{\partial \dot{Q}_{12}}{\partial \dot{Q}_{21}} + \frac{1}{2}\dot{Q}_{21}\right) \\ &= -\frac{1}{2}\left(\frac{1}{2}\dot{Q}_{12} - \frac{1}{2}\dot{Q}_{21}\right) + \frac{1}{2}\left(-\frac{1}{2}\dot{Q}_{12} + \frac{1}{2}\dot{Q}_{21}\right) \\ &= \dot{q}_3 \end{aligned} \tag{B.7}$$

Therefore either method can be used to achieve the N-dimensional rotational equations of motion given in the text.

APPENDIX C

DERIVATION OF THE TRANSPORT THEOREM

This appendix describes an alternative approach to the derivation of Poisson's equation and the transport theorem based largely on the ideas used in this dissertation. Poisson's equation, which was used extensively throughout the text, describes the time evolution of rotation matrices which perform transformations between coordinatization of a vector in two different frames. The transport theorem relates the first time derivatives of a vector taken with respect to two different frames.

Any approach to deriving these equations requires slightly more attention than was required in the rest of the dissertation be paid to the notation for tensors, matrix representations of tensors, and the derivatives of tensors. If one starts with a vector $\boldsymbol{r}$, then a matrix representation of this vector can be found in any desired coordinate system. The matrix representation of $\boldsymbol{r}$ in the $\boldsymbol{a}$ frame is denoted as $[\boldsymbol{r}]_a$. Additionally, the derivative of the vector can be taken with respect to any desired reference frame. The time derivative of $\boldsymbol{r}$ with respect to the $\boldsymbol{a}$ frame is denoted as $\frac{{}^a\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r})$. Finally, the two choices above are completely independent; however, when a vector derivative is coordinatized in the same frame that the derivative was taken with respect to, the following simplification occurs.

$$\left[\frac{{}^a\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r})\right]_a = \frac{\mathrm{d}}{\mathrm{d}t}\left([\boldsymbol{r}]_a\right) \tag{C.1}$$

In words, the coordinatization in a frame of the time derivative of a vector with respect to the same frame is equal to the time derivative of the scalar components of the vector in that frame. In general, taking the derivative of a vector and a matrix are completely different operations. Whereas the derivative of a matrix is found by simply differentiating the scalar components, a vector must be differentiated with respect to a particular reference frame. Equation (C.1) is a critical relationship because it provides a link between these two operations. This is necessary for deriving Poisson's equation and the transport theorem because one is a matrix expression and the other is a vector expression.

In traditional treatments of three-dimensional rotational mechanics, the angular-velocity vector is defined as a limit in time of an infinitesimal rotation. From this definition the transport theorem is typically derived. Out of notational convenience the angular-velocity matrix can then be defined. Finally, from the transport theorem, Poisson's equation is typically derived. See for example Baruh, Section 2.5, pp. 107–12, and Section 7.4, pp. 365–6 [44].

In this dissertation, however, an alternative definition was used for the angular velocity. This definition is based solely on the properties of the orthonormal rotation matrix $[\boldsymbol{C}]$, which will be repeated here for completeness. The rotation matrix performs a transformation from a vector parameterized in the reference coordinate system, the $\boldsymbol{n}$ frame, to a coordinatization in a rotated frame, the $\boldsymbol{b}$ frame.

$$[\boldsymbol{r}]_b = [\boldsymbol{C}]\,[\boldsymbol{r}]_n \tag{C.2}$$

The orthogonality of $\boldsymbol{C}$ can be used to investigate its derivative.

$$\boldsymbol{C}\boldsymbol{C}^T = \boldsymbol{I} \tag{C.3}$$

$$\dot{\boldsymbol{C}}\boldsymbol{C}^T + \boldsymbol{C}\dot{\boldsymbol{C}}^T = \boldsymbol{0} \tag{C.4}$$

$$-\dot{\boldsymbol{C}}\boldsymbol{C}^T = \boldsymbol{C}\dot{\boldsymbol{C}}^T = (\dot{\boldsymbol{C}}\boldsymbol{C}^T)^T \tag{C.5}$$

In the current approach this becomes the definition of the angular-velocity matrix.

$$\boldsymbol{\Omega} = -\dot{\boldsymbol{C}}\boldsymbol{C}^T \tag{C.6}$$

The geometric interpretation of this definition is that the angular-velocity components are the projection of the rotated-frame coordinate axes onto their derivatives. For example, Ω_{ij} is the projection of the derivative of $\boldsymbol{b}_i$ with respect to the $\boldsymbol{n}$ frame onto $\boldsymbol{b}_j$. This is seen by recognizing that the rows of $[\boldsymbol{C}]$ are the $\boldsymbol{b}$ vectors coordinatized in the $\boldsymbol{n}$ frame and then substituting into the definition of angular velocity.

$$[\boldsymbol{C}] = \begin{bmatrix} [\boldsymbol{b}_1]_n & [\boldsymbol{b}_2]_n & \dots & [\boldsymbol{b}_N]_n \end{bmatrix}^T \tag{C.7}$$

$$\Omega_{ij} = -\frac{\mathrm{d}}{\mathrm{d}t}\left([\boldsymbol{b}_i]_n^T\right)[\boldsymbol{b}_j]_n = -\left[\frac{{}^n\mathrm{d}}{\mathrm{d}t}(\boldsymbol{b}_i)\right]_n^T [\boldsymbol{b}_j]_n = -\frac{{}^n\mathrm{d}}{\mathrm{d}t}(\boldsymbol{b}_i)\cdot\boldsymbol{b}_j \tag{C.8}$$

This is clearly related to the rotational rate in the $(\boldsymbol{b}_i, \boldsymbol{b}_j)$ plane. Equation (C.8) reveals a subtle fact that the components Ω_{ij} are the angular velocity of the $\boldsymbol{b}$ frame relative to the $\boldsymbol{n}$ frame coordinatized in the rotated $\boldsymbol{b}$ frame. The components are clearly the angular velocity of the $\boldsymbol{b}$ frame relative to the $\boldsymbol{n}$ frame because Eq. (C.8) is related to the derivative of the $\boldsymbol{b}$ vectors with respect to the $\boldsymbol{n}$ frame. The components are said to be coordinatized in the $\boldsymbol{b}$ frame because Ω_{ij} is simply related to $\boldsymbol{b}_i$ and $\boldsymbol{b}_j$. Thus the quantity referred to in the text as $\boldsymbol{\Omega}$ is technically $[\boldsymbol{\Omega}]_b$. Like any other tensor, the angular velocity of the $\boldsymbol{b}$ frame relative to the $\boldsymbol{n}$ frame, $\boldsymbol{\Omega}$, can be coordinatized in any desired reference frame (e.g., the principal frame). For other coordinatizations the component Ω_{ij} will still be related to the $\boldsymbol{b}$ vectors and their derivatives with respect to the $\boldsymbol{n}$ frame, however, they will not simply depend on just $\boldsymbol{b}_i$ and $\boldsymbol{b}_j$. Instead, other coordinatizations of Ω_{ij} will be related to a linear combination of all $\boldsymbol{b}$ vectors.

Based on these concepts, the definition of angular velocity can be rearranged to give Poisson's equation.

$$\left[\dot{\boldsymbol{C}}\right] = \frac{\mathrm{d}}{\mathrm{d}t}([\boldsymbol{C}]) = -[\boldsymbol{\Omega}]_b[\boldsymbol{C}] \tag{C.9}$$

A converse approach can be used to progress from Poisson's equation to the transport theorem. First, the derivative of Eq. (C.2) is taken.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([\boldsymbol{r}]_b\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left([\boldsymbol{C}]\right)[\boldsymbol{r}]_n + [\boldsymbol{C}]\frac{\mathrm{d}}{\mathrm{d}t}\left([\boldsymbol{r}]_n\right) \tag{C.10}$$

The first term on the right-hand side is the change in $[\boldsymbol{r}]_b$ due to the rotational motion whereas the second term is the change in $[\boldsymbol{r}]_b$ due to the explicit change in $[\boldsymbol{r}]_n$. The property in Eq. (C.1) and Poisson's equation can be used to rewrite this relationship.

$$\begin{aligned}\left[\frac{{}^b\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r})\right]_b &= -\left[\boldsymbol{\Omega}\right]_b[\boldsymbol{C}]\left[\boldsymbol{r}\right]_n + [\boldsymbol{C}]\left[\frac{{}^n\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r})\right]_n \\ &= -\left[\boldsymbol{\Omega}\right]_b\left[\boldsymbol{r}\right]_b + \left[\frac{{}^n\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r})\right]_b\end{aligned} \tag{C.11}$$

$$\left[\frac{{}^n\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r})\right]_b = \left[\frac{{}^b\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r})\right]_b + \left[\boldsymbol{\Omega}\right]_b\left[\boldsymbol{r}\right]_b \tag{C.12}$$

The above expression is a matrix equation where both sides are coordinatized in the $\boldsymbol{b}$ frame. The expression, however, could be coordinatized in any frame equally as well or "elevated" to a tensor expression.

$$\frac{{}^n\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \frac{{}^b\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) + \boldsymbol{\Omega}\boldsymbol{r} \tag{C.13}$$

This is the transport theorem relating the first derivative of the vector $\boldsymbol{r}$ with respect to the $\boldsymbol{n}$ and $\boldsymbol{b}$ frames and the angular velocity between the frames. Of course, for the three-dimensional special case multiplication by the angular-velocity tensor can be interpreted as cross multiplication with an angular-velocity vector. Many conventional derivations of the transport theorem essentially use as a starting point the idea that the change in $[\boldsymbol{r}]_b$ due to rotational motion is $[\boldsymbol{\omega}\times\boldsymbol{r}]_b$. This is accepted using physical intuition with three-dimensional rotations. Considering N-dimensional rotations, however, removes this physical intuition and forces the realization that this "starting point" is really just an implication of the coordinate-transformation and angular-velocity definitions.

The attractiveness of the approach outlined above is that it is based only on the orthonormality of $\boldsymbol{C}$, it provides a geometric interpretation of angular velocity, and it produces a result that can be applied to rotations in any dimension. These advantages are largely the product of recognizing the difference between a tensor and its matrix representation (e.g., a vector and an associated column matrix). Whereas this difference can seem subtle and can often be neglected, Poisson's equation and the transport theorem are exactly the expressions that demonstrate the precise difference. Therefore their derivations are greatly clarified by paying careful attention to this issue.

VITA

Andrew James Sinclair is the son of Thomas and Carol Sinclair of Gainesville, Florida. After graduating from Eastside High School, Andrew attended the University of Florida receiving B.S. and M.S. degrees in aerospace engineering in 2000 and 2002 respectively. He entered Texas A&M University in 2003 and has worked under the supervision of Dr. John L. Junkins and Dr. John E. Hurtado. Published journal papers include:

A. Sinclair and N. Fitz-Coy, "Comparison of obstacle avoidance strategies for Mars landers," *Journal of Spacecraft and Rockets*, vol. 40, no. 3, pp. 388-395, 2003.

J. Hurtado and A. Sinclair, "Hamel coefficients for the rotational motion of an N-dimensional rigid body," *Proceedings of the Royal Society of London Series A*, vol. 460, no. 2052, pp. 3613-3630, 2004.

A. Sinclair and J. Hurtado, "Cayley kinematics and the Cayley form of dynamic equations," *Proceedings of the Royal Society of London Series A*, vol. 461, no. 2055, pp. 761-781, 2005.

Contact Address: Dr. John E. Hurtado; 3141 TAMU; College Station, TX 77843-3141