NONLINEAR ANALYSIS OF
SMART COMPOSITE PLATE AND SHELL STRUCTURES

A Dissertation

by

SEUNG JOON LEE

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2004

Major Subject: Civil Engineering
NONLINEAR ANALYSIS OF
SMART COMPOSITE PLATE AND SHELL STRUCTURES

A Dissertation

by

SEUNG JOON LEE

Submitted to Texas A&M University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Approved as to style and content by:

J. N. Reddy
(Chair of Committee)

Ravinder Chona
(Member)

Harry L. Jones
(Member)

Hayes E. Ross, Jr.
(Member)

Paul N. Roschke
(Head of Department)

May 2004

Major Subject: Civil Engineering
ABSTRACT

Nonlinear Analysis of Smart Composite Plate and Shell Structures. (May 2004)

Seung Joon Lee, B.S., Yeungnam University;
M.S., Yeungnam University

Chair of Advisory Committee: Dr. J. N. Reddy

Theoretical formulations, analytical solutions, and finite element solutions for laminated composite plate and shell structures with smart material laminae are presented in the study. A unified third-order shear deformation theory is formulated and used to study vibration/deflection suppression characteristics of plate and shell structures. The von Kármán type geometric nonlinearity is included in the formulation. Third-order shear deformation theory based on Donnell and Sanders nonlinear shell theories is chosen for the shell formulation. The smart material used in this study to achieve damping of transverse deflection is the Terfenol-D magnetostrictive material. A negative velocity feedback control is used to control the structural system with the constant control gain.

The Navier solutions of laminated composite plates and shells of rectangular planeform are obtained for the simply supported boundary conditions using the linear theories. Displacement finite element models that account for the geometric nonlinearity and dynamic response are developed. The conforming element which has eight degrees of freedom per node is used to develop the finite element model. Newmark's time integration scheme is used to reduce the ordinary differential equations in time to algebraic equations. Newton-Raphson iteration scheme is used to solve the resulting nonlinear finite element equations.

A number of parametric studies are carried out to understand the damping characteristics of laminated composites with embedded smart material layers.
To my beloved family
ACKNOWLEDGMENTS

First of all, my utmost gratitude and thanks are extended to my advisor, Dr. J. N. Reddy, for his encouragement, counsel, confidence, patience and constructive criticism throughout the research. His vast knowledge and insight into the area of mechanics and materials have helped me get through some obstacles along the way.

I wish to thank Dr. Ravinder Chona and Dr. Harry L. Jones for serving on my committee and for their valuable remarks. I also wish to thank Dr. Harry Hogan from the Department of Mechanical Engineering for his technical advices.

I would like to express special thanks to Dr. Hayes E. Ross for his sincere friendship and support. Special thanks are also expressed to Dr. Roger P. Bligh at Texas Transportation Institute (TTI) for his guidance and support during my graduate study at Texas A&M University. I thank the fellow members in the Advanced Computational Mechanics Laboratory (ACML) for their friendship during my stay at ACML.

I am also grateful for the financial support from the Army Research Office (ARO) through the Grant DAAD19-01-1-0483 during this work.

Last but not least, I would like to thank my wife, Young A Park, and my son, Yoon Dong Lee, for their unwavering support, patience, encouragement and love which provided me the strength to complete this study.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>v</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>vi</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>ix</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>xvi</td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1. General</td>
<td>1</td>
</tr>
<tr>
<td>1.2. Smart Materials</td>
<td>2</td>
</tr>
<tr>
<td>1.3. Background Literature</td>
<td>4</td>
</tr>
<tr>
<td>1.4 Present Study</td>
<td>7</td>
</tr>
<tr>
<td>2. THEORETICAL FORMULATIONS</td>
<td>9</td>
</tr>
<tr>
<td>2.1. Methodology</td>
<td>9</td>
</tr>
<tr>
<td>2.2. Displacement Field and Strains</td>
<td>9</td>
</tr>
<tr>
<td>2.2.1. Displacement Field and Strains for Plates</td>
<td>9</td>
</tr>
<tr>
<td>2.2.2. Kinetics of Shells</td>
<td>12</td>
</tr>
<tr>
<td>2.2.3. Displacement Field and Strains for Shells</td>
<td>14</td>
</tr>
<tr>
<td>2.3. Equations of Motion</td>
<td>19</td>
</tr>
<tr>
<td>2.3.1. Equations of Motion for TSDT Plates</td>
<td>20</td>
</tr>
<tr>
<td>2.3.2. Equations of Motion for Special Cases</td>
<td>23</td>
</tr>
<tr>
<td>2.3.3. Equations of Motion for TSDT Donnell Shell Theory</td>
<td>27</td>
</tr>
<tr>
<td>2.3.4. Equations of Motion for TSDT Sanders Shell Theory</td>
<td>28</td>
</tr>
<tr>
<td>2.4. Constitutive Relations</td>
<td>30</td>
</tr>
<tr>
<td>2.5. Velocity Feedback Control</td>
<td>32</td>
</tr>
<tr>
<td>2.6. Laminated Constitutive Equations</td>
<td>34</td>
</tr>
<tr>
<td>3. ANALYTICAL SOLUTIONS</td>
<td>36</td>
</tr>
<tr>
<td>3.1. Analytical Solutions for Laminated Composite Plates</td>
<td>36</td>
</tr>
<tr>
<td>3.1.1. Introduction</td>
<td>36</td>
</tr>
<tr>
<td>3.1.2. Boundary Conditions</td>
<td>36</td>
</tr>
<tr>
<td>3.1.3. Navier Solution</td>
<td>37</td>
</tr>
<tr>
<td>3.2. Analytical Solutions for Laminated Composite Shells</td>
<td>40</td>
</tr>
</tbody>
</table>
3.2.1. Modified Sanders Shell Theory ................................................................. 40
3.2.2. Navier Solutions of Shell Theories ............................................................ 41
3.3. Vibration Control.......................................................................................... 47
3.4. Analytical Results for Laminated Composite Plates....................................... 48
  3.4.1. Vibration Suppression of Different Modes .................................................. 49
  3.4.2. Effect of Lamina Material Properties......................................................... 49
  3.4.3. Effect of Plate Theories.............................................................................. 50
  3.4.4. Effect of Smart Material Position.............................................................. 56
  3.4.5. Effect of Lamina Thickness ..................................................................... 60
  3.4.6. Effect of Feedback Coefficients............................................................... 61
  3.4.7. Variation of $s_T$ and $e_r$ for Different Laminates .................................... 64
3.5. Analytical Results for Laminated Composite Shells ....................................... 66
  3.5.1. The General............................................................................................. 66
  3.5.2. Effect of Shell Theories ......................................................................... 71
  3.5.3. Effect of $R_z/a$ .................................................................................... 74
  3.5.4. Effect of Shell Types ............................................................................ 78
  3.5.5. Effect of Material Properties .................................................................. 78
4. FINITE ELEMENT FORMULATIONS .................................................................. 86
  4.1. Virtual Work Statements ............................................................................ 86
    4.1.1. Virtual Work Statements for Laminated Composite Plates .................... 86
    4.1.2. Virtual Work Statements for Laminated Composite Shells.................... 89
  4.2. Finite Element Model .............................................................................. 94
  4.3. Transient Analysis ................................................................................... 97
  4.4. Nonlinear Analysis .................................................................................. 99
  4.5. Computer Implementation ...................................................................... 100
  4.6. Preliminary Linear Finite Element Results ............................................... 100
    4.6.1. Simply Supported Laminated Composite Plates ..................................... 105
    4.6.2. Deflection Suppression Time ................................................................. 108
    4.6.3. Effect of Lamina Thickness .................................................................. 108
    4.6.4. Effect of Feedback Coefficients........................................................... 109
    4.6.5. Other Effects on Deflection Control .................................................... 111
5. RESULTS OF NONLINEAR ANALYSIS .............................................................. 118
  5.1. Nonlinear Static Results .......................................................................... 120
  5.2. Nonlinear Transient Results for Laminated Composite Plates.................... 127
    5.2.1. Load and Time Increments .................................................................. 127
    5.2.2. Effect of Plate Theories ..................................................................... 127
    5.2.3. Effect of Lamination Schemes ............................................................ 132
5.2.4. Effect of Loading Conditions ................................................................. 135
5.2.5. Effect of Boundary Conditions .............................................................. 139
5.3. Nonlinear Transient Results for Laminated Composite Shells ................. 144
  5.3.1. Effect of Shell Theories ......................................................................... 144
  5.3.2. Effect of $R/a$ ...................................................................................... 149
  5.3.3. Effect of Shell Types ............................................................................. 149
  5.3.4. Effect of Lamination Schemes ............................................................. 156
  5.3.5. Effect of Boundary Conditions ............................................................ 157
  5.3.6. Effect of Loading Conditions ............................................................... 157
5.4. Nonlinear Results under Thermomechanical Loads .................................. 157
  5.4.1. Static Results under Thermomechanical Loads .................................... 167
  5.4.2. Laminated Composite Plates under Thermomechanical Loads ............ 171
  5.4.3. Laminated Composite Shells under Thermomechanical Loads .............. 191

6. CONCLUSIONS ................................................................................................. 199
  6.1. Concluding Remarks .................................................................................. 199
  6.2. Recommendations for Future Work .......................................................... 201

REFERENCES ........................................................................................................ 203

APPENDIX A  EULER-LAGRANGE EQUATIONS OF TSDT SANDERS SHELL
                      THEORY IN TERMS OF DISPLACEMENTS ......................................... 210

APPENDIX B  FINITE ELEMENT COEFFICIENTS BY TSDT DONNELL (DMV)
                        SHELL THEORY ........................................................................... 216

APPENDIX C  FINITE ELEMENT COEFFICIENTS BY TSDT SANDERS
                         (SANDERS-KOITER) SHELL THEORY .................................................. 226

VITA ...................................................................................................................... 238
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. 1</td>
<td>Geometry and coordinate system of laminated plate</td>
<td>10</td>
</tr>
<tr>
<td>2. 2</td>
<td>Geometry and coordinate system of laminated shell</td>
<td>12</td>
</tr>
<tr>
<td>2. 3</td>
<td>Force and moment resultants on a plate element</td>
<td>22</td>
</tr>
<tr>
<td>2. 4</td>
<td>Closed-loop feedback control system</td>
<td>33</td>
</tr>
<tr>
<td>3. 1</td>
<td>Simply supported boundary conditions (SS-1 and SS-2) for TSDT</td>
<td>36</td>
</tr>
<tr>
<td>3. 2</td>
<td>Suppression characteristics for different modes of vibration</td>
<td>51</td>
</tr>
<tr>
<td>3. 3</td>
<td>Effect of material properties on vibration suppression</td>
<td>52</td>
</tr>
<tr>
<td>3. 4</td>
<td>Effect of the different plate theories on vibration suppression</td>
<td>54</td>
</tr>
<tr>
<td>3. 5</td>
<td>Vibration suppression characteristics for different smart material location</td>
<td>57</td>
</tr>
<tr>
<td>3. 6</td>
<td>Vibration suppression times for CFRP laminated plate</td>
<td>59</td>
</tr>
<tr>
<td>3. 7</td>
<td>Effect of lamina thickness on vibration suppression</td>
<td>61</td>
</tr>
<tr>
<td>3. 8</td>
<td>Effect of feedback coefficient on the suppression time</td>
<td>63</td>
</tr>
<tr>
<td>3. 9</td>
<td>Variation of $T_s$ and $e_r$ with respect to $Z_m$</td>
<td>65</td>
</tr>
<tr>
<td>3. 10</td>
<td>Vibration suppression characteristics of CFRP spherical shells ($R_z/a = 10$) for the different mode; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)</td>
<td>68</td>
</tr>
<tr>
<td>3. 11</td>
<td>Vibration suppression characteristics of CFRP spherical shells ($R_z/a = 3$) for the different lamination; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)</td>
<td>70</td>
</tr>
<tr>
<td>3. 12</td>
<td>Vibration suppression characteristics of CFRP cylindrical shells for different shell theories; (a) Thin shell ($R_z/a = 2$), (b) Thick shell ($R_z/a = 2$)</td>
<td>73</td>
</tr>
<tr>
<td>3. 13</td>
<td>Effect of $R_z/a$ on vibration suppression characteristics of CFRP spherical shells; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)</td>
<td>75</td>
</tr>
</tbody>
</table>
3. 14 Effect of $R_z/a$ on vibration suppression characteristics of CFRP cylindrical shells; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)............................. 76

3. 15 Effect of $R_z/a$ on vibration suppression characteristics of CFRP doubly curved shells; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)............. 77

3. 16 Vibration suppression characteristics for different CFRP shells; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)........................................ 79

3. 17 Effect of elastic material properties on vibration suppression characteristics for thin ($a/h = 100$) shells; (a) spherical shell, (b) cylindrical shell, (c) doubly curved shell......................................................................................... 84

4. 1 Conforming rectangular element with eight degrees of freedom per node........ 95

4. 2 Flow chart of the nonlinear transient analysis of a problem............................. 101

4. 3 Finite element meshes of laminated composite plates...................................... 102

4. 4 Effect of finite element modeling of CFRP composite plates; (a) Simply supported antisymmetric laminates, (b) Clamped antisymmetric laminates, (c) Simply supported symmetric laminates, (d) Clamped symmetric laminates103

4. 5 Center deflection predicted by the analytical and finite element methods for the case of symmetric cross-ply CFRP laminated plate........................................ 105

4. 6 Center displacements by the different plate theories for simply supported cross-ply CFRP laminated plates; (a) (m,90,0,90,0)$_s$, (b) (0,90,0,90,m)$_s$ ....... 106

4. 7 Effect of the lamina material properties on the damping of deflection in symmetric cross-ply laminated plate (m,90,0,90,0)$_s$ ........................................ 107

4. 8 Effect of the smart material layer position on the deflection for the symmetric cross-ply CFRP laminated plate................................................................. 107

4. 9 Effect of the thickness of smart material layers on the deflection damping characteristics of symmetric cross-ply laminated plate (m,90,0,90,0)$_s$............. 110

4. 10 Effect of the magnitude of the feedback coefficients on the suppression time for symmetric cross-ply CFRP laminated plate (m,90,0,90,0)$_s$ ...................... 111
<table>
<thead>
<tr>
<th>FIGURE</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.11</td>
<td>Center displacement for symmetric angle-ply CFRP laminated plate</td>
<td>113</td>
</tr>
<tr>
<td>4.12</td>
<td>Effect of the smart layer position for symmetric general angle-ply CFRP laminated plate</td>
<td>114</td>
</tr>
<tr>
<td>4.13</td>
<td>Effect of boundary conditions (a) Comparison of simply supported and clamped laminates, (b) smart layer positions</td>
<td>115</td>
</tr>
<tr>
<td>4.14</td>
<td>Nondimensionalized center deflection under uniform load</td>
<td>116</td>
</tr>
<tr>
<td>4.15</td>
<td>Nondimensionalized center deflection under suddenly applied uniform load</td>
<td>116</td>
</tr>
<tr>
<td>4.16</td>
<td>Nondimensionalized center displacement for simply supported laminated plates under sinusoidal and uniformly distributed loads</td>
<td>117</td>
</tr>
<tr>
<td>5.1</td>
<td>Six boundary conditions used in this study</td>
<td>119</td>
</tr>
<tr>
<td>5.2</td>
<td>Load-deflection curve for the different lamination schemes with SSSS boundary conditions under uniformly distributed loading: (a) $a/h=10$, (b) $a/h=100$</td>
<td>125</td>
</tr>
<tr>
<td>5.3</td>
<td>Load-deflection curve for the cross-ply laminates with the different boundary conditions under uniformly distributed sinusoidal loading: (a) $a/h=10$, (b) $a/h=100$</td>
<td>126</td>
</tr>
<tr>
<td>5.4</td>
<td>Effect of load intensity on the nonlinear transient responses for the symmetric cross-ply laminates with SSSS boundary conditions under uniformly distributed loading using TSDT: (a) $a/h=10$, (b) $a/h=100$</td>
<td>128</td>
</tr>
<tr>
<td>5.5</td>
<td>Effect of time increments on the nonlinear transient analyses for the symmetric cross-ply laminates with SSSS boundary conditions under uniformly distributed loading using TSDT: (a) $a/h=10$, (b) $a/h=100$</td>
<td>129</td>
</tr>
<tr>
<td>5.6</td>
<td>Effect of plate theories on the nonlinear transient responses for the symmetric cross-ply thick ($a/h=10$) laminates with SSSS boundary conditions under uniformly distributed loading, $q_0 = 5 \times 10^7$: (a) Details, (b) Nonlinear responses</td>
<td>130</td>
</tr>
</tbody>
</table>
5. 7 Effect of lamination schemes on the transient responses with SSSS boundary conditions under uniformly distributed loading: (a) $a/h = 10$ ($q_0 = 5 \times 10^3$), (b) $a/h = 100$ ($q_0 = 5 \times 10^7$) ........................................... 134

5. 8 Effect of loading conditions on the transient responses with SSSS boundary conditions: (a) cross-ply laminates ($a/h = 10$) with $q_0 = 5 \times 10^7$, (b) cross-ply laminates ($a/h = 100$) with $q_0 = 10^4$, (c) angle-ply laminates ($a/h = 10$) with $q_0 = 5 \times 10^7$, (d) angle-ply laminates ($a/h = 100$) with $q_0 = 10^4$, (e) general angle-ply laminates ($a/h = 10$) with $q_0 = 5 \times 10^7$, (f) the asymmetric general angle-ply laminates ($a/h = 100$) with $q_0 = 10^4$ ........................................... 135

5. 9 Effect of boundary conditions on the transient responses under uniformly distributed loading: (a) cross-ply laminates ($a/h = 10$) with $q_0 = 10^7$, (b) cross-ply laminates ($a/h = 100$) $q_0 = 10^4$, (c) angle-ply laminates ($a/h = 10$) with $q_0 = 10^7$, (d) angle-ply laminates ($a/h = 100$) $q_0 = 10^4$, (e) general angle-ply laminates ($a/h = 10$) with $q_0 = 10^7$, (f) the asymmetric general angle-ply laminates ($a/h = 100$) $q_0 = 10^4$ ........................................... 139

5. 10 Effect of shell theory on the nonlinear transient analysis for the symmetric cross-ply cylindrical shell: (a) thin shell ($a/h = 100$), (b) thick shell ($a/h = 10$) ........................................... 145

5. 11 Effect of time increments on the nonlinear transient analysis for the symmetric cross-ply cylindrical shell ($R_2/a = 200$): (a) thin shell ($a/h = 100$), (b) thick shell ($a/h = 10$) ........................................... 148

5. 12 Effect of $R_2/a$ for cylindrical cross-ply shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h = 100$), (b) thick shell ($a/h = 10$) ......... 151

5. 13 Effect of shell type for the cross-ply thick ($a/h = 10$) shell under uniform load in SSSS boundary condition; (a) $R_2/a = 5$, (b) $R_2/a = 50$ ........................................... 152

5. 14 Effect of shell type for the cross-ply thin ($a/h = 100$) shell under uniform load in SSSS boundary condition; (a) $R_2/a = 5$, (b) $R_2/a = 50$ ........................................... 153
<table>
<thead>
<tr>
<th>FIGURE</th>
<th>Effect of $R_2/a$ for spherical cross-ply shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h=100$), (b) thick shell ($a/h=10$)</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. 15</td>
<td>Effect of $R_2/a$ for doubly-curved cross-ply shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h=100$), (b) thick shell ($a/h=10$)</td>
<td>155</td>
</tr>
<tr>
<td>5. 16</td>
<td>Effect of lamination schemes for cylindrical shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h=100$), (b) thick shell ($a/h=10$)</td>
<td>158</td>
</tr>
<tr>
<td>5. 17</td>
<td>Effect of lamination schemes for spherical shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h=100$), (b) thick shell ($a/h=10$)</td>
<td>159</td>
</tr>
<tr>
<td>5. 18</td>
<td>Effect of lamination schemes for doubly-curved shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h=100$), (b) thick shell ($a/h=10$)</td>
<td>160</td>
</tr>
<tr>
<td>5. 19</td>
<td>Effect of boundary conditions for cross-ply cylindrical shell under uniform load; (a) thin shell ($a/h=100$, $R_2/a=100$), (b) thick shell ($a/h=10$, $R_2/a=50$)</td>
<td>162</td>
</tr>
<tr>
<td>5. 20</td>
<td>Effect of boundary conditions for cross-ply spherical shell under uniform load; (a) thin shell ($a/h=100$, $R_2/a=20$), (b) thick shell ($a/h=10$, $R_2/a=10$)</td>
<td>163</td>
</tr>
<tr>
<td>5. 21</td>
<td>Effect of boundary conditions for cross-ply doubly-curved shell under uniform load; (a) thin shell ($a/h=100$, $R_2/a=10$), (b) thick shell ($a/h=10$, $R_2/a=20$)</td>
<td>164</td>
</tr>
<tr>
<td>5. 22</td>
<td>Effect of loading conditions for cross-ply cylindrical shell in SSSS boundary condition; (a) thin shell ($a/h=100$, $R_2/a=5$), (b) thick shell ($a/h=10$, $R_2/a=5$)</td>
<td>166</td>
</tr>
<tr>
<td>5. 23</td>
<td>Effect of the plate theories on static behaviors of simply supported thick cross-ply laminates under mechanical and thermal load; (a) CLPT, (b) FSDT, (c) TSDT</td>
<td>169</td>
</tr>
<tr>
<td>5. 24</td>
<td>Thermomechanical behavior of simply supported thin cross-ply laminates</td>
<td>170</td>
</tr>
<tr>
<td>FIGURE</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>5.26</td>
<td>Effect of the laminations on static behaviors of simply supported laminated plates under mechanical and thermal load; (a) Thick laminate, (b) Thin laminate</td>
<td>172</td>
</tr>
<tr>
<td>5.27</td>
<td>Effect of $R_2/a$ on static behavior of cross-ply cylindrical shell under thermomechanical load: (a) Thick shell ($a/h=10$, $\Delta T=100{}^\circ C$), (b) Thin shell ($a/h=100$, $\Delta T=10{}^\circ C$)</td>
<td>173</td>
</tr>
<tr>
<td>5.28</td>
<td>Temperature effect on static behavior of cross-ply cylindrical shell under thermomechanical load: (a) Thick shell ($a/h=10$, $R_2/a=2$), (b) Thin shell ($a/h=100$, $R_2/a=2$)</td>
<td>174</td>
</tr>
<tr>
<td>5.29</td>
<td>Temperature effect on static behavior of cross-ply doubly-curved shell under thermomechanical load: (a) Thick shell ($a/h=10$, $R_2/a=10$), (b) Thin shell ($a/h=100$, $R_2/a=10$)</td>
<td>175</td>
</tr>
<tr>
<td>5.30</td>
<td>Temperature effect on static behavior of cross-ply spherical shell under thermomechanical load: (a) Thick shell ($a/h=10$, $R_2/a=2$), (b) Thin shell ($a/h=100$, $R_2/a=2$)</td>
<td>176</td>
</tr>
<tr>
<td>5.31</td>
<td>Transient behavior of the cross-ply laminates with simply supported boundary condition; (a) Thick laminate under impact load, (b) Thick laminate under uniform load, (c) Thin laminate under impact load, (d) Thin laminate under uniform load</td>
<td>178</td>
</tr>
<tr>
<td>5.32</td>
<td>Transient behavior of the angle-ply laminates with simply supported boundary condition; (a) Thick laminate under impact load, (b) Thick laminate under uniform load, (c) Thin laminate under impact load, (d) Thin laminate under uniform load</td>
<td>180</td>
</tr>
<tr>
<td>5.33</td>
<td>Transient behavior of the general laminates with simply supported boundary condition; (a) Thick laminate under impact load, (b) Thick laminate under uniform load, (c) Thin laminate under impact load, (d) Thin laminate under uniform load</td>
<td>182</td>
</tr>
<tr>
<td>5.34</td>
<td>Effect of the elastic materials on the transient behavior of the thick cross-ply laminates with simply supported boundary condition: (a) Thick laminate under uniform load, (b) GrEp case, (c) GlEp case, (d) BrEp case</td>
<td>185</td>
</tr>
</tbody>
</table>
5. 35 Effect of the elastic materials on the transient behavior of the thin cross-ply laminates with simply supported boundary condition; (a) Thin laminate under uniform load, (b) GrEp case, (c) GlEp case, (d) BrEp case.................... 187

5. 36 Effect of the boundary conditions on the transient behavior of the cross-ply laminates; (a) Thick laminate under thermomechanical load, (b) Thin laminate under thermomechanical load ............................................... 190

5. 37 Temperature effect on nonlinear transient behavior of cross-ply cylindrical shell ($R_2/a = 2$) under thermomechanical load: (a) Thick shell ($a/h = 10$), (b) Thin shell ($a/h = 100$)........................................................................... 192

5. 38 Temperature effect on nonlinear transient behavior of cross-ply cylindrical shell ($R_2/a = 200$) under thermomechanical load: (a) Thick shell ($a/h = 10$), (b) Thin shell ($a/h = 100$)........................................................................... 193

5. 39 Effect of $R_2/a$ on nonlinear transient behavior of cross-ply shell under thermomechanical load: (a) Cylindrical shell ($a/h = 100$, $\Delta T = 10^\circ C$), (b) Spherical shell ($a/h = 100$, $\Delta T = 10^\circ C$)........................................................................... 195

5. 40 Effect of shell types on nonlinear transient behavior of cross-ply shell under thermomechanical load: (a) Thick shell ($a/h = 10$, $\Delta T = 100^\circ C$), (b) Thin shell ($a/h = 100$, $\Delta T = 10^\circ C$) .............................................................. 197

5. 41 Effect of boundary conditions on nonlinear transient behavior of cross-ply shell under thermomechanical load: (a) Thick shell ($a/h = 10$, $\Delta T = 150^\circ C$, $R_2/a = 100$), (b) Thin shell ($a/h = 100$, $\Delta T = 10^\circ C$, $R_2/a = 20$)................................. 198
# LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. 1</td>
<td>Material properties of magnetostrictive and elastic composite materials</td>
</tr>
<tr>
<td>3. 2</td>
<td>Inertial coefficients of symmetric cross-ply (m,90,0,90,0)s laminated plate</td>
</tr>
<tr>
<td>3. 3</td>
<td>Damping coefficients and frequencies for different materials</td>
</tr>
<tr>
<td>3. 4</td>
<td>Eigenvalue $(-\alpha_d \pm \omega_d)$ obtained through the different plate theories</td>
</tr>
<tr>
<td>3. 5</td>
<td>Suppression times for different CFRP laminates</td>
</tr>
<tr>
<td>3. 6</td>
<td>Vibration suppression characteristics for the different lamina thickness</td>
</tr>
<tr>
<td>3. 7</td>
<td>Vibration suppression ratio for the different laminates</td>
</tr>
<tr>
<td>3. 8</td>
<td>Suppression time for two control gains for different laminates</td>
</tr>
<tr>
<td>3. 9</td>
<td>$T_\alpha$ and $e_r$ parameter for different CFRP laminates</td>
</tr>
<tr>
<td>3. 10</td>
<td>Inertial coefficients of symmetric cross-ply (m,90,0,90,0)s laminated shell</td>
</tr>
<tr>
<td>3. 11</td>
<td>Effect of the mode on the Eigenvalues of the symmetric cross-ply (m,90,0,90,0)s CFRP spherical shells with $R_z/a = 10$</td>
</tr>
<tr>
<td>3. 12</td>
<td>Effect of the lamination scheme on the Eigenvalues of the symmetric cross-ply CFRP spherical shells with $R_z/a = 3$</td>
</tr>
<tr>
<td>3. 13</td>
<td>Eigenvalues of the symmetric cross-ply (m,90,0,90,0)s CFRP laminated shells by Donnell shell theory</td>
</tr>
<tr>
<td>3. 14</td>
<td>Eigenvalues of the symmetric cross-ply (m,90,0,90,0)s CFRP laminated shells by Sanders shell theory</td>
</tr>
<tr>
<td>3. 15</td>
<td>Selected center displacement versus time of the symmetric cross-ply (m,90,0,90,0)s CFRP laminated shells with $R_z/a = 5$ by Sanders shell theory</td>
</tr>
<tr>
<td>3. 16</td>
<td>Maximum transverse deflection and vibration suppression time for the symmetric cross-ply (m,90,0,90,0)s CFRP laminated shells</td>
</tr>
<tr>
<td>TABLE</td>
<td>Page</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>3.17</td>
<td>Vibration suppression characteristics for the thin symmetric cross-ply (m,90,0,90,0)s laminated shells with the different composite materials</td>
</tr>
<tr>
<td>3.18</td>
<td>Vibration suppression characteristics for the thick symmetric cross-ply (m,90,0,90,0)s laminated shells with the different composite materials</td>
</tr>
<tr>
<td>4.1</td>
<td>Deflection suppression time for the different smart layer positions on the symmetric cross-ply CFRP laminated plate (m,90,0,90,0)s</td>
</tr>
<tr>
<td>4.2</td>
<td>Suppression times for the different smart layer thicknesses in symmetric cross-ply laminated plate (m,90,0,90,0)s</td>
</tr>
<tr>
<td>5.1</td>
<td>Nondimensional center deflection (w/h) for symmetric cross-ply and angle-ply thick (a/h = 10) laminates under different load and boundary conditions</td>
</tr>
<tr>
<td>5.2</td>
<td>Nondimensional center deflection (w/h) for symmetric cross-ply and angle-ply thin (a/h = 100) laminates under different load and boundary conditions</td>
</tr>
<tr>
<td>5.3</td>
<td>Nondimensional center deflection (w/h) for symmetric and asymmetric general angle-ply thick (a/h = 10) laminates under different load and boundary conditions</td>
</tr>
<tr>
<td>5.4</td>
<td>Nondimensional center deflection (w/h) for symmetric and asymmetric general angle-ply thin (a/h = 100) laminates under different load and boundary conditions</td>
</tr>
<tr>
<td>5.5</td>
<td>Nondimensionalized transverse deflections versus time of the simply supported cross-ply laminates subjected to uniformly distributed load (a/h = 10; (\Delta t=0.0001); 4x4L mesh)</td>
</tr>
<tr>
<td>5.6</td>
<td>Nondimensionalized transverse deflections versus time of the simply supported cross-ply laminates subjected to uniformly distributed load (a/h = 100; (\Delta t=0.0005); 4x4L mesh)</td>
</tr>
<tr>
<td>5.7</td>
<td>Nondimensionalized maximum transverse deflections and deflection suppression time for different lamination schemes</td>
</tr>
<tr>
<td>5.8</td>
<td>Nondimensionalized maximum transverse deflections and deflection suppression time by nonlinear TSDT</td>
</tr>
<tr>
<td>TABLE</td>
<td>Page</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>5. 9</td>
<td>Selected center displacement of the symmetric cross-ply thick ($a/h = 10$) shell with $R_2/a = 2$ by Donnell and Sanders shell theories</td>
</tr>
<tr>
<td>5. 10</td>
<td>Selected center displacement of the symmetric cross-ply thin ($a/h = 100$) shell with $R_2/a = 2$ by Donnell and Sanders shell theories</td>
</tr>
<tr>
<td>5. 11</td>
<td>Maximum transverse deflection and deflection suppression time for the symmetric cross-ply cylindrical shells</td>
</tr>
<tr>
<td>5. 12</td>
<td>Maximum transverse deflection and deflection suppression time for the symmetric cross-ply shells by Sanders shell theory</td>
</tr>
<tr>
<td>5. 13</td>
<td>Vibration suppression characteristics for the symmetric cross-ply laminated shells with the different lamination</td>
</tr>
<tr>
<td>5. 14</td>
<td>Nondimensionalized maximum transverse deflections and deflection suppression time by nonlinear TSDT</td>
</tr>
<tr>
<td>5. 15</td>
<td>Nondimensionalized center displacements of the simply supported thick cross-ply laminate under thermomechanical loading by CLPT, FSDT, and TSDT</td>
</tr>
<tr>
<td>5. 16</td>
<td>Nonlinear nondimensionalized center displacements of the simply supported thin cross-ply laminate under thermomechanical loading by TSDT</td>
</tr>
<tr>
<td>5. 17</td>
<td>Nondimensionalized maximum center displacements and vibration suppression time of cross-ply laminates for different elastic materials</td>
</tr>
<tr>
<td>5. 18</td>
<td>Nondimensionalized maximum center displacements and vibration suppression time of CFRP cross-ply laminates under thermomechanical loading</td>
</tr>
<tr>
<td>5. 19</td>
<td>Maximum transverse deflection and vibration suppression time for the symmetric cross-ply cylindrical shells</td>
</tr>
<tr>
<td>5. 20</td>
<td>Nondimensionalized maximum center displacements and vibration suppression time of cross-ply laminates for different shell types</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

1.1. General

Composite materials are widely used in a variety of structures, including army and aerospace vehicles, buildings and smart highways (i.e. civil infrastructure applications) as well in sports equipment and medical prosthetics. Laminated composite structures consist of several layers of different fiber-reinforced laminae bonded together to obtain desired structural properties (e.g. stiffness, strength, wear resistance, damping, and so on). The desired structural properties are achieved by varying the lamina thickness, lamina material properties, and stacking sequence (Reddy 2004 b).

The increased use of laminated composites in various types of structures led to considerable interests in their analysis. Composite materials exhibit high strength-to-weight and stiffness-to-weight ratios, which make them ideally suited for use in weight-sensitive structures. This weight reduction of structures leads to improvement of their structural performance, especially in space applications.

With the availability of functional materials and feasibility of embedding or bonding them to composite structures, new smart structural concepts are emerging to be attractive for potentially high-performance structural applications (Maugin 1988, Gandhi and Thompson 1992, and Srinivasan and McFarland 2001). A smart structure is the structure that has surface mounted or embedded sensors and actuators so that it has a capability to sense and take corrective action. Numerous conferences, workshops, and journals dedicated to smart materials and structures stand testimony to this growth. The technological implications of this class of materials and structures are immense: structures that monitor their own health, process monitoring, vibration isolation and control, medical applications, damage detection, noise control and shape control.

This dissertation follows the style and format of Journal of Structural Engineering.
As applications of active vibration/deflection controls in aerospace, automobile industries and building applications, the smart structures have received considerable attention (See Lowey 1997 for example). Vibration and shape control of structures is essential to achieve the desirable performance in modern structural systems. Advances made in design and manufacturing of smart structure systems improve the efficiency of the structural performance.

1.2. Smart Materials

Two of the basic elements of a smart structural system are actuators and sensors. These sensors and actuators may be either mounted on the flexible passive structure or embedded inside it. The sensing and control of flexible structures are primarily performed with the help of sensors and actuators which are made of smart materials. Smart materials can be divided into two main types: Passive smart materials are those that respond to external change without assistance. These materials are useful when there is only one correct response. Active smart materials utilize the feedback loop and recognize the change and response to the actuator circuit.

The commonly used smart materials are piezoelectric materials, magnetostrictive materials, electrostrictive materials, shape memory alloys, fiber optics, and electrorheological fluids. Each smart material has a unique advantage of its own. Piezoelectric materials deform by mechanical loads, and deformation occurs due to the application of electric potential by a converse effect (Bailey and Hubbard 1985, Uchino 1986, Crawley and Luis 1987, and Yellin and Shen 1996). Examples of piezoelectric materials are Rochelle salt, quartz, and PZT (Pb(Zr,Ti)O₃). Piezoelectric materials exhibit a linear relationship between the electric field and strains up to 100 V/mm. However, the relationship becomes nonlinear for large fields, and the material exhibits hysteresis (Uchino 1986). Furthermore, piezoelectric materials show dielectric aging and hence lack reproducibility of strains, i.e., a drift from zero state of strain is observed under cyclic electric field conditions (Cross and Jang 1988). Magnetostrictive materials produce
deformation (displacement) under magnetic field. Magnetostriction is the development of large mechanical deformations due to the rotations of small magnetic domains when subjected to an external magnetic field (Pratt and Flatau 1995). The shape memory alloys are suitable for static shape control and low frequency dynamic applications (Baz, Imam and McCoy 1990, and Anders, Rogers and Fuller 1991). The electro-rheologic fluids are a class of specially formulated suspensions which undergo a change in the resistance to flow (i.e. viscosity) due to an applied electric field (Choi, Sprecher and Conrad 1990). Among these materials, the piezoelectric and magnetostrictive materials have the capability to serve as sensing and actuation materials.

The magnetostrictive material selected for this study, Terfenol-D, has some dominant advantages as actuators and sensors over other materials. Terfenol-D is a commercially available magnetostrictive material in the form of particles deposited on thin sheets and it is an alloy of terbium, iron, and dysprosium. It can serve both as actuator and sensor and produce strains up to $2500 \, \mu m$, which is 10 times more than a piezoceramic material (Newnham 1993 and Kleinke and Unas 1994 a, b). It also has high energy density, negligible weight, and point excitation with a wide frequency bandwidth (Goodfriend and Shoop 1992, Dapino, Flatau, and Calkins 1997, Flatau, Dapino, and Calkins 1998, and Duenas and Carman 2000).

Mechanics of smart material systems involve coupling between electric, magnetic, thermal, and mechanical effects. In addition to this coupling, it may be necessary to account for geometric and material nonlinearities. Toupin (1956) was the first one to consider the material nonlinearity in electro-elastic formulations. Knops (1963) presented a two-dimensional theory of electrostriction and solved a simplified boundary value problem using complex potentials. Rotationally invariant nonlinear thermo-electro-elastic equations were derived by Tiersten (1971, 1993), Baumhauer and Tiersten (1973), and by Nelson (1978). Tiersten (1993) has stressed the importance of including nonlinear terms in the constitutive relations, particularly at large fields. Joshi (1991) presented nonlinear constitutive relations for piezoceramic materials.
Among the currently available sensors and actuators, the smallest ones are of the order of few millimeters. The reduction in size has tremendous technological benefits; however, clear understanding of reliability and system integrity is vital to the efficient and optimum use of these material systems. As dimensions get smaller, induced electro-thermo-mechanical fields get larger. Therefore, the material and geometric nonlinearities should be accounted for (Tzou, Bao, and Ye 1994, Carman and Mitrovic 1995, Kannan and Dasgupta 1997, Smith 1998, and Armstrong 2000).

1.3. Background Literature

The analysis of laminated composites is difficult due to the anisotropic structural behavior and complicated constituent interactions (Reddy 2004 b). Since the transverse shear modulus of composite material is usually very low compared to the in-plane modulus, the shear deformation effects are more pronounced in laminated composites subjected to transverse loads than in the isotropic plates under similar loading. A number of methodologies for the analysis of laminated composite plates and shells are available. The three-dimensional elasticity theory provides the most accurate solutions while the traditional plate and shell theories; namely, the classical theory and first-order and third-order shear deformation theories provide simpler and but adequate solutions for most applications.

The equivalent single-layer (ESL) theories are derived from the 3-D elasticity theory by making suitable assumptions concerning the kinematics of deformation through the thickness of the laminate. These assumptions allow the reduction of a 3-D problem to a 2-D problem. The simplest equivalent single-layer theories are the classical laminate plate theory (CLPT) (Whitney and Leissa 1969 and Whitney 1970), and the first-order shear deformation theory (FSDT) (Whitney and Pagano 1970, Reddy and Chao 1981, Reddy 1997, 2004 b). These theories adequately describe the kinematics of most laminated plates. However, for better inter-laminar stress distributions, higher-order theories need to be used. Reddy (1984a, b, 1997, 1999a, b) developed the equations of motion for the third-order
shear deformation theory for laminated composites using the principle of virtual displacements. The third-order shear deformation theory (TSDT) represents the plate kinematics better and yields better inter-laminar stress distributions. Quadratic variations of the transverse shear strains and stresses through the thickness of the laminate avoid the need for shear correction coefficients that are required in the first-order shear deformation theory.

Surveys of various shell theories can be found in the work of Naghdi (1956) and Bert (1980). Many of these theories were developed originally for thin shells, and are based on the Kirchhoff-Love kinematic hypothesis (classical shell theory) that straight lines normal to the undeformed midsurface remain straight and normal to the midsurface after deformation (i.e. the transverse shear strains are neglected). These theories often yield sufficiently accurate results when the material anisotropy is not severe. However, the classical theories are inadequate for a deformation description of shallow layered composite shells.

Dong and Tso (1972) and Dong, Pister, and Taylor (1962) presented the theory of laminated thin shells with orthotropic and anisotropic materials. Ambartsumyan (1964) analyzed the laminated orthotropic shell and considered the bending-stretching coupling effects. Gulati and Essenber (1967) and Zukas and Vinson (1971) considered the effect of transverse shear deformation and transverse isotropy in the cylindrical shell. The first-order shear deformation theory of general shells can be found in Sanders (1959), Koiter (1960), and Kraus (1967). Reddy (1984c, 2004b) presented a shear deformation shell theory for laminated composite shells.

Higher-order shear deformation shell theories are also developed by Whitney and Sun 1974, Reddy and Liu (1985), Librescu, Khdeir, and Frederick (1989). They used the principle of virtual displacements to derive the equations of motion.

For nonlinear shell theory, Donnell (1935) presented a set of shell equations for cylindrical shells and his approximations were extended to shallow shells of general geometry which is known as Donnell-Mushtari-Vlasov (DMV) equations. Sanders (1959,
1963) and Koiter (1966) developed more refined nonlinear theory of shell, the Sanders-Koiter equations (also referred to as Sanders theory; Reddy 2004 b).

Bailey and Hubbard (1985) and Crawley and Luis (1987) demonstrated the feasibility of using piezoelectric actuators for free vibration reduction of cantilever beams. A self-sensing active constrained damping layer treatment for Euler-Bernoulli beam was studied by Yellin and Shen (1996). Baz, Iman and McCoy (1990) have investigated vibration control using shape memory alloy actuators and their characterization. Anders, Rogers and Fuller (1991) have analytically demonstrated their control of sound radiation from shape memory alloy hybrid composite panels. By changing the elastic properties of the host structure, Choi, Sprecher and Conrad (1990) demonstrated the vibration reduction effects of electrorheological fluid actuators in a composite beam.

Compared to other smart materials, the magnetostrictive material has significant advantages as actuators. A commercially available magnetostrictive material Terfenol-D is an alloy of terbium, iron, and dysprosium. The use of Terfenol-D particle sheets for vibration suppression has some advantages over other smart materials. In particular, it has easy embedability into host materials, such as the modern Carbon Fiber-Reinforced Polimeric (CFRP) composites, without significantly effecting the structural integrity.

Considerable effort is spent to understand the interaction between magnetostrictive layers and structural composite laminae, and the feasibility of using magnetostrictive materials for active vibration suppressions. Goodfriend and Shoop (1992), Hudson, Busbridge, and Piercy (1999, 2000), and Lim et al. (1999) reviewed the material properties of magnetostrictive material, Terfenol-D, with regard to its use in static and dynamic applications. Anjanaappa and Bi (1993, 1994 a, b) investigated the feasibility of using embedded magnetostrictive mini actuators for smart structural applications, such as vibration suppression of beams. Bryant, Fernandez and Wang (1993) presented results of an experiment in which a rod of magnetostrictive Terfenol-D was used in the dual capacity passive structural support element and an active vibration control actuator. A self-sensing magnetostrictive actuator design based on a linear model of magnetostrictive transduction
for Terfenol-D was developed and analyzed by Pratt and Flatau (1995) and Jones and Garcia (1996). Eda, et al. (1995), Anjanappa and Wu (1996), Krishna Murty, Anjanappa and Wu (1997) and Krishna Murty, et al. (1998) proposed magnetostrictive actuators that take advantage of easy embedability and remote excitation capability of magnetostrictive particle sheets as new actuators. Using a combination of magnetostrictive and ferromagnetic alloys, the combined passive and active damping strategy was proposed by Bhattacharya et al. (2000). Recently, Pulliam, Mc Knight, and Carman (2002) provided magnetostrictive particulate technology in damping applications.


1.4. Present Study

The main goal of this research is to develop efficient computational procedures for analyzing laminated composite plate and shell structures with embedded smart layers, while accounting for geometric nonlinearity (von Kármán sense) and thermo-mechanical effects. The temperature field is assumed to be uniformly distributed through the surface but vary through the thickness of the laminate, and the material properties are not dependent on temperature. Both nonlinear Donnell shell theory and Sanders nonlinear shell theory are used to derive the strains and the equations of motion by third-order shear deformation theory. Three different shell types, spherical, cylindrical, and doubly-curved shells, are
considered. A simple negative velocity feedback control in a closed loop is used to actively control the dynamic response of the structure. Newton-Raphson iteration method is used to solve the nonlinear algebraic equations resulting from the finite element approximation in space and Newmark’s time integration scheme.

The present study is divided into three parts. First, the theoretical formulations for laminated composite plates and shells are developed. In the second part, the finite element formulations based on the previously developed theories are developed, and in the third part, various parametric studies are carried out to determine the response of laminated composite structures with smart material layers.

The development of theoretical formulations of composite plates and shells with smart material layers is based on a unified shear deformation theory that includes the CLPT, FSDT, and TSDT. The von Kármán strains, constitutive relations, equations of motions and negative velocity feedback are presented in Section 2. In Section 3, analytical solutions for simply supported laminated composite plate/shell structures are presented using the linear version of the plate and shell theories.

Development of the general finite element model for laminated composite plates and shells is discussed in Section 4; linear and nonlinear displacement finite element models for composite plate and shell structures are presented and computer implementation is discussed. The numerical results of linear analysis of laminated composite plates and shells are reported at the end of Section 4.

Section 5 is devoted to numerical results from the various parametric studies of laminated composite plate and shell structures, respectively. The parametric studies conducted include the effect of material properties, lamination schemes, smart layer position, boundary conditions, loading conditions, plate and shell theories, and so on.

Finally, the conclusions and recommendations for the future study of this study are presented in Section 6.
2. THEORETICAL FORMULATIONS

2.1. Methodology

The equivalent single-layer (ESL) models provide sufficiently accurate descriptions of global response for the thin to moderately thick laminates, e.g., gross deflection and fundamental vibration frequencies and associated mode shapes (Reddy 2004b). The third-order shear deformation theory, which is based on the same assumptions as the classical and first-order plate theories, except for the assumptions on the straightness and normality of a transverse normal during deformation, is used mainly in this study. The reason for expanding the displacements up to the cubic term in the thickness coordinate is to have quadratic variation of the transverse shear stresses and strains through the thickness. This avoids the need for shear correction coefficients as required in the first-order shear deformation plate theory to account for the parabolic variation of the actual shear stress through the thickness of the laminate.

2.2. Displacement Field and Strains

2.2.1. Displacement Field and Strains for Plates

The plate under consideration is composed of a finite number of orthotropic layers of uniform thickness (see Figure 2.1). We begin with the following displacement field (Reddy, 2004):

\[ u(x, y, z, t) = u_0 + z\phi_z + z^2\psi_1 + z^3\theta_1 \]  \hspace{1cm} (2.1.a)

\[ v(x, y, z, t) = v_0 + z\phi_y + z^2\psi_2 + z^3\theta_2 \]  \hspace{1cm} (2.1.b)

\[ w(x, y, z, t) = w_0 \]  \hspace{1cm} (2.1.c)

where \( t \) is time, \((u, v, w)\) are the displacements along the \((x, y, z)\) coordinates, \((u_0, v_0, w_0)\) are the displacements of a point on the middle surface and \( \phi_z(x, y, 0, t) = \frac{\partial u}{\partial z} \) and
\( \phi_y(x, y, 0, t) = \partial v / \partial z \) are the rotations at \( z = 0 \) of normals to the mid-surface about the \( y \) and \( x \) axes, respectively.

The particular choice of the displacement field in the equation is dictated by the desire to represent the transverse shear strains by quadratic functions of the thickness coordinate, \( z \), and by the requirement that the transverse normal strain be zero. The function \( \psi_i \) and \( \theta_j \) are determined using the condition that the transverse shear stresses, \( \sigma_{xz} \) and \( \sigma_{yz} \) vanish on the top and bottom surfaces of the plate.

The displacement field for the third-order shear deformation theory (TSDT) of plate now can be expressed in the form (Reddy 2004 b)

\[
\begin{align*}
    u(x, y, z, t) &= u_0(x, y, t) + z\phi_x(x, y, t) - c_1z^3 \left( \phi_x + \frac{\partial w_0}{\partial x} \right) \\
    v(x, y, z, t) &= v_0(x, y, t) + z\phi_y(x, y, t) - c_1z^3 \left( \phi_y + \frac{\partial w_0}{\partial y} \right) \\
    w(x, y, z, t) &= w_0(x, y, t)
\end{align*}
\]  

(2.2.a, 2.2.b, 2.2.c)

where the constant \( c_1 \) is given by \( c_1 = 4/3h^2 \), \( h \) being the total thickness of the laminate.

![Figure 2.1 Geometry and coordinate system of laminated plate](image)
The nonzero von Kármán nonlinear strains in the TSDT are

\[
\begin{align*}
\{ \varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy} \} &= \{ \varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{xy}^0 \} + z \left\{ \varepsilon_{xx}^1, \varepsilon_{yy}^1, \gamma_{xy}^1 \right\} + z^3 \left\{ \varepsilon_{xx}^3, \varepsilon_{yy}^3, \gamma_{xy}^3 \right\} \\
&= \mathbf{\varepsilon}^0 + z \mathbf{\varepsilon}^1 + z^3 \mathbf{\varepsilon}^3 
\end{align*}
\]

(2.3.a)

\[
\begin{align*}
\{ \gamma_{yz}, \gamma_{xz} \} &= \{ \gamma_{yz}^0, \gamma_{xz}^0 \} + z^2 \left\{ \gamma_{yz}^2, \gamma_{xz}^2 \right\} \\
&= \mathbf{\gamma}^0 + z^2 \mathbf{\gamma}^2 
\end{align*}
\]

(2.3.b)

where

\[
\begin{align*}
\{ \varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{xy}^0 \} &= \begin{align*}
\frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \\
\frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} 
\end{align*} \\
&= \mathbf{\varepsilon}^0 
\end{align*}
\]

(2.4.a)

\[
\begin{align*}
\{ \varepsilon_{xx}^1, \varepsilon_{yy}^1, \gamma_{xy}^1 \} &= \begin{align*}
\frac{\partial \phi_x}{\partial x} \\
\frac{\partial \phi_y}{\partial y} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}
\end{align*} \\
&= \mathbf{\varepsilon}^1 
\end{align*}
\]

(2.4.b)

\[
\begin{align*}
\{ \varepsilon_{xx}^3, \varepsilon_{yy}^3, \gamma_{xy}^3 \} &= -c_1 \begin{align*}
\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\
\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w_0}{\partial x \partial y}
\end{align*} \\
&= \mathbf{\varepsilon}^3 
\end{align*}
\]

(2.4.c)
\[ \begin{align*}
\{\gamma_{yz}^0\} &= \left\{ \frac{\phi_y}{\partial y} + \frac{\partial w_0}{\partial y} \right\} \\
\{\gamma_{xz}^0\} &= \left\{ \frac{\phi_x}{\partial x} + \frac{\partial w_0}{\partial x} \right\}
\end{align*} \]  

(2.4.d) \hspace{1cm} (2.4.e)

and \( c_2 = \frac{4}{h^2}. \)

Figure 2.2 Geometry and coordinate system of laminated shell

2.2.2. Kinetics of Shells

Let \((\xi_1, \xi_2, \zeta)\) denote the orthogonal curvilinear coordinates such that the \(\xi_1\) – and \(\xi_2\) – curves are lines of curvature on the mid-surface \(\zeta = 0\), and \(\zeta\) – curves are straight lines perpendicular to the surface \(\zeta = 0\) (see Figure 2.2). For cylindrical, spherical, and doubly-curved shells discussed in this study, the lines of principal curvature coincide with
the coordinate lines. The values of the principal radii of curvature of the middle surface are denoted by $R_1$ and $R_2$. For additional details consult the textbook by Reddy (2004 b).

The total tensile force on the differential element in the $\xi_1$ direction is $N_1 \alpha_2 d\xi_2$. This force can be computed by integrating $\sigma_1 dA$ over the thickness of the shell.

$$N_1 \alpha_2 d\xi_2 = \int_{-h/2}^{h/2} \sigma_1 dA_2 = \alpha_2 \int_{-h/2}^{h/2} \sigma_1 \left(1 + \frac{\xi}{R_2}\right) d\xi_2 d\zeta$$

(2.5)

where $N_1$ is the tensile force per unit length along a $\xi_1$ coordinate line, $\alpha_2$ is the surface metric, and $h$ is the thickness of the shell.

$$N_1 = \int_{-h/2}^{h/2} \sigma_1 \left(1 + \frac{\xi}{R_2}\right) d\zeta$$

(2.6)

The remaining stress resultants per unit length can be derived in the similar way (Reddy 2004 b). The complete set of force and moment resultants is given by

$$\begin{bmatrix} N_1 \\ N_2 \\ N_{12} \\ N_{21} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_1 \left(1 + \frac{\xi}{R_2}\right) \\ \sigma_2 \left(1 + \frac{\xi}{R_1}\right) \\ \sigma_6 \left(1 + \frac{\xi}{R_2}\right) \\ \sigma_6 \left(1 + \frac{\xi}{R_1}\right) \end{bmatrix} d\zeta$$

(2.7.a)

and

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_5 \left(1 + \frac{\xi}{R_2}\right) \\ \sigma_4 \left(1 + \frac{\xi}{R_1}\right) \end{bmatrix} d\zeta$$

(2.7.b)
The shear stress resultants \( N_{12} \) and \( N_{21} \), and the twisting moments \( M_{12} \) and \( M_{21} \) are, in general, not equal. However, for shallow shell (\( h/R_1, h/R_2 \) less than 1/20) \( \zeta/R_1 \) and \( \zeta/R_2 \) can be neglected in comparison with unity so that one has \( N_{12} = N_{21} = N \) and \( M_{12} = M_{21} = M \).

### 2.2.3. Displacement Field and Strains for Shells

The following form of the displacement field is assumed consistent with moderate thick shell assumptions (Reddy 2004 b):

\[
\begin{align*}
\sigma \phi \xi, \zeta, \Theta = & \left( u_0 + \zeta \phi_1 + \zeta^2 \psi_1 + \zeta^3 \Theta_1 \right) \\
\sigma \psi \xi, \zeta, \Theta = & \left( v_0 + \zeta \phi_2 + \zeta^2 \psi_2 + \zeta^3 \Theta_2 \right) \\
\sigma \theta \xi, \zeta, \Theta = & w_0
\end{align*}
\]

where \( (\phi_1, \phi_2), (\psi_1, \psi_2), \) and \( (\Theta_1, \Theta_2) \) are functions to be determined. The function \( \psi_1 \) and \( \psi_2 \) can be determined by imposing traction free boundary conditions on the top and bottom surfaces of the laminated shell:

\[
\sigma_\xi \left( \xi_1, \xi_2, \pm \frac{h}{2}, t \right) = 0
\]
The middle surface kinematic relations for Donnell shell theory are

$$\sigma_4 \left( \xi_1, \xi_2, \frac{h}{2}, t \right) = 0$$

(2.9.b)

The middle surface kinematic relations for Donnell shell theory are

$$\varepsilon_1 = \frac{\partial u}{\partial X_1} + \frac{w}{R_1} + \frac{1}{2} \left( \frac{\partial w}{\partial X_1} \right)^2$$

(2.10.a)

$$\varepsilon_2 = \frac{\partial v}{\partial X_2} + \frac{w}{R_2} + \frac{1}{2} \left( \frac{\partial w}{\partial X_2} \right)^2$$

(2.10.b)

$$\varepsilon_6 = \frac{\partial u}{\partial X_2} + \frac{\partial v}{\partial X_1} + \left( \frac{\partial w}{\partial X_1} \right) \left( \frac{\partial w}{\partial X_2} \right)$$

(2.10.c)

$$\varepsilon_3 = \frac{\partial w}{\partial X_1} + \frac{\partial u}{\partial z}$$

(2.10.d)

$$\varepsilon_4 = \frac{\partial w}{\partial X_2} + \frac{\partial v}{\partial z}$$

(2.10.e)

The Sanders kinematic relations can be written as

$$\varepsilon_1 = \frac{\partial u}{\partial X_1} + \frac{w}{R_1} + \frac{1}{2} \left( \frac{\partial w}{\partial X_1} - \frac{u}{R_1} \right)^2$$

(2.11.a)

$$\varepsilon_2 = \frac{\partial v}{\partial X_2} + \frac{w}{R_2} + \frac{1}{2} \left( \frac{\partial w}{\partial X_2} - \frac{v}{R_2} \right)^2$$

(2.11.b)

$$\varepsilon_6 = \frac{\partial u}{\partial X_2} + \frac{\partial v}{\partial X_1} + \left( \frac{\partial w}{\partial X_1} - \frac{u}{R_1} \right) \left( \frac{\partial w}{\partial X_2} - \frac{v}{R_2} \right)$$

(2.11.c)

$$\varepsilon_3 = \frac{\partial w}{\partial X_1} + \frac{\partial u}{\partial z} - \frac{u}{R_1}$$

(2.11.d)
According to the Donnell’s shell kinematics, the displacement field for the third-order shear deformation theory (TSDT) that satisfies the traction free boundary conditions can be expressed such as

\[
\begin{align*}
\mathbf{u}_1 &= u(\xi, \eta, \zeta, t) = u_0 + \zeta \phi_1 - c_1 \zeta^3 \left( \phi_1 + \frac{\partial w_0}{\partial \xi} \right) \quad (2.12.a) \\
\mathbf{u}_2 &= v(\xi, \eta, \zeta, t) = v_0 + \zeta \phi_2 - c_1 \zeta^3 \left( \phi_2 + \frac{\partial w_0}{\partial \eta} \right) \quad (2.12.b) \\
\mathbf{u}_3 &= w(\xi, \eta, \zeta, t) = w_0 \quad (2.12.c)
\end{align*}
\]

where \((u, v, w)\) are the displacements along the orthogonal curvilinear coordinates, \((u_0, v_0, w_0)\) are the displacements of a point on the middle surface and \(\phi_1\) and \(\phi_2\) are the rotations at \(\zeta = 0\) of normals to the mid-surface with respect to the \(\xi\) and \(\eta\) axes, respectively. The constant \(c_1\) is given by \(c_1 = \frac{4}{3} h^2\), \(h\) being the total thickness of the laminate.

The consistent Donnell’s strains in the third-order shear deformation are

\[
\begin{align*}
\{ \varepsilon_1 \} &= \{ \varepsilon_1^0 \} + \zeta \{ \kappa_1 \} + \zeta^3 \{ \kappa_2^2 \} \\
\{ \varepsilon_2 \} &= \{ \varepsilon_2^0 \} + \zeta \{ \kappa_2 \} + \zeta^3 \{ \kappa_2^2 \} \\
\{ \varepsilon_3 \} &= \{ \varepsilon_3^0 \} + \zeta \{ \kappa_3 \} + \zeta^3 \{ \kappa_3^2 \}
\end{align*}
\]

where

\[
\varepsilon_4 = \frac{\partial w}{\partial X_1} + \frac{\partial v}{\partial z} - \frac{v}{R_2} \quad (2.11.e)
\]
\[
\begin{align*}
\begin{cases}
\varepsilon_1^0 \\
\varepsilon_2^0 \\
\varepsilon_6^0
\end{cases} &= \begin{cases}
\frac{\partial u_0}{\partial X_1} + \frac{w}{R_1} + \frac{1}{2} \left( \frac{\partial w_0}{\partial X_1} \right)^2 \\
\frac{\partial v_0}{\partial X_2} + \frac{w}{R_2} + \frac{1}{2} \left( \frac{\partial w_0}{\partial X_2} \right)^2 \\
\frac{\partial v_0}{\partial X_1} + \frac{\partial u_0}{\partial X_2} + \frac{\partial w_0}{\partial X_1} \frac{\partial w_0}{\partial X_2}
\end{cases} \\
\kappa_1^0 &= \begin{cases}
\frac{\partial \phi}{\partial X_1} \\
\frac{\partial \phi}{\partial X_2} \\
\frac{\partial \phi}{\partial X_1} + \frac{\partial \phi}{\partial X_2}
\end{cases} \\
\kappa_2^0 &= -c_1 \begin{cases}
\frac{\partial \phi_2}{\partial X_1} + \frac{\partial^2 w_0}{\partial X_1^2} \\
\frac{\partial \phi_2}{\partial X_2} + \frac{\partial^2 w_0}{\partial X_2^2} \\
\frac{\partial \phi_2}{\partial X_1} + \frac{\partial \phi_2}{\partial X_2} + 2 \frac{\partial^2 w_0}{\partial X_1 \partial X_2}
\end{cases} \\
\kappa_6^0 &= \begin{cases}
\phi_2 + \frac{\partial w_0}{\partial X_2} \\
\phi_1 + \frac{\partial w_0}{\partial X_1}
\end{cases} \\
\kappa_4^0 &= -c_2 \begin{cases}
\phi_2 + \frac{\partial w_0}{\partial X_2} \\
\phi_1 + \frac{\partial w_0}{\partial X_1}
\end{cases}
\end{align*}
\]

where \( c_2 = 3c_1 \) and \( X_i \) denote the Cartesian coordinates \( (dX_i = \alpha_i d\xi_i, \ i = 1, 2) \).
The equivalent displacement field for the third-order shear deformation theory for Sanders kinematics is

\[ u_1 = u_0 + \zeta \phi_1 - c_1 \zeta^3 \left( \phi_1 + \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1} \right) \]  

(2.15.a)

\[ u_2 = v_0 + \zeta \phi_2 - c_1 \zeta^3 \left( \phi_2 + \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2} \right) \]  

(2.15.b)

\[ u_3 = w_0 \]  

(2.15.c)

and the strains are

\[ \varepsilon_1^0 = \frac{\partial u_0}{\partial X_1} + \frac{w}{R_1} + \frac{1}{2} \left( \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1} \right)^2 \]

\[ \varepsilon_2^0 = \frac{\partial v_0}{\partial X_2} + \frac{w}{R_2} + \frac{1}{2} \left( \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2} \right)^2 \]

\[ \varepsilon_6^0 = \frac{\partial v_0}{\partial X_1} + \frac{\partial u_0}{\partial X_2} + \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1} \left( \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2} \right) \]

(2.16.a)

\[ \kappa_1^0 = \frac{\partial \phi_1}{\partial X_1} \]

\[ \kappa_2^0 = \frac{\partial \phi_2}{\partial X_2} \]

\[ \kappa_6^0 = \frac{\partial \phi_2}{\partial X_1} + \frac{\partial \phi_1}{\partial X_2} \]

(2.16.b)

\[ \kappa_1^2 = -c_1 \left\{ \frac{\partial \phi_1}{\partial X_1} + \frac{\partial^2 w_0}{\partial X_1^2} - \frac{1}{R_1} \frac{\partial u}{\partial X_1} \right\} \]

\[ \kappa_2^2 = -c_2 \left\{ \frac{\partial \phi_2}{\partial X_2} + \frac{\partial^2 w_0}{\partial X_2^2} - \frac{1}{R_2} \frac{\partial v}{\partial X_2} \right\} \]

\[ \kappa_6^2 = \frac{\partial \phi_2}{\partial X_1} + \frac{\partial \phi_1}{\partial X_2} + \frac{\partial^2 w_0}{\partial X_1 \partial X_2} - \frac{1}{R_1} \frac{\partial u}{\partial X_1} - \frac{1}{R_2} \frac{\partial v}{\partial X_2} \]

(2.16.c)
2.3. Equations of Motion

The governing equations of motion are derived using the dynamic version of the principle of virtual displacements (Hamilton’s principle).

\[ 0 = \int_0^T (\delta U + \delta V - \delta K) dt \]  

where \( \delta U \), \( \delta V \), and \( \delta K \) denote the virtual strain energy, virtual work done by external applied forces, and the virtual kinetic energy, respectively. For shell or plate structures, laminated or not, the integration over the domain is represented as the product of integration over the plane and integration over the thickness,

\[ \int_{\Omega} dV = \int_{-h/2}^{h/2} \int_{\Omega} d\Omega \, dz . \]

Thus, the above Equation (2.17) can be written to

\[ 0 = \int_0^{h/2} \left[ \int_{-h/2}^{h/2} \left[ \int_{\Omega} (\sigma_{1}\delta\varepsilon_{1}^{(k)} + \sigma_{2}\delta\varepsilon_{2}^{(k)} + \sigma_{6}\delta\varepsilon_{6}^{(k)} + \sigma_{4}\delta\varepsilon_{4}^{(k)} + \sigma_{5}\delta\varepsilon_{5}^{(k)})dx_1 dx_2 \right] dz 
- \int_{\Omega} q \delta w dx_1 dx_2 \right] \right] dt \]  

(2.18)
The governing equations of motion can be derived from Equation (2.18) by integrating the displacement gradients by parts to relieve the virtual displacements and setting the coefficients of $\delta u$, $\delta v$, $\delta w$, $\delta \phi_x$ and $\delta \phi_y$ to zero separately (i.e. use the Fundamental Lemma of calculus of variations; see Reddy 2002).

In the derivation, thermal and magnetostrictive effects are taken into consideration with the understanding that the material properties are independent of temperature and magnetic fields, and that the temperature $T$ is a known function of position. Thus temperature and magnetic field enter the formulation only through constitutive equations.

2.3.1. Equations of Motion for TSDT Plates

The equations of motion of the third-order shear deformation theory (TSDT) are

$$\delta u : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \frac{\partial^2 u_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_x}{\partial t^2} - c_1 I_3 \frac{\partial^2}{\partial x^2} \left( \frac{\partial w_0}{\partial x} \right)$$  (2.19.a)

$$\delta v : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = \frac{\partial^2 v_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_y}{\partial t^2} - c_1 I_3 \frac{\partial^2}{\partial y^2} \left( \frac{\partial w_0}{\partial y} \right)$$  (2.19.b)

$$\delta w : \frac{\partial \overline{Q}_x}{\partial x} + \frac{\partial \overline{Q}_y}{\partial y} + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right) + c_1 \left( \frac{\partial^2 P_{xx}}{\partial x^2} + 2 \frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_{yy}}{\partial y^2} \right) + q = \frac{\partial^2 w_0}{\partial t^2} - c_1 I_6 \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2 w_0}{\partial x^2} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial^2 w_0}{\partial y^2} \right) + 2 \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial^2 w_0}{\partial x \partial y} \right)$$  (2.19.c)

$$\delta \phi_x : \frac{\partial \overline{M}_{xx}}{\partial x} + \frac{\partial \overline{M}_{xy}}{\partial y} - \overline{Q}_x = \frac{\partial^2}{\partial t^2} \left( J_1 u_0 + K_2 \phi_x - c_4 J_4 \frac{\partial w_0}{\partial x} \right)$$  (2.19.d)

$$\delta \phi_y : \frac{\partial \overline{M}_{xy}}{\partial x} + \frac{\partial \overline{M}_{yy}}{\partial y} - \overline{Q}_y = \frac{\partial^2}{\partial t^2} \left( J_1 v_0 + K_2 \phi_y - c_4 J_4 \frac{\partial w_0}{\partial y} \right)$$  (2.19.e)

where
\[
\overline{M}_{a\beta} = M_{a\beta} - c_1 P_{a\beta}, \quad \overline{Q}_\alpha = Q_\alpha - c_2 R_\alpha \quad (\alpha, \beta = x, y)
\]

\[
I_i = \sum_{k=1}^{N} \int_{k}^{k+1} \rho^{(k)}(z) \, dz \quad (i = 0, 1, 2, ..., 6),
\]

\[
J_i = I_i - c_1 I_{i+2} \quad (i = 1, 4)
\]

\[
K_2 = I_2 - 2c_1 I_4 + c_1^2 I_6
\]

(2.20)

The primary and secondary variables of the third-order theory are

Primary variables: \( u_n, u_x, w_0, \frac{\partial w_0}{\partial n}, \phi_n, \phi_x \)

Secondary variables: \( N_{nn}, N_{nx}, \overline{V}_n, P_{nn}, \overline{M}_{nn}, \overline{M}_{nx} \)

where \( \overline{V}_n \) and \( p \) are defined as

\[
\overline{V}_n = c_1 \left[ \left( \frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right) n_x + \left( \frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} \right) n_y \right] - c_1 \left[ \left( I_3 \overline{u}_0 + J_4 \overline{\phi}_x - c_1 I_6 \overline{\phi}_{xx} \right) n_x + \left( I_3 \overline{v}_0 + J_4 \overline{\phi}_y - c_1 I_6 \overline{\phi}_{yy} \right) n_y \right]
\]

(2.22.a)

\[
P(w_0) = \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x + \left( N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right) n_y
\]

(2.22.b)

\[
\begin{bmatrix}
N_{nn} \\
N_{nx}
\end{bmatrix} = \begin{bmatrix}
n_x^2 & n_y^2 & 2n_x n_y \\
-n_x n_y & n_x n_y & n_x^2 - n_y^2
\end{bmatrix} \begin{bmatrix}
N_{xx} \\
N_{xy}
\end{bmatrix}
\]

(2.22.c)

\[
\begin{bmatrix}
M_{nn} \\
M_{nx}
\end{bmatrix} = \begin{bmatrix}
n_x^2 & n_y^2 & 2n_x n_y \\
-n_x n_y & n_x n_y & n_x^2 - n_y^2
\end{bmatrix} \begin{bmatrix}
M_{xx} \\
M_{xy}
\end{bmatrix}
\]

(2.22.d)
and \((N_{xx}, N_{yy}, N_{xy})\) denote the total in-plane forces, \((M_{xx}, M_{yy}, M_{xy})\) the moments, and \((P_{xx}, P_{yy}, P_{xy})\) and \((R_{x}, R_{y})\) denote the higher-order stress resultants (see Figure 2.3).

\[
\begin{align*}
\begin{bmatrix}
N_{xx} \\
N_{yy} \\
N_{xy}
\end{bmatrix} &= \int h z^2 \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} dz \\
&= \frac{h}{2} \begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} \frac{h}{2} dz \\
&= \frac{h}{2} \begin{bmatrix}
M_{xx} \\
M_{yy} \\
M_{xy}
\end{bmatrix} \frac{h}{2} dz \\
&= \frac{h}{2} \begin{bmatrix}
P_{xx} \\
P_{yy} \\
P_{xy}
\end{bmatrix} \frac{h}{2} dz \\
&= \frac{h}{2} \begin{bmatrix}
Q_{y} \\
Q_{x}
\end{bmatrix} \frac{h}{2} dz \\
&= \frac{h}{2} \begin{bmatrix}
R_{y} \\
R_{x}
\end{bmatrix} \frac{h}{2} dz
\end{align*}
\]

Figure 2.3 Force and moment resultants on a plate element
2.3.2. Equations of Motion for Special Cases

The current third-order shear deformation theory contains the classical and first-order shear deformation theories as special cases. By setting $c_1 = 0$ and the rotation become the slopes of the transverse deflection, one can obtain the classical laminated plate theory and by setting $c_1 = 0$, the first-order shear deformation theory can be obtained.

(1) The Classical Laminated Plate Theory

The displacement field for the classical laminated plate theory (CLPT) is of the form

$$u(x, y, z, t) = u_0(x, y, t) - z \frac{\partial w_0}{\partial x}$$

(2.24.a)

$$v(x, y, z, t) = v_0(x, y, t) - z \frac{\partial w_0}{\partial y}$$

(2.24.b)

$$w(x, y, z, t) = w_0(x, y, t)$$

(2.24.c)

where $(u_0, v_0, w_0)$ denote the displacement of a material point at $(x, y, 0)$. Note that $(u_0, v_0)$ are associated with extensional deformation of the plate while $w_0$ denotes the bending deflection.

The corresponding von Kármán strains of classical laminated plate theory are the following. The transverse strains $(\varepsilon_{xz}, \varepsilon_{yz}, \varepsilon_{zz})$ are identically zero in the classical laminated plate theory.

$$\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{xy}^0
\end{bmatrix} + z \begin{bmatrix}
\varepsilon_{xx}^1 \\
\varepsilon_{yy}^1 \\
\gamma_{xy}^1
\end{bmatrix}$$

(2.25)

where
\[
\begin{align*}
\{ \varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{xy}^0 \} &= \left\{ \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2, \frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2, \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \right\} \\
\{ \varepsilon_{xx}^1, \varepsilon_{yy}^1, \gamma_{xy}^1 \} &= \left\{ \frac{-\partial^2 w_0}{\partial x^2}, \frac{-\partial^2 w_0}{\partial y^2}, \frac{-2 \partial^2 w_0}{\partial x \partial y} \right\}
\end{align*}
\]

(2.26.a)

(2.26.b)

where \((\varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{xy}^0)\) are the membrane stains, and \((\varepsilon_{xx}^1, \varepsilon_{yy}^1, \gamma_{xy}^1)\) are the flexural (bending) strains, known as the curvatures.

The equations of motion of the classical theory of laminated plates are given by

\[
\begin{align*}
\delta u: \quad & \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_1 \frac{\partial^2 w_0}{\partial t^2} \left( \frac{\partial w_0}{\partial x} \right) \\
\delta v: \quad & \frac{\partial^2 M_{yy}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{xx}}{\partial y^2} + \frac{\partial}{\partial y} \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_1 \frac{\partial^2 w_0}{\partial t^2} \left( \frac{\partial w_0}{\partial y} \right) \\
\delta w: \quad & + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right) + q = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_1 \frac{\partial^2 w_0}{\partial t^2} \left( \frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial y} \right) + I_2 \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} + I_1 \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial^2 w_0}{\partial x \partial y}
\end{align*}
\]

(2.27.a)

(2.27.b)

(2.27.c)
(2) The First-order Shear Deformation Laminated Plate Theory

The displacement field for the first-order shear deformation theory (FSDT) can be expressed in the form

\[ u(x, y, z, t) = u_0(x, y, t) + z\phi_z(x, y, t) \]  
\[ v(x, y, z, t) = v_0(x, y, t) + z\phi_y(x, y, t) \]  
\[ w(x, y, z, t) = w_0(x, y, t) \]

where \((u_0, v_0, w_0)\) denote the displacement of a point on the plane \(z = 0\) and \((\phi_z, \phi_y)\) are the rotations of a transverse normal about the \(y\) – and \(x\) – axis, respectively.

The von Kar’ma’n Strains associated with the displacement field are \((\varepsilon_{zz} = 0)\)

\[
\begin{align*}
\begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{pmatrix} &= 
\begin{pmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{yz}^0 \\
\gamma_{xz}^0 \\
\gamma_{xy}^0
\end{pmatrix} + 
\begin{pmatrix}
\varepsilon_{xx}^1 \\
\varepsilon_{yy}^1 \\
0 \\
0 \\
\gamma_{xy}^1
\end{pmatrix} \\
\end{align*}
\]

where,

\[
\begin{align*}
\begin{pmatrix}
\varepsilon_{xx}^0 \\
\varepsilon_{yy}^0 \\
\gamma_{yz}^0 \\
\gamma_{xz}^0 \\
\gamma_{xy}^0
\end{pmatrix} &= 
\begin{pmatrix}
\frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \\
\frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 \\
\frac{\partial w_0}{\partial y} + \phi_y \\
\frac{\partial w_0}{\partial x} + \phi_x \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y}
\end{pmatrix} \\
\end{align*}
\]
\[
\begin{bmatrix}
\varepsilon_{xx}^1 \\
\varepsilon_{yy}^1 \\
0 \\
0 \\
\gamma_{xy}^1
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \phi_x}{\partial x} \\
\frac{\partial \phi_y}{\partial y} \\
0 \\
0 \\
\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}
\end{bmatrix}
\]  
(2.30.b)

Note that \((\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy})\) are linear through the plate thickness, while the transverse shear strains \((\gamma_{xz}, \gamma_{yz})\) are constant.

The equations of motion of FSDT are derived using the dynamic version of the principle of virtual displacements.

\[
\delta u : \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2} 
\]  
(2.31.a)

\[
\delta v : \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} 
\]  
(2.31.b)

\[
\delta w : \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right) + q = I_0 \frac{\partial^2 w_0}{\partial t^2} 
\]  
(2.31.c)

\[
\delta \phi_x : \quad \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u_0}{\partial t^2} 
\]  
(2.31.d)

\[
\delta \phi_y : \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v_0}{\partial t^2} 
\]  
(2.31.e)
2.3.3. Equations of Motion for TSDT Donnell Shell Theory

The Euler-Lagrange equations of nonlinear Donnell shell theory for TSDT are

\[ \delta u : \frac{\partial N_1}{\partial X_1} + \frac{\partial N_2}{\partial X_2} = I_0 \frac{\partial^2 u_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_1}{\partial t^2} - c_i I_3 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial X_1} \right) \] (2.32.a)

\[ \delta v : \frac{\partial N_5}{\partial X_1} + \frac{\partial N_6}{\partial X_2} = I_0 \frac{\partial^2 v_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_2}{\partial t^2} - c_i I_3 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial X_2} \right) \] (2.32.b)

\[ \delta w : \frac{\partial \overline{Q}_1}{\partial X_1} + \frac{\partial \overline{Q}_2}{\partial X_2} + N(w_0) + c_i \left( \frac{\partial^2 P_1}{\partial X_1^2} + 2 \frac{\partial^2 P_0}{\partial X_1 \partial X_2} + \frac{\partial^2 P_2}{\partial X_2^2} \right) \] (2.32.c)

\[ \delta \phi : \frac{\partial \overline{M}_1}{\partial X_1} + \frac{\partial \overline{M}_6}{\partial X_2} - \overline{Q}_1 = J_1 \frac{\partial^2 u_0}{\partial t^2} + K_2 \frac{\partial^2 \phi_1}{\partial t^2} - c_i J_4 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial X_1} \right) \] (2.32.d)

\[ \delta \phi_2 : \frac{\partial \overline{M}_6}{\partial X_1} + \frac{\partial \overline{M}_2}{\partial X_2} - \overline{Q}_2 = J_1 \frac{\partial^2 v_0}{\partial t^2} + K_2 \frac{\partial^2 \phi_2}{\partial t^2} - c_i J_4 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial X_2} \right) \] (2.32.e)

where

\[ N(w_0) = \frac{\partial}{\partial X_1} \left( N_1 \frac{\partial w_0}{\partial X_1} + N_6 \frac{\partial w_0}{\partial X_2} \right) + \frac{\partial}{\partial X_2} \left( N_6 \frac{\partial w_0}{\partial X_1} + N_2 \frac{\partial w_0}{\partial X_2} \right) \]

\[ \overline{M}_\alpha = M_\alpha - c_i P_\alpha \quad (\alpha = 1, 2, 6), \quad \overline{Q}_\beta = Q_\beta - c_2 K_\beta \quad (\beta = 1, 2) \] (2.33)

\[ I_i = \sum_{k=1}^{N} k^{k+1} \rho^{(k)}(\zeta) d\zeta \quad (i = 0, 1, 2, ..., 6), \]

\[ J_i = I_i - c_i I_{i+2} \quad (i = 1, 4), \quad K_2 = I_2 - 2c_1 I_4 + c_1^2 I_6 \]
2.3.4. Equations of Motion for TSDT Sanders Shell Theory

The equations of motion are

\[
\begin{align*}
\delta u : & \quad \frac{\partial N_1}{\partial X_1} + \frac{\partial N_6}{\partial X_2} + \frac{\partial Q_1}{R_1} + N_1(u_0, v_0, w_0) + \frac{c_1}{R_1} \left( \frac{\partial P_2}{\partial X_1} + \frac{\partial P_6}{\partial X_2} \right) \\
& \quad = I_0 \frac{\partial^2 u_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_1}{\partial t^2} - c_1 T_3 \frac{\partial^3}{\partial t^3} \left( \frac{\partial w_0}{\partial X_1} \right) \\
\delta v : & \quad \frac{\partial N_6}{\partial X_1} + \frac{\partial N_2}{\partial X_2} + \frac{\partial Q_2}{R_2} + N_2(u_0, v_0, w_0) + \frac{c_1}{R_2} \left( \frac{\partial P_2}{\partial X_2} + \frac{\partial P_6}{\partial X_1} \right) \\
& \quad = I_0 \frac{\partial^2 v_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_2}{\partial t^2} - c_1 T_3 \frac{\partial^3}{\partial t^3} \left( \frac{\partial w_0}{\partial X_2} \right) \\
\delta w : & \quad \frac{\partial Q_1}{\partial X_1} + \frac{\partial Q_2}{\partial X_2} + N_3(u_0, v_0, w_0) + \frac{c_1}{R_1} \left( \frac{\partial^2 P_2}{\partial X_1^2} + 2 \frac{\partial^2 P_6}{\partial X_1 \partial X_2} + \frac{\partial^2 P_2}{\partial X_2^2} \right) \\
& \quad = q - \frac{N_1}{R_1} - \frac{N_2}{R_2} = I_0 \frac{\partial^2 w_0}{\partial t^2} - c_1 I_6 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w_0}{\partial X_1^2} + \frac{\partial^2 w_0}{\partial X_2^2} \right) + \\
& \quad c_1 \left[ T_3 \frac{\partial^2}{\partial t^2} \left( \frac{\partial u_0}{\partial X_1} \right) + T_3 \frac{\partial^2}{\partial t^2} \left( \frac{\partial v_0}{\partial X_2} \right) + J_4 \frac{\partial^2}{\partial t^2} \left( \frac{\partial \phi_1}{\partial X_1} + \frac{\partial \phi_2}{\partial X_2} \right) \right] \\
\delta \phi : & \quad \frac{\partial \overline{M}_1}{\partial X_1} + \frac{\partial \overline{M}_6}{\partial X_2} - \overline{Q}_1 = J_1 \frac{\partial^2 u_0}{\partial t^2} + K_2 \frac{\partial^2 \phi_1}{\partial t^2} - c_1 J_4 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial X_1} \right) \\
\delta \phi : & \quad \frac{\partial \overline{M}_6}{\partial X_1} + \frac{\partial \overline{M}_2}{\partial X_2} - \overline{Q}_2 = J_1 \frac{\partial^2 v_0}{\partial t^2} + K_2 \frac{\partial^2 \phi_2}{\partial t^2} - c_1 J_4 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial X_2} \right)
\end{align*}
\]

where

\[
\begin{align*}
N_1(u_0, v_0, w_0) & = - \frac{1}{R_1} \left[ N_1 \left( \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1} \right) + N_6 \left( \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2} \right) \right] \\
N_2(u_0, v_0, w_0) & = - \frac{1}{R_2} \left[ N_6 \left( \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1} \right) + N_2 \left( \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2} \right) \right]
\end{align*}
\]
\( N_3(u_0, v_0, w_0) = \frac{\partial}{\partial X_1} \left[ N_1 \left( \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1} \right) + N_6 \left( \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2} \right) \right] + \frac{\partial}{\partial X_2} \left[ N_6 \left( \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1} \right) + N_2 \left( \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2} \right) \right] \) \tag{2.35.c}

\( M_\alpha = M_\alpha - c_1 P_\alpha \)  \( (\alpha = 1, 2, 6), \quad Q_\beta = Q_\beta - c_2 K_\beta \)  \( (\beta = 1, 2) \)

\( I_i = \sum_{k=1}^{N} \int_{k}^{k+1} \rho^{(k)}(\zeta) d\zeta \quad (i = 0, 1, 2, ..., 6), \)

\( J_i = I_i - c_4 I + 2 \quad (i = 1, 4), \quad K_2 = I_2 - 2 c_1 I_4 + c_1^2 I_6, \)

\( c_1 = \frac{4}{3h^2}, \quad c_2 = \frac{4}{h^2} \) \tag{2.35.d}

\( \bar{I}_0 = I_0 + 2 \frac{c_1}{R_1} I_3 + \left( \frac{c_1}{R_1} \right)^2 I_6, \quad \bar{T}_0 = I_0 + 2 \frac{c_1}{R_2} I_3 + \left( \frac{c_1}{R_2} \right)^2 I_6, \)

\( \bar{J}_1 = J_1 + \frac{c_1}{R_1} J_4, \quad \bar{J}_1 = J_1 + \frac{c_1}{R_1} J_4, \)

\( \bar{T}_3 = I_3 + \frac{c_1}{R_1} I_6, \quad \bar{T}_3 = I_3 + \frac{c_1}{R_2} I_6 \)

\( (N_1, N_2, N_6) \) denote the total in-plane force resultants, \((M_1, M_2, M_6)\) the moment resultants, and \((P_1, P_2, P_6)\) and \((K_1, K_2)\) denote the higher-order stress resultants.

\[
\begin{bmatrix}
N_1 \\
N_2 \\
N_6
\end{bmatrix} = \int_{\frac{h}{2}}^{\frac{b}{2}} \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix} \zeta \, d\zeta \tag{2.36.a}
\]

\[
\begin{bmatrix}
M_1 \\
M_2 \\
M_6
\end{bmatrix} = \int_{\frac{h}{2}}^{\frac{b}{2}} \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix} \, d\zeta \tag{2.36.b}
\]

\[
\begin{bmatrix}
P_1 \\
P_2 \\
P_6
\end{bmatrix} = \int_{\frac{h}{2}}^{\frac{b}{2}} \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix} \zeta^2 \, d\zeta \tag{2.36.c}
\]
\[ \left\{ \begin{array}{c} K_2 \\ K_1 \end{array} \right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ \begin{array}{c} \sigma_5 \\ \sigma_4 \end{array} \right\} \zeta^2 \, d\zeta \]  
\hspace{1cm} (2.36.d)

\[ \left\{ \begin{array}{c} Q_2 \\ Q_1 \end{array} \right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ \begin{array}{c} \sigma_5 \\ \sigma_4 \end{array} \right\} \, d\zeta \]  
\hspace{1cm} (2.36.e)

### 2.4. Constitutive Relations

In Sections 2.4 - 2.6, the equations are based on shell coordinate system. By setting \((\xi_1, \xi_2, \zeta)\) to \((x, y, z)\), the structural system returns to plate coordinates.

The plate and shell structure under consideration is composed of a finite number of orthotropic layers of uniform thickness. Each composite lamina of the shell is assumed to behave as an orthotropic material, with its material axes oriented arbitrarily with respect to the laminate coordinates. The smart material layer is assumed to be orthotropic in deriving the relations, but taken to be isotropic in actual calculations.

From the constitutive relations of the magnetostrictive layer (IEEE standard 319, 1976, 1990, Clark 1980), \(\varepsilon = S\sigma + dH\), the constitutive relations for the \(k^{th}\) lamina (magnetostrictive layer) when referred to the shell laminate coordinates can be written

\[ \left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{array} \right\}^{(k)} = \left[ \begin{array}{ccc} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{array} \right]^{(k)} \left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{array} \right\} - \left\{ \begin{array}{c} \overline{\alpha}_1 \\ \overline{\alpha}_2 \\ \overline{\alpha}_6 \end{array} \right\} \Delta T - \left[ \begin{array}{ccc} 0 & 0 & \varepsilon_{31}^{(k)} \\ 0 & 0 & \varepsilon_{32}^{(k)} \\ 0 & 0 & \varepsilon_{36}^{(k)} \end{array} \right] \left\{ \begin{array}{c} \overline{H} \\ \zeta \end{array} \right\} \]  
\hspace{1cm} (2.37.a)

\[ \left\{ \begin{array}{c} \sigma_4 \\ \sigma_5 \end{array} \right\}^{(k)} = \left[ \begin{array}{ccc} \overline{Q}_{44} & \overline{Q}_{45} & \overline{Q}_{46} \\ \overline{Q}_{45} & \overline{Q}_{55} & \overline{Q}_{56} \end{array} \right]^{(k)} \left\{ \begin{array}{c} \varepsilon_4 \\ \varepsilon_5 \end{array} \right\} - \left[ \begin{array}{ccc} \varepsilon_{44} & \varepsilon_{45} & 0 \\ \varepsilon_{54} & \varepsilon_{55} & 0 \end{array} \right]^{(k)} \left\{ \begin{array}{c} \overline{H} \\ \zeta \end{array} \right\} \]  
\hspace{1cm} (2.37.b)

For the structural part of the composite structure, the part including the electric field intensity \(\overline{H}_\zeta\) should be excluded in the constitutive relations.
Equation (2.37) can also be written as

\[ \{\sigma\}^{(k)} = [\bar{Q}]^{(k)} \{\varepsilon - \bar{e}\Delta T\} - [\bar{H}]^{(k)} \{H_s\} \] (2.38)

where \( \Delta T \) is temperature rise from a reference state, \( \Delta T = T - T_0 \), and \( \bar{\alpha}_1 \), \( \bar{\alpha}_2 \), and \( \bar{\alpha}_6 \) are the transformed thermal expansion coefficients.

\[
\begin{align*}
\bar{\alpha}_1 &= \alpha_1 \cos^2 \theta + \alpha_2 \sin^2 \theta \quad (2.39.a) \\
\bar{\alpha}_2 &= \alpha_1 \sin^2 \theta + \alpha_2 \cos^2 \theta \quad (2.39.b) \\
\bar{\alpha}_6 &= 2(\alpha_1 - \alpha_2) \sin \theta \cos \theta \quad (2.39.c)
\end{align*}
\]

The transformed stiffnesses \( \bar{Q}_{ij} \) are calculated from the plane stress-reduced stiffnesses \( Q_{ij} \) using the transformation relations,

\[
\begin{align*}
\bar{Q}_{11} &= Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^2 \theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12} (\sin^4 \theta + \cos^4 \theta) \\
\bar{Q}_{22} &= Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^2 \theta \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \theta \cos \theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin \theta \cos^3 \theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66} (\sin^4 \theta + \cos^4 \theta) \\
\bar{Q}_{44} &= Q_{44} \cos^2 \theta + Q_{55} \sin^2 \theta \\
\bar{Q}_{45} &= (Q_{55} - Q_{44}) \cos \theta \sin \theta \\
\bar{Q}_{55} &= Q_{55} \cos^2 \theta + Q_{44} \sin^2 \theta
\end{align*}
\]
Here $\theta$ is the angle measured counter-clockwise from the shell $x_j$ coordinate to the material 1-coordinate. The coefficients $Q^{(k)}_{ij}$ are known in terms of engineering constants of the $k$th layer:

$$
Q_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}}, \quad Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{12} \nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}} \quad (2.41)
$$

and $\bar{e}^{(k)}_{ij}$ are the transformed magnetostrictive coupling moduli of the $k$th lamina.

$$
\bar{e}_{31} = e_{31} \cos^2 \theta + e_{32} \sin^2 \theta
$$

$$
\bar{e}_{32} = e_{31} \sin^2 \theta + e_{32} \cos^2 \theta
$$

$$
\bar{e}_{36} = (e_{31} - e_{32}) \sin \theta \cos \theta
$$

$$
\bar{e}_{14} = (e_{15} - e_{24}) \sin \theta \cos \theta \quad (2.42)
$$

$$
\bar{e}_{24} = e_{24} \cos^2 \theta + e_{14} \sin^2 \theta
$$

$$
\bar{e}_{25} = (e_{15} - e_{24}) \sin \theta \cos \theta
$$

$$
\bar{e}_{15} = e_{15} \cos^2 \theta + e_{24} \sin^2 \theta
$$

2.5. Velocity Feedback Control

The smart layer produces an actuating force required to control vibration and deflection in a smart plate or shell structure, based on a control law. From a structural point of view, the two fundamental types for realizing control are the open-loop control and the closed-loop control known as the feedback control. A sensor to measure the output is not required for the open-loop control. The feedback strategy requires sensors for control system design and has the potential to give much better performance than the open-loop control (Franklin, Powell and Emani-Naeini 2000, Bishop and Dorf 2001). Because of the
simplicity, direct output measurement feedback control is a promising solution to the practical problems (Balas 1979, Chung, Liu and Chu 1993).

Magnetostrictive material is selected to actively control the structural system by the simple control algorithm, a negative velocity feedback control, where the feedback amplitude varies by the negative velocity. The constant control gain is assumed in this study.

Considering velocity proportional closed-loop feedback control shown in Figure 2.4, the magnetic field intensity $H_c$ can be expressed in terms of coil constant $k_c$ and coil current $I(\xi_1, \xi_2, t)$ which is related to the velocity as

$$H_c(\xi_1, \xi_2, t) = k_c I(\xi_1, \xi_2, t)$$  \hspace{1cm} (2.43)

$$k_c = \frac{n_c}{\sqrt{b_c^2 + 4r_c^2}}$$  \hspace{1cm} (2.44)

$$I(\xi_1, \xi_2, t) = c(t) \frac{\partial w_0}{\partial t}$$  \hspace{1cm} (2.45)

where $b_c$ is the coil width, $r_c$ is coil radius, $n_c$ is number of turns in the coil, and $c(t)$ is the control gain (Krishna Murty, Anjanappa and Wu 1977)

![Figure 2.4 Closed-loop feedback control system](image-url)
2.6. Laminated Constitutive Equations

The force resultants can be expressed in terms of the strain components as follows:

\[
\begin{align*}
\{N\} &= [A] \{\varepsilon^0\} - \{N^M\} - \{N^T\} \\
\{M\} &= [B] \{\kappa^0\} - \{M^M\} - \{M^T\} \\
\{P\} &= [E] \{\kappa^2\} - \{P^M\} - \{P^T\}
\end{align*}
\] \quad (2.46.a)

\[
\begin{align*}
\{Q\} &= [A] \{\varepsilon^0\} \\
\{R\} &= [D] \{\kappa^3\}
\end{align*}
\] \quad (2.46.b)

The stiffnesses \(A_{ij}, D_{ij}, \text{ and } F_{ij}\) are defined for \(i, j = 1, 2, 6\) as well as \(i, j = 4, 5\). The stiffnesses \(B_{ij}, E_{ij}, \text{ and } H_{ij}\) are defined only for \(i, j = 1, 2, 6\). The coefficients of \(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, \text{ and } H_{ij}\) are given in terms of the layer coordinates \(\zeta_{k+1}\) and \(\zeta_k\) and lamina stiffnesses \(\overline{Q}_{ij}\).

\[
(A_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} \overline{Q}_{ij}^{(k)} (1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5) d\zeta \quad (2.47)
\]

The magnetostrictive stress resultants \(\{N^M\}, \{M^M\}, \text{ and } \{P^M\}\) are defined by

\[
\begin{align*}
\{N^M_{1}\} &= \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} \left[ -\frac{\varepsilon_{31}}{\varepsilon_{32}} \right] \quad (2.48.a) \\
N^M_{2} &= \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} H_{\zeta} \frac{e_{31}}{e_{32}} d\zeta \\
N^M_{6} &= \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} \frac{1}{H_{\zeta}} d\zeta \\
M^M_{1} &= \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} \left[ -\frac{\varepsilon_{31}}{\varepsilon_{32}} \right] \quad (2.48.b) \\
M^M_{2} &= \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} H_{\zeta} \zeta d\zeta \\
M^M_{6} &= \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} \frac{1}{H_{\zeta}} \zeta d\zeta \\
P^M_{1} &= \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} \left[ -\frac{\varepsilon_{31}}{\varepsilon_{32}} \right] \quad (2.48.c) \\
P^M_{2} &= \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} H_{\zeta} \zeta^2 d\zeta \\
P^M_{6} &= \sum_{k=1}^{N} \int_{\xi_{ki}}^{\zeta_{k+1}} \frac{1}{H_{\zeta}} \zeta^2 d\zeta
\end{align*}
\]
again,

\[
\begin{align*}
\begin{bmatrix} N^M \end{bmatrix} &= \begin{bmatrix} A^M \end{bmatrix} H \begin{bmatrix} M^M \end{bmatrix} = \begin{bmatrix} \tilde{A}^M \end{bmatrix} \frac{\partial W}{\partial t} \\
\begin{bmatrix} P^M \end{bmatrix} &= \begin{bmatrix} B^M \end{bmatrix} \begin{bmatrix} E^M \end{bmatrix} \end{align*}
\] (2.49)

The magnetostrictive stiffnesses \( A^M, B^M, E^M \) are defined for \( l = 1, 2, 6 \).

\[
\begin{align*}
\begin{bmatrix} \tilde{A}^M, \tilde{B}^M, \tilde{E}^M \end{bmatrix} &= k_c c(t) \begin{bmatrix} A^M, B^M, E^M \end{bmatrix} = k_c c(t) \sum_{k=1}^{N_k} \tilde{e}_{ij} \bar{\zeta}^k (1, \zeta, \zeta^3) d \zeta
\end{align*}
\] (2.50)

where \( \tilde{e}_{ij} \) is the transformed moduli of the actuating/sensing material, and \( H \) is the electric field intensity, which should be excluded in the constitutive relations for the structural part of the composite structures.

The thermal stress resultants \( N^T, M^T, P^T \) are defined by

\[
\begin{bmatrix} N^T \end{bmatrix} = \begin{bmatrix} N_1^T & M_1^T & P_1^T \\ N_2^T & M_2^T & P_2^T \\ N_6^T & M_6^T & P_6^T \end{bmatrix} = \sum_{k=1}^{N_k} \int_{\zeta_4}^{\zeta_{4+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \bar{\alpha}_6 \end{bmatrix} (1, \zeta, \zeta^3) d \zeta \Delta T
\] (2.51)

where \( \Delta T \) is temperature rise from the reference state and \( \bar{\alpha}_1, \bar{\alpha}_2, \) and \( \bar{\alpha}_6 \) are the transformed thermal expansion coefficients which are stated in Equation (2.39).
3. ANALYTICAL SOLUTIONS

3.1. Analytical Solutions for Laminated Composite Plates

3.1.1. Introduction

The exact solutions of partial differential equations on arbitrary domains and for general conditions are not always possible. However, for the simply supported boundary conditions, the linear version of Equation (2.19) can be solved exactly, provided the lamination scheme is cross-ply or anti-symmetric angle-ply laminates. Equations of motion for the third-order shear deformation theory describe five second-order, nonlinear, partial differential equations in terms of five generalized displacements \((u, v, w, \phi_x, \phi_y)\). For linear case, the in-plane displacements \((u, v)\) are uncoupled from the bending deflections, \((w, \phi_x, \phi_y)\), and it is sufficient to consider only the bending equations, Equations (2.19.c), (2.19.d), and (2.19.e).

3.1.2. Boundary Conditions

For simply supported rectangular plates, it is possible to obtain the Navier solutions. There are two types of simply supported boundary conditions as shown in Figure 3.1. The geometry and coordinate system of the rectangular plate are shown in Figure 2.1.

![Figure 3.1 Simply supported boundary conditions (SS-1 and SS-2) for TSDT](image)

Figure 3.1 Simply supported boundary conditions (SS-1 and SS-2) for TSDT
The first type of simply supported boundary condition is denoted by SS-1 (used for cross-ply laminates), and they are

\[ u_0(x,0,t) = 0, \quad u_0(x,b,t) = 0, \quad v_0(o,y,t) = 0, \quad v_0(a,y,t) = 0 \]
\[ \phi_x(x,0,t) = 0, \quad \phi_x(x,b,t) = 0, \quad \phi_y(o,y,t) = 0, \quad \phi_y(a,y,t) = 0 \]
\[ w_0(x,0,t) = 0, \quad w_0(x,b,t) = 0, \quad w_0(0,y,t) = 0, \quad w_0(a,y,t) = 0 \] (3.1)
\[ N_{xx}(0,y,t) = 0, \quad N_{xx}(a,y,t) = 0, \quad N_{yy}(x,0,t) = 0, \quad N_{yy}(x,b,t) = 0 \]
\[ \overline{M}_{xx}(0,y,t) = 0, \quad \overline{M}_{xx}(a,y,t) = 0, \quad \overline{M}_{yy}(x,0,t) = 0, \quad \overline{M}_{yy}(x,b,t) = 0 \]

The second type of simply supported boundary conditions are denoted by SS-2 (used for anti-symmetric angle-ply laminates), and they are

\[ u_0(0,y,t) = 0, \quad u_0(a,y,t) = 0, \quad v_0(x,0,t) = 0, \quad v_0(x,b,t) = 0 \]
\[ \phi_x(x,0,t) = 0, \quad \phi_x(x,b,t) = 0, \quad \phi_y(o,y,t) = 0, \quad \phi_y(a,y,t) = 0 \]
\[ w_0(x,0,t) = 0, \quad w_0(x,b,t) = 0, \quad w_0(0,y,t) = 0, \quad w_0(a,y,t) = 0 \] (3.2)
\[ N_{xy}(0,y,t) = 0, \quad N_{xy}(a,y,t) = 0, \quad N_{yx}(x,0,t) = 0, \quad N_{yx}(x,b,t) = 0 \]
\[ \overline{M}_{xx}(0,y,t) = 0, \quad \overline{M}_{xx}(a,y,t) = 0, \quad \overline{M}_{yy}(x,0,t) = 0, \quad \overline{M}_{yy}(x,b,t) = 0 \]

For the case in which the in-plane displacements are uncoupled from the bending deflections, both the SS-1 and SS-2 boundary conditions reduce to the same, and they are given by

\[ w_0(x,0,t) = 0, \quad w_0(x,b,t) = 0, \quad w_0(0,y,t) = 0, \quad w_0(a,y,t) = 0 \]
\[ \phi_x(x,0,t) = 0, \quad \phi_x(x,b,t) = 0, \quad \phi_y(o,y,t) = 0, \quad \phi_y(a,y,t) = 0 \] (3.3)
\[ \overline{M}_{xx}(0,y,t) = 0, \quad \overline{M}_{xx}(a,y,t) = 0, \quad \overline{M}_{yy}(x,0,t) = 0, \quad \overline{M}_{yy}(x,b,t) = 0 \]

3.1.3 Navier Solution

The above boundary conditions can be satisfied by the following expansions of the displacements for both anti-symmetric cross-ply and angle-ply laminates:
\begin{equation}
    w_0(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn}(t) \sin \alpha x \sin \beta y \tag{3.4.a}
\end{equation}

\begin{equation}
    \phi_x(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} X_{mn}(t) \cos \alpha x \sin \beta y \tag{3.4.b}
\end{equation}

\begin{equation}
    \phi_y(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{mn}(t) \sin \alpha x \cos \beta y \tag{3.4.c}
\end{equation}

where \( \alpha = \frac{m\pi}{a} \) and \( \beta = \frac{n\pi}{b} \), \((W_{mn}, X_{mn}, Y_{mn})\) are unknowns to be determined.

The loads are also expanded in double Fourier series

\begin{equation}
    M^M_{xx}(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M^1_{mn}(t) \sin \alpha x \sin \beta y \tag{3.5.a}
\end{equation}

\begin{equation}
    M^M_{yy}(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M^2_{mn}(t) \sin \alpha x \sin \beta y \tag{3.5.b}
\end{equation}

\begin{equation}
    P^M_{xx}(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P^1_{mn}(t) \sin \alpha x \sin \beta y \tag{3.5.c}
\end{equation}

\begin{equation}
    P^M_{yy}(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P^2_{mn}(t) \sin \alpha x \sin \beta y \tag{3.5.d}
\end{equation}

\begin{equation}
    M^1_{mn}(t) = \frac{4}{ab} \int_0^a \int_0^b M^M_{xx}(x, y, t) \sin \alpha x \sin \beta y \, dx \, dy \tag{3.6.a}
\end{equation}

\begin{equation}
    M^2_{mn}(t) = \frac{4}{ab} \int_0^a \int_0^b M^M_{yy}(x, y, t) \sin \alpha x \sin \beta y \, dx \, dy \tag{3.6.b}
\end{equation}

\begin{equation}
    P^1_{mn}(t) = \frac{4}{ab} \int_0^a \int_0^b P^M_{xx}(x, y, t) \sin \alpha x \sin \beta y \, dx \, dy \tag{3.6.c}
\end{equation}

\begin{equation}
    P^2_{mn}(t) = \frac{4}{ab} \int_0^a \int_0^b P^M_{yy}(x, y, t) \sin \alpha x \sin \beta y \, dx \, dy \tag{3.6.d}
\end{equation}

Substituting the above expansions into the governing equations expressed in terms of the generalized displacements, we obtain
where

\[ S_{33} = \bar{A}_{55} \alpha^2 + \bar{A}_{44} \beta^2 + c_1^2 [H_{11} \alpha^4 + 2(H_{12} + 2H_{66}) \alpha^2 \beta^2 + H_{22} \beta^4] \]

\[ S_{34} = \bar{A}_{55} \alpha - c_1 [\hat{F}_{11} \alpha^3 + (\hat{F}_{12} + 2 \hat{F}_{66}) \alpha \beta^2] = S_{43} \]

\[ S_{35} = \bar{A}_{44} \beta - c_1 [\hat{F}_{22} \beta^3 + (\hat{F}_{12} + 2 \hat{F}_{66}) \alpha^2 \beta] = S_{53} \]

\[ S_{45} = (\bar{D}_{12} + \bar{D}_{66}) \alpha \beta = S_{54} \]

\[ S_{44} = \bar{A}_{55} + \bar{D}_{11} \alpha^2 + \bar{D}_{66} \beta^2 \]

\[ S_{55} = \bar{A}_{44} + \bar{D}_{66} \alpha^2 + \bar{D}_{22} \beta^2 \]

\[ M_{33} = I_0 + c_1^2 I_6 (\alpha^2 + \beta^2) \]

\[ M_{34} = \beta (\hat{B}_{32} - c_1 \hat{E}_{32}^M) \]

\[ M_{35} = -c_1 J_4 \beta = M_{53} \quad \text{for cross-ply laminates} \]

\[ M_{35} = 0 = M_{53} \quad \text{for Antisymmetric angle-ply laminates} \]

\[ C_{34} = C_{35} = C_{44} = C_{45} = C_{54} = C_{55} = 0 \]

\[ C_{33} = -c_1 (\alpha^2 \hat{E}_{31}^M + \beta^2 \hat{E}_{32}^M) \]

\[ C_{43} = \alpha (\tilde{B}_{31} - c_1 \tilde{E}_{31}^M) \]

\[ C_{53} = \beta (\tilde{B}_{32} - c_1 \tilde{E}_{32}^M) \]

\[ I_i = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \rho^{(k)}(z)_i dz \quad (i = 0, 1, 2, ..., 6) \]

\[ J_{i} = I_i - c_1 I_{i+2}, \quad K_{2} = I_2 - 2c_1 I_4 + c_1^2 I_6 \]

\[ \bar{A}_{ij} = A_{ij} - 3c_1 D_{ij}, \quad \bar{D}_{ij} = D_{ij} - 3c_1 F_{ij}, \quad \hat{F}_{ij} = F_{ij} - c_1 H_{ij} \]
3.2. Analytical Solutions for Laminated Composite Shells

3.2.1. Modified Sanders Shell Theory

In this study, Donnell and Sanders nonlinear strain-displacement relations are utilized to derive the equation of motion. Sanders nonlinear shell theory is modified for shallow shells by omitting the following terms in the strain-displacement relations:

\[
\begin{aligned}
\left(\frac{u_0}{R_1}\right)^2, & \quad \left(\frac{v_0}{R_2}\right)^2, \quad \left(\frac{u_0}{R_1} \cdot \frac{v_0}{R_2}\right), \quad \left(\frac{u_0}{R_1} \frac{\partial w_0}{\partial X_1}\right), \quad \left(\frac{v_0}{R_2} \frac{\partial w_0}{\partial X_2}\right), \\
\left(\frac{u_0}{R_1} \frac{\partial w_0}{\partial X_2}\right), & \quad \left(\frac{v_0}{R_2} \frac{\partial w_0}{\partial X_1}\right)
\end{aligned}
\]  

(3.9)

The equations of motions for modified Sanders shell theory are

\[
\begin{aligned}
\frac{\partial N_1}{\partial X_1} + \frac{\partial N_6}{\partial X_2} + \frac{Q_1}{R_1} + c_1 \left(\frac{\partial P_2}{\partial X_1} + \frac{\partial P_6}{\partial X_2}\right) = & \quad T_0 \frac{\partial^2 u_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_1}{\partial t^2} - c_1 T_3 \frac{\partial^2 \frac{\partial w_0}{\partial X_1}}{\partial t^2} \\
\frac{\partial N_6}{\partial X_1} + \frac{\partial N_2}{\partial X_2} + \frac{Q_2}{R_2} + c_1 \left(\frac{\partial P_2}{\partial X_2} + \frac{\partial P_6}{\partial X_1}\right) = & \quad T_0 \frac{\partial^2 v_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_2}{\partial t^2} - c_1 T_3 \frac{\partial^2 \frac{\partial w_0}{\partial X_2}}{\partial t^2} \\
\frac{\partial Q_1}{\partial X_1} + \frac{\partial Q_2}{\partial X_2} + Q(w_0) + c_1 \left(\frac{\partial^2 P_2}{\partial X_1^2} + 2 \frac{\partial^2 P_6}{\partial X_1 \partial X_2} + \frac{\partial^2 P_2}{\partial X_2^2}\right) + & \quad q - \frac{N_1}{R_1} - \frac{N_2}{R_2} = I_0 \frac{\partial^2 w_0}{\partial t^2} - c_1 I_6 \frac{\partial^2 \frac{\partial w_0}{\partial X_1}}{\partial t^2} + \frac{\partial^2 \frac{\partial w_0}{\partial X_2}}{\partial t^2} \\
+ & \quad c_1 \left[T_3 \frac{\partial^2 u_0}{\partial X_1} + T_3 \frac{\partial^2 v_0}{\partial X_2} + J_4 \frac{\partial^2 \phi_1}{\partial X_1} + J_4 \frac{\partial^2 \phi_2}{\partial X_2}\right] \\
\frac{\partial M_1}{\partial X_1} + \frac{\partial M_6}{\partial X_2} - Q_1 = & \quad J_1 \frac{\partial^2 u_0}{\partial t^2} + K_2 \frac{\partial^2 \phi_1}{\partial t^2} - c_1 J_4 \frac{\partial^2 \frac{\partial w_0}{\partial X_1}}{\partial t^2}.
\end{aligned}
\]  

(3.10.a-b-c-d)
\[
\frac{\partial M_6}{\partial X_1} + \frac{\partial M_2}{\partial X_2} - \overline{Q}_2 = J_1 \frac{\partial^2 \nu_0}{\partial t^2} + R_2 \frac{\partial^2 \phi_2}{\partial t^2} - c_I J_4 \frac{\partial^2 \left( \frac{\partial w_0}{\partial X_2} \right)}{\partial t^2} \quad (3.10.e)
\]

where

\[
\begin{align*}
N(w_0) &= \frac{\partial}{\partial X_1} \left( N_1 \frac{\partial w_0}{\partial X_1} + N_6 \frac{\partial w_0}{\partial X_2} \right) + \frac{\partial}{\partial X_2} \left( N_6 \frac{\partial w_0}{\partial X_1} + N_2 \frac{\partial w_0}{\partial X_2} \right) \\
\overline{M}_\alpha &= M_\alpha - c_I P_\alpha \quad (\alpha = 1, 2, 6), \quad \overline{Q}_\beta = Q_\beta - c_\beta K_\beta \quad (\beta = 1, 2)
\end{align*}
\]

\[
I_i = \sum_{k=1}^{N} \int_{k}^{k+1} \rho^{(k)}(\zeta) d\zeta \quad (i = 0, 1, 2, ..., 6),
\]

\[
J_i = I_i - c_1 I_{i+2} \quad (i = 1, 2, 4), \quad R_2 = I_2 - 2c_1 I_4 + c_1^2 I_6,
\]

\[
\overline{I}_0 = I_0 + 2 \frac{c_1}{R_1} I_3 + \left( \frac{c_1}{R_1} \right)^2 I_6, \quad \overline{I}_0' = I_0 + 2 \frac{c_1}{R_2} I_3 + \left( \frac{c_1}{R_2} \right)^2 I_6, \quad (3.11)
\]

\[
\overline{J}_1 = J_1 + \frac{c_1}{R_1} J_4, \quad \overline{J}_1' = J_1 + \frac{c_1}{R_2} J_4,
\]

\[
\overline{I}_3 = I_3 + \frac{c_1}{R_1} I_6, \quad \overline{I}_3' = I_3 + \frac{c_1}{R_2} I_6
\]

The Sanders nonlinear equations of motion with smart material layers for shallow shells which is expressed in terms of displacements by substituting for the force and moment resultants are summarized in the Appendix A.

3.2.2. Navier Solutions of Shell Theories

For simply-supported shells whose projection in the \(x_1, x_2\) plane is a rectangle, the linear version of Equation (2.32) for Donnell shell theory and Equation (2.34) for Sanders shell theory can be solved exactly, provided the lamination scheme is of antisymmetric cross-ply or symmetric cross-ply type. The Navier solution exists if the following stiffness coefficients are zero:

\[
A_{i6} = B_{i6} = D_{i6} = E_{i6} = F_{i6} = H_{i6} = 0, \quad (i = 1, 2)
\]

\[
A_{45} = D_{45} = F_{45} = 0 \quad (3.12)
\]
The simply supported boundary condition (SS1) for the third-order shear deformation theory is assumed to be of the form

\[
\begin{align*}
\phi_1(x_1,0,t) &= 0, \quad \phi_1(x_1,b,t) = 0, \quad \phi_2(a,x_2,t) = 0, \quad \phi_2(a,x_2,t) = 0 \\
N_1(0,x_2,t) &= 0, \quad N_1(a,x_2,t) = 0, \quad N_2(x_1,0,t) = 0, \quad N_2(x_1,b,t) = 0 \\
\bar{M}_1(0,x_2,t) &= 0, \quad \bar{M}_1(a,x_2,t) = 0, \quad \bar{M}_2(x_1,0,t) = 0, \quad \bar{M}_2(x_1,b,t) = 0
\end{align*}
\]

(3.13)

where \(a\) and \(b\) denote the lengths along the \(x_1\) – and \(x_2\) – directions, respectively.

Following the Navier solution procedure, we assume the following solution form that satisfies the boundary conditions.

\[
\begin{align*}
u_0(x_1,x_2,t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{mn}(t) \cos \alpha x_1 \sin \beta x_2 \\
v_0(x_1,x_2,t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{mn}(t) \sin \alpha x_1 \cos \beta x_2 \\
w_0(x_1,x_2,t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{mn}(t) \sin \alpha x_1 \sin \beta x_2 \\
\phi_1(x_1,x_2,t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} X_{mn}(t) \cos \alpha x_1 \sin \beta x_2 \\
\phi_2(x_1,x_2,t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Y_{mn}(t) \sin \alpha x_1 \cos \beta x_2
\end{align*}
\]

(3.14.a, b, c, d, e)

where \(\alpha = \frac{m\pi}{a}\) and \(\beta = \frac{n\pi}{b}\). The transverse load \(q\) is also expanded in double Fourier sine series

\[
q(x_1,x_2,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Q_{mn}(t) \sin \alpha x_1 \sin \beta x_2
\]

(3.15.a)

\[
Q_{mn}(t) = \frac{4}{ab} \int_0^a \int_0^b q(x_1,x_2,t) \sin \alpha x_1 \sin \beta x_2 \, dx_1 \, dx_2
\]

(3.15.b)
Substituting Equations (3.14) and (3.15) into Equation (2.30) for Donnell shell theory and Equation (2.34) for Sanders’ shell theory, we obtain

\[
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55}
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn} \\
X_{mn} \\
Y_{mn}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & C_{13} & 0 & 0 \\
0 & 0 & C_{23} & 0 & 0 \\
0 & 0 & C_{33} & 0 & 0 \\
0 & 0 & C_{43} & 0 & 0 \\
0 & 0 & C_{53} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{U}_{mn} \\
\dot{V}_{mn} \\
\dot{W}_{mn} \\
\dot{X}_{mn} \\
\dot{Y}_{mn}
\end{bmatrix}
\]  

(3.16)

where coefficients $S_{ij}$, $C_{ij}$, and $M_{ij}$ are defined below for the two theories.

(1) Donnell shell theory

\[
S_{11} = -\alpha^2 A_{11} - \beta^2 A_{66}, \quad S_{12} = -\alpha \beta (A_{12} + A_{66})
\]

\[
S_{13} = \alpha \left( \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2} \right) + c_i \alpha (\alpha^2 E_{11} + \beta^2 E_{12} + 2\beta^2 E_{66})
\]

\[
S_{14} = -\alpha^2 \hat{B}_{11} - \beta^2 \hat{B}_{66}, \quad S_{15} = -\alpha \beta (\hat{B}_{12} + \hat{B}_{66})
\]

\[
S_{22} = -\alpha^2 A_{66} - \beta^2 A_{22}
\]

\[
S_{23} = \beta \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) + c_i \beta (\alpha^2 E_{12} + \beta^2 E_{22} + 2\alpha^2 E_{66})
\]

\[
S_{24} = -\alpha \beta (\hat{B}_{12} + \hat{B}_{66}), \quad S_{25} = -\alpha^2 \hat{B}_{66} - \beta^2 \hat{B}_{22}
\]
\[ S_{33} = -c_i^2\left[H_{11}\alpha^4 + 2(H_{12} + 2H_{66})\alpha^2\beta^2 + H_{22}\beta^4\right] - \alpha^2 \hat{A}_{45} \]
\[ - \beta^2 \hat{A}_{44} - 2\alpha^2 c_i \left( \frac{E_{11}}{R_1} + \frac{E_{12}}{R_2} \right) - 2\beta^2 c_i \left( \frac{E_{12}}{R_1} + \frac{E_{22}}{R_2} \right) \]
\[ - \frac{1}{R_1} \left( \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2} \right) - \frac{1}{R_2} \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right), \]

\[ S_{34} = c_i [\alpha^3 \hat{F}_{11} + \alpha \beta^2(\hat{F}_{12} + 2\hat{F}_{66})] - \alpha \hat{A}_{55} + \alpha \left( \frac{\hat{B}_{11}}{R_1} + \frac{\hat{B}_{12}}{R_2} \right) \]

\[ S_{55} = c_i [\beta^3 \hat{F}_{22} + \alpha^2 \beta(\hat{F}_{12} + 2\hat{F}_{66})] - \beta \hat{A}_{44} + \beta \left( \frac{\hat{B}_{12}}{R_1} + \frac{\hat{B}_{22}}{R_2} \right) \]

\[ S_{44} = -\alpha^2 \overline{D}_{11} - \beta^2 \overline{D}_{66} - \hat{A}_{55}, \quad S_{45} = -\alpha \beta(\overline{D}_{12} + \overline{D}_{66}) \]

\[ S_{55} = -\alpha^2 \overline{D}_{66} - \beta^2 \overline{D}_{22} - \hat{A}_{44} \]

\[ \hat{B}_y = B_{y} - c_i E_{y}, \quad \hat{D}_y = D_{y} - c_i F_{y}, \quad \hat{F}_y = F_{y} - c_i H_{y} \]

\[ \overline{D}_{ij} = \hat{D}_{ij} = D_{ij} - 2c_i F_{ij} + c_i^2 H_{ij} \quad (i, j = 1, 2, 6) \]

\[ \overline{A}_{ij} = A_{ij} - 3c_i D_{ij}, \quad \overline{D}_{ij} = D_{ij} - 3c_i F_{ij} \]

\[ \hat{A}_y = \overline{A}_y - 3c_i \overline{D}_y = A_y - 6c_i D_y + 9c_i^2 F_y \quad (i, j = 4, 5) \]

\[ M_{11} = -I_0, \quad M_{13} = \alpha c_i I_3, \quad M_{14} = -J_1, \quad M_{22} = -I_0 \]

\[ M_{23} = \beta c_i I_3, \quad M_{25} = -J_1, \quad M_{33} = -I_0 - c_i^2 I_6(\alpha^2 + \beta^2) \]

\[ M_{34} = c_i J_4 \alpha, \quad M_{35} = c_i J_4 \beta, \quad M_{44} = -K_2, \quad M_{55} = -K_2 \]

\[ \begin{align*}
C_{31} &= \alpha \hat{A}_{31}, \quad C_{32} = \beta \hat{A}_{32}, \quad C_{33} = -c_i (\alpha^2 \hat{E}_{31} + \beta^2 \hat{E}_{32}) + \frac{\hat{A}_{31}}{R_1} + \frac{\hat{A}_{32}}{R_2} \\
C_{43} &= \alpha (\hat{B}_{31} - c_i \hat{E}_{31}), \quad C_{53} = \beta (\hat{B}_{32} - c_i \hat{E}_{32})
\end{align*} \]
(2) Sanders shell theory

\[ S_{11} = -\alpha^2 A_{11} - \beta^2 A_{66} \]
\[-\frac{1}{R_1} \left[ \frac{1}{R_1} \hat{A}_{55} + 2c_1 (\alpha^2 E_{11} + \beta^2 E_{66}) - \frac{c_1^2}{R_1} (\alpha^2 H_{11} + \beta^2 H_{66}) \right] \]

\[ S_{12} = -\alpha \beta (A_{12} - A_{66}) \]
\[-\alpha \beta \left( \frac{c_1}{R_1} + \frac{c_1}{R_2} \right) (E_{12} + E_{66}) - \alpha \beta \frac{c_1}{R_1} \frac{c_1}{R_2} (H_{12} + H_{66}) \]

\[ S_{13} = \alpha \left( \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2} \right) + c_1 \alpha (\alpha^2 E_{11} + \beta^2 E_{12} + 2 \beta^2 E_{66}) \]
\[ + \frac{\alpha}{R_1} \left[ \hat{A}_{55} + c_1^2 (\alpha^2 H_{11} + \beta^2 H_{12} + 2 \beta^2 H_{66}) + c_1 \left( \frac{E_{11}}{R_1} + \frac{E_{12}}{R_2} \right) \right] \]

\[ S_{14} = -\alpha^2 \hat{B}_{11} - \beta^2 \hat{B}_{66} + \frac{1}{R_1} \left[ \hat{A}_{55} - c_1 (\alpha^2 \hat{F}_{11} + \beta^2 \hat{F}_{66}) \right] \]

(3.18)

\[ S_{15} = -\alpha \beta (\hat{B}_{12} + \hat{B}_{66}) - c_1 \frac{1}{R_1} \alpha \beta (\hat{F}_{12} + \hat{F}_{66}) \]

\[ S_{22} = -\alpha^2 A_{66} - \beta^2 A_{22} \]
\[-\frac{1}{R_2} \left[ \frac{1}{R_2} \hat{A}_{44} + 2c_1 (\alpha^2 E_{66} + \beta^2 E_{22}) - \frac{c_1^2}{R_2} (\alpha^2 H_{66} + \beta^2 H_{22}) \right] \]

\[ S_{23} = \beta \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) + c_1 \beta (\alpha^2 E_{12} + \beta^2 E_{22} + 2 \alpha^2 E_{66}) \]
\[ + \frac{\beta}{R_2} \left[ \hat{A}_{44} + c_1^2 (\alpha^2 H_{12} + \beta^2 H_{22} + 2 \alpha^2 H_{66}) + c_1 \left( \frac{E_{12}}{R_1} + \frac{E_{22}}{R_2} \right) \right] \]

\[ S_{24} = -\alpha \beta (\hat{B}_{12} + \hat{B}_{66}) - c_1 \frac{1}{R_2} \alpha \beta (\hat{F}_{12} + \hat{F}_{66}) \]

\[ S_{25} = -\alpha^2 \hat{B}_{66} - \beta^2 \hat{B}_{22} + \frac{1}{R_2} \left[ \hat{A}_{44} - c_1 (\alpha^2 \hat{F}_{66} + \beta^2 \hat{F}_{22}) \right] \]
\[ S_{33} = -c_1^2 [H_1\alpha^4 + 2(H_{12} + 2H_{66})\alpha^2 \beta^2 + H_{22}\beta^4] - \alpha^2 \hat{A}_{55} \\
- \beta^2 \hat{A}_{44} - 2\alpha^2 c_1 \left( \frac{E_{11}}{R_1} + \frac{E_{12}}{R_2} \right) - 2\beta^2 c_1 \left( \frac{E_{12}}{R_1} + \frac{E_{22}}{R_2} \right) \\
- \frac{1}{R_1} \left( \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2} \right) - \frac{1}{R_2} \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) \]

\[ S_{34} = c_1 [\alpha^3 \hat{F}_{11} + \alpha \beta^2 (\hat{F}_{12} + 2\hat{F}_{66})] - \alpha \hat{A}_{55} + \alpha \left( \frac{\hat{B}_{11}}{R_1} + \frac{\hat{B}_{12}}{R_2} \right) \]

\[ S_{35} = c_1 [\beta^3 \hat{F}_{22} + \alpha^2 \beta (\hat{F}_{12} + 2\hat{F}_{66})] - \beta \hat{A}_{44} + \beta \left( \frac{\hat{B}_{12}}{R_1} + \frac{\hat{B}_{22}}{R_2} \right) \]

\[ S_{44} = -\alpha^2 \hat{D}_{11} - \beta^2 \hat{D}_{66} - \hat{A}_{55}, \quad S_{45} = -\alpha \beta (\hat{D}_{12} + \hat{D}_{66}) \]

\[ S_{55} = -\alpha^2 \hat{D}_{66} - \beta^2 \hat{D}_{22} - \hat{A}_{44} \]

\[ \hat{B}_y = B_y - c_1 E_y, \quad \hat{D}_y = D_y - c_1 F_y, \quad \hat{F}_y = F_y - c_1 H_y \]

\[ \hat{D}_{ij} = \hat{D}_{ij} - c_1 \hat{F}_{ij} = D_{ij} - 2c_1 F_{ij} + c_1^2 H_{ij} \quad (i, j = 1, 2, 6) \]

\[ \hat{A}_{ij} = A_{ij} - 3c_1 D_{ij}, \quad \hat{D}_{ij} = D_{ij} - 3c_1 F_{ij} \]

\[ \hat{A}_{ij} = \bar{A}_{ij} - 3c_1 \bar{D}_{ij} = A_{ij} - 6c_1 D_{ij} + 9c_1^2 F_{ij} \quad (i, j = 4, 5) \]

\[ M_{11} = -\bar{T}_0, \quad M_{13} = \alpha c_1 \bar{T}_3, \quad M_{14} = -\bar{J}_1 \]

\[ M_{22} = -\bar{T}_0, \quad M_{23} = \beta c_1 \bar{T}_3, \quad M_{25} = -\bar{J}_1' \]

\[ M_{33} = -I_0 - c_1^2 I_6 (\alpha^2 + \beta^2), \quad M_{34} = c_1 J_4 \alpha, \quad M_{35} = c_1 J_4 \beta \]

\[ M_{44} = -\bar{K}_2, \quad M_{55} = -\bar{K}_2 \]

\[ C_{31} = \alpha \left( \frac{\bar{A}_{31}}{R_1} + \frac{c_1 \bar{E}_{31}}{R_1} \right), \quad C_{32} = \beta \left( \frac{\bar{A}_{32}}{R_2} + \frac{c_1 \bar{E}_{32}}{R_2} \right) \]
\[ C_{33} = -c_1 \left( \alpha^2 \ddot{E}_{31} + \beta^2 \ddot{E}_{32} \right) + \frac{\ddot{A}_{31}}{R_1} + \frac{\ddot{A}_{32}}{R_2} \]

\[ C_{43} = \alpha \left( \ddot{B}_{31} - c_1 \ddot{E}_{31} \right), \quad C_{53} = \beta \left( \ddot{B}_{32} - c_1 \ddot{E}_{32} \right) \]

### 3.3. Vibration Control

For vibration control, the solution of the governing equations are sought in the form

\[ U_{mn}(t) = U_0 \ e^{jt}, \quad V_{mn}(t) = V_0 \ e^{jt}, \quad W_{mn}(t) = W_0 \ e^{jt} \]

\[ X_{mn}(t) = X_0 \ e^{jt}, \quad Y_{mn}(t) = Y_0 \ e^{jt} \]

and obtain, for non-trivial solution, the result

\[
\begin{bmatrix}
\ddot{S}_{11} & \ddot{S}_{12} & \ddot{S}_{13} & \ddot{S}_{14} & \ddot{S}_{15} \\
\ddot{S}_{21} & \ddot{S}_{22} & \ddot{S}_{23} & \ddot{S}_{24} & \ddot{S}_{25} \\
\ddot{S}_{31} & \ddot{S}_{32} & \ddot{S}_{33} & \ddot{S}_{34} & \ddot{S}_{35} \\
\ddot{S}_{41} & \ddot{S}_{42} & \ddot{S}_{43} & \ddot{S}_{44} & \ddot{S}_{45} \\
\ddot{S}_{51} & \ddot{S}_{52} & \ddot{S}_{53} & \ddot{S}_{54} & \ddot{S}_{55}
\end{bmatrix} = 0
\]  

(3.20)

where

\[
\ddot{S}_{ij} = S_{ij} + \lambda C_{ij} + \lambda^2 M_{ij}
\]

(3.21)

For the plate case, the Equation (3.20) reduces to

\[
\begin{bmatrix}
\ddot{S}_{33} & \ddot{S}_{34} & \ddot{S}_{35} \\
\ddot{S}_{43} & \ddot{S}_{44} & \ddot{S}_{45} \\
\ddot{S}_{53} & \ddot{S}_{54} & \ddot{S}_{55}
\end{bmatrix} = 0
\]  

(3.22)

Equations (3.20) and (3.22) give five sets of eigenvalues and three sets of eigenvalues, respectively. The lowest imaginary part corresponds to the transverse motion. The eigenvalue can be written as \( \lambda = -\alpha_d + i \omega_d \), so that the damped motion is given by
\[
\begin{align*}
 w_0(x_1, x_2, t) &= \frac{1}{\omega_d} e^{-\alpha_d t} \sin \omega_d t \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b} \\
(3.23)
\end{align*}
\]

In arriving at the last solution, the following initial conditions are used:

\[
\begin{align*}
 u_0(x_1, x_2, 0) &= 0, & \dot{u}_0(x_1, x_2, 0) &= 0 \\
 v_0(x_1, x_2, 0) &= 0, & \dot{v}_0(x_1, x_2, 0) &= 0 \\
 w_0(x_1, x_2, 0) &= 0, & \dot{w}_0(x_1, x_2, 0) &= 1 \\
\phi_1(x_1, x_2, 0) &= 0, & \dot{\phi}_1(x_1, x_2, 0) &= 0 \\
\phi_2(x_1, x_2, 0) &= 0, & \dot{\phi}_2(x_1, x_2, 0) &= 0 \\
(3.24)
\end{align*}
\]

### 3.4. Analytical Results for Laminated Composite Plates

Using the analytical solutions developed in the previous sections, numerical studies are carried out. In particular, the effect of the position of the smart material layer, the thickness of the smart material layer and the elastic composite layer, and the material properties of the elastic layer, on the frequency and vibration suppression time of the laminated composite plate. Here, the vibration suppression time is defined as the time required to reducing the uncontrolled vibration amplitude to one-tenth of its initial amplitude. Various values of the vibration suppression time ratio \( T_s \) (suppression time divided by the maximum suppression time) are obtained as the distance between the magnetostrictive layers and the neutral axis is varied. The vibration suppression time ratio can be shown to be \( T_s = \frac{h_m}{2z_m} \), where \( h_m \) is the thickness of the magnetostrictive layer and \( z_m \) is the distance between the mid plane of the magnetostrictive layer and the mid plane of the plate. Studies involving different lamination schemes, layer thickness, and control gain values have also been carried out.

Four different kinds of the elastic composite material are used. CFRP [composite fiber reinforced polymer], Gr-Ep (AS) [graphite-epoxy], Gl-Ep [glass epoxy], Br-Ep [boron
The material properties of magnetostrictive material, Terfenol-D, and typical composite materials are listed in Table 3.1.

Table 3.1 Material properties of magnetostrictive and elastic composite materials

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_1$ [GPa]</th>
<th>$E_2$ [GPa]</th>
<th>$G_{13}$ [GPa]</th>
<th>$G_{23}$ [GPa]</th>
<th>$G_{12}$ [GPa]</th>
<th>$\nu_{12}$</th>
<th>$\rho$ [Kg m$^{-3}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Terfenol-D</td>
<td>26.5</td>
<td>26.5</td>
<td>13.25</td>
<td>13.25</td>
<td>13.25</td>
<td>0.00</td>
<td>9250</td>
</tr>
<tr>
<td>CFRP</td>
<td>138.6</td>
<td>8.27</td>
<td>4.96</td>
<td>4.12</td>
<td>4.96</td>
<td>0.26</td>
<td>1824</td>
</tr>
<tr>
<td>Gr-Ep(AS)</td>
<td>137.9</td>
<td>8.96</td>
<td>7.20</td>
<td>6.21</td>
<td>7.20</td>
<td>0.30</td>
<td>1450</td>
</tr>
<tr>
<td>Gl-Ep</td>
<td>53.78</td>
<td>17.93</td>
<td>8.96</td>
<td>3.45</td>
<td>8.96</td>
<td>0.25</td>
<td>1900</td>
</tr>
<tr>
<td>Br-Ep</td>
<td>206.9</td>
<td>20.69</td>
<td>6.9</td>
<td>4.14</td>
<td>6.9</td>
<td>0.30</td>
<td>1950</td>
</tr>
</tbody>
</table>

3.4.1. Vibration Suppression of Different Modes

The vibration suppression characteristics of the first five vibration modes of the composite plates are also studied. Displacement versus time is presented in Figures 3.2(a) and 3.2(b). From the figures, it can be seen that the vibration suppression time decreases very rapidly as mode number increases. This is because the amplitude of vibration that has to be suppressed decreases as the mode number increases. The results have been obtained for a CFRP $(m/90/0/90/0)_S$ and Gr-Ep(AS) $(m/90/0/90/0)_S$ laminated composites with $h_e = 1$ mm and $h_m = 1$ mm. Here $m$ stands for the magnetostrictive material layer and the subscript “S” stands for symmetric.

3.4.2. Effect of Lamina Material Properties

The influence of lamina material properties on the amplitude of vibration and vibration suppression times has been studied and the results are tabulated. Table 3.2 lists the inertial coefficients of the different lamina materials used. The lamination scheme used in all the materials is $(m/90/0/90/0)_S$. This lamination scheme means that the laminated plate consists of 10 laminae, the fiber orientation being $(m/90/0/90/0/0/90/0/90/0/m)$. 


Table 3.2 Inertial coefficients of symmetric cross-ply \((m,90,0,90,0)s\) laminated plate

<table>
<thead>
<tr>
<th>Material</th>
<th>Lamination Scheme</th>
<th>(I_0) ([kg m^{-2}])</th>
<th>(I_2 (\times 10^{-3})) ([kg])</th>
<th>(I_4 (\times 10^{-9})) ([kg m^2])</th>
<th>(I_6 (\times 10^{-14})) ([kg m^4])</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFRP</td>
<td>((m/90/0/90/0)s)</td>
<td>33.092</td>
<td>4.5399</td>
<td>8.5208</td>
<td>17.171</td>
</tr>
<tr>
<td>Gr-Ep(AS)</td>
<td>((m/90/0/90/0)s)</td>
<td>30.100</td>
<td>4.3803</td>
<td>8.3676</td>
<td>16.996</td>
</tr>
<tr>
<td>Gl-Ep</td>
<td>((m/90/0/90/0)s)</td>
<td>33.700</td>
<td>4.5723</td>
<td>8.5519</td>
<td>17.207</td>
</tr>
<tr>
<td>Br-Ep</td>
<td>((m/90/0/90/0)s)</td>
<td>34.100</td>
<td>4.5937</td>
<td>8.5724</td>
<td>17.230</td>
</tr>
</tbody>
</table>

Table 3.3 contains the frequencies \(\omega_d\) and damping coefficient \(\alpha\) obtained using different composite materials. Figures 3.3(a) - 3.3(d) shows the vibration suppression characteristics of composite plates made up of different materials. Where the thickness of the lamina \(h_e\) is taken to be 1mm and the thickness of the smart material layer \(h_m\) is taken to be 1mm. It is observed that materials having almost same \(E_1/E_2\) ratios have similar vibration suppression characteristics.

3.4.3. Effect of Plate Theories

As mentioned in the introduction, the CLPT and FSDT theories are mere special cases of the TSDT. The comparison of the eigenvalues obtained by using all three theories is presented in Table 3.4. Figure 3.4 shows the vibration suppression behavior obtained from each plate theory. It is observed that the CLPT theory gives higher frequencies of

Table 3.3 Damping coefficients and frequencies for different materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Lamination Scheme</th>
<th>(-\alpha_d \pm \omega_d) ([rad s^{-1}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFRP</td>
<td>((m/90/0/90/0)s)</td>
<td>11.86 ± 184.508</td>
</tr>
<tr>
<td>Gr-Ep(AS)</td>
<td>((m/90/0/90/0)s)</td>
<td>13.043 ± 197.095</td>
</tr>
<tr>
<td>Gl-Ep</td>
<td>((m/90/0/90/0)s)</td>
<td>11.516 ± 162.968</td>
</tr>
<tr>
<td>Br-Ep</td>
<td>((m/90/0/90/0)s)</td>
<td>11.507 ± 212.05</td>
</tr>
</tbody>
</table>
Figure 3.2 Suppression characteristics for different modes of vibration
Figure 3.3 Effect of material properties on vibration suppression
(c) Gl-Ep (m/90/0/90/0)s

(d) Br-Ep (m/90/0/90/0)s

Figure 3.3 Continued
Figure 3.4 Effect of the different plate theories on vibration suppression
Figure 3.4 Continued

(c) Gr-Ep(AS) a/h=10 (m/90/0/90/0)s

(d) Gr-Ep(AS) a/h=20 (m/90/0/90/0)s
vibration. This is expected, since the CLPT theory renders the plate stiffer compared to the other two theories. Where the thickness of the lamina \( h_e \) is taken to be 1mm and the thickness of the smart material layer \( h_m \) is taken from 1 to 5 mm based on the ratio of \( a/h \).

### Table 3.4 Eigenvalues \((-\alpha_d \pm \omega_d)\) obtained through the different plate theories

<table>
<thead>
<tr>
<th>Material</th>
<th>Lamination Scheme</th>
<th>( a/h )</th>
<th>CLPT</th>
<th>FSDT (k=5/6)</th>
<th>TSDT</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFRP</td>
<td>(0/90/0/90/m)(_s)</td>
<td>10</td>
<td>13.199 ± 2049</td>
<td>11.570 ± 1911</td>
<td>11.512 ± 1935</td>
</tr>
<tr>
<td></td>
<td>(0/90/0/90/m)(_s)</td>
<td>20</td>
<td>6.599 ± 1024</td>
<td>6.372 ± 1006</td>
<td>6.368 ± 1009</td>
</tr>
<tr>
<td>Gr-Ep(AS)</td>
<td>(m/90/0/90/0)(_s)</td>
<td>10</td>
<td>126.95 ± 1945</td>
<td>117.32 ± 1866</td>
<td>116.592 ± 1848</td>
</tr>
<tr>
<td></td>
<td>(m/90/0/90/0)(_s)</td>
<td>20</td>
<td>65.299 ± 986.2</td>
<td>63.449 ± 971.8</td>
<td>63.324 ± 969.2</td>
</tr>
</tbody>
</table>

#### 3.4.4. Effect of Smart Material Position

Next, suppression times for CFRP laminates are studied. The lamina thickness \( h_e \) is taken to be 1mm and the thickness of the smart material layer \( h_m \) is taken to be 1mm. The natural frequencies and the damping coefficients for different lamination schemes are obtained. The maximum deflection \((W_{\text{max}})\) of the composite plate and the suppression times have been calculated.

Figure 3.5 shows the displacement versus time for various laminates. It is observed that as the smart material layer is moved farther from the mid-plane the suppression time decreases. This is expected since the moment generated by the actuation of the smart material is more as the smart material is moved away from the mid-plane. Figure 3.6 shows the effect of the smart material layer location on the vibration suppression time. It is observed that the vibration suppression time does not show appreciable change when the distance of the smart material layer from the mid-plane is reduced from 0.045m to 0.015m, but then increases by almost an order of magnitude when the smart material layer is moved from 0.0015m to 0.0005m from the mid-plane. The eigenvalues and suppression time for each lamination schemes is tabulated in Table 3.5, where \( z_m \) denotes the positive distance from the center of the laminate to the center of the smart layer.
Figure 3.5 Vibration suppression characteristics for different smart material location
Figure 3.5 Continued

(c) CFRP (0/90/m/90/0)s

(d) CFRP (90/m/90/0/90)s
Figure 3.5 Continued

(e) CFRP (m/90/0/90/0)s

Figure 3.6 Vibration suppression times for CFRP laminated plate
Table 3.5 Suppression times for different CFRP laminates

<table>
<thead>
<tr>
<th>Lamination Scheme</th>
<th>( z_m ) (m)</th>
<th>(-\alpha_d)</th>
<th>( \pm \omega_d)</th>
<th>( W_{\text{max}} ) (10(^{-3}) m)</th>
<th>t at ( W_{\text{max}}/10)</th>
<th>( T_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0/90/0/90/m)(_S)</td>
<td>0.0005</td>
<td>1.318</td>
<td>204.721</td>
<td>4.823</td>
<td>1.728</td>
<td>1.000</td>
</tr>
<tr>
<td>(90/0/90/m/90)(_S)</td>
<td>0.0015</td>
<td>3.954</td>
<td>202.800</td>
<td>4.771</td>
<td>0.563</td>
<td>0.333</td>
</tr>
<tr>
<td>(0/90/m/90/0)(_S)</td>
<td>0.0025</td>
<td>6.589</td>
<td>198.881</td>
<td>4.769</td>
<td>0.355</td>
<td>0.200</td>
</tr>
<tr>
<td>(90/m/90/0/90)(_S)</td>
<td>0.0035</td>
<td>9.224</td>
<td>192.823</td>
<td>4.815</td>
<td>0.253</td>
<td>0.143</td>
</tr>
<tr>
<td>(m/90/0/90/0)(_S)</td>
<td>0.0045</td>
<td>11.86</td>
<td>184.507</td>
<td>4.907</td>
<td>0.194</td>
<td>0.111</td>
</tr>
</tbody>
</table>

3.4.5. Effect of Lamina Thickness

The vibration characteristics are obtained for the CFRP laminate with the fiber orientation of (0/90/0/90/m)\(_S\) for different thicknesses of the lamina and smart material layers, keeping the control gain constant. The vibration frequencies, damping factors and suppression times are presented in Figure 3.7 and Table 3.6 for the different lamina thickness. Where \( h_e \) and \( h_m \) are the thicknesses of the elastic material layer and smart material layer, respectively.

Table 3.7 shows the effect of the position and thickness of the smart material layer and the thickness of the composite material laminae on the vibration suppression time ratio. It is observed that thinner smart material layers result in better attenuation of the vibration. This is due to a higher mass matrix that is caused by the large increase in the moment of inertia of the system when thickness of the smart material layer is increased. This increase is because the smart material layer has a density of almost five times to that of the composite material.
Table 3.6 Vibration suppression characteristics for the different lamina thickness

<table>
<thead>
<tr>
<th>Thickness (mm)</th>
<th>$-\alpha_d$</th>
<th>$\pm \omega_d$</th>
<th>$W_{\text{max}}$ ($10^{-3}$ m)</th>
<th>t at $W_{\text{max}}/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_e = 1, h_m = 1$</td>
<td>1.318</td>
<td>204.721</td>
<td>4.823</td>
<td>1.728</td>
</tr>
<tr>
<td>$h_e = 5, h_m = 2$</td>
<td>1.533</td>
<td>1023.00</td>
<td>0.947</td>
<td>1.499</td>
</tr>
<tr>
<td>$h_e = 2, h_m = 5$</td>
<td>8.910</td>
<td>442.249</td>
<td>2.140</td>
<td>0.259</td>
</tr>
<tr>
<td>$h_e = 5, h_m = 5$</td>
<td>6.368</td>
<td>1009.00</td>
<td>0.920</td>
<td>0.368</td>
</tr>
</tbody>
</table>

3.4.6. Effect of Feedback Coefficients

The value of the feedback coefficient $c(t)k_c$ influences the vibration suppression characteristics. The study is performed on a CFRP laminate with thickness of the elastic layer to be 1 mm and the thickness of the smart material to be 1 mm. Different lamination schemes are used and the position of the smart material layer is varied. Two different
values of the feedback coefficient are used: \(c(t)k_c = 10^4\) and \(c(t)k_c = 10^3\). The results obtained are presented in Table 3.8.

<table>
<thead>
<tr>
<th>Lamination Scheme</th>
<th>Thickness (mm)</th>
<th>(h_e/h_m)</th>
<th>(z_m(\text{m}))</th>
<th>(T_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0/90/0/90/m)S</td>
<td>(h_e = 1, h_m = 1)</td>
<td>1.0</td>
<td>0.0005</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 2)</td>
<td>2.5</td>
<td>0.0010</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>(h_e = 2, h_m = 5)</td>
<td>0.4</td>
<td>0.0025</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 5)</td>
<td>1.0</td>
<td>0.0025</td>
<td>1.0000</td>
</tr>
<tr>
<td>(90/0/90/m/90)S</td>
<td>(h_e = 1, h_m = 1)</td>
<td>1.0</td>
<td>0.0015</td>
<td>0.3333</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 2)</td>
<td>2.5</td>
<td>0.0060</td>
<td>0.1667</td>
</tr>
<tr>
<td></td>
<td>(h_e = 2, h_m = 5)</td>
<td>0.4</td>
<td>0.0045</td>
<td>0.5556</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 5)</td>
<td>1.0</td>
<td>0.0075</td>
<td>0.3333</td>
</tr>
<tr>
<td>(0/90/m/90/0)S</td>
<td>(h_e = 1, h_m = 1)</td>
<td>1.0</td>
<td>0.0025</td>
<td>0.2000</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 2)</td>
<td>2.5</td>
<td>0.0110</td>
<td>0.0909</td>
</tr>
<tr>
<td></td>
<td>(h_e = 2, h_m = 5)</td>
<td>0.4</td>
<td>0.0065</td>
<td>0.3846</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 5)</td>
<td>1.0</td>
<td>0.0125</td>
<td>0.2000</td>
</tr>
<tr>
<td>(90/m/90/0/90)S</td>
<td>(h_e = 1, h_m = 1)</td>
<td>1.0</td>
<td>0.0035</td>
<td>0.1429</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 2)</td>
<td>2.5</td>
<td>0.0160</td>
<td>0.0476</td>
</tr>
<tr>
<td></td>
<td>(h_e = 2, h_m = 5)</td>
<td>0.4</td>
<td>0.0085</td>
<td>0.2941</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 5)</td>
<td>1.0</td>
<td>0.0175</td>
<td>0.1429</td>
</tr>
<tr>
<td>(m/90/0/90/0)S</td>
<td>(h_e = 1, h_m = 1)</td>
<td>1.0</td>
<td>0.0045</td>
<td>0.1111</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 2)</td>
<td>2.5</td>
<td>0.0210</td>
<td>0.0476</td>
</tr>
<tr>
<td></td>
<td>(h_e = 2, h_m = 5)</td>
<td>0.4</td>
<td>0.0105</td>
<td>0.2381</td>
</tr>
<tr>
<td></td>
<td>(h_e = 5, h_m = 5)</td>
<td>1.0</td>
<td>0.0225</td>
<td>0.1111</td>
</tr>
</tbody>
</table>

The variation of the vibration suppression time for different laminates and for two different values of the feedback control coefficient are presented in Figure 3.8. It can be seen that the suppression time increases when the value of the feedback coefficient decreases. This is because the damping coefficients decrease, thereby resulting in less damping. However, from Table 3.7, it can be noted that there is no appreciable change in
the natural frequency of vibration for different lamination schemes when the value of the feedback coefficient changes.

Table 3.8 Suppression time for two control gains for different laminates

<table>
<thead>
<tr>
<th>Lamination Scheme</th>
<th>$z_m$ (m)</th>
<th>$c(t)k_c = 10^3$</th>
<th>$c(t)k_c = 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\alpha_d \pm \omega_d$</td>
<td>$W_{\text{max}}$ (10⁻³ m)</td>
</tr>
<tr>
<td>(0/90/0/90/m)ₘ</td>
<td>0.0005</td>
<td>0.132</td>
<td>204.726</td>
</tr>
<tr>
<td>(90/0/90/m/90)ₘ</td>
<td>0.0015</td>
<td>0.395</td>
<td>202.838</td>
</tr>
<tr>
<td>(0/90/m/90/0)ₘ</td>
<td>0.0025</td>
<td>0.659</td>
<td>198.989</td>
</tr>
<tr>
<td>(90/m/90/0/90)ₘ</td>
<td>0.0035</td>
<td>0.922</td>
<td>193.041</td>
</tr>
<tr>
<td>(m/90/0/90/0)ₘ</td>
<td>0.0045</td>
<td>1.186</td>
<td>184.884</td>
</tr>
</tbody>
</table>

Figure 3.8 Effect of feedback coefficient on the suppression time
3.4.7. Variation of $T_s$ and $e_r$, for Different Laminates

The variation of the vibration suppression ratio $T_s$ and the normalized damping parameter $e_r = \frac{\varepsilon_3}{\varepsilon_{\text{max}}}$ is studied for different positions of the smart material layer in the laminated composite plate. The results are tabulated in Table 3.9 and presented in Figure 3.9. From the earlier discussions, the vibration suppression ratio decreases when the smart material is moved away from the laminate. The normalized damping parameter increases as the smart material layer is moved away from the neutral axis. This is explained by the increased damping that is achieved when the smart material layer is moved away from the neutral axis. The variation is studied on a CFRP laminate with the thickness of the elastic composite layer and the thickness of the smart material layer being 1mm.

Table 3.9 $T_s$ and $e_r$ parameter for different CFRP laminates

<table>
<thead>
<tr>
<th>Laminate Scheme</th>
<th>$z_m$ (m)</th>
<th>$T_s$</th>
<th>$\varepsilon_{31}$</th>
<th>$e_r = \frac{\varepsilon_3}{\varepsilon_{\text{max}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0/90/0/90/m)_S$</td>
<td>0.0005</td>
<td>1.000</td>
<td>0.0295</td>
<td>0.0027</td>
</tr>
<tr>
<td>$(90/0/90/m/90)_S$</td>
<td>0.0015</td>
<td>0.333</td>
<td>0.4426</td>
<td>0.0407</td>
</tr>
<tr>
<td>$(0/90/m/90/0)_S$</td>
<td>0.0025</td>
<td>0.200</td>
<td>1.9177</td>
<td>0.1762</td>
</tr>
<tr>
<td>$(90/m/90/0/90)_S$</td>
<td>0.0035</td>
<td>0.143</td>
<td>5.1631</td>
<td>0.4743</td>
</tr>
<tr>
<td>$(m/90/0/90/0)_S$</td>
<td>0.0045</td>
<td>0.111</td>
<td>10.8867</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Figure 3.9 Variation of $T_s$ and $e_r$ with respect to $z_m$. 

(a) Relations between $T_s$ and $z_m$ 

(b) Relations between $e_r$ and $z_m$
3.5. Analytical Results for Laminated Composite Shells

Using the previously developed analytical solutions based on Donnell and Sanders shell theory, numerical parametric studies are carried out. Symmetric cross-ply laminated square shell \((a/b = 1)\) with both the upper and lower surfaces embedded magnetostrictive material, Terfenol-D, is considered under the initial unit velocity in \(\zeta\) direction for simply supported boundary condition. The laminated composite shells are composed of total 10 layers and all the layers are assumed to be of the same thickness. Three different shell types, spherical \((R_1 = R_2 = R)\), cylindrical \((R_1 = \infty)\), and doubly curved shell \((R_1 = 2R_2\) for this study), are considered with two side-to-thickness ratios \(a/h = 10\) and 100 for the thick and thin shells. The elastic composite materials considered in here to study the effect of the elastic material are CFRP (composite fiber reinforced polymer), Gr-Ep(AS) (graphite-epoxy), Gl-Ep (glass-epoxy), and Br-Ep (boron-epoxy). The material properties of these materials are tabulated in Table 3.1.

The inertial coefficients with different elastic material are shown in Table 3.10 for thin and thick cross-ply \((m,90,0,90,0)s\) laminated shells. Here \(m\) represents magnetostrictive layer, 90 and 0 for the angles of elastic material layer in degree and subscript \(s\) stands for symmetric.

3.5.1. The General

The symmetric cross-ply laminated CFRP spherical shells \((R_1 = R_2 = R)\) are considered to study the effect of mode and smart layer position on the vibration suppression characteristics. Table 3.11 shows the frequency \(\omega_d\) and damping coefficient \(-\alpha_d\) of the symmetric cross-ply \((m,90,0,90,0)s\) spherical shell with \(R_2/a = 10\) for different modes in the case of thin and thick shells. It is observed that the maximum amplitude of vibration and vibration suppression can be found in the mode \((1,1)\). This is because the amplitude of vibration that has to be suppressed decreases as the mode number increases. It can be seen clearly in Figure 3.10.
Table 3.10 Inertial coefficients of symmetric cross-ply (m,90,0,90,0)s laminated shell

<table>
<thead>
<tr>
<th>Thickness</th>
<th>Material</th>
<th>( I_0 ) [kg m(^{-2})]</th>
<th>( I_2 ) [kg]</th>
<th>( I_4 ) [kg m(^{-2})]</th>
<th>( I_6 ) [kg m(^4)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin Shell</td>
<td>CFRP</td>
<td>33.092</td>
<td>0.45399((\times 10^{-3}))</td>
<td>8.5208((\times 10^{-9}))</td>
<td>17.171((\times 10^{-14}))</td>
</tr>
<tr>
<td></td>
<td>Br-Ep</td>
<td>34.100</td>
<td>0.45937((\times 10^{-3}))</td>
<td>8.5724((\times 10^{-9}))</td>
<td>17.230((\times 10^{-14}))</td>
</tr>
<tr>
<td></td>
<td>Gr-Ep(AS)</td>
<td>30.100</td>
<td>0.43803((\times 10^{-3}))</td>
<td>8.3676((\times 10^{-9}))</td>
<td>16.996((\times 10^{-14}))</td>
</tr>
<tr>
<td></td>
<td>Gl-Ep</td>
<td>33.700</td>
<td>0.45723((\times 10^{-3}))</td>
<td>8.5519((\times 10^{-9}))</td>
<td>17.207((\times 10^{-14}))</td>
</tr>
<tr>
<td>Thick Shell</td>
<td>CFRP</td>
<td>330.92</td>
<td>0.45399</td>
<td>8.5208((\times 10^{-4}))</td>
<td>17.171((\times 10^{-7}))</td>
</tr>
<tr>
<td></td>
<td>Br-Ep</td>
<td>341.00</td>
<td>0.45937</td>
<td>8.5724((\times 10^{-4}))</td>
<td>17.230((\times 10^{-7}))</td>
</tr>
<tr>
<td></td>
<td>Gr-Ep(AS)</td>
<td>301.00</td>
<td>0.43803</td>
<td>8.3676((\times 10^{-4}))</td>
<td>16.996((\times 10^{-7}))</td>
</tr>
<tr>
<td></td>
<td>Gl-Ep</td>
<td>337.00</td>
<td>0.45723</td>
<td>8.5519((\times 10^{-4}))</td>
<td>17.207((\times 10^{-7}))</td>
</tr>
</tbody>
</table>

Table 3.11 Effect of the mode on the Eigenvalues of the symmetric cross-ply (m,90,0,90,0)s CFRP spherical shells with \( R_z/a = 10 \)

<table>
<thead>
<tr>
<th>Theory</th>
<th>Mode ((m, n))</th>
<th>Thin Shell ((a/h = 100))</th>
<th>Thick Shell ((a/h = 10))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(-\alpha_d)</td>
<td>(\omega_d)</td>
<td>(-\alpha_d)</td>
</tr>
<tr>
<td>Donnell</td>
<td>mode (1, 1)</td>
<td>16.2933</td>
<td>316.8491</td>
</tr>
<tr>
<td></td>
<td>mode (1, 3)</td>
<td>66.3169</td>
<td>1200.4960</td>
</tr>
<tr>
<td></td>
<td>mode (3, 1)</td>
<td>66.5636</td>
<td>979.4863</td>
</tr>
<tr>
<td></td>
<td>mode (3, 3)</td>
<td>109.7777</td>
<td>1667.9786</td>
</tr>
<tr>
<td></td>
<td>mode (1, 5)</td>
<td>159.3204</td>
<td>3112.0733</td>
</tr>
<tr>
<td></td>
<td>mode (5, 1)</td>
<td>161.1095</td>
<td>2462.9380</td>
</tr>
<tr>
<td></td>
<td>mode (5, 5)</td>
<td>289.8144</td>
<td>4520.7199</td>
</tr>
<tr>
<td>Sanders</td>
<td>mode (1, 1)</td>
<td>16.2814</td>
<td>316.7589</td>
</tr>
<tr>
<td></td>
<td>mode (1, 3)</td>
<td>66.2975</td>
<td>1200.3622</td>
</tr>
<tr>
<td></td>
<td>mode (3, 1)</td>
<td>66.5476</td>
<td>979.3728</td>
</tr>
<tr>
<td></td>
<td>mode (3, 3)</td>
<td>109.7507</td>
<td>1667.8271</td>
</tr>
<tr>
<td></td>
<td>mode (1, 5)</td>
<td>159.2856</td>
<td>3112.0733</td>
</tr>
<tr>
<td></td>
<td>mode (5, 1)</td>
<td>161.0844</td>
<td>2462.8316</td>
</tr>
<tr>
<td></td>
<td>mode (5, 5)</td>
<td>289.7587</td>
<td>4520.5710</td>
</tr>
</tbody>
</table>
Figure 3.10 Vibration suppression characteristics of CFRP spherical shells ($R_z/a = 10$) for the different mode; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)
The effect of the position of smart material layer is studied with the symmetric cross-ply spherical shell with $R_z/a = 3$. Four different cross-ply laminates, $(m,90,0,90,0)s$, $(90,m,0,90,0)s$, $(90,0,m,90,0)s$, and $(90,0,90,m,0)s$ are considered. It is observed that the damping coefficient $-\alpha_d$ increases with the distance between the smart layer and mid-plane of the shell. Thus, the vibration suppression time decreases as the smart material layers are moved from the mid-plane. The damping coefficients and frequencies for different smart layer positions are shown in Table 3.12. Figure 3.11 shows the vibration suppression characteristics of each lamination case. In case of thin shell, frequencies are close to each other so that the deviations are not obvious in the Figure. It is clear that $(m,90,0,90,0)s$ lamination has the maximum vibration suppression and lowest frequency for thin and thick shells.

Table 3.12 Effect of the lamination scheme on the Eigenvalues of the symmetric cross-ply CFRP spherical shells with $R_z/a = 3$

<table>
<thead>
<tr>
<th>Theory</th>
<th>Laminations</th>
<th>Thin Shell $(a/h = 100)$</th>
<th>Thick Shell $(a/h = 10)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-\alpha_d$</td>
<td>$\omega_d$</td>
<td>$-\alpha_d$</td>
</tr>
<tr>
<td>Donnell</td>
<td>$(m,90,0,90,0)s$</td>
<td>26.0966</td>
<td>872.7906</td>
</tr>
<tr>
<td></td>
<td>$(90,m,0,90,0)s$</td>
<td>23.5016</td>
<td>874.6319</td>
</tr>
<tr>
<td></td>
<td>$(90,0,m,90,0)s$</td>
<td>20.9074</td>
<td>876.0250</td>
</tr>
<tr>
<td></td>
<td>$(90,0,90,m,0)s$</td>
<td>18.3127</td>
<td>876.9696</td>
</tr>
<tr>
<td>Sanders</td>
<td>$(m,90,0,90,0)s$</td>
<td>25.9807</td>
<td>872.4346</td>
</tr>
<tr>
<td></td>
<td>$(90,m,0,90,0)s$</td>
<td>23.4096</td>
<td>874.2427</td>
</tr>
<tr>
<td></td>
<td>$(90,0,m,90,0)s$</td>
<td>20.8393</td>
<td>875.6105</td>
</tr>
<tr>
<td></td>
<td>$(90,0,90,m,0)s$</td>
<td>18.2687</td>
<td>876.5381</td>
</tr>
</tbody>
</table>
Figure 3.11 Vibration suppression characteristics of CFRP spherical shells \((R_z/a = 3)\) for the different lamination; (a) Thin shell \((a/h = 100)\), (b) Thick shell \((a/h = 10)\)
3.5.2. Effect of Shell Theories

First, it should be mentioned that the difference between Donnell and Sanders third-order shear deformation shell theories is not significant in linear analysis. Tables 3.13 and 3.14 show the eigenvalues of symmetric cross-ply laminated shells by Donnell and Sanders shell theories. It is observed that Donnell shell theory gives higher frequency and damping coefficients than Sanders shell theory. As total thickness of shell structures is increasing and $R_z/a$ value is decreasing, the numerical differences in eigenvalues become larger.

Table 3.13 Eigenvalues of the symmetric cross-ply $(m,90,0,90,0)s$ CFRP laminated shells by Donnell shell theory

<table>
<thead>
<tr>
<th>Thickness $R_z/a$</th>
<th>Spherical Shell</th>
<th>Cylindrical Shell</th>
<th>Doubly Curved Shell</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-\alpha_d$</td>
<td>$\omega_d$</td>
<td>$-\alpha_d$</td>
</tr>
<tr>
<td>Thin Shell $a/h=100$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>N/A*</td>
<td>N/A*</td>
<td>N/A*</td>
</tr>
<tr>
<td>2</td>
<td>32.2630</td>
<td>1280.8428</td>
<td>22.4376</td>
</tr>
<tr>
<td>3</td>
<td>26.0966</td>
<td>872.7906</td>
<td>19.0867</td>
</tr>
<tr>
<td>4</td>
<td>22.7193</td>
<td>667.9063</td>
<td>17.3330</td>
</tr>
<tr>
<td>5</td>
<td>20.6192</td>
<td>546.3928</td>
<td>16.2604</td>
</tr>
<tr>
<td>10</td>
<td>16.2933</td>
<td>316.8491</td>
<td>14.0782</td>
</tr>
<tr>
<td>20</td>
<td>14.0853</td>
<td>224.9884</td>
<td>12.9724</td>
</tr>
<tr>
<td>50</td>
<td>12.7514</td>
<td>191.5289</td>
<td>12.3055</td>
</tr>
<tr>
<td>100</td>
<td>12.3057</td>
<td>186.2710</td>
<td>12.0827</td>
</tr>
<tr>
<td>$10^{\text{99}}$</td>
<td>11.8596</td>
<td>184.5068</td>
<td>11.8596</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Thick Shell $a/h=10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>50</td>
</tr>
<tr>
<td>100</td>
</tr>
<tr>
<td>$10^{\text{99}}$</td>
</tr>
</tbody>
</table>

* Since $R_z/h < 20$, the current formulation cannot be applied to this case
However, when one considers the vibration control behavior it is hard to see the differences (see Figure 3.12). Thus, the following numerical results and discussion regarding the vibration control characteristics in linear analysis are common in both Donnell and Sanders shell theories.

Table 3.14 Eigenvalues of the symmetric cross-ply \((m,90,0,90,0)\)s CFRP laminated shells by Sanders shell theory

<table>
<thead>
<tr>
<th>Thickness ( \frac{R_z}{a} )</th>
<th>Spherical Shell (-\alpha_d)</th>
<th>Cylindrical Shell (-\alpha_d)</th>
<th>Doubly Curved Shell (-\alpha_d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin Shell (a/\bar{h} = 100)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>N/A*</td>
<td>N/A*</td>
<td>N/A*</td>
</tr>
<tr>
<td>2</td>
<td>32.0114</td>
<td>1280.3174</td>
<td>22.9742</td>
</tr>
<tr>
<td>3</td>
<td>25.9807</td>
<td>872.4346</td>
<td>19.0224</td>
</tr>
<tr>
<td>4</td>
<td>22.6525</td>
<td>667.6420</td>
<td>17.2959</td>
</tr>
<tr>
<td>5</td>
<td>20.5756</td>
<td>546.1846</td>
<td>16.2362</td>
</tr>
<tr>
<td>10</td>
<td>16.2814</td>
<td>316.7589</td>
<td>14.0715</td>
</tr>
<tr>
<td>20</td>
<td>14.0818</td>
<td>224.9568</td>
<td>12.9705</td>
</tr>
<tr>
<td>50</td>
<td>12.3054</td>
<td>186.2695</td>
<td>12.0825</td>
</tr>
<tr>
<td>100</td>
<td>11.8596</td>
<td>184.5068</td>
<td>11.8596</td>
</tr>
<tr>
<td>Thick Shell (a/\bar{h} = 10)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>N/A*</td>
<td>N/A*</td>
<td>N/A*</td>
</tr>
<tr>
<td>2</td>
<td>115.7878</td>
<td>2060.5069</td>
<td>109.1022</td>
</tr>
<tr>
<td>3</td>
<td>113.1522</td>
<td>1870.9359</td>
<td>107.8476</td>
</tr>
<tr>
<td>4</td>
<td>111.1736</td>
<td>1798.0837</td>
<td>106.9091</td>
</tr>
<tr>
<td>5</td>
<td>109.7873</td>
<td>1763.0312</td>
<td>106.2478</td>
</tr>
<tr>
<td>10</td>
<td>106.5941</td>
<td>1714.9065</td>
<td>104.7078</td>
</tr>
<tr>
<td>20</td>
<td>104.8007</td>
<td>1702.6525</td>
<td>103.8315</td>
</tr>
<tr>
<td>50</td>
<td>103.6659</td>
<td>1699.2348</td>
<td>103.2723</td>
</tr>
<tr>
<td>100</td>
<td>103.2782</td>
<td>1698.7575</td>
<td>103.0805</td>
</tr>
<tr>
<td>10(^9)</td>
<td>102.8859</td>
<td>1698.6124</td>
<td>102.8859</td>
</tr>
</tbody>
</table>

* Since \(R_z/\bar{h} < 20\), the current formulation cannot be applied to this case.
Figure 3.12 Vibration suppression characteristics of CFRP cylindrical shells for different shell theories; (a) Thin shell \((R_z/a = 2)\), (b) Thick shell \((R_z/a = 2)\)
3.5.3. Effect of $R_2/a$

Next, the effect of $R_2/a$ on the vibration suppression characteristics is studied for spherical, cylindrical, and doubly curved CFRP shells. In this study doubly curved shell is chosen as $R_i = 2R_2$ for specific example. Tables 3.13 for Donnell and 3.14 for Sanders shell theories show the frequency $\sigma_d$ and damping coefficient $-\alpha_d$ of the thin and thick symmetric cross-ply (m,90,0,90,0)s shells with various $R_2/a$ values. Here the cases of $R_2/a = 10^{99}$ pretends to a laminated composite plate. Note that the case of $R_2/a = 1$ is not available because the basic assumption of shallow shell ($h/R_i, h/R_2$ less than 1/20) for current theory is not valid. The center displacements versus time for spherical, cylindrical, and doubly curved shells are shown in Figures 3.13 to 3.15, respectively. The Figures clearly show the vibration suppression characteristics for the selected $R_2/a$ values for thin shell case.

It is observed that the damping coefficient $-\alpha_d$ and the frequency $\sigma_d$ both increase with decreasing $R_2/a$ value. Thus the shell with the smallest $R_2/a$ shows the maximum vibration suppression. The same trend should hold for all shell types. It is also observed that the spherical shell has the largest damping coefficient and frequency and the cylindrical shell has the smallest damping coefficient and frequency. It is because of the fact that spherical shell has the smallest $R_i$ and doubly curved shell (in this study $R_i = 2R_2$), and cylindrical shell has the largest $R_i$ ($R_i = \infty$).
Figure 3.13 Effect of $R_2 / a$ on vibration suppression characteristics of CFRP spherical shells; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)
Figure 3.14 Effect of $R_2/a$ on vibration suppression characteristics of CFRP cylindrical shells; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)
Figure 3.15 Effect of $R_2/a$ on vibration suppression characteristics of CFRP doubly curved shells; (a) Thin shell ($a/h = 100$), (b) Thick shell ($a/h = 10$)
3.5.4. Effect of Shell Types

The comparison of vibration suppression characteristics between three different shell types could be found in Figure 3.16. Figure 3.16 shows the center displacements in case of $R_z/a = 5$ for thin shell, $R_z/a = 4$ for thick shell. It is clear that the spherical shell shows the maximum vibration suppression results from the Figure. Selected center displacement values versus time are tabulated in Table 3.15 for Sanders shell theory. The numerical results are for the thin and thick symmetric cross-ply $(m,90,0,90,0)s$ shells with $R_z/a = 5$.

The maximum deflection and vibration suppression time have been tabulated in Table 3.16 for CFRP laminated shells. In this study, vibration suppression time is defined as the time required to reduce the center displacements to 10% of its uncontrolled magnitude. The maximum transverse deflection and vibration suppression time are occurred in cylindrical shell with the largest $R_z/a$. The spherical shell with the smallest $R_z/a$ shows the minimum deflection and vibration suppression time.

3.5.5. Effect of Material Properties

Finally, the effect of elastic material property on the vibration suppression characteristics is studied for spherical, cylindrical, and doubly curved shells. Gr-Ep(AS), Gl-Ep, and Br-Ep are considered in addition to CFRP as elastic materials of laminated composite shells. The frequency $\omega$, damping coefficient $-\alpha$, the maximum deflection $w_{\text{max}}$, and vibration suppression time of the thin and thick shells with different elastic materials are tabulated in Tables 3.17 and 3.18. Figure 3.17 shows the vibration suppression of different elastic materials for the thin shells. The same trend of vibration suppression characteristics of CFRP shells could be found in the shells with different elastic materials. It is observed from the tables that Gl-Ep shows the largest damping coefficients, maximum deflections and the minimum vibration suppression time regardless of shell type and thickness.
Figure 3.16 Vibration suppression characteristics for different CFRP shells; (a) Thin shell \((a/h = 100)\), (b) Thick shell \((a/h = 10)\)
Table 3.15 Selected center displacement versus time of the symmetric cross-ply (m,90,0,90,0)s CFRP laminated shells with $R_s/a = 5$ by Sanders shell theory

<table>
<thead>
<tr>
<th>Centre Displacement ($\times 10^4$ m)</th>
<th>Thin Shell ($a/h = 100$)</th>
<th>Thick Shell ($a/h = 10$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Spherical shell</td>
<td>Spherical shell</td>
</tr>
<tr>
<td></td>
<td>Cylindrical shell</td>
<td>Cylindrical shell</td>
</tr>
<tr>
<td></td>
<td>Doubly curved shell</td>
<td>Doubly curved shell</td>
</tr>
<tr>
<td>t ($\times 10^4$ sec)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.9974</td>
<td>0.9940</td>
</tr>
<tr>
<td>2</td>
<td>1.9880</td>
<td>1.9200</td>
</tr>
<tr>
<td>3</td>
<td>2.9680</td>
<td>2.7690</td>
</tr>
<tr>
<td>4</td>
<td>3.9360</td>
<td>3.5190</td>
</tr>
<tr>
<td>5</td>
<td>4.8880</td>
<td>4.1430</td>
</tr>
<tr>
<td>7</td>
<td>6.7330</td>
<td>4.9580</td>
</tr>
<tr>
<td>10</td>
<td>9.3160</td>
<td>4.9890</td>
</tr>
<tr>
<td>15</td>
<td>12.9700</td>
<td>2.2940</td>
</tr>
<tr>
<td>20</td>
<td>15.6000</td>
<td>-1.7080</td>
</tr>
<tr>
<td>30</td>
<td>17.1700</td>
<td>-3.4200</td>
</tr>
<tr>
<td>40</td>
<td>13.7800</td>
<td>2.5420</td>
</tr>
<tr>
<td>50</td>
<td>6.5950</td>
<td>1.8760</td>
</tr>
<tr>
<td>70</td>
<td>-9.9890</td>
<td>-0.5872</td>
</tr>
<tr>
<td>90</td>
<td>-14.9000</td>
<td>-0.3350</td>
</tr>
<tr>
<td>110</td>
<td>-3.9670</td>
<td>-0.0524</td>
</tr>
<tr>
<td>130</td>
<td>10.2200</td>
<td>-0.3655</td>
</tr>
<tr>
<td>150</td>
<td>12.6800</td>
<td>0.6022</td>
</tr>
<tr>
<td>200</td>
<td>-12.1000</td>
<td>-0.4081</td>
</tr>
<tr>
<td>250</td>
<td>9.6960</td>
<td>0.0340</td>
</tr>
<tr>
<td>300</td>
<td>-6.1910</td>
<td>-0.3837</td>
</tr>
<tr>
<td>400</td>
<td>1.1520</td>
<td>0.0044</td>
</tr>
<tr>
<td>500</td>
<td>5.3800</td>
<td>-0.0191</td>
</tr>
<tr>
<td>700</td>
<td>2.2070</td>
<td>-0.0020</td>
</tr>
<tr>
<td>900</td>
<td>-2.5720</td>
<td>0.0003</td>
</tr>
<tr>
<td>1000</td>
<td>-2.1900</td>
<td>-0.0001</td>
</tr>
</tbody>
</table>
Table 3.16 Maximum transverse deflection and vibration suppression time for the symmetric cross-ply (m,90,0,90,0)s CFRP laminated shells

<table>
<thead>
<tr>
<th>$R_z/a$</th>
<th>Shell Type</th>
<th>Thin Shell ($a/h = 100$)</th>
<th>Thick Shell ($a/h = 10$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$w_{max}$</td>
<td>$t$(sec) at $w_{max}/10$</td>
</tr>
<tr>
<td>3</td>
<td>Spherical Shell</td>
<td>0.001094</td>
<td>0.0887</td>
</tr>
<tr>
<td></td>
<td>Cylindrical Shell</td>
<td>0.002018</td>
<td>0.1247</td>
</tr>
<tr>
<td></td>
<td>Doubly Curved Shell</td>
<td>0.001422</td>
<td>0.1059</td>
</tr>
<tr>
<td>5</td>
<td>Spherical Shell</td>
<td>0.001727</td>
<td>0.1125</td>
</tr>
<tr>
<td></td>
<td>Cylindrical Shell</td>
<td>0.002920</td>
<td>0.1451</td>
</tr>
<tr>
<td></td>
<td>Doubly Curved Shell</td>
<td>0.002187</td>
<td>0.1223</td>
</tr>
<tr>
<td>10</td>
<td>Spherical Shell</td>
<td>0.002916</td>
<td>0.1444</td>
</tr>
<tr>
<td></td>
<td>Cylindrical Shell</td>
<td>0.004039</td>
<td>0.1756</td>
</tr>
<tr>
<td></td>
<td>Doubly Curved Shell</td>
<td>0.003430</td>
<td>0.1591</td>
</tr>
<tr>
<td>50</td>
<td>Spherical Shell</td>
<td>0.004713</td>
<td>0.1895</td>
</tr>
<tr>
<td></td>
<td>Cylindrical Shell</td>
<td>0.004850</td>
<td>0.1943</td>
</tr>
<tr>
<td></td>
<td>Doubly Curved Shell</td>
<td>0.004790</td>
<td>0.1914</td>
</tr>
<tr>
<td>100</td>
<td>Spherical Shell</td>
<td>0.004850</td>
<td>0.1943</td>
</tr>
<tr>
<td></td>
<td>Cylindrical Shell</td>
<td>0.004890</td>
<td>0.1975</td>
</tr>
<tr>
<td></td>
<td>Doubly Curved Shell</td>
<td>0.004873</td>
<td>0.1956</td>
</tr>
</tbody>
</table>
Table 3.17 Vibration suppression characteristics for the thin symmetric cross-ply (m,90,0,90,0)s laminated shells with the different composite materials

<table>
<thead>
<tr>
<th>Thickness</th>
<th>Material</th>
<th>$R_c/a$</th>
<th>Shell Type</th>
<th>$-\alpha_d$</th>
<th>$\omega_d$</th>
<th>$w_{\text{max}}$</th>
<th>$t(\text{sec})$</th>
<th>$w_{\text{max}}/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Br-Ep</td>
<td></td>
<td>Spherical</td>
<td>22.7268</td>
<td>972.4400</td>
<td>0.000992</td>
<td>0.0987</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>17.2074</td>
<td>520.6025</td>
<td>0.001824</td>
<td>0.1237</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>20.0674</td>
<td>744.5307</td>
<td>0.001288</td>
<td>0.1123</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Br-Ep</td>
<td>10</td>
<td>Spherical</td>
<td>15.0417</td>
<td>356.6537</td>
<td>0.002626</td>
<td>0.1553</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>13.2753</td>
<td>255.8587</td>
<td>0.003607</td>
<td>0.1778</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>14.1640</td>
<td>302.0174</td>
<td>0.003079</td>
<td>0.1623</td>
<td></td>
</tr>
<tr>
<td>$a/h = 100$</td>
<td>Gr-Ep (AS)</td>
<td>3</td>
<td>Spherical</td>
<td>31.9361</td>
<td>1006.8317</td>
<td>0.000945</td>
<td>0.0702</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>22.6085</td>
<td>532.1451</td>
<td>0.001759</td>
<td>0.1031</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>27.4022</td>
<td>767.8548</td>
<td>0.001232</td>
<td>0.0838</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Gr-Ep (AS)</td>
<td>10</td>
<td>Spherical</td>
<td>18.9261</td>
<td>357.2369</td>
<td>0.002579</td>
<td>0.1273</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>15.9857</td>
<td>246.9154</td>
<td>0.003666</td>
<td>0.1588</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>17.4624</td>
<td>297.9251</td>
<td>0.003066</td>
<td>0.1316</td>
<td></td>
</tr>
<tr>
<td>$a/h = 100$</td>
<td>Gl-Ep</td>
<td>3</td>
<td>Spherical</td>
<td>40.5446</td>
<td>933.7043</td>
<td>0.001001</td>
<td>0.0560</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>26.1841</td>
<td>487.8336</td>
<td>0.001887</td>
<td>0.0876</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>33.4922</td>
<td>709.4313</td>
<td>0.001309</td>
<td>0.0733</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Gl-Ep</td>
<td>10</td>
<td>Spherical</td>
<td>20.5195</td>
<td>321.6349</td>
<td>0.002818</td>
<td>0.1219</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>16.0851</td>
<td>214.3368</td>
<td>0.004158</td>
<td>0.1532</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>18.3081</td>
<td>264.4086</td>
<td>0.003400</td>
<td>0.1261</td>
<td></td>
</tr>
</tbody>
</table>
Table 3.18 Vibration suppression characteristics for the thick symmetric cross-ply (m,90,0,90,0)s laminated shells with the different composite materials

<table>
<thead>
<tr>
<th>Thickness</th>
<th>Material</th>
<th>$R_c/a$</th>
<th>Shell Type</th>
<th>$-\alpha_d$</th>
<th>$\omega_d$</th>
<th>$w_{max}$</th>
<th>$t(\text{sec})$ at $w_{max}/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Br-Ep</td>
<td>3</td>
<td>Spherical</td>
<td>105.8611</td>
<td>2115.7754</td>
<td>0.000437</td>
<td>0.0213</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>101.9342</td>
<td>1965.9601</td>
<td>0.000469</td>
<td>0.0230</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>104.3097</td>
<td>2033.2345</td>
<td>0.000452</td>
<td>0.0222</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>Spherical</td>
<td>101.1661</td>
<td>1947.3458</td>
<td>0.000474</td>
<td>0.0233</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>99.6938</td>
<td>1932.9803</td>
<td>0.000478</td>
<td>0.0235</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>100.4647</td>
<td>1939.2729</td>
<td>0.000476</td>
<td>0.0235</td>
<td></td>
</tr>
<tr>
<td>Gr-Ep (AS)</td>
<td>3</td>
<td>Spherical</td>
<td>131.2207</td>
<td>2063.4504</td>
<td>0.000438</td>
<td>0.0189</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>123.6701</td>
<td>1892.5972</td>
<td>0.000478</td>
<td>0.0191</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>127.9278</td>
<td>1968.9198</td>
<td>0.000458</td>
<td>0.0189</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>Spherical</td>
<td>121.6644</td>
<td>1868.2618</td>
<td>0.000484</td>
<td>0.0209</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>119.0876</td>
<td>1851.7913</td>
<td>0.000489</td>
<td>0.0195</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>120.4154</td>
<td>1858.9472</td>
<td>0.000487</td>
<td>0.0195</td>
<td></td>
</tr>
<tr>
<td>Gl-Ep</td>
<td>3</td>
<td>Spherical</td>
<td>131.0681</td>
<td>1782.7453</td>
<td>0.000500</td>
<td>0.0185</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>118.0475</td>
<td>1604.9161</td>
<td>0.000556</td>
<td>0.0205</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>124.9809</td>
<td>1684.1377</td>
<td>0.000530</td>
<td>0.0195</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>Spherical</td>
<td>113.8741</td>
<td>1575.5661</td>
<td>0.000566</td>
<td>0.0209</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cylindrical</td>
<td>109.7757</td>
<td>1558.2955</td>
<td>0.000575</td>
<td>0.0212</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Doubly Curved</td>
<td>111.8580</td>
<td>1565.7502</td>
<td>0.000571</td>
<td>0.0210</td>
<td></td>
</tr>
</tbody>
</table>
Figure 3.17 Effect of elastic material properties on vibration suppression characteristics for thin ($a/h = 100$) shells; (a) spherical shell, (b) cylindrical shell, (c) doubly curved shell
Figure 3.17 Continued
4. FINITE ELEMENT FORMULATIONS*

4.1. Virtual Work Statements

In this Section, the linear and nonlinear finite element formulations and numerical results of the linear analysis of laminated composite plate and shell structures are presented. Finite element models are developed using the weak forms of governing differential equations (see Reddy 2004 b).

4.1.1. Virtual Work Statements for Laminated Composite Plates

The virtual work statements of the TSDT over a typical finite element domain $\Omega$ are given by

$$
0 = \int_{\Omega} \left\{ \frac{\partial \delta u_0}{\partial x} N_{xx} + \frac{\partial \delta u_0}{\partial y} N_{xy} + \delta u_0 \left[ I_0 \frac{\partial^2 u_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_0}{\partial t^2} - c_1 I_s \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial x} \right) \right] \right\} dxdy - \oint_{\Gamma} \{\delta u_0 (N_{xx} n_x + N_{xy} n_y)\} ds
$$

(4.1.a)

$$
0 = \int_{\Omega} \left\{ \frac{\partial \delta v_0}{\partial x} N_{xy} + \frac{\partial \delta v_0}{\partial y} N_{yy} + \delta v_0 \left[ I_0 \frac{\partial^2 v_0}{\partial t^2} + J_1 \frac{\partial^2 \phi_0}{\partial t^2} - c_1 I_s \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial y} \right) \right] \right\} dxdy - \oint_{\Gamma} \{\delta v_0 (N_{xy} n_x + N_{yy} n_y)\} ds
$$

(4.1.b)

$$
0 = \int_{\Omega} \left\{ \frac{\partial \delta w_0}{\partial x} \bar{Q}_x + \frac{\partial \delta w_0}{\partial y} \bar{Q}_y - c_1 \left( \frac{\partial^2 \delta w_0}{\partial x^2} P_{xx} + 2 \frac{\partial^2 \delta w_0}{\partial x \partial y} P_{xy} + \frac{\partial^2 \delta w_0}{\partial y^2} P_{yy} \right) - \delta w_0 q + \frac{\partial \delta w_0}{\partial x} \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial \delta w_0}{\partial y} \right\} dxdy
$$

(4.1.c)

As a special case of TSDT, the virtual statements for classical laminated plate theory are

\[ 0 = \int_{\Omega} \left( \frac{\partial \delta \phi_{x}}{\partial x} M_{x x} + \frac{\partial \delta \phi_{y}}{\partial y} M_{x y} + \delta \phi_{x} \overline{Q}_{x} + \delta \phi_{y} \overline{Q}_{y} \right) \frac{1}{\partial t^2} (J_{1} u_{0} + K_{2} \phi_{x} - c_{1} J_{4} \frac{\partial w_{0}}{\partial x}) \right) dx dy - \oint_{\Gamma} \{ \delta u_{0} (N_{x x} n_{x} + N_{x y} n_{y}) \} ds \]  

(4.1.d)

\[ 0 = \int_{\Omega} \left( \frac{\partial \delta \phi_{y}}{\partial x} M_{x y} + \frac{\partial \delta \phi_{y}}{\partial y} M_{y y} + \delta \phi_{y} \overline{Q}_{y} + \delta \phi_{y} \overline{Q}_{y} \right) \frac{1}{\partial t^2} (J_{1} v_{0} + K_{2} \phi_{y} - c_{1} J_{4} \frac{\partial w_{0}}{\partial y}) \right) dx dy - \oint_{\Gamma} \{ \delta v_{0} (N_{y x} n_{x} + N_{y y} n_{y}) \} ds \]  

(4.1.e)
\[ 0 = \int_{\Omega} \left[ -\frac{\partial^2 \delta w_0}{\partial x^2} M_{xx} - 2 \frac{\partial^2 \delta w_0}{\partial x \partial y} M_{xy} - \frac{\partial^2 \delta w_0}{\partial y^2} M_{yy} - \delta w_0 q + \frac{\partial \delta w_0}{\partial x} \right. \]
\[ \left. \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial \delta w_0}{\partial y} \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right) + \delta w_0 I_0 \right. \]
\[ \frac{\partial^2 w_0}{\partial t^2} + I_2 \frac{\partial^2}{\partial t^2} \left( \frac{\partial \delta w_0}{\partial x} \frac{\partial w_0}{\partial x} \frac{\partial \delta w_0}{\partial y} \frac{\partial w_0}{\partial y} \right) - I_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial \delta w_0}{\partial x} \frac{\partial w_0}{\partial x} \right) (4.2.c) \]
\[ + \frac{\partial \delta w_0}{\partial y} v_0 \right] \right) dx dy - \oint_{\Gamma} (\delta w_0 V_n + P_n) ds \]

where
\[ V_n = \left[ \left( \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x + \left( \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right) n_y \right] \]
\[ \left[ I_1 \ddot{u}_0 - I_2 \frac{\partial \ddot{w}_0}{\partial x} \right] n_x + \left[ I_1 \ddot{v}_0 - I_2 \frac{\partial \ddot{w}_0}{\partial y} \right] n_y \]
\[ P_n = \left( M_{xx} n_x + M_{xy} n_y \right) \frac{\partial \delta w_0}{\partial x} + \left( M_{xy} n_x + M_{yy} n_y \right) \frac{\partial \delta w_0}{\partial y} \] (4.3.b)

For the first-order shear deformation theory, the virtual work statements are
\[ 0 = \int_{\Omega} \left[ \frac{\partial \delta u_0}{\partial x} N_{xx} + \frac{\partial \delta u_0}{\partial y} N_{xy} + \delta u_0 \left[ I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2} \right] \right] dx dy \] (4.4.a)
\[ - \oint_{\Gamma} \left\{ \delta u_0 \left( N_{xx} n_x + N_{xy} n_y \right) \right\} ds \]
\[ 0 = \int_{\Omega} \left[ \frac{\partial \delta v_0}{\partial x} N_{xy} + \frac{\partial \delta v_0}{\partial y} N_{yy} + \delta v_0 \left[ I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} \right] \right] dx dy \] (4.4.b)
\[ - \oint_{\Gamma} \left\{ \delta v_0 \left( N_{xy} n_x + N_{yy} n_y \right) \right\} ds \]
\[ 0 = \int_{\alpha} \left( \frac{\partial \delta w_0}{\partial x} Q_x + \frac{\partial \delta w_0}{\partial y} Q_y - \delta w_0 q + \frac{\partial \delta w_0}{\partial x} \left( N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) \right) \, dx \, dy + \delta w_0 I_0 \frac{\partial^2 w_0}{\partial t^2} \, ds \]

\[ (4.4.c) \]

\[ 0 = \int_{\alpha} \left( \frac{\partial \delta \phi_x}{\partial x} M_{xx} + \frac{\partial \delta \phi_x}{\partial y} M_{xy} + \delta \phi_x Q_x + \delta \phi_x \left[ I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u_0}{\partial t^2} \right] \right) \, dx \, dy \]

\[- \oint_r \left\{ \delta \phi_x \left( M_{xx} \eta_x + M_{xy} \eta_y \right) \right\} ds \]

\[ (4.4.d) \]

\[ 0 = \int_{\alpha} \left( \frac{\partial \delta \phi_y}{\partial x} M_{xx} + \frac{\partial \delta \phi_y}{\partial y} M_{xy} + \delta \phi_y Q_y + \delta \phi_y \left[ I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v_0}{\partial t^2} \right] \right) \, dx \, dy \]

\[- \oint_r \left\{ \delta \phi_y \left( M_{xx} \eta_x + M_{xy} \eta_y \right) \right\} ds \]

\[ (4.4.e) \]

where

\[ V_n = \left[ \left( Q_x + N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) n_x + \left( Q_y + N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y} \right) n_y \right] \]

\[ (4.5) \]

### 4.1.2. Virtual Work Statements for Laminated Composite Shells

(1) **Donnell nonlinear shell theory**

\[ 0 = \int_{\alpha} \left( \frac{\partial \delta u_0}{\partial X_1} N_1 + \frac{\partial \delta u_0}{\partial X_2} N_6 + \delta u_0 \left[ I_0 \frac{\partial^2 u_0}{\partial t^2} + J_1 \frac{\partial^2 \phi}{\partial t^2} \right] \right) \, dX_1 \, dX_2 - \oint_r \{ \delta u_0 (N_1 n_1 + N_6 n_2) \} ds \]

\[ (4.6.a) \]

\[ 0 = \int_{\alpha} \left( \frac{\partial \delta v_0}{\partial X_1} N_6 + \frac{\partial \delta v_0}{\partial X_2} N_2 + \delta v_0 \left[ I_0 \frac{\partial^2 v_0}{\partial t^2} + J_1 \frac{\partial^2 \phi}{\partial t^2} \right] \right) \, dX_1 \, dX_2 - \oint_r \{ \delta v_0 (N_6 n_1 + N_2 n_2) \} ds \]

\[ (4.6.b) \]
\[
0 = \int_\Omega \left\{ \frac{\partial \delta w_0}{\partial X_1} \tilde{Q}_1 + \frac{\partial \delta w_0}{\partial X_2} \tilde{Q}_2 - c_1 \left( \frac{\partial^2 \delta w_0}{\partial X_1^2} P + 2 \frac{\partial^2 \delta w_0}{\partial X_1 \partial X_2} P_6 + \frac{\partial^2 \delta w_0}{\partial X_2^2} p_2 \right) \\
- \delta w_0 q + \delta w_0 \left( \frac{N_1}{R_1} + \frac{N_2}{R_2} \right) + \delta w_0 I_0 \frac{\partial^2 w_0}{\partial t^2} + c_1 I_6 \left( \frac{\partial \delta w_0}{\partial X_1} \frac{\partial^3 w_0}{\partial X_1 \partial t^2} + \frac{\partial \delta w_0}{\partial X_2} \frac{\partial^3 w_0}{\partial X_2 \partial t^2} \right) \right\} dX_1 dX_2 - \oint_{t_r} \delta w_0 \tilde{V}_n ds - \oint_{t_r} \frac{\partial \delta w_0}{\partial n} p_m ds
\] (4.6.c)

\[
0 = \int_\Omega \left\{ \frac{\partial \delta \phi_1}{\partial X_1} \tilde{M}_1 + \frac{\partial \delta \phi_1}{\partial X_2} \tilde{M}_6 + \delta \phi_1 \tilde{Q}_1 + \delta \phi_1 \left[ I_3 \left( \frac{\partial \delta u_0}{\partial t^2} + \frac{\partial \delta v_0}{\partial t^2} \right) \right] + \frac{\partial \delta w_0}{\partial X_1} \frac{\partial^2 \phi_2}{\partial t^2} \right\} dX_1 dX_2 - \oint_{t_r} \{ \delta \phi_1 (\tilde{M}_1 n_1 + \tilde{M}_6 n_2) \} ds
\] (4.6.d)

\[
0 = \int_\Omega \left\{ \frac{\partial \delta \phi_2}{\partial X_1} \tilde{M}_6 + \frac{\partial \delta \phi_2}{\partial X_2} \tilde{M}_2 + \delta \phi_2 \tilde{Q}_2 + \delta \phi_2 \left[ I_4 \frac{\partial^2 v_0}{\partial t^2} + K_2 \frac{\partial^2 \phi_2}{\partial t^2} \right] \right\} dX_1 dX_2 - \oint_{t_r} \{ \delta \phi_2 (\tilde{M}_6 n_1 + \tilde{M}_2 n_2) \} ds
\] (4.6.e)

where

\[
\tilde{V}_n = c_1 \left[ \left( \frac{\partial P}{\partial X_1} + \frac{\partial P_n}{\partial X_1} \right) n_1 + \left( \frac{\partial P}{\partial X_2} + \frac{\partial P_n}{\partial X_2} \right) n_2 \right]
- c_1 \left[ \left( T_3 \dot{u}_0 + J_4 \ddot{\phi}_1 - c_1 I_6 \frac{\partial \dot{w}_0}{\partial X_1} \right) n_1 + \left( T_3 \dot{v}_0 + J_4 \ddot{\phi}_2 - c_1 I_6 \frac{\partial \dot{w}_0}{\partial X_2} \right) n_2 \right]
+ \left( \tilde{Q}_1 n_1 + \tilde{Q}_2 n_2 \right) + N(w_0) + c_1 \frac{\partial P_n}{\partial s}
\] (4.7.a)

\[
N(w_0) = \left( N_1 \frac{\partial w_0}{\partial X_1} + N_6 \frac{\partial w_0}{\partial X_2} \right) n_1 + \left( N_6 \frac{\partial w_0}{\partial X_1} + N_2 \frac{\partial w_0}{\partial X_2} \right) n_2
\] (4.7.b)
(2) Sanders nonlinear shell theory

\[ 0 = \int_{\Gamma} \left[ \frac{\partial^2 u_0}{\partial X_1^2} + \frac{\partial^2 u_0}{\partial X_2^2} - c_I \frac{\partial^2 \phi_1}{\partial t^2} \right] dX_1 dX_2 - \hat{\phi}_r \delta u_0 \left[ \left( N_1 + \frac{c_I}{R_1} P_1 \right) n_1 + \left( N_6 + \frac{c_I}{R_1} P_6 \right) n_2 \right] ds \]  

\[ 0 = \int_{\Gamma} \left[ \frac{\partial^2 w_0}{\partial X_1^2} + \frac{\partial^2 w_0}{\partial X_2^2} - c_I \frac{\partial^2 \phi_0}{\partial t^2} \right] dX_1 dX_2 - \hat{\phi}_r \delta w_0 \left[ \left( N_6 + \frac{c_I}{R_2} P_6 \right) n_1 + \left( N_2 + \frac{c_I}{R_2} P_2 \right) n_2 \right] ds \]  

\[ 0 = \int_{\Gamma} \left[ \frac{\partial^2 \phi_0}{\partial X_1 \partial t} + \frac{\partial^2 \phi_0}{\partial X_2 \partial t} \right] dX_1 dX_2 - \hat{\phi}_r \delta w_0 \left[ \left( N_6 + \frac{c_I}{R_3} P_6 \right) n_1 + \left( N_2 + \frac{c_I}{R_3} P_2 \right) n_2 \right] ds \]  

\[ - \frac{\partial \delta w_0}{\partial n} p_{mn} ds \]
\begin{align}
0 &= \int_{\alpha} \left\{ \delta \phi_1 \frac{\partial}{\partial X_1} M_1 + \delta \phi_2 \frac{\partial}{\partial X_2} M_6 + \delta \phi_1 \tilde{Q}_1 + \delta \phi_2 \left[ J_1 \frac{\partial^2 u_0}{\partial t^2} + K_2 \frac{\partial^2 \phi_1}{\partial t^2} - \frac{c_1 J_4}{\partial t^2} \left( \frac{\partial w_0}{\partial X_1} \right) \right] \right\} dX_1 dX_2 - \oint_{\alpha} \{ \delta \phi_1 (\overline{M}_1 n_1 + \overline{M}_6 n_2) \} ds \tag{4.8.d}
\end{align}

\begin{align}
0 &= \int_{\alpha} \left\{ \delta \phi_2 \frac{\partial}{\partial X_1} M_6 + \delta \phi_1 \frac{\partial}{\partial X_2} M_2 + \delta \phi_2 \tilde{Q}_2 + \delta \phi_1 \left[ J_1 \frac{\partial^2 v_0}{\partial t^2} + K_2 \frac{\partial^2 \phi_2}{\partial t^2} - \frac{c_1 J_4}{\partial t^2} \left( \frac{\partial w_0}{\partial X_2} \right) \right] \right\} dX_1 dX_2 - \oint_{\alpha} \{ \delta \phi_2 (\overline{M}_6 n_1 + \overline{M}_2 n_2) \} ds \tag{4.8.e}
\end{align}

where

\begin{align}
\overline{V}_n &= c_1 \left[ \left( \frac{\partial P_1}{\partial X_1} + \frac{\partial P_2}{\partial X_2} \right) n_1 + \left( \frac{\partial P_2}{\partial X_1} + \frac{\partial P_1}{\partial X_2} \right) n_2 \right] \\
- c_1 \left[ T_3 \ddot{u}_0 + J_4 \dot{\phi}_1 - c_1 l_6 \frac{\partial \tilde{w}_0}{\partial X_1} \right] n_1 + \left[ T_3 \ddot{v}_0 + J_4 \dot{\phi}_2 - c_1 l_6 \frac{\partial \tilde{w}_0}{\partial X_2} \right] n_2 \tag{4.9.a}
\end{align}

\begin{align}
\frac{\partial P_m}{\partial s} + N_n (u_0, v_0, w_0)
\end{align}

\begin{align}
N_n (u_0, v_0, w_0) &= \left[ N_1 \left( \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1} \right) + N_6 \left( \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2} \right) \right] n_1 \\
+ \left[ N_6 \left( \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1} \right) + N_2 \left( \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2} \right) \right] n_2 \tag{4.9.b}
\end{align}
For the modified Sanders shell theory for shallow shell, the virtual work statements are

\[ 0 = \int_{\Gamma'} \left[ \frac{\partial \delta u_0}{\partial X_1} N_1 + \frac{\partial \delta u_0}{\partial X_2} N_6 - \delta u_0 \frac{Q_1}{R_1} + \frac{c_1}{R_1} \left( \frac{\partial \delta u_0}{\partial X_1} P_1 + \frac{\partial \delta u_0}{\partial X_2} P_6 \right) + \delta u_0 \left[ \overline{T}_0 \frac{\partial^2 u_0}{\partial t^2} + \overline{J}_1 \frac{\partial^2 \phi_1}{\partial t^2} - c_1 \overline{T}_3 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial X_1} \right) \right] \right] dX_1 dX_2 \]

\[ - \oint_{\Gamma'} \delta u_0 \left[ \left( N_1 + \frac{c_1}{R_1} P_1 \right) n_1 + \left( N_6 + \frac{c_1}{R_1} P_6 \right) n_2 \right] ds \]

\[ 0 = \int_{\Gamma'} \left[ \frac{\partial \delta v_0}{\partial X_1} N_6 + \frac{\partial \delta v_0}{\partial X_2} N_2 - \delta v_0 \frac{Q_2}{R_2} + \frac{c_1}{R_2} \left( \frac{\partial \delta v_0}{\partial X_1} P_2 + \frac{\partial \delta v_0}{\partial X_2} P_6 \right) + \delta v_0 \left[ \overline{T}_0 \frac{\partial^2 v_0}{\partial t^2} + \overline{J}_1 \frac{\partial^2 \phi_2}{\partial t^2} - c_1 \overline{T}_3 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial X_2} \right) \right] \right] dX_1 dX_2 \]

\[ - \oint_{\Gamma'} \delta v_0 \left[ \left( N_6 + \frac{c_1}{R_2} P_6 \right) n_1 + \left( N_2 + \frac{c_1}{R_2} P_2 \right) n_2 \right] ds \]

\[ 0 = \int_{\Gamma'} \left[ \frac{\partial \delta w_0}{\partial X_1} Q_1 + \frac{\partial \delta w_0}{\partial X_2} Q_2 - c_1 \left( \frac{\partial^2 \delta w_0}{\partial X_1^2} P_1 + \frac{\partial^2 \delta w_0}{\partial X_1 \partial X_2} P_2 + \frac{\partial^2 \delta w_0}{\partial X_2^2} P_2 \right) - \delta w_0 q + \delta w_0 \left( \frac{N_1}{R_1} + \frac{N_2}{R_2} \right) + \delta w_0 \overline{T}_0 \frac{\partial^2 w_0}{\partial t^2} + c_1 \overline{T}_6 \left( \frac{\partial \delta w_0}{\partial X_1} \frac{\partial^3 w_0}{\partial X_1^2} \right) + \frac{\partial \delta w_0}{\partial X_1} \frac{\partial^3 w_0}{\partial X_1^3} \right] dX_1 dX_2 - \oint_{\Gamma'} \delta w_0 \overline{\nabla} \cdot \delta \mathbf{n} ds - \oint_{\Gamma'} \frac{\partial \delta w_0}{\partial n} \overline{p}_m ds \]

\[ 0 = \int_{\Gamma'} \left[ \frac{\partial \delta \phi}{\partial X_1} \overline{M}_1 + \frac{\partial \delta \phi}{\partial X_2} \overline{M}_6 + \delta \phi \overline{Q}_1 + \delta \phi \left[ \overline{J}_1 \frac{\partial^2 u_0}{\partial t^2} + \overline{K}_2 \frac{\partial^2 \phi}{\partial t^2} - c_1 \overline{T}_4 \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_0}{\partial X_1} \right) \right] \right] dX_1 dX_2 - \oint_{\Gamma'} \{ \delta \phi (\overline{M}_1 n_1 + \overline{M}_6 n_2) \} ds \]
\[ 0 = \int_{\Omega} \left\{ \frac{\partial}{\partial t} \frac{\partial \delta \phi_2}{\partial X_1} \overline{M_6} + \frac{\partial}{\partial X_1} \frac{\partial \delta \phi_2}{\partial t} \overline{M_2} + \frac{\partial \delta \phi_2}{\partial X_2} \overline{Q_2} + \delta \phi_2 \left[ J_1' \frac{\partial^2 v_0}{\partial t^2} + K_2 \frac{\partial^2 \phi_2}{\partial t^2} \right] \right\} dX_1 dX_2 - \int \phi_2 \{ \delta \phi_2 (\overline{M_6 n_1} + \overline{M_2 n_2}) \} ds \]  

where

\[ \overline{V_n} = c_1 \left[ \left( \frac{\partial P_1}{\partial X_1} + \frac{\partial P_2}{\partial X_2} \right) n_1 + \left( \frac{\partial P_3}{\partial X_1} + \frac{\partial P_4}{\partial X_2} \right) n_2 \right] \]

\[ - c_1 \left[ \left( \bar{T}_1 u_0 + J_4 \ddot{\phi}_2 - c_1 I_6 \frac{\partial \ddot{w}_0}{\partial X_1} \right) n_1 + \left( \bar{T}_2 v_0 + J_4 \ddot{\phi}_2 - c_1 I_6 \frac{\partial \ddot{w}_0}{\partial X_2} \right) n_2 \right] \]

\[ + \left( \bar{Q}_1 n_1 + \bar{Q}_2 n_2 \right) + N(w_0) + c_1 \frac{\partial P_m}{\partial s} \]

\[ N(w_0) = \left( N_1 \frac{\partial \ddot{w}_0}{\partial X_1} + N_6 \frac{\partial \ddot{w}_0}{\partial X_2} \right) n_1 + \left( N_2 \frac{\partial \ddot{w}_0}{\partial X_1} + N_5 \frac{\partial \ddot{w}_0}{\partial X_2} \right) n_2 \]  

4.2. Finite Element Model

The generalized displacements are approximated over the domain \( \Omega \) by the expressions

\[ u_0(x, y, t) = \sum_{i=1}^{m} u_i^e(t) \psi_i^e(x, y) \]  

\[ v_0(x, y, t) = \sum_{i=1}^{m} v_i^e(t) \psi_i^e(x, y) \]  

\[ w_0(x, y, t) = \sum_{i=1}^{m} \Delta_i^e(t) \phi_i^e(x, y) \]  

\[ \phi_i(x, y, t) = \sum_{i=1}^{m} X_i^e(t) \psi_i^e(x, y) \]  

\[ \phi_i(x, y, t) = \sum_{i=1}^{m} Y_i^e(t) \psi_i^e(x, y) \]
where \( \psi_i \) denotes the Lagrange interpolation functions and \( \phi_i \) denotes the Hermite interpolation functions. The same Lagrange linear rectangular element for in-plane displacements \((u_0, v_0)\) and rotations \((\phi_x, \phi_y)\) and the conforming rectangular elements for bending deflections are used in this study. The conforming element is one of \( C^1 \) plate bending elements in which the interelement continuity of \( w_0, w_{0,x}, w_{0,y} \) are satisfied (see Reddy 1997, 2004b). The combined conforming rectangular element, which has eight degrees of freedom \( u_0, v_0, w_0, w_{0,x}, w_{0,y}, \phi_x, \phi_y \) per node, is shown in Figure 4.1. The four nodal values associated with \( w_0 \) are \( \bar{\Delta}_1 = w_0, \bar{\Delta}_2 = \frac{\partial w_0}{\partial x}, \bar{\Delta}_3 = \frac{\partial w_0}{\partial y}, \bar{\Delta}_4 = \frac{\partial^2 w_0}{\partial x \partial y}. \)

Substitution of the approximation, Equation (4.12) for \( u_0, v_0, w_0, \phi_x, \phi_y \), into the weak forms yields the semidiscrete finite element model of the third-order shear deformation theory.
or
\[
\sum_{\beta=1}^{5} \sum_{j=1}^{n_\beta} \left( K_{y}^{\alpha\beta} \Delta_{j}^{\beta} + C_{y}^{\alpha\beta} \dot{\Delta}_{j}^{\beta} + M_{y}^{\alpha\beta} \ddot{\Delta}_{j}^{\beta} \right) - F_{j}^{\alpha} = 0 \quad i = 1, 2, \ldots, n
\]  

(4.13.b)

where \( \alpha = 1, 2, 3, 4, 5 \); \( n_1 = n_2 = n_4 = n_5 = 4 \) and \( n_3 = 16 \) for the conforming element, and the nodal values \( \Delta_{j}^{\beta} \) are \( \Delta_{j}^{1} = u_{j} \), \( \dot{\Delta}_{j}^{1} = v_{j} \), \( \ddot{\Delta}_{j}^{1} = \ddot{X}_{j} \), \( \Delta_{j}^{i} = X_{j} \), \( \dot{\Delta}_{j}^{i} = Y_{j} \).

Since the equations of motion are expressed in terms of the displacements and the generalized displacements are the primary modal degrees of freedom, this finite element models are called displacement finite element models (Reddy 1997, 2004b). The shear and membrane locking problem (Reddy 2004a) in this displacement finite element model can be overcome by using the reduced integration for evaluating the shear stiffness coefficients.
The thermal source term corresponding to the deflection is nonlinear and this nonlinearity becomes significant at high temperature. In this study, the nodal values of deflection are treated as unknowns and the resulting force term is transferred to the left hand side of the Equation (4.13). Thus the term is included into the direct stiffness matrix and this avoids re-calculation of tangent stiffness matrix in Newton-Raphson iteration method.

The stiffness coefficients $K_{ij}^{\alpha\beta} = (K_{ij}^{\alpha\beta})_L + (K_{ij}^{\alpha\beta})_{NL} + (K_{ij}^{\alpha\beta})^T_{NL}$, mass coefficients $M_{ij}^{\alpha\beta}$, and active damping coefficients $C_{ij}^{\alpha\beta} = (C_{ij}^{\alpha\beta})_L + (C_{ij}^{\alpha\beta})_{NL}$ for Donnell and Sanders nonlinear shell theory are well defined in the Appendices B and C. where $(K_{ij}^{\alpha\beta})_L$ is the linear stiffness coefficients, $(K_{ij}^{\alpha\beta})_{NL}$ is the geometric and $(K_{ij}^{\alpha\beta})^T_{NL}$ is the thermal nonlinear stiffness coefficients, $(C_{ij}^{\alpha\beta})_L$ is the linear and $(C_{ij}^{\alpha\beta})_{NL}$ is the nonlinear damping coefficients. Note by setting $R_1 = R_2 = \infty$ from the finite element coefficients of Donnell nonlinear theory, the coefficients of laminated composite plates by Third-order shear deformation theory can be obtained. In the case of the first-order shear deformation theory, the resulting finite element model requires only $C^0$ continuity for generalized displacements.

4.3. Transient Analysis

The linear equations of motion can be solved using analytical methods, but those are algebraically complicated and require the determination of eignevalues and eigenfunctions, as in the state-space approach. Newmark’s numerical integration method (Reddy 1983, 1993) that takes advantage of the static solution form for spatial variation and uses a numerical method to solve the resulting differential equations in time is used to determine the transient response of laminated composites in this study. The constant-average-acceleration scheme is used for linear transient problem.
Using the Newmark’s scheme, a second-order differential equation of the form,

\[
[M\{\ddot{\Delta}\}] + [C\{\dot{\Delta}\}] + [K\{\Delta\}] = \{F\}
\]  \hspace{1cm} (4.14)

can be reduced to the fully discretized form:

\[
[K(\{\Delta\}_{s+1})]\{\Delta\}_{s+1} = \{\dot{F}\}_{s+1}
\]  \hspace{1cm} (4.15)

where the subscript \(s + 1\) refers to the time \(t_{s+1}\) at which the solution is sought, and

\[
[K(\{\Delta\}_{s+1})] = [K(\{\Delta\}_{s+1})] + a_3[M]_{s+1} + a_6[C]_{s+1}
\]  \hspace{1cm} (4.16.a)

\[
\{\dot{F}\}_{s+1} = \{F\}_{s+1} + [M]_{s+1} \{\dot{\Delta}\} + [C]_{s+1} \{\ddot{\Delta}\}
\]  \hspace{1cm} (4.16.b)

\[
\{\dot{\Delta}\} = a_3 \{\Delta\} + a_4 \{\dot{\Delta}\} + a_5 \{\ddot{\Delta}\}
\]  \hspace{1cm} (4.16.c)

\[
\{\ddot{\Delta}\} = a_6 \{\Delta\} + a_7 \{\dot{\Delta}\} + a_8 \{\ddot{\Delta}\}
\]  \hspace{1cm} (4.16.d)

where \(a_i\) are the parameters

\[
a_1 = (1 - \alpha)\Delta t, \hspace{0.5cm} a_2 = \alpha\Delta t, \hspace{0.5cm} a_3 = \frac{2}{\gamma(\Delta t)^2}, \hspace{0.5cm} a_4 = a_3 \Delta t
\]

\[
a_5 = \frac{(1 - \gamma)}{\gamma}, \hspace{0.5cm} a_6 = \frac{2\alpha}{\gamma \Delta t}, \hspace{0.5cm} a_7 = \frac{2\alpha}{\gamma} - 1, \hspace{0.5cm} a_8 = \Delta t \left(\frac{\alpha}{\gamma} - 1\right)
\]  \hspace{1cm} (4.17)

At the end of each time step, the new velocity vector \(\{\dot{\Delta}\}_{s+1}\) and acceleration vector \(\{\ddot{\Delta}\}_{s+1}\) are computed using

\[
\{\dot{\Delta}\}_{s+1} = a_3 (\{\Delta\}_{s+1} - \{\Delta\}_s) - a_4 \{\dot{\Delta}\}_s - a_5 \{\ddot{\Delta}\}_s
\]  \hspace{1cm} (4.18.a)

\[
\{\ddot{\Delta}\}_{s+1} = \{\dot{\Delta}\}_s + a_1 \{\ddot{\Delta}\}_s + a_2 \{\ddot{\Delta}\}_{s+1}
\]  \hspace{1cm} (4.18.b)
4.3. Nonlinear Analysis

The resulting nonlinear algebraic equations must be solved by an iterative method. In iterative methods, the nonlinear equations are linearized by evaluating the nonlinear terms with the known solution from preceding iteration. The Newton-Raphson iteration method, which is based on the Taylor series expansion and uses the tangent stiffness matrix, is selected in this study. This Newton-Raphson iteration method yields a symmetric tangent stiffness matrix for all structural problems.

Solution of Equation (4.15) by the Newton-Raphson iteration method results in the following linearized equations for the incremental solution at the \((r+1)\)st iteration (see Reddy, 2004a).

\[
\{\delta \Delta\} = -[\hat{K}^T(\{\Delta\}_{r+1})]^{-1}\{R\}_{r+1}
\]  

(4.19)

where the tangent stiffness matrix is defined by

\[
[\hat{K}^r(\{\Delta\}_{r+1})] \equiv \left[ \frac{\partial \{R\}}{\partial \{\Delta\}} \right]_{r+1}
\]  

(4.20)

\[
\{R\}_{r+1} = [\hat{K}(\{\Delta\}_{r+1})]\{\Delta\}_{r+1}^r - \{\hat{F}\}_{r+1}
\]  

(4.21)

The total solution is obtained from

\[
\{\Delta\}_{r+1} = \{\Delta\}_{r+1}^r + \{\delta \Delta\}
\]  

(4.22)

Note that the tangent stiffness matrix is evaluated using the latest known solution, while the residual vector contains contributions from the latest known solution in computing \([K(\{\Delta\}_{r+1})]\{\Delta\}_{r+1}^r\) and previous time step solution in computing \(\{\hat{F}\}_{r+1}\).

The iteration process is continued until the difference between \(\{\Delta\}_{r+1}^r\) and \(\{\Delta\}_{r+1}^{r+1}\) is reduced to a preselected error tolerance. The error criterion used in this study is of the form
where \( n \) is the total number of nodal generalized displacements in the finite element mesh, and \( \varepsilon \) is the error tolerance. The velocity and acceleration vectors are updated using Equation (4.18) only after convergence is reached for a given time step. The details of the tangent stiffness coefficients could be found in the Appendices B and C.

4.5. Computer Implementation

Computer implementation of nonlinear time-dependent problems is complicated by the fact that one must keep track of the solution vectors at different loads, times, and iteration. Thus, there are three levels of calculations (see Reddy 2004a). Often, for a fixed value of load, one wishes to obtain the transient solution. Thus the outermost loop is on the number of load steps, followed by a loop on the number of time steps, and the innermost loop being on nonlinear equilibrium iterations. A flow chart of the general scheme is shown in Figure 4.2. In the present study the load loop is suppressed as we are dealing with a single load.

4.6. Preliminary Linear Finite Element Results

Linear finite element analysis is carried out to analyze the deflection suppression characteristics. The baseline of the simulations is the simply supported square laminate \((a/b = 1, a/h = 10)\) under sinusoidal distributions of the initial velocity field

\[
\frac{\partial w}{\partial t}(x, y, t = 0) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}
\]  
(4.24)
The time step selected in the linear transient study is $t = 0.0005\text{sec}$. The notation for lamination scheme $(\theta_1, \theta_2, \theta_3, \theta_4, m)_s$ means that there are 10 layers symmetrically placed about the midplane with the fiber orientations being $(\theta_1, \theta_2, \theta_3, \theta_4, m, m, \theta_4, \theta_2, \theta_1)$, where $m$ stands for the magnetostrictive layer and subscript $s$ stands for symmetric and anti – $s$ stands for anti-symmetric lamination. The lamination schemes which are used in this study are symmetric cross-ply $(m, 90, 0, 90, 0)_s$, angle-ply $(m, 30, -30, 30, 30)_s$, general angle-ply $(m, 45, -45, 90, 0)_s$ laminates, anti-symmetric cross-ply $(m, 90, 0, 90, 0)_{anti-s}$, angle-ply $(m, 30, -30, 30, 30)_{anti-s}$, and general angle-ply $(m, 45, -45, 90, 0)_{anti-s}$ laminates. Simply supported and
clamped boundary conditions are selected to study the effect of boundary condition on the deflection control. The material properties of smart material, Terfenol-D, and the elastic composite materials which is used in this numerical example are listed in Table 3.1.

In finite element analysis, solution symmetries should be taken advantage of by identifying the computational domain to reduce computational effort. For a laminated composite plate with all edges simply-supported or clamped, a quadrant of the plate may be used as the computational domain as shown in Figure 4.3. Figures 4.4(a)-(d) shows the effects of the finite element results for the different laminations and boundary conditions. Quarter plate models with proper boundary conditions can be used in the antisymmetric laminates with simply supported boundary condition, but not for laminated plates with the clamped edges. For symmetric laminates, the simply supported cross-ply laminates can be modeled as a quarter plate. The boundary conditions along a line of symmetry should be correctly identified and imposed in the finite element model. When one is not sure of the solution symmetry, it is advised that the whole plate be modeled.
Figure 4.4 Effect of finite element modeling of CFRP composite plates; (a) Simply supported antisymmetric laminates, (b) Clamped antisymmetric laminates, (c) Simply supported symmetric laminates, (d) Clamped symmetric laminates
Figure 4.4 Continued
4.6.1. Simply Supported Laminated Composite Plates

To compare with the analytical results, the SS-1 boundary conditions and quarter plate model have been used for symmetric cross-ply laminates. The deflections predicted from the analytical (eigenvalue) analysis and transient finite element analysis are within the reasonable agreement, as shown in Figure 4.5 for cross-ply \((m,90,0,90,0)_s\) laminated plates.

Figure 4.6 shows the central displacements using the different plate theories (CLPT, FSDT, and TSDT) for two different lamination schemes. It is observed that CLPT shows higher deflection suppression capacity in both cases. This is expected because the CLPT renders plate stiffer compared to the other theories. After studying the influence of lamina material properties on the amplitude of deflection and deflection suppression times, it is observed that materials having the almost same \(E_1/E_2\) ratio have similar deflection suppression characteristics under the same lamination, loading and boundary conditions. Figure 4.7 shows the deflection damping characteristics for the different laminate materials.

![Figure 4.5 Center deflection predicted by the analytical and finite element methods for the case of symmetric cross-ply CFRP laminated plate](image)

**Figure 4.5** Center deflection predicted by the analytical and finite element methods for the case of symmetric cross-ply CFRP laminated plate.
Figure 4.6 Center displacements by the different plate theories for simply supported cross-ply CFRP laminated plates; (a) $(m,90,0,90,0)_s$, (b) $(0,90,0,90,m)_s$
Figure 4.7 Effect of the lamina material properties on the damping of deflection in symmetric cross-ply laminated plate (m,90,0,90,0)$_s$

Figure 4.8 Effect of the smart material layer position on the deflection for the symmetric cross-ply CFRP laminated plate
4.6.2. Deflection Suppression Time

As stated earlier, the deflection suppression time is the time required to reduce the uncontrolled center deflection to one-tenth of its magnitude. The deflection suppression time ratio (suppression time divided by the maximum suppression time) can be shown to be

\[ T_s = \frac{h_m}{2z_m} \]

where \( h_m \) is the thickness of the magnetostrictive layer and \( z_m \) is the positive distance between the mid plane of the magnetostrictive layer and the mid plane of the plate.

The effect of the smart layer positions on the deflection suppression can be shown in the Figure 4.8. It is observed that as the smart material layer is moved farther from the mid-plane the suppression time decreases, as may be expected because of the moment effect by smart layer actuations. The maximum deflections \( W_{\text{max}} \) of the composite plate and the suppression times for the different position of smart layers are presented in Table 4.1.

<table>
<thead>
<tr>
<th>Lamination Scheme</th>
<th>( z_m ) (m)</th>
<th>( T_s )</th>
<th>( W_{\text{max}}) (10^{-4}) m</th>
<th>( t ) at ( \frac{W_{\text{max}}}{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m,90,0,90,0)s</td>
<td>0.045</td>
<td>0.111</td>
<td>5.21</td>
<td>0.0285</td>
</tr>
<tr>
<td>(90,m,90,0,90)s</td>
<td>0.035</td>
<td>0.143</td>
<td>5.09</td>
<td>0.0350</td>
</tr>
<tr>
<td>(0,90,m,90,0)s</td>
<td>0.025</td>
<td>0.200</td>
<td>4.90</td>
<td>0.0480</td>
</tr>
<tr>
<td>(90,0,90,m,90)s</td>
<td>0.015</td>
<td>0.333</td>
<td>4.85</td>
<td>0.0850</td>
</tr>
<tr>
<td>(0,90,0,90,m)s</td>
<td>0.005</td>
<td>1.000</td>
<td>4.87</td>
<td>0.2560</td>
</tr>
</tbody>
</table>

4.6.3. Effect of Lamina Thickness

The effect of the thickness of smart-material layer on deflection damping characteristics is studied next. It is observed that thicker smart material layers result in better attenuation of the deflection. This is due to a larger mass inertia that is caused by the large increase in the moment of inertia of the system when thickness of the smart material
layer is increased. We note that the smart material layer has a density of five times that of the composite material. The suppression times and characteristics for different smart layer thicknesses are shown in Table 4.2 and Figure 4.9.

Table 4.2 Suppression times for the different smart layer thicknesses in symmetric cross-ply laminated plate (m,90,0,90,0).

<table>
<thead>
<tr>
<th>Lamina Thickness (mm)</th>
<th>$z_m$ (m)</th>
<th>$T_s$</th>
<th>$W_{max}$ $(10^{-4} m)$</th>
<th>$t$ at $W_{max}$ /10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_e = 10, h_m = 2$</td>
<td>0.0410</td>
<td>0.0244</td>
<td>4.77</td>
<td>0.0780</td>
</tr>
<tr>
<td>$h_e = 10, h_m = 4$</td>
<td>0.0420</td>
<td>0.0476</td>
<td>4.98</td>
<td>0.0485</td>
</tr>
<tr>
<td>$h_e = 10, h_m = 5$</td>
<td>0.0425</td>
<td>0.0588</td>
<td>5.05</td>
<td>0.0420</td>
</tr>
<tr>
<td>$h_e = 10, h_m = 6$</td>
<td>0.0430</td>
<td>0.0698</td>
<td>5.10</td>
<td>0.0350</td>
</tr>
<tr>
<td>$h_e = 10, h_m = 8$</td>
<td>0.0440</td>
<td>0.0909</td>
<td>5.17</td>
<td>0.0320</td>
</tr>
<tr>
<td>$h_e = 10, h_m = 10$</td>
<td>0.0450</td>
<td>0.1111</td>
<td>5.21</td>
<td>0.0285</td>
</tr>
<tr>
<td>$h_e = 5, h_m = 5$</td>
<td>0.0225</td>
<td>0.1111</td>
<td>9.12</td>
<td>0.0310</td>
</tr>
<tr>
<td>$h_e = 5, h_m = 10$</td>
<td>0.0250</td>
<td>0.2</td>
<td>9.45</td>
<td>0.0400</td>
</tr>
</tbody>
</table>

4.6.4. Effect of Feedback Coefficients

Figure 4.10 shows the effect of the feedback coefficient $c(t)k_e$ on the deflection suppression characteristics. Two different values of the feedback coefficient are used; $10^4$ and $10^3$. It can be seen that the suppression time increases when the value of the feedback coefficient decrease. This is because the coefficients of the damping matrix decrease, thereby resulting in less damping.
Figure 4.9 Effect of the thickness of smart material layers on the deflection damping characteristics of symmetric cross-ply laminated plate (m,90,0,90,0)s
4.6.5. Other Effects on Deflection Control

The deflection damping characteristics of symmetric angle-ply and general angle-ply laminated composites are studied using full plate F.E. models. Observations made earlier on various characteristics such as the effects of smart layer position, its thickness, and magnitude of the feedback coefficient are also valid for these laminates, as shown in Figures 4.11 and 4.12.

Next, fully clamped laminated plates are analyzed using $8 \times 8$ mesh in a full plate. The effect of the boundary conditions on the deflection is shown in the Figure 4.13. In Figure 4.13, ‘S’ represents all edges simply supported and ‘C’ represents all edges clamped boundary conditions. The maximum displacements of the simply supported plate are
greater than those of the clamped case, which is expected. Simply supported laminates, which have larger displacements, take less suppression time compared to the clamped laminates.

Since laminated composite structures are subjected to a variety of loading conditions during their service life, understanding of the response of these structures for various loading conditions is necessary. Numerical studies are also carried out to analyze smart laminated composites under uniformly distributed load $q_o$ instead of specified initial velocity field. Figure 4.14 shows the center deflection for selected simply supported and clamped laminates under continuously applied uniformly distributed loading, while Figure 4.15 shows the case under suddenly applied step loading. The effect of sinusoidal loading, 

$$q(x, y) = q_o \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right),$$

on the central displacement has been studied. The results of symmetric cross-ply laminates with simply supported boundary conditions and subjected to sinusoidal and uniformly distributed loads are shown in Figure 4.16. Figures under the mechanical loading cases, the following nondimensionalized form is used for the transverse center displacements as $w_o (100) \frac{E_h^3}{b^4 q_o}$. 

Figure 4.11 Center displacement for symmetric angle-ply CFRP laminated plate
Figure 4.12 Effect of the smart layer position for symmetric general angle-ply CFRP laminated plate
Figure 4.13 Effect of boundary conditions (a) Comparison of simply supported and clamped laminates, (b) smart layer positions
Figure 4.14 Nondimensionalized center deflection under uniform load

Figure 4.15 Nondimensionalized center deflection under suddenly applied uniform load
Figure 4.16 Nondimensionalized center displacement for simply supported laminated plates under sinusoidal and uniformly distributed loads
5. RESULTS OF NONLINEAR ANALYSIS*

In this Section, the numerical studies using the nonlinear finite element models developed in Section 4 for laminated composite plate and shell structures under mechanical loading are carried out. Recall that the nonlinearity accounted for is that of the von Kármán type.

Laminated composite square plate and shell \((a/b = 1)\) with both the upper and lower surfaces embedded magnetostrictive materials is considered. The plate and shell structures considered here are made of composite fiber reinforced polymer (CFRP) and for magnetostrictive material, Terfenol-D. The material properties are presented in Table 3.1. The adhesive used to bond the structural layers or smart-material layers are neglected in the analysis. The laminated composite structures are composed of total 10 layers and all the layers are assumed to be of the same thickness. Two side-to-thickness ratios \(a/h = 10\) and \(a/h = 100\) are considered to represent the thick and thin laminated composites. Four different lamination schemes, symmetric cross-ply \(m,90,0,90,0\)s, symmetric angle-ply \(m,45,−45,45,−45\)s, symmetric general angle-ply \(m,45,−45,0,90\)s and asymmetric general angle-ply \(m,45,−45,15,−15,0,90,30,−30,m\) are considered to study the effect of lamination schemes on the deflection control. Six boundary conditions are considered to study the effects on deflection control. They are SSSS (SS), CCCC (CC), CCSS (CS), CCFF (CF), SSFF (SF) and CFSS (CFS) which are well shown in Figure 5.1. As a baseline of computer simulation, unless otherwise specified, symmetric cross-ply laminates with simply supported boundary condition are mainly used.

Three different plate theories, CLPT, FSDT, TSDT, are used to analyze plate structures. The shear correction factor used in FSDT is 5/6. For laminated composite shells, Donnell and Sanders nonlinear shell theories are used. Three shell types, spherical \((R_1 = R_2)\), cylindrical \((R_1 = \infty)\), and doubly curved shell \((R_1 = 2R_2)\), are considered with various \(R_2/a\) values.

As shown in the previous section, the biaxial symmetry may not be assumed even if the geometry and loading are symmetric but laminates are not symmetric (due to the bending-stretching coupling). In this study solution symmetries are considered only for the simply supported cross-ply laminated composite plate and shell to reduce the computational efforts and \(4 \times 4\) meshes in a quadrant are used. For all other cases \(8 \times 8\) meshes of the full models are used for the computational domains. Since locating smart material layers farthest from the mid-plane has the best effect on deflection suppression as shown in Sections 3 and 4, the smart layer position in this study is limited to both top and bottom layers. Feedback coefficient \(k_c c(t)\) is assumed to be a constant, \(10^4\), in this nonlinear analysis. The feedback coefficient effect could be found in section 4.6.4.

![Figure 5.1 Six boundary conditions used in this study](image)
Throughout numerical examples in this study, the center displacements are nondimensionalized as \( \bar{w}_c = 100 \times w_0 E_2 h^3 / a^4 q_0 \) for transient results. In static analysis center deflections are nondimensionalized as \( w/h \) and load parameter \( \bar{p} = q_0 a^4 / E_2 h^4 \) has been used. A tolerance of \( \varepsilon = 10^{-2} \) is selected for convergence in the Newton-Raphson iteration scheme to check for convergence of the nodal displacements.

### 5.1. Nonlinear Static Results

Four different laminations schemes and six different boundary conditions are considered. The effect of plate thickness is also investigated. Results are presented tabular and/or graphical form.

In Figure 5.2, nondimensionalized deflections from the linear and nonlinear analyses have been plotted for different lamination schemes under uniformly distributed load. It is observed that symmetric angle-ply and asymmetric angle-ply laminates show very similar behavior in nonlinear analysis. The effect of the nonlinearity is apparent with increasing load intensity from the results presented in Figures.

The effect of different boundary conditions on nonlinear deflections is shown in Figure 5.3 and Tables 5.1 - 5.4. Nondimensionalized deflections have been plotted for the six different boundary conditions under uniformly distributed sinusoidal load. The magnitude of deflections in nonlinear analysis is in order of SSFF, CFSS, SSSS, CCFF, CCSS, and CCCC from the large value, which could be expected, for the thick and thin plates. Numerical values of nondimensional center deflection as function of the load parameter and loading condition for the three boundary conditions and two plate thicknesses are tabulated in Tables 5.1 and 5.2 for symmetric cross-ply and angle-ply laminates and Tables 5.3 and 5.4 for symmetric and asymmetric general angle-ply laminates.
Table 5.1 Nondimensional center deflection ($w/h$) for symmetric cross-ply and angle-ply thick ($a/h = 10$) laminates under different load and boundary conditions

<table>
<thead>
<tr>
<th>Loading condition</th>
<th>Load parameter</th>
<th>Symmetric cross-ply</th>
<th>Symmetric angle-ply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SSSS</td>
<td>CCSS</td>
</tr>
<tr>
<td>Uniformly distributed load</td>
<td>1</td>
<td>0.01360</td>
<td>0.00615</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.04078</td>
<td>0.01844</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.06791</td>
<td>0.03073</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.13519</td>
<td>0.06137</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.20128</td>
<td>0.09187</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.38866</td>
<td>0.18183</td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>0.55658</td>
<td>0.26838</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.70614</td>
<td>0.35070</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.88245</td>
<td>0.45357</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>1.03880</td>
<td>0.54900</td>
</tr>
<tr>
<td>Sinusoidal load</td>
<td>1</td>
<td>0.00869</td>
<td>0.00422</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.02606</td>
<td>0.01265</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.04341</td>
<td>0.02107</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.08666</td>
<td>0.04213</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.12958</td>
<td>0.06313</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.25519</td>
<td>0.12567</td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>0.37402</td>
<td>0.18707</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.48504</td>
<td>0.24690</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.62070</td>
<td>0.32378</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.74426</td>
<td>0.39713</td>
</tr>
</tbody>
</table>
Table 5.2 Nondimensional center deflection \((w/h)\) for symmetric cross-ply and angle-ply thin \((a/h = 100)\) laminates under different load and boundary conditions

<table>
<thead>
<tr>
<th>Loading condition</th>
<th>Load parameter</th>
<th>Symmetric cross-ply</th>
<th>Symmetric angle-ply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SSSS</td>
<td>CCSS</td>
</tr>
<tr>
<td>Uniformly distributed load</td>
<td>10</td>
<td>0.11888</td>
<td>0.03977</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.23513</td>
<td>0.07946</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.34668</td>
<td>0.11899</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.45242</td>
<td>0.15827</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.55200</td>
<td>0.19725</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.64560</td>
<td>0.23585</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>0.73368</td>
<td>0.27403</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.81676</td>
<td>0.31174</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>0.89541</td>
<td>0.34896</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.97013</td>
<td>0.38566</td>
</tr>
<tr>
<td>Sinusoidal load</td>
<td>10</td>
<td>0.07518</td>
<td>0.02750</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.14969</td>
<td>0.05497</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.22294</td>
<td>0.08239</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.29445</td>
<td>0.10973</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.36386</td>
<td>0.13698</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.43098</td>
<td>0.16411</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>0.49572</td>
<td>0.19109</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.55807</td>
<td>0.21792</td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>0.61811</td>
<td>0.24457</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.67592</td>
<td>0.27103</td>
</tr>
</tbody>
</table>
Table 5.3 Nondimensional center deflection ($w/h$) for symmetric and asymmetric general angle-ply thick ($a/h = 10$) laminates under different load and boundary conditions

<table>
<thead>
<tr>
<th>Loading condition</th>
<th>Load parameter</th>
<th>Symmetric general angle-ply</th>
<th>Asymmetric general angle-ply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SSSS</td>
<td>CCSS</td>
</tr>
<tr>
<td>Uniformly distributed load</td>
<td>1</td>
<td>0.01079</td>
<td>0.00677</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.03236</td>
<td>0.02031</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.05390</td>
<td>0.03385</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.10737</td>
<td>0.06759</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.16006</td>
<td>0.10113</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.31071</td>
<td>0.19974</td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>0.44778</td>
<td>0.29398</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.57170</td>
<td>0.38302</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.72004</td>
<td>0.49362</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.85376</td>
<td>0.59593</td>
</tr>
<tr>
<td>Sinusoidal load</td>
<td>1</td>
<td>0.00697</td>
<td>0.00463</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.02090</td>
<td>0.01389</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.03482</td>
<td>0.02314</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.06952</td>
<td>0.04626</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.10401</td>
<td>0.06931</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.20531</td>
<td>0.13782</td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>0.30186</td>
<td>0.20480</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.39279</td>
<td>0.26977</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.50523</td>
<td>0.35275</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.60858</td>
<td>0.43147</td>
</tr>
</tbody>
</table>
Table 5.4 Nondimensional center deflection ($w/h$) for symmetric and asymmetric general angle-ply thin ($a/h = 100$) laminates under different load and boundary conditions

<table>
<thead>
<tr>
<th>Loading condition</th>
<th>Load parameter</th>
<th>Symmetric general angle-ply</th>
<th>Asymmetric general angle-ply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SSSS</td>
<td>CCSS</td>
</tr>
<tr>
<td>Uniformly distributed load</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.09090</td>
<td>0.04826</td>
<td>0.03450</td>
</tr>
<tr>
<td>20</td>
<td>0.18035</td>
<td>0.09633</td>
<td>0.06894</td>
</tr>
<tr>
<td>30</td>
<td>0.26716</td>
<td>0.14405</td>
<td>0.10326</td>
</tr>
<tr>
<td>40</td>
<td>0.35056</td>
<td>0.19127</td>
<td>0.13742</td>
</tr>
<tr>
<td>50</td>
<td>0.43019</td>
<td>0.23786</td>
<td>0.17135</td>
</tr>
<tr>
<td>60</td>
<td>0.50602</td>
<td>0.28372</td>
<td>0.20502</td>
</tr>
<tr>
<td>70</td>
<td>0.57819</td>
<td>0.32877</td>
<td>0.23839</td>
</tr>
<tr>
<td>80</td>
<td>0.64696</td>
<td>0.37296</td>
<td>0.27142</td>
</tr>
<tr>
<td>90</td>
<td>0.71261</td>
<td>0.41628</td>
<td>0.30410</td>
</tr>
<tr>
<td>100</td>
<td>0.77546</td>
<td>0.45871</td>
<td>0.33639</td>
</tr>
<tr>
<td>Sinusoidal load</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.05799</td>
<td>0.03302</td>
<td>0.02518</td>
</tr>
<tr>
<td>20</td>
<td>0.11560</td>
<td>0.06598</td>
<td>0.05035</td>
</tr>
<tr>
<td>30</td>
<td>0.17252</td>
<td>0.09883</td>
<td>0.07547</td>
</tr>
<tr>
<td>40</td>
<td>0.22843</td>
<td>0.13154</td>
<td>0.10053</td>
</tr>
<tr>
<td>50</td>
<td>0.28313</td>
<td>0.16403</td>
<td>0.12550</td>
</tr>
<tr>
<td>60</td>
<td>0.33646</td>
<td>0.19629</td>
<td>0.15037</td>
</tr>
<tr>
<td>70</td>
<td>0.38834</td>
<td>0.22827</td>
<td>0.17513</td>
</tr>
<tr>
<td>80</td>
<td>0.43871</td>
<td>0.23446</td>
<td>0.19975</td>
</tr>
<tr>
<td>90</td>
<td>0.48759</td>
<td>0.29126</td>
<td>0.22422</td>
</tr>
<tr>
<td>100</td>
<td>0.53501</td>
<td>0.32224</td>
<td>0.24853</td>
</tr>
</tbody>
</table>
Figure 5.2 Load-deflection curve for the different lamination schemes with SSSS boundary conditions under uniformly distributed loading: (a) $a/h = 10$, (b) $a/h = 100$
Figure 5.3 Load-deflection curve for the cross-ply laminates with the different boundary conditions under uniformly distributed sinusoidal loading: (a) $a/h = 10$, (b) $a/h = 100$
5.2. Nonlinear Transient Results for Laminated Composite Plates

5.2.1. Load and Time Increments

First, suitable load intensity and time increments are selected to achieve desired the accuracy and convergence of the solutions. The magnitude of the applied load is selected such that the problem can be solved without considering the load loop in nonlinear analysis. Numerical results are presented for simply supported cross-ply laminated plates. Note that the amplitude and period of the nondimensionalized center displacements decrease with increasing value of the load as shown in Figures 5.4(a) and 5.4(b). Load values of \( q = 5.0 \times 10^7 q_0 \) for the thick plates \((a/h = 10)\) and \( q = 1.0 \times 10^4 q_0 \) for the thin plates \((a/h = 100)\) are selected for the simply supported boundary conditions.

Newmark’s time scheme with \( \alpha = 0.5 \) and \( \gamma = 0.5 \) (the constant-average acceleration method) is unconditionally stable for the linear analysis; however such stability is not available for nonlinear problems. Convergence studies were conducted to select a time increment that yielded a stable and accurate solution while keeping the computational time to a minimum. As shown in Figures 5.5(a) and 5.5(b), \( \Delta t = 0.0001 \text{ sec} \) for the thick composite plates \((a/h = 10)\) and \( \Delta t = 0.0005 \text{ sec} \) for the thin composite plates \((a/h = 100)\) is usable for the time step to satisfy the above conditions.

5.2.2. Effect of Plate Theories

Next, the effects of the different plate theories on the transient responses are considered. Linear and nonlinear responses obtained by the three different theories are presented in Figure 5.6(a). Figure 5.6(b) shows the controlled and uncontrolled motions for each theory. It is observed that the effect of nonlinearity on the transient responses is to decrease the amplitude and increase the frequency. Note that due to the large geometric nonlinearity effects the nonlinear transient behaviors between TSDT and other two theories are apparent. It is also observed that the CLPT theory gives higher frequencies and lower amplitudes. It is because CLPT theory renders the plate stiffer compared to the other two theories.
Figure 5.4 Effect of load intensity on the nonlinear transient responses for the symmetric cross-ply laminates with SSSS boundary conditions under uniformly distributed loading using TSDT: (a) $a/h=10$, (b) $a/h=100$
Figure 5.5 Effect of time increments on the nonlinear transient analyses for the symmetric cross-ply laminates with SSSS boundary conditions under uniformly distributed loading using TSDT: (a) $a/h=10$, (b) $a/h=100$
Figure 5.6 Effect of plate theories on the nonlinear transient responses for the symmetric cross-ply thick \((a/h = 10)\) laminates with SSSS boundary conditions under uniformly distributed loading, \(q_0 = 5 \times 10^7\): (a) Details, (b) Nonlinear responses
Selected linear and nonlinear numerical results in transient responses are presented in Tables 5.5 and 5.6 for the composite plates under the uniformly distributed transverse load of intensity \( q_0 = 5.0 \times 10^7 \) and \( q_0 = 1.0 \times 10^4 \), respectively. The 4×4 meshes of the quadrant plate are used for all three different plate theories.

Table 5.5 Nondimensionalized transverse deflections versus time of the simply supported cross-ply laminates subjected to uniformly distributed load (\( a/h = 10; \Delta t = 0.0001; 4\times4L \) mesh)

<table>
<thead>
<tr>
<th>( t ) (( \times 10^3 ) sec)</th>
<th>Linear Analysis</th>
<th>Nonlinear Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CLPT</td>
<td>FSDT</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3300</td>
<td>0.3179</td>
</tr>
<tr>
<td>1.0</td>
<td>1.2578</td>
<td>1.3249</td>
</tr>
<tr>
<td>1.5</td>
<td>2.0185</td>
<td>2.1768</td>
</tr>
<tr>
<td>2.0</td>
<td>2.0672</td>
<td>2.2983</td>
</tr>
<tr>
<td>3.0</td>
<td>0.7099</td>
<td>0.8962</td>
</tr>
<tr>
<td>4.0</td>
<td>0.6494</td>
<td>0.5573</td>
</tr>
<tr>
<td>6.0</td>
<td>1.3321</td>
<td>1.6739</td>
</tr>
<tr>
<td>8.0</td>
<td>1.2684</td>
<td>1.1432</td>
</tr>
<tr>
<td>10.0</td>
<td>0.9250</td>
<td>1.2147</td>
</tr>
<tr>
<td>15.0</td>
<td>1.2394</td>
<td>1.1407</td>
</tr>
<tr>
<td>20.0</td>
<td>1.1732</td>
<td>1.4209</td>
</tr>
<tr>
<td>30.0</td>
<td>1.1962</td>
<td>1.3103</td>
</tr>
<tr>
<td>40.0</td>
<td>1.1817</td>
<td>1.2636</td>
</tr>
<tr>
<td>60.0</td>
<td>1.1695</td>
<td>1.2750</td>
</tr>
<tr>
<td>80.0</td>
<td>1.1691</td>
<td>1.2772</td>
</tr>
<tr>
<td>100.0</td>
<td>1.1692</td>
<td>1.2849</td>
</tr>
</tbody>
</table>
Table 5.6 Nondimensionalized transverse deflections versus time of the simply supported cross-ply laminates subjected to uniformly distributed load ($a/h = 100$; $\Delta t = 0.0005$; 4x4L mesh)

<table>
<thead>
<tr>
<th>$t$ ($\times 10^3$ sec)</th>
<th>Nondimensionalized Center Displacement ($\bar{w}_c$) of the thin laminated plate</th>
<th>Linear Analysis</th>
<th>Nonlinear Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CLPT</td>
<td>FSDT</td>
<td>TSDT</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0016</td>
<td>0.0015</td>
<td>0.0016</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0077</td>
<td>0.0072</td>
<td>0.0079</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0200</td>
<td>0.0186</td>
<td>0.0207</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0386</td>
<td>0.0385</td>
<td>0.0408</td>
</tr>
<tr>
<td>6.0</td>
<td>0.5738</td>
<td>0.5865</td>
<td>0.5499</td>
</tr>
<tr>
<td>10.0</td>
<td>1.3316</td>
<td>1.3558</td>
<td>1.3376</td>
</tr>
<tr>
<td>15.0</td>
<td>2.0564</td>
<td>2.0410</td>
<td>2.0894</td>
</tr>
<tr>
<td>30.0</td>
<td>0.6271</td>
<td>0.5591</td>
<td>0.6741</td>
</tr>
<tr>
<td>50.0</td>
<td>1.7982</td>
<td>1.7905</td>
<td>1.8194</td>
</tr>
<tr>
<td>70.0</td>
<td>0.6552</td>
<td>0.7000</td>
<td>0.6324</td>
</tr>
<tr>
<td>100.0</td>
<td>0.8561</td>
<td>0.8036</td>
<td>0.9235</td>
</tr>
<tr>
<td>150.0</td>
<td>1.3147</td>
<td>1.3429</td>
<td>1.2786</td>
</tr>
<tr>
<td>200.0</td>
<td>1.1082</td>
<td>1.0487</td>
<td>1.1903</td>
</tr>
<tr>
<td>300.0</td>
<td>1.1651</td>
<td>1.1218</td>
<td>1.2202</td>
</tr>
<tr>
<td>400.0</td>
<td>1.1727</td>
<td>1.1434</td>
<td>1.2074</td>
</tr>
<tr>
<td>500.0</td>
<td>1.1700</td>
<td>1.1483</td>
<td>1.2084</td>
</tr>
</tbody>
</table>

5.2.3. Effect of Lamination Schemes

The deflection suppression characteristics are studied for the different lamination schemes. The differences between the lamination schemes can be seen in Figure 5.7 for thick and thin plate cases. The maximum deflections and deflection suppression times of the linear and nonlinear analyses for the different lamination schemes with simply supported boundary condition under uniformly distributed loading by TSDT ($q = 10^7 q_o$ for $a/h = 10$, $q = 10^4 q_o$ for $a/h = 100$) have been tabulated in Table 5.7. Since the converged...
transient solutions for each lamination scheme under the uniformly distributed loads are different the maximum deflection $\hat{w}_{\text{max}}$ are defined as $\hat{w}_{\text{max}} = \bar{w}_{\text{max}} - \bar{w}_{\text{converged}}$. Deflection suppression time is defined as the time required to reduce the center displacements to 10% of its uncontrolled magnitude. It is observed that it takes less deflection suppression time with increasing plate thickness and the effect of nonlinear analysis reduces the deflection suppression time. It is also reported that symmetric cross-ply lamination shows the bigger amplitude and period under the same boundary and loading conditions.

<table>
<thead>
<tr>
<th>Plate Thickness</th>
<th>Lamination Schemes</th>
<th>Linear Analysis</th>
<th>Nonlinear Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\hat{w}_{\text{max}}$</td>
<td>$t$(sec) at $\hat{w}_{\text{max}}/10$</td>
</tr>
<tr>
<td>$a/h = 10$</td>
<td>Symmetric cross-ply</td>
<td>1.1837</td>
<td>0.0245</td>
</tr>
<tr>
<td></td>
<td>Symmetric angle-ply</td>
<td>0.8536</td>
<td>0.0285</td>
</tr>
<tr>
<td></td>
<td>Symmetric general angle-ply</td>
<td>0.8825</td>
<td>0.0290</td>
</tr>
<tr>
<td></td>
<td>Asymmetric general angle-ply</td>
<td>0.8385</td>
<td>0.0275</td>
</tr>
<tr>
<td>$a/h = 100$</td>
<td>Symmetric cross-ply</td>
<td>1.1984</td>
<td>0.1770</td>
</tr>
<tr>
<td></td>
<td>Symmetric angle-ply</td>
<td>0.8804</td>
<td>0.2120</td>
</tr>
<tr>
<td></td>
<td>Symmetric general angle-ply</td>
<td>0.9034</td>
<td>0.2150</td>
</tr>
<tr>
<td></td>
<td>Asymmetric general angle-ply</td>
<td>0.9224</td>
<td>0.2170</td>
</tr>
</tbody>
</table>
Figure 5.7 Effect of lamination schemes on the transient responses with SSSS boundary conditions under uniformly distributed loading: (a) \( a/h = 10 \ (q_0 = 5 \times 10^7) \), (b) \( a/h = 100 \ (q_0 = 5 \times 10^3) \)
5.2.4. Effect of Loading Conditions

The effect of applied loading conditions on the deflection suppression can be seen from Figures 5.8(a)-(f). The four different loading conditions are considered to study their effect on the response. They are uniformly distributed load (UMD), uniformly distributed sinusoidal load (SUMD), uniformly distributed impact load (IMD) and uniformly distributed sinusoidal impact load (SIMD). Since the first two loadings are continuously applied over the computational domain during the analysis the converged transient solution is different for each case. The figures show the differences between the loading conditions on transient response effects for the different lamination schemes.

Figure 5.8 Effect of loading conditions on the transient responses with SSSS boundary conditions: (a) cross-ply laminates \( a/h = 10 \) with \( q_0 = 5 \times 10^7 \), (b) cross-ply laminates \( a/h = 100 \) with \( q_0 = 10^4 \), (c) angle-ply laminates \( a/h = 10 \) with \( q_0 = 5 \times 10^7 \), (d) angle-ply laminates \( a/h = 100 \) with \( q_0 = 10^4 \), (e) general angle-ply laminates \( a/h = 10 \) with \( q_0 = 5 \times 10^7 \), (f) the asymmetric general angle-ply laminates \( a/h = 100 \) with \( q_0 = 10^4 \)
Figure 5.8 Continued
Figure 5.8 Continued
Figure 5.8 Continued
5.2.5. Effect of Boundary Conditions

The effect of boundary conditions on the deflection suppression is studied using the six different boundary conditions: SSSS (SS), CCCC (CC), CCSS (CS), CCFF (CF), SSFF (SF) and CFSS (CFS). Figure 5.9 contains the center displacements for various boundary conditions under different loading. The effect of the boundary conditions on amplitude and frequency is apparent from this Figure. Regardless of plate thickness and loading condition, SSFF boundary conditions show the largest amplitude values. The maximum deflections and deflection suppression times for each boundary condition have been tabulated in Table 5.8.

![Figure 5.9](image)

Figure 5.9 Effect of boundary conditions on the transient responses under uniformly distributed loading: (a) cross-ply laminates \((a/h=10)\) with \(q_0 = 10^7\), (b) cross-ply laminates \((a/h=100)\) \(q_0 = 10^4\), (c) angle-ply laminates \((a/h=10)\) with \(q_0 = 10^7\), (d) angle-ply laminates \((a/h=100)\) \(q_0 = 10^4\), (e) general angle-ply laminates \((a/h=10)\) with \(q_0 = 10^7\), (f) the asymmetric general angle-ply laminates \((a/h=100)\) \(q_0 = 10^4\)
Figure 5.9 Continued
Figure 5.9 Continued
Figure 5.9 Continued
Table 5.8 Nondimensionalized maximum transverse deflections and deflection suppression time by nonlinear TSDT

<table>
<thead>
<tr>
<th>Plate Thickness Schemes</th>
<th>Loading Condition</th>
<th>Boundary Condition</th>
<th>( \tilde{w}_{\text{max}} )</th>
<th>( t(\text{sec}) ) at ( \frac{\tilde{w}_{\text{max}}}{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Symmetric cross-ply</strong></td>
<td>Uniformly distributed load, ( q = 10^7 q_0 )</td>
<td>SSSS</td>
<td>1.1433</td>
<td>0.0245</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.5183</td>
<td>0.0290</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.4472</td>
<td>0.0280</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.5983</td>
<td>0.0745</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>1.8126</td>
<td>0.0475</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>1.5007</td>
<td>0.0475</td>
</tr>
<tr>
<td><strong>Symmetric angle-ply</strong></td>
<td>Uniformly distributed Sinusoidal load, ( q = 10^7 q_0 )</td>
<td>SSSS</td>
<td>0.5320</td>
<td>0.0265</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.3962</td>
<td>0.0270</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.2997</td>
<td>0.0280</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.4947</td>
<td>0.0615</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>1.3937</td>
<td>0.0575</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>0.7511</td>
<td>0.0570</td>
</tr>
<tr>
<td><strong>Symmetric general angle-ply</strong></td>
<td>Uniformly distributed Sinusoidal load, ( q = 10^7 q_0 )</td>
<td>SSSS</td>
<td>0.5349</td>
<td>0.0285</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.3914</td>
<td>0.0280</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.2926</td>
<td>0.0605</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.4907</td>
<td>0.0575</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>1.4437</td>
<td>0.0540</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>0.7149</td>
<td>0.0555</td>
</tr>
<tr>
<td><strong>Symmetric cross-ply</strong></td>
<td>Uniformly distributed Impact load, ( q = 10^4 q_0 )</td>
<td>SSSS</td>
<td>0.1044</td>
<td>0.2020</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.0679</td>
<td>0.1870</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.0588</td>
<td>0.1820</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.0532</td>
<td>0.4840</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>0.1153</td>
<td>0.3890</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>0.0782</td>
<td>0.4380</td>
</tr>
<tr>
<td><strong>Symmetric angle-ply</strong></td>
<td>Uniformly distributed Sinusoidal Impaict load, ( q = 10^4 q_0 )</td>
<td>SSSS</td>
<td>0.0547</td>
<td>0.2210</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.0451</td>
<td>0.1950</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.0425</td>
<td>0.1850</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.0438</td>
<td>0.4410</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>0.0696</td>
<td>0.5320</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>0.0479</td>
<td>0.4630</td>
</tr>
<tr>
<td><strong>Asymmetric general angle-ply</strong></td>
<td>Uniformly distributed Sinusoidal Impact load, ( q = 10^4 q_0 )</td>
<td>SSSS</td>
<td>0.0555</td>
<td>0.2240</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.0459</td>
<td>0.2070</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.0421</td>
<td>0.1910</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.0448</td>
<td>0.4270</td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>0.0752</td>
<td>0.4910</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>0.0502</td>
<td>0.4350</td>
</tr>
</tbody>
</table>
5.3. Nonlinear Transient Results for Laminated Composite Shells

5.3.1. Effect of Shell Theories

The effect of shell theories on the deflection control is studied using the nonlinear finite element analysis. Figure 5.10 shows the deference between Donnell and Sanders shell theories on the nonlinear transient analysis. For thin \( (a/h =100) \) and thick \( (a/h =10) \) symmetric cross-ply laminated shells, transient behavior from the selected \( R_2/a \) values are shown. Tables 5.9 and 5.10 show the numerical values of nondimensionalized center displacements for thick and thin cross-ply shells, respectively. These tabular values contain the Donnell and Sanders shell theory results for the cylindrical, spherical and doubly-curved shell types. It is observed that as \( a/h \) and \( R_2/a \) value is decreasing, the numerical differences in deflection control become larger. However, as mentioned in Section 3 for linear case, it is hard to see the big differences in the deflection control effect even in nonlinear analysis. Thus, Sanders shell theory is selected in this study to show the numerical results for shallow shell case unless otherwise stated.

As mentioned in the previous section 5.2.1, the time derivatives are approximated by using the Newmark’s direction integration method. Since no estimate on the time step for the stable nonlinear analysis is available, a convergence study has been conducted to select the appropriate time step that yields a stable and accurate solution while keeping the computational time to a minimum. \( \Delta t = 0.0001 \) sec for the thick \( (a/h = 10) \) and \( \Delta t = 0.0005 \) sec for the thin \( (a/h = 100) \) shells are selected the end of the convergence study. Figure 5.11 shows the effect of time increments on the nonlinear transient results of thick and thin cross-ply cylindrical shell with \( R_2/a = 200 \) under uniform loading.
Figure 5.10 Effect of shell theory on the nonlinear transient analysis for the symmetric cross-ply cylindrical shell: (a) thin shell ($a/h = 100$), (b) thick shell ($a/h = 10$)
Table 5.9 Selected center displacement of the symmetric cross-ply thick \((a/h = 10)\) shell with \(R_x/a = 2\) by Donnell and Sanders shell theories

<table>
<thead>
<tr>
<th>(t \times 10^3) sec</th>
<th>Cylindrical shell</th>
<th>Spherical shell</th>
<th>Doubly curved shell</th>
<th>Cylindrical shell</th>
<th>Spherical shell</th>
<th>Doubly curved shell</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.29520</td>
<td>1.16180</td>
<td>1.24010</td>
<td>1.28720</td>
<td>1.15010</td>
<td>1.23140</td>
</tr>
<tr>
<td>2</td>
<td>2.36660</td>
<td>1.48640</td>
<td>1.95170</td>
<td>2.30500</td>
<td>1.42130</td>
<td>1.89360</td>
</tr>
<tr>
<td>3</td>
<td>1.04570</td>
<td>0.29116</td>
<td>0.59766</td>
<td>0.97178</td>
<td>0.26968</td>
<td>0.55154</td>
</tr>
<tr>
<td>4</td>
<td>0.47827</td>
<td>0.90837</td>
<td>0.65336</td>
<td>0.49720</td>
<td>0.94202</td>
<td>0.68648</td>
</tr>
<tr>
<td>5</td>
<td>1.66100</td>
<td>1.37280</td>
<td>1.69060</td>
<td>1.68060</td>
<td>1.29920</td>
<td>1.66800</td>
</tr>
<tr>
<td>6</td>
<td>1.83100</td>
<td>0.52019</td>
<td>1.14260</td>
<td>1.73420</td>
<td>0.47227</td>
<td>1.05110</td>
</tr>
<tr>
<td>7</td>
<td>0.82918</td>
<td>0.78598</td>
<td>0.56680</td>
<td>0.76024</td>
<td>0.83772</td>
<td>0.57554</td>
</tr>
<tr>
<td>8</td>
<td>0.92622</td>
<td>1.26090</td>
<td>1.27600</td>
<td>0.97759</td>
<td>1.19950</td>
<td>1.30600</td>
</tr>
<tr>
<td>9</td>
<td>1.71240</td>
<td>0.69047</td>
<td>1.39810</td>
<td>1.70300</td>
<td>0.62236</td>
<td>1.31220</td>
</tr>
<tr>
<td>10</td>
<td>1.44620</td>
<td>0.74337</td>
<td>0.77151</td>
<td>1.33000</td>
<td>0.79441</td>
<td>0.72574</td>
</tr>
<tr>
<td>15</td>
<td>1.01510</td>
<td>0.87520</td>
<td>1.22950</td>
<td>1.02650</td>
<td>0.79435</td>
<td>1.23580</td>
</tr>
<tr>
<td>20</td>
<td>1.40130</td>
<td>0.95693</td>
<td>1.02040</td>
<td>1.38980</td>
<td>0.95405</td>
<td>0.96610</td>
</tr>
<tr>
<td>25</td>
<td>1.27050</td>
<td>0.83950</td>
<td>1.10450</td>
<td>1.20140</td>
<td>0.82265</td>
<td>1.10640</td>
</tr>
<tr>
<td>30</td>
<td>1.23210</td>
<td>0.92218</td>
<td>1.07920</td>
<td>1.22930</td>
<td>0.88134</td>
<td>1.03530</td>
</tr>
<tr>
<td>40</td>
<td>1.26530</td>
<td>0.89282</td>
<td>1.08750</td>
<td>1.22780</td>
<td>0.86167</td>
<td>1.05380</td>
</tr>
<tr>
<td>50</td>
<td>1.28010</td>
<td>0.88995</td>
<td>1.08600</td>
<td>1.24660</td>
<td>0.86765</td>
<td>1.05810</td>
</tr>
<tr>
<td>60</td>
<td>1.27390</td>
<td>0.89237</td>
<td>1.08420</td>
<td>1.24630</td>
<td>0.86994</td>
<td>1.05890</td>
</tr>
<tr>
<td>70</td>
<td>1.27300</td>
<td>0.89318</td>
<td>1.08340</td>
<td>1.24400</td>
<td>0.86948</td>
<td>1.05900</td>
</tr>
<tr>
<td>80</td>
<td>1.27370</td>
<td>0.89310</td>
<td>1.08300</td>
<td>1.24390</td>
<td>0.86922</td>
<td>1.05900</td>
</tr>
<tr>
<td>90</td>
<td>1.27390</td>
<td>0.89300</td>
<td>1.08290</td>
<td>1.24440</td>
<td>0.86925</td>
<td>1.05890</td>
</tr>
<tr>
<td>100</td>
<td>1.27390</td>
<td>0.89300</td>
<td>1.08280</td>
<td>1.24450</td>
<td>0.86928</td>
<td>1.05890</td>
</tr>
<tr>
<td>110</td>
<td>1.27370</td>
<td>0.89301</td>
<td>1.08280</td>
<td>1.24420</td>
<td>0.86928</td>
<td>1.05890</td>
</tr>
<tr>
<td>120</td>
<td>1.27360</td>
<td>0.89301</td>
<td>1.08290</td>
<td>1.24420</td>
<td>0.86929</td>
<td>1.05890</td>
</tr>
<tr>
<td>150</td>
<td>1.27360</td>
<td>0.89298</td>
<td>1.08280</td>
<td>1.24410</td>
<td>0.86926</td>
<td>1.05880</td>
</tr>
<tr>
<td>200</td>
<td>1.26760</td>
<td>0.88692</td>
<td>1.07670</td>
<td>1.23820</td>
<td>0.86320</td>
<td>1.05270</td>
</tr>
</tbody>
</table>
Table 5.10 Selected center displacement of the symmetric cross-ply thin \((a/h = 100)\) shell with \(R_z/a = 2\) by Donnell and Sanders shell theories

<table>
<thead>
<tr>
<th>(t \times 10^3) sec</th>
<th>Center Displacement ((\bar{w}_c))</th>
<th>Cylindrical shell</th>
<th>Spherical shell</th>
<th>Doubly curved shell</th>
<th>Cylindrical shell</th>
<th>Spherical shell</th>
<th>Doubly curved shell</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Donnell</td>
<td>0.03714</td>
<td>0.02650</td>
<td>0.03186</td>
<td>0.03713</td>
<td>0.02649</td>
<td>0.03185</td>
</tr>
<tr>
<td>3</td>
<td>Sanders</td>
<td>0.08376</td>
<td>0.03788</td>
<td>0.05981</td>
<td>0.08372</td>
<td>0.03785</td>
<td>0.05978</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.12634</td>
<td>0.02106</td>
<td>0.06521</td>
<td>0.12624</td>
<td>0.02101</td>
<td>0.06513</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0.14099</td>
<td>-0.00204</td>
<td>0.03840</td>
<td>0.14077</td>
<td>-0.00203</td>
<td>0.03831</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>0.12226</td>
<td>0.00678</td>
<td>0.00560</td>
<td>0.12192</td>
<td>0.00683</td>
<td>0.00556</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>0.07968</td>
<td>0.03217</td>
<td>-0.00051</td>
<td>0.07928</td>
<td>0.03216</td>
<td>-0.00046</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>0.02954</td>
<td>0.03372</td>
<td>0.02274</td>
<td>0.02928</td>
<td>0.03364</td>
<td>0.02285</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>-0.00167</td>
<td>0.01511</td>
<td>0.05272</td>
<td>-0.00165</td>
<td>0.01506</td>
<td>0.05274</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>0.00676</td>
<td>0.00412</td>
<td>0.06352</td>
<td>0.00705</td>
<td>0.00414</td>
<td>0.06339</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>0.10740</td>
<td>0.00355</td>
<td>0.04161</td>
<td>0.10689</td>
<td>0.00357</td>
<td>0.04166</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0.04428</td>
<td>0.00668</td>
<td>0.01046</td>
<td>0.04463</td>
<td>0.00672</td>
<td>0.01055</td>
</tr>
<tr>
<td>25</td>
<td></td>
<td>0.07721</td>
<td>0.00765</td>
<td>0.01919</td>
<td>0.07643</td>
<td>0.00768</td>
<td>0.01907</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>0.07849</td>
<td>0.00921</td>
<td>0.04312</td>
<td>0.07893</td>
<td>0.00927</td>
<td>0.04288</td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>0.09343</td>
<td>0.01137</td>
<td>0.02776</td>
<td>0.09370</td>
<td>0.01143</td>
<td>0.02797</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>0.09606</td>
<td>0.01317</td>
<td>0.03168</td>
<td>0.09578</td>
<td>0.01323</td>
<td>0.03148</td>
</tr>
<tr>
<td>60</td>
<td></td>
<td>0.08870</td>
<td>0.01455</td>
<td>0.03577</td>
<td>0.08806</td>
<td>0.01462</td>
<td>0.03588</td>
</tr>
<tr>
<td>70</td>
<td></td>
<td>0.07596</td>
<td>0.01560</td>
<td>0.02762</td>
<td>0.07526</td>
<td>0.01566</td>
<td>0.02753</td>
</tr>
<tr>
<td>80</td>
<td></td>
<td>0.06530</td>
<td>0.01637</td>
<td>0.03681</td>
<td>0.06471</td>
<td>0.01642</td>
<td>0.03678</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>0.05989</td>
<td>0.01735</td>
<td>0.03496</td>
<td>0.05995</td>
<td>0.01739</td>
<td>0.03485</td>
</tr>
<tr>
<td>150</td>
<td></td>
<td>0.07261</td>
<td>0.01820</td>
<td>0.03277</td>
<td>0.07241</td>
<td>0.01820</td>
<td>0.03277</td>
</tr>
<tr>
<td>200</td>
<td></td>
<td>0.06823</td>
<td>0.01831</td>
<td>0.03212</td>
<td>0.06819</td>
<td>0.01830</td>
<td>0.03211</td>
</tr>
<tr>
<td>250</td>
<td></td>
<td>0.06972</td>
<td>0.01830</td>
<td>0.03217</td>
<td>0.06959</td>
<td>0.01829</td>
<td>0.03214</td>
</tr>
<tr>
<td>300</td>
<td></td>
<td>0.06922</td>
<td>0.01830</td>
<td>0.03223</td>
<td>0.06914</td>
<td>0.01828</td>
<td>0.03220</td>
</tr>
<tr>
<td>400</td>
<td></td>
<td>0.06933</td>
<td>0.01829</td>
<td>0.03223</td>
<td>0.06924</td>
<td>0.01828</td>
<td>0.03221</td>
</tr>
<tr>
<td>500</td>
<td></td>
<td>0.06783</td>
<td>0.01690</td>
<td>0.03077</td>
<td>0.06774</td>
<td>0.01689</td>
<td>0.03075</td>
</tr>
</tbody>
</table>
Figure 5.11 Effect of time increments on the nonlinear transient analysis for the symmetric cross-ply cylindrical shell \( R_2/a = 200 \): (a) thin shell \( a/h = 100 \), (b) thick shell \( a/h = 10 \)
5.3.2. Effect of $R_2/a$

In this section, the effect of $R_2/a$ on the deflection suppression is studied. Figure 5.12 shows the effect of $R_2/a$ for cylindrical cross-ply shell. The uniform loads $q_o = 10^4$ for thick shell $a/h =10$, and $q_o = 10^3$ for thin shell $a/h =100$ are applied with simply supported boundary condition. The maximum transverse deflection and the deflection suppression time is tabulated in Table 5.11 for each $R_2/a$ value. Here the maximum deflection $\tilde{w}_{\text{max}}$ is defined as $\tilde{w}_{\text{max}} = \tilde{w}_{\text{max}} - \tilde{w}_{\text{converged}}$, where $\tilde{w}_{\text{max}}$ is the maximum nondimensionalized center displacement and $\tilde{w}_{\text{converged}}$ is the converged displacement value.

It is observed that it takes more deflection suppression time with increasing $R_2/a$ value in both thin and thick shells. For the thin shell, the maximum deflection suppression time is 5 times faster than the others depending on $R_2/a$ value. In the thick shell, $R_2/a$ values bigger than $R_2/a = 50$ cases have the little differences between the transient behaviors.

5.3.3. Effect of Shell Types

Next, the effect of shell types on the deflection suppression characteristics is studied for spherical, cylindrical, and doubly curved CFRP shells. In this study doubly curved shell is chosen as $R_i = 2R_2$ for specific example. The center displacements versus time for spherical, cylindrical, and doubly curved shells are shown in Figures 5.13 to 5.16. Figures 5.13 and 5.14 show the comparison of transient behavior of three shell types for thin and thin cases. For the thick shell with bigger $R_2/a$ values, the deference between the shell types is hard to recognize. Selected maximum deflection and deflection suppression time for spherical and doubly curved shells are shown in Table 5.12. It is observed that it takes less deflection suppression time with decreasing $R_2/a$ value for spherical and doubly-curved shells. Thus the shell with the smallest $R_2/a$ shows the maximum deflection suppression. It is also observed that the spherical shell has the biggest deflection
suppression and smallest maximum deflection and the cylindrical shell has the smallest deflection suppression and largest maximum deflection. It is because of the fact that spherical shell has the smallest $R_1$ and doubly curved shell (in this study $R_1 = 2R_2$), and cylindrical shell has the largest $R_1$ ($R_1 = \infty$) with the same $R_2$ value.

Table 5.11 Maximum transverse deflection and deflection suppression time for the symmetric cross-ply cylindrical shells

<table>
<thead>
<tr>
<th>Thickness</th>
<th>$R_2/a$</th>
<th>$\tilde{w}_{\text{max}}$</th>
<th>$t$ (sec) at $\tilde{w}_{\text{max}}/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin Shell</td>
<td>$a/h = 100$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.071646</td>
<td>0.114000</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.293180</td>
<td>0.161500</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.611730</td>
<td>0.176000</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.845500</td>
<td>0.206500</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.983200</td>
<td>0.225500</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1.033500</td>
<td>0.228000</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>1.056300</td>
<td>0.232000</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>1.066600</td>
<td>0.234000</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>1.069600</td>
<td>0.234000</td>
<td></td>
</tr>
<tr>
<td>$10^{99}$</td>
<td>1.072400</td>
<td>0.235500</td>
<td></td>
</tr>
<tr>
<td>Thick Shell</td>
<td>$a/h = 10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.093000</td>
<td>0.024000</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.238800</td>
<td>0.025000</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.262900</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1.270300</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>1.273200</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1.274000</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>1.274300</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>1.274500</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>1.274600</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>$10^{99}$</td>
<td>1.274600</td>
<td>0.025500</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.12 Effect of $R_2/a$ for cylindrical cross-ply shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h=100$), (b) thick shell ($a/h=10$)
Figure 5.13 Effect of shell type for the cross-ply thick \((a/h=10)\) shell under uniform load in SSSS boundary condition; (a) \(R_2/a = 5\), (b) \(R_2/a = 50\)
Figure 5.14 Effect of shell type for the cross-ply thin \((a/h=100)\) shell under uniform load in SSSS boundary condition; (a) \(R_2/a = 5\), (b) \(R_2/a = 50\)
Figure 5.15 Effect of $R_2/a$ for spherical cross-ply shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h=100$), (b) thick shell ($a/h=10$)
Figure 5.16 Effect of $R_2/a$ for doubly-curved cross-ply shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h=100$), (b) thick shell ($a/h=10$)
Table 5.12 Maximum transverse deflection and deflection suppression time for the symmetric cross-ply shells by Sanders shell theory

<table>
<thead>
<tr>
<th>Thickness</th>
<th>$R_z/a$</th>
<th>Spherical Shell</th>
<th></th>
<th>Doubly Curved Shell</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin Shell</td>
<td>$a/h = 100$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.019592</td>
<td>0.077000</td>
<td>0.034506</td>
<td>0.093500</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.123990</td>
<td>0.119000</td>
<td>0.191890</td>
<td>0.137000</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.312520</td>
<td>0.164000</td>
<td>0.424780</td>
<td>0.172000</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.892100</td>
<td>0.212000</td>
<td>0.946900</td>
<td>0.218500</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.978200</td>
<td>0.225000</td>
<td>1.011100</td>
<td>0.227000</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>1.059900</td>
<td>0.234000</td>
<td>1.073100</td>
<td>0.234500</td>
<td></td>
</tr>
<tr>
<td>Thick Shell</td>
<td>$a/h = 10$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.726790</td>
<td>0.023500</td>
<td>0.868800</td>
<td>0.022500</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.146000</td>
<td>0.024500</td>
<td>1.199200</td>
<td>0.025000</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.237100</td>
<td>0.025000</td>
<td>1.251600</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>1.271300</td>
<td>0.025500</td>
<td>1.272300</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1.273200</td>
<td>0.025500</td>
<td>1.273500</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>1.274500</td>
<td>0.025500</td>
<td>1.274400</td>
<td>0.025500</td>
<td></td>
</tr>
</tbody>
</table>

5.3.4. Effect of Lamination Schemes

The effect of lamination schemes on the deflection suppression characteristics is studied. Symmetric cross-ply ($m, 90, 0, 90, 0$), symmetric angle-ply ($m, 45, -45, 45, -45$) and symmetric general angle-ply ($m, 45, -45, 0, 90$) laminations are considered for spherical, cylindrical, and doubly curved shells. Figures 5.17 – 5.19 show the effect of lamination schemes for cylindrical ($R_z/a = 2$), spherical ($R_z/a = 5$) and doubly-curved ($R_z/a = 200$) shells respectively. All the examples are under uniformly distributed load and SSSSS boundary condition. The effect of lamination on the deflection suppression time and maximum deflection can be seen in Table 5.13 as numerical values. It is reported that symmetric cross-ply shell has the biggest maximum deflection for all shell types and angle-ply shell shows the smaller deflection values than other lamination schemes.
5.3.5. *Effect of Boundary Conditions*

The effect of boundary condition of laminated composite shells on the deflection suppression is shown in Figures 5.20 – 5.22. Table 5.14 has the maximum deflection and deflection suppression time for each boundary condition. Each shell type, cylindrical, spherical, doubly-curved shell has the different $R_1$ and $R_2$ values, the maximum deflection can be found in SSFF boundary condition and the maximum deflection suppression occurs in CCCC boundary condition.

5.3.6. *Effect of Loading Conditions*

The effect of applied loading conditions on the deflection suppression is shown in Figure 5.23. Uniformly distributed load, uniformly distributed sinusoidal load, uniformly distributed impact load and uniformly distributed sinusoidal impact load are considered to study the loading effect. The deflection under the uniformly distributed load has the largest value regardless of shell type or thickness.

5.4. *Nonlinear Results under Thermomechanical Loads*

In this section, parametric studies for laminated composite plate and shell structures that are subjected to a temperature field in addition to the mechanical loading are conducted. Temperature changes often represented a significant factor, and sometimes the predominant cause of failure of composite structures subject to severe environmental loads. In fiber reinforced laminated composites such as CFRP, the thermal expansion coefficients in the direction of fibers are usually much smaller than those in the transverse direction.
Figure 5.17 Effect of lamination schemes for cylindrical shell under uniform load in SSSS boundary condition; (a) thin shell ($a/h = 100$), (b) thick shell ($a/h = 10$)
Figure 5.18 Effect of lamination schemes for spherical shell under uniform load in SSSS boundary condition; (a) thin shell \((a/h=100)\), (b) thick shell \((a/h=10)\)
Figure 5.19 Effect of lamination schemes for doubly-curved shell under uniform load in SSSS boundary condition; (a) thin shell \( (a/h=100) \), (b) thick shell \( (a/h=10) \)
Table 5.13 Vibration suppression characteristics for the symmetric cross-ply laminated shells with the different lamination

<table>
<thead>
<tr>
<th>Thickness</th>
<th>Shell Type</th>
<th>$R_z/a$</th>
<th>Lamination</th>
<th>$\ddot{w}_{\text{max}}$</th>
<th>$t$ (sec) at $\ddot{w}_{\text{max}}/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin Shell</td>
<td>Cylindrical</td>
<td>2</td>
<td>Cross-ply</td>
<td>0.071648</td>
<td>0.114000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Angle-ply</td>
<td>0.020724</td>
<td>0.066500</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>General Angle-ply</td>
<td>0.026657</td>
<td>0.081500</td>
</tr>
<tr>
<td></td>
<td>Spherical</td>
<td>5</td>
<td>Cross-ply</td>
<td>0.123960</td>
<td>0.119000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Angle-ply</td>
<td>0.032041</td>
<td>0.074500</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>General Angle-ply</td>
<td>0.050940</td>
<td>0.084000</td>
</tr>
<tr>
<td>Doubly</td>
<td>Cylindrical</td>
<td>200</td>
<td>Cross-ply</td>
<td>1.048900</td>
<td>0.231000</td>
</tr>
<tr>
<td>Curved</td>
<td></td>
<td></td>
<td>Angle-ply</td>
<td>0.534070</td>
<td>0.305500</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>General Angle-ply</td>
<td>0.584150</td>
<td>0.335500</td>
</tr>
<tr>
<td>Thick</td>
<td>Cylindrical</td>
<td>2</td>
<td>Cross-ply</td>
<td>1.092700</td>
<td>0.024000</td>
</tr>
<tr>
<td>Shell</td>
<td></td>
<td></td>
<td>Angle-ply</td>
<td>0.687550</td>
<td>0.019500</td>
</tr>
<tr>
<td>$a/h = 10$</td>
<td></td>
<td></td>
<td>General Angle-ply</td>
<td>0.738630</td>
<td>0.020500</td>
</tr>
<tr>
<td>Spherical</td>
<td>5</td>
<td></td>
<td>Cross-ply</td>
<td>1.146000</td>
<td>0.024500</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Angle-ply</td>
<td>0.744220</td>
<td>0.023500</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>General Angle-ply</td>
<td>0.778160</td>
<td>0.024000</td>
</tr>
<tr>
<td>Doubly</td>
<td>200</td>
<td></td>
<td>Cross-ply</td>
<td>1.274100</td>
<td>0.025500</td>
</tr>
<tr>
<td>Curved</td>
<td></td>
<td></td>
<td>Angle-ply</td>
<td>0.937800</td>
<td>0.026000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>General Angle-ply</td>
<td>0.965800</td>
<td>0.026000</td>
</tr>
</tbody>
</table>
Figure 5.20 Effect of boundary conditions for cross-ply cylindrical shell under uniform load; (a) thin shell \((a/h=100, \ R_2/a=100)\), (b) thick shell \((a/h=10, \ R_2/a=50)\)
Figure 5.21 Effect of boundary conditions for cross-ply spherical shell under uniform load; (a) thin shell \((a/h=100, \ R_2/a=20)\), (b) thick shell \((a/h=10, \ R_2/a=10)\)
Figure 5.22 Effect of boundary conditions for cross-ply doubly-curved shell under uniform load; (a) thin shell \((a/h=100, \ R_2/a=10)\), (b) thick shell \((a/h=10, \ R_2/a=20)\)
Table 5.14 Nondimensionalized maximum transverse deflections and deflection suppression time by nonlinear TSDT

<table>
<thead>
<tr>
<th>Thickness</th>
<th>Shell Type</th>
<th>$R_z/a$</th>
<th>Boundary Condition</th>
<th>$\tilde{w}_{\text{max}}$</th>
<th>$t(\text{sec})$ at $\tilde{w}_{\text{max}}/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylindrical</td>
<td>100</td>
<td>SSSS</td>
<td>1.024900</td>
<td>0.229500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.307740</td>
<td>0.176000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.402020</td>
<td>0.179500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.410440</td>
<td>0.392500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>1.594400</td>
<td>0.436500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>1.355100</td>
<td>0.468500</td>
<td></td>
</tr>
<tr>
<td>Spherical</td>
<td>20</td>
<td>SSSS</td>
<td>0.618830</td>
<td>0.202500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.197590</td>
<td>0.111000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.278060</td>
<td>0.132000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.260770</td>
<td>0.234500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>0.836260</td>
<td>0.317000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>0.744180</td>
<td>0.444000</td>
<td></td>
</tr>
<tr>
<td>Doubly curved</td>
<td>10</td>
<td>SSSS</td>
<td>0.425350</td>
<td>0.171000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.133960</td>
<td>0.089000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.166440</td>
<td>0.112000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.129180</td>
<td>0.150500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>0.550670</td>
<td>0.248500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>0.556810</td>
<td>0.398500</td>
<td></td>
</tr>
<tr>
<td>Cylindrical</td>
<td>50</td>
<td>SSSS</td>
<td>1.273200</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.449170</td>
<td>0.026500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.561410</td>
<td>0.027500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.651010</td>
<td>0.059000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>2.050600</td>
<td>0.046000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>1.341900</td>
<td>0.048500</td>
<td></td>
</tr>
<tr>
<td>Spherical</td>
<td>10</td>
<td>SSSS</td>
<td>1.237100</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.438030</td>
<td>0.024000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.530840</td>
<td>0.024500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.632780</td>
<td>0.052000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>1.942700</td>
<td>0.045000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>1.337900</td>
<td>0.048500</td>
<td></td>
</tr>
<tr>
<td>Doubly curved</td>
<td>10</td>
<td>SSSS</td>
<td>1.266800</td>
<td>0.025500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCC</td>
<td>0.389790</td>
<td>0.026500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCSS</td>
<td>0.554230</td>
<td>0.027500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CCFF</td>
<td>0.638720</td>
<td>0.056000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>SSFF</td>
<td>2.029700</td>
<td>0.046000</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>CFSS</td>
<td>1.339900</td>
<td>0.048500</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.23 Effect of loading conditions for cross-ply cylindrical shell in SSSS boundary condition; (a) thin shell \((a/h=100, R_2/a=5)\), (b) thick shell \((a/h=10, R_2/a=5)\)
As mentioned in the previous section, since thermal effects are taken into consideration with the understanding that the material properties are independent of temperature, temperature enters the formulation only through constitutive equations. The temperature field considered is assumed to be a constant distribution over the plate/shell structure surface and thickness. Laminated composite plate and shell models \((a/b = 1)\) used in this section are the same as the previous ones. All numerical results are presented in terms of nondimensionalized. The nondimensionalized parameters used in this study are

\[
\text{Load parameter } \quad p' = \frac{q_0 a^4}{E_z h^4} \quad (5.1)
\]

\[
\text{Center deflection } \quad w' = \frac{w}{h}, \quad \bar{w}_c = 100 \times \frac{w_0 E_z h^3}{a^4 q_0} \quad (5.2)
\]

The following thermal expansion coefficients are used for this study,

\[
\begin{align*}
\alpha_1 &= 12.0 \times 10^{-6} / c \quad \text{and} \quad \alpha_2 = 12.0 \times 10^{-6} / c \quad \text{for Terfenol-D} \\
\alpha_1 &= 0.1 \times 10^{-6} / c \quad \text{and} \quad \alpha_2 = 22.0 \times 10^{-6} / c \quad \text{for CFRP} \\
\alpha_1 &= 1.8 \times 10^{-6} / c \quad \text{and} \quad \alpha_2 = 54.0 \times 10^{-6} / c \quad \text{for Gr-Ep (AS)} \\
\alpha_1 &= 6.3 \times 10^{-6} / c \quad \text{and} \quad \alpha_2 = 20.5 \times 10^{-6} / c \quad \text{for Gl-Ep} \\
\alpha_1 &= 4.5 \times 10^{-6} / c \quad \text{and} \quad \alpha_2 = 14.4 \times 10^{-6} / c \quad \text{for Br-Ep}
\end{align*}
\]

5.4.1. Static Results under Thermomechanical Loads

The static analysis is performed for different mechanical and thermal loads. First, the behavior of the cross-ply laminate with simply supported boundary condition is studied under uniformly distributed mechanical load and thermal load. To see the effect of shear deformation on the response of thick laminated plates, CLPT, FSDT, and TSDT solutions are obtained for side-to-thickness ratio \(a/h = 10\). Figure 5.24 shows the nondimensionalized center displacements of thick laminates by three different plate theories. It is observed that CLPT theory renders the plate stiffer compared to the other two theories. For selected mechanical and thermal loads, the center displacements for each theory are tabulated in Table 5.15. Next, the behavior of thin laminated plates under
thermomechanical loading is considered. Using TSDT, the nonlinearities of center deflection due to the temperature effect are clearly shown in Figure 5.25 and Table 5.16.

Table 5.15 Nondimensionalized center displacements of the simply supported thick cross-ply laminate under thermomechanical loading by CLPT, FSDT, and TSDT

<table>
<thead>
<tr>
<th>Temperature Rise, $\Delta T$</th>
<th>60 °C</th>
<th>100 °C</th>
<th>300 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plate Theory</td>
<td>CLPT</td>
<td>FSDT</td>
<td>TSDT</td>
</tr>
<tr>
<td>$p' = 0.5$</td>
<td>0.0565</td>
<td>0.0615</td>
<td>0.0658</td>
</tr>
<tr>
<td>$p' = 1$</td>
<td>0.1127</td>
<td>0.1225</td>
<td>0.1310</td>
</tr>
<tr>
<td>$p' = 3$</td>
<td>0.3265</td>
<td>0.3526</td>
<td>0.3780</td>
</tr>
<tr>
<td>$p' = 5$</td>
<td>0.5146</td>
<td>0.5513</td>
<td>0.5944</td>
</tr>
<tr>
<td>$p' = 10$</td>
<td>0.8777</td>
<td>0.9269</td>
<td>1.0228</td>
</tr>
<tr>
<td>$p' = 15$</td>
<td>1.1417</td>
<td>1.1964</td>
<td>1.3564</td>
</tr>
<tr>
<td>$p' = 25$</td>
<td>1.5234</td>
<td>1.5847</td>
<td>1.8924</td>
</tr>
<tr>
<td>$p' = 35$</td>
<td>1.8063</td>
<td>1.8722</td>
<td>2.3406</td>
</tr>
<tr>
<td>$p' = 50$</td>
<td>2.1374</td>
<td>2.2081</td>
<td>2.9286</td>
</tr>
<tr>
<td>$p' = 65$</td>
<td>2.4043</td>
<td>2.4779</td>
<td>3.4551</td>
</tr>
</tbody>
</table>

Table 5.16 Nonlinear nondimensionalized center displacements of the simply supported thin cross-ply laminate under thermomechanical loading by TSDT

<table>
<thead>
<tr>
<th>Load Parameter, $p'$</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 (Linear)</td>
<td>0.11933</td>
<td>0.09656</td>
<td>0.08200</td>
<td>0.07188</td>
<td>0.06446</td>
<td>0.05876</td>
<td>0.05426</td>
<td>0.05062</td>
</tr>
<tr>
<td>10</td>
<td>0.11888</td>
<td>0.08339</td>
<td>0.06606</td>
<td>0.05581</td>
<td>0.04903</td>
<td>0.04421</td>
<td>0.04061</td>
<td>0.03784</td>
</tr>
<tr>
<td>20</td>
<td>0.23513</td>
<td>0.16626</td>
<td>0.13194</td>
<td>0.11154</td>
<td>0.09803</td>
<td>0.08839</td>
<td>0.08124</td>
<td>0.07567</td>
</tr>
<tr>
<td>30</td>
<td>0.34668</td>
<td>0.24817</td>
<td>0.19754</td>
<td>0.16712</td>
<td>0.14696</td>
<td>0.13260</td>
<td>0.12182</td>
<td>0.11349</td>
</tr>
<tr>
<td>40</td>
<td>0.45242</td>
<td>0.32869</td>
<td>0.26267</td>
<td>0.22524</td>
<td>0.19580</td>
<td>0.17670</td>
<td>0.16233</td>
<td>0.15121</td>
</tr>
<tr>
<td>50</td>
<td>0.55200</td>
<td>0.40751</td>
<td>0.32723</td>
<td>0.27772</td>
<td>0.24447</td>
<td>0.22075</td>
<td>0.20284</td>
<td>0.18897</td>
</tr>
<tr>
<td>60</td>
<td>0.64560</td>
<td>0.48441</td>
<td>0.39109</td>
<td>0.33260</td>
<td>0.29304</td>
<td>0.26471</td>
<td>0.24331</td>
<td>0.22671</td>
</tr>
<tr>
<td>70</td>
<td>0.73368</td>
<td>0.55925</td>
<td>0.45416</td>
<td>0.38713</td>
<td>0.34144</td>
<td>0.30849</td>
<td>0.28382</td>
<td>0.26452</td>
</tr>
<tr>
<td>80</td>
<td>0.81676</td>
<td>0.63198</td>
<td>0.51636</td>
<td>0.44127</td>
<td>0.38964</td>
<td>0.35224</td>
<td>0.32415</td>
<td>0.30206</td>
</tr>
<tr>
<td>90</td>
<td>0.89541</td>
<td>0.70258</td>
<td>0.57763</td>
<td>0.49497</td>
<td>0.43763</td>
<td>0.39588</td>
<td>0.36429</td>
<td>0.33965</td>
</tr>
<tr>
<td>100</td>
<td>0.97013</td>
<td>0.77110</td>
<td>0.63793</td>
<td>0.54821</td>
<td>0.48537</td>
<td>0.43938</td>
<td>0.40464</td>
<td>0.37721</td>
</tr>
</tbody>
</table>
Figure 5.24 Effect of the plate theories on static behaviors of simply supported thick cross-ply laminates under mechanical and thermal load; (a) CLPT, (b) FSDT, (c) TSDT
Figure 5.24 Continued

Figure 5.25 Thermomechanical behavior of simply supported thin cross-ply laminates
The typical nonlinear results of center deflection of thick and thin cross-ply and angle-ply plates are given in Figure 5.26. The effect of temperature rise $\Delta T = 200^\circ C$ for thick laminate and $\Delta T = 20^\circ C$ for thin laminate is considered, respectively. The effect of the thermal loading is apparent in the thin laminate with small temperature rise. The magnitude of center deflections of cross-ply laminate is bigger than that of angle-ply laminate.

The effects of $R_2/a$ on the nonlinear deflection under thermomechanical loads are shown in Figure 5.27. The nondimensionalized center displacements of thick ($a/h=10, \Delta T=100^\circ C$) and thin ($a/h=100, \Delta T=10^\circ C$) cross-ply cylindrical shells are plotted. Figures 5.28 – 5.30 show the temperature effects of cross-ply cylindrical, doubly-curved and spherical shells, respectively. As thermal loads increase, the center deflections increase for any shell type or $R_2/a$ value.

5.4.2. Laminated Composite Plates under Thermomechanical Loads

Nonlinear transient results of laminated composites under thermomechanical loadings are presented in this section. The critical time step for Newmark’s scheme in nonlinear problem is selected to satisfy the stable and accurate solution condition. Time interval 0.1 and 0.5 milliseconds are chosen for the thick ($a/h=10$) and the thin ($a/h=100$) laminates.
Figure 5.26 Effect of the laminations on static behaviors of simply supported laminated plates under mechanical and thermal load; (a) Thick laminate, (b) Thin laminate.
Figure 5.27 Effect of $R_2/a$ on static behavior of cross-ply cylindrical shell under thermomechanical load: (a) Thick shell ($a/h=10$, $\Delta T=100^\circ C$), (b) Thin shell ($a/h=100$, $\Delta T=10^\circ C$)
Figure 5.28 Temperature effect on static behavior of cross-ply cylindrical shell under thermomechanical load: (a) Thick shell \((a/h = 10, R_2/a = 2)\), (b) Thin shell \((a/h = 100, R_2/a = 2)\)
Figure 5.29 Temperature effect on static behavior of cross-ply doubly-curved shell under thermomechanical load: (a) Thick shell \((a/h=10, R_2/a=10)\), (b) Thin shell \((a/h=100, R_2/a=10)\)
Figure 5.30 Temperature effect on static behavior of cross-ply spherical shell under thermomechanical load: (a) Thick shell \((h/a =10, R_2/a =2)\), (b) Thin shell \((h/a =100, R_2/a =2)\)
First, the thermomechanical effect with different laminations is studied. The transient behavior of the laminated composites under thermomechanical loadings is shown in Figures 5.31, 5.32 and 5.33. The applied mechanical load parameters are $10^7$ for thick laminates and $5 \times 10^3$ for thin laminates, respectively. Each mechanical load is selected to show the nonlinear behavior of laminates and that selected load makes the nonlinear problem can be solved without considering the load loop. The thermal load is applied to the opposite direction of the mechanical load in the thermomechanical simulations of laminated composite plates.

Uniformly distributed load and suddenly applied uniform load are considered with simply supported boundary condition in this study. The Figure includes the behaviors of uncontrolled case, the case without thermal load, and the case with thermal load. The considered thermal rise values are $\Delta T = 100, 200^\circ C$ for all thick laminates, $\Delta T = 20, 50^\circ C$ for thin cross-ply and angle-ply laminates, and $\Delta T = 20, 40^\circ C$ for thin general laminates. The effect of the thermal load is to reduce the amplitude and vibration suppression time. Here vibration suppression time is the time required to reduce the center deflections to 10% of its uncontrolled magnitude. The amplitude and period of the center deflections decrease with increasing thermal loading. The results of cross-ply, angle-ply and general laminates under thermomechanical loads can be seen in Figures 5.31, 5.32 and 5.33, respectively.
Figure 5.31 Transient behavior of the cross-ply laminates with simply supported boundary condition; (a) Thick laminate under impact load, (b) Thick laminate under uniform load, (c) Thin laminate under impact load, (d) Thin laminate under uniform load
Figure 5.31 Continued
Figure 5.32 Transient behavior of the angle-ply laminates with simply supported boundary condition; (a) Thick laminate under impact load, (b) Thick laminate under uniform load, (c) Thin laminate under impact load, (d) Thin laminate under uniform load.
Figure 5.32 Continued
Figure 5.33 Transient behavior of the general laminates with simply supported boundary condition; (a) Thick laminate under impact load, (b) Thick laminate under uniform load, (c) Thin laminate under impact load, (d) Thin laminate under uniform load
Figure 5.33 Continued
The effect of elastic material property on the vibration suppression response is studied under thermomechanical loading. Gr-Ep(AS), Gl-Ep, and Br-Ep are considered in addition to CFRP material, where the thermal expansion coefficients are shown in the beginning of this section. Figures 5.34 and 5.35 show the transient responses of thick and thin laminates, respectively. The mechanical load parameter $10^6$ and thermal rise $\Delta T = 150^\circ C$ are used for thick laminates and $p' = 10^3$ and $\Delta T = 20^\circ C$ for thin laminates.

The maximum deflections and vibration suppression times of cross-ply laminates with different elastic materials under uniformly distributed mechanical loadings have been tabulated with and without thermal load in Table 5.17. Since the converged transient solutions for each material are different, the maximum deflection $\tilde{w}_{max}$ are defined as $\tilde{w}_{max} = \tilde{w}_{max} - \tilde{w}_{converged}$. It is observed that Gl-Ep shows the largest maximum deflection and the minimum vibration suppression time for thick laminates regardless of thermal effect.

<table>
<thead>
<tr>
<th>Plate Thickness</th>
<th>Elastic Material</th>
<th>Temperature Rise</th>
<th>$\tilde{w}_{max}$</th>
<th>$t$(sec) at $\tilde{w}_{max}$/10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a/h = 10$</td>
<td>Gr-Ep (AS)</td>
<td>0</td>
<td>1.0874</td>
<td>0.0265</td>
</tr>
<tr>
<td></td>
<td></td>
<td>150</td>
<td>0.9423</td>
<td>0.0240</td>
</tr>
<tr>
<td></td>
<td>Gl-Ep</td>
<td>0</td>
<td>2.9160</td>
<td>0.0230</td>
</tr>
<tr>
<td></td>
<td></td>
<td>150</td>
<td>2.5015</td>
<td>0.0245</td>
</tr>
<tr>
<td></td>
<td>Br-Ep</td>
<td>0</td>
<td>2.0213</td>
<td>0.0285</td>
</tr>
<tr>
<td></td>
<td></td>
<td>150</td>
<td>1.8231</td>
<td>0.0255</td>
</tr>
<tr>
<td>$a/h = 100$</td>
<td>Gr-Ep (AS)</td>
<td>0</td>
<td>1.0366</td>
<td>0.2110</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>0.3231</td>
<td>0.2630</td>
</tr>
<tr>
<td></td>
<td>Gl-Ep</td>
<td>0</td>
<td>2.5370</td>
<td>0.2130</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>0.9914</td>
<td>0.2480</td>
</tr>
<tr>
<td></td>
<td>Br-Ep</td>
<td>0</td>
<td>1.9293</td>
<td>0.2280</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>0.7680</td>
<td>0.2730</td>
</tr>
</tbody>
</table>
Figure 5.34 Effect of the elastic materials on the transient behavior of the thick cross-ply laminates with simply supported boundary condition; (a) Thick laminate under uniform load, (b) GrEp case, (c) GlEp case, (d) BrEp case.
Figure 5.34 Continued
Figure 5.35 Effect of the elastic materials on the transient behavior of the thin cross-ply laminates with simply supported boundary condition; (a) Thin laminate under uniform load, (b) GrEp case, (c) GlEp case, (d) BrEp case
Simply supported cross-ply thin laminate with uniform load $P' = 10^3$

(c)

(d)

Figure 5.35 Continued
The following six boundary conditions are considered to see the differences in transient responses; SSSS, CCCC, CCSS, CCFF, SSFF, and CFSS. Where ‘S’ represents simply supported, ‘C’ clamped, and ‘F’ free edge conditions. Figure 5.36 contains the center displacements of cross-ply laminate for different boundary conditions under thermomechanical loading. The maximum deflections and vibration suppression times for each boundary condition have been tabulated in Table 5.18. It is observed that SSFF boundary condition shows the maximum deflection and CCCC shows the minimum deflection for thin and thick laminate under the thermal load in addition to the uniformly distributed mechanical load. The order of the boundary conditions with respect to the maximum center deflection value under thermomechanical loads is SSFF, CFSS, SSSS, CCFF, CCSS, and CCCC boundary conditions for all thin and thick laminated composite plates.

Table 5.18 Nondimensionalized maximum center displacements and vibration suppression time of CFRP cross-ply laminates under thermomechanical loading

<table>
<thead>
<tr>
<th>Plate Thickness</th>
<th>Boundary Condition</th>
<th>$\tilde{w}_{\text{max}}$</th>
<th>$t$(sec) at $\tilde{w}_{\text{max}}/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a/h=10$</td>
<td>CCCC</td>
<td>0.4489</td>
<td>0.0300</td>
</tr>
<tr>
<td></td>
<td>CCFF</td>
<td>0.6493</td>
<td>0.0990</td>
</tr>
<tr>
<td></td>
<td>CCSS</td>
<td>0.4788</td>
<td>0.0315</td>
</tr>
<tr>
<td></td>
<td>CFSS</td>
<td>1.2685</td>
<td>0.0565</td>
</tr>
<tr>
<td></td>
<td>SSFF</td>
<td>1.7966</td>
<td>0.0525</td>
</tr>
<tr>
<td></td>
<td>SSSS</td>
<td>1.1163</td>
<td>0.0240</td>
</tr>
<tr>
<td>$a/h=100$</td>
<td>CCCC</td>
<td>0.2774</td>
<td>0.2000</td>
</tr>
<tr>
<td></td>
<td>CCFF</td>
<td>0.3980</td>
<td>0.6920</td>
</tr>
<tr>
<td></td>
<td>CCSS</td>
<td>0.3473</td>
<td>0.2210</td>
</tr>
<tr>
<td></td>
<td>CFSS</td>
<td>0.7736</td>
<td>0.5000</td>
</tr>
<tr>
<td></td>
<td>SSFF</td>
<td>1.2023</td>
<td>0.4970</td>
</tr>
<tr>
<td></td>
<td>SSSS</td>
<td>0.7318</td>
<td>0.2200</td>
</tr>
</tbody>
</table>
Figure 5.36 Effect of the boundary conditions on the transient behavior of the cross-ply laminates; (a) Thick laminate under thermomechanical load, (b) Thin laminate under thermomechanical load.
5.4.3. Laminated Composite Shells under Thermomechanical Loads

First, thermomechanical effect with different $R_z/a$ value is studied. Nonlinear transient analysis of laminated composite shells under thermomechanical loadings is presented. The critical time 0.1 and 0.5 milliseconds are chosen for the thick ($a/h = 10$) and the thin ($a/h = 100$) shell laminates. The applied mechanical loadings are $10^4$ for thick laminates and $10^3$ for thin laminates. The thermal loading is applied in the same direction as the mechanical loading in laminated composite shell simulation. Figures 5.37 and 5.38 show the temperature effects on the deflection control of cross-ply cylindrical shell with different $R_z/a$ value, 2, 200, respectively. The maximum deflection and the deflection suppression time are tabulated in Table 5.19 for different thermal increase. The deflection suppression characteristics of $R_z/a$ effects are shown in Figure 5.39 for cylindrical and spherical thin shells. It is observed that the maximum deflection value increase with increasing thermal loading $\Delta T$ and takes more deflection suppression time with increasing $R_z/a$ value in both thin and thick shells.

Figure 5.40 shows the effect of the shell types on the deflection suppression characteristics under thermomechanical load. Selected maximum deflection and deflection suppression time for each shell theory are tabulated in Table 5.20. Regardless of shell types, the effect of thermal load in the deflection suppression characteristics leads the increased maximum deflection and deflection suppression time.

The effect of boundary condition on the deflection suppression under thermomechanical loadings is shown in Figure 5.41. SSSS, CCCC, CCSS, CCFF and SSFF boundary conditions are considered with the selected $\Delta T$ and $R_z/a$. It is observed that the SSFF boundary condition gives the maximum deflection.
Figure 5.37 Temperature effect on nonlinear transient behavior of cross-ply cylindrical shell \((R_z/a = 2)\) under thermomechanical load: (a) Thick shell \((a/h = 10)\), (b) Thin shell \((a/h = 100)\)
Figure 5.38 Temperature effect on nonlinear transient behavior of cross-ply cylindrical shell \((R_2/a = 200)\) under thermomechanical load: (a) Thick shell \((a/h = 10)\), (b) Thin shell \((a/h = 100)\)
Table 5.19 Maximum transverse deflection and vibration suppression time for the symmetric cross-ply cylindrical shells

<table>
<thead>
<tr>
<th>Thickness</th>
<th>$\Delta T$</th>
<th>$R_2/a = 2$</th>
<th>$R_2/a = 200$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{w}_{\text{max}}$</td>
<td>$t(\text{sec})$ at $\bar{w}_{\text{max}}/10$</td>
<td>$\bar{w}_{\text{max}}$</td>
</tr>
<tr>
<td>Thin Shell</td>
<td>$a/h = 100$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.090926</td>
<td>0.105500</td>
<td>1.130600</td>
</tr>
<tr>
<td>4</td>
<td>0.110380</td>
<td>0.114500</td>
<td>1.246300</td>
</tr>
<tr>
<td>6</td>
<td>0.130000</td>
<td>0.114500</td>
<td>1.389800</td>
</tr>
<tr>
<td>8</td>
<td>0.151150</td>
<td>0.115000</td>
<td>1.543600</td>
</tr>
<tr>
<td>10</td>
<td>0.171970</td>
<td>0.115000</td>
<td>1.751000</td>
</tr>
<tr>
<td>12</td>
<td>0.192860</td>
<td>0.115500</td>
<td>1.975500</td>
</tr>
<tr>
<td>14</td>
<td>0.213850</td>
<td>0.115500</td>
<td>2.242000</td>
</tr>
<tr>
<td>16</td>
<td>0.234930</td>
<td>0.107000</td>
<td>2.475200</td>
</tr>
<tr>
<td>18</td>
<td>0.256120</td>
<td>0.107000</td>
<td>2.804500</td>
</tr>
<tr>
<td>20</td>
<td>0.277430</td>
<td>0.116500</td>
<td>3.125300</td>
</tr>
<tr>
<td>Thick Shell</td>
<td>$a/h = 10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2.388200</td>
<td>0.024000</td>
<td>1.294900</td>
</tr>
<tr>
<td>20</td>
<td>3.698700</td>
<td>0.024500</td>
<td>1.316600</td>
</tr>
<tr>
<td>30</td>
<td>5.024100</td>
<td>0.028000</td>
<td>1.338300</td>
</tr>
<tr>
<td>40</td>
<td>6.364100</td>
<td>0.028000</td>
<td>1.360100</td>
</tr>
<tr>
<td>50</td>
<td>7.719700</td>
<td>0.028000</td>
<td>1.382000</td>
</tr>
<tr>
<td>60</td>
<td>9.089000</td>
<td>0.024500</td>
<td>1.403800</td>
</tr>
<tr>
<td>70</td>
<td>10.474000</td>
<td>0.024500</td>
<td>1.425700</td>
</tr>
<tr>
<td>80</td>
<td>11.873000</td>
<td>0.024500</td>
<td>1.447400</td>
</tr>
<tr>
<td>90</td>
<td>13.288000</td>
<td>0.028000</td>
<td>1.469200</td>
</tr>
<tr>
<td>100</td>
<td>14.717000</td>
<td>0.024500</td>
<td>1.490900</td>
</tr>
</tbody>
</table>
Figure 5.39 Effect of $R_2/a$ on nonlinear transient behavior of cross-ply shell under thermomechanical load: (a) Cylindrical shell ($a/h=100$, $\Delta T=10^\circ C$), (b) Spherical shell ($a/h=100$, $\Delta T=10^\circ C$)
Table 5.20 Nondimensionalized maximum center displacements and vibration suppression time of cross-ply laminates for different shell types

<table>
<thead>
<tr>
<th>Thickness</th>
<th>Shell Type</th>
<th>$R_z/a$</th>
<th>$\Delta T$</th>
<th>$\bar{w}_{\text{max}}$</th>
<th>$t$(sec) at $\bar{w}_{\text{max}}/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a/h = 10$</td>
<td>Cylindrical</td>
<td>0</td>
<td>1.262900</td>
<td>0.025500</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>4.381400</td>
<td>0.026000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Spherical</td>
<td>0</td>
<td>1.237100</td>
<td>0.025000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>7.373700</td>
<td>0.026000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Doubly curved</td>
<td>0</td>
<td>1.251600</td>
<td>0.025500</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>5.876900</td>
<td>0.026000</td>
<td>0</td>
</tr>
<tr>
<td>$a/h = 100$</td>
<td>Cylindrical</td>
<td>0</td>
<td>0.611730</td>
<td>0.176000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>1.709800</td>
<td>0.248500</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Spherical</td>
<td>0</td>
<td>0.312520</td>
<td>0.164000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>1.586700</td>
<td>0.235500</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Doubly curved</td>
<td>0</td>
<td>0.424780</td>
<td>0.172000</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>1.643300</td>
<td>0.240500</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 5.40 Effect of shell types on nonlinear transient behavior of cross-ply shell under thermomechanical load: (a) Thick shell \((a/h=10, \Delta T=100^\circ C)\), (b) Thin shell \((a/h=100, \Delta T=10^\circ C)\)
Figure 5.41 Effect of boundary conditions on nonlinear transient behavior of cross-ply shell under thermomechanical load: (a) Thick shell \((a/h=10, \Delta T=150^\circ C, R_2/a=100)\), (b) Thin shell \((a/h=100, \Delta T=10^\circ C, R_2/a=20)\)
6. CONCLUSIONS

6.1. Concluding Remarks

Theoretical formulations, analytical solutions for the linear case and finite element analysis results for laminated composite plate and shell structures with smart material laminae are presented in the study. A unified third-order shear deformation formulation that includes the classical plate theory and the first-order shear deformation theory as special cases is used to study vibration/deflection suppression characteristics. The von Kármán type geometric nonlinearity is accounted for in formulations of laminated composite plates. Third-order shear deformation theory based on Donnell and Sanders nonlinear shell kinematics is chosen for the laminated composite shell formulations.

The smart material used in this study to achieve damping of transverse deflection is the Terfenol-D magnetostrictive material, although in principle any other actuating material can be used. The simple control algorithm, a negative velocity feedback control, where the feedback amplitude varies by the negative velocity is used to control the structural system. The constant control gain is assumed in this study.

The exact solution of the linear equations of motion is based on the Navier solution procedure for the simply supported boundary condition. Displacement finite element formulation that considers the geometric nonlinearity and thermal loading through the third-order shear deformation theory is presented. The conforming element which has eight degrees of freedom, \((u_0, v_0, w_0, w_{0,x}, w_{0,y}, w_{0,xy}, \varphi_x, \varphi_y)\), per node is used to develop the finite element model.

Newton-Raphson iteration method which has the symmetric tangent stiffness matrix for all structural problems is used to solve the nonlinear problem. Newmark's time integration method, the constant average acceleration scheme, is selected to determine the transient response.
A number of parametric studies are carried out to understand the damping characteristics of laminated composites with embedded smart material layers. The effect of the different thickness, loading condition, material properties, boundary conditions, \( R_2 / a \) and so on are presented. The observations are summarized below.

From the analytical result, it can be seen that the vibration suppression time decreases very rapidly as mode number increases. This is because the amplitude of vibration that has to be suppressed decreases as the mode number increases.

It is observed that the damping coefficient \(-\alpha_d\) increases with the distance between the smart layer and mid-plane of the structures. This is because as the effect of the moment applied by the actuation of the smart material on the structure is more as the smart material is moved away from the mid-plane section. Thus, the vibration suppression time decreases as the smart material layers are moved from the mid-plane.

The effect of using different values of the feedback coefficient has also been studied. It is observed that for a lower value of the feedback coefficient the time taken to suppress the vibration is longer. This is because as the amount of actuation done by the smart material layer onto the laminated composites becomes less as the feedback value is less.

Use of quarter or full finite element models is validated first. It is found that for antisymmetric cross-ply, angle-ply, and general angle-ply laminates and symmetric cross-ply laminates with simply supported boundary conditions, a quadrant model of the plate and shell structures with proper symmetry boundary conditions may be used to reduce the computational effort.

It is observed that the CLPT theory gives higher frequencies of vibration than shear deformation theories. This is expected, as the CLPT theory renders the plate stiffer compared to the other two theories, FSDT and TSDT.

It is observed that Donnell shell theory gives higher frequency and damping coefficients than Sanders shell theory. As total thickness of shell structures is increasing
and $R_2/a$ value is decreasing, the numerical differences in eigenvalue and deflection become larger. However, when one considers the vibration control behavior it is hard to see the differences.

The effect of $R_2/a$ value in the laminated composite shell structure on the deflection suppression characteristics is studied. It is found that it takes more deflection suppression time with increasing $R_2/a$ value in both thin and thick shells. It is observed that the spherical shell has the largest damping coefficient and frequency and the cylindrical shell has the smallest damping coefficient and frequency from the eigenvalue analysis. It is also observed that the spherical shell has the biggest deflection suppression and smallest maximum deflection and the cylindrical shell has the smallest deflection suppression and largest maximum deflection. It is because of the fact that spherical shell has the smallest $R_i$ and doubly curved shell (in this study $R_i = 2R_2$), and cylindrical shell has the largest $R_i$ ($R_i = \infty$).

The behavior of laminated composites with embedded magnetostrictive layers for different kinds of structural material has been studied. CFRP, Gr-Ep (AS), Gl-Ep and Br-Ep are the elastic materials used in this study. It is found that the suppression characteristics are similar for elastic materials having similar $E_1/E_2$ ratios. It is also observed that Gl-Ep shows the minimum vibration suppression time.

The nonlinear thermal effects are also investigated for various simulation conditions. The amplitude and period of the center deflections decrease with increasing thermal loading for all cases.

### 6.2. Recommendations for Future Work

The result of this study is a general nonlinear formulation and analytical and finite element solutions for the laminated composite plate and shell structures with linear constitutive relations of smart material.
As a possible continuation of this work is development of nonlinear constitutive relations for the magnetostrictive material followed by computer implementation of nonlinear finite element analysis of laminated composite plate/shell structures including the nonlinear constitutive relations of smart materials. Also, studies relating to the placement of smart material patches, instead of full layers, thereby achieving discreet actuation could be performed.
REFERENCES


Institute of Electrical and Electronics Engineers (IEEE), Standard 319 (1976), *IEEE Standard on Magnetostrictive Materials; Piezo-magnetic Nomenclature*, New York.


Euler-Lagrange Equations of TSDT Sanders Shell Theory in Terms of Displacements

A.1 Equation of motion by TSDT Sanders shell theory for $\delta u$

$$A_{11} \left( \frac{\partial^2 u_0}{\partial X_1^2} + \frac{1}{R_i} \frac{\partial w_0}{\partial X_1} + \frac{\partial w_0}{\partial X_1} \frac{\partial^2 w_0}{\partial X_1^2} \right) + A_{12} \left( \frac{\partial^2 v_0}{\partial X_1 \partial X_2} + \frac{1}{R_i} \frac{\partial w_0}{\partial X_1} + \frac{\partial w_0}{\partial X_1} \frac{\partial^2 w_0}{\partial X_1 \partial X_2} \right) + A_{16} \left( \frac{\partial^2 v_0}{\partial X_1^2} + \frac{\partial^2 u_0}{\partial X_1^2} \right)$$

$$\frac{\partial^2 w_0}{\partial X_1^2} + \frac{\partial^2 w_0}{\partial X_1^2} + B_{11} \frac{\partial^2 \phi}{\partial X_1^2} + B_{12} \frac{\partial^2 \phi}{\partial X_1 \partial X_2} + B_{16} \left( \frac{\partial^2 \phi}{\partial X_1^2} + \frac{\partial^2 \phi}{\partial X_1 \partial X_2} \right) - c_{1} E_{11} \left( \frac{\partial^2 \phi}{\partial X_1^2} + \frac{\partial^2 \phi}{\partial X_1^2} \right)$$

$$-\frac{1}{R_i} \frac{\partial^2 u_0}{\partial X_1 \partial X_2} - c_{1} E_{12} \left( \frac{\partial^2 \phi}{\partial X_1 \partial X_2} + \frac{\partial^2 \phi}{\partial X_1 \partial X_2} \right) - c_{1} E_{16} \left( \frac{\partial^2 \phi}{\partial X_1^2} + \frac{\partial^2 \phi}{\partial X_1 \partial X_2} \right)$$

$$A_{66} \left( \frac{\partial^2 v_0}{\partial X_2 \partial X_1} + \frac{\partial^2 w_0}{\partial X_2 \partial X_1} + \frac{\partial^2 w_0}{\partial X_2 \partial X_1} + \frac{\partial^2 w_0}{\partial X_2 \partial X_1} + B_{16} \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + B_{26} \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + B_{66} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi}{\partial X_2 \partial X_1} \right)$$

$$c_{1} E_{16} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi}{\partial X_2 \partial X_1} - \frac{1}{R_i} \frac{\partial^2 u_0}{\partial X_2 \partial X_1} - c_{1} E_{26} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi}{\partial X_2 \partial X_1} - \frac{1}{R_i} \frac{\partial^2 v_0}{\partial X_2 \partial X_1} \right)$$

$$c_{1} E_{66} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi}{\partial X_2 \partial X_1} \right)$$

$$\frac{1}{R_i} \frac{\partial^2 u_0}{\partial X_2 \partial X_2} - c_{1} H_{12} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_2} + \frac{\partial^2 \phi}{\partial X_2 \partial X_2} - \frac{1}{R_i} \frac{\partial^2 v_0}{\partial X_2 \partial X_2} \right) - c_{1} H_{16} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_2} + \frac{\partial^2 \phi}{\partial X_2 \partial X_2} \right)$$

$$c_{1} H_{26} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi}{\partial X_2 \partial X_1} \right)$$

$$c_{1} H_{66} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + 2 \right)$$

$$\left( \frac{\partial^2 w_0}{\partial X_1 \partial X_2} + \frac{\partial^2 w_0}{\partial X_1 \partial X_2} - \frac{1}{R_i} \frac{\partial^2 u_0}{\partial X_1 \partial X_2} \right) - c_{1} H_{26} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi}{\partial X_2 \partial X_1} \right) - c_{1} H_{66} \left( \frac{\partial^2 \phi}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi}{\partial X_2 \partial X_1} \right)$$

$$\left( \frac{\partial^3 w_0}{\partial X_1 \partial X_2} + \frac{\partial^3 w_0}{\partial X_1 \partial X_2} - \frac{1}{R_i} \frac{\partial^3 u_0}{\partial X_1 \partial X_2} \right)$$

$$\left( \frac{\partial^3 w_0}{\partial X_1 \partial X_2} + \frac{\partial^3 w_0}{\partial X_1 \partial X_2} - \frac{1}{R_i} \frac{\partial^3 u_0}{\partial X_1 \partial X_2} \right)$$

$$= 0$$
A.2 Equation of motion by TSDT Sanders shell theory for $\delta v$

$$A_{16} \left( \frac{\partial^2 u_0}{\partial x_1^2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1^2} \right) + A_{26} \left( \frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{1}{R_2} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) + A_{66} \left( \frac{\partial^2 v_0}{\partial x_1^2} + \frac{\partial^2 w_0}{\partial x_1^2} \right)$$

$$+ \frac{\partial^2 w_0}{\partial x_1^2} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + B_{16} \frac{\partial^2 \phi}{\partial x_1^2} + B_{26} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + B_{66} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \right) - c_i E_{16} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 w_0}{\partial x_1^2} \right) - \frac{1}{R_1} \frac{\partial^2 u_0}{\partial x_1^2} - c_i E_{26} \left( \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} - \frac{1}{R_2} \frac{\partial^2 v_0}{\partial x_1 \partial x_2} \right) - c_i E_{66} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 w_0}{\partial x_1^2} \right) + \frac{2}{R_2} \frac{\partial^2 v_0}{\partial x_1 \partial x_2} - 1 \frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{1}{R_2} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} - \frac{1}{R_1} \frac{\partial^2 u_0}{\partial x_1^2} + A_{12} \left( \frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) + A_{22} \frac{\partial^2 \phi}{\partial x_1^2} + B_{12} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + B_{22} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \frac{c_i E_{12}}{R_1} \left( \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) - c_i E_{22} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 w_0}{\partial x_1^2} \right) - \frac{c_i E_{26}}{R_2} \left( \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) + 2 \frac{\partial^2 w_0}{\partial x_1 \partial x_2} - 1 \frac{\partial^2 v_0}{\partial x_1 \partial x_2} - 1 \frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{1}{R_2} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} - \frac{1}{R_1} \frac{\partial^2 u_0}{\partial x_1^2} + A_{44} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 w_0}{\partial x_1^2} \right) + A_{45} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 w_0}{\partial x_1^2} \right) + A_{12} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 w_0}{\partial x_1^2} \right) + A_{22} \frac{\partial^2 \phi}{\partial x_1^2} + B_{12} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + B_{22} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \frac{c_i H_{12}}{R_2} \left( \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) - c_i H_{22} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 w_0}{\partial x_1^2} \right) - c_i H_{26} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 w_0}{\partial x_1^2} \right) + 2 \frac{\partial^2 w_0}{\partial x_1 \partial x_2} - 1 \frac{\partial^2 v_0}{\partial x_1 \partial x_2} - 1 \frac{\partial^2 u_0}{\partial x_1 \partial x_2} + E_{16} \left( \frac{\partial^2 u_0}{\partial x_1^2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) + E_{26} \left( \frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{1}{R_2} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) + E_{66} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \right) + F_{16} \frac{\partial^2 \phi}{\partial x_1^2} + F_{26} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + F_{66} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \right) + c_i H_{16} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_1 \partial x_2} - \frac{c_i H_{26}}{R_2} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \right) - c_i H_{66} \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \right) + 2 \frac{\partial^2 w_0}{\partial x_1 \partial x_2} - 1 \frac{\partial^2 v_0}{\partial x_1 \partial x_2} - 1 \frac{\partial^2 u_0}{\partial x_1 \partial x_2} \right) \right\} - \left( \frac{J_{32} M + c_i E_{32}}{R_2} \right) \frac{\partial^2 w_0}{\partial x_1 \partial x_2} - \frac{1}{\partial x_2 \partial t^2} J_{32} \frac{\partial^2 v_0}{\partial t^2} - \frac{1}{\partial x_2 \partial t^2} J_{32} \frac{\partial^2 \phi}{\partial t^2} + c_i T_{12} \frac{\partial^2 \phi}{\partial x_1 \partial x_2} = 0
A.3 Equation of motion by TSDT sanders shell theory for $\partial w$

\[
\hat{A}_{45} \left( \frac{\partial \phi}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_2 \partial x_2} - \frac{1}{R_1} \frac{\partial v_0}{\partial x_2} \right) + \hat{A}_{43} \left( \frac{\partial \phi}{\partial x_1} + \frac{\partial^2 w_0}{\partial x_1 \partial x_1} - \frac{1}{R_1} \frac{\partial u_0}{\partial x_1} \right) + \hat{A}_{44} \left( \frac{\partial \phi}{\partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_2} - \frac{1}{R_2} \frac{\partial v_0}{\partial x_2} \right) + \hat{A}_{42} \left( \frac{\partial \phi}{\partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_2} - \frac{1}{R_1} \frac{\partial u_0}{\partial x_2} \right)
\]

\[
\hat{A}_{45} \left( \frac{\partial \phi}{\partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_1} - \frac{1}{R_1} \frac{\partial u_0}{\partial x_2} \right) + c_{1} \left[ E_{11} \left( \frac{\partial^3 u_0}{\partial x_1^3} + \frac{1}{R_1} \frac{\partial v_0}{\partial x_1} \right) + E_{12} \left( \frac{\partial^3 u_0}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u_0}{\partial x_2 \partial x_1^2} \right) + E_{16} \left( \frac{\partial^3 v_0}{\partial x_1^3} + \frac{\partial^3 v_0}{\partial x_1^2 \partial x_2} + \frac{\partial^3 v_0}{\partial x_2 \partial x_1^2} \right) \right] + \hat{A}_{42} \left( \frac{\partial \phi}{\partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_2} - \frac{1}{R_1} \frac{\partial u_0}{\partial x_2} \right)
\]

\[
= \frac{1}{R_2} \frac{\partial^2 w_0}{\partial x_2 \partial x_2} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_1} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_2} + \hat{A}_{12} \left( \frac{\partial \phi}{\partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_1} - \frac{1}{R_1} \frac{\partial u_0}{\partial x_2} \right)
\]

\[
\hat{A}_{43} \left( \frac{\partial^3 v_0}{\partial x_1^3} + \frac{\partial^3 v_0}{\partial x_1^2 \partial x_2} + \frac{\partial^3 v_0}{\partial x_2 \partial x_1^2} \right)
\]

\[
+ \frac{\partial^2 w_0}{\partial x_2 \partial x_2} + \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_1} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_2} + \hat{A}_{12} \left( \frac{\partial \phi}{\partial x_2} + \frac{\partial^2 w_0}{\partial x_2 \partial x_1} - \frac{1}{R_1} \frac{\partial u_0}{\partial x_2} \right)
\]
\[
\begin{align*}
&\left(\frac{\partial \phi_1}{\partial X_1} + \frac{\partial \phi_1}{\partial X_2}\right) - c_1 E_{11} \left(\frac{\partial \phi_1}{\partial X_1} + \frac{\partial^2 w_0}{\partial X_1^2} - \frac{1}{R_1} \frac{\partial u}{\partial X_1} - c_1 E_{12} \left(\frac{\partial \phi_2}{\partial X_2} + \frac{\partial^2 w_0}{\partial X_2^2} - \frac{1}{R_2} \frac{\partial v}{\partial X_2}\right) - c_1 E_{16} \left(\frac{\partial \phi_2}{\partial X_1} + \frac{\partial \phi_1}{\partial X_2}\right) + 2 \frac{\partial^2 w_0}{\partial X_1 \partial X_2} - \frac{1}{R_2} \frac{\partial v}{\partial X_1} - \frac{1}{R_1} \frac{\partial u}{\partial X_2}\right) + \frac{\partial \phi_1}{\partial X_2}
\end{align*}
\]
\[ I_0 \frac{\partial^2 \omega_0}{\partial t^2} + c_1 I_0 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 \omega_0}{\partial x_1^2} + \frac{\partial^2 \omega_0}{\partial x_2^2} \right) - c_i \left[ T_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial u_0}{\partial x_1} \right) + T_3 \frac{\partial^2}{\partial t^2} \left( \frac{\partial v_0}{\partial x_2} \right) + J_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} \right) \right] = 0 \]

A.4 Equation of motion by TSDT sanders shell theory for \( \delta \phi \)

\[
\begin{aligned}
\hat{B}_{11} \left( \frac{\partial^2 u_0}{\partial x_1^2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} \frac{\partial^2 w_0}{\partial x_1^2} \right) + \hat{B}_{12} \left( \frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{1}{R_2} \frac{\partial w_0}{\partial x_2} + \frac{\partial w_0}{\partial x_2} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) + \hat{B}_{16} \left( \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) + \\
\frac{\partial^2 w_0}{\partial x_1^2} + \frac{\partial w_0}{\partial x_1} \frac{\partial^2 w_0}{\partial x_1^2} + \hat{D}_{12} \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} + \hat{D}_{16} \left( \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) - c_i \hat{F}_{11} \left( \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) - \\
\frac{1}{R_1} \frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \hat{B}_{16} \left( \frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{1}{R_2} \frac{\partial w_0}{\partial x_2} + \frac{\partial w_0}{\partial x_2} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) + \hat{B}_{26} \left( \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) + \\
\left( \frac{\partial^2 v_0}{\partial x_1^2} + \frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{\partial w_0}{\partial x_1} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \hat{D}_{16} \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} + \hat{D}_{26} \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \hat{D}_{66} \left( \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) - \\
c_i \hat{F}_{16} \left( \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) - c_i \hat{F}_{26} \left( \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) - \frac{2 \partial^2 w_0}{\partial x_1 \partial x_2} \right) - A_{45} \left( \phi_1 + \frac{\partial w_0}{\partial x_2} \right) - A_{55} \left( \phi_1 + \frac{\partial w_0}{\partial x_1} \right) + (B_{31} - c_i \hat{F}_{31}) \left( \frac{\partial^2 u_0}{\partial x_1 \partial t} - J_1 \frac{\partial^2 u_0}{\partial t^2} - R_1 \frac{\partial^2 \phi}{\partial t^2} + c_i J_4 \frac{\partial^2}{\partial t^2} \left( \frac{\partial \phi_1}{\partial x_1} \right) \right) = 0
\end{aligned}
\]

A.5 Equation of motion by TSDT sanders shell theory for \( \delta \phi_2 \)

\[
\begin{aligned}
\hat{B}_{16} \left( \frac{\partial^2 u_0}{\partial x_1^2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} \frac{\partial^2 w_0}{\partial x_1^2} \right) + \hat{B}_{26} \left( \frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{1}{R_2} \frac{\partial w_0}{\partial x_2} + \frac{\partial w_0}{\partial x_2} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) + \hat{B}_{66} \left( \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) + \\
\frac{\partial^2 w_0}{\partial x_1^2} + \frac{\partial w_0}{\partial x_1} \frac{\partial^2 w_0}{\partial x_1^2} + \hat{D}_{16} \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} + \hat{D}_{26} \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \hat{D}_{66} \left( \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) - c_i \hat{F}_{16} \left( \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) - \\
\frac{1}{R_1} \frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{1}{R_1} \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_1} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \hat{B}_{26} \left( \frac{\partial^2 v_0}{\partial x_1 \partial x_2} + \frac{1}{R_2} \frac{\partial w_0}{\partial x_2} + \frac{\partial w_0}{\partial x_2} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} \right) + \hat{B}_{26} \left( \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) + \\
\left( \frac{\partial^2 v_0}{\partial x_1^2} + \frac{\partial^2 u_0}{\partial x_1 \partial x_2} + \frac{\partial w_0}{\partial x_1} \frac{\partial^2 w_0}{\partial x_1 \partial x_2} + \hat{D}_{16} \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} + \hat{D}_{26} \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \hat{D}_{66} \left( \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) - \\
c_i \hat{F}_{26} \left( \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) - \frac{2 \partial^2 w_0}{\partial x_1 \partial x_2} \right) - A_{45} \left( \phi_1 + \frac{\partial w_0}{\partial x_2} \right) - A_{55} \left( \phi_1 + \frac{\partial w_0}{\partial x_1} \right) + (B_{31} - c_i \hat{F}_{31}) \left( \frac{\partial^2 u_0}{\partial x_1 \partial t} - J_1 \frac{\partial^2 u_0}{\partial t^2} - R_1 \frac{\partial^2 \phi}{\partial t^2} + c_i J_4 \frac{\partial^2}{\partial t^2} \left( \frac{\partial \phi_2}{\partial x_1} \right) \right) = 0
\end{aligned}
\]
\[
\begin{align*}
&\left(\frac{\partial^2 v_0}{\partial X_1 \partial X_2} + \frac{\partial^2 u_0}{\partial X_2^2} + \frac{\partial^2 w_0}{\partial X_1 \partial X_2} + \frac{\partial^2 w_0}{\partial X_1 \partial X_2^2}\right) + \dot{D}_{12} \frac{\partial^2 \phi_1}{\partial X_1 \partial X_2} + \dot{D}_{22} \frac{\partial^2 \phi_2}{\partial X_2^2} + \dot{D}_{26} \frac{\partial^2 \phi_2}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi_1}{\partial X_2^2} + c_1 \\
&\hat{F}_{12} \left(\frac{\partial^2 \phi_1}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi_0}{\partial X_2 \partial X_1^2} \right) - c_1 \hat{F}_{22} \left(\frac{\partial^2 \phi_2}{\partial X_2^2} + \frac{\partial^2 \phi_0}{\partial X_2 \partial X_1} \right) - c_1 \hat{F}_{26} \left(\frac{\partial^2 \phi_2}{\partial X_2 \partial X_1} + \frac{\partial^2 \phi_1}{\partial X_2^2} \right) \\
&\frac{3}{2} \frac{\partial^2 w_0}{\partial X_1 \partial X_2^2} - \frac{1}{R_2} \frac{\partial^2 v_0}{\partial X_2 \partial X_1} - \frac{1}{R_2} \frac{\partial^2 u_0}{\partial X_2^2} - \hat{A}_{44} \left(\phi_2 + \frac{\partial w_0}{\partial X_2} - \frac{v_0}{R_2}\right) - \hat{A}_{45} \left(\phi_1 + \frac{\partial w_0}{\partial X_1} - \frac{u_0}{R_1}\right) + (\bar{B}_{12} - c_1 \bar{E}_{12}) \\
&\frac{\partial^2 w_0}{\partial X_2 \partial t} - \bar{J} \left(\frac{\partial^2 v_0}{\partial t^2} - \bar{K}_2 \frac{\partial^2 \phi_2}{\partial t^2} \right) + c_1 J_4 \frac{\partial^2 \left(\frac{\partial w_0}{\partial X_2}\right)}{\partial t^2} = 0
\end{align*}
\]
APPENDIX B

FINITE ELEMENT COEFFICIENTS BY TSĐT DONNELL (DMV) SHELL THEORY

B.1 Linear coefficients of laminated composite shells by Donnell shell theory

\[ K_y^{11} = \int_\alpha \left[ A_{11} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + A_{16} \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + A_{16} \left( \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} \right) \right] dX_1 dX_2 \]

\[ K_y^{12} = \int_\alpha \left[ A_{12} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + A_{16} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + A_{26} \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_1} + A_{16} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} \right] dX_1 dX_2 \]

\[ K_y^{13} = \int_\alpha \left[ \left( \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left( \frac{A_{16}}{R_1} + \frac{A_{26}}{R_2} \right) \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \left( -c_1 \right) \left[ \frac{E_{11}}{\partial X_1} \frac{\partial^2 \phi_j}{\partial X_1^2} \right] \right] dX_1 dX_2 \]

\[ K_y^{14} = \int_\alpha \left[ \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} \right] \right] dX_1 dX_2 \]

\[ K_y^{15} = \int_\alpha \left[ \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} \right] dX_1 dX_2 \]

\[ K_y^{16} = \int_\alpha \left[ \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} \right] dX_1 dX_2 \]

\[ K_y^{17} = \int_\alpha \left[ \left( \frac{A_{22}}{R_1} + \frac{A_{12}}{R_2} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left( \frac{A_{16}}{R_1} + \frac{A_{26}}{R_2} \right) \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \left( -c_1 \right) \left[ \frac{E_{11}}{\partial X_1} \frac{\partial^2 \phi_j}{\partial X_1^2} \right] \right] dX_1 dX_2 \]

\[ K_y^{18} = \int_\alpha \left[ \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} \right] dX_1 dX_2 \]
\[ K_{y}^{33} = \int_{\Omega} \left[ \left( \frac{1}{R_{1}} A_{11} + \frac{2}{R_{1} R_{2}} A_{12} + \frac{1}{R_{2}} A_{22} \right) \frac{\partial^{2} \phi_{j}}{\partial x_{1} \partial x_{2}} \phi_{j} + \left( \frac{1}{R_{1}} E_{11} + \frac{2}{R_{1} R_{2}} E_{12} + \frac{1}{R_{2}} E_{22} \right) \frac{\partial^{2} \phi_{j}}{\partial x_{2}^{2}} \phi_{j} \right] \, dX_{1} \, dX_{2} \]

\[ K_{y}^{34} = \int_{\Omega} \left[ \left( \frac{1}{R_{1}} B_{11} + \frac{2}{R_{1} R_{2}} B_{12} + \frac{1}{R_{2}} B_{22} \right) \frac{\partial^{2} \phi_{j}}{\partial x_{1} \partial x_{2}} \phi_{j} + \left( \frac{1}{R_{1}} \hat{F}_{11} + \frac{2}{R_{1} R_{2}} \hat{F}_{12} + \frac{1}{R_{2}} \hat{F}_{22} \right) \frac{\partial^{2} \phi_{j}}{\partial x_{2}^{2}} \phi_{j} \right] \, dX_{1} \, dX_{2} \]

\[ K_{y}^{35} = \int_{\Omega} \left[ \left( \frac{1}{R_{1}} B_{11} + \frac{2}{R_{1} R_{2}} B_{12} + \frac{1}{R_{2}} B_{22} \right) \frac{\partial^{2} \phi_{j}}{\partial x_{1} \partial x_{2}} \phi_{j} + \left( \frac{1}{R_{1}} \hat{F}_{11} + \frac{2}{R_{1} R_{2}} \hat{F}_{12} + \frac{1}{R_{2}} \hat{F}_{22} \right) \frac{\partial^{2} \phi_{j}}{\partial x_{2}^{2}} \phi_{j} \right] \, dX_{1} \, dX_{2} \]
\[ K_{y}^{44} = \int_{\mathcal{T}} \left[ \hat{A}_{55,\psi_j,\psi_j} + \left( \hat{D}_{51} - c_{i,\hat{F}_{51}} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left( \hat{D}_{60} - c_{i,\hat{F}_{60}} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + \left( \hat{D}_{60} - c_{i,\hat{F}_{60}} \right) \frac{\partial \psi_j}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \left( \hat{D}_{66} - c_{i,\hat{F}_{66}} \right) \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} \right] dX_1 dX_2 \]

\[ K_{y}^{44} = \int_{\mathcal{T}} \left[ \hat{A}_{55,\psi_j,\psi_j} + \left( \hat{D}_{51} - c_{i,\hat{F}_{51}} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left( \hat{D}_{60} - c_{i,\hat{F}_{60}} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + \left( \hat{D}_{60} - c_{i,\hat{F}_{60}} \right) \frac{\partial \psi_j}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \left( \hat{D}_{66} - c_{i,\hat{F}_{66}} \right) \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} \right] dX_1 dX_2 \]

\[ K_{y}^{51} = \int_{\mathcal{T}} \left[ \hat{B}_{16} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \hat{B}_{26} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + \hat{B}_{26} \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_1} + \hat{B}_{26} \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} \right] dX_1 dX_2 \]

\[ K_{y}^{52} = \int_{\mathcal{T}} \left[ \hat{B}_{22} \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \hat{B}_{26} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + \hat{B}_{26} \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_1} \right] dX_1 dX_2 \]

\[ K_{y}^{53} = \int_{\mathcal{T}} \left[ \hat{B}_{22} \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \hat{B}_{26} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + \hat{B}_{26} \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_1} \right] dX_1 dX_2 \]

\[ K_{y}^{54} = \int_{\mathcal{T}} \left[ \hat{A}_{55,\psi_j,\psi_j} + \left( \hat{D}_{51} - c_{i,\hat{F}_{51}} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left( \hat{D}_{60} - c_{i,\hat{F}_{60}} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + \left( \hat{D}_{60} - c_{i,\hat{F}_{60}} \right) \frac{\partial \psi_j}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \left( \hat{D}_{66} - c_{i,\hat{F}_{66}} \right) \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} \right] dX_1 dX_2 \]

\[ K_{y}^{55} = \int_{\mathcal{T}} \left[ \hat{A}_{55,\psi_j,\psi_j} + \left( \hat{D}_{51} - c_{i,\hat{F}_{51}} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left( \hat{D}_{60} - c_{i,\hat{F}_{60}} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + \left( \hat{D}_{60} - c_{i,\hat{F}_{60}} \right) \frac{\partial \psi_j}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \left( \hat{D}_{66} - c_{i,\hat{F}_{66}} \right) \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} \right] dX_1 dX_2 \]

\[ M_{y}^{11} = \int_{\mathcal{T}} I_{0,\psi_j,\psi_j} dX_1 dX_2 \]

\[ M_{y}^{22} = \int_{\mathcal{T}} I_{1,\psi_j,\psi_j} dX_1 dX_2 \]

\[ M_{y}^{31} = M_{y}^{11} = -c_{1} \int_{\mathcal{T}} I_{3,\psi_j,\psi_j} dX_1 dX_2 \]
\[ M_{ij}^{32} = M_{ij}^{53} = -c_i \int_{\Gamma} \frac{\partial \varphi_i}{\partial X_2} \psi_j dX_1 dX_2 \]

\[ M_{ij}^{33} = \int_{\Gamma} \left[ I_0 \varphi_i \varphi_j + c_i^2 I_0 \left( \frac{\partial \varphi_i}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} + \frac{\partial \varphi_i}{\partial X_2} \frac{\partial \varphi_j}{\partial X_2} \right) \right] dX_1 dX_2 \]

\[ M_{ij}^{41} = M_{ij}^{34} = \int_{\Gamma} J_i \psi_j dX_1 dX_2 \]

\[ M_{ij}^{42} = M_{ij}^{35} = -c_i J_i \int_{\Gamma} \frac{\partial \varphi_i}{\partial X_1} dX_1 dX_2 \]

\[ M_{ij}^{52} = M_{ij}^{25} = \int_{\Gamma} J_i \psi_j dX_1 dX_2 \]

\[ M_{ij}^{53} = M_{ij}^{26} = -c_i J_i \int_{\Gamma} \frac{\partial \varphi_i}{\partial X_2} dX_1 dX_2 \]

\[ M_{ij}^{44} = \int_{\Gamma} K_i \psi_j dX_1 dX_2 \]

\[ M_{ij}^{55} = \int_{\Gamma} K_i \psi_j dX_1 dX_2 \]

\[ M_{ij}^{12} = M_{ij}^{21} = M_{ij}^{42} = M_{ij}^{52} = M_{ij}^{21} = M_{ij}^{35} = M_{ij} = 0 \]

\[ C_{ij}^{33} = \int_{\Gamma} A_{ij}^M \frac{\partial \psi_j}{\partial X_2} \varphi_i dX_1 dX_2 \]

\[ C_{ij}^{23} = \int_{\Gamma} A_{ij}^M \frac{\partial \psi_j}{\partial X_2} \varphi_i dX_1 dX_2 \]

\[ C_{ij}^{33} = -c_i \int_{\Gamma} \left( E_{ij}^M \frac{\partial^2 \varphi_i}{\partial X_1^2} + E_{ij}^M \frac{\partial^2 \varphi_i}{\partial X_2^2} \right) \varphi_j dX_1 dX_2 + \int_{\Gamma} \left( \frac{A_{ij}^M}{R_i} + \frac{A_{ij}^M}{R_j} \right) \psi_j \varphi_i dX_1 dX_2 \]

\[ C_{ij}^{33} = \int_{\Gamma} \left( B_{ij}^M - c_i \tilde{E}_{ij}^M \right) \frac{\partial \psi_j}{\partial X_2} \varphi_i dX_1 dX_2 \]

\[ C_{ij}^{33} = \int_{\Gamma} \left( B_{ij}^M - c_i \tilde{E}_{ij}^M \right) \frac{\partial \psi_j}{\partial X_2} \varphi_i dX_1 dX_2 \]

\[ C_{ij}^{11} = C_{ij}^{22} = C_{ij}^{33} = C_{ij}^{44} = C_{ij}^{55} = C_{ij}^{21} = C_{ij}^{22} = C_{ij}^{25} = C_{ij}^{31} = C_{ij}^{32} = C_{ij}^{34} = C_{ij}^{35} = C_{ij}^{42} = C_{ij}^{44} = C_{ij}^{45} = C_{ij}^{52} = C_{ij}^{54} = C_{ij}^{55} = 0 \]

\[ F_i^1 = \oint_{\Gamma_i} \left( N_i n_i + N_i n_i \right) \psi_i ds \]

\[ F_i^2 = \oint_{\Gamma_i} \left( N_i n_i + N_i n_i \right) \psi_i ds \]

\[ F_i^3 = \int_{\Gamma} g \varphi_i dX_1 dX_2 + \int_{\Gamma} \left( \tilde{V}_i \psi_i + p_{ij} \frac{\partial \varphi_i}{\partial n} \right) ds \]

\[ F_i^4 = \oint_{\Gamma_i} \left( M_i n_i + M_i n_i \right) \psi_i ds \]

\[ F_i^5 = \oint_{\Gamma_i} \left( M_i n_i + M_i n_i \right) \psi_i ds \]
\[ \begin{align*}
F_{i}^{T1} &= \int_{\Omega} \left( \frac{\partial \psi_{i}}{\partial X_{1}} N_{i} + \frac{\partial \psi_{i}}{\partial X_{2}} N_{2} \right) dX_{1} dX_{2} \\
F_{i}^{T2} &= \int_{\Omega} \left( \frac{\partial \psi_{i}}{\partial X_{1}} N_{i} + \frac{\partial \psi_{i}}{\partial X_{2}} N_{2} \right) dX_{1} dX_{2} \\
F_{i}^{T3} &= -c_{i} \int_{\Omega} \left( \frac{\partial^{2} \varphi_{i}}{\partial X_{1}^{2}} P_{1} + 2 \frac{\partial^{2} \varphi_{i}}{\partial X_{1}^{2} \partial X_{2}} P_{6} + \frac{\partial^{2} \varphi_{i}}{\partial X_{2}^{2}} P_{2} \right) dX_{1} dX_{2} + \int_{\Omega} \left( \frac{N_{i}}{R_{1}} + \frac{N_{2}}{R_{2}} \right) \varphi_{i} dX_{1} dX_{2} \\
F_{i}^{T4} &= \int_{\Omega} \left[ \frac{\partial \psi_{i}}{\partial X_{1}} \left( M_{i}^{T} - c_{i} P_{1} \right) + \frac{\partial \psi_{i}}{\partial X_{2}} \left( M_{i}^{T} - c_{i} P_{2} \right) \right] dX_{1} dX_{2} \\
F_{i}^{T5} &= \int_{\Omega} \left[ \frac{\partial \psi_{i}}{\partial X_{1}} \left( M_{i}^{T} - c_{i} P_{1} \right) + \frac{\partial \psi_{i}}{\partial X_{2}} \left( M_{i}^{T} - c_{i} P_{2} \right) \right] dX_{1} dX_{2}
\end{align*} \]

B.2 Additional nonlinear coefficients of laminated composite shells by Donnell shell theory

\[
(K_{i}^{\nu})_{NL} = \left( \frac{1}{2} \right) \int_{\Omega} \left[ \frac{\partial \psi_{i}}{\partial X_{1}} \left( A_{i1} \frac{\partial \varphi_{i}}{\partial X_{1}} + A_{i2} \frac{\partial \varphi_{i}}{\partial X_{2}} + A_{i6} \frac{\partial \varphi_{i}}{\partial X_{1} \partial X_{2}} + A_{i6} \frac{\partial \varphi_{i}}{\partial X_{2} \partial X_{1}} \right) \\
\quad + A_{i6} \frac{\partial \varphi_{i}}{\partial X_{1} \partial X_{2}} \left( A_{i1} \frac{\partial \varphi_{i}}{\partial X_{1} \partial X_{2}} + A_{i2} \frac{\partial \varphi_{i}}{\partial X_{2} \partial X_{1}} + A_{i6} \frac{\partial \varphi_{i}}{\partial X_{1} \partial X_{2}} + A_{i6} \frac{\partial \varphi_{i}}{\partial X_{2} \partial X_{1}} \right) \right] dX_{1} dX_{2}
\]

\[
(K_{y}^{\nu})_{NL} = \left( \frac{1}{2} \right) \int_{\Omega} \left[ \frac{\partial \psi_{y}}{\partial X_{1}} \left( A_{y1} \frac{\partial \varphi_{y}}{\partial X_{1}} + A_{y2} \frac{\partial \varphi_{y}}{\partial X_{2}} + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{1} \partial X_{2}} + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{2} \partial X_{1}} \right) \\
\quad + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{1} \partial X_{2}} \left( A_{y1} \frac{\partial \varphi_{y}}{\partial X_{1} \partial X_{2}} + A_{y2} \frac{\partial \varphi_{y}}{\partial X_{2} \partial X_{1}} + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{1} \partial X_{2}} + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{2} \partial X_{1}} \right) \right] dX_{1} dX_{2}
\]

\[
(K_{z}^{\nu})_{NL} = \int_{\Omega} \left[ \frac{\partial \varphi}{\partial X_{1}} \left( A_{z1} \frac{\partial \varphi_{z}}{\partial X_{1}} + A_{z6} \frac{\partial \varphi_{z}}{\partial X_{1} \partial X_{2}} + A_{z6} \frac{\partial \varphi_{z}}{\partial X_{2} \partial X_{1}} \right) + A_{z6} \frac{\partial \varphi_{z}}{\partial X_{1} \partial X_{2}} \left( A_{z1} \frac{\partial \varphi_{z}}{\partial X_{1} \partial X_{2}} + A_{z6} \frac{\partial \varphi_{z}}{\partial X_{2} \partial X_{1}} + A_{z6} \frac{\partial \varphi_{z}}{\partial X_{1} \partial X_{2}} + A_{z6} \frac{\partial \varphi_{z}}{\partial X_{2} \partial X_{1}} \right) \right] dX_{1} dX_{2}
\]

\[
(K_{y}^{\nu})_{NL} = \int_{\Omega} \left[ \frac{\partial \varphi}{\partial X_{1}} \left( A_{y1} \frac{\partial \varphi_{y}}{\partial X_{1}} + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{1} \partial X_{2}} + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{2} \partial X_{1}} \right) + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{1} \partial X_{2}} \left( A_{y1} \frac{\partial \varphi_{y}}{\partial X_{1} \partial X_{2}} + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{2} \partial X_{1}} + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{1} \partial X_{2}} + A_{y6} \frac{\partial \varphi_{y}}{\partial X_{2} \partial X_{1}} \right) \right] dX_{1} dX_{2}
\]
\begin{align}
(K_{ij}^{33})_{NL} &= \frac{1}{2} \int \left[ (-c_t) \left\{ \frac{\partial^3 \phi_i}{\partial x^3} \left[ \frac{\partial \phi_j}{\partial x_1} + E_{11} \frac{\partial \phi_j}{\partial x_2} + E_{16} \frac{\partial \phi_j}{\partial x_2} \right] + \frac{\partial \phi_i}{\partial x_1} \left( E_{11} \frac{\partial \phi_j}{\partial x_2} + E_{16} \frac{\partial \phi_j}{\partial x_2} \right) \right\} + 2 \frac{\partial^2 \phi_i}{\partial x_1 \partial x_2} \right]
+ \frac{\partial \phi_i}{\partial x_1} \left( \frac{\partial \phi_j}{\partial x_2} \right) \bigg] dX_1 dX_2
\end{align}
B.3 Tangent stiffness coefficients of laminated composite shells by Donnell shell theory

\[
(K_{ij}^{43})_{NL} = \left(\frac{1}{2}\right) \int_{\Gamma} \left[ \frac{\partial \psi_i}{\partial X_j} \left[ \hat{B}_{11} \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} + \hat{B}_{12} \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_j}{\partial X_2} + \hat{B}_{16} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_2} + \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_j}{\partial X_1} \right) \right] + \frac{\partial \psi_j}{\partial X_i} \left[ \hat{B}_{16} \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_i}{\partial X_1} + \hat{B}_{26} \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_i}{\partial X_2} + \hat{B}_{66} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_i}{\partial X_2} + \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_i}{\partial X_1} \right) \right] \right] dX_i dX_j
\]

\[
(K_{ij}^{53})_{NL} = \left(\frac{1}{2}\right) \int_{\Gamma} \left[ \frac{\partial \psi_i}{\partial X_j} \left[ \hat{B}_{12} \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} + \hat{B}_{22} \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_j}{\partial X_2} + \hat{B}_{26} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_2} + \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_j}{\partial X_1} \right) \right] + \frac{\partial \psi_j}{\partial X_i} \left[ \hat{B}_{16} \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_i}{\partial X_1} + \hat{B}_{26} \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_i}{\partial X_2} + \hat{B}_{66} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_i}{\partial X_2} + \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_i}{\partial X_1} \right) \right] \right] dX_i dX_j
\]

\[
(C_{ij}^{53})_{NL} = \int_{\Gamma} \left( A_{31} \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_i}{\partial X_1} + A_{32} \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_i}{\partial X_2} \right) \varphi_j dX_i dX_j
\]
\[
\sum_{k=1}^{n^*} \frac{\partial K_{ik}^{23}}{\partial X_j} \Delta_k = \frac{1}{2} \int_{\Gamma^t} \left[ \frac{\partial \psi_j}{\partial X_2} \left( A_{12} \frac{\partial \phi_i}{\partial X_1} + A_{22} \frac{\partial \phi_i}{\partial X_2} + A_{26} \left( \frac{\partial \phi_i}{\partial X_1} \frac{\partial \phi_j}{\partial X_2} - \frac{\partial \phi_j}{\partial X_1} \frac{\partial \phi_i}{\partial X_2} \right) \right) + \frac{\partial \psi_j}{\partial X_1} \left( A_{16} \frac{\partial \phi_i}{\partial X_1} + A_{26} \frac{\partial \phi_i}{\partial X_2} + A_{66} \left( \frac{\partial \phi_i}{\partial X_1} \frac{\partial \phi_j}{\partial X_2} - \frac{\partial \phi_j}{\partial X_1} \frac{\partial \phi_i}{\partial X_2} \right) \right) \right] dX_1 dX_2
\]

\[
T_{yj}^{24} = \sum_{\gamma=1}^{n^*} \sum_{k=1}^{n^*} \frac{\partial K_{ik}^{2y}}{\partial X_j} \Delta_{\gamma} + K_{ij}^{24} = K_{ij}^{24}
\]

\[
T_{yj}^{25} = \sum_{\gamma=1}^{n^*} \sum_{k=1}^{n^*} \frac{\partial K_{ik}^{2y}}{\partial Y_j} \Delta_{\gamma} + K_{ij}^{25} = K_{ij}^{25}
\]

\[
T_{yj}^{31} = \sum_{\gamma=1}^{n^*} \sum_{k=1}^{n^*} \frac{\partial K_{ik}^{3y}}{\partial u_j} \Delta_{\gamma} + K_{ij}^{31} = K_{ij}^{31}
\]

\[
T_{yj}^{32} = \sum_{\gamma=1}^{n^*} \sum_{k=1}^{n^*} \frac{\partial K_{ik}^{3y}}{\partial v_j} \Delta_{\gamma} + K_{ij}^{32} = K_{ij}^{32}
\]

\[
T_{yj}^{33} = \sum_{\gamma=1}^{n^*} \sum_{k=1}^{n^*} \frac{\partial K_{ik}^{3y}}{\partial \Delta_j} \Delta_{\gamma} + K_{ij}^{33}
\]

\[
\sum_{k=1}^{n^*} \frac{\partial K_{ik}^{31}}{\partial \Delta_j} u_k = \int_{\Gamma^t} \left[ \frac{\partial u}{\partial X_1} \left( A_{11} \frac{\partial \phi_j}{\partial X_1} + A_{12} \frac{\partial \phi_j}{\partial X_2} + A_{16} \left( \frac{\partial \phi_j}{\partial X_1} \frac{\partial \phi_i}{\partial X_2} - \frac{\partial \phi_i}{\partial X_1} \frac{\partial \phi_j}{\partial X_2} \right) \right) + \frac{\partial u}{\partial X_2} \left( A_{26} \frac{\partial \phi_j}{\partial X_1} + A_{66} \left( \frac{\partial \phi_j}{\partial X_1} \frac{\partial \phi_i}{\partial X_2} - \frac{\partial \phi_i}{\partial X_1} \frac{\partial \phi_j}{\partial X_2} \right) \right) \right] dX_1 dX_2
\]

\[
\sum_{k=1}^{n^*} \frac{\partial K_{ik}^{32}}{\partial \Delta_j} v_k = \int_{\Gamma^t} \left[ \frac{\partial v}{\partial X_1} \left( A_{16} \frac{\partial \phi_j}{\partial X_1} + A_{26} \frac{\partial \phi_j}{\partial X_2} + A_{66} \left( \frac{\partial \phi_j}{\partial X_1} \frac{\partial \phi_i}{\partial X_2} - \frac{\partial \phi_i}{\partial X_1} \frac{\partial \phi_j}{\partial X_2} \right) \right) + \frac{\partial v}{\partial X_2} \left( A_{26} \frac{\partial \phi_j}{\partial X_1} + A_{66} \left( \frac{\partial \phi_j}{\partial X_1} \frac{\partial \phi_i}{\partial X_2} - \frac{\partial \phi_i}{\partial X_1} \frac{\partial \phi_j}{\partial X_2} \right) \right) \right] dX_1 dX_2
\]

\[
\sum_{k=1}^{n^*} \frac{\partial K_{ik}^{33}}{\partial \Delta_j} \Delta_k = \left( \frac{1}{2} \right) \int_{\Gamma^t} \left( -c_1 \right) \left[ \frac{\partial^2 \phi_i}{\partial X_1^2} \left( E_{11} \frac{\partial \phi_j}{\partial X_1} + E_{16} \frac{\partial \phi_j}{\partial X_2} \right) + \frac{\partial \phi_i}{\partial X_1} \left( E_{12} \frac{\partial \phi_j}{\partial X_2} + E_{16} \frac{\partial \phi_j}{\partial X_1} \right) \right] + 2 \frac{\partial^2 \phi_i}{\partial X_1 \partial X_2} \left[ \frac{\partial \phi_i}{\partial X_1} \left( E_{11} \frac{\partial \phi_j}{\partial X_1} + E_{16} \frac{\partial \phi_j}{\partial X_2} \right) + \frac{\partial \phi_i}{\partial X_2} \left( E_{12} \frac{\partial \phi_j}{\partial X_2} + E_{16} \frac{\partial \phi_j}{\partial X_1} \right) \right]
\]

\[
+ \frac{\partial \phi_i}{\partial X_1} \left( E_{26} \frac{\partial \phi_j}{\partial X_2} + E_{66} \frac{\partial \phi_j}{\partial X_1} \right) + \frac{\partial \phi_i}{\partial X_2} \left( E_{26} \frac{\partial \phi_j}{\partial X_2} + E_{66} \frac{\partial \phi_j}{\partial X_1} \right) \right] + \frac{\partial \phi_i}{\partial X_1} \left( A_{11} \frac{\partial \phi_j}{\partial X_1} + A_{12} \frac{\partial \phi_j}{\partial X_2} \right)
\]

\[
+ \frac{\partial \phi_i}{\partial X_2} \left( B_{12} \frac{\partial \phi_j}{\partial X_2} + B_{16} \frac{\partial \phi_j}{\partial X_1} \right) + \frac{\partial \phi_i}{\partial X_1} \left( A_{26} \frac{\partial \phi_j}{\partial X_1} + A_{66} \frac{\partial \phi_j}{\partial X_2} \right)
\]

\[
+ \frac{\partial \phi_i}{\partial X_2} \left( B_{26} \frac{\partial \phi_j}{\partial X_2} + B_{66} \frac{\partial \phi_j}{\partial X_1} \right) \right] + \frac{\partial \phi_i}{\partial X_1} \left( 2 A_{11} \frac{\partial \phi_j}{\partial X_1} + A_{12} \frac{\partial \phi_j}{\partial X_1} \right)
\]

\[
+ \frac{\partial \phi_i}{\partial X_2} \left( 2 A_{26} \frac{\partial \phi_j}{\partial X_2} + A_{66} \frac{\partial \phi_j}{\partial X_2} \right) \right]
\]

\[
+ \frac{\partial \phi_i}{\partial X_1} \left( 2 A_{11} \frac{\partial \phi_j}{\partial X_1} + A_{12} \frac{\partial \phi_j}{\partial X_1} \right)
\]

\[
+ \frac{\partial \phi_i}{\partial X_2} \left( 2 A_{26} \frac{\partial \phi_j}{\partial X_2} + A_{66} \frac{\partial \phi_j}{\partial X_2} \right) \right]
\]
\[
\sum_{k=1}^{\alpha} \frac{\partial \mathbf{K}^{34}}{\partial \mathbf{A}_k^i} \cdot dX_k = \int_{\Omega} \left[ \frac{\partial \varphi_i}{\partial \mathbf{X}_1} \left( \frac{\partial \varphi_i}{\partial \mathbf{X}_1} + \frac{\partial \varphi_i}{\partial \mathbf{X}_2} + \frac{\partial \varphi_i}{\partial \mathbf{X}_3} \right) + \frac{\partial \varphi_i}{\partial \mathbf{X}_2} \left( \frac{\partial \varphi_i}{\partial \mathbf{X}_1} + \frac{\partial \varphi_i}{\partial \mathbf{X}_2} + \frac{\partial \varphi_i}{\partial \mathbf{X}_3} \right) + \frac{\partial \varphi_i}{\partial \mathbf{X}_3} \left( \frac{\partial \varphi_i}{\partial \mathbf{X}_1} + \frac{\partial \varphi_i}{\partial \mathbf{X}_2} + \frac{\partial \varphi_i}{\partial \mathbf{X}_3} \right) \right] dX_i dX_2
\]

\[
\sum_{k=1}^{\alpha} \frac{\partial \mathbf{K}^{35}}{\partial \mathbf{A}_k^j} \cdot dX_k = \int_{\Omega} \left[ \frac{\partial \varphi_j}{\partial \mathbf{X}_1} \left( \frac{\partial \varphi_j}{\partial \mathbf{X}_1} + \frac{\partial \varphi_j}{\partial \mathbf{X}_2} + \frac{\partial \varphi_j}{\partial \mathbf{X}_3} \right) + \frac{\partial \varphi_j}{\partial \mathbf{X}_2} \left( \frac{\partial \varphi_j}{\partial \mathbf{X}_1} + \frac{\partial \varphi_j}{\partial \mathbf{X}_2} + \frac{\partial \varphi_j}{\partial \mathbf{X}_3} \right) + \frac{\partial \varphi_j}{\partial \mathbf{X}_3} \left( \frac{\partial \varphi_j}{\partial \mathbf{X}_1} + \frac{\partial \varphi_j}{\partial \mathbf{X}_2} + \frac{\partial \varphi_j}{\partial \mathbf{X}_3} \right) \right] dX_i dX_2
\]

\[
T_y^{34} = \sum_{k=1}^{\alpha} \sum_{k=1}^{\alpha} \frac{\partial \mathbf{K}^{34}}{\partial \mathbf{X}_f} \cdot dX_f + K_y^{34} = K_y^{34}
\]

\[
T_y^{35} = \sum_{k=1}^{\alpha} \sum_{k=1}^{\alpha} \frac{\partial \mathbf{K}^{35}}{\partial \mathbf{Y}_f} \cdot dX_f + K_y^{35} = K_y^{35}
\]

\[
T_y^{41} = \sum_{k=1}^{\alpha} \sum_{k=1}^{\alpha} \frac{\partial \mathbf{K}^{41}}{\partial \mathbf{U}_f} \cdot dX_f + K_y^{41} = K_y^{41}
\]
\[ T_{y42} = \sum_{j=1}^{\gamma} \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{4y}}{\partial Y_j} \Delta_k + K_{y}^{42} = K_{y}^{42} \]

\[ T_{y43} = \sum_{j=1}^{\gamma} \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{4y}}{\partial Y_j} \Delta_k + K_{y}^{43} = \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{4y}}{\partial X_j} \Delta_k + K_{y}^{43} \]

\[ \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{4y}}{\partial X_j} \Delta_k = \frac{1}{2} \int_{V} \left[ \frac{\partial \psi}{\partial X_1} \left[ \hat{B}_{11} \frac{\partial \varphi}{\partial X_1} \frac{\partial \varphi}{\partial X_2} \frac{\partial \varphi}{\partial X_1} + \hat{B}_{12} \frac{\partial \varphi}{\partial X_1} \frac{\partial \varphi}{\partial X_2} + \hat{B}_{16} \left( \frac{\partial \varphi}{\partial X_1} + \frac{\partial \varphi}{\partial X_2} \right) \right] \right. \]

\[ + \left. \frac{\partial \psi}{\partial X_2} \left[ \hat{B}_{16} \frac{\partial \varphi}{\partial X_1} \frac{\partial \varphi}{\partial X_1} + \hat{B}_{26} \frac{\partial \varphi}{\partial X_1} \frac{\partial \varphi}{\partial X_2} + \hat{B}_{66} \left( \frac{\partial \varphi}{\partial X_1} + \frac{\partial \varphi}{\partial X_2} \right) \right] \right] dX_1dX_2 \]

\[ T_{y44} = \sum_{j=1}^{\gamma} \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{4y}}{\partial X_j} \Delta_k + K_{y}^{44} = K_{y}^{44} \]

\[ T_{y45} = \sum_{j=1}^{\gamma} \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{4y}}{\partial Y_j} \Delta_k + K_{y}^{45} = K_{y}^{45} \]

\[ T_{y51} = \sum_{j=1}^{\gamma} \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{5y}}{\partial Y_j} \Delta_k + K_{y}^{51} = K_{y}^{51} \]

\[ T_{y52} = \sum_{j=1}^{\gamma} \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{5y}}{\partial X_j} \Delta_k + K_{y}^{52} = K_{y}^{52} \]

\[ T_{y53} = \sum_{j=1}^{\gamma} \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{5y}}{\partial Y_j} \Delta_k + K_{y}^{53} = \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{53}}{\partial X_j} \Delta_k + K_{y}^{53} \]

\[ \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{53}}{\partial X_j} \Delta_k = \frac{1}{2} \int_{V} \left[ \frac{\partial \psi}{\partial X_1} \left[ \hat{B}_{11} \frac{\partial \varphi}{\partial X_1} \frac{\partial \varphi}{\partial X_2} \frac{\partial \varphi}{\partial X_1} + \hat{B}_{12} \frac{\partial \varphi}{\partial X_1} \frac{\partial \varphi}{\partial X_2} + \hat{B}_{16} \left( \frac{\partial \varphi}{\partial X_1} + \frac{\partial \varphi}{\partial X_2} \right) \right] \right. \]

\[ + \left. \frac{\partial \psi}{\partial X_2} \left[ \hat{B}_{16} \frac{\partial \varphi}{\partial X_1} \frac{\partial \varphi}{\partial X_1} + \hat{B}_{26} \frac{\partial \varphi}{\partial X_1} \frac{\partial \varphi}{\partial X_2} + \hat{B}_{66} \left( \frac{\partial \varphi}{\partial X_1} + \frac{\partial \varphi}{\partial X_2} \right) \right] \right] dX_1dX_2 \]

\[ T_{y54} = \sum_{j=1}^{\gamma} \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{5y}}{\partial X_j} \Delta_k + K_{y}^{54} = K_{y}^{54} \]

\[ T_{y55} = \sum_{j=1}^{\gamma} \sum_{k=1}^{n^*} \frac{\partial K_{jk}^{5y}}{\partial Y_j} \Delta_k + K_{y}^{55} = K_{y}^{55} \]
APPENDIX C

FINITE ELEMENT COEFFICIENTS BY TSDT SANDERS (SANDERS-KOITER)

SHELL THEORY

C.1 Linear coefficients of laminated composite shells by Sanders shell theory

\[ K_{yy}^{11} = \int_{\Omega} \left\{ A_{1i} + 2 \frac{c_i}{R_i} E_{1i} + \left( \frac{c_i}{R_i} \right)^2 H_{1i} \right\} \frac{\partial \psi_i}{\partial X_i} \frac{\partial \psi_j}{\partial X_j} + \left[ A_{66} + 2 \frac{c_i}{R_i} E_{66} + \left( \frac{c_i}{R_i} \right)^2 H_{66} \right] \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \right. \]
\[ + \left[ A_{16} + 2 \frac{c_i}{R_i} E_{16} + \left( \frac{c_i}{R_i} \right)^2 H_{16} \right] \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + \left( \frac{1}{R_i} \right) \hat{A}_{yi} \psi_j \right\} dX_1 dX_2 \]

\[ K_{yy}^{12} = \int_{\Omega} \left\{ A_{1i} + \frac{c_i}{R_i} E_{1i} + \left( \frac{c_i}{R_i} \right)^2 H_{1i} \right\} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left[ A_{26} + \frac{c_i}{R_i} E_{26} + \left( \frac{c_i}{R_i} \right)^2 H_{26} \right] \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \right. \]
\[ + \left[ A_{66} + \frac{c_i}{R_i} E_{66} + \frac{1}{R_i} \hat{A}_{yi} \psi_j \right\} dX_1 dX_2 \]

\[ K_{yy}^{13} = \int_{\Omega} \left\{ \left( \frac{A_{1i}}{R_i} \right) + \frac{c_i}{R_i} E_{1i} + \left( \frac{c_i}{R_i} \right)^2 H_{1i} \right\} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left[ \frac{A_{16} + A_{26}}{R_i} + \frac{c_i}{R_i} \right] \frac{E_{16} + E_{26}}{R_i} \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} - \frac{1}{R_i} \hat{A}_{yi} \psi_j \right\} dX_1 dX_2 \]

\[ K_{yy}^{14} = \int_{\Omega} \left\{ \left( \frac{\hat{B}_{1i}}{R_i} \right) + \frac{c_i}{R_i} E_{1i} + \frac{1}{R_i} \hat{B}_{1i} \frac{E_{1i}}{R_i} \right\} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left[ \frac{\hat{B}_{16} + \frac{c_i}{R_i} \hat{F}_{16}}{R_i} \right] \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \left[ \frac{\hat{B}_{66} + \frac{c_i}{R_i} \hat{F}_{66}}{R_i} \right] \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_1} \right\} dX_1 dX_2 \]

\[ K_{yy}^{15} = \int_{\Omega} \left\{ \left( \frac{\hat{B}_{1i}}{R_i} \right) + \frac{c_i}{R_i} E_{1i} + \frac{1}{R_i} \hat{B}_{1i} \frac{E_{1i}}{R_i} \right\} \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + \left[ \frac{\hat{B}_{16} + \frac{c_i}{R_i} \hat{F}_{16}}{R_i} \right] \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + \left[ \frac{\hat{B}_{66} + \frac{c_i}{R_i} \hat{F}_{66}}{R_i} \right] \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_1} \right\} dX_1 dX_2 \]
\[ K_{y}^{21} = \int_{\Omega} \left\{ A_{12} + \left( \frac{c_{1}}{R_{1}} + \frac{c_{1}}{R_{2}} \right) E_{12} + \left( \frac{c_{1}}{R_{1}} \right) \left( \frac{c_{1}}{R_{2}} \right) H_{12} \right\} \frac{\partial \psi_{i}}{\partial X_{2}} \frac{\partial \psi_{j}}{\partial X_{1}} + \left[ A_{26} + \left( \frac{c_{1}}{R_{1}} + \frac{c_{1}}{R_{2}} \right) E_{26} + \left( \frac{c_{1}}{R_{1}} \right) \left( \frac{c_{1}}{R_{2}} \right) H_{26} \right] \frac{\partial \psi_{i}}{\partial X_{2}} \frac{\partial \psi_{j}}{\partial X_{2}} + \\
\left[ A_{66} + \left( \frac{c_{1}}{R_{1}} + \frac{c_{1}}{R_{2}} \right) E_{66} + \left( \frac{c_{1}}{R_{1}} \right) \left( \frac{c_{1}}{R_{2}} \right) H_{66} \right] \frac{\partial \psi_{i}}{\partial X_{1}} \frac{\partial \psi_{j}}{\partial X_{2}} dX_{1}dX_{2} \]

\[ K_{y}^{22} = \int_{\Omega} \left\{ A_{66} + 2 \left( \frac{c_{1}}{R_{2}} \right) E_{66} + \left( \frac{c_{1}}{R_{2}} \right)^{2} H_{66} \right\} \frac{\partial \psi_{i}}{\partial X_{1}} \frac{\partial \psi_{j}}{\partial X_{2}} + \left[ A_{26} + 2 \left( \frac{c_{1}}{R_{2}} \right) E_{26} \right] \frac{\partial \psi_{i}}{\partial X_{2}} \frac{\partial \psi_{j}}{\partial X_{2}} + \left[ \frac{1}{R_{1}} \right] \hat{A}_{44} \frac{\partial \psi_{i}}{\partial X_{1}} \frac{\partial \psi_{j}}{\partial X_{1}} dX_{1}dX_{2} \]

\[ K_{y}^{23} = \int_{\Omega} \left\{ \left( \frac{A_{12}}{R_{1}} + \frac{A_{22}}{R_{2}} \right) + \left( \frac{c_{1}}{R_{1}} \left( \frac{c_{1}}{R_{2}} + \frac{c_{1}}{R_{2}} \right) \right) E_{12} + \left( \frac{c_{1}}{R_{1}} \right) \left( \frac{c_{1}}{R_{2}} \right) H_{12} \right\} \frac{\partial \psi_{i}}{\partial X_{2}} \frac{\partial \psi_{j}}{\partial X_{1}} + \left[ \left( \frac{A_{16}}{R_{1}} + \frac{A_{26}}{R_{2}} \right) \right] \frac{\partial \psi_{i}}{\partial X_{1}} \frac{\partial \psi_{j}}{\partial X_{1}} \frac{\partial \psi_{i}}{\partial X_{1}} \frac{\partial \psi_{j}}{\partial X_{1}} + \left( \frac{c_{1}}{R_{1}} \right) \left( \frac{c_{1}}{R_{2}} \right) H_{16} \right] \frac{\partial \psi_{i}}{\partial X_{1}} \frac{\partial \psi_{j}}{\partial X_{1}} dX_{1}dX_{2} \]

\[ K_{y}^{24} = \int_{\Omega} \left\{ \left( \frac{A_{16}}{R_{1}} + \frac{A_{26}}{R_{2}} \right) + \left( \frac{c_{1}}{R_{1}} \right) \left( \frac{c_{1}}{R_{2}} + \frac{c_{1}}{R_{2}} \right) \right\} \frac{\partial \psi_{i}}{\partial X_{1}} \frac{\partial \psi_{j}}{\partial X_{1}} + \left[ \left( \frac{A_{26}}{R_{1}} + \frac{A_{26}}{R_{2}} \right) \right] \frac{\partial \psi_{i}}{\partial X_{2}} \frac{\partial \psi_{j}}{\partial X_{2}} + \left( \frac{c_{1}}{R_{1}} \right) \left( \frac{c_{1}}{R_{2}} \right) H_{26} \right] \frac{\partial \psi_{i}}{\partial X_{1}} \frac{\partial \psi_{j}}{\partial X_{1}} dX_{1}dX_{2} \]

\[ K_{y}^{25} = \int_{\Omega} \left\{ \left( \frac{A_{22}}{R_{1}} + \frac{A_{22}}{R_{2}} \right) + \left( \frac{c_{1}}{R_{1}} \right) \left( \frac{c_{1}}{R_{2}} + \frac{c_{1}}{R_{2}} \right) \right\} \frac{\partial \psi_{i}}{\partial X_{2}} \frac{\partial \psi_{j}}{\partial X_{2}} + \left[ \left( \frac{A_{26}}{R_{1}} + \frac{A_{26}}{R_{2}} \right) \right] \frac{\partial \psi_{i}}{\partial X_{2}} \frac{\partial \psi_{j}}{\partial X_{2}} + \left( \frac{c_{1}}{R_{1}} \right) \left( \frac{c_{1}}{R_{2}} \right) H_{26} \right] \frac{\partial \psi_{i}}{\partial X_{2}} \frac{\partial \psi_{j}}{\partial X_{2}} dX_{1}dX_{2} \]
\[ K^{32}_y = \int_\alpha \left( \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) \partial \psi_j + \frac{A_{16} + A_{26}}{R_1} + \frac{A_{16} + A_{26}}{R_2} \right) \partial \psi_j - c_1 \right) \right) dX_1 dX_2 \]

\[ K^{33}_y = \int_\alpha \left( \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) \partial \psi_j + \frac{A_{16} + A_{26}}{R_1} + \frac{A_{16} + A_{26}}{R_2} \right) \partial \psi_j \right) \right) dX_1 dX_2 \]

\[ K^{34}_y = \int_\alpha \left( \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) \partial \psi_j + \frac{A_{16} + A_{26}}{R_1} + \frac{A_{16} + A_{26}}{R_2} \right) \partial \psi_j \right) \right) dX_1 dX_2 \]

\[ K^{35}_y = \int_\alpha \left( \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) \partial \psi_j + \frac{A_{16} + A_{26}}{R_1} + \frac{A_{16} + A_{26}}{R_2} \right) \partial \psi_j \right) \right) dX_1 dX_2 \]

\[ K^{36}_y = \int_\alpha \left( \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) \partial \psi_j + \frac{A_{16} + A_{26}}{R_1} + \frac{A_{16} + A_{26}}{R_2} \right) \partial \psi_j \right) \right) dX_1 dX_2 \]
\[ K_{y}^{32} = \int_{\Omega} \left[ \left( \hat{B}_{12} + \frac{c_{1}}{R_{2}} \hat{F}_{12} \right) \frac{\partial \psi_{i}}{\partial X_{12}} \frac{\partial \psi_{j}}{\partial X_{1}} + \left( \hat{B}_{16} + \frac{c_{1}}{R_{2}} \hat{F}_{16} \right) \frac{\partial \psi_{i}}{\partial X_{16}} \frac{\partial \psi_{j}}{\partial X_{1}} + \left( \hat{B}_{26} + \frac{c_{1}}{R_{2}} \hat{F}_{26} \right) \frac{\partial \psi_{i}}{\partial X_{26}} \frac{\partial \psi_{j}}{\partial X_{1}} + \left( \hat{B}_{66} + \frac{c_{1}}{R_{2}} \hat{F}_{66} \right) \frac{\partial \psi_{i}}{\partial X_{66}} \frac{\partial \psi_{j}}{\partial X_{1}} \right] \ dX_{1} dX_{2} \]

\[ K_{y}^{33} = \int_{\Omega} \left[ \left( \hat{B}_{11} + \frac{\hat{B}_{16} + \hat{B}_{26}}{R_{2}} \right) \frac{\partial \psi_{i}}{\partial X_{11}} \frac{\partial \psi_{j}}{\partial X_{1}} + \hat{B}_{66} \frac{\partial \psi_{i}}{\partial X_{66}} \frac{\partial \psi_{j}}{\partial X_{1}} + \hat{A}_{4s} \psi_{i} \frac{\partial \psi_{j}}{\partial X_{1}} - c_{1} \frac{\partial \psi_{i}}{\partial X_{1}} \frac{\partial^{2} \psi_{j}}{\partial X_{1}^{2}} \right] \ dX_{1} dX_{2} \]

\[ K_{y}^{34} = \int_{\Omega} \left[ \hat{A}_{4s} \psi_{i} \psi_{j}\left( \hat{D}_{11} - c_{1} \hat{F}_{11} \right) \frac{\partial \psi_{i}}{\partial X_{11}} \frac{\partial \psi_{j}}{\partial X_{1}} \left( \hat{D}_{16} - c_{1} \hat{F}_{16} \right) \frac{\partial \psi_{i}}{\partial X_{16}} \frac{\partial \psi_{j}}{\partial X_{1}} \left( \hat{D}_{26} - c_{1} \hat{F}_{26} \right) \frac{\partial \psi_{i}}{\partial X_{26}} \frac{\partial \psi_{j}}{\partial X_{1}} \right] \ dX_{1} dX_{2} \]

\[ K_{y}^{35} = \int_{\Omega} \left[ \hat{A}_{4s} \psi_{i} \psi_{j} + \left( \hat{D}_{12} - c_{1} \hat{F}_{12} \right) \frac{\partial \psi_{i}}{\partial X_{12}} \frac{\partial \psi_{j}}{\partial X_{1}} + \left( \hat{D}_{16} - c_{1} \hat{F}_{16} \right) \frac{\partial \psi_{i}}{\partial X_{16}} \frac{\partial \psi_{j}}{\partial X_{1}} + \left( \hat{D}_{26} - c_{1} \hat{F}_{26} \right) \frac{\partial \psi_{i}}{\partial X_{26}} \frac{\partial \psi_{j}}{\partial X_{1}} \right] \ dX_{1} dX_{2} \]
\[ K_{ij}^{25} = \int_{\Omega} \left[ \dot{A}_{44} \psi_i \psi_j + (\dot{D}_{66} - c_1 \dot{F}_{66}) \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_1} + (\dot{D}_{22} - c_1 \dot{F}_{22}) \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_2} + (\dot{D}_{26} - c_1 \dot{F}_{26}) \left( \frac{\partial \psi_i}{\partial X_1} \frac{\partial \psi_j}{\partial X_2} + \frac{\partial \psi_i}{\partial X_2} \frac{\partial \psi_j}{\partial X_1} \right) \right] dX_1 dX_2 \]

\[ M_{ij}^{31} = \int_{\Omega} \bar{T}_{ij} \psi_i \psi_j dX_1 dX_2 \]
\[ M_{ij}^{32} = \int_{\Omega} \bar{T}_{ij} \psi_i \psi_j dX_1 dX_2 \]
\[ M_{ij}^{33} = -c_1 \int_{\Omega} \bar{T}_{ij} \frac{\partial \psi_i}{\partial X_1} \psi_j dX_1 dX_2 \]
\[ M_{ij}^{34} = -c_1 \int_{\Omega} \bar{T}_{ij} \psi_i \frac{\partial \psi_j}{\partial X_1} dX_1 dX_2 \]
\[ M_{ij}^{43} = -c_1 \int_{\Omega} \bar{T}_{ij} \psi_i \psi_j dX_1 dX_2 \]
\[ M_{ij}^{44} = -c_1 \int_{\Omega} \bar{T}_{ij} \psi_i \psi_j dX_1 dX_2 \]
\[ M_{ij}^{52} = \int_{\Omega} \bar{T}_{ij} \psi_i \psi_j dX_1 dX_2 \]
\[ M_{ij}^{53} = -c_1 \int_{\Omega} \bar{T}_{ij} \psi_i \psi_j dX_1 dX_2 \]
\[ M_{ij}^{55} = -c_1 \int_{\Omega} \bar{T}_{ij} \psi_i \psi_j dX_1 dX_2 \]
\[ C_{yy}^{11} = C_{yy}^{12} = C_{yy}^{14} = C_{yy}^{21} = C_{yy}^{22} = C_{yy}^{24} = C_{yy}^{25} = C_{yy}^{31} = C_{yy}^{32} = C_{yy}^{34} = C_{yy}^{35} = 0 \]

\[ C_{yy}^{41} = C_{yy}^{42} = C_{yy}^{44} = C_{yy}^{51} = C_{yy}^{52} = C_{yy}^{54} = C_{yy}^{55} = 0 \]

\[ F_i^1 = \frac{1}{2} \int_{\Gamma'} \left[ (N_1 + \frac{c_1}{R_1} P_i) n_1 + (N_6 + \frac{c_1}{R_1} P_6) n_2 \right] \psi_i ds \]

\[ F_i^2 = \frac{1}{2} \int_{\Gamma'} \left[ (N_6 + \frac{c_1}{R_2} P_6) n_1 + (N_2 + \frac{c_1}{R_2} P_2) n_2 \right] \psi_i ds \]

\[ F_i^3 = \int_{\Gamma'} q \phi_i dX_1 dX_2 + \frac{1}{2} \int_{\Gamma} \left\{ \phi_i - \frac{\partial \phi_i}{\partial n} \right\} ds \]

\[ F_i^4 = \frac{1}{2} \int_{\Gamma'} (\bar{M}_i n_1 + \bar{M}_6 n_2) \psi_i ds, \quad F_i^5 = \frac{1}{2} \int_{\Gamma'} (\bar{M}_i n_1 + \bar{M}_2 n_2) \psi_i ds \]

\[ F_i^{11} = \int_{\Gamma'} \left[ \frac{\partial \psi_i}{\partial X_1} \left( N_1^{T} + \frac{c_1}{R_1} P_1^{T} \right) + \frac{\partial \psi_i}{\partial X_2} \left( N_6^{T} + \frac{c_1}{R_1} P_6^{T} \right) \right] dX_1 dX_2 \]

\[ F_i^{12} = \int_{\Gamma'} \left[ \frac{\partial \psi_i}{\partial X_1} \left( N_6^{T} + \frac{c_1}{R_2} P_6^{T} \right) + \frac{\partial \psi_i}{\partial X_2} \left( N_2^{T} + \frac{c_1}{R_2} P_2^{T} \right) \right] dX_1 dX_2 \]

\[ F_i^{13} = \int_{\Gamma'} \left[ \frac{N_1^{T} + N_6^{T}}{R_1 + R_2} \phi_i - c_i \left( \frac{\partial^2 \phi_i}{\partial X_1^2} P_1^{T} + 2 \frac{\partial^2 \phi_i}{\partial X_1 \partial X_2} P_6^{T} + \frac{\partial^2 \phi_i}{\partial X_2^2} P_2^{T} \right) \right] dX_1 dX_2 \]

\[ F_i^{14} = \int_{\Gamma'} \left[ \frac{\partial \psi_i}{\partial X_1} \left( M_1^{T} - c_1 P_1^{T} \right) + \frac{\partial \psi_i}{\partial X_2} \left( M_6^{T} - c_1 P_6^{T} \right) \right] dX_1 dX_2 \]

\[ F_i^{15} = \int_{\Gamma'} \left[ \frac{\partial \psi_i}{\partial X_1} \left( M_6^{T} - c_1 P_6^{T} \right) + \frac{\partial \psi_i}{\partial X_2} \left( M_2^{T} - c_1 P_2^{T} \right) \right] dX_1 dX_2 \]

C.2 Additional nonlinear coefficients of laminated composite shells by Sanders shell theory

\[ (K_{yy}^{13})_{NL} = \left( \frac{1}{2} \right) \int_{\Gamma'} \left[ \frac{\partial \psi_i}{\partial X_1} \left[ \left( A_{11} + \frac{c_1}{R_1} E_{11} \right) \frac{\partial w_0}{\partial X_1} + \frac{c_1}{R_1} E_{11} \frac{\partial w_0}{\partial X_1} \right] + \left( A_{12} + \frac{c_1}{R_1} E_{12} \right) \frac{\partial \psi_i}{\partial X_2} \frac{\partial w_0}{\partial X_1} + \left( A_{16} + \frac{c_1}{R_1} E_{16} \right) \frac{\partial \psi_i}{\partial X_2} \frac{\partial w_0}{\partial X_1} \right] \right] dX_1 dX_2 \]

\[ (K_{yy}^{23})_{NL} = \left( \frac{1}{2} \right) \int_{\Gamma'} \left[ \frac{\partial \psi_i}{\partial X_2} \left[ \left( A_{12} + \frac{c_1}{R_1} E_{12} \right) \frac{\partial w_0}{\partial X_2} + \frac{c_1}{R_1} E_{12} \frac{\partial w_0}{\partial X_2} \right] + \left( A_{16} + \frac{c_1}{R_1} E_{16} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial w_0}{\partial X_2} + \left( A_{26} + \frac{c_1}{R_1} E_{26} \right) \frac{\partial \psi_i}{\partial X_1} \frac{\partial w_0}{\partial X_2} \right] \right] dX_1 dX_2 \]
\[
\begin{align*}
(K_{y3})_{NL} &= \int d\Omega \left( \frac{\partial \rho_i}{\partial x_2} \frac{\partial \phi_j}{\partial x_2} \right) + \left( A_{66} + \frac{c_l}{R_2} E_{66} \right) \left( \frac{\partial \phi_i}{\partial x_1} \frac{\partial \phi_j}{\partial x_1} + \frac{\partial \phi_i}{\partial x_2} \frac{\partial \phi_j}{\partial x_2} \right)
\end{align*}
\]

\[
(K_{y3})_{NL} = \int d\Omega \left( \frac{\partial \rho_i}{\partial x_2} \frac{\partial \phi_j}{\partial x_2} \right) + \left( A_{66} + \frac{c_l}{R_2} E_{66} \right) \left( \frac{\partial \phi_i}{\partial x_1} \frac{\partial \phi_j}{\partial x_1} + \frac{\partial \phi_i}{\partial x_2} \frac{\partial \phi_j}{\partial x_2} \right)
\]

\[
\begin{align*}
(K_{y3})_{NL} &= \left( \frac{1}{2} \right) \int d\Omega \left( \frac{\partial^2 \rho_i}{\partial x_1^2} \frac{\partial \phi_j}{\partial x_2} + \frac{\partial \rho_i}{\partial x_1} \frac{\partial^2 \phi_j}{\partial x_1 \partial x_2} \right) + \left( \frac{\partial \rho_i}{\partial x_2} \frac{\partial \phi_j}{\partial x_1} + \frac{\partial \rho_i}{\partial x_2} \frac{\partial \phi_j}{\partial x_2} \right) + \left( \frac{\partial \rho_i}{\partial x_2} \frac{\partial \phi_j}{\partial x_2} \right)
\end{align*}
\]
\[
\frac{A_{12}}{R} w_0 \varphi_j + A_{12} \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} + A_{22} \frac{\partial w_0}{\partial X_2} \frac{\partial \varphi_j}{\partial X_2} + A_{26} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_2} + \frac{\partial \varphi_j}{\partial X_1} \frac{\partial w_0}{\partial X_2} \right) - 2c_1 \left( \frac{\partial^2 \varphi_j}{\partial X_1^2} + \frac{\partial^2 \varphi_j}{\partial X_2^2} \right) + 2 E_{26} \frac{\partial^2 \varphi_j}{\partial X_2} \right) \right] dX_1, dX_2 \\

(K_{y}^{33}\text{NL}) = \int_{\Omega} \left[ \frac{\partial \varphi_j}{\partial X_1} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} + \frac{\partial \varphi_j}{\partial X_1} \frac{\partial w_0}{\partial X_1} \right) + \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} \right] dX_1, dX_2 \\

(K_{y}^{34}\text{NL}) = \int_{\Omega} \left[ \frac{\partial \varphi_j}{\partial X_1} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} + \frac{\partial \varphi_j}{\partial X_1} \frac{\partial w_0}{\partial X_1} \right) + \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} \right] dX_1, dX_2 \\

(K_{y}^{35}\text{NL}) = \int_{\Omega} \left[ \frac{\partial \varphi_j}{\partial X_1} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} + \frac{\partial \varphi_j}{\partial X_1} \frac{\partial w_0}{\partial X_1} \right) + \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} \right] dX_1, dX_2 \\

(K_{y}^{36}\text{NL}) = \int_{\Omega} \left[ \frac{\partial \varphi_j}{\partial X_1} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} + \frac{\partial \varphi_j}{\partial X_1} \frac{\partial w_0}{\partial X_1} \right) + \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} \right] dX_1, dX_2 \\

(C_{y}^{33}\text{NL}) = \int_{\Omega} \left( \frac{\partial \varphi_j}{\partial X_1} \left( \frac{\partial w_0}{\partial X_1} \frac{\partial \varphi_j}{\partial X_1} + \frac{\partial \varphi_j}{\partial X_1} \frac{\partial w_0}{\partial X_1} \right) \right) dX_1, dX_2 \\

C.3 Tangent stiffness coefficients of laminated composite shells by Sanders shell theory

\[
T_{ij}^{11} = \sum_{y=1}^{5} \sum_{k=1}^{n^*} \frac{\partial K_{y}^{11}}{\partial u_j} \Delta_k + K_{ij}^{11} = K_{ij}^{11} \\
T_{ij}^{12} = \sum_{y=1}^{5} \sum_{k=1}^{n^*} \frac{\partial K_{y}^{12}}{\partial v_j} \Delta_k + K_{ij}^{12} = K_{ij}^{12}
\]
\[
T_{y}^{13} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{y}}{\partial \Delta_{j}} \Delta_{i} + K_{y}^{13} = \sum_{k=1}^{n} \frac{\partial K_{ik}^{13}}{\partial \Delta_{j}} \Delta_{i} + K_{y}^{13}
\]

\[
= \sum_{k=1}^{n} \frac{\partial K_{ik}^{13}}{\partial \Delta_{j}} \Delta_{i} = (1/2) \int \left( \frac{\partial \psi_{j}}{\partial X_{2}} \left[ \left( A_{11} + \frac{c_{1}}{R_{1}} \right) \frac{\partial w_{0}}{\partial \Delta_{j}} \frac{\partial \varphi_{j}}{\partial X_{1}} + \left( A_{12} + \frac{c_{1}}{R_{2}} \right) \frac{\partial w_{0}}{\partial \Delta_{j}} \frac{\partial \varphi_{j}}{\partial X_{2}} + \left( A_{16} + \frac{c_{1}}{R_{1}} \right) \frac{\partial w_{0}}{\partial \Delta_{j}} \frac{\partial \varphi_{j}}{\partial X_{1}} \right] \right) dX_{i} dX_{2}
\]

\[
T_{y}^{14} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{y}}{\partial X_{j}} \Delta_{i} + K_{y}^{14} = K_{y}^{14}
\]

\[
T_{y}^{15} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{y}}{\partial Y_{j}} \Delta_{i} + K_{y}^{15} = K_{y}^{15}
\]

\[
T_{y}^{21} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{2y}}{\partial u_{j}} \Delta_{i} + K_{y}^{21} = K_{y}^{21}
\]

\[
T_{y}^{22} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{2y}}{\partial v_{j}} \Delta_{i} + K_{y}^{22} = K_{y}^{22}
\]

\[
T_{y}^{23} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{2y}}{\partial \Delta_{j}} \Delta_{i} + K_{y}^{23} = \sum_{k=1}^{n} \frac{\partial K_{ik}^{23}}{\partial \Delta_{j}} \Delta_{i} + K_{y}^{23}
\]

\[
= \sum_{k=1}^{n} \frac{\partial K_{ik}^{23}}{\partial \Delta_{j}} \Delta_{i} = \frac{1}{2} \int \left( A_{11} + \frac{c_{1}}{R_{1}} \right) \frac{\partial w_{0}}{\partial \Delta_{j}} \frac{\partial \varphi_{j}}{\partial X_{1}} + A_{12} + \frac{c_{1}}{R_{2}} \frac{\partial w_{0}}{\partial \Delta_{j}} \frac{\partial \varphi_{j}}{\partial X_{2}} + A_{16} + \frac{c_{1}}{R_{1}} \frac{\partial w_{0}}{\partial \Delta_{j}} \frac{\partial \varphi_{j}}{\partial X_{1}} \right) dX_{i} dX_{2}
\]

\[
T_{y}^{24} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{2y}}{\partial X_{j}} \Delta_{i} + K_{y}^{24} = K_{y}^{24}
\]

\[
T_{y}^{25} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{2y}}{\partial Y_{j}} \Delta_{i} + K_{y}^{25} = K_{y}^{25}
\]

\[
T_{y}^{31} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{3y}}{\partial u_{j}} \Delta_{i} + K_{y}^{31} = K_{y}^{31}
\]

\[
T_{y}^{32} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{3y}}{\partial v_{j}} \Delta_{i} + K_{y}^{32} = K_{y}^{32}
\]

\[
T_{y}^{33} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial K_{ik}^{3y}}{\partial \Delta_{j}} \Delta_{i} + K_{y}^{33} = \sum_{k=1}^{n} \left( \frac{\partial K_{ik}^{13}}{\partial \Delta_{j}} u_{k} + \frac{\partial K_{ik}^{23}}{\partial \Delta_{j}} v_{k} + \frac{\partial K_{ik}^{33}}{\partial \Delta_{j}} \Delta_{k} + \frac{\partial K_{ik}^{43}}{\partial \Delta_{j}} X_{k} + \frac{\partial K_{ik}^{53}}{\partial \Delta_{j}} Y_{k} \right) + K_{y}^{33}
\]
\[
\sum_{i=1}^{n} \frac{\partial K_{ji}^{31}}{\partial \Delta_j} \cdot u_i = \int_{\alpha} \left[ \frac{\partial}{\partial x_1} \left( A_1 + \frac{c_i}{R_i} E_{11} \right) \frac{\partial \phi_j}{\partial x_1} + \frac{A_2 + \frac{c_i}{R_i} E_{12}}{\partial x_1} \frac{\partial \phi_j}{\partial x_1} + \left( A_1 + \frac{c_i}{R_i} E_{16} \right) \frac{\partial \phi_j}{\partial x_1} \right] dX_1 dX_2
\]

\[
\sum_{i=1}^{n} \frac{\partial K_{ji}^{32}}{\partial \Delta_j} \cdot v_k = \int_{\alpha} \left[ \frac{\partial}{\partial x_1} \left( A_1 + \frac{c_i}{R_i} E_{16} \right) \frac{\partial \phi_j}{\partial x_1} + \frac{A_2 + \frac{c_i}{R_i} E_{26}}{\partial x_1} \frac{\partial \phi_j}{\partial x_1} + \left( A_1 + \frac{c_i}{R_i} E_{66} \right) \frac{\partial \phi_j}{\partial x_1} \right] dX_1 dX_2
\]

\[
\sum_{i=1}^{n} \frac{\partial K_{ji}^{33}}{\partial \Delta_j} \cdot \Delta_k = \left( \frac{1}{2} \right) \int_{\alpha} \left( -c_i \right) \left[ \frac{\partial^2 \phi_j}{\partial x_1^2} \left( \frac{\partial w_0}{\partial x_1} + \frac{\partial w_0}{\partial x_2} \frac{\partial \phi_j}{\partial x_1} + \frac{\partial w_0}{\partial x_2} \frac{\partial \phi_j}{\partial x_2} \right) \right] dX_1 dX_2
\]
\[
\sum_{k=1}^{n} \frac{\partial K_{ik}^{34}}{\partial \Delta_j} \Delta_k + K_{ij}^{34} = K_{ij}^{34} \\
T_{ij}^{35} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial K_{ik}^{35}}{\partial Y_j} \Delta_k + K_{ij}^{35} = K_{ij}^{35} \\
T_{ij}^{41} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial K_{ik}^{41}}{\partial A_j} \Delta_k + K_{ij}^{41} = K_{ij}^{41} \\
T_{ij}^{42} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial K_{ik}^{42}}{\partial \psi_j} \Delta_k + K_{ij}^{42} = K_{ij}^{42} \\
T_{ij}^{43} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial K_{ik}^{43}}{\partial \Delta_j} \Delta_k + K_{ij}^{43} = \sum_{k=1}^{n} \frac{\partial K_{ik}^{43}}{\partial \Delta_j} \Delta_k + K_{ij}^{43} \\
\sum_{k=1}^{n} \frac{\partial K_{ik}^{44}}{\partial \Delta_j} \Delta_k = \frac{1}{2} \int_x \left[ \frac{\partial \psi_j}{\partial \Delta_j} \left( \hat{B}_{i1} \frac{\partial w_0}{\partial \Delta_j} \frac{\partial \varphi_j}{\partial \Delta_j} + \hat{B}_{i2} \frac{\partial w_0}{\partial \Delta_j} \frac{\partial \varphi_j}{\partial \Delta_j} + \hat{B}_{i6} \left( \frac{\partial w_0}{\partial \Delta_j} \frac{\partial \varphi_j}{\partial \Delta_j} + \frac{\partial w_0}{\partial \Delta_j} \frac{\partial \varphi_j}{\partial \Delta_j} \right) \right) \right] dX_1 dX_2 \\
T_{ij}^{44} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial K_{ik}^{44}}{\partial \psi_j} \Delta_k + K_{ij}^{44} = K_{ij}^{44} \\
T_{ij}^{45} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial K_{ik}^{45}}{\partial Y_j} \Delta_k + K_{ij}^{45} = K_{ij}^{45} 
\]
\[ T_{y}^{51} = \sum_{y=1}^{5} \sum_{k=1}^{n_{k}} \frac{\partial K_{ik}^{5y}}{\partial u_{j}} \Delta_{k}^{i} + K_{y}^{51} = K_{y}^{51} \]

\[ T_{y}^{52} = \sum_{y=1}^{5} \sum_{k=1}^{n_{k}} \frac{\partial K_{ik}^{5y}}{\partial v_{j}} \Delta_{k}^{i} + K_{y}^{52} = K_{y}^{52} \]

\[ T_{y}^{53} = \sum_{y=1}^{5} \sum_{k=1}^{n_{k}} \frac{\partial K_{ik}^{5y}}{\partial \Delta_{j}} \Delta_{k}^{i} + K_{y}^{53} = \sum_{k=1}^{n_{k}} \frac{\partial K_{ik}^{5y}}{\partial \Delta_{j}} \Delta_{k}^{i} + K_{y}^{53} \]

\[ \sum_{y=1}^{n_{k}} \frac{\partial K_{ik}^{5y}}{\partial \Delta_{j}} \Delta_{k}^{i} = \frac{1}{2} \int_{\alpha} \left[ \frac{\partial \psi_{i}}{\partial X_{1}} \left[ \dot{B}_{12} \frac{\partial \omega_{y}}{\partial X_{1}} \frac{\partial \varphi_{j}}{\partial X_{1}} + \dot{B}_{22} \frac{\partial \omega_{y}}{\partial X_{1}} \frac{\partial \varphi_{j}}{\partial X_{2}} + \dot{B}_{26} \left( \frac{\partial \omega_{y}}{\partial X_{1}} \frac{\partial \varphi_{j}}{\partial X_{2}} + \frac{\partial \omega_{y}}{\partial X_{2}} \frac{\partial \varphi_{j}}{\partial X_{1}} \right) \right] \right] \\
+ \frac{\partial \psi_{i}}{\partial X_{1}} \left[ \dot{B}_{16} \frac{\partial \omega_{y}}{\partial X_{1}} + \dot{B}_{26} \frac{\partial \omega_{y}}{\partial X_{1}} + \dot{B}_{66} \left( \frac{\partial \omega_{y}}{\partial X_{1}} + \frac{\partial \omega_{y}}{\partial X_{2}} \right) \right] dX_{1}dX_{2} \]

\[ T_{y}^{54} = \sum_{y=1}^{5} \sum_{k=1}^{n_{k}} \frac{\partial K_{ik}^{5y}}{\partial Y_{j}} \Delta_{k}^{i} + K_{y}^{54} = K_{y}^{54} \]

\[ T_{y}^{55} = \sum_{y=1}^{5} \sum_{k=1}^{n_{k}} \frac{\partial K_{ik}^{5y}}{\partial Y_{j}} \Delta_{k}^{i} + K_{y}^{55} = K_{y}^{55} \]
VITA

Mr. Seung Joon Lee received his Bachelor of Science degree in February 1995 and Master of Science degree in February 1997 in civil engineering from Yeungnam University, Korea. His master’s thesis is “Evaluation of Occupant Risk by Flail Space Model for Roadside Guardrails”. In Fall 1998, he pursued his studies in the Department of Civil Engineering at Texas A&M University. Mr. Lee strengthened his study in the field of Roadside Safety Structures through his working in Safety and Structural System Division of Texas Transportation Institute (TTI) as a Graduate Research Assistant for his first three years at Texas A&M University.

Mr. Lee worked as a Graduate Research Assistant from Fall 2001 while working toward his doctoral degree in the Department of Civil Engineering at Texas A&M University, College Station, TX. under Professor J. N. Reddy, University Distinguished Professor and holder of the Oscar S. Wyatt Endowed Chair. He received his Ph. D. degree in May 2004.

His areas of interest are Composite Materials and Structures, Smart Structures, Computational Mechanics, Roadside Safety Structure Design, and Computer Simulation of Vehicle Dynamics.

He can be reached at his permanent address: Seung Joon Lee
245-16 Naedang-Dong Seo-Gu
Taegu, KOREA (R.O.K.) 703-803