ON CONDITIONS FOR SUB-RIEMANNIAN METRICS TO ADMIT THE PRODUCT STRUCTURE

A Dissertation

by

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ABSTRACT

The classical result of Eisenhart states that if a Riemannian metric g admits a Riemanian metric that is not constantly proportional to g and has the same (parameterized) geodesics as g in a neighborhood of a given point, then g is a direct product of two Riemannian metrics in this neighbourhood. We extend this result to sub-Riemannian metrics on a class of step 2 distributions.

The thesis is devoted to study of the properties of sub-Riemannian metrics that ensure that it admits the product structure. It consist of two parts devoted to two different set of properties.

In the first part, inspired by the classical Eisenhart and De Rham decomposition theorem in Riemannian geometry, we make an attempt to show that the property that a sub-Riemannian metric admits a nonconstantly proportional metric with the same geodesics (this property is called affine non-rigidity) implies that it admits the product structure.

In the second part, we replace affine non-rigidity by a weaker set of the following two properties (both of which follows from affine non-rigidity):

1. the Tanaka symbol of the underlying distribution is decomposable in the natural sense (as a fundamental graded Lie algebra) ;

2. the Jacobi equation along generic extremals are decoupled.

Our work in the first part solves the main conjecture and shows that a step 2 distribution with its *Tanaka symbol* decomposable into 2 nonzero ad-*surjective* indecomposable fundamental graded Lie algebras, along with the affine non-rigidity of a sub-Riemannian metric on it, must admit a product structure.

In the second part, first, we prove that the decomposability of *Jacobi curve* is necessary for affine non-rigidity on (D, M, g), and second, we found a combination of necessary conditions for the affine non-rigidity which is actually sufficient for (D, M, g) to admit a product structure in the case study of $(4, 6)$ distribution. To show this sufficiency, we use Zelenko-Li's theory of a *normal moving frames* of the corresponding Jacobi curves, The analysis of the structure equations

for such frames allows us to conclude the existence of a product structure for distributions under consideration.

DEDICATION

To my mother, father, grandfather, grandmother and all these years being an Aggie.

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NOMENCLATURE

(g, D, M)	sub-Riemannian manifold
m	dimension of the distribution D
n	dimension of the manifold M
TM	tangent bundle
T^*M	cotangent bundle
$[X, Y]$	the Lie brackets of X and Y, which are either vector fields or elements of a Lie algebra
σ	canonical symplectic structure on T^*M
h	sub-Riemannian Hamiltonian
$\vec{h}$	Hamiltonian vector field of h
$e^{t\vec{h}}$	the flow generated by the vector field $\vec{h}$
$\mathfrak{m}(q)$	Tanaka symbol at a point $q \in M$
c_{ij}^k	structure functions of a moving frame
$[1 : m]$	set of integers from 1 to m
$[u_i]$	linear form of u_i with u_i's coefficient equal to 1
$g_1 \overset{a}{\sim} g_2$	two metrics g_1 and g_2 are affinely equivalent
Φ	orbital diffeomorphism

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1. INTRODUCTION

1.1 Affine equivalence in Riemannian geometry: nonrigidity and product structure

The thesis is devoted to a problem in sub-Riemannian geometry but we start with a historical overview of the same problem in Riemannian geometry. Recall that two Riemannian metrics g_1 and g_2 on a manifold M are called projectively equivalent if they have the same geodesics, as unparametrized curves, namely, for every geodesic $\gamma(t)$ of g_1 there exists a reparametrization $t = \varphi(\tau)$ such that $\gamma\big(\varphi(\tau)\big)$ is a geodesic of g_2. They are called affinely equivalent, if they are projective equivalent and the reparametrizations $\varphi(\tau)$ above are affine functions, i.e., they are of the form $\varphi(\tau) = a\tau + b$. We will write $g_1 \overset{p}{\sim} g_2$ and $g_1 \overset{a}{\sim} g_2$ in the case of projective and affine equivalence, respectively. We also need the local version of the same definitions for germs of Riemannian metrics at a point when conditions on the coincidence of geodesics hold in a neighborhood of this point.

From the form of the equation for Riemannian geodesics, it follows immediately that two Riemannian metrics are affinely equivalent if and only if they have the same geodesics as parametrized curves, which in turn is equivalent to the condition that they have the same Levi-Civita connection, i.e. one metric is parallel with respect to the Levi-Civita connection of the other.

Obviously given any Riemannian metric g and a positive constant c the metrics cg and g are affinely equivalent. The metric cg will be said a *constantly proportional metric* to the metric g. The Riemannian metric g is called *affinely rigid* if metrics constantly proportional to it are the only affinely equivalent metric to it.

A class of Riemannian metrics g that are not affinely rigid are the metrics admitting a product structure, i.e. when the ambient manifold M can be represented as $M = M_1 \times M_2$, where each M_1 and M_2 are of positive dimension and there exist Riemannian metrics g_1 and g_2 on M_1 and M_2, respectively, such that if $\pi_i : M \to M_i$, $i = 1, 2$, are canonical projections, then

$$g = \pi_1^* g_1 + \pi_2^* g_2. \tag{1.1}$$

Then obviously for every positive constants C_1 and C_2

$$g \overset{a}{\sim} (C_1\pi_1^*g_1 + C_2\pi_2^*g_2)$$

and the metric $(C_1\pi_1^*g_1 + C_2\pi_2^*g_2)$ is not constantly proportional to g if $C_1 \neq C_2$, i.e. the metric g is not affinely rigid. In 1923 L. P. Eisenhart proved that locally the converse is true, i.e. the following theorem holds

Theorem 1.1.1 ([11])**.** *If a Riemannian metric g is not affinely rigid near a point q_0, i.e. admits a locally affinely equivalent non-constantly proportional Riemannian metric in a neighborhood of a point q_0, then the metric g is the direct product of two Riemannian metrics in a neighborhood of q_0.*

This theorem is closely related to (and in fact is a local version of) the De Rham decomposition theorem ([7]) on the direct product structure of a simply connected complete Riemannian manifolds in terms of the decomposition of the tangent bundle with respect to the action of holonomy group, as once $g_1 \overset{a}{\sim} g_2$ and they are not constantly proportional, then the eigenspaces of the transition operators between these metrics form such a decomposition of the tangent bundle.

1.2 Affine equivalence of sub-Riemannian metrics: the main conjecture

Sub-Riemannian manifolds is a far going generalization of Riemannian one. To define it we need a new ingredient, a distribution D is a distribution on M (i.e. a subbundle of TM), which is assumed to be bracket-generating. To define the latter notion, first using iterative Lie brackets tangent to a distribution D (i.e. sections of D) one can define a filtration

$$D = D^1 \subset D^2 \subset \dots D^j(q) \subset \dots \tag{1.2}$$

of the tangent bundle, called a *weak derived flag*. More precisely, set $D = D^1$ and define recursively

$$D^j = D^{j-1} + [D, D^{j-1}], \quad j > 1 \tag{1.3}$$

If $X_1, \dots, X_m$ is m vector fields constituting a local basis of a distribution D, then $D^j(q)$ is the linear span of all iterated Lie brackets of these vector fields, of length not greater than j, evaluated at a point q,

$$D^j(q) = \operatorname{span}\{\Big[X_{i_1}(q), \dots [X_{i_{s-1}}, X_{i_s}](q) \dots\Big] : (i_1, \dots, i_s) \in [m]^s, s \in [j]\} \tag{1.4}$$

(here given a positive integer n we denote by $[n]$ the set $\{1, \dots, n\}$). A distribution D is called *bracket-generating* (or *completely nonholonomic*) if for any q there exists $\mu(q) \in \mathbb{N}$ such that $D^{\mu(q)}(q) = T_qM$. The number $\mu(q)$ is called the *degree of nonholonomy* of D at a point q. If the degree of nonholonomy is equal to a constant μ at every point, one say that D is *step μ distribution*.

The assumption of bracket-genericity is not restrictive: if a distribution is not bracket generating and there exist a positive integer μ such that $D^{\mu+1} = D^{\mu} \subsetneq TM$ in some neighborhood U then D^{μ} is a proper involutive subbundle of TU and the distribution D is bracket generating on each integral submanifold of D^{μ} in U. So, we can restrict ourselves to this integral submanifolds instead of U.

A sub-Riemannian manifold/structure is a triple (M, D, g), where M is a smooth manifold, D is a bracket-generating distribution, and g is a Riemannian metric on D. We say that g is a sub-Riemannian metric on (M, D). Riemannian geometry appears as the particular case where $D = TM$.

First, what is a sub-Riemannian geodesics? There are at least two different approaches to this concept. One approach is variational: a geodesic is seen as an extremal trajectory, i.e. a candidate for the “shortest path" connecting its endpoints, w.r.t. the corresponding energy functional. The other approach is differential-geometric: geodesic is the “straightest path" i.e. the curves for which the vector field of velocities is parallel along the curve, w.r.t. a natural connection. while in Riemannian geometry these two approaches lead to the same set of trajectories, in sub-Riemannian geometry they lead to different sets of trajectories (see [6] for details), and in general for the second approach the natural connection only exist under additional (and rather restrictive) assumptions of constancy of sub-Riemannian symbol ([14]).

In the present thesis, we consider the geodesic defined by the variational approach. A horizontal curve $\gamma : [a,b] \to M$ is an absolutely continuous curve tangent to D, i.e. $\gamma'(t) \in D(\gamma(t))$. In the sequel the manifold M is assumed to be connected. By the Rashevskii-Chow theorem the assumption that D is bracket-generating guaranties that the space of horizontal curves connecting two given points q_0 and q_1 is not empty. The following energy minimizing problem:

$$
\begin{aligned}
&E(\gamma) = \int_a^b g\left(\gamma'(t), \gamma'(t)\right)\, dt \Rightarrow \text{min.}\\
&\gamma'(t) \in D\left(\gamma(t)\right) \qquad \text{a.e. } t \\
&\gamma(a) = q_0, \quad \gamma(b) = q_1
\end{aligned}
\tag{1.5}
$$

can be solved using the Pontryagin Maximum Principle ([4, 15]) in optimal control theory that defines special curves in the cotangent bundle T^*M, called the *Pontryagin extremals*, so that for a minimizer of the optimal control problem (1.5) is a projection from T^*M to M of some *Pontryagin extremal*. To describe them, we need to recall some basic constructions in the cotangent bundle T^*M. Let $\pi : T^*M \to M$ denote the canonical projection. The *tautological Liouville* 1*-form* s on T^*M is defined as follows: if $\lambda = (p,q) \in T^*M$, where $q \in M$ and $p \in T_q^*M$, and $v \in T_\lambda T^*M$, then

$$
s(\lambda)(v) := p\Big(\pi_*(v)\Big).
$$

Given a differential 2-form ω on a manifold N its kernel $\mathrm{Ker}\omega_z$ at a point $z \in N$ is defined as follows:

$$
\mathrm{Ker}\ \omega_z = \{X \in T_zN, \omega_z(X,Y) = 0, \forall Y \in T_zN\}.
$$

A differential 2-form ω is called nondegenerate at a point z if $\mathrm{Ker}\,\omega_z = 0$ and it is degenerate otherwise. Recall that a symplectic manifold is a manifold endowed with a closed nondegenerate differential 2-form, called a symplectic structure. It turns out that the exterior derivative of the tautological 1-form s:

$$
\sigma := ds. \tag{1.6}
$$

is nondegenerate. The form σ is called the *canonical symplectic structure on* T^*M and it makes T^*M symplectic manifold.

A Hamiltonian is any smooth function on T^*M. To any Hamiltonian H one can assign a vector field $\vec{H}$ on T^*M as the unique vector field satisfying

$$i_{\vec{H}}\sigma = -dH, \tag{1.7}$$

where $i_{\vec{H}}$ denotes the operation of the interior product, $(i_{\vec{H}})\sigma_\lambda(Y) := \sigma(\vec{H}, Y)$ for all $Y \in T_\lambda T^*M$. The existence and uniqueness of $\vec{H}$ satisfying (1.7) is the direct consequence of the nondegeneracy of σ.

Now we are ready two describe the Pontryagin extremals for sub-Riemannian energy-minimizing problem (1.5). In general, there are two types of Pontryagin extremals for optimal control problem (1.5) ([2, 4, 15]):

- *normal extremals* that are the integral curves of the Hamiltonian vector field $\vec{h}$ on T^*M corresponding to the Hamiltonian

  $$h(p, q) = ||\, p|_{D(q)}||^2, \quad q \in M, p \in T_q^*M, \tag{1.8}$$

  and lying on a nonzero level set of h. Here $||\, p|_{D(q)}||$ the operator norm of the functional $p|_{D(q)}$, i.e.

  $$||\, p|_{D(q)}|| = \max\{p(v) : v \in D(q), g(q)(v, v) = 1\}.$$

  The Hamiltonian h, defined by (1.8), is called the *Hamiltonian , associated with the metric* g or shortly the *sub-Riemannian Hamiltonian*.

- *abnormal extremals* that are special curves (with the trace different from a point) in the annihilator

  $$D^\perp = \{(p, q) \in T^*M : p|_{D(q)} = 0\} \tag{1.9}$$

of the distribution D (that coincides with the zero level set of h from (1.8)) that are also tangent a.e. to the Hamiltonian lift $\vec{H}_D$ of the distribution D to $D^\perp$,

$$\overrightarrow{H}_D = \mathrm{span}\{\overrightarrow{H}_X, X \text{ is a vector field tangent to } D\},$$

where

$$H_X(p,q) = p\left(X(q)\right), \quad q \in M, p \in T_q^*M \tag{1.10}$$

and $\overrightarrow{H}_X$ is the Hamiltonian vector field of H_X.

Note that abnormal extremals, as unparametrized curves, depend on the distribution D only and not on a metric g on it.

Definition 1.2.1. The *(variational) sub-Riemannian geodesics* are projection of the Pontryagin extremals of the optimal control problem (1.5).

Note that in the Riemannian case geodesics given by Definition 1.2.1 coincides with the usual Riemannian geodesics, as for Riemannian metrics abnormal extremals do not exist and the Riemannian geodesics are exactly the projections of the corresponding normal Pontryagin extremals. We thus extend the definition of equivalences of metrics in the following way.

Definition 1.2.2. Let M be a manifold and D be a bracket generating distribution on M. Two sub-Riemannian metrics g_1 and g_2 on (M, D) are called *projectively equivalent* at $q_0 \in M$ if they have the same geodesics, up to a reparameterization, in a neighborhood of q_0. They are called *affinely equivalent* at q_0 if they have the same geodesics, up to affine reparameterization, in a neighborhood of q_0.

We will write $g_1 \overset{p}{\sim} g_2$ and $g_1 \overset{a}{\sim} g_2$ in the case of projective and affine equivalence, respectively.

By complete analogy with the Riemannian case, for a sub-Riemannian metric g on (M, D) and a positive constant c the metrics cg and g are affinely equivalent. The metric cg will be said a *constantly proportional metric* to the metric g.

Definition 1.2.3. A sub- Riemannian metric g on (M, D) is called *affinely rigid* if the sub-Riemannian metrics constantly proportional to it is the only sub-Riemannian metrics on (M, D) that is affinely equivalent to g.

As in the Riemannian case, examples of affinely nonrigid sub-Riemannian structure can be constructed with the help of the appropriate notion of product structure. For this we first have to define distributions admitting product structure as follows:

Definition 1.2.1. A distribution D on a manifold M *admits a product structure* if there exist two manifold M_1 and M_2 of positive dimension endowed with two distributions D_1 and D_2 of positive rank (on M_1 and M_2, respectively) such that the following two conditions holds:

1. $M = M_1 \times M_2$;

2. If $\pi_i : M \to M_i$, $i = 1, 2$, are the canonical projections and $\pi_i^* D_i$ denotes the pullback of the distribution D_i form M_i to M, i.e.

$$\pi_i^* D_i(q) = \{v \in T_q M : d\pi_i(q)v \in D_i\Big(\pi_i(q)\Big)\},$$

then

$$D(q) = \pi_1^* D_1(q) \cap \pi_2^* D_2(q)$$

In this case, we will write that $(M, D) = (M_1, D_1) \times (M_2, D_2)$.

Definition 1.2.4. A sub-Riemannian structure (M, D, g) *admits a product structure* if there exist (nonempty) sub-Riemannian structures (M_1, D_1, g_1) and (M_2, D_2, g_2) such that $(M, D) = (M_1, D_1) \times (M_2, D_2)$ and if $\pi_i : M \mapsto M_1$ are the canonical projections then identity (1.1) holds. In this case we will write that $(M, D, g) = (M_1, D_1, g_1) \times (M_2, D_2, g_2)$

It is easy to see that if $(M, D, g) = (M_1, D_1, g_1) \times (M_2, D_2, g_2)$ then this sub-Riemannian metric is affinely equivalent to

$$(M_1, D_1, c_1 g_1) \times (M_2, D_2, c_2 g_2),$$

for every two positive constants c_1 and c_2, but the latter metric is not constantly proportional to (M, D, g) , if $c_1 \neq c_2$, i.e. a *sub-Riemannian metric admitting product structure* is not affinely rigid. The main question is whether or not the converse of this statement , at least in a local setting, i.e. the analog of the Eisenhart theorem (Theorem 1.1.1) holds.

Conjecture 1.2.5 ([13])**.** *If a sub-Riemannian metric g is not affinely rigid near a point q_0, i.e. admits a locally affinely equivalent non-constantly proportional sub-Riemannian metric in a neighborhood of a point q_0, then the metric g is the direct product of two sub-Riemannian metrics in a neighborhood of q_0.*

In chapter 2 we prove this conjecture for sub-Riemannian metrics on a class of step 2 distributions, see Theorem 2.1.11.

1.3 Product structure from Jacobi curve's perspective

In [13] the following necessary conditions for a sub-Riemannian metric to be affinely non-rigid were proved:

1. The Tanaka symbol of the distribution D is decomposable (see Theorem 2.1.6 and references therein);

2. There exists a first integral of the flow of normal extremals, which is compatible with the decomposition of the Tanaka symbol in a sense of Definition 3.2.1 (see Theorem 3.2.2 and references thereafter) .

In Chapter 3, subsection 3.1.2 we found the new necessary condition for affine non-rigidity (Theorem 3.1.4) in terms of decoupling of Jacobi equations along generic normal extremals, or, in more geometric but equivalent language, in terms of product structure for the corresponding Jacobi curves, see section 3.1.1 for the definition of Jacobi curves and their relations to Jacobi fields.

In section 3.2 we remove the assumption of affine non-rigidity on the sub-Riemannian manifold, and examine what combinations of three necessary conditions for affine non-rigidity listed above are in fact sufficient for a sub-Riemannian metric to admit a product structure. Our main conjecture in this regard is the following

Conjecture 1.3.1. *If a sub-Riemannian manifold is such that the reduced Jacobi curve of its generic normal extremal can be represented as a direct product of two curves in Lagrangian Grassmannians in symplectic spaces of smaller dimensions then it admits a product structure.*

The Conjecture 1.3.1 is widely open. Alternatively, we perform a case study for sub-Riemannian structures on $(4, 6)$-distributions. Assuming that in addition to the assumptions of Conjecture 1.3.1 the necessary conditions 1 and 2 above hold as well, we were able to prove the conclusion of this conjecture.

In addition, in subsection 3.1.2 we give another necessary condition for affine nonrigidity (Theorem 3.1.3) in terms of the Young diagram of Jacobi curve of generic normal extremal.

2. AFFINE EQUIVALENCE OF SUB-RIEMANNIAN METRICS ON STEP 2 DISTRIBUTIONS WITH TANAKA SYMBOL DECOMPOSABLE INTO TWO AD-SURJECTIVE COMPONENTS

2.1 Direct product structure on the level of Tanaka symbol/nilpotent approximation and the main result

Conjecture 1.2.5 is still widely open. In the present chapter, we prove it for sub-Riemannian metrics on a particular, but still rather abundant class of distributions (see Theorem 2.1.11 below). To motivate and formulate our result properly we need to introduce some terminology and motivation.

In [13] we proved, among other things, a weaker product structure result for affinely non-rigid sub-Riemannian structures , in which the product structure necessarily occurs on the level of Tanaka symbol/ nilpotent approximation of the the sub-Riemannian structure.

To define the Tanaka symbol of the distribution D at a point q we need further assumption D near q, called *equiregularity*. A distribution D is called *equiregular* at a point q if there is a neighborhood U of q in M for each $j > 0$ the dimensions of subspaces $D^j(y)$ are constant for all $y \in U$, where D^j is in (1.3) (equivalently, as in (1.4)). Note that a bracket generic distribution is equiregular at a generic point.

From now on we assume that D is an equiregular bracket-generating distribution with degree of nonholonomy μ. Set

$$\mathfrak{m}_{-1}(q) := D(q), \quad \mathfrak{m}_{-j}(q) := D^j(q)/D^{j-1}(q), \ \forall j > 1$$

and consider the graded space

$$\mathfrak{m}(q) = \bigoplus_{j=-\mu}^{-1} \mathfrak{m}_j(q), \tag{2.1}$$

associated with the filtration (1.2).

The space $\mathfrak{m}(q)$ is endowed with the natural structure of a graded Lie algebra, i.e. with the natural Lie product $[\cdot,\cdot]$ such that

$$[\mathfrak{m}_{i_1}(q), \mathfrak{m}_{i_2}(q)] \subset \mathfrak{m}_{i_1+i_2},$$

defined as follows:

Let $\mathfrak{p}_j : D^j(q) \mapsto \mathfrak{m}^{-j}(q)$ be the canonical projection to a factor space. Take $Y_1 \in \mathfrak{m}^{-i_1}(q)$ and $Y_2 \in \mathfrak{m}^{-i_2}(q)$. To define the Lie bracket $[Y_1, Y_2]$ take a local section $\widetilde{Y}_1$ of the distribution D^{i_1} and a local section $\widetilde{Y}_2$ of the distribution D^{i_2} such that

$$\mathfrak{p}_{i_1}\Big(\widetilde{Y}_1(q)\Big) = Y_1, \quad \mathfrak{p}_{i_2}\Big(\widetilde{Y}_2(q)\Big) = Y_2.$$

It is clear from definitions of the spaces D^j that $[Y_1, Y_2] \in \mathfrak{m}^{i_1+i_2}(q)$. Then set

$$[Y_1, Y_2] := \mathfrak{p}_{i_1+i_2}\Big([\widetilde{Y}_1, \widetilde{Y}_2](q)\Big). \tag{2.2}$$

It can be shown ([16, 18]) that the right-hand side of (2.2) does not depend on the choice of sections $\widetilde{Y}_1$ and $\widetilde{Y}_2$.

Definition 2.1.1. The graded Lie algebra $\mathfrak{m}(q)$ from (2.1) is called the symbol of the distribution D at the point q.

By constructions, it is clear that the Lie algebra $\mathfrak{m}(q)$ is nilpotent. The Tanaka symbol is the infinitesimal version of the so-called *nilpotent approximation* of the distribution D at q, which can be defined as the left-invariant distribution $\widehat{D}$ on the simply connected Lie group with the Lie algebra $\mathfrak{m}(q)$ and the identity e, such that $\widehat{D}(e) = \mathfrak{m}_{-1}(q)$.

Further, since D is bracket generating, its Tanaka symbol $\mathfrak{m}(q)$ at any point is generated by the component $\mathfrak{m}_{-1}(q)$.

Definition 2.1.2. A (nilpotent) $\mathbb{Z}_-$- graded Lie algebra

$$\mathfrak{m} = \bigoplus_{j=-\mu}^{-1} \mathfrak{m}_j \tag{2.3}$$

is called a *fundamental* graded Lie algebra (here $\mathbb{Z}_-$ denotes the set of all negative integers).

The following notion is crucial for the sequel:

Definition 2.1.3. A fundamental graded Lie algebra $\mathfrak{m}$ is called *decomposable* if it can be represented as a direct sum of two nonzero fundamental graded Lie algebras $\mathfrak{m}^1$ and $\mathfrak{m}^2$ and it is called *indecomposable* otherwise . Here the jth component of $\mathfrak{m}$ is the direct sum of the jth components of $\mathfrak{m}^1$ and $\mathfrak{m}^2$.

Obviously, if a distribution D admits product structure then its Tanaka symbol at any point is decomposable.

Example 2.1.4 (contact and even contact distributions). Assume that D is a corank 1 distributions, $\dim D(q) = \dim M - 1$, and assume α is its defining 1-form, i.e. $D = \ker \alpha$.

- Recall that the distribution D is called *contact* if $\dim M$ is odd and the form $d\alpha|_D$ is non-degenerate. In this case the Tanaka symbol at a point q is isomorphic to the Heisenberg algebra $\mathfrak{m}_{-1}(q) \oplus \mathfrak{m}_{-2}(q)$ of dimension equal to $\dim M$, where $\mathfrak{m}_{-2}(q)$ is the (one-dimensional) center and the brackets on $\mathfrak{m}_{-1}(q)$ ($\cong D(q)$) are given by $[X, Y] := d\alpha(X, Y)Z$, where Z is the generator of $\mathfrak{m}_{-1}$ so that $\alpha(Z) = 1$. Note that the Heisenberg algebra is indecomposable as the fundamental graded Lie algebra. Otherwise, since $\dim \mathfrak{m}_{-2}(q) = 1$ one of the components in the nontrivial decomposition of $\mathfrak{m}(q)$ will be commutative and belong to $\mathfrak{m}_{-1}(q)$ and hence to the kernel of $d\alpha|_D$, which contradicts the condition of contactness.

- Recall that the distribution D is called *even contact* if $\dim M$ is even and the form $d\alpha|_D$ has an one-dimensional kernel (i.e. of the minimal possible dimension for a skew-symmetric form on an odd-dimensional vector space). In this case by the arguments similar to the previous item the Tanaka symbol is the direct some of the Heisenberg algebra (of dimension $\dim M - 1$) and $\mathbb{R}$ (the kernel of $d\alpha|_D$), i.e. the Tanaka symbol is decomposable.

Remark 2.1.5. *It is easy to show that the decomposition of a fundamental graded* $\mathfrak{m}$ *Lie algebra into indecomposable fundamental Lie algebra is unique modulo the center of* $\mathfrak{m}$*.*

The following theorem is a consequence of the results proved in [13] and it is a weak version of Conjecture 1.2.5:

Theorem 2.1.6. *[13, a consequence of Theorem 7.1, Proposition 4.7, and Corollary 4.9 there] If a sub-Riemanian metric on an equiregular distribution* D *is not affinely rigid near a point* q_0 *then its Tanaka symbol at* q_0 *is decomposable.*

In other words, the problem of affine equivalence is nontrivial only on the distributions with decomposable Tanaka symbols (at points where the distribution is equiregular).

Now we are almost ready to formulate the main result of the chapter. We restrict ourselves here to step 2 distributions, i.e. when $D^2 = TM$. Such distributions are automatically equiregular (at any point). Then it is clear that the components in the decomposition of the Tanaka symbols of such distribution are of step not greater than 2 (i.e. with $\mu \leq 2$ in (2.3)) . So, they are either of step 2 or commutative.

Definition 2.1.7. We say that a step 2 fundamental graded Lie algebra $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ is ad-*surjective* if there exists $X \in \mathfrak{m}_{-1}$ such that the map $\mathrm{ad}X : \mathfrak{m}_{-1} \to \mathfrak{m}_{-2}$,

$$Y \mapsto [X, Y], \quad Y \in \mathfrak{m}_{-1},$$

is surjective. An element $X \in \mathfrak{m}_{-1}$ for which $\mathrm{ad}X$ is surjective is called an ad-*generating element* of the algebra $\mathfrak{m}$.

Remark 2.1.8. *Note that the direct sum* $\mathfrak{m}^1 \oplus \mathfrak{m}^2$ *of two* ad*-surjective Lie algebras* $\mathfrak{m}^i = \mathfrak{m}^i_{-1} \oplus \mathfrak{m}^i_{-2}$*,* $i = 1, 2$*, is* ad*-surjective. Indeed, if* $X_i \in \mathfrak{m}^i_{-1}$*,* $i = 1, 2$*, are such that* $\mathrm{ad}\, X_i : \mathfrak{m}^i_{-1} \to \mathfrak{m}^i_{-2}$ *is* *surjective, then*

$$\mathrm{ad}(X_1 + X_2) : \mathfrak{m}^1_{-1} \oplus \mathfrak{m}^2_{-1} \to \mathfrak{m}^1_{-2} \oplus \mathfrak{m}^2_{-2}$$

is surjective as well.

The following will be shown in the Appendix A:

Proposition 2.1.9. *Any step* 2 *fundamental graded Lie algebra* $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ *such that the following three condition holds*

1. $\dim\, \mathfrak{m}_{-2} \leq 3$;
2. $\dim\, \mathfrak{m}_{-2} < \dim \mathfrak{m}_{-1}$;
3. *the intersection of* $\mathfrak{m}_{-1}$ *with the center of* $\mathfrak{m}$ *is trivial*

is ad*-surjective.*

Note that if $\dim\, \mathfrak{m}_{-2} \leq 2$ then item (2) of the previous proposition holds automatically.

Corollary 2.1.10. *The only non-*ad*-surjective step* 2 *fundamental graded Lie algebra with* $\mathfrak{m}_{-2} \leq 3$ *is the truncated step* 2 *free Lie algebra with* 3 *generators.*

Note that Proposition 2.1.9 does not hold if one drops item (1), see Appendix A, Example A.0.3 for a counter-example with $\dim \mathfrak{m}_{-2} = 4$ and $\dim \mathfrak{m}_{-1} = 5$. The main result of the present chapter is the following

Theorem 2.1.11. *Assume that* D *is a step* 2 *distribution such that its Tanaka symbol is decomposable into* 2 *nonzero* ad*-surjective indecomposable fundamental graded Lie algebras. If a sub-Riemanian metric* (M, D, g_1) *is not affinely rigid near a point* q_0 *then it is a product of* 2 *sub-Riemannian structures each of which are affinely rigid (in the neighborhood of the projection of* q_0 *to the corresponding manifold).*

Remark 2.1.12. *Note that by Remark 2.1.5, if a step* 2 *fundamental graded Lie algebra* $\mathfrak{m}$ *is decomposable into* 2 *nonzero* ad*-surjective indecomposable fundamental graded Lie algebras, then* *in any other decomposition of* $\mathfrak{m}$ *into indecomposable fundamental graded Lie algebras all components are* ad-*surjective.*

The rest of the chapter is devoted to the proof of Theorem 2.1.11. This theorem confirms the Conjecture 1.2.5 for sub-Riemannian metrics on step 2 distributions with Tanaka symbol being decomposed into k nonzero ad-surjective indecomposable fundamental graded Lie algebras with $k \geq 2$. As a direct consequence of Theorem 2.1.11 and Corollary 2.1.10 we get the following

Corollary 2.1.13. *Assume that* D *is a step 2 distribution such that its Tanaka symbol is decomposed into two nonzero indecomposable fundamental graded Lie algebras with degree* -2 *components of dimension not greater than* 3 *and such that among them there is no truncated step* 2 *free Lie algebra with* 3 *generators. If a sub-Riemanian metric* (M, D, g_1) *is not affinely rigid near a point* q_0 *then it admits a product of two sub-Riemannian structures each of which are affinely rigid (in the neighborhood of the projection of* q_0 *to the corresponding manifold).*

2.2 Orbital equivalence and fundamental algebraic system

In [13], following [17], the problems of projectively and affine equivalence of sub-Riemannian metric were reduced to the study of the orbital equivalence of the corresponding sub-Riemannian Hamiltonian systems (for normal Pontryagin extremals of the energy minimzing problem (1.5)), which in turn is reduced to the study of solvability of a special linear algebraic system with coefficients being polynomial in the fibers, called the *fundamental algebraic system* ([13, Proposition 3.10]). In this section we summarize all information from [13] we need for the proof of Theorem 2.1.11.

As before, fix a connected manifold M and a bracket generating equiregular distribution D on M, and consider two sub-Riemannian metrics g_1 and g_2 on (M, D). We denote by h_1 and h_2 the respective sub-Riemannian Hamiltonians of g_1 and g_2, as defined in (1.8). Let the annihilator $D^\perp$ of D in T^*M is defines as in (1.9).

Definition 2.2.1. We say that $\vec{h}_1$ and $\vec{h}_2$ are *orbitally diffeomorphic* on an open subset V_1 of $T^*M\backslash D^\perp$ if there exists an open subset V_2 of $T^*M\backslash D^\perp$ and a diffeomorphism $\Phi : V_1 \to V_2$ such that Φ is fiber-preserving, i.e. $\pi(\Phi(\lambda)) = \pi(\lambda)$, and Φ sends the integral curves of $\vec{h}_1$ to the reparameterized integral curves of $\vec{h}_2$, i.e., there exists a smooth function $s = s(\lambda, t)$ with $s(\lambda, 0) = 0$ such that $\Phi\Big(e^{t\vec{h}_1}\lambda\Big) = e^{s\vec{h}_2}\Big(\Phi(\lambda)\Big)$ for all $\lambda \in V_1$ and $t \in \mathbb{R}$ for which $e^{t\vec{h}_1}\lambda$ is well defined. Equivalently, there exists a smooth function $\alpha(\lambda)$ such that

$$d\Phi\,\vec{h}_1(\lambda) = \alpha(\lambda)\vec{h}_2(\Phi(\lambda)). \tag{2.4}$$

The map Φ is called an *orbital diffeomorphism* between the extremal flows of g_1 and g_2.

The reduction of projective (respectively, affine) equivalence of sub-Riemannian metrics to the orbital (respectively, special form of orbital) equivalence of the corresponding sub-Riemannian Hamiltonian systems is given by the following:

Proposition 2.2.2. *[13, a combination of Proposition 3.4 and Theorem 2.10 there] Assume that the sub-Riemannian metrics g_1 and g_2 are projectively equivalent in a neighborhood $U \subset M$ and* *let $\pi : T^*M \to M$ be the canonical projection. Then, for generic point $\lambda_1 \in \pi^{-1}(U)\backslash D^{\perp}$, $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic on a neighborhood V_1 of λ_1 in T^*M. Moreover, if g_1 and g_2 are affinely equivalent in a neighborhood $U \subset M$, then the function $\alpha(\lambda)$ in (2.4) satisfies $\vec{h}_1(\alpha) = 0$.*

Further, the differential equation (2.4) can be written ([13, Lemma 3.8]) and transformed to the algebraic system via a sequence of prolongations ([13, Proposition 3.9]) in a special moving frame adapted to the sub-Riemannian structures g_1 and g_2. For this, first we need the following

Definition 2.2.3. The *transition operator* at a point $q \in M$ of the pair of metrics (g_1, g_2) is the linear operator $S_q : D(q) \to D(q)$ such that

$$g_2(q)(v_1, v_2) = g_1(q)(S_q v_1, v_2), \quad \forall v_1, v_2 \in D(q).$$

Obviously S_q is a positive g_1-self-adjoint operator and its eigenvalues $\alpha_1^2(q), \ldots, \alpha_m^2(q)$ are positive real numbers (we choose $\alpha_1(q), \ldots, \alpha_m(q)$ as positive numbers as well). The field S of transition operators is a $(1,1)$-tensor field that will be called the *transition tensor*.

The important simplification in the case of the affine equivalence compared to the projective equivalence is given in the following

Proposition 2.2.4. *[13, Propostion 4.7] If two sub-Riemannian metrics g_1, g_2 on (M, D) are affinely equivalent on an open connected subset $U \subset M$, then all the eigenvalues $\alpha_1^2, \ldots, \alpha_m^2$ of the transition operator are constant.*

This proposition implies that the number of the distinct eigenvalues $k(q)$ of the operators S_q is independent of $q \in U$, i.e. $k(q) \equiv k$ on U for some positive integer q. Also, there is k distributions D_i such that

$$D(q) = \bigoplus_{i=1}^{k} D_i(q) \tag{2.5}$$

is the eigenspace decomposition of $D(q)$ with respect to the eigenspaces of the operator S_q. Now let

$$\mathfrak{m}^i_{-1}(q) = D_i(q), \quad \mathfrak{m}^i_{-j}(q) = (D_i)^j(q)/\left((D_i)^j(q) \cap D^{j-1}(q)\right), \ \forall j > 1^{1} \tag{2.6}$$

Set

$$\mathfrak{m}^i(q) = \bigoplus_{j=1}^{\mu} \mathfrak{m}^i_{-j}(q). \tag{2.7}$$

By construction $\mathfrak{m}^i$, $i = 1, \dots k$, are fundamental graded Lie algebras.

Remark 2.2.5. *Note that in general $\mathfrak{m}^i(q)$ is not equal/isomorphic to the Tanaka symbol of the distribution D_i at q, as when defining the components $m^i_{-j}(q)$ with $j > 1$ we also make the quotient by the powers of D. In fact, the proof that $m^i(q)$ is isomorphic to the Tanaka symbol of the distribution D_i under the assumption of affine nonrigidity is one of the main steps in the proof of Theorem 2.1.11.*

Proposition 2.2.6. *[13, Theorem 6.2 and Theorem 7.1] If sub-Riemannian metrics g_1, g_2 are affinely equivalent and not constantly proportional to each other near a point q_0, then the Tanaka symbol of the distribution D at q_0 is the direct sum of the fundamental graded Lie algebras $\mathfrak{m}^i$, $i = 1, \dots k$, defined by* (2.6) *and* (2.7)

Further, in a neighborhood U_1 of any point $q_0 \in U$ we can choose a g_1-orthonormal local frame $X_1, \dots, X_m$ of D whose values at any $q \in U_1$ diagonalizes S_q, i.e. $X_i(q)$ is an eigenvectors of S_q associated with the eigenvalues $\alpha_i^2(q)$, $i = 1, \dots m$. Note that $\frac{1}{\alpha_1}X_1, \dots, \frac{1}{\alpha_m}X_m$ form a g_2-orthonormal frame of D. We then complete $X_1, \dots, X_m$ into a frame $\{X_1, \dots, X_n\}$ of TM adapted to the distribution D near q_0, i..e such that for every positive integer j this frame contain a a local frame of D^j. We call such a set of vector fields $\{X_1, \dots, X_n\}$ a *(local) frame adapted to the (ordered) pair of metrics* (g_1, g_2). The *structure coefficients* of the frame $\{X_1, \dots, X_n\}$ are the real-valued functions c_{ij}^k, $i, j, k \in \{1, \dots, n\}$, defined near q by

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k. \tag{2.8}$$

[1]Since $(D_i)^j \subset D^j$, the space $\mathfrak{m}^i_{-j}$ is the subspace of $\mathfrak{m}_{-j}$.

Let $u = (u_1, \dots, u_n)$ be the coordinates on the fibers T_q^*M induced by this frame, i.e.

$$u_i(q,p) = p\left(X_i(q)\right) \tag{2.9}$$

or, equivalently, $u_i = H_{X_i}$ in the notation of (1.10). These coordinates in turn induce a basis $\partial_{u_1}, \dots, \partial_{u_n}$ of $T_\lambda(T_q^*M)$ for any $\lambda \in \pi^{-1}(q)$. For $i = 1, \dots, n$, we define the lift Y_i of X_i as the (local) vector field on T^*M such that $\pi_* Y_i = X_i$ and $du_j(Y_i) = 0 \;\; \forall 1 \leq j \leq n$. The family of vector fields $\{Y_1, \dots, Y_n, \partial_{u_1}, \dots, \partial_{u_n}\}$ obtained in this way is called a *frame of* $T(T^*M)$ *adapted at* q_0. By a standard calculation, we obtain the expression for the sub-Riemannian Hamiltonian h_1 of the metric g_1 and the corresponding Hamiltonian vector field $\vec{h}_1$:

$$h_1 = \frac{1}{2}\sum_{i=1}^{m} u_i^2 \tag{2.10}$$

$$\vec{h}_1 = \sum_{i=1}^{m} u_i \vec{u}_i = \sum_{i=1}^{m} u_i Y_i + \sum_{i=1}^{m}\sum_{j,k=1}^{n} c_{ij}^k u_i u_k \partial_{u_j}. \tag{2.11}$$

Indeed, to prove (2.11), recall that given two vector fields X and Z on M and the corresponding Hamiltonians H_X and H_Z as in (1.10) (where X is replaced by Z in the case of Z) we have the following identities

$$\overrightarrow{H_X}(H_Z) = d\,H_Z\left(\overrightarrow{H_X}\right) = H_{[X,Z]}. \tag{2.12}$$

From this and (2.8) it follows immediately that

$$\vec{u_i} = Y_i + \sum_{j=1}^{n} \vec{u_i}(u_j)\partial_{u_j} = Y_i + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ij}^k u_k \partial_{u_j},$$

which immediately implies (2.11).

Assume now that $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic near $\lambda_0 \in H_1 \cap \pi^{-1}(q_0)$ and let Φ be the corresponding orbital diffeomorphism. Let us denote by $\Phi_i,\ i = 1, \dots, n$, the coordinates u_i of Φ on the fiber, i.e. $u \circ \Phi(\lambda) = (\Phi_1(\lambda), \Phi_2(\lambda), \dots, \Phi_n(\lambda))$. Then first it is easy to see [17, Lemma

1] that the function α from (2.4) in (2.4) satisfies

$$\alpha = \sqrt{\frac{\sum_{i=1}^m \alpha_i^2 u_i^2}{\sum_{i=1}^m u_i^2}}$$

and

$$\Phi_k = \frac{\alpha_k^2 u_k}{\alpha}, \forall 1 \leq k \leq m.$$

In [13] in order to find the equations for the rest of the components $\Phi_{m+1}, \dots \Phi_n$ of Φ we first plugged into (2.4) the expression (2.11) for $\vec{h}_1$ and similar expression for $\vec{h}_2$ and then we made the "prolongation" of the resulting differential equation by recursively differentiating it in the direction of $\vec{h}_1$ and replacing the derivatives of Φ_i's in the direction of $\vec{h}_1$ by their expressions from the first step. The resulting system of algebraic equation for $\Phi_{m+1}, \dots \Phi_n$ is summarized in the following

Proposition 2.2.7. [2]*[13, a combination of Propositon 3.4, Proposition 4.3 , Proposition 3.10 applied to the case of affine equivalence] Assume that the sub-Riemannian metrics g_1 and g_2 are projectively equivalent in a neighborhood $U \subset M$ and let Φ be the corresponding orbital diffeomorphism between the normal extremal flows of g_1 and g_2 with coordinates $(\Phi_1, \dots, \Phi_n)$. Set*

$$\widetilde{\Phi} = \alpha(\Phi_{m+1}, \dots, \Phi_n).$$

Let also

$$q_{jk} = \sum_{i=1}^{m} c_{ij}^k u_i \tag{2.13}$$

and

$$R_j = \alpha_j^2 \vec{h}_1(u_j) - \sum_{1 \leq i,k \leq m} c_{ij}^k \alpha_k^2 u_i u_k, \tag{2.14}$$

[2]Since in the present paper we mainly work with the affine equivalence only, for which α_i's are constant and $\vec{h}_1(\alpha) = 0$, the expressions in (2.14) and (2.18) are significantly simpler than in [13], where the more general case of the projective equivalence is considered.

Then $\widetilde{\Phi}$ satisfies a linear system of equations,

$$A\widetilde{\Phi} = b, \tag{2.15}$$

where A is a matrix with $(n-m)$ columns and an infinite number of rows, and b is a column vector with an infinite number of rows. These infinite matrices can be decomposed in layers of m rows each as

$$A = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^s \\ \vdots \end{pmatrix} \qquad \textit{and} \qquad b = \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^s \\ \vdots \end{pmatrix}, \tag{2.16}$$

where the coefficients a^s_{jk} of the $(m \times (n-m))$ matrix A^s, $s \in \mathbb{N}$, are defined by induction as

$$\begin{cases} a^1_{j,k} = q_{jk}, & 1 \le j \le m,\ m < k \le n, \\ a^{s+1}_{j,k} = \vec{h}_1(a^s_{j,k}) + \displaystyle\sum_{l=m+1}^{n} a^s_{j,l} q_{lk}, & 1 \le j \le m,\ m < k \le n, \end{cases} \tag{2.17}$$

(note that the columns of A are numbered from $m+1$ to n according to the indices of $\widetilde{\Phi}$) and the coefficients b^s_j, $1 \le j \le m$, of the vector $b^s \in \mathbb{R}^m$ are defined by

$$\begin{cases} b^1_j = \ R_j, \\ b^{s+1}_j = \ \vec{h}_1(b^s_j) - \displaystyle\sum_{k=m+1}^{n} a^s_{j,k} \sum_{i=1}^{m} u_i \alpha_i^2 q_{ki} \end{cases} \tag{2.18}$$

Definition 2.2.8. The system (2.15) with A and b defined recursively by (2.17) and (2.18) is called the *fundamental algebraic system* for the affine equivalence of the sub-Riemannian metrics g_1 and

g_2 [3] The subsystem

$$A^i\widetilde{\Phi} = b^i,$$

with A^i and b^i as in (2.16) is called the *ith layer of the fundamental algebraic system* (2.15).

The matrix A has $n-m$ columns and infinite many rows and b is the infinite-dimensional column vector. So, the fundamental algebraic system (2.15) is an over-determined linear system on $(\Phi_{m+1},\ldots,\Phi_n)$, and all entries of A and b are polynomials (2.17) and (2.18) in u_j's. Therefore, all $(n-m+1)\times(n-m+1)$ minors of the augmented matrix $[A|b]$ must be equal to zero, and since all of these minors are polynomials in u_j's, the coefficient of every monomial of these polynomials is equal to zero. It results in a huge collection of constraints on the structure coefficient c_{ij}^k. By discovering and analyzing the monomials with the "simplest" coefficients we were able to prove our main Theorem 2.1.11. This analysis is given in Lemmas 2.3.8, 2.3.13, and 2.3.15.

[3]The column vector b in (2.15) here corresponds to αb in the notations of the fundamental algebraic system in [13]. It is more convenient in the context of affine equivalence as all components of b becomes polynomial in u_j's, i.e. the polynomials on the fibers of T^*M.

2.3 Proof of Theorem 2.1.11

Let (M, D, g_1) be a sub-Riemannian metric satisfying the assumptions of Theorem 2.1.11. Assume that g_2 is a sub-Riemannian metric that is affinely and non-constantly proportional to g_1, sub-distributions D_i are defined by (2.5), and the algebras $\mathfrak{m}^i(q)$ are as in (2.6)-(2.7). Then by Proposition 2.2.6 and the assumption that the Tanaka symbol is decomposed into 2-component, the number of distinct eigenvalues of the transition operator is equal to 2, i.e.

$$D = D_1 \oplus D_2.$$

The proof consists of several steps.

2.3.1 Step 1: Distribution D_i^2 is involutive of rank equal to $\dim \mathfrak{m}^i$

Observe that in general

$$D_i(q) \subset D(q) \cap (D_i)^2(q). \tag{2.19}$$

The main goal of this section is to prove the following two propositions:

Proposition 2.3.1. *The following identity holds*

$$D(q) \cap (D_i)^2(q) = D_i(q) \tag{2.20}$$

Proposition 2.3.2. *The following identity holds*

$$(D_i)^3(q) = (D_i)^2(q).$$

As a direct consequence of the previous two propositions and the assumption that D is a step 2 distribution one gets the following

Corollary 2.3.3. *The Tanaka symbol of D_i at q is isomorphic to $\mathfrak{m}^i(q)$ and the distribution D_i^2 is of rank equal to $\dim \mathfrak{m}^i$.*

Proof of Proposition 2.3.1 The proof of Proposition consists of several Lemmas.

First, one can define the canonical projection of quotient spaces

$$\mathrm{pr}_i : (D_i)^2(q)/D_i(q) \to (D_i)^2(q)/\left(D(q) \cap (D_i)^2(q)\right). \tag{2.21}$$

Further, given $X \in D_i(q)$ we can define two different operators

$$(\mathrm{ad}\, X)_{\mathrm{mod}\, D} : D(q) \to D^2(q)/D(q),$$
$$(\mathrm{ad}\, X)_{\mathrm{mod}\, D_i} : D_i(q) \to (D_i)^2(q)/D_i(q).$$

where in the first case we apply the Lie brackets with X as in the Tanaka symbol of the distribution D at q and in the second case we apply the Lie brackets with X as in the Tanaka symbol of the distribution D_i at q.

Proposition 2.3.4. *Assume that $X \in D_i(q)$ is such that the restriction of the map $(\mathrm{ad}\, X)_{\mathrm{mod}\, D}$ to $D_i(q)$ is onto $(D_i)^2(q)/\left(D(q) \cap (D_i)^2(q)\right)$. Then the projection pr_i as in (2.21) defines the bijection between the image of the map $(\mathrm{ad}\, X)_{\mathrm{mod}\, D_i}$ and the image of the restriction of the map $(\mathrm{ad}\, X)_{\mathrm{mod}\, D}$ to $D_i(q)$.*

Proposition 2.3.5. *Assume that $X \in D_i(q)$ satisfies the assumption of the previous lemma. Then $(D_i)^2(q)/D_i(q)$ coincides with the image of the map $(\mathrm{ad}\, X)_{\mathrm{mod}\, D_i}$.*

Before proving Propositions 2.3.4 and 2.3.5 let us that they imply identity (2.20) of Proposition 2.3.1. Indeed, we have the following chain of inequalities/ equalities:

$$\dim (D_i)^2(q)/D_i(q) \overset{(2.19)}{\leq} \dim (D_i)^2(q)/\left(D(q) \cap (D_i)^2(q)\right) \overset{\text{Prop. 2.3.5}}{=}$$
$$\mathrm{rank}\left((\mathrm{ad} X)_{\mathrm{mod} D_i}\right) \overset{\text{Prop. 2.3.4}}{=} \mathrm{rank}\left((\mathrm{ad} X)_{\mathrm{mod} D}|_{D_i(q)}\right) \leq \dim (D_i)^2(q)/D_i(q)$$

Hence, $\dim (D_i)^2(q)/D_i(q) = \dim (D_i)^2(q)/\left(D(q) \cap (D_i)^2(q)\right)$, which implies (2.20).

2.3.2 Proof of Proposition 2.3.4

Assume that $\mathfrak{m}^i(q)$, $i = 1, 2$ be as in (2.6)-(2.7). Note that by the paragraph after Proposition 2.2.4 and the fact that the graded algebras $\mathfrak{m}^i(q)$ are of step not greater than 2, $\dim \mathfrak{m}^i_j(q)$ are independent of q. Let

$$\begin{aligned} &m_i = \dim \mathfrak{m}^i_{-1}(q), \quad d_i = \dim \mathfrak{m}^i_{-2}(q) \\ &\mathcal{I}^1_1 = [1 : m_1], \quad \mathcal{I}^2_1 = [(m+1) : (m+d_1)] \\ &\mathcal{I}^1_2 = [(m_1+1) : m], \quad \mathcal{I}^2_2 = [(m+d_1+1) : n] \end{aligned}$$

Since $\mathfrak{m}^i$ is ad-surjective for $i \in \{1, 2\}$, we can choose a local g_1-orthogonal basis $(X_1, \ldots, X_m)$ of D such that the following conditions hold

1. $D_i = \mathrm{span}\{X_j\}_{j \in \mathcal{I}^1_i}$;
2. $X_1(q)$, and $X_{m_1+1}(q)$ are an ad-generating (in a sense of Definition 2.1.7) elements of $\mathfrak{m}^1(q)$ and $\mathfrak{m}^2(q)$, respectively.

Then one can complete $(X_1, \ldots, X_m)$ to the local frame $(X_1, \ldots, X_n)$ of TM by setting

$$\begin{aligned} &X_{m+1} := [X_1, X_2], X_{m+2} := [X_1, X_3], \ldots, X_{m+d_1} := [X_1, X_{d_1+1}] = \\ &X_{m+d_1+1} := [X_{m_1+1}, X_{m_1+2}], X_{m+d_1+2} := [X_{m_1+1}, X_{m_1+3}] \ldots, X_n := [X_{m_1+1}, X_{m_1+d_2+1}]. \end{aligned}$$

A local frame $(X_1, \ldots, X_n)$ of TM constructed in this way will be called *quasi-normal frame adapted to* X_1.

By construction, quasi-normality implies the following conditions for the structure function of the frame:

$$\begin{aligned} &c^k_{1,j} = \delta_{k,m+j-1}, \quad \forall j \in [: d_1+1], k \in [1 : n], \\ &c^k_{m_1+1,j} = \delta_{k,m+d_1-m_1+j-1}, \quad \forall j \in [m_1+2 : m_1+d_2+1], k \in [1 : n], \end{aligned} \tag{2.22}$$

where $\delta_{s,t}$ stands for the Kronecker symbol.

In the sequel, we will work with a quasi-normal frame. The statement of Proposition 2.3.4 is true if one shows that pr_i restricted to $\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_i}$ is injective while the surjectivity follows automatically from the definition of the projection. Without loss of generality we can assume that $i = 1$, as the proof for $i = 2$ is completely analogous. The injectivity of $\mathrm{pr}_1|_{\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_1}}$ is equivalent to

$$\ker(\mathrm{pr}_1|_{\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_1}}) = 0. \tag{2.23}$$

Clearly,

$$\ker(\mathrm{pr}_1) = \left((D_1)^2(q) \cap D_2(q)\right)/D_1(q). \tag{2.24}$$

We may assume that $X = X_1$, where X_1 is the first element of a quasi-normal frame. Then by (2.24)

$$\ker(\mathrm{pr}_1|_{\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_1}}) = \left(\left((D_1)^2(q) \cap D_2(q)\right)/D_1(q)\right) \cap \left(\mathrm{Im}(\mathrm{ad}X)_{\mathrm{mod}D_1}\right).$$

So, the desired relation (2.23) is equivalent to

$$c_{1l}^k = 0, \text{ for } l \in \mathcal{I}_1^1, k \in \mathcal{I}_2^1. \tag{2.25}$$

To begin with, we will give more explicit expressions for the vector b in the fundamental algebraic system (2.15), which will be helpful in the sequel:

Lemma 2.3.6. *The entries b_j^1 in (2.18) with $j \in [1 : m_1]$, are given by*

$$b_j^1 = \alpha_1^2 \sum_{k=m+1}^{m+d_1} q_{jk} u_k + \left(\alpha_1^2 - \alpha_{m_1+1}^2\right) \sum_{k=m_1+1}^{m} q_{jk} u_k, \tag{2.26}$$

where q_{jk} are defined by (2.13). While entries b_j^1 with $j \in [m_1 + 1 : m]$ are given by

$$b_j^1 = \alpha_{m_1+1}^2 \sum_{k=m+d_1+1}^{n} q_{jk} u_k + \left(\alpha_{m_1+1}^2 - \alpha_1^2\right) \sum_{k=1}^{m_1} q_{jk} u_k. \tag{2.27}$$

Proof. Let us prove the equality (2.26), while (2.27) will follow by the symmetry of indices. Using (2.11), (2.12), and (2.8), we get

$$\vec{h}_1(u_j) = \sum_{i=1}^{m} u_i \vec{u}_i(u_j) = \sum_{k=1}^{n} \sum_{i=1}^{m} u_i u_k c_{ij}^k = \sum_{k=1}^{n} q_{jk} u_k. \tag{2.28}$$

By the definition of b_j^1, recalling that α_1^2 and $\alpha_{m_1+1}^2$ are the only distinct values of the transition operator, we have

$$\begin{aligned}
b_j^1 &= \alpha_1^2 \vec{h}_1(u_j) - \sum_{1 \le i,k \le m} \alpha_k^2 u_i u_k c_{ij}^k \\
&= \alpha_1^2 \sum_{k=1}^{n} q_{jk} u_k - \sum_{k=1}^{m} \sum_{i=1}^{m} \alpha_k^2 u_i u_k c_{ij}^k \\
&= \alpha_1^2 \left(\sum_{k=1}^{m_1} q_{jk} u_k + \sum_{k=m_1+1}^{n} q_{jk} u_k \right) - \sum_{k=1}^{m} \alpha_k^2 q_{jk} u_k \\
&= \alpha_1^2 \left(\sum_{k=1}^{m_1} q_{jk} u_k + \sum_{k=m_1+1}^{n} q_{jk} u_k \right) - \alpha_1^2 \sum_{k=1}^{m_1} q_{jk} u_k - \alpha_{m_1+1}^2 \sum_{k=m_1+1}^{m} q_{jk} u_k \\
&= \alpha_1^2 \sum_{k=m+1}^{n} q_{jk} u_k + \left(\alpha_1^2 - \alpha_{m_1+1}^2 \right) \sum_{k=m_1+1}^{m} q_{jk} u_k \\
&= \alpha_1^2 \sum_{k=m+1}^{m+d_1} q_{jk} u_k + \left(\alpha_1^2 - \alpha_{m_1+1}^2 \right) \sum_{k=m_1+1}^{m} q_{jk} u_k.
\end{aligned}$$

The last equality holds due to $q_{jk} = 0$ for $k \in [m + d_1 + 1 : n]$, and thus it completes the proof of the lemma. □

In case of affine equivalence, the eigenvalues α_1 and α_{m_1+1} are constant, and this implies important condition on the structure coefficients.

Remark 2.3.7. *Note that we can assume that* $d_1 > 0$ *as from* $d_1 = 0$ *and indecomposability of* $\mathfrak{m}^1$ *we have* $m_1 = 1$ *and relation* (2.25) *is trivial in this case.*

Now we present a long analysis of coefficients of specific monomials in specific $(n-m+1) \times (n-m+1)$ minor of the the augmented matrix $[A|b]$.

Further, to achieve (2.25), one may consider a submatrix M_1 consisting of rows with indices

$$[1 : d_1 + 1] \cup [m_1 + 1 : m_1 + d_2] \tag{2.29}$$

from the first layer of the fundamental algebraic system (2.16). By assumption on the decomposition of the Tanaka symbol, M_1 is a block-diagonal matrix,

$$M_1 = \begin{pmatrix} M_{1,1} & 0 \\ 0 & M_{1,2} \end{pmatrix},$$

where $M_{1,1}$ is of the size $(d_1 + 1) \times d_1$ and $M_{1,2}$ is of the size $d_2 \times d_2$. Let $b^{1,1}$ be the subvector of b^1 consisting of the same rows as in M_1, i.e., the rows of b from the set (2.29). Since the fundamental algebraic system is an overdetermined linear system admitting a solution, the determinant $\det(M_1|b^{1,1})$ must vanish, as a polynomial with respect to u_i's. It implies that the coefficients of each monomial w.r.t u_i's in $\det(M_1|b^{1,1})$ must vanish as well. We have the following

Lemma 2.3.8.

1. *The coefficient of the monomial*

$$u_1^{d_1} u_l u_{m_1+1}^{d_2} u_{m_1+d_2+1}, \quad l \in \mathcal{I}_1^1 \backslash \{1\} \tag{2.30}$$

in $\det(M_1|b^{1,1})$ *is , up to a sign, equal to* $\left(\alpha_1^2 - \alpha_{m_1+1}^2\right) c_{1,l}^{m_1+1}$. *Hence*

$$c_{1,l}^{m_1+1} = 0, \quad l \in \mathcal{I}_1^1. \tag{2.31}$$

2. *Assuming that* (2.31) *holds, the coefficient of the monomial*

$$u_1^{d_1} u_l u_{m_1+1}^{d_2-1} u_k u_{m_1+d_2+1} \quad l \in \mathcal{I}_1^1, k \in \mathcal{I}_2^1 \backslash \{m_1 + 1\} \tag{2.32}$$

in $\det(M_1|b^{1,1})$ *is* $\left(\alpha_1^2-\alpha_{m_1+1}^2\right)c_{1,l}^k$*. Hence*

$$c_{1,l}^k=0,\quad l\in\mathcal{I}_1^1,k\in\mathcal{I}_2^1\backslash\{m_1+1\}. \tag{2.33}$$

Proof. Let us take into account only those u_i's that appear in (2.32) (or, equivalently, set all other u_i's equal to zero). Then, using (2.13), the first line of (2.17), and (2.22) one can write each block of M_1 as

$$M_{1,1}=\begin{pmatrix} -c_{1l}^{m+1}u_l & -c_{1l}^{m+2}u_l & \dots & \dots & \dots & -c_{1l}^{m+d_1}u_l \\ u_1-c_{2l}^{m+1}u_l & -c_{2l}^{m+2}u_l & \dots & \dots & & -c_{2l}^{m+d_1}u_l \\ -c_{3l}^{m+1}u_l & u_1-c_{3l}^{m+2}u_l & \dots & \dots & & \vdots \\ -c_{4l}^{m+1}u_l & \dots & u_1-c_{4l}^{m+3}u_l & \dots & & \vdots \\ \vdots & & \vdots & & \vdots & \vdots \\ -c_{d_1+1l}^{m+1}u_l & \dots & \dots & \dots & \dots & u_1-c_{d_1+1,l}^{m+d_1}u_l \end{pmatrix} \tag{2.34}$$

$$M_{1,2}=$$

$$\begin{pmatrix} -c_{m_1+1,\rho}^{m+d_1+1}u_\rho-c_{m_1+1,k}^{m+d_1+1}u_k & -c_{m_1+1,\rho}^{m+d_1+2}u_\rho-c_{m_1+1,k}^{m+d_1+2}u_k & \dots & \dots & -u_\rho-c_{m_1+1,k}^{n}u_k \\ u_{m_1+1}-c_{m_1+2,\rho}^{m+d_1+1}u_\rho- & -c_{m_1+2,\rho}^{m+d_1+2}u_\rho-c_{m_1+2,k}^{m+d_1+2}u_k & \dots & \dots & -c_{m_1+2,\rho}^{n}u_\rho-c_{m_1+2,k}^{n}u_k \\ -c_{m_1+2,k}^{m+d_1+1}u_k & & & & \\ -c_{m_1+3,\rho}^{m+d_1+1}u_\rho-c_{m_1+3,k}^{m+d_1+1}u_k & u_{m_1+1}-c_{m_1+3,\rho}^{m+d_1+2}u_\rho- & \dots & \dots & \vdots \\ & c_{m_1+3,k}^{m+d_1+2}u_k & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -c_{m_1+d_2,\rho}^{m+d_1+1}u_\rho-c_{m_1+d_2,k}^{m+d_1+1}u_k & \dots & \dots & u_{m_1+1}-c_{m_1+d_2,\rho}^{n-1}u_\rho- & -c_{m_1+d_2,\rho}^{n}u_\rho-c_{m_1+d_2,k}^{n}u_k \\ & & & c_{m_1+d_2,k}^{n-1}u_k & \end{pmatrix} \tag{2.35}$$

where $\rho=m_1+d_2+1$, $M_{1,1}$ has size $(d_1+1)\times d_1$, and $M_{1,2}$ has size $d_2\times d_2$.

In the sequel, we will refer to the classical formula for determinants in terms of permutations of the matrix elements as the *Leibniz formula for determinants*.

From the form of (2.35) the variable u_{m_1+1} appears in the following d_2 columns of the augmented matrix $(M_1|b^{1,1})$ only:

(A1) The d_2-1 columns containing the first d_2-1 columns of the matrix $M_{1,2}$. Moreover, in each of these columns u_{m_1+1} appears exactly in one entry situated right below the diagonal entry of the matrix $M_{1,2}$.

(A2) the last column of $(M_1|b^{1,1})$.

From item (A1) above and the fact that u_{m_1+1} appears in the power not less than $d_2 - 1$ in the monomials (2.30) or (2.32) it follows that in the Leibniz formula for the determinant of the matrix $(M_1|b^{1,1})$ the contribution to the monomials (2.30) or (2.32) $M_{1,2}$ comes only from the following terms:

(B1) Terms containing $\prod_{j=1}^{d_2-1}(M_{1,2})_{j+1\,j}$. Moreover, from the fact that in this case, we use all rows of $M_{2,1}$ except the first one and from the block-diagonal structure of M_1 it follows that the terms giving the desired contribution must contain the entry

$$(M_{1,2})_{1\,d_2} = -u_\rho - c^n_{m_1+1,k}u_k = -u_\rho - c^n_{m_1+1,k}u_k. \tag{2.36}$$

(B2) (possible only if $k \neq m_1 + 1$) Terms containing all factors of the form $(M_{1,2})_{j+1\,j}$, $j \in [1 : d_2 - 1]$ except one. Then these terms also contain a factor from the column $b^{1,1}$ containing the variable u_{m_1+1}.

Further, from the form of (2.34) the variable u_1 appears in the following $d_1 + 1$ columns of the augmented matrix $(M_1|b^{1,1})$ only:

(C1) The d_1 columns containing the first columns of the matrix $M_{1,1}$. Moreover, in each of these columns, u_1 appears exactly in one entry situated right below the diagonal entry of the matrix $M_{1,1}$.

(C2) The last column of of $(M_1|b^{1,1})$.

So, From item (C1) above and the fact that u_1 appears in the power d_1 in the monomials (2.30) or (2.32), it follows that in the Leibntz formula for the determinant of the matrix $(M_1|b^{1,1})$ the contribution to the monomials (2.30) or (2.32) $M_{1,2}$ comes only from the following terms

(D1) Terms containing $\prod_{j=1}^{d_1}(M_{1,1})_{j+1\,j}$. Then from the block-diagonal structure of the matrix M_1 these terms must contain the factor b_1^1 as the only possible factor from the first row of the augmented matrix $(M_1|b^{1,1})$.

(D2) Terms for which one of the entries of the form $(M_{1,1})_{j+1\,j}$, $j \in [1 : d_1]$, is excluded and replaced by the term from the last column $b^{1,1}$ containing u_1.

Now consider four possible cases separately:

(E1) Assume that (B2) and (D2) occur simultaneously. Let us show that such term does not contribute to the monomials (2.30) or (2.32). Indeed, assuming the converse, the participating factor from $b^{1,1}$ must contain the monomial $u_1 u_{m_1+1}$. Let us analyze b^1. As before, for our purpose, we set variables u_i not appearing in (2.32) to 0. In this case, since variables u_k with $k > m$ do not appear in (2.32) the first sum in (2.26) is equal to zero, so the jth entry b_j^1 of b^1 has the following form:

$$b_j^1 = \begin{cases} (\alpha_1^2 - \alpha_{m_1+1}^2) \displaystyle\sum_{k=m_1+1}^{m} q_{jk} u_k, & j \in [1 : m_1] \\ (\alpha_{m_1+1}^2 - \alpha_1^2) \displaystyle\sum_{k=1}^{m_1} q_{jk} u_k, & j \in [m_1 + 1 : m]. \end{cases} \tag{2.37}$$

For the participating factor b_j^1 from $b^{1,1}$ we have that either $j \in [1 : d_1+1]$ or $j \in [m_1+1 : m_1+d_2]$. In the former case, using (2.13) and the first line of (2.37), the coefficient near $u_1 u_{m_1+1}$ in $b_j^1$is equal to $(\alpha_1^2 - \alpha_{m_1+1}^2) c_{1j}^{m_1+1}$, while in the latter case, using (2.13) and the second line of (2.37), the coefficient near $u_1 u_{m_1+1}$ in $b_j^1$is equal to $(\alpha_{m_1+1}^2 - \alpha_1^2) c_{m_1+1,j}^1$. In both cases, this coefficient vanishes by normalization condition (2.22).

(E2) Assume that (B1) and (D2) occur simultaneously. Similarly to the previous case ,let us show that such term does not contribute to the monomials (2.30) or (2.32). Indeed, assuming the converse and using (2.36), the participating factor b_j^1 from $b^{1,1}$ must contain either the monomial $u_1 u_k$ or $u_1 u_\rho$. Moreover in the considered case $j \in [1 : d_1 + 1]$, therefore by the firs line of the coefficients near $u_1 u_k$ and $u_1 u_\rho$ in b_j^1 are equal to $(\alpha_1^2 - \alpha_{m_1+1}^2) c_{1j}^k$ and $(\alpha_1^2 - \alpha_{m_1+1}^2) c_{1j}^\rho$, respectively, so it vanishes by normalization condition (2.22).

(E3) Assume that (B2) and (D1) occur simultaneously. In this case the contribution to the monomials (2.30) or (2.32) is from the coefficient of the monomial $u_l u_{m_1+1}$ in the factor b_1^1, which is equal to $(\alpha_{m_1+1}^2 - \alpha_1^2)c_{1l}^{m_1+1}$.

(E4). Assume that (B1) and (D1) occur simultaneously. In this case the contribution to the monomials (2.30) or (2.32) is from the coefficient of the monomial $u_l u_k$ in the factor b_1^1, which is equal to $(\alpha_{m_1+1}^2 - \alpha_1^2)c_{1l}^{k}$.

Now assume first that $k = m_1 + 1$. Then the case (B2) and therefore (E3) is impossible. So from (E4) the coefficient of the monomial (2.30) in $\det(M_1|b^{1,1})$ is, up to the sign, equal to the coefficient of the monomial $u_l u_{m_1+1}$ in b_1^1, which is equal to $(\alpha_{m_1+1}^2 - \alpha_1^2)c_{1l}^{m_1+1}$. Consequently, (2.31). This proves the part 1 of Lemma 2.3.8.

For part (2), since (2.31) is assumed, the case (E3) does not contribute to the monomial (2.32), so the only contribution is from the case (E4) and based on the conclusion of this case we get part 2 of the lemma. □

By symmetric arguments for the second component instead of the first one we get the following corollary of Lemma 2.3.8:

Corollary 2.3.9. *The following identities hold*

$$c_{m_1+1,l}^{k} = 0, \quad l \in \mathcal{I}_2^1, k \in \mathcal{I}_1^1.$$

Now assume that X_1 is a local section of D_1 such that $X_1(q)$ is an ad-generating element of $\mathfrak{m}^1(q)$ in the sense of Definition (2.1.7) for every q. Let

$$K := \ker\left((\mathrm{ad}\, X_1)_{\mathrm{mod} D_1}\right) \tag{2.38}$$

$$H := (\mathrm{span}\{X_1\})^{\perp} \cap D_1, \tag{2.39}$$

where $^{\perp}$ stands for the g_1-orthogonal complement. As a direct consequence of Lemmas 2.3.8, we get the following

Corollary 2.3.10. *For any d_1-dimensional subspace F of H with*

$$F \cap K = 0 \tag{2.40}$$

the image of the restriction of the map $(\operatorname{ad} X_1)_{\operatorname{mod} D_1}$ to the subspace F coincides with the entire image of the map $(\operatorname{ad} X_1)_{\operatorname{mod} D_1}$,

$$\operatorname{Im}\Big(\big(\operatorname{ad} X_1)_{\operatorname{mod} D_1}\big)|_F\Big) = \operatorname{Im}\big((\operatorname{ad} X_1)_{\operatorname{mod} D_1}\big)$$

In particular, this image of the restriction is independent of F.

Indeed, previously we used $\operatorname{span}\{X_2, \dots, X_{d_1+1}\}$ as a subspace F but relation (2.40) was the only property we actually used to get the conclusion of Lemma 2.3.8.

2.3.3 Proof of Proposition 2.3.5

The statement of Proposition 2.3.5 is equivalent to showing

$$D_1^2(q) \cap D_2(q) = \{0\},$$
$$D_2^2(q) \cap D_1(q) = \{0\}.$$

Due to the symmetry of indices, we only need to verify the first equality that $D_1^2(q) \cap D_2(q) = \{0\}$, which is equivalent to show

$$c_{i,j}^k = 0, \text{ for } i, j \in \mathcal{I}_1^1,\ k \in \mathcal{I}_2^1. \tag{2.41}$$

Remark 2.3.11. *Note that (2.41) trivially holds for $d_1 = 0$. Further, if $d_1 = 1$, then from indecomposability assumption $\mathfrak{m}^1$ is isomorphic to the Heisenberg algebra and one can find a quasi-normal frame $(X_1, \dots, X_n)$ such that $[X_1, X_i] \notin D$ for every $i \in [2 : m_1]$. Then by Lemma 2.3.8 we have*

$$\operatorname{Im}\big((\operatorname{ad} X_1)_{\operatorname{mod} D_1}\big) = \operatorname{Im}\big((\operatorname{ad} X_i)_{\operatorname{mod} D_1}\big), \quad i \in [2 : m_1]$$

which in turn, by the same lemma, implies (2.41).

In the sequel we need the following

Proposition 2.3.12 ([13], Proposition 4.1 and Proposition 4.2)**.** *If g_1 and g_2 are affinely equivalent but not constantly proportional to each other, then the following properties hold:*

1. $c_{ij}^{j} = c_{ji}^{i} = 0$, *for any* $i \in \mathcal{I}_1^1$, $j \in \mathcal{I}_2^1$;
2. $c_{jk}^{i} = -c_{ji}^{k}$, *for any* $i, k \in \mathcal{I}_1^1$, $j \in \mathcal{I}_2^1$;
3. $c_{jk}^{i} = -c_{ji}^{k}$, *for any* $i, k \in \mathcal{I}_2^1$, $j \in \mathcal{I}_1^1$.

To show (2.41), we first construct a submatrix M_2 of A with row indices

$$[1 : d_1] \cup [m_1 + 1 : m_1 + d_2].$$

Then M_2 has the following form:

$$M_2 = \left(\begin{array}{cc} M_{2,1} & 0 \\ 0 & M_{2,2} \\ \hline \multicolumn{2}{c}{a_1^2} \end{array} \right),$$

where $M_{2,1}$ is of size $d_1 \times d_1$, and $M_{2,2}$ is of size $d_2 \times d_2$, and a_1^2 is the 1st row of the matrix A^2 from the second layer of the fundamental algebraic system. Denote by $b^{1,2}$ is the sub-vector of b with the same row indices as M_2.

Lemma 2.3.13. *Assume that* (2.31)*,* (2.33)*, and Proposition 2.3.12 hold. Then the coefficient of the monomial*

$$u_1^{d_1-1} u_j u_{d_1+1} u_{m_1+1}^{d_2} u_{m_1+d_2+1} u_{m+l}, \quad j \in [2 : d_1 + 1], l \in [1 : d_1] \tag{2.42}$$

in $\det([M_2|b^{1,2}])$ *is given by*

$$\left(\alpha_1^2 - \alpha_{m_1+1}^2\right) \sum_{s=1}^{d_1+1} \operatorname{sgn}(\sigma_s) c_{js}^{m_1+1} \sum_{i=2}^{m_1} c_{1i}^{m+l} c_{1i}^{m+s-1}, \tag{2.43}$$

where $\mathrm{sgn}(\sigma_s)$ *in* (2.43) *is either* 1 *or* -1*. Further, if all coefficients in* (2.43) *vanish, then*

$$c_{js}^{m_1+1} = 0, \quad j \in [2 : d_1 + 1], \text{ and } s \in [1 : d_1 + 1]. \tag{2.44}$$

Proof. For simplicity, we only consider u_i's involved in (2.42) plugging zero to all other variables. As column operations would not affect the determinant of a matrix, we perform them on $[M_{2,1}|b^{1,2}]$ such that a new column $\tilde{b}^{1,2}$ is obtained by

$$\tilde{b}^{1,2} = b^{1,2} - \alpha_1^2 \sum_{j=1}^{d_1} (M_2)_j u_{m+j} - \alpha_{m_1+1}^2 \sum_{j=d_1+1}^{d_1+d_2} (M_2)_j u_{m+j}, \tag{2.45}$$

where $(M_2)_j$ represents the jth column of M_2. In this way, using the first line of (2.17), one gets that the first term in (2.26) and (2.27) is canceled by the column operations, namely

$$\tilde{b}_i^{1,2} = \begin{cases} (\alpha_1^2 - \alpha_{m_1+1}^2) \displaystyle\sum_{k=m_1+1}^{m} q_{ik} u_k, & i \in [1 : d_1] \\ (\alpha_{m_1+1}^2 - \alpha_1^2) \displaystyle\sum_{k=1}^{m_1} q_{i-d_1+m_1,k} u_k, & i \in [d_1 + 1 : d_1 + d_2], \end{cases} \tag{2.46}$$

where $\tilde{b}_i^{1,2}$ is the ith component (from the top) of the column vector $\tilde{b}^{1,2}$. Hence, $\tilde{b}^{1,2}$ has no term of u_i's, with $i \in \mathcal{I}_1^2 \cup \mathcal{I}_2^2$.

Using (2.31) and normalization conditions (2.22), we have

$$M_{2,1} = \begin{pmatrix} 0 & \dots & 0 & -u_j & 0 & \dots & -u_{d_1+1} \\ [u_1] & \dots & \dots & \dots & \dots & \dots & c_{j,2}^{m+d_1} u_j - c_{2,d_1+1}^{m+d_1} u_{d_1+1} \\ \vdots & [u_1] & \dots & \dots & \dots & \dots & c_{j,3}^{m+d_1} u_j - c_{3,d_1+1}^{m+d_1} u_{d_1+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & & [u_1] & c_{j,d_1}^{m+d_1} u_j - c_{d_1,d_1+1}^{m+d_1} u_{d_1+1} \end{pmatrix}, \tag{2.47}$$

where $[u_1]$ is a linear form in $u_1, \dots, u_n$ with u_1's coefficient equal to 1 and the entry $-u_j$ in the first row appears in the $(j-1)$st column. The normalization conditions (2.22) are responsible for the form of the first row in (2.47). We will show that the omitted terms (in all columns except the last one) do not contribute to the coefficient of the monomial (2.42). Note that $M_{2,2}$ is identical to $M_{2,1}$ and has similar form as (2.47).

The following observation are helpful in our calculations:

(F1) $M_{2,1}$ does not depend on u_{m_1+1} and $M_{2,2}$ does not depend on u_1 due to our assumption on the decomposition of the Tanaka symbol,

(F2) The variable u_1 in $M_{2,1}$ appears in the sub-diagonal only. In the same way, the variable u_{m_1+1} in $M_{2,2}$ appears in the sub-diagonal only. These conclusions comes from normalization conditions (2.22).

(F3) The first d_1 components (from the top) of the column vector $\tilde{b}^{1,2}$ do not contain u_1 and the last d_2 components of the column vector $\tilde{b}^{1,2}$ do not contain u_{m_1+1}. This follows from applying the normalization conditions (2.22) to the first line of (2.46).

Since the only non-zero entries in the first row of the augmented matrix $[M_2|b^{1,2}]$ are the ones in the entries $(1, j-1)$, $(1, d_1)$, and $(1, d_1+d_2+1)$, the *cofactor expansion of the determinant with respect to the first row consists of three terms*.

1. The term of the cofactor expansion of the determinant with respect to the first row containing the $(1, j-1)$**st entry.** This term automatically contains the factor u_j. The removal of $(j-1)$st column implies the removal of one u_1 from the sub-diagonal of $M_{2,1}$, so only $d_1 - 2$ u_1's are in our disposal from this sub-diagonal. Thus, in order to get the factor $u_1^{d_1-1}u_{m_1+1}^{d_2}$ in our monomial and from items (F1)-(F3) it follows that we have to use all remaining u_1 from this sub-diagonal together with another u_1 taken from one of the following three entries of the matrix ($[M_2|b^{1,2}]$)

(G1) The entry in the last column and (d_1+1)st row, i.e. $\tilde{b}^{1,2}_{d_1+1}$;

(G2) The entry in the last column and the last row, i.e. $\tilde{b}^{1,2}_{d_1+d_2+1}$;

(G3) The entry in the last row and d_1st column, i.e. $a^2_{1,m+d_1}$ as in the second row of (2.17).

Let us consider cases (G)-(G2) separately:

(H1) The entry in (G1) does not contain variable u_1. Indeed, by second line of (2.46) and (2.13), we have

$$\frac{1}{\alpha^2_{m_1+1}-\alpha^2_1}\tilde{b}^{1,2}_{d_1+1} = \sum_{k=1}^{m_1} q_{m_1+1,k}u_k = \sum_{k=1}^{m_1}\sum_{i=1}^{m} c^k_{i,m_1+1}u_iu_k = \sum_{k=1}^{m_1}\sum_{i=m_1+1}^{m} c^k_{i,m_1+1}u_iu_k + \sum_{1\leq i<k\leq m_1}(c^k_{i,m_1+1}+c^i_{k,m_1+1})u_iu_k + \sum_{i=1}^{m_1} c^i_{i,m_1+1}u_i^2 = 0. \tag{2.48}$$

Indeed, by Proposition 2.3.12 the second and the third terms of the right hand side of (2.48) vanish, while the first term vanishes by Corollary 2.3.9.

(H2) The entry in (G2) does not contain variable u_1. Indeed, by column operations (2.45)

$$\tilde{b}^{1,2}_{d_1+d_2+1} = b^2_1 - \alpha^2_1\sum_{j=1}^{d_1} a^2_{1,m+j}u_{m+j} - \alpha^2_{m_1+1}\sum_{j=d_1+1}^{d_1+d_2} a^2_{1,m+j}u_{m+j}, \tag{2.49}$$

where by (2.17) and (2.18)

$$a^2_{1,m+j} = \vec{h}_1(q_{1,m+j}) + \sum_{k=m+1}^{n} q_{1,k}q_{k,m+j}, \tag{2.50}$$

$$b^2_1 = \vec{h}_1(b^1_1) - \alpha^2_1\sum_{k=m+1}^{n}\sum_{i=1}^{m_1} q_{1k}q_{ki}u_i - \alpha^2_{m_1+1}\sum_{k=m+1}^{n}\sum_{i=m_1+1}^{m} q_{1k}q_{ki}u_i \tag{2.51}$$

and q_{jk} and b^1_1 is as in (2.13), (2.26) and (2.18), respectively. Note that from the decomposition of the Tanaka symbol it follows that

$$q_{jk} = 0, \quad (j,k) \in (\mathcal{I}^1_1 \times \mathcal{I}^2_2) \times (\mathcal{I}^1_2 \times \mathcal{I}^2_1). \tag{2.52}$$

In particular,

$$q_{1k} = 0, \quad k \in I^2_2. \tag{2.53}$$

Substituting (2.53) into (2.50), we get

$$a_{1,m+j}^{2}=\begin{cases}\vec{h}_{1}(q_{1,m+j})+\sum\limits_{k=m+1}^{m+d_1} q_{1k}q_{km+j}, & j\in[1:d_1]\\ \sum\limits_{k=m+1}^{m+d_1} q_{1k}q_{km+j}, & j\in[d_1+1:d_2].\end{cases}\tag{2.54}$$

By (2.49), (2.51), and (2.54), we have

$$\begin{aligned}\tilde{b}_{d_1+d_2+1}^{1,2}&=b_1^2-\alpha_1^2\sum_{j=1}^{d_1}a_{1,m+j}^2u_{m+j}-\alpha_{m_1+1}^2\sum_{j=d_1+1}^{d_1+d_2}a_{1,m+j}^2u_{m+j}\\&=\vec{h}_1(b_1^1)-\alpha_1^2\sum_{k=m+1}^{m+d_1}\sum_{i=1}^{m_1}q_{1k}q_{ki}u_i-\alpha_{m_1+1}^2\sum_{k=m+1}^{m+d_1}\sum_{i=m_1+1}^{m}q_{1k}q_{ki}u_i\\&\quad-\alpha_1^2\sum_{j=1}^{d_1}a_{1,m+j}^2u_{m+j}-\alpha_{m_1+1}^2\sum_{j=d_1+1}^{d_1+d_2}a_{1,m+j}^2u_{m+j}\end{aligned}\tag{2.55}$$

Using (2.26) and (2.28), we get

$$\begin{aligned}\vec{h}(b_1^1)&=\vec{h}\left(\alpha_1^2\sum_{k=m+1}^{m+d_1}q_{1k}u_k+\left(\alpha_1^2-\alpha_{m_1+1}^2\right)\sum_{k=m_1+1}^{m}q_{ik}u_k\right)\\&=\alpha_1^2\sum_{k=m+1}^{m+d_1}\vec{h}(q_{1k})u_k+\alpha_1^2\sum_{k=m+1}^{m+d_1}q_{1k}\vec{h}(u_k)+\\&\quad\left(\alpha_1^2-\alpha_{m_1+1}^2\right)\sum_{k=m_1+1}^{m}\vec{h}(q_{ik})u_k+\left(\alpha_1^2-\alpha_{m_1+1}^2\right)\sum_{k=m_1+1}^{m}q_{ik}\vec{h}(u_k)=\\&=\alpha_1^2\sum_{k=m+1}^{m+d_1}\vec{h}(q_{1k})u_k+\alpha_1^2\sum_{k=m+1}^{m+d_1}\sum_{s=1}^{n}q_{1k}q_{ks}u_s+\\&\quad\left(\alpha_1^2-\alpha_{m_1+1}^2\right)\sum_{k=m_1+1}^{m}\vec{h}(q_{ik})u_k+\left(\alpha_1^2-\alpha_{m_1+1}^2\right)\sum_{k=m_1+1}^{m}\sum_{s=1}^{n}q_{1k}q_{ks}u_s.\end{aligned}\tag{2.56}$$

Substituting (2.50), and (2.56) into (2.55), and taking into account (2.54) again, we get the following cancellations:

$$
\begin{aligned}
\tilde{b}^{1,2}_{d_1+d_2+1} &= \cancel{\alpha_1^2 \sum_{k=m+1}^{m+d_1} \vec{h}(q_{1k})}u_k + \alpha_1^2 \sum_{k=m+1}^{m+d_1} \sum_{s=1}^{m+d_1} q_{1k} q_{ks} u_s + \\
&\left(\alpha_1^2 - \alpha_{m_1+1}^2\right) \sum_{k=m_1+1}^{m} \vec{h}(q_{1k}) u_k + \left(\alpha_1^2 - \alpha_{m_1+1}^2\right) \sum_{k=m_1+1}^{m} \sum_{s=1}^{n} q_{1k} q_{ks} u_s - \\
&\alpha_1^2 \sum_{k=m+1}^{m+d_1} \sum_{i=1}^{m_1} q_{1k} q_{ki} u_i - \alpha_{m_1+1}^2 \sum_{k=m+1}^{m+d_1} \sum_{i=m_1+1}^{m} q_{1k} q_{ki} u_i \\
&\cancel{-\alpha_1^2 \sum_{j=m+1}^{m+d_1} \vec{h}_1(q_{1,j})} u_j - \alpha_1^2 \sum_{j=m+1}^{m+d_1} \sum_{k=m+1}^{m+d_1} q_{1k} q_{kj} u_j - \alpha_{m_1+1}^2 \sum_{j=m+d_1+1}^{n} \sum_{k=m+1}^{m+d_1} q_{1k} q_{kj} u_j.
\end{aligned}
\tag{2.57}
$$

Summing up the terms with factor α_1^2 in (2.57) we get

$$
\alpha_1^2 \Big(\sum_{k=m+1}^{m+d_1} \sum_{s=1}^{m+d_1} q_{1k} q_{ks} u_s - \sum_{k=m+1}^{m+d_1} \sum_{i=1}^{m_1} q_{1k} q_{ki} u_i - \sum_{j=m+1}^{m+d_1} \sum_{k=m+1}^{m+d_1} q_{1k} q_{kj} u_j \Big) = \alpha_1^2 \sum_{k=m+1}^{m+d_1} \sum_{s=m_1+1}^{m} q_{1k} q_{ks} u_s,
\tag{2.58}
$$

while the sum of the terms with factor $\alpha_{m_1+1}^2$ in (2.57) is :

$$
-\alpha_{m_1+1}^2 \left(\sum_{k=m+1}^{m+d_1} \sum_{i=m_1+1}^{m} q_{1k} q_{ki} u_i + \sum_{j=m+d_1+1}^{n} \sum_{k=m+1}^{m+d_1} q_{1k} q_{kj} u_j \right)
\tag{2.59}
$$

After adding and subtracting $\alpha_1^2 \sum_{j=m+d_1+1}^{n} \sum_{k=m+1}^{m+d_1} q_{1k} q_{kj} u_j$, the right hand side of (2.58) can be written as follows

$$
\begin{aligned}
\alpha_1^2 \sum_{k=m+1}^{m+d_1} \sum_{s=m_1+1}^{m} q_{1k} q_{ks} u_s &= \alpha_1^2 \left(\sum_{k=m+1}^{m+d_1} \sum_{i=m_1+1}^{m} q_{1k} q_{ki} u_i + \sum_{j=m+d_1+1}^{n} \sum_{k=m+1}^{m+d_1} q_{1k} q_{kj} u_j \right) - \\
&\alpha_1^2 \sum_{j=m+d_1+1}^{n} \sum_{k=m+1}^{m+d_1} q_{1k} q_{kj} u_j
\end{aligned}
\tag{2.60}
$$

so that the first term of (2.60) has the same second factor as in (2.59).

Combining (2.58) with (2.60) and plugging it together with (2.59) into (2.57) we get

$$\begin{aligned}\tilde{b}^{1,2}_{d_1+d_2+1} = \left(\alpha_1^2 - \alpha_{m_1+1}^2\right)\left(\sum_{k=m_1+1}^{m} \vec{h}_1(q_{1k})u_k + \sum_{k=m+1}^{m+d_1}\sum_{j=m_1+1}^{m} q_{1k}q_{ki}u_j + \sum_{j=m+d_1+1}^{n}\sum_{k=m+1}^{m+d_1} q_{1k}q_{kj}u_j\right) \\ - \alpha_1^2 \sum_{j=m+d_1+1}^{n}\sum_{k=m+1}^{m+d_1} q_{1k}q_{kj}u_j\end{aligned} \tag{2.61}$$

Since $M_{2,1}$ and $M_{2,2}$ do not contain u_{m+l}, the u_1 taken from $\tilde{b}^{1,2}_{d_1+d_2+1}$ must come with u_{m+l}. The crucial observation here is that the variable u_{m+l} appears in the following term of (2.61) only:

$$\left(\alpha_1^2 - \alpha_{m_1+1}^2\right)\left(\sum_{k=m_1+1}^{m} \vec{h}_1(q_{1k})u_k\right) \tag{2.62}$$

Here we strongly use that by (2.13) all q_{jk} depend on u_i's with $i \in [1:m]$ only. Using (2.13) and (2.28) we get

$$\begin{aligned}\vec{h}_1(q_{1k}) &= \sum_{i=1}^{m}\left(c_{i1}^k \vec{h}(u_i) + \vec{h}(c_{i1}^k)u_i\right) \\ &= \sum_{i=1}^{m}\sum_{s=1}^{n} c_{i1}^k q_{is}u_s + \sum_{i=1}^{m}\vec{h}(c_{i1}^k)u_i.\end{aligned} \tag{2.63}$$

The second term in (2.63) does not have u_{m+l}, while to get u_{m+l}, in the first term the index s must be equal to $m+l$. So, the terms containing u_{m+l} in (2.62) are

$$u_{m+l}\sum_{i=1}^{m}\sum_{k=m_1+1}^{m} c_{i1}^k q_{i,m+l}.$$

Recall that

$$\begin{cases} c_{i1}^k = 0, & \text{if } i \in \mathcal{I}_1^1 \; k \in \mathcal{I}_2^1 \\ q_{i,m+l} = 0, & \text{if } i \in I_2^1. \end{cases}$$

Here the first line comes from (2.31) and (2.33) and the second line comes from (2.52). So, u_{m+l} does not appear in $\tilde{b}^{1,2}_{d_1+d_2+1}$ and we got a contradiction , which completes the proof of the claim in (H2). As a biproduct of the calculations and arguments above we get the following corollary which will be used in the sequel:

Corollary 2.3.14. *The entry in the last column and the last row, i.e.* $\tilde{b}^{1,2}_{d_1+d_2+1}$, *does not depend on* u_{m+l} *for* $l \in [1 : d_1]$.

(H3) The analysis of (G3). Since again $M_{2,1}$ and $M_{2,2}$ do not contain u_{m+l}, we need to analyze the coefficient of the monomial $u_1 u_{m+l}$ in $a^2_{1,m+d_1}$. Note that $a^2_{1,m+d_1}$ is given by (2.50) with $j = d_1$. From (2.13) it follows that the second term of (2.50) depends only on u_i's with $i \in [1 : m]$, so it does not contribute to the required monomial. Therefore, we have to find the contribution of the second term, i.e. of $\vec{h}_1(q_{1,m+d_1})$. From (2.13), the decomposition of the Tanaka symbol, and the normalization conditions (2.22) it follows that

$$q_{1,m+d_1} = -u_{d_1+1} - \sum_{i=d_1+2}^{m_1} c^{m+d_1}_{1,i} u_i$$

Then, using (2.28)

$$\begin{aligned} \vec{h}_1(q_{1,m+d_1}) &= -\vec{h}_1(u_{d_1+1} + \sum_{i=d_1+2}^{m_1} c^{m+d_1}_{1,i} u_i) = \\ &- \Big(\sum_{k=1}^{n} q_{d_1+1k} u_k + \sum_{i=d_1+2}^{m_1} \sum_{k=1}^{n} c^{m+d_1}_{1,i} q_{ik} u_k + \sum_{i=d_1+2}^{m_1} \vec{h}_1(c^{m+d_1}_{1,i}) u_i\Big). \end{aligned} \tag{2.64}$$

The last term in (2.64) depends on u_i's with $i \in [1 : m]$ only and does not contribute to the coefficient of the monomial $u_1 u_{m+l}$ in $\vec{h}_1(q_{1,m+d_1})$. As q_{ik} depends on u_i's with $i \in [1 : m]$ only (see (2.13)), in the first two terms of (2.64) only summands with $k = m + l$ contribute to the coefficient of monomial $u_1 u_{m+l}$ in $\vec{h}_1(q_{1,m+d_1})$. So, again by (2.13) this coefficient is equal to

$$-c^{m+l}_{1d_1+1} - \sum_{i=d_1+2}^{m_1} c^{m+l}_{1i} c^{m+d_1}_{1i} = -\sum_{i=1}^{m} c^{m+l}_{1i} c^{m+d_1}_{1i}, \tag{2.65}$$

where the last equality follows from the decomposition of the Tanaka symbol, and the normalization conditions (2.22).

Further, as $(j-1)$st column had been removed by the corresponding cofactor expansion of the determinant, the only possible factor from the jth row of the matrix $[M_2|b^{1,2}]$ participating in the term of the Leibniz decomposition lies in the last column, i.e. it is the (j, d_1+d_2+1)st entry equal to b^1_j. The only factor from the monomial (2.42) that we are interested in this entry is $u_{d_1+1}u_{m_1+1}$ and by (2.26) the coefficient of $u_{d_1+1}u_{m_1+1}$ in b^1_j is equal to

$$-\left(\alpha_1^2-\alpha_{m_1+1}^2\right)c_{j,d_1+1}^{m_1+1}. \tag{2.66}$$

Finally, the contribution of the considered term of the cofactor expansion of $\det[M_2|b^{1,2}]$ with respect to the first row to the coefficient of the monomial (2.42) is the product of (2.65) and (2.66), i.e.

$$-\left(\alpha_1^2-\alpha_{m_1+1}^2\right)c_{j,d_1+1}^{m_1+1}\sum_{i=1}^{m}c_{1i}^{m+l}c_{1i}^{m+d_1}. \tag{2.67}$$

2. The term of the cofactor expansion of the determinant with respect to the first row containing the $(1, d_1)$**st entry.** This term automatically contains the factor u_{d_1+1}. First by (2.46) and Corollary 2.3.14 the column $\tilde{b}^{1,2}$ of $[M_2|b^{1,2}]$ does not contain the variable u_{m+l} Besides by (2.13) neither $M_{2,1}$ nor $M_{2,2}$ contain this variable . Hence we have the folowing fact

(I1) u_{m+l} must be selected in the last row of $[M_2|b^{1,2}]$ except the very last entry of the row, i.e. in a_1^2.

Using property (I1) and properties (F1)-(F3) above the only way to get the factor

$$u_1^{d_1-1}u_{m_1+1}^{d_2}u_{m_1+d_2+1} \tag{2.68}$$

in a term of the Leibniz decompostion of $\det([M_2|b^{1,2}])$ is to choose, among the factors of this term, all d_2 sub-diagonal entries of $M_{2,2}$, the last entry in the first row of $M_{2,2}$, and (d_1-2) many entries from the sub-diagonal of $M_{2,1}$ with one exception. If the exception takes place in the $(s, s-1)$st entry, $s\in[2:d_1]$, then entries (s, d_1+d_2+1)st and (d_1+d_2+1, s)th entries of M_2 must be among factors of the desired term of the Leibniz decompostion of $\det([M_2|b^{1,2}])$. Besides, the product of this entries must contain the factor $u_1 u_j u_{m_1+1} u_{m+l}$, completing the monomial (2.68) to the monomial (2.42). Further, by property (I1) above the variable u_{m+l} must be taken from (d_1+d_2+1, s)th entry, and by property (F3) above the variable u_1 is not contained in the (s, d_1+d_2+1)st entry. Hence, the contribution of the factor $u_1 u_{m+l}$ is taken from the (d_1+d_2+1, s)th entry while $u_j u_{m_1+1}$ is taken from the (s, d_1+d_2+1)st entry.

The rest is by complete analogous with the calculations in item (H3) above. By (2.26), the coefficient of $u_j u_{m_1+1}$ from (s, d_1+d_2+1)st entry is equal to

$$\left(\alpha_1^2 - \alpha_{m_1+1}^2\right) u_j u_{m_1+1} \cdot c_{j,s}^{m_1+1}.$$

Further, by analogy with (2.65) the coefficient of $u_1 u_{m+l}$ from the (d_1+d_2+1, s)th entry is equal to

$$-\left(\alpha_1^2 - \alpha_{m_1+1}^2\right) u_i u_{m+l} \cdot \left(c_{1s}^{m+l} + \sum_{i=d_1+2}^{m_1} c_{1i}^{m+l} c_{1i}^{m+s-1}\right).$$

Finally, by multiplying all coefficients above and soming over all s, taking into account corresponding signs, we get the contribution of the considered terms in the cofactor decomposition to the coefficient of the monomial (2.42) is equal to

$$\left(\alpha_1^2 - \alpha_{m_1+1}^2\right) \sum_{s=1}^{d_1} \operatorname{sgn}(\sigma_s) c_{js}^{m_1+1} \sum_{i=2}^{m_1} c_{1i}^{m+l} c_{1i}^{m+s-1}, \tag{2.69}$$

where $\operatorname{sgn}(\sigma_s)$ stands for the sign of the permutation related to each entry that appears in the summation. We will see later in (2.72) that the sign $\operatorname{sgn}(\sigma_s)$ is actually not important to the conclusion, so its explicit expression is not written out here.

3. The term of the cofactor expansion of the determinant with respect to the first row containing the $(1, d_1+d_2+1)$st entry Note that $\tilde{b}_1^{1,2} = 0$ by complete analogy with (2.48). So, the considered term in the cofactor expansion is in fact equal to zero.

Now, after considering all terms in the cofactor expansion one can combinie (2.67) and (2.69) to get (2.43).

Further, let

$$v^s = (c^s_{1,2}, c^s_{1,3}, \ldots, c^s_{1,m_1}) \quad s \in [m+1 : m+d_1],$$

The coefficient (2.43) can be rewritten as

$$\begin{aligned} &\left(\alpha_1^2 - \alpha_{m_1+1}^2\right) \sum_{s=1}^{d_1+1} \operatorname{sgn}(\sigma_s) c_{js}^{m_1+1} \langle v^{m+s-1}, v^{m+l} \rangle \\ &= \left(\alpha_1^2 - \alpha_{m_1+1}^2\right) \left\langle \sum_{s=1}^{d_1+1} \operatorname{sgn}(\sigma_s) c_{js}^{m_1+1} v^{m+s-1}, v^{m+l} \right\rangle . \end{aligned} \tag{2.70}$$

Since the coefficient (2.70) vanishes everywhere, we have

$$\left\langle \sum_{s=1}^{d_1+1} \operatorname{sgn}(\sigma_s) c_{js}^{m_1+1} v^{m+s-1}, v^{m+l} \right\rangle = 0 \tag{2.71}$$

implies the vector $\sum_{s=1}^{d_1+1} \operatorname{sgn}(\sigma_s) c_{js}^{m_1+1} v^{m+s-1}$, which belongs to $\operatorname{span}\{v^{m+l} : 1 \leq l \leq d_1\}$ is orthogonal to all v^{m+l}. Hence, by (2.71),

$$\sum_{s=1}^{d_1+1} \operatorname{sgn}(\sigma_s) c_{js}^{m_1+1} v^{m+s-1} = 0, \tag{2.72}$$

Note that from ad-surjecitvity, and, more precisely, since X_1 is chosen as ad-generating element, the tuple of vectors $(v^{m+1}, v^{m+2}, \ldots, v^{m+d_1})$ form a basis in $\mathbb{R}^{d^1}$. Therefore,

$$c_{js}^{m_1+1} = 0, \text{ for } j \in [2 : d_1+1], \text{ and } s \in [1 : d_1+1],$$

So, (2.44) is proved. □

Lemma 2.3.15. *Assume that* (2.31)*,* (2.33)*,* (2.44)*, and Proposition* 2.3.12 *hold. Then the coefficient of the monomial*

$$u_1^{d_1-1} u_j u_{d_1+1} u_{m_1+1}^{d_2-1} u_i u_{m_1+d_2+1} u_{m+l}, \quad i \in \mathcal{I}_2^1 \backslash \{m_1+1\}, j \in [2 : d_1+1], l \in [1 : d_1] \tag{2.73}$$

in $\det([M_2|b^{1,2}])$ *is given by*

$$\left(\alpha_1^2 - \alpha_{m_1+1}^2\right) \sum_{s=1}^{d_1} \operatorname{sgn}(\sigma_s) c_{js}^i \sum_{k=2}^{m_1} c_{1k}^{m+l} c_{1k}^{m+s-1}, \tag{2.74}$$

and again $\operatorname{sgn}(\sigma_s)$ *is either* 1 *or* -1*. Further, vanishing of all coefficients in* (2.74) *imply that*

$$c_{js}^i = 0, \quad 1 \leq s \leq d_1 + 1, i \in \mathcal{I}_2^1 \backslash \{m_1 + 1\}, j \in [2 : d_1 + 1]. \tag{2.75}$$

Proof. To begin with, observe that (2.73) differs from (2.42) by a u_i. First, let us figure out which entries of the matrix $[M_2|b^{1,2}]$ contain u_i . Similar to above, denote by $[u_i]$ a linear form in u's with coefficient of u_j being equal to 1.

$$M_{2,2} = \begin{pmatrix} \cdots & & & [u_i] & \cdots & [u_\rho] \\ [u_{m_1+1}] & \cdots & & & & \\ & [u_{m_1+1}] & & & & \vdots \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & [u_{m_1+1}] & \end{pmatrix},$$

where $[u_i]$ is located in the $(i - m_1 - 1)$th column and $[u_{m_1+1}]$'s are along the sub-diagonal of $M_{2,2}$. If we assume u_i is selected in $M_{2,2}$, one instance of $[u_{m_1+1}]$ on the sub-diagonal needs to be excluded from consideration, i.e. one term of u_{m_1+1} left to be fixed outside of $M_{2,2}$, but it does not affect the decisions made in $M_{2,1}$ and a_1^2, and therefore the coefficient of this case will be a multiple of (2.43), which is assume to be zero by (2.44). Therefore u_i should be selected in $M_{2,1}$ and a_1^2, and in this case, all u_{m_1+1} along the sub-diagonal of $M_{2,2}$ need to be chosen in order to get the factor $u_{m_1+1}^{d_2-1}$.

In this way, u_i takes place of u_{m_1+1} and similar to the arguments in the part (H3) in the proof of Lemma 2.3.13 we can conclude that the selection of u_i has to be made in the last column of M_2, so we can follow similar arguments as in the derivation of (2.67) to conclude that the coefficient of the monomial (2.73) in $\det([M_2|b^{1,2}])$ is as in (2.74). Writing expressions (2.74) in the vector form an equating them to zero we get

$$\begin{aligned}&\left(\alpha_1^2-\alpha_{m_1+1}^2\right)\sum_{s=1}^{d_1+1}\operatorname{sgn}(\sigma_s)c_{js}^i\langle v^{m+s-1},v^{m+l}\rangle\\&=\left(\alpha_1^2-\alpha_{m_1+1}^2\right)\left\langle\sum_{s=1}^{d_1+1}\operatorname{sgn}(\sigma_s)c_{js}^i v^{m+s-1},v^{m+l}\right\rangle=0,\end{aligned}$$

and by the same argument as at the end of the proof of Lemma 2.3.13 we get (2.75). □

Lemmas 2.3.13 and 2.3.15 prove only a subset of relations from (2.41). Using the flexibility given by Corollary 2.3.10 one can show that (2.41) holds not for the original quasi-normal frame but for its perturbation adapted to the same X_1, which will be enough to finish the proof of Proposition 2.3.5.

Corollary 2.3.16. *Let K and H be as in (2.38) and (2.39), respectively. If Y and Z are sections of H satisfying*

$$\operatorname{span}\{Y,Z\}\cap K=0, \tag{2.76}$$

then

$$[Y,Z]\in\operatorname{Im}\Big((\operatorname{ad}X_1)_{\operatorname{mod}D_1}\Big)\quad\operatorname{mod}D_1. \tag{2.77}$$

Proof. Indeed, from (2.76) it follows that we can find the quasi-normal frame $(X_1,\dots,X_n)$ such that $X_2=Y$ and $X_3=Z$. Then (2.77) follows from Lemmas 2.3.13 and 2.3.15 and Corollary 2.3.10. □

By Remark 2.3.11, we can assume that $d_1\geq 2$. Then the set of planes in H having a trivial intersection with K is generic. Therefore by a finite number of consecutive small perturbations, one can build a quasi-normal frame $(X_1,\dots,X_n)$ such that for every $2\leq i<j\leq m_1$, the pair $(Y,Z)=(X_i,X_j)$ satisfy (2.76) and so by Corollary 2.3.16 this frame satisfies (2.41). By symmetric arguments for the second component instead of the first one we also get

$$c_{i,j}^k=0,\ \text{for } i,j\in\mathcal{I}_2^1,\ k\in\mathcal{I}_1^1. \tag{2.78}$$

The proof of Proposition 2.3.5 is completed.

2.3.4 Proof of Proposition 2.3.2

Plugging (2.41) and (2.78) into (2.13) one gets that

$$q_{jk} = 0, \quad \forall (j,k) \in (\mathcal{I}_1^1 \times \mathcal{I}_2^1) \cup (\mathcal{I}_2^1 \times \mathcal{I}_1^1) \tag{2.79}$$

Plugging (2.79) into (2.26) and (2.27) we get

$$b_j^1 = \begin{cases} \alpha_1^2 \sum\limits_{k=m+1}^{m+d_1} q_{jk} u_k & j \in [1 : m_1], \\ \alpha_{m_1+1}^2 \sum\limits_{k=m+d_1+1}^{n} q_{jk} u_k & j \in [m_1 + 1 : m] \end{cases} \tag{2.80}$$

This implies that the tuple $\widetilde{\Phi} = (\Phi_{m+1}, \dots \Phi_n)$ satisfying

$$\Phi_j = \begin{cases} \alpha_1^2 u_j, & j \in [m+1 : m+d_1]. \\ \alpha_{m_1+1}^2 u_j, & j \in [m+d_1+1 : n] \end{cases} \tag{2.81}$$

is the solution to the first layer $A^1 \widetilde{\Phi} = b^1$ of the Fundamental Algebraic System (2.15). This can be verified directly by plugging (2.81) into $A^1 \widetilde{\Phi} = b^1$ and using the first line of (2.17) and (2.80).

Further, note that the $m \times (n-m)$-matrix A^1 has the maximal rank $n - m$ at a generic point, as from the normalization conditions (2.22) the coefficient of the monomial $u_1^{d_1} u_{m_1+1}^{d_2}$ in its maximal minor consisting of rows from the set $[2 : d_1 + 1] \cup [m_1 + 2, m_1 + d_2 + 1]$ is equal to 1. Hence, (2.81) defines the unique (rational in u's) solution of the the system $A^1 \widetilde{\Phi} = b^1$ and therefore it must coincide with the solution of the whole Fundamental Algebraic System (2.15), i.e. must satisfy other layers of it. Consequently,

$$\sum_{k=m+1}^{n} a_{j,k}^s \alpha_k^2 u_k = b_j^s, \text{ for } s \geq 1,$$

where $a^s_{j,k}$ and b^s_j satisfy (2.17) and (2.18), respectively. in particular, for $j = 1$ and $s = 2$, we have

$$b^2_1 - \sum_{k=m+1}^{n} \alpha^2_k u_k a^2_{1,k} = 0, \tag{2.82}$$

The coefficients of the monomials

$$u_{i+1} u_{j+1} u_{m_1+s}, \quad i, j \in [0 : d_1], s \in [1 : d_2]$$

of the left side of (2.82) are given by

$$\begin{cases} \left(\alpha^2_1 - \alpha^2_{m_1+1}\right) \left(c^{m_1+s}_{i+1,m+j} + c^{m_1+s}_{j+1,m+i}\right), & i \neq j \in [1 : d_1], s \in [1 : d_2], \\ \left(\alpha^2_1 - \alpha^2_{m_1+1}\right) c^{m_1+s}_{i+1,m+i}, & i = j \in [1 : d_1], s \in [1 : d_2]. \end{cases}$$

which directly implies

$$\begin{cases} c^{m_1+s}_{i+1,m+j} + c^{m_1+s}_{j+1,m+i} = 0, & i \neq j \in [1 : d_1], s \in [1 : d_2], \\ c^{m_1+s}_{i+1,m+i} = 0, & i = j \in [1 : d_1], s \in [1 : d_2]. \end{cases} \tag{2.83}$$

Note that the second line of (2.83) indicates that $\mathrm{ad}(X_1)(D^2_1) \subset D^2_1$.

For $i \neq j \in [1 : d_1], s \in [1 : d_2]$, Jacobi identity shows that

$$\begin{aligned} [X_{i+1}, X_{m+j}] &= [X_{i+1}, [X_1, X_{j+1}]] \\ &= [[X_{i+1}, X_1], X_{j+1}] + [X_1, [X_{i+1}, X_{j+1}]] \\ &= [-X_{m+i}, X_{j+1}] + [X_1, [X_{i+1}, X_{j+1}]]. \end{aligned}$$

Hence

$$[X_{i+1}, X_{m+j}] - [X_{j+1}, X_{m+i}] = [X_1, [X_{i+1}, X_{j+1}]] \in \mathrm{ad}(X_1)(D^2_1) \subset D^2_1, \tag{2.84}$$

by examining the coefficient of the basis vector X_k, $k \in \mathcal{I}_2^1$, for which $X_k \notin D_1^2$, on the left side of (2.84), one can get

$$c_{i+1,m+j}^{m_1+s} - c_{j+1,m+i}^{m_1+s} = 0, \quad i \neq j \in [1:d_1], s \in [1:d_2]. \tag{2.85}$$

Combining (2.83) and (2.85), it follows that

$$c_{i+1,m+j}^{m_1+s} = c_{j+1,m+i}^{m_1+s} = 0, \quad i \neq j \in [1:d_1], s \in [1:d_2].$$

In other words, for $j \in [1:d_1+1]$

$$\mathrm{ad}(X_j)(D_1^2) \subset D_1^2,$$

and moreover, when $i \in [d_1+1:m_1]$ and $j \in [1:d_1+1]$, Jacobi identity gives

$$\begin{aligned}
[X_i, X_{m+j}] &= [X_i, [X_1, X_{j+1}]] \\
&= [[X_i, X_1], X_{j+1}] + [X_1, [X_i, X_{j+1}]] \\
&\in \mathrm{ad}(X_{j+1})(D_1^2) + \mathrm{ad}(X_1)(D_1^2) \\
&\subset (D_1^2) + (D_1^2) = D_1^2.
\end{aligned}$$

Therefore,

$$[X_i, X_j] \in (D_1)^2, \text{ for } i \in \mathcal{I}_1^1, j \in \mathcal{I}_1^2$$

in other words

$$(D_1)^3(q) = (D_1)^2(q).$$

While it follows from the symmetry that $(D_2)^3(q) = (D_2)^2(q)$, the proof of Proposition 2.3.2 is completed.

2.3.5 Rotating the frames on D_1 and D_2

By Proposition 2.3.2, we have 2 involutive distributions D_1^2 and D_2^2, and moreover, $D_1^2 \oplus D_2^2 = TM$. Due to this involutivity, there exist foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ with leaves being integral submanifolds of D_1^2 and D_2^2 respectively. For any point $q \in M$, the integral manifold $\mathcal{F}_1(q)$ is transversal to $\mathcal{F}_2(q)$. By the transversality and the Inverse Function Theorem, there exists a neighborhood $\mathcal{U}$ of q_0, such that for any $q_1 \in \mathcal{U}$ and $q_2 \in \mathcal{U}$, their foliations $\mathcal{F}_1(q_1)$ intersects with $\mathcal{F}_2(q_2)$ at exactly one point. Then the following projection maps $\pi_1 : \mathcal{U} \to \mathcal{F}_1(q_0)$ and $\pi_2 : \mathcal{U} \to \mathcal{F}_2(q_0)$ are well defined:

$$\pi_1(q) = \mathcal{F}_1(q_0) \cap \mathcal{F}_2(q)$$
$$\pi_2(q) = \mathcal{F}_2(q_0) \cap \mathcal{F}_1(q)$$

for any $q \in \mathcal{U}$. Actually, π_1 and π_2 are local diffeomorphisms between $\mathcal{F}_1(q_0) \cong \mathcal{F}_1(q)$, and between $\mathcal{F}_2(q_0) \cong \mathcal{F}_2(q)$. In light of this, for any $i \in \mathcal{I}_1^1$, there exist a unique vector field $\tilde{X}_i$ on $\mathcal{U}$ such that

- $\tilde{X}_i(q) = \mathrm{d}\pi_1^{-1}\Big|_{\pi_1(q)} X_i(\pi_1(q))$;
- $\tilde{X}_i(q) \in D_1^2(q)$.

By construction, $\pi_1 \circ e^{tV_2} = \pi_1$, for any $V_2 \in D_2^2$. And by the property of Lie derivative, one gets

$$\begin{aligned}(\pi_1)_*[V_2, \tilde{X}_i](q) &= \frac{\mathrm{d}}{\mathrm{d}t}(\pi_1)_*(e^{-tV_2})_*\tilde{X}_i(e^{tV_2}(q))\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\pi_1)_*\tilde{X}_i(e^{tV_2}(q))\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}X_i(\pi_1(q)) = 0,\end{aligned}$$

where $(\pi_1)_*$ denotes the pushforward of the map π_1. The above calculation shows that $[V_2, \tilde{X}_i] \in D_2^2$, since $\ker(\pi_1)_* = D_2^2$. Similarly, one can also verify that $[V_1, \tilde{X}_j] \in D_1^2$ for any $j \in \mathcal{I}_2^1$ and

$V_1 \in D_1^2$. Thus,

$$[\tilde{X}_i, \tilde{X}_j] \in D_1^2 \cap D_2^2 = \{0\},$$

the vectors $\tilde{X}_i$ and $\tilde{X}_j$ commute,

$$[\tilde{X}_i, \tilde{X}_s] = 0, \text{ for } i \in \mathcal{I}_1^1, s \in \mathcal{I}_2^1. \tag{2.86}$$

Further, the pushforward map preserves the grading structure of the distributions D_1^2 and D_2^2. It follows from the decomposition of Tanaka symbol that for any $1 \leq i \leq m_1$ and $m_1 + 1 \leq j \leq m$, $[X_j, X_i] \in D_1 \oplus D_2$, which implies

$$(e^{tX_j})_* X_i(q) \in (D_1 \oplus D_2^2)(e^{tX_j} q)$$

and let V_2 be a vector field tangent to D_2^2 such that $\mathrm{d}\pi_{1|q}\Big((e^{tV_2})_* X_i\Big)(q) = X_i(\pi_1(q))$, which is equivalent to

$$\begin{aligned}
&\mathrm{d}\pi_{1|q}\Big((e^{tV_2})_* X_i\Big)(q)) = \mathrm{d}\pi_{1|q} \tilde{X}_i(q) \\
\Longrightarrow &\Big((e^{tV_2})_* X_i - \tilde{X}_i\Big)(q) \in \ker \mathrm{d}\pi_{1|q} = D_2^2(q) \\
\Longrightarrow &(e^{tV_2})_* X_i - \tilde{X}_i \in D_2^2
\end{aligned}$$

and since $(e^{tV_2})_* X_i \in D_1 + D_2^2$, $\tilde{X}_i$ must be in $D_1 + D_2^2$ as well. Therefore $\tilde{X}_i \in (D_1 + D_2^2) \cap D_1^2 = D_1$. Similarly, for $m_1 + 1 \leq j \leq m$, we have $\tilde{X}_j \in D_2$. Therefore the moving frame $(X_i)_{i \in \mathcal{I}_1^1}$ and$(\tilde{X}_i)_{i \in \mathcal{I}_1^1}$are related by a transition matrix

$$T_1 := (a_{ij})_{i,j=1}^{m_1} \in \mathrm{GL}(m_1),$$

so that

$$\tilde{X}_i = \sum_{j \in \mathcal{I}_1^1} a_{ij} X_j. \tag{2.87}$$

Similarly,

$$T_2 := (b_{ij})_{i,j=1}^{m_1} \in \mathrm{GL}(m_2)$$

is a transition matrix from the moving frame $(X_j)_{j\in\mathcal{I}_2^1}$ to $(\tilde{X}_j)_{j\in\mathcal{I}_2^1}$,

$$\tilde{X}_s = \sum_{l\in\mathcal{I}_2^1} b_{sl}X_l.$$

Our final goal is to show that by an appropriate selection of initial conditions, T_1 and T_2 can be chosen from $\mathrm{SO}(m_1)$ and $\mathrm{SO}(m_2)$,respectively, which is crucial as we have to stay in the class of g_1-orthogonal moving frames.

First, we have the following

Lemma 2.3.17. *The transition matrices $T_1 = (a_{ij})_{i,j=1}^{m_1} \in \mathrm{GL}(m_1)$ and $T_2 = (b_{ij})_{i,j=1}^{m_2} \in \mathrm{GL}(m_2)$ are the solution to the following partial equation system:*

$$\begin{aligned} X_l(a_{ir}) &= \sum_{j=1}^{m_1} c_{jl}^r a_{ij} = -\sum_{j\in\mathcal{I}_1^1} c_{lj}^r a_{ij}, \quad \textit{for } l\in\mathcal{I}_2^1, \\ X_j(b_{sr}) &= -\sum_{l\in\mathcal{I}_2^1} c_{jl}^r b_{sl}, \quad \textit{for } j\in\mathcal{I}_1^1. \end{aligned} \tag{2.88}$$

Proof. Previously, we constructively found a pair of new basis $(\tilde{X}_i)_{i\in\mathcal{I}_1^1}$ and $(\tilde{X}_j)_{j\in\mathcal{I}_2^1}$, so the their transition matrices T_1 and T_2 belong to $\mathrm{GL}(m_1)$ and $\mathrm{GL}(m_2)$, respectively. Using the commutativity relations (2.86) and s (2.87)-(2.87), direct computations of the Lie bracket give

$$\begin{aligned} 0 &= [\tilde{X}_j, \tilde{X}_s] = [\sum_{j\in\mathcal{I}_1^1} a_{ij}X_j, \sum_{l\in\mathcal{I}_2^1} b_{sl}X_l] = \\ &\sum_{j\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1} a_{ij}b_{sl}[X_j, X_l] + \sum_{j\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1} a_{ij}X_j(b_{sl})X_l - \sum_{j\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1} b_{sl}X_l(a_{ij})X_j = \\ &\sum_{j\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1}\sum_{r=1}^{m} a_{ij}b_{sl}c_{jl}^r X_r + \sum_{j\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1} a_{ij}X_j(b_{sl})X_l - \sum_{j\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1} b_{sl}X_l(a_{ij})X_j \end{aligned}$$

Replace l by r and j by r in the second and third terms of the last term

$$
\begin{aligned}
0 &= \sum_{j\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1}\sum_{r=1}^{m} a_{ij}b_{sl}c_{jl}^r X_r + \sum_{j\in\mathcal{I}_1^1}\sum_{r\in\mathcal{I}_2^1} a_{ij}X_j(b_{sr})X_r - \sum_{r\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1} b_{sl}X_l(a_{ir})X_r \\
&= \sum_{r\in\mathcal{I}_2^1}\sum_{j\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1} a_{ij}\Big(b_{sl}c_{jl}^r + X_j(b_{sr})\Big)X_r + \sum_{r\in\mathcal{I}_1^1}\sum_{j\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1} b_{sl}\Big(a_{ij}c_{jl}^r - X_l(a_{ir})\Big)X_r \\
&= \sum_{r\in\mathcal{I}_2^1}\sum_{j\in\mathcal{I}_1^1} a_{ij}\Big(\sum_{l\in\mathcal{I}_2^1} b_{sl}c_{jl}^r + X_j(b_{sr})\Big)X_r + \sum_{r\in\mathcal{I}_1^1}\sum_{l\in\mathcal{I}_2^1} b_{sl}\Big(\sum_{j\in\mathcal{I}_1^1} a_{ij}c_{jl}^r - X_l(a_{ir})\Big)X_r.
\end{aligned}
$$

Then it is equivalent to that the coefficient of each vector X_r must be zero,

$$
\begin{aligned}
&\sum_{j\in\mathcal{I}_1^1} a_{ij}\Big(\sum_{l\in\mathcal{I}_2^1} b_{sl}c_{jl}^r + X_j(b_{sr})\Big) = 0 \\
&\sum_{l\in\mathcal{I}_2^1} b_{sl}\Big(\sum_{j\in\mathcal{I}_1^1} a_{ij}c_{jl}^r - X_l(a_{ir})\Big) = 0.
\end{aligned} \tag{2.89}
$$

The first equation of (2.89) can be written as

$$\langle \boldsymbol{a}_i, \boldsymbol{v}\rangle = 0,$$

where $\boldsymbol{a}_i$ is the i-th row vector of T_1, and

$$\boldsymbol{v} = \Big[\sum_{l\in\mathcal{I}_2^1} b_{sl}c_{jl}^r + X_j(b_{sr})\Big]_{j=1}^{m_1}.$$

As i varies through $\mathcal{I}_1^1$, $\boldsymbol{v}$ is orthogonal to all row vectors of the invertible matrix T_1, and hence it must be a zero vector $\boldsymbol{v} = 0$ with

$$\sum_{l\in\mathcal{I}_2^1} b_{sl}c_{jl}^r + X_j(b_{sr}) = 0, \text{ for } j \in \mathcal{I}_2^1$$

Similarly,

$$\sum_{j\in\mathcal{I}_1^1} a_{ij}c_{jl}^r - X_l(a_{ir}) = 0, \text{ for } i \in \mathcal{I}_1^1.$$

Therefore, T_1 and T_2 are the actual solution to (2.88). □

Lemma 2.3.18. *The transition matrices can be selected such that* $T_1 = (a_{ij})_{i,j=1}^{m_1} \in \mathrm{SO}(m_1)$ *and* $T_2 = (b_{ij})_{i,j=1}^{m_2} \in \mathrm{SO}(m_2)$*, if* $T_1 = (a_{ij})_{i,j=1}^{m_1}$, $T_2 = (b_{ij})_{i,j=1}^{m_2}$ *satisfy the following equalities:*

- *Orthogonality:*

$$\begin{aligned}
&\sum_{s \in \mathcal{I}_1^1} a_{is} a_{js} = \delta_{ij}, \text{ for } i, j \in \mathcal{I}_1^1, \\
&\sum_{s \in \mathcal{I}_2^1} b_{is} b_{js} = \delta_{ij}, \text{ for } i, j \in \mathcal{I}_2^1
\end{aligned} \tag{2.90}$$

- *Commutativity:*

$$\begin{aligned}
X_l(a_{ir}) &= \sum_{j=1}^{m_1} c_{jl}^r a_{ij} = -\sum_{j \in \mathcal{I}_1^1} c_{lj}^r a_{ij}, \text{ for } l \in \mathcal{I}_2^1, \\
X_j(b_{sr}) &= -\sum_{l \in \mathcal{I}_2^1} c_{jl}^r b_{sl}, \text{ for } j \in \mathcal{I}_1^1.
\end{aligned}$$

Proof. Relations (2.90) are just the condition for matrices T_1 and T_2 to be orthogonal. We claim that the orthogonality condition (2.90) is indeed compatible with (2.88) with the help of the consequences of Proposition 2.3.12 above. Indeed, by taking derivative over X_l on both sides of the equations in (2.90), one gets

$$\begin{aligned}
X_l\left(\sum_{s=1} a_{is} a_{js}\right) &= X_l(\delta_{ij}) = 0 \\
&= \sum_{s=1} \left(X_l(a_{is}) a_{js} + a_{is} X_l(a_{js})\right) \\
&= -\sum_{s=1} \left(\sum_{r=1} c_{lr}^s a_{ir} a_{js} + \sum_{r=1} c_{ls}^r a_{jr} a_{is}\right) \\
&= -\sum_{r,s=1} \left(c_{lr}^s a_{ir} a_{js} + c_{ls}^r a_{jr} a_{is}\right) \\
&= -\sum_{r,s=1} \left(c_{lr}^s + c_{ls}^r\right) a_{ir} a_{js} \\
&= 0
\end{aligned} \tag{2.91}$$

where the second to last equality is obtained from a swap of indices r, s in the second term, and the last equality is due to $c_{lr}^s + c_{sl}^r = 0$ in Proposition 2.3.12. Moreover, if we let $i = j$ in (2.91), it follows immediately that

$$X_l \left(\sum_{s=1} a_{is}^2 \right) = 0.$$

This means the orthogonal constraints (2.90) on (a_{ij}) are constant along any vector field X_l, and thus they can be considered as constraints on the initial data. In other words, if one chooses an initial condition in $\mathrm{SO}(m_1)$, e.g. the identity matrix I_{m_1}, the solution of the PDE system (2.88) with initial condition (2.90) will always stay within $\mathrm{SO}(m_1)$. Hence T_1 and T_2 exist as wanted.

Thus, the transition matrix T_1 and T_2 can be selected as a rotation matrix, and we have constructively proved the existence of the solution to the system of PDEs.

□

3. CONDITIONS FOR PRODUCT STRUCTURE VIA JACOBI CURVES

3.1 Affine equivalence and Jacobi curves

The goals of this chapter is first to prove the new necessary conditions for affine non-rigidity in terms of Jacobi curves (Theorems 3.1.3 and 3.1.4) and then to show that in a certain case study this necessary condition in combination with other necessary condition obtained in [13] ensure that a sub-Riemannian metric admits a product structure (Theorem 3.2.3). In the next subsection we relate Jacobi curves with the classical notion of Jacoibi fields and Jacobi equations.

3.1.1 Jacobi curves along normal sub-Riemannian extremals

In Riemannian geometry and, more generally, in Calculus of Variations, Jacobi fields are variational vector fields along a geodesic/an extremal trajectory corresponding to a variation (i.e. a one parametric family containing the original trajectory) of curves consisting of geodesics/ extremal trajectories ([8, 12]). Equivalently, a Jacobi field along a geodesic/extremal trajectory is a solution to the linearization of the geodesic/Euler-Lagrange equation along this geodesic/extremal trajectory. The main role of Jacobi fields is that the important notion of conjugate points (times) is defined in terms of them and the Morse Index theorem relates the index of second variation along the geodesic/ extremal trajectory (and therefore optimality property of the extremal trajectory) with the number of conjugate points along it.

The notions of Jacobi fields, conjugate points, and the Morse Index Theorem can be extended to extremals of a rather general class of optimal control problems, and in particular to Sub-Riemannian geometry ([1, 2, 4]). Since in sub-Riemannian geometry, normal geodesics are images under the canonical projection $\pi : T^*M \to M$ of integral curves of the Hamiltonian system corresponding to the sub-Riemannian Hamiltonian in the cotangent bundle, Jacobi fields along a normal sub-Riemanian extremal can be seen as solutions of the linearization of this Hamiltonian system along this extremal, so it is a special vector field along this extremal. The classical Jacobi field in the Riemannian case is obtained from this vector field along the extremal in the cotangent bundle by applying to the latter the push-forward π_* of π.

In more detail, let $\lambda = (p, q) \in T^*M$, where $q \in M$ and $p \in T_q^*M$. Let $\lambda(t) = e^{t\vec{h}}(\lambda)$ be the normal extremal starting from q, where $e^{t\vec{h}}$ denotes the flow generated by the vector field $\vec{h}$. Let $\mathcal{V}(\lambda)$ be the vertical subspace of $T_\lambda(T^*M)$, i.e. the tangent spaces to the fibers of T^*M passing through λ. A *Jacobi field along the extremal* $\lambda(t)$ is a vector field of the form $X(t) := e_*^{t\vec{h}}\xi$ for some $\xi \in \mathcal{V}(\lambda)$. If $s \mapsto \lambda^s$ is a smooth curve in $\pi^{-1}(q)$ so that $\lambda^0 = \lambda$ and $\frac{d}{ds}\lambda^s|_{s=0} = \xi$, then

$$X(t) = \frac{\partial}{\partial s} e^{t\vec{h}}(\lambda^s)|_{s=0},$$

i.e. $X(t)$ is the variational vector field on the extremal $\lambda(t)$ for the one-parametric family of extremals $e^{t\vec{h}}(\lambda^s)$ at $s = 0$ all of which correspond to sub-Riemannian geodesics starting at the same point $q = \pi(\lambda)$.

The classical notion of conjugate time along the extremal $\lambda(t)$ can be reformulated as follows: A time moment t_1 is called conjugate to the time 0 along $\lambda(t)$ if there exists a *nonzero* element $\xi \in \mathcal{V}(\lambda)$ such that

$$X(t_1) = e_*^{t_1\vec{h}}\xi \in \mathcal{V}\Big(\lambda(t_1)\Big). \tag{3.1}$$

Indeed, in this case, the classical Jacobi field $\pi_* X$ along the geodesic $\pi \circ \lambda(t)$ is not identically equal to zero but it is equal to zero at $t = 0$ and $t = t_1$, so t_1 is the conjugate time to 0 in the classical sense. Note that (3.1) is equivalent to

$$e_*^{t_1\vec{h}}\mathcal{V}(\lambda) \cap \mathcal{V}\Big(\lambda(t_1)\Big) \neq 0. \tag{3.2}$$

or, equivalently, after applying $e_*^{-t_1\vec{h}}$ to both sides of (3.2),

$$\mathcal{V}(\lambda) \cap e_*^{-t_1\vec{h}}\mathcal{V}\Big(\lambda(t_1)\Big) \neq 0. \tag{3.3}$$

Definition 3.1.1 ([3, 5])**.** Given $\lambda \in T^*M$, the *Jacobi curve* $J_\lambda(\cdot)$ attached to λ (or along the extremal $\lambda(t) = e^{t\vec{h}}\lambda)$ is the curve of subspaces of $T_\lambda(T^*M)$ given by

$$J_\lambda(t) = e_*^{-t\vec{h}}\mathcal{V}\Big(\lambda(t)\Big)$$

Let σ be the canonical symplectic form in T^*M, as defined in (1.6). The vector space $T_\lambda T^*M$, endowed with the nondegenerate skew-symmetric form $\sigma(\lambda)$, is an example of a linear symplectic space. Not that the space $\mathcal{V}(\lambda)$ is a half-dimensional subspace of $T_\lambda T^*M$ annihilated by $\sigma(\lambda)$, i.e. $\sigma(\lambda)|_{\mathcal{V}(\lambda)} = 0$. Subspaces of a linear symplectic space, satisfying the last two properties, are called *Lagrangian* and the space of all Lagrangian subspaces is called the *Lagrangian Grassmannian* of $T_\lambda T^*M$. Note also that a Hamiltonian flow preserves the canonical symplectic form, so the push-forward of a Hamiltonian flow sends Lagrangian subspaces to Lagrangian subspaces (of the corresponding target symplectic space). Consequently, *the Jacobi curve is a curve in the Lagrangian Grassmannian of* $T_\lambda T^*M$.

Directly from Definition 3.1.1, the condition (3.3) that t_1 is a conjugate time to zero along the extremal $\lambda(t)$ can be written in terms of Jacobi curve along as follows:

$$J_\lambda(0) \cap J_\lambda(t_1) \neq 0.$$

so that the Jacobi curve along an extremal contains all information on the conjugate points along this extremal. From the point of view of differential geometry Jacobi curves has a big advantage over Jacobi fields, because the former are curve of subspaces in a given vector space, so any invariant of it under the action of the symplectic group will be automatically an invariant of sub-Riemannian structure, while Jacobi fields consist of vectors belonging to different tangent spaces so that in order to find invariants of sub-Riemannian structures from them one has to identify those vector spaces. In fact, the push-forward of the Hamiltonian flow provides such identification, but the latter is already encoded in the definition of the Jacobi curve. To summarize, studying Jacobi curves is a more geometric way of studying the space of Jacobi fields in Riemannian geometry and Calculus of Variation, which can be applied to much more general optimal control problems.

3.1.2 Separation on the level of Jacobi curves in affine equivalence problem

Let E be the Euler field on T^*M, i.e. the generator of homotheties of the fibers and set

$$W_1(\lambda) := \text{span}\{\vec{h}(\lambda), E(\lambda)\}.$$

Using the Euler identity for homogeneous functions it is easy to see that

$$[\vec{h}, E] = -\vec{h}. \tag{3.4}$$

This implies that for every t, if

$$J^1_\lambda(t) := (e^{-t\vec{H}})_* \text{span}\left\{E\left(e^{t\vec{h}}\lambda\right)\right\},$$

then

$$J^1_\lambda(t) \subset W_1(\lambda).$$

Moreover, from (3.5) it follows that

$$J^1_\lambda(t) = J_\lambda(t) \cap W_1(\lambda).$$

Now let $W_2(\lambda) := W_1(\lambda)^\angle$, the skew-orthogonal complement of $W_1(\lambda)$ in $T_\lambda(T^*M)$ with respect to the form σ at λ. Note that if $\lambda \notin H^{-1}(0)$ (which is exactly the case of a normal extremal) then the restriction of σ_λ to $W_1(\lambda)$ is non-degenerate, so that

$$T_\lambda(T^*M) = W_1(\lambda) \oplus W_2(\lambda).$$

Let $J^2_\lambda(t) = J_\lambda(t) \cap W_2(\lambda)$. Then by construction we proved the following

Lemma 3.1.2. The Jacobi curve J_λ is the direct product of curves J^1_λ and J^2_λ.

The curve J^2_λ is called the *reduced Jacobi curve*.

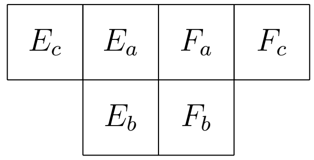

Figure 3.1: Reduced Young diagram Δ related to J_λ in th case of $(4, 6)$-distributions. The left half is the reflection of the right half. Label the first row of the right half (from left to right) as a, c, and the second row of the right half as b. Each superbox contains two basis vectors. Denote $E_l = (E_l^1, E_l^2)$, $F_l = (F_l^1, F_l^2)$, for $l \in \Delta$.

Given a Jacobi curve J_λ and an integer $i \geq 0$, the *i-th extension* of the Jacobi curve $J_\lambda(\cdot)$ is defined as

$$J_\lambda^{(i)} = \left\{ \frac{d^j}{dt^j} l(0) : l(s) \in J_\lambda(s), \forall s \in [0, T], l(\cdot) \text{ smooth}, 0 \leq j \leq i \right\}. \tag{3.5}$$

The flag

$$J_\lambda \subset J_\lambda^{(1)} \subset J_\lambda^{(2)} \subset \dots$$

is called the associated flag of the curve J_λ. We assign the Young diagram in the following way: the number of boxes in the ith column of this Young diagram is equal to $\dim J_\lambda^i - \dim J_\lambda^{(i-1)}$ and it is called the *Young diagram of the curve* J_λ. In particular, the number of boxes in the first column is equal to the rank of the curve.

Observe that any Young diagram $\mathfrak{D}$ can be uniquely represented as a union of d rectangular diagram $\mathfrak{D}_i$ of sizes $r_i \times p_i$, $1 \leq i \leq d$ such that the sequence $\{p_i\}_{i=1}^d$ is strictly decreasing. The Young diagram Δ, consisting of d rows s.t. the ith row has p_i boxes, will be called *reduced diagram* or the reduction of the diagram, $\mathfrak{D}$. In order to distinguish between boxes and rows of the diagram $\mathfrak{D}$ and its reduction Δ, the boxes of Δ will be called *superboxes* and the rows of Δ will be called *levels*. In the case of $(4, 6)$-distributions satisfying the conditions in Theorem 3.2.3, we will see later in section 3.2.6 and specifically in Remark 3.2.18 that the i-th extension of the Jacobi curve has dimension:

$$\dim J_\lambda = 6, \quad \dim J_\lambda^{(1)} = 10, \quad \dim J_\lambda^{(2)} = 12. \tag{3.6}$$

It implies the Young diagram $\mathfrak{D}$ associated with the distribution D is of the form in Figure 3.1.

Now we are ready to give a new necessary condition for affine non-rigidity in terms of in terms of the Young diagram of Jacobi curve of generic normal extremal. In fact we formulate it in more general context of conformally projective nonrigid sub-Riemannian metrics. A sub-Riemannian metric g_1 is called *conformally projectively rigid* if from the fact that a sub-Riemannian metric g_2 is projectively equivalent to it, $g_2 \overset{p}{\sim} g_1$, it follows that implies that g_2 is conformal to g_1, i.e. there is a function α such that $g_2 = \alpha^2 g_1$. By [13][Corollary 4.9] if a sub-Riemannian metric is conformally projectively rigid, then it is affinely rigid.

Theorem 3.1.3. *If a sub-Riemannian metric is not conformally projectively rigid (and, in particular, if it is not affinely rigid), then the Young diagram of any its extremal the Jacobi Young contains at least two rows of length one.*

Proof. By Lemma 3.1.2 the Young diagram already of any Jacobi curve has a row of length 1, coming from the J_λ^1.

Recall [13, Theorem 1.4] that if the metric g is not conformally projectively rigid, then its flow of normal extremals admits at least one nontrivial (i.e. not equal to the sub-Riemannian Hamiltonian h) integral quadratic on the fibers. In the notation of chapters 3 and 4 of [13] this additional integral, denoted by F, satisfies

$$F = \left(\prod_{\ell=1}^{N} \alpha_\ell^2\right)^{-\frac{2}{N+1}} \sum_{i=1}^{m} \alpha_i^2 u_i^2.$$

Note that $\pi_* \vec{F} \in D$ and that the first osculating subspace $J_\lambda^{(1)}(0)$ satisfies $\pi_* \left(J_\lambda^{(1)}(0)\right) = D$. Hence there exists the vertical vector field Z such that

$$J'_\lambda(0)(Z(\lambda)) = \vec{F}(\lambda) \tag{3.7}$$

Here we consider $J'_\lambda(0)$ as an element of $\mathrm{Hom}\left(V(\lambda), J_\lambda^{(1)}(0)/V(\lambda)\right)$. The equation (3.7) is equivalent to

$$[\vec{h}, Z](\lambda) = \vec{F}(\lambda) \mod V(\lambda)$$

Since F is the first integral for the flow of $\vec{h}$, i.e. $[\vec{h}, \vec{F}] = 0$, we have

$$[\vec{h}, [\vec{h}, Z]](\lambda) \subset J_\lambda^{(1)}(0).$$

This together with (3.4) implies that $\dim J_\lambda^{(2)}(0) - \dim J_\lambda^{(1)} \leq m-2$, where $\dim J_\lambda^{(2)}$ is the second osculation space of the Jacobi curve at $t = 0$. This completes the proof of our theorem. □

Theorem 3.1.4. *If a sub-Riemannian metric is not affinely rigid, then the reduced Jacobi curve of a generic normal extremal can be represented as a direct product of two curves in Lagrangian Grassmannians in symplectic spaces of smaller dimensions.*

Proof. Let g and $\tilde{g}$ be two affinely equivalent metrics. Assume that Φ is the fiber-preserving orbital diffeomorphism between the Hamiltonian flows of the first and the second metric, reparametrizing the extremals up to an affine reparameterization . Also let J_λ and $\widetilde{J}_\lambda$ be the Jacobi curves at $T_\lambda T^*M$ corresponding to the extremals of the first and the second metric, respectively, starting at λ. From the construction of Jacobi curves and the fact that Φ is fiber-preserving it follows that

$$d\Phi(\lambda)J_\lambda(t) = \widetilde{J}_{\Phi(\lambda)}(at), a \in \mathbb{R} \tag{3.8}$$

One of the consequences of the theory of canonical frames for parametrized curves in Lagrangian Grassmannians developed in [20, 21] or even earlier construction of derivative curves [3, 5] using the Laurent expansion in the affine chart in Grassmannians of half-dimensions [1] that

[1]Note that the construction in [20, 21] and [3, 5] give different canonical complements, but it does not really matter

for a generic λ in T^*M one can assign the canonical horizontal subspace $\mathrm{Hor}(\lambda)$ of $T_\lambda T^*M$. Here horizontal means that it is transversal to the tangent space $V(\lambda)$ to the fiber of T^*M at λ (also called vertical spaces), i.e. such that $T_\lambda T^*M$ has the following canonical splitting:

$$T_\lambda T^*M = V(\lambda) \oplus \mathrm{Hor}(\lambda). \tag{3.9}$$

This construction follows from the following

Proposition 3.1.5. *[21] For any monotonic ample parameterized curve Λ in a Lagrangian Grassmannian with an equiregular osculating flag we can assign the canonical curve $\Lambda^{\pitchfork}$ such that $\Lambda(t)$ nd $\Lambda^{\pitchfork}(t)$ are transversal.*

Then the canonical horizontal distribution in T^*M for the first sub-Riemannian metric is defined by

$$\mathrm{Hor}(\lambda) = J_\lambda^{\pitchfork}(0).$$

Let $\widetilde{\mathrm{Hor}}\big(\lambda\big)$ denotes the horizontal distribution on T^*M corresponding to the second metric, then we have the following

Lemma 3.1.6. *The following relation holds:*

$$d\Phi(\lambda)\mathrm{Hor}(\lambda) = \widetilde{\mathrm{Hor}}\Big(\Phi(\lambda)\Big). \tag{3.10}$$

Proof Note that the map $d\Phi : T_\lambda T^*M \to T_{\Phi(\lambda)}T^*M$ does not have to preserve the symplectic structures. Also in the relation (3.8) in the target curve a reparameterization by an affine reparameterization is used. So, Proposition 3.10 will be proved if we will prove Proposition 3.1.5 for a wider class of curves in half-dimensional Grassmannians defined up to affine reparameterizations. This in fact immediately follows from the construction of [3, 5] with Laurent expansion in the affine chart as this construction works in the Grassmannians of half dimension and the free term of the Laurent expansion defining the horizontal distribution is independent of the affine change of

and the approach in [3, 5] is in fact sufficient for our purposes of Lemma 3.1.6

parameter). It can be done also using the general theory described in my recent preprint [19], but it needs additional calculations. □

Remark 3.1.7. *If for two points λ_1 and λ_2 on the same fiber of T^*M the horizontal complements Hor_i are chosen in $T_{\lambda_i}(T^*M)$, $i = 1, 2$, then we can identify the spaces $T_{\lambda_i}(T^*M)$ canonically. Indeed, since T^*M is a vector bundle, the vertical spaces V_λ for all λ's from the same fiber of T^*M are canonically identified, as the tangent spaces to points of the same vector space. Note that the symplectic form defines the canonical identification of Hor_i with the dual space $\Big(V(\lambda_i)\Big)^*$ of $V(\lambda_i)$ and hence the spaces Hor_1 and Hor_2 are canonically identified . So, using the splitting* (3.9) *the spaces $T_{\lambda_1}(T^*M)$ and $T_{\lambda_2}(T^*M)$ are indeed canonically identified.*

Remark 3.1.8. *Note that there is no statement analogous to Proposition 3.1.5 in the case of unparametrized curve but the theory of unparametrized curves in Grassmannians and Lagrangian Grassmannians is well understood ([10, 9]), hence we still optimistic that something similar but more involved can be done in the projective equivalence case.*

Using the canonical identification of Remark 3.1.7 with

$$\lambda_1 = \lambda, \quad \lambda_2 = \Phi(\lambda), \quad \mathrm{Hor}_1 = \mathrm{Hor}(\lambda), \quad \mathrm{Hor}_2 = \widetilde{\mathrm{Hor}}\Big(\Phi(\lambda)\Big), \tag{3.11}$$

and the fact that the map Φ is fiber-preserving we can look on the curve J_λ and $\widetilde{J}_{\Phi(\lambda)}$ as the curve on the same symplectic space and the map $d\Phi$ as acting on this space. Also from the identity (3.10) it follows that in a symplectic (Darboux) basis compatible with the splitting (3.9) the matrix of $d\Phi(\lambda)$ has the form

$$\left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & D(\lambda) \end{array}\right) \tag{3.12}$$

for some matrix $D(\lambda)$, where the first half of the vectors in the basis spans the horizontal subspace and the second half spans the vertical one. The reason for the form of the matrix in (3.12) is that the vertical and the horizontal subspaces at λ and $\Phi(\lambda)$, respectively, are the same under the canonical identification of $T_\lambda T^*M$ and $T_{\Phi(\lambda)}T^*M$ corresponding to the choice in (3.11).

Moreover, if $S_\lambda(t)$ and $\widetilde{S}_\lambda(t)$ are matrix-coordinates with respect to the chosen symplectic basis of the germs at 0 of the curves $J_\lambda(t)$ and $\widetilde{J}_{\Phi(\lambda)}(t)$ in the affine chart $\mathrm{Hor}(\lambda)^{\pitchfork}$ of the Lagrangian subspaces transversal to $\mathrm{Hor}(\lambda)$ with the origin (in this affine space) chosen at $V(\lambda)$, then by (3.8) and (3.10) we have that there exists $a > 0$ such that

$$D(\lambda) = \widetilde{S}_\lambda(at)S_\lambda(t)^{-1} \Leftrightarrow \widetilde{S}_\lambda(at) = D(\lambda)S_\lambda(t), \quad \forall t \neq 0 \text{ close } \text{ to } 0. \tag{3.13}$$

Note that the matrices $S_\lambda(t)$ and $\widetilde{S}_\lambda(t)$ are symmetric as they correspond to Lagrangian subspaces and they correspond to the positive definite quadratic forms for sufficiently small $t \neq 0$, as Jacobi curves are monotonic. Hence from the elementary linear algebra it follows that the matrix $D(\lambda)$ is diagonizable.

Now, if we assume that the metrics g and $\tilde{g}$ are not constantly proportional, then there is λ (actually generic) for which the matrix $D(\lambda)$ is not a scalar multiple of identity, hence in an appropriate basis $D(\lambda)$ splits into more than one diagonal block and each block is the multiple of identities (the scalars in different blocks are different).

Let us work in a basis of eigenvectors of $D(\lambda)$, so $D(\lambda) = \mathrm{diag}(d_1, \ldots d_n)$, with not all d_i equal one to another. Then (3.13) implies that $\widetilde{S}_{ij}(at) = d_i S_{ij}(t)$. Taking into account that both $\widetilde{S}$ and S are symmetric matrices, we get that $d_i S_{ij}(t) = d_j S_{ji}(t)$. Therefore $S_{ij}(t) = 0$ for $d_i \neq d_j$, and the matrices $S_{ij}(t)$ are simultaneously block-diagonal.

This exactly means that the curves J_λ and $\widetilde{J}_\lambda$ are direct products of curves in Lagrangian Grassmannians of symplectic spaces of smaller dimensions.

This completes the proof of Theorem 3.1.4. □

3.2 A weaker version of Conjecture 1.3.1 for (4,6)-distributions

In this section, we perform a case study of Conjecture 1.3.1 when D is a $(4, 6)$ distribution so that its Tanaka symbols is decomposed into the sum of two copies of 3-dimensional Heisenberg algebras.

3.2.1 Existence of first integral compatible with the decomposition of Tanaka symbol

We will introduce the definition of the *compatibility* of a Hamiltonian with the decomposition of an arbitrary Tanaka symbol.

Definition 3.2.1. Given a decomposition of a Tanaka symbol $\mathfrak{m}(q) = \oplus_{i=1}^{\mu}\mathfrak{m}_i(q)$ of (D, M, g) at a point $q \in M$ and D_i as the sub-bundle of D such that D_i is the span of the g-orthogonal basis corresponding to $\mathfrak{m}_i$, we call a Hamiltonian $\mathcal{P}$ *compatible with a decomposition of the Tanaka symbol* if

$$\mathcal{P}(p, q) = \sum_{i=1}^{\mu} \alpha_i ||p|_{D_i(q)}||^2, \quad q \in M, p \in T_q^*M,$$

for some positive constants $\alpha_1, \ldots, \alpha_\mu$. As in (1.8), $||p|_{D_i(q)}||^2$ is the sub-Riemannian Hamiltonian corresponding to the restriction of g to each D_i.

Theorem 3.2.2. *If a sub-Riemannian metric is not affinely rigid, then the flow of its normal extremals admits a nontrivial (i.e. different from the sub-Riemannian Hamiltonian) integral which is compatible with a g-orthogonal decomposition of the Tanaka symbol.*

Theorem 3.2.2 is a direct application of Proposition 4.5 in [13] in a case of affine non-rigidity. If there exists a pair of affine equivalent metrics (g_1, g_2), which are not constantly conformal to each other, the Hamiltonian $\mathcal{P}$ defined as

$$\mathcal{P} = \sum_{i=1}^{m} \alpha_i^2 u_i^2$$

is compatible with a g-orthogonal decomposition of the Tanaka symbol, where $\alpha_1, \ldots, \alpha_m$ are the eigenvalues of the transition map between g_1 and g_2.

3.2.2 Sub-Riemannian manifolds on $(4, 6)$-distributions

Throughout the rest of Chapter 3, we restrict (D, M) to a $(4, 6)$ sub-Riemannian manifold, with Tanaka symbol at a point $q_0 \in M$

$$\mathfrak{m} = \eta_1 \oplus \eta_2,$$

where η_1 and η_2 are three-dimensional Heisenberg algebras. In other words, D is a 4-dimensional distribution spanned by (X_1, X_2, X_3, X_4) and a moving frame $(X_1, X_2, X_3, X_4, X_5, X_6)$ of TM can be constructed such that

$$X_1(q), X_2(q) \text{ generate } \eta_1, \text{ and } [X_1, X_2](q) = X_5(q),$$

$$X_3(q), X_4(q) \text{ generate } \eta_2, \text{ and } [X_3, X_4](q) = X_6(q).$$

The main result of this section is formulated as the following

Theorem 3.2.3. *If a sub-Riemannian structure on $(4, 6)$-distribution D satisfies each of the following three conditions:*

1. *The Tanaka symbol of the distribution D decomposes into direct sum of two 3-dimensional Heisenberg algebras;*

2. *The reduced Jacobi curve of a generic normal extremal can be represented as a direct product of two curves in Lagrangian Grassmannians in symplectic spaces of smaller dimensions;*

3. *The flow of its normal extremals admits a nontrivial integral which is compatible with a g-orthogonal decomposition of the Tanaka symbol,*

then D will admit a product structure.

Remark 3.2.4. *In the case of the Tanaka symbol consisting of 2 indecomposable components, the existence of the first integral compatible with the decomposition of such Tanaka symbol implies that sub-Riemannian Hamiltonians of the restriction on each component D_1 and D_2 are also first integrals of the original flow of normal extremals. In the local moving frame $(X_1, \dots X_6)$ as above, it is equivalent to the fact that $u_1^2 + u_2^2$ and $u_3^2 + u_4^2$ are first integrals of the flow of normal extremals, where u_i's are as in (2.9).*

3.2.3 Normal moving frame for Jacobi curves: general case and consequences of splitting

Let J_λ be the Jacobi curve with the reduced Young diagram Δ. Let $\Delta \times \Delta$ we mean the set of all pairs of superboxes of Δ. Also denote by Mat the set of matrices of all sizes. The mapping $R : \Delta \times \Delta \mapsto \mathrm{Mat}$ is called the *compatible with the Young diagram* $\mathfrak{D}$, if to any pair (a_1, a_2) of superboxes of size s_1 and s_2 respectively the matrix $R(a_1, a_2)$ is of the size $s_2 \times s_1$. The compatible mapping R is called *symmetric* if for any pair (a_1, a_2) of supberboxes the following identity holds

$$R(a_2, a_1) = R(a_1, a_2)^T$$

Also denote by σ_i the last superboxes of the ith level and by $r : \Delta \backslash \{\sigma_i\}_{i=1}^{d} \mapsto \Delta$ the right shift on the diagram Δ. The last supberbox of any level will be called *special*. Consider the following tuple of pairs of superboxes

$$(b, a), (b, c) \tag{3.14}$$

we are now able to introduce two important notions:

Definition 3.2.5. A symmetric compatible mapping $R : \Delta \times \Delta \mapsto \mathrm{Mat}$ is called *quasi-normal* if the following two conditions hold:

1. Among all matrices $R(a_1, a_2)$, where the superbox a_2 is not higher than the superbox than the supberbox a_1 in the diagram Δ, the only possible nonzero matrices are the following: the matrices $R(a, a)$ for all $a \in \Delta$, the matrices $R(a, r(a))$, $R(r(a), a)$ for all non-special boxes, and the matrices, corresponding to the tuple (3.14)

2. The matrix $R(a, r(a))$ is anti-symmetric for any non-special superbox a.

Remark 3.2.6. *In the case of* $(4, 6)$ *distribution with the associated Young diagram Figure 3.1 (or other distributions having the same Young diagram as Figure 3.1, e.g.* $(2, 3)$ *distribution), the first condition in Definition 3.2.5 is always satisfied, so it can be interpreted as: a symmetric compatible mapping* R *is quasi-normal if and only if* $R(a, c)$ *is anti-symmetric.*

Definition 3.2.7. A quasi-normal mapping $R : \Delta \times \Delta \mapsto \mathrm{Mat}$ is called *normal* if it satisfies that for any $1 \leq j < i \leq d$, the matrices corresponding to the first $p_j - p_i - 1$ pairs of the tuple (3.14) are equal to zero.

Remark 3.2.8. *In the case of* $(4, 6)$ *distribution (or other distributions having the same Young diagram, e.g.* $(2, 3)$ *distribution), we have* $p_1 = 2$ *and* $p_2 = 1$ *for each row, then* $p_2 - p_1 - 1 = 0$*. Hence a quasi-normal mapping* R *is always a normal one.*

A frame $(\{e_a\}_{a\in D}, \{f_a\}_{a\in D}$ of $T_\lambda T^*M$ is called *Darboux* or *symplectic* if they satisfies

$$\sigma(e_a, e_b) = \sigma(f_a, f_b) = \sigma(f_a, e_b) - \delta_{a,b} = 0.$$

Definition 3.2.9. The moving Darboux frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ is called the normal(quasi-normal) moving frame of a monotonically non-decreasing curve J_λ with the Young diagram D, if

$$J_\lambda(t) = \mathrm{span}\{E_a(t)\}_{a\in\Delta}$$

for any t and there exists an one-parametric family of normal(quasi-normal) mappings $R_t : \Delta \times \Delta \mapsto \mathrm{Mat}$ such that the moving frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ satisfies the following structural equation:

$$\begin{cases} E_a'(t) = E_{l(a)}(t) & \text{if } a \in \Delta\backslash\mathcal{F}_1 \\ E_a'(t) = F_a(t) & \text{if } a \in \mathcal{F}_1 \\ F_a'(t) = \sum_{b\in\Delta} E_b(t)R_t(a,b) - F_{r(a)}(t) & \text{if } a \in \Delta\backslash\mathcal{S} \\ F_a'(t) = \sum_{b\in\Delta} E_b(t)R_t(a,b) & \text{if } a \in \mathcal{S}. \end{cases}$$

where $\mathcal{F}_1$ is the first column of the diagram Δ, $\mathcal{S}$ is the set of special superboxes, and l, r are the left and right shifts respectively.

It is stated in [21] that given a monotonically non-decreasing curve J_λ one can uniquely determine a unique normal moving frame.

Theorem 3.2.10 ([21]). *For any monotonically non-decreasing curve $\Lambda(t)$ with the Young diagram D in the Lagrange Grassmannian there exists a normal moving frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$. A moving frame*

$$(\{\tilde{E}_a(t)\}_{a\in\Delta}, \{\tilde{F}_a(t)\}_{a\in\Delta})$$

is a normal moving frame of the curve $J_\lambda(t)$ if and only if for any $1 \leq i \leq d$ there exists a constant orthogonal matrix U_i of size $r_i \times r_i$ such that for all t

$$\tilde{E}_a(t) = E_a(t)U_i, \quad \tilde{F}_a(t) = F_a(t)U_i, \quad \forall a \text{ in ith level.}$$

In light of Theorem 3.2.10, the rest of the section will implement the strategy carried out in [21] and find such moving frame of the $(4, 6)$ distribution D, and from there, by using the fact that J_λ is separable, we eventually arrive at the conclusion of (3.2.3).

The next lemma gives us the motivation of the calculation in the following sections:

Lemma 3.2.11. *Assume the Jacobi curve J_λ defined in* (3.5) *and* (3.6) *is a product of two smaller curves, i.e. $J_\lambda = J_{1\lambda} \oplus J_{2\lambda}$. Then for any normal moving frame of $J_\lambda(t)$, the corresponding curvature map $R_t : \Delta \times \Delta \mapsto \mathrm{Mat}$ must vanish at (a, c), i.e.*

$$R_t(a, c) = 0.$$

Proof. As a consequence of the separation of the Jacobi curve where $J_\lambda = J_{1\lambda} \oplus J_{2\lambda}$, the Young diagram Δ can split into 2 smaller diagrams Δ_1 and Δ_2 associated with $J_{1\lambda}$ and $J_{2\lambda}$ respectively. Δ_1 and Δ_2 have the exactly same shape as Δ shown in Figure 3.1 except each box represents a single vector instead of a tuple of vectors. Due to Theorem 3.2.10, there exists a normal moving frame on $J_{1\lambda}$. Further, denote by R_{1t} the associated curvature map with the frame. As discussed in Remark 3.2.6 and 3.2.8, R_{1t} at (a, c) takes value in the space of all 1×1 skew-symmetric matrices, so $R_{1t}(a, c) = 0$ as 1×1 skew-symmetric is a zero matrix. Since the Young diagram is decomposable, the value of $R_t(a, c)$ is a block-diagonal 2×2 skew-symmetric matrix, and each block is a 1×1 zero matrix, in other words, $R_t(a, c) = 0$. □

Lemma 3.2.11 put a very strong constraint on the curvature map in the case of $(4, 6)$ distribution. In the next few pages, we perform a series of calculation to evaluate $R_t(a, c)$ and examine what it could bring us with the condition that $R_t(a, c) = 0$.

3.2.4 Auxiliary lemmas on structural functions

In this section, we carried out some technical results based on a direct calculation for $(4, 6)$-distributions.

Lemma 3.2.12. *If the $(4, 6)$ distribution admits a decomposable Tanaka symbol $\mathfrak{m}$ at $q \in M$ consisting of two 3-dimensional Heisenberg algebras $\{\eta_i\}_{i=1,2}$, i.e.*

$$\mathfrak{m} = \eta_1 \oplus \eta_2,$$

then the structural function c_{ij}^k's satisfy

$$\begin{cases} c_{12}^k = \delta_{k,5} & \textit{for } 1 \leq k \leq 6 \\ c_{34}^k = \delta_{k,6} & \textit{for } 1 \leq k \leq 6 \\ c_{13}^k = c_{23}^k = 0 & \textit{for } k = 5, 6 \\ c_{14}^k = c_{24}^k = 0 & \textit{for } k = 5, 6 \end{cases},$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

Proof. Since the Tanaka symbol is a direct sum of two Heisenberg algebras, WLOG, up to a re-index, we may assume X_1, X_2 generate η_1 while X_3, X_4 generate η_2 and further $X_5 = [X_1, X_2]$, $X_6 = [X_3, X_4]$, which leads to

$$\begin{cases} c_{12}^k = \delta_{k,5} & \text{for } 1 \leq k \leq 6 \\ c_{34}^k = \delta_{k,6} & \text{for } 1 \leq k \leq 6 \end{cases}.$$

From the decomposition of $\mathfrak{m} = \eta_1 \oplus \eta_2$, we know that

$$[X_i, X_j] = 0 \in \mathfrak{m}^{-1}(q),$$

for $i \in \{1, 2\}$, $j \in \{3, 4\}$, in other words, $[X_i, X_j] \in \mathfrak{m}^0 = \operatorname{span}(X_1, \dots, X_4)$, and it implies that

$$\begin{cases} c_{13}^k = c_{23}^k = 0 & \text{for } k = 5, 6 \\ c_{14}^k = c_{24}^k = 0 & \text{for } k = 5, 6 \end{cases}$$

□

Before getting into the normal frame, the following observations are significantly helpful to simplify the calculation.

Lemma 3.2.13 ([17], Proposition 6)**.** *If there exists a first integral compatible with the decomposition of the Tanaka symbol, then the structural coefficients of the sub-Riemannian manifold* (D, M) *satisfy*

$$\begin{aligned} &c_{13}^1 = c_{14}^1 = c_{13}^3 = c_{14}^4 = 0; \\ &c_{23}^2 = c_{24}^2 = c_{23}^3 = c_{24}^4 = 0; \\ &c_{23}^1 = -c_{13}^2; c_{24}^1 = -c_{14}^2; \\ &c_{14}^3 = -c_{13}^4; c_{24}^3 = -c_{23}^4; \end{aligned} \tag{3.15}$$

Proof. The existence of first integral compatible with the decomposition of such Tanaka symbol is equivalent to the fact that $u_1^2 + u_2^2$ and $u_3^2 + u_4^2$ are first integrals of the flow of normal extremals, i.e. $\vec{h}(u_1^2 + u_2^2) = \vec{h}(u_3^2 + u_4^2) = 0$, where as in (2.10)

$$\vec{h} = \sum_{i=1}^{4} u_i Y_i + \sum_{i=1}^{4} \sum_{j,k=1}^{6} c_{ij}^k u_i u_k \partial_{u_j},$$

hence

$$\vec{h}(u_1^2 + u_2^2) = \vec{h}(u_1^2) + \vec{h}(u_2^2) = 2u_1\vec{h}(u_1) + 2u_2\vec{h}(u_2) = 0, \tag{3.16}$$

similarly,

$$2u_3\vec{h}(u_3) + 2u_4\vec{h}(u_4) = 0. \tag{3.17}$$

As a special case to (2.28) when $m = 4$, $n = 6$, we have

$$\vec{h}(u_j) = \sum_{k=1}^{6}\sum_{i=1}^{4} u_i u_k c_{ij}^k, \text{ for } j \in [1:4]$$

By comparing the coefficients of $u_i u_j u_k$ in the (3.16) and (3.17) with 0, where $1 \leq i, j, k \leq 4$, it follows immediately that (3.15) holds. □

Lemma 3.2.14. *If the $(4, 6)$ distribution admits a decomposable Tanaka symbol at $q \in M$ consisting of two 3-dimensional Heisenberg algebras and a first integral compatible with the decomposition of such Tanaka symbol, then the following structural coefficients satisfy*

$$c_{16}^5 = c_{26}^5 = c_{16}^6 = c_{26}^6 = 0,$$

$$c_{35}^5 = c_{45}^5 = c_{35}^6 = c_{45}^6 = 0,$$

and further

$$c_{56}^6 = c_{56}^5 = 0.$$

Proof. A direct application of Jacobi identity and Lemma 3.2.12 gives

$$\begin{aligned}
[X_1, X_6] &= [X_1, [X_3, X_4]] \\
&= [[X_1, X_3], X_4] + [X_3, [X_1, X_4]] \\
&= \left[\sum_{k=1}^{4} c_{13}^k X_k, X_4\right] + \left[X_3, \sum_{k=1}^{4} c_{14}^k X_k\right] \\
&= \sum_{k=1}^{4} \Big(X_3(c_{14}^k) X_k + c_{14}^k [X_3, X_k] - c_{13}^k [X_4, X_k] - X_4(c_{13}^k) Y_k\Big) \\
&= (c_{13}^2 c_{24}^1 - c_{14}^2 c_{23}^1) X_1 + \Big(X_3(c_{14}^2) - X_4(c_{13}^2)\Big) X_2 \\
&\quad + \Big(X_3(c_{14}^3) + c_{13}^2 c_{24}^3\Big) X_3 + \Big(-X_4(c_{13}^4) - c_{14}^2 c_{23}^4\Big) X_4
\end{aligned}$$

Then we have $c_{16}^5 = c_{16}^6 = 0$ immediately and recall that from Lemma 3.2.13

$$c_{24}^1 = -c_{14}^2 \text{ and } c_{23}^1 = -c_{13}^2,$$

and hence $c_{16}^1 = c_{13}^2 c_{24}^1 - c_{14}^2 c_{23}^1 = 0$. In the same way

$$\begin{aligned}
[X_2, X_6] = &\Big(X_3(c_{24}^1) - X_4(c_{23}^1)\Big) X_1 + \Big(c_{23}^1 c_{14}^2 - c_{24}^1 c_{13}^2\Big) X_2 \\
&+ \Big(X_3(c_{24}^3) + c_{23}^1 c_{14}^3\Big) X_3 + \Big(-X_4(c_{23}^4) - c_{24}^1 c_{13}^4\Big) X_4,
\end{aligned}$$

then

$$\begin{aligned}
c_{26}^5 &= c_{26}^6 = 0 \\
c_{26}^2 &= c_{23}^1 c_{14}^2 - c_{24}^1 c_{13}^2 = 0.
\end{aligned}$$

If we repeat similar analysis on $[X_3, X_5]$ and $[X_4, X_5]$, it's not hard to find

$$c_{35}^5 = c_{45}^5 = c_{35}^6 = c_{45}^6 = 0.$$

For $[X_5, X_6]$, we write it as

$$\begin{aligned}
[X_5, X_6] &= [[X_1, X_2], X_6] \\
&= [[X_1, X_6], X_2] + [X_1, [X_2, X_6]] \\
&= \left[\sum_{k=1}^{4} c_{16}^k X_k, X_2\right] + \left[X_1, \sum_{k=1}^{4} c_{26}^k X_k\right] \\
&= [c_{16}^1 X_1, X_2] + [X_1, c_{26}^2 X_2] \mod \operatorname{span}(X_1, X_2, X_3, X_4) \\
&= 0 \mod \operatorname{span}(X_1, X_2, X_3, X_4),
\end{aligned}$$

Since $[X_5, X_6]$ does not have X_5, X_6 components, it concludes that

$$c_{56}^6 = c_{56}^5 = 0.$$

□

3.2.5 The flag associated with Jacobi curves

Now we perform the calculations of the normal moving frame for Jacobi curves of extremals of the considered sub-Riemannian structures on $(4, 6)$-distributions in terms of the moving frame $(Y_1, \ldots, Y_6, \partial u_1, \ldots, \partial u_6)$. As in (2.10) the sub-Riemannian Hamiltonian vector field on $T(T^*M)$ is

$$\vec{h} = \sum_{i=1}^{4} u_i Y_i + \sum_{i=1}^{4} \sum_{j,k=1}^{6} c_{ij}^k u_i u_k \partial u_j.$$

Furthermore, if $\omega_1, \ldots, \omega_6$ constitute the coframe dual to the frame $Y_1, \ldots, Y_6$, then canonical symplectic form σ on T^*M takes the form

$$\sigma = -\sum_{k=1}^{6} du_k \wedge \omega_k + \sum_{k=1}^{6} \sum_{1 \leq i < j \leq 6} c_{ij}^k \omega_i \wedge \omega_j.$$

Lemma 3.2.15. *In the frame of $(Y_1, \ldots, Y_6, \partial u_1, \ldots, \partial u_6)$, the extensions of the Jacobi curve take* *the following form:*

$$J_\lambda^{(k)} = J_\lambda^{(k-1)} + \operatorname{span}\left\{(\mathrm{ad}\vec{h})^{k-1}Y_1(\lambda), \ldots, (\mathrm{ad}\vec{h})^{k-1}Y_6(\lambda)\right\},$$

where $J_\lambda^{(0)} = \mathcal{V}_\lambda$.

Lemma 3.2.15 allows us to calculate the expression of the $i-$th extension $J_\lambda^{(k)}$ by taking Lie bracket with $\vec{h}$ for $k-1$ times and adding up to $J_\lambda^{(k-1)}$ in an iterative way.

Given a Jacobi curve $J_\lambda(t)$ in $T(T^*M)$, we will construct a monotonic sequence of subspaces of $J_\lambda(\cdot)$ in addition to the extensions $J_\lambda^{(i)}$. For this, let $J_{\lambda(0)}(t) = J_\lambda(t)$ and recursively

$$\begin{aligned} J_{\lambda(i)}(t) = \Big\{ & v \in J_{\lambda(i-1)} : \exists l \in \Theta(J_{\lambda(i-1)}) \\ & \text{with } l(\lambda) = v \text{ such that } l'(t) \in J_{\lambda(i-1)}(t)\Big\}, \end{aligned}$$

where $\Theta(J_{\lambda(i)})$ is the set of all smooth curves $l(t)$ in $T(T^*M)$ such that $l(t) \in J_{\lambda(i-1)}(t)$ for any t. The subspaces $J_{\lambda(i)}(t)$ are called *the ith contraction of the curve* J_λ at a point t.

For $J_{\lambda(1)}(t)$, set $E_c(t)$ as a tuple of vectors $E_c(t) := (E_c^1(t), E_c^2(t))$ with

$$E_c^1(0) = \frac{\partial u_5}{\sqrt{u_1^2 + u_2^2}}$$
$$E_c^2(0) = \frac{\partial u_6}{\sqrt{u_3^2 + u_4^2}},$$

where u_i's are defined in (2.9), then (E_c^1, E_c^2) forms a basis of $J_{\lambda(1)}(0)$. For the simplicity of notations, we denote $E_c(0)$ by E_c for short to avoid multiple zeros in equations and apply this notation to all vectors $\{E_i, F_i\}_{i\in\Delta}$ unless specified. In the second row of the Young diagram, set

$$E_b(t) := (E_b^1, E_b^2)(t),$$

with

$$E_b^1 = \sum_{i=1}^{6} u_i \partial u_i$$

$$E_b^2 = u_1\partial u_1 + u_2\partial u_2 - u_3\partial u_3 - u_4\partial u_4 + u_5\partial u_5 - u_6\partial u_6.$$

Then (E_b^1, E_b^2) spans a subspace complement to the subspace $(J_{\lambda(1)})^{(1)}(0)$ in $J_\lambda(0)$, i.e.

$$J_\lambda(0) = (J_{\lambda(1)})^{(1)}(0) \oplus \operatorname{span}\{E_b^1, E_b^2\}.$$

Lemma 3.2.16. *The tuple $E_c = (E_c^1, E_c^2)$ forms a canonical basis of $J_{\lambda(1)}(0)$, and $E_b = (E_b^1, E_b^2)$ generates the canonical complement of $(J_{\lambda(1)})^{(1)}(0)$ in $J_\lambda(0)$.*

Proof. As a consequence of Lemma 5 in [21], tuples of vectors $E_{\sigma_i}(t)$ constitute bases of the canonical complement of $(J_{\lambda(p_i)})^{(1)}(t)$ in $J_{\lambda(p_i-1)}(t)$ if and only if

$$\sigma\left(E_{\sigma_i}^{(p_i-1)}(t), E_{\sigma_j}^{(p_j-1+k)}(t)\right) = 0,$$

for $1 \le j < i \le d$ and $1 \le k \le p_j - p_i + 1$.

Indeed, by substituting E_c and E_b into E_{σ_1} and E_{σ_2} respectively, and taking account of the fact that $u_1^2 + u_2^2$ and $u_3^2 + u_4^2$ are first integrals of the flow of normal extremals as mentioned in Remark 3.2.4, we obtain

$$\sigma\left(E_b, E_c^{(k)}\right) = 0,$$

for $k = 1, 2$. Therefore, E_c and E_b are the canonical bases of $J_\lambda(0)$. □

As in (1.8), h is the sub-Riemannian Hamiltonian on $T_\lambda(T^*M)$ and $\operatorname{ad}\vec{h}$ acts on a tuple of vectors component-wise, i.e. $\operatorname{ad}\vec{h}(E_c) = (\operatorname{ad}\vec{h}(E_c^1), \operatorname{ad}\vec{h}(E_c^2))$.

In this way, set

$$E_a(t) = (E_a^1, E_a^2)(t) := \operatorname{ad}\vec{h}(E_c)(t) = (\operatorname{ad}\vec{h}(E_c^1), \operatorname{ad}\vec{h}(E_c^2))(t)$$

with

$$E_a^1 = \frac{-1}{\sqrt{u_1^2+u_2^2}} \left(-u_2 \partial u_1 + u_1 \partial u_2 + \sum_{i=1}^{4} u_i c_{i,5}^5 \partial u_5 + \sum_{i=1}^{4} u_i c_{i,6}^5 \partial u_6 \right)$$
$$E_a^2 = \frac{-1}{\sqrt{u_3^2+u_4^2}} \left(-u_4 \partial u_3 + u_3 \partial u_4 + \sum_{i=1}^{4} u_i c_{i,5}^6 \partial u_5 + \sum_{i=1}^{4} u_i c_{i,6}^6 \partial u_6 \right)$$

It is obvious that the span of the vectors in the tuples E_a, E_b and E_c is equal to $\mathcal{V}_\lambda$, hence $\dim J_\lambda(0) = 6$ in (3.6).

3.2.6 Computing the normal moving frames of Jacobi curves

In this subsection, we will fill the Young diagram Δ with the canonical moving frame of the curve J_λ. Denote by $V_1(t)$ and $V_2(t)$ the span of vectors in $E_c(t)$ and $E_b(t)$ respectively. It is known that the symplectic form σ induces a canonical quadratic form $Q_{i,\tau}$ on $V_i(\tau)$ by

$$Q_{i,\tau} = \sigma\left(\epsilon^{(p_i)}(\tau), \epsilon^{(p_i-1)}(\tau)\right).$$

The quadratic forms $Q_{i,\tau}$ are not degenerated, positive definite, and hence they also induce the *canonical Euclidean structure* on $V_i(\tau)$.

For any $i \in \{1,2\}$, let $\mathcal{B}_i$ be a fiber bundle over the curve $J_\lambda(t)$ such that the fiber of $\mathcal{B}_i$ over the point $J_\lambda(t)$ consists of all orthonormal bases of the space $V_i(\tau)$ with respect to the canonical Euclidean structure on $V_i(\tau)$.

Proposition 3.2.17 (Proposition 1. [21])**.** *Each bundle $\mathcal{B}_i$ is endowed with the canonical principal connection uniquely characterized by the following condition: the section $E_{\sigma_i}(t)$ of $\mathcal{B}_i$ is horizontal w.r.t. this connection if and only if $\text{span}\{E_{\sigma_i}^{(p_i)}(t)\}$ are isotropic subspaces of J_λ for any t. Given any two horizontal sections $E_{\sigma_i}(t)$ and $\tilde{E}_{\sigma_i}(t)$ of $\mathcal{B}_i$ there exists a constant orthogonal matrix U_i such that*

$$\tilde{E}_{\sigma_i}(t) = E_{\sigma_i}(t) U_i$$

Proposition 3.2.17 leads to an important observation of the curvature maps related to any normal moving frames of the curve J_λ.

Define $F_b(t) := \mathrm{ad}\vec{h}(E_b)(t) = E_b^{(1)}(t)$ and then a quick calculation verifies that

$$\sigma(F_b, F_b) = \sigma(E_b^{(1)}, E_b^{(1)}) = 0,$$

hence V_2 is an isotropic subspace.

For the sake of $\sigma(E_c^{(2)}, E_c^{(2)})$, we follow the construction of the curve J_λ, and another calculation gives

$$F_a(t) := \mathrm{ad}\vec{h}(E_a)(t) = E_a^{(1)}(t).$$

with

$$\begin{aligned} F_a^1 &= -\frac{u_2}{\sqrt{u_1^2+u_2^2}}Y_1 + \frac{u_1}{\sqrt{u_1^2+u_2^2}}Y_2 \bmod \mathcal{V}_\lambda \\ F_a^2 &= -\frac{u_4}{\sqrt{u_3^2+u_4^2}}Y_3 + \frac{u_3}{\sqrt{u_3^2+u_4^2}}Y_4 \bmod \mathcal{V}_\lambda. \end{aligned}$$

However, V_1 is not isotropic since

$$\begin{aligned} \sigma(E_2^{(c)}, E_2^{(c)}) &= \sigma(F_a, F_a) \\ &= \begin{pmatrix} 0 & \sigma(F_a^1, F_a^2) \\ -\sigma(F_a^1, F_a^2) & 0 \end{pmatrix} \\ &\neq 0, \end{aligned}$$

where

$$\sigma(F_a^1, F_a^2) = \frac{(u_1^2+u_2^2)(u_1 c_{15}^6 + u_2 c_{25}^6) - (u_3^2+u_4^2)\,(u_3 c_{36}^5 + u_4 c_{46}^5)}{\sqrt{(u_1^2+u_2^2)(u_3^2+u_4^2)}}.$$

Thus we turn to find a new frame $\tilde{E}_c$ such that

$$\tilde{E}_c(t) = E_c(t) \cdot U(t) \tag{3.18}$$

for some transition matrix $U(t)$ with $U(0) = \mathrm{Id}$ and $\mathrm{span}\{\tilde{E}_c^{(2)}\}$ forms an isotropic subspace. Notice that

$$\sigma\left(E_c^{(2)}(t), E_c^{(1)}(t)\right) = \mathrm{Id},$$

so it not hard to get

$$\sigma\left(\tilde{E}_c^{(2)}(t), \tilde{E}_c^{(2)}(t)\right) = U(t)\left(4 \cdot U'(t) + \sigma(E_c^{(2)}, E_c^{(2)}) \cdot U(t)\right).$$

It implies that the subspace $\mathrm{span}\{\tilde{E}_c^{(2)}(t)\}$ being isotropic is equivalent to

$$4 \cdot U'(t) + \sigma(E_c^{(2)}, E_c^{(2)}) \cdot U(t) = 0,$$

which is formula (3.36) in [21].

In general, it is difficult to recover $U(t)$ for all $t \geq 0$, but it seems more feasible to just focus on $\tilde{E}_c^{(k)}$ at $t = 0$ for different $k \in \{1, 2\}$.

It follows that

$$\begin{aligned} U'(0) &= -\frac{1}{4}\sigma(E_c^{(2)}, E_c^{(2)})(0) = -\frac{1}{4}\sigma(F_a, F_a)(0) \\ &= \begin{pmatrix} 0 & \sigma(F_a^1, F_a^2)(0) \\ -\sigma(F_a^1, F_a^2)(0) & 0 \end{pmatrix}, \end{aligned} \tag{3.19}$$

Using the chain rule, we are able to recover the remaining horizontal moving frame in the first row of figure (3.1):

$$\begin{aligned} \tilde{E}_a := \left(\tilde{E}_a^1, \tilde{E}_a^2\right) &= (E_c \cdot U)'(0) \\ &= E_c^{(1)} \cdot U(0) + E_c \cdot U'(0) \\ &= E_a - \frac{1}{4}E_a \cdot \sigma(F_a, F_a). \end{aligned}$$

Thus

$$\tilde{E}_a^1 = \frac{1}{\sqrt{u_1^2+u_2^2}}\,(u_2\partial u_1 - u_1\partial u_2) \text{ mod span}(\partial u_5, \partial u_6),$$

$$\tilde{E}_a^2 = \frac{1}{\sqrt{u_3^2+u_4^2}}(u_4\partial u_3 - u_3\partial u_4) \text{ mod span}(\partial u_5, \partial u_6).$$

By taking derivative on (3.19), we get

$$4\cdot U''(0) + \left(\sigma(E_c^{(2)}, E_c^{(2)})\right)'(0) + \sigma(E_c^{(2)}, E_c^{(2)})\cdot U'(0) = 0.$$

So

$$\begin{aligned} U''(0) &= -\frac{1}{4}\left(\sigma(E_c^{(3)}, E_c^{(2)}) - \sigma(E_c^{(3)}, E_c^{(2)})^T + \sigma(E_c^{(2)}, E_c^{(2)})\cdot U'\right)(0) \\ &= -\frac{1}{4}\left(\sigma(F_c, F_a) - \sigma(F_c, F_a)^T + \sigma(F_a, F_a)\cdot U'\right)(0) \end{aligned}$$

To find $\tilde{F}_a(0)$, we take derivative twice on (3.18), and $\tilde{F}_a(0)$ is given by

$$\tilde{F}_a = 2\cdot\tilde{E}_a\cdot U'(0) + E_c\cdot U''(0) + F_a,$$

with

$$\begin{aligned} \tilde{F}_a^1 &= \frac{-u_2Y_1 + u_1Y_2}{\sqrt{u_1^2+u_2^2}} \text{ mod span}(\partial u_1, \dots, \partial u_6) \\ \tilde{F}_a^2 &= \frac{-u_4Y_3 + u_3Y_4}{\sqrt{u_3^2+u_4^2}} \text{ mod span}(\partial u_1, \dots, \partial u_6). \end{aligned}$$

For $U'''(t)$ at 0, we take the third derivative on (3.19), then

$$\begin{aligned} U'''(0) = &-\frac{1}{2}\left(\sigma(F_c, F_a) - \sigma(F_a, F_c)^T\right)\cdot U'(0) - \frac{1}{4}\sigma(F_a, F_a)\cdot U''(0) \\ &-\frac{1}{4}\left(\sigma(E_c^{(4)}, E_c^{(2)}) - \sigma(E_c^{(4)}, E_c^{(2)})^T + 2\sigma(E_c^{(3)}, E_c^{(3)})\right). \end{aligned}$$

Set $F_c(t) := \mathrm{ad}\vec{h}(F_a)(t) = F_a^{(1)}(t) = E_a^{(2)}(t) = E_c^{(3)}(t)$, then $\tilde{F}_c$ is given by the third derivative of

(3.18) as

$$\tilde{F}_c = 3E_a \cdot U''(0) + 3F_a \cdot U'(0) + E_c \cdot U'''(0) + F_c,$$

with

$$\tilde{F}_c^1 = \left(\frac{u_1^2}{\sqrt{u_1^2+u_2^2}} + \frac{u_2^2}{\sqrt{u_1^2+u_2^2}}\right) Y_5 \text{ mod } \operatorname{span}(Y_1,\dots,Y_4,\partial u_1,\dots,\partial u_6)$$
$$\tilde{F}_c^2 = \left(\frac{u_3^2}{\sqrt{u_3^2+u_4^2}} + \frac{u_4^2}{\sqrt{u_3^2+u_4^2}}\right) Y_6 \text{ mod } \operatorname{span}(Y_1,\dots,Y_4,\partial u_1,\dots,\partial u_6).$$

Remark 3.2.18. Note that

$$J_\lambda^{(1)}(0) = \operatorname{span}\{E_a^i, E_b^i, E_c^i, F_a^i, F_b^i\}_{i=1,2} \text{ and } J_\lambda^{(2)}(0) = J_\lambda^{(1)}(0) + \operatorname{span}\{F_c^i\}_{i=1,2},$$

so the second and third dimensions in (3.6) are $\dim J_\lambda^{(1)} = 10$ and $\dim J_\lambda^{(2)} = 12$.

For the next few steps, we start with the tuple

$$\left\{\tilde{E}_c, \tilde{E}_b, \tilde{E}_a, \tilde{F}_a, \tilde{F}_b\right\}$$

and complete it into a quasi-normal moving frame by adding a vector $\hat{F}_c$ to it

$$\left\{\tilde{E}_c, \tilde{E}_b, \tilde{E}_a, \tilde{F}_a, \tilde{F}_b, \hat{F}_c\right\}, \tag{3.20}$$

which is equivalent to finding the solution to a system of equations of $\hat{F}_c$ that

$$\begin{cases} \sigma\left(\tilde{E}_i, \hat{F}_c\right) = \delta_{i,c}, \text{ for } i \in \Delta, \\ \sigma\left(\tilde{F}_i, \hat{F}_c\right) = \delta_{i,c}, \text{ for } i \neq c \end{cases}.$$

Thus, in the coordinate with respect to the basis $Y_1, \dots, Y_6$ of $T_\lambda(T^*M)/\mathcal{V}_\lambda$, $\hat{F}_c = (\hat{F}_c^1, \hat{F}_c^2)$ has a form

$$
\hat{F}_c^1 = \begin{pmatrix} \frac{u_1\left(u_5-u_2c_{15}^5\right)-u_2^2c_{25}^5}{\sqrt{u_1^2+u_2^2}} \\ \\ \frac{u_2\left(u_1c_{25}^5+u_5\right)+u_1^2c_{15}^5}{\sqrt{u_1^2+u_2^2}} \\ \\ \frac{u_4\left(u_3c_{36}^5+u_4c_{46}^5\right)}{4\sqrt{u_1^2+u_2^2}} - \frac{5\sqrt{u_1^2+u_2^2}u_4\left(u_1c_{15}^6+u_2c_{25}^6+u_3\right)}{4\left(u_3^2+u_4^2\right)} \\ \\ -\frac{u_3\left(u_3c_{36}^5+u_4c_{46}^5\right)}{4\sqrt{u_1^2+u_2^2}} + \frac{5\sqrt{u_1^2+u_2^2}u_3\left(u_1c_{15}^6+u_2c_{25}^6\right)}{4\left(u_3^2+u_4^2\right)} \\ \\ -\sqrt{u_1^2+u_2^2} \\ \\ 0 \end{pmatrix}
$$

and

$$
\hat{F}_c^2 = \begin{pmatrix} -\frac{5u_2\sqrt{u_3^2+u_4^2}\left(u_3c_{36}^5+u_4c_{46}^5\right)}{4\left(u_1^2+u_2^2\right)} + \frac{u_2\left(u_1c_{15}^6+u_2c_{25}^6\right)}{4\sqrt{u_3^2+u_4^2}} \\ \\ \frac{5u_1\sqrt{u_3^2+u_4^2}\left(u_3c_{36}^5+u_4c_{46}^5\right)}{4\left(u_1^2+u_2^2\right)} - \frac{u_1\left(u_1c_{15}^6+u_2c_{25}^6\right)}{4\sqrt{u_3^2+u_4^2}} \\ \\ \frac{u_3\left(u_6-u_4c_{36}^6\right)-u_4^2c_{46}^6}{\sqrt{u_3^2+u_4^2}} \\ \\ \frac{u_3^2c_{36}^6+u_4\left(u_3c_{46}^6+u_6\right)}{\sqrt{u_3^2+u_4^2}} \\ \\ 0 \\ \\ -\sqrt{u_3^2+u_4^2} \end{pmatrix}
$$

From the Proposition 2, there exists a corresponding mapping

$$
\tilde{R}_t : \Delta \times \Delta \mapsto \mathrm{Mat} \quad \text{for } t \geq 0,
$$

which satisfies the structural equations for this moving frame, i.e.

$$\begin{aligned}
\tilde{F}_a' &= \tilde{E}_a \tilde{R}_t(a,a) + \tilde{E}_b \tilde{R}_t(a,b) + \tilde{E}_c \tilde{R}_t(a,c) - \hat{F}_c \\
\hat{F}_c' &= \tilde{E}_a \tilde{R}_t(c,a) + \tilde{E}_b \tilde{R}_t(c,b) + \tilde{E}_c \tilde{R}_t(c,c) \\
\tilde{F}_b' &= \tilde{E}_a \tilde{R}_t(b,a) + \tilde{E}_b \tilde{R}_t(b,b) + \tilde{E}_c \tilde{R}_t(b,c)
\end{aligned} \tag{3.21}$$

Since we are interested in $\tilde{R}_t$ at $t = 0$, denote $\tilde{R} := \tilde{R}_0$ for short.

Proposition 3.2.19. *The mapping $\tilde{R}_t$ defined from above satisfies*

$$\tilde{R}(a,c) = \left(\tilde{R}(a,c)\right)^T$$

Proof. We will work with the normal moving frame of the Jacobi curve to prove this proposition and also utilize the fact that (3.20) is a good candidate for such normal moving frame.

Case 1. Assume the quasi-normal frame (3.20) is indeed a normal one. It follows from Lemma 3.2.11 that it must be a zero matrix. Hence

$$\tilde{R}(a,c) = \left(\tilde{R}(a,c)\right)^T = 0.$$

Case 2. Assume the quasi-normal frame (3.20) is not normal. From the argument done in Proposition 3 in [21], there is another vector F_c such that

$$\left\{\tilde{E}_c, \tilde{E}_b, \tilde{E}_a, \tilde{F}_a, \tilde{F}_b, F_c\right\} \tag{3.22}$$

forms a normal moving frame with a normal mapping R_t. As shown in Case 1, the normality of R_t gives

$$R_0(a,c) = \mathbf{0}. \tag{3.23}$$

As a consequence of Proposition 2 in [21], if (3.20) and (3.22) are both quasi-normal, there exists a symmetric mapping

$$\Gamma_t : \Delta \times \Delta \mapsto \mathrm{Mat}$$

such that the only non-trivial value is $\Gamma_t(c, c)$ defined as

$$\Gamma_t(c, c) = -\frac{1}{2}\left(\tilde{R}_t(a, c) + \tilde{R}_t(a, c)^T\right)$$

and Γ_t satisfies

$$R_t(a, c) = \tilde{R}_t(a, c) + \Gamma_t(c, c).$$

Together with (3.23) at $t = 0$, it yields

$$0 = \tilde{R}(a, c) - \frac{1}{2}\left(\tilde{R}(a, c) + \tilde{R}(a, c)^T\right),$$

which is equivalent to

$$\tilde{R}(a, c) = \left(\tilde{R}(a, c)\right)^T$$

□

With Proposition 3.21, we are now able to explicitly establish the constraint imposed by Lemma 3.2.11.

Lemma 3.2.20. *Let $\tilde{F}_c := \tilde{F}'_a$, then the matrix*

$$\sigma(\tilde{F}_c, \hat{F}_c) = \begin{pmatrix} \sigma(\tilde{F}_c^1, \hat{F}_c^1) & \sigma(\tilde{F}_c^1, \hat{F}_c^2) \\ \sigma(\tilde{F}_c^2, \hat{F}_c^1) & \sigma(\tilde{F}_c^2, \hat{F}_c^2) \end{pmatrix}$$

is symmetric, i.e.

$$\sigma(\tilde{F}_c^1, \hat{F}_c^2) - \sigma(\tilde{F}_c^2, \hat{F}_c^1) = 0. \tag{3.24}$$

Proof. Evaluate the sympletic form σ on both sides of the first line in the structural equation (3.21).

A direct calculation gives

$$\begin{aligned}\sigma(\tilde{F}'_a, \hat{F}_c) &= \sigma(\tilde{E}_a\tilde{R}(a,a) + \tilde{E}_b\tilde{R}(a,b) + \tilde{E}_c\tilde{R}(a,c) - \hat{F}_c, \hat{F}_c) \\ &= 0 + 0 + \sigma(\tilde{E}_c, \hat{F}_c)\tilde{R}(a,c) - 0 \\ &= \tilde{R}(a,c).\end{aligned}$$

Due to Proposition 3.21, $\tilde{R}(a,c)$ is symmetric, so does $\sigma(\tilde{F}_c, \hat{F}_c)$. Therefore the difference between skew-diagonal elements must be zero, i.e. $\sigma(\tilde{F}_c^1, \hat{F}_c^2) - \sigma(\tilde{F}_c^2, \hat{F}_c^1) = 0$. □

Given the condition established in Lemma 3.2.20, we proceed to finding the difference between $\sigma(\tilde{F}_c^1, \hat{F}_c^2)$ and $\sigma(\tilde{F}_c^2, \hat{F}_c^1)$ in the next section, to see what constraints will the structural coefficients c_{ij}^k satisfy.

3.2.7 Completing the proof of product structure

The next lemma shows us the explicit constraints on the structural coefficients c_{ij}^k given in Lemma 3.2.20.

Lemma 3.2.21. The following structural coefficients of the sub-Riemannian manifold (D, M) vanish:

$$c_{15}^6 = c_{46}^5 = c_{25}^6 = c_{36}^5 = 0.$$

Proof. As a result of Lemma 3.2.20, the left hand side of (3.24) is a rational expression with respect to u_i's with denominator $(u_1^2 + u_2^2)^{5/2}(u_3^2 + u_4^2)^{5/2}$. Next, we will compare the coefficients of the following monomials of the numerators of the left side of (3.24) with 0:

$$u_1^{11}u_3^2 \tag{3.25}$$

$$u_2^{11}u_3^2, \quad u_2^2u_4^{11}, \quad u_2^2u_3^{11} \tag{3.26}$$

In the coordinate of $(Y_1, \ldots, Y_6, \partial u_1, \ldots, \partial u_6)$,

$$\tilde{F}_c^1 = \begin{pmatrix} g_{c,1,1} \\ \vdots \\ g_{c,6,1} \\ f_{c,1,1} \\ \vdots \\ f_{c,6,1} \end{pmatrix} \quad \hat{F}_c^2 = \begin{pmatrix} t_{1,2} \\ \vdots \\ t_{6,2} \\ r_{1,2} \\ \vdots \\ r_{6,2} \end{pmatrix}.$$

In order to find the coefficient of (3.25), we analyze each term in $\sigma(\tilde{F}_c^1, \hat{F}_c^2)$, which is given by

$$\sigma(\tilde{F}_c^1, \hat{F}_c^2) = \sum_{j>i}^{6} \sum_{k=1}^{6} u_k c_{i,j}^k \left(t_{j,2} g_{c,i,1} - t_{i,2} g_{c,j,1}\right) + \sum_{i=1}^{6} \left(r_{i,2} g_{c,i,1} - t_{i,2} f_{c,i,1}\right).$$

The first summand on the left does not give $u_1^{11} u_3^2$ in its numerator. The numerator of $\sum_{i=1}^{6} r_i g_{c,i,1}$ gives

$$-\frac{9}{64} u_1^{11} u_3^2 (c_{15}^6)^3, \tag{3.27}$$

and meanwhile, $-\sum_{i=1}^{6} t_i f_{c,i,1}$ gives

$$\frac{7}{32} u_1^{11} u_3^2 (c_{15}^6)^3.$$

On the other side,

$$\tilde{F}_c^2 = \begin{pmatrix} g_{c,1,2} \\ \vdots \\ g_{c,6,2} \\ f_{c,1,2} \\ \vdots \\ f_{c,6,2} \end{pmatrix} \quad \hat{F}_c^1 = \begin{pmatrix} t_{1,1} \\ \vdots \\ t_{6,1} \\ r_{1,1} \\ \vdots \\ r_{6,1} \end{pmatrix},$$

$\sigma(\tilde{F}_c^2, \hat{F}_c^1)$ gives

$$\sigma(\tilde{F}_c^2, \hat{F}_c^1) = \sum_{j>i}^{6} \sum_{k=1}^{6} u_k c_{i,j}^k \left(t_{j,1} g_{c,i,2} - t_{i,1} g_{c,j,2}\right) + \sum_{i=1}^{6} \left(r_{i,1} g_{c,i,2} - t_{i,1} f_{c,i,2}\right)$$

The first term in the summand $\sum_{i=1}^{6} r_{i,1} g_{c,i,2}$ yields

$$\frac{7}{64} u_1^{11} u_3^2 (c_{15}^6)^3$$

while the second summand $-\sum_{i=1}^{6} t_{i,1} f_{c,i,2}$ yields

$$\frac{1}{32} u_1^{11} u_3^2 (c_{6,1,5})^3.$$

By summing up these two expressions, $-\sigma(\tilde{F}_c^2, \hat{F}_c^1)$ gives

$$\frac{9}{64} u_1^{11} u_3^2 (c_{15}^6)^3,$$

which cancels with (3.27). As a result, the coefficient of $u_1^{11} u_3^2$ is given by

$$\frac{7}{32} (c_{15}^6)^3.$$

Therefore, we just show that $c_{15}^6 = 0$.

By repeating the same process for monomials appearing in (3.26), one can show that

$$c_{15}^6 = c_{46}^5 = c_{25}^6 = c_{36}^5 = 0.$$

□

With Lemma 3.2.21, we just are one step away to showing the product structure of (D, M).

Lemma 3.2.22. *The following structural coefficients of the sub-Riemannian manifold (D, M) van-*

ish:

$$c_{15}^3 = 0, \quad c_{15}^4 = 0$$

$$c_{25}^3 = 0, \quad c_{25}^4 = 0$$

$$c_{36}^1 = 0, \quad c_{36}^2 = 0$$

$$c_{46}^1 = 0, \quad c_{46}^2 = 0$$

Proof. By Jacobi identity,

$$[X_1, [X_3, X_5]] = [[X_1, X_3], X_5] + [X_3, [X_1, X_5]]$$
$$\Longrightarrow \left[X_1, \sum_{k=1}^{4} c_{35}^k X_k\right] = \left[\sum_{k=1}^{4} c_{13}^k X_k, X_5\right] + \left[X_3, \sum_{k=1}^{6} c_{15}^k X_k\right]$$

The left hand side belongs to D and thereby the right hand side does not have X_5, X_6 components, as $X_5, X_6 \notin D$. On the right side, X_6 has a coefficient as $c_{23}^1 c_{15}^6 + c_{25}^4 + c_{25}^6 c_{36}^6 + X_3(c_{25}^6)$, so we must have

$$c_{23}^1 c_{15}^6 + c_{25}^4 + c_{25}^6 c_{36}^6 + X_3(c_{25}^6) = 0.$$

Repeat the same process on $[X_2, [X_3, X_5]]$, $[X_1, [X_4, X_5]]$, $[X_2, [X_4, X_5]]$, $[X_3, [X_1, X_6]]$, $[X_4, [X_1, X_6]]$, $[X_3, [X_2, X_6]]$ and $[X_4, [X_2, X_6]]$ we arrive at following constraints

$$c_{23}^1 c_{15}^6 + c_{25}^4 + c_{25}^6 c_{36}^6 + X_3(c_{25}^6) = 0$$

$$c_{13}^2 c_{25}^6 + c_{15}^4 + c_{15}^6 c_{36}^6 + X_3(c_{15}^6) = 0$$

$$c_{14}^2 c_{25}^6 - c_{15}^3 + c_{15}^6 c_{36}^6 + X_4(c_{15}^6) = 0$$

$$c_{24}^1 c_{15}^6 - c_{25}^3 + c_{25}^6 c_{36}^6 + X_4(c_{25}^6) = 0$$

$$-c_{13}^4 c_{46}^5 + c_{36}^2 + c_{36}^5 c_{15}^5 + X_1(c_{36}^5) = 0$$

$$-c_{14}^3 c_{36}^5 + c_{46}^2 + c_{46}^5 c_{15}^5 + X_1(c_{46}^5) = 0$$

$$-c_{23}^4 c_{46}^5 - c_{36}^1 + c_{36}^5 c_{25}^5 + X_2(c_{36}^5) = 0$$

$$-c_{24}^3 c_{36}^5 - c_{46}^1 + c_{46}^5 c_{25}^5 + X_2(c_{46}^5) = 0.$$

Taking into account that fact in Lemma (3.2.21), it turns out that

$$c_{15}^3, c_{15}^4, c_{25}^3, c_{25}^4, c_{36}^1, c_{36}^2, c_{46}^1, c_{46}^2$$

would vanish □

With auxiliary lemmas in section 3.2.4, Lemma 3.2.20 and Lemma 3.2.21 we verify that subspaces $\mathcal{V}_1 := \mathrm{span}(X_1, X_2, X_5)$ and $\mathcal{V}_2 := \mathrm{span}(X_3, X_4, X_6)$ are involutive. Analogy to section 2.3.5, one can construct a new basis $\tilde{X}_1, \ldots, \tilde{X}_6$, such that

$$[\tilde{X}_i, \tilde{X}_j] = 0, \text{ for } i \in \{1, 2\}, j \in \{3, 4\},$$

and

$$[\tilde{X}_1, \tilde{X}_2] = \tilde{X}_5,$$
$$[\tilde{X}_3, \tilde{X}_4] = \tilde{X}_6.$$

Thus the distribution D admits a product structure $D = D_1 \times D_2$. This completes the proof of Theorem 3.2.3.

As a consequence of the product structure, there exist sub-Riemannian metrics g_1 and g_2 on D_1 and D_2 and the metric g defined in (1.1) is affine equivalent to $(C_1\pi_1^* g_1 + C_2\pi_2^* g_2)$ but the latter is not constantly proportional to g if $C_1 \neq C_2$. In the end, the metric g is not affinely rigid.

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APPENDIX A

PROOF OF PROPOSITION 2.1.9

Assume by contradiction that $\mathfrak{m}$ is not ad-surjective. Let $m := \dim\, \mathfrak{m}_{-1}$ and $d := \dim\, \mathfrak{m}_{-1}$. We start with consideration for general d. Assume that

$$r := \max_{X \in \mathfrak{m}_{-1}} \operatorname{rank}(\mathrm{ad}X) \tag{A.1}$$

Then from non-ad-surjectivity assumption $r < d$. Take X_1 such that

$$\operatorname{rank}(\mathrm{ad}X_1) = r \tag{A.2}$$

Then the rank-nullity theorem implies that

$$\dim\, \ker(\mathrm{ad}X_1) = m - r$$

Obviously, $X_1 \in \ker(\mathrm{ad}X_1)$. Let us complete it to the basis $(X_1 \dots X_m)$ of $\mathfrak{m}_{-1}$ such that

$$\ker(\mathrm{ad}X_1) = \operatorname{span}\{X_1, X_{r+2}, \dots, X_m\}. \tag{A.3}$$

Let

$$Y_l := [X_1, X_{l+1}], l \in [1 : r].$$

By constrictions, $Y_1, \dots, Y_r$ are linearly independent, and

$$\operatorname{Im}(\mathrm{ad}X_1) = \operatorname{Im}(\mathrm{ad}X_1|_{\operatorname{span}\{X_2,\dots,X_{r+1}\}}) = \operatorname{span}\{Y_1, \dots, Y_r\}. \tag{A.4}$$

Since $\mathfrak{m}$ is step 2 and fundamental , there exist $i < j \in [2, m]$ such that

$$[X_i, X_j] \notin \mathrm{Im}(\mathrm{ad} X_1). \tag{A.5}$$

Set

$$Y_{r+1} := [X_i, X_j] \tag{A.6}$$

Lemma A.0.1. *The index j (and therefore also i) in (A.5) does not exceed $r+1$.*

Proof. Assume by contradiction that $j \geq r+2$. From maximality of r in (A.1), (A.2), and (A.4) it follows that for sufficiently small t

$$\mathrm{rank}(\mathrm{ad}(X_1 + tX_i|_{\mathrm{span}\{X_2,\dots,X_{r+1}\}}) = r$$

and the spaces $\mathrm{Im}(\mathrm{ad}(X_1 + tX_i|_{\mathrm{span}\{X_2,\dots,X_{r+1}\}})$ are sufficiently closed to $\mathrm{Im}(\mathrm{ad}X_1)$ so that

$$Y_{r+1} \notin \mathrm{Im}(\mathrm{ad}(X_1 + tX_i|_{\mathrm{span}\{X_2,\dots,X_{r+1}\}}). \tag{A.7}$$

On the other hand, from (A.3) and (A.6) it follows that

$$[X_1 + tX_i, X_j] = tY_{r+1},$$

which implies that $\mathrm{rank}(\mathrm{ad}(X_1 + tX_i) > r$ for sufficiently small $t \neq 0$, This contradicts the maximality of r in (A.1) and completes the proofof the lemma. □

In the proof of the previous lemma, based on (A.3) and (A.4) we actually have shown that

$$[\ker(\mathrm{ad}X_1), \mathfrak{m}_{-1}] \subset \mathrm{Im}(\mathrm{ad}X_1) \tag{A.8}$$

After permuting indices we can assume that $(i, j) = (2, 3)$, i.e. that

$$[X_2, X_3] \notin \mathrm{Im}(\mathrm{ad}X_1). \tag{A.9}$$

Now, given X and $\tilde{X}$ from $\mathfrak{m}_{-1}$, set

$$L_{X,\tilde{X}} := \ker(\mathrm{ad}X) \cap \ker(\mathrm{ad}\tilde{X})$$

Let

$$k := \min_{X,\tilde{X}\in\mathfrak{m}_{-1}} \dim L_{X,\tilde{X}} \tag{A.10}$$

By genericity of (A.2), (A.9) and (A.10) we can choose X_1, X_2, and X_3, maybe after small perturbation, such that (A.2), (A.9), and

$$\dim\Big(\ker(\mathrm{ad}X_1) \cap \ker(\mathrm{ad}X_2)\Big) = k \tag{A.11}$$

hold simultaneously.

Now, by item 1 of Proposition 2.1.9 $d \geq 3$. Therefore either $r = 1$ or $r = 2$. Consider these two cases separately.

Case 1: $r = 1$**.** By (A.3)

$$\ker(\mathrm{ad}X_1) = \mathrm{span}\{X_1, X_3, \ldots, X_m\}$$

and by this and (A.8)

$$[X_i, X_j] \in \mathrm{Im}(\mathrm{ad}X_1), \quad \forall i \in [2 : m], j \in [3, m]$$

or, equivalently, from fundamentality of $\mathfrak{m}$,

$$\mathfrak{m}_{-2} = \mathrm{Im}(\mathrm{ad}X_1),$$

so in fact $d = r$ $(= 1)$ and this case is done.

Case 2: $r = 2$**.** By the previous constructions,

$$[X_1, X_2] = Y_1, \quad [X_2, X_3] = Y_3, \tag{A.12}$$

where

$$Y_3 \notin \operatorname{Im}(\operatorname{ad}X_1) \tag{A.13}$$

as a particular case of (A.7) for $r = 2$. Then from (A.12) and (A.13) by maximality of $r = 2$ in (A.1),

$$\operatorname{Im}(\operatorname{ad}X_2) = \operatorname{span}\{Y_1, Y_3\}$$

From this, (A.4), and (A.8) it follows that

$$[X_2, X_i] \in \operatorname{Im}(\operatorname{ad}X_1) \cap \operatorname{Im}(\operatorname{ad}X_2) = \operatorname{span}\{Y_1\}, \quad \forall i \in [4 : m].$$

Since $[X_1, X_2] = Y_1$ for any $i \in [4 : m]$ one can replace X_i by

$$\tilde{X}_i \equiv X_i \mod \operatorname{span}\{X_1\} \tag{A.14}$$

such that $[X_2, \tilde{X}_i] = 0$, i.e.

$$\ker(\operatorname{ad}X_2) = \operatorname{span}\{X_2, \tilde{X}_4, \ldots \tilde{X}_m\}.$$

This together with (A.3) and (A.14) implies that

$$L_{X_1,X_2} = \operatorname{span}\{\tilde{X}_4, \ldots, \tilde{X}_m\}$$

Therefore , by (A.11)

$$k = m - 3.$$

The following Lemma will give a contradiction with item 3 of the assumptionsof Proposition 2.1.9 and therefore will complete the proof of it in the considered case of $r = 2$:

Lemma A.0.2. *The space L_{X,X_1} is the same for all $X \in \mathfrak{m}_{-1}$ for which $\dim L_{X,X_1} = m-3$ and so, the space L_{X,X_1} lies in the center of $\mathfrak{m}$* [1].

Proof. Assume by contradiction that there exist X_2 and X_3 such that $\dim L_{X_i,X_1} = m-3$, $i = 2, 3$ but

$$L_{X_2,X_1} \neq L_{X_3,X_1} \tag{A.15}$$

Obviously, X_1, X_2 and X_3 are linearly independent. By openness of condition (A.15) we can always assume that

$$\mathrm{ad}X_1(\mathrm{span}X_2) \neq \mathrm{ad}X_1(\mathrm{span}X_3) \tag{A.16}$$

We claim that

$$L_{X_2,X_3} = L_{X_2,X_1} \cap L_{X_3,X_1}. \tag{A.17}$$

Before proving (A.17), note that if it holds then by (A.15) it will follow that $\dim L_{X_2 m X_3} < m-3$ which will contradicts the minimality of $k = m-3$ in (A.10).

It remains to prove (A.17). First, it is clear that

$$L_{X_2,X_1} \cap L_{X_3,X_1} = \bigcap_{i=1}^{3} \ker(\mathrm{ad}X_i) \subset L_{X_2,X_3}. \tag{A.18}$$

On the other hand, from the dimension assumptions it follows that

$$\ker(\mathrm{ad}X_i) = \mathrm{span}\{X_i\} \oplus L_{X_i,X_1}, \quad i = 2, 3$$

[1] The latter conclusion follows from the fact that the set of such X is generic in $\mathfrak{m}_{-1}$.

So if $v \in L_{X_2,X_3}$ then

$$v \equiv \alpha_2 X_2 \mod \ker(\mathrm{ad}X_1) \equiv \alpha_3 X_3 \mod \ker(\mathrm{ad}X_1) \tag{A.19}$$

Note that (A.16) means that X_2 and X_3 are linearly independent modulo $\ker(\mathrm{ad}X_1)$. Hence, (A.19) implies that $\alpha_2 = \alpha_3 = 0$, i.e. $v \in \ker(\mathrm{ad}X_1)$. This implies that $v \in L_{X_2,X_1} \cap L_{X_3,X_1}$, i.e.

$$L_{X_2,X_3} \subset L_{X_2,X_1} \cap L_{X_3,X_1}.$$

This and inclusion (A.18) complete the proof of (A.17) and therefore of Lemma A.0.2. □

Finally note that in the case of $\dim \mathfrak{m}_{-2} = 4$, even if the assumptions 2 and 3 holds Proposition 2.1.9 is wrong. Here is the counterexample:

Example A.0.3. Let $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ be the step 2 graded Lie algebra such that

$$\mathfrak{m}_{-1} = \mathrm{span}\{X_1, \dots, X_5\}$$
$$\mathfrak{m}_{-2} = \mathrm{span}\{Y_1, \dots, Y_4\}$$

so that, up to skew-symmetricity, the following brackets of the chosen basis are the only nonzero ones:

$$[X_1, X_i] = Y_{i-1}, \;\; \forall i \in [2:4],$$
$$[X_2, X_3] = Y_4, [X_2, X_5] = \beta Y_3,$$
$$[X_3, X_5] = \delta Y_3, [X_4.X_5] = \lambda Y_3,$$

where β, δ, λ are nonzero constants. It can be checked by straightforward computations that here r, defined by (A.1), is equal to $3 < d = 4$ and that $\mathfrak{m}_{-1}$ meets the center trivially, i.e. it is indeed a counter-example.

Indeed, if $X \in \mathfrak{m}_{-1}$,

$$X = \sum_{i=1}^{5} C_i X_i$$

then the map $\mathrm{ad}X$ has the following matrix with respect to the bases $(X_1, \ldots, X_5)$ and $Y_1, \ldots, Y_4)$ of $\mathfrak{m}_{-1}$ and $\mathfrak{m}_{-2}$:

$$\mathrm{ad}X = \begin{pmatrix} -C_2 & -C_3 & -C_4 & 0 \\ C_1 & 0 & -\beta C_5 & C_3 \\ 0 & C_1 & \delta C_5 & C_2 \\ 0 & 0 & C_1 - \lambda C_3 & 0 \\ 0 & 0 & \beta C_2 + \delta C_3 + \lambda C_4 & 0 \end{pmatrix} \tag{A.20}$$

It is easy to check that the maximal rank of this matrix (as a function of C's) is equal to 3, which implies that $r = 3$. Also, this matrix is not equal to zero if $(C_1, \ldots C_5) \neq 0$, which means that $\mathfrak{m}_{-1}$ meets the center trivially.

Finally note that $\mathfrak{m}$ is not decomposable. Assuming the converse, i.e. that $\mathfrak{m} = \mathfrak{m}^1 \oplus \mathfrak{m}^2$ for some nonzero fundamental graded Lie algebra $\mathfrak{m}^1$ and $\mathfrak{m}^2$. Without loss of generality assume that

$$\dim\ \mathfrak{m}^1_{-1} \geq \dim\ \mathfrak{m}^2_{-1}. \tag{A.21}$$

Since $\mathfrak{m}_{-1}$ meets the center trivially, it is impossible that $\dim\ \mathfrak{m}^2_{-1} = 1$. Hence by (A.21) and the fact that $\dim \mathfrak{m}_{-1} = 5$, we have that $\dim\ \mathfrak{m}^2_{-1} = 2$ and the algebra $\mathfrak{m}_{-1}$ is nothing but the 3-dimensional Heisenberg algebra. Therefore for every nonzero $X \in \mathfrak{m}^2_{-1}$, the rank of $\mathrm{ad}X$ is equak to 1. However, it is straightforward to show that if the rank of the matrix (A.20) is not greater than 1, then $(C_1, \ldots C_5) = 0$, which leads to the contradiction. So $\mathfrak{m}$ is indecomposable.

An alternative, more conceptual way to prove indecomposability of $\mathfrak{m}$ is to observe that otherwise, each component in its decomposition will have -2 degree part of dimension not greater than 3 (but not equal to 0) and by Proposition 2.1.9 each component is ad-surjective. Then by Remark 2.1.8, $\mathfrak{m}$ is ad-surjective, which is not the case.